Chair of Automatic Control

Technische Universität München

Model Order Reduction of Linear Time-Varying Systems: Some straightforward approaches

MOR 4 MEMS 2015

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Motivation for Model Order Reduction

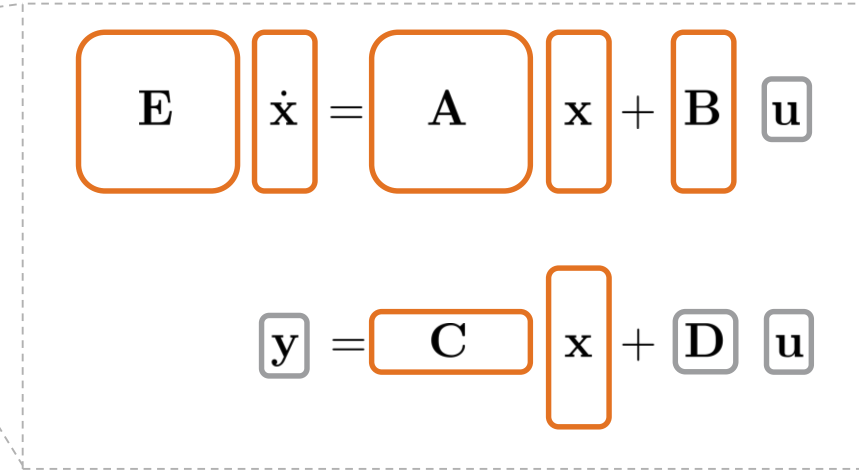
Linear time-invariant system in state-space representation

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t)$$

$\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$

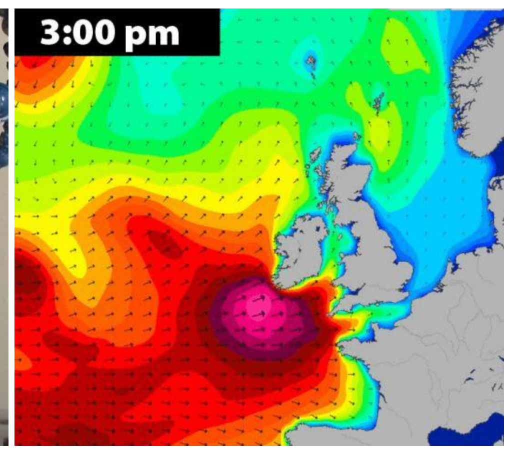
$\mathbf{u}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^q$

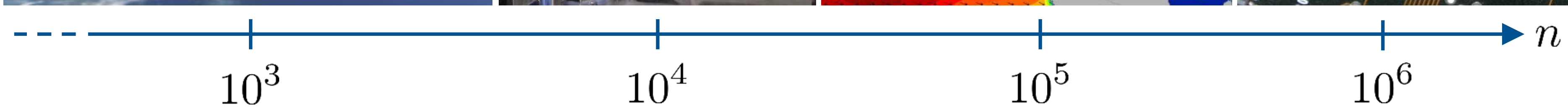
$m, q \ll n$



$\mathbf{x}(t) \in \mathbb{R}^n$

$\det(\mathbf{E}) \neq 0$





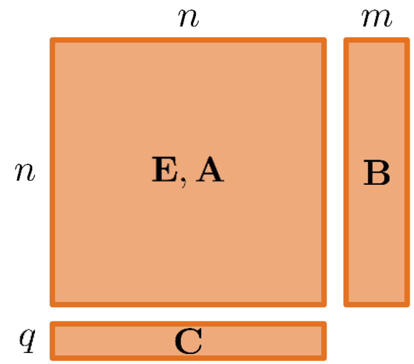
Model Order Reduction (MOR)

Linear time-invariant (LTI) system

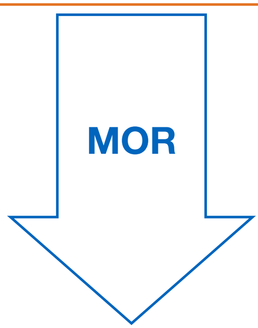
$$\mathbf{G}(s):\begin{cases}\mathbf{E}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)\end{cases}$$

$\mathbf{E},\ \mathbf{A}\in\mathbb{R}^{n\times n}$

$\mathbf{B}\in\mathbb{R}^{n\times m},\ \mathbf{C}\in\mathbb{R}^{q\times n}$



$r \ll n$



Projection

$\mathbf{V},\ \mathbf{W}\in\mathbb{R}^{n\times r}$

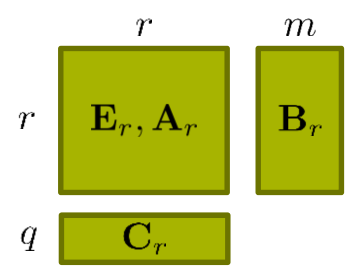
$\mathbf{E}_r=\mathbf{W}^T\,\mathbf{E}\,\mathbf{V},\ \mathbf{A}_r=\mathbf{W}^T\,\mathbf{A}\,\mathbf{V},\ \mathbf{B}_r=\mathbf{W}^T\,\mathbf{B},\ \mathbf{C}_r=\mathbf{C}\,\mathbf{V}$

Reduced order model (ROM)

$$\mathbf{G}_r(s):\begin{cases}\mathbf{E}_r\dot{\mathbf{x}}_r(t)=\mathbf{A}_r\mathbf{x}_r(t)+\mathbf{B}_r\mathbf{u}(t)\\ \mathbf{y}_r(t)=\mathbf{C}_r\mathbf{x}_r(t)\end{cases}$$

$\mathbf{E}_r,\ \mathbf{A}_r\in\mathbb{R}^{r\times r}$

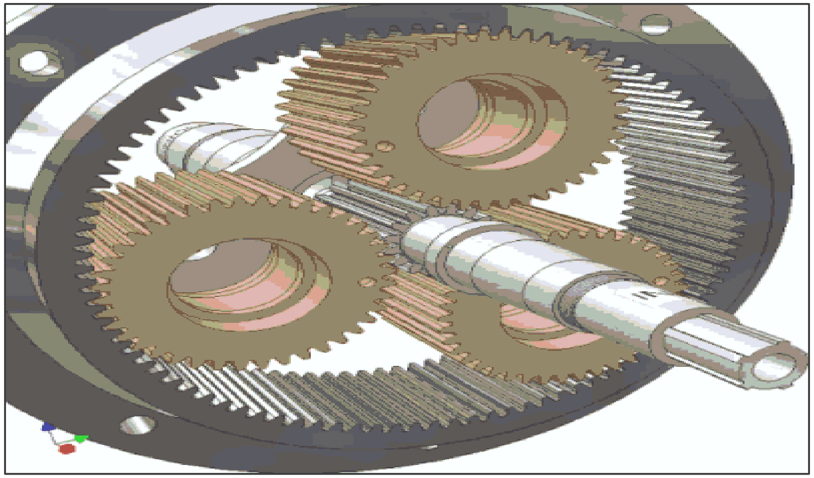
$\mathbf{B_r}\in\mathbb{R}^{r\times m},\ \mathbf{C}_r\in\mathbb{R}^{q\times r}$



Outline

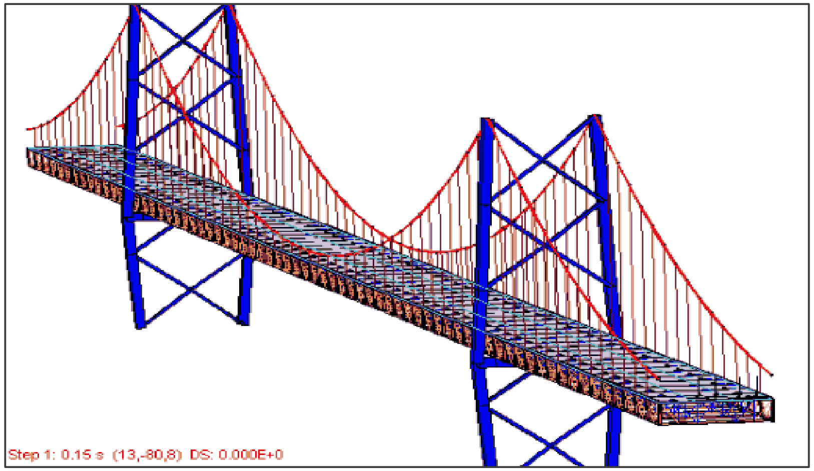
1. **Systems with Moving Loads**
   - Motivation & Examples
   - State-of-the-art: system representation and reduction
2. **Linear Time-Varying Model Order Reduction (LTV-MOR)**
   - Projection-based tMOR
   - Procedure
3. **Straightforward approaches for LTV-MOR**
   - Special cases: Moving Loads, Moving Sensors, Moving Loads + Sensors
   - Numerical example: Timoshenko beam with moving load / moving sensor
4. **Summary and Outlook**
   - Discussion

Systems with Moving Loads



gearing wheels

cable railways



bridge with moving vehicles

circular milling machine

Systems with Moving Loads

- **Applications:** structural dynamics, multibody systems, turning/milling processes
- Position of the load varies over time
- Moving load causes time-varying dynamic behaviour

Moving Loads

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$

$\mathbf{B}(t) \in \mathbb{R}^{n\times m}, \ \mathbf{C} \in \mathbb{R}^{q\times n}$

Moving Sensors

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$

$\mathbf{B} \in \mathbb{R}^{n\times m}, \ \mathbf{C}(t) \in \mathbb{R}^{q\times n}$

Linear time-varying (LTV) system

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

$\mathbf{E}(t), \mathbf{A}(t) \in \mathbb{R}^{n\times n}$

$\mathbf{B}(t) \in \mathbb{R}^{n\times m}, \ \mathbf{C}(t) \in \mathbb{R}^{q\times n}$

Reduction of Systems with Moving Loads

LTV System

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

Balanced Truncation for LTV systems
[Shokoohi '83, Sandberg '04]

- Solution of two Lyapunov-Differential Equations (LDE)
- high storage and computational effort

Two-step approach
[Stykel/Vasilyev '15]

I) Low-rank approximation of the input matrix
II) Application of LTI-MOR (BT, Krylov)

Some straightforward approaches
[Cruz/Lohmann]

Switched Linear System

$$\mathbf{E}_\alpha\dot{\mathbf{x}}(t) = \mathbf{A}_\alpha\mathbf{x}(t) + \mathbf{B}_\alpha\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_\alpha\mathbf{x}(t)$$

Switched Linear System + BT
[Lang et al. '14]

- Representation as switched linear system
- Application of BT for each subsystem

LPV System

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t)$$

Parametric LTI system + IRKA
[Lang et al. '14]

- Time-independ. parameter
- Concatenation of the local bases calculated by IRKA

Parametric LTI system + MatrInt
[Fischer '14, Fischer et al. '15]

- Time-independ. parameter
- Application of pMOR by Matrix Interpolation

LPV System + MatrInt
[Cruz/Geuss/Lohmann '15]

- Time-dependent parameter
- Adapted MatrInt with additional time-derivatives

Linear Time-Varying Model Order Reduction: tMOR

Linear time-varying (LTV) system

$$\mathbf{E}(t)\,\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{x}(t) \in \mathbb{R}^n$$
$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t)$$

$r \ll n$ **tMOR**

Approximation of the full state vector:

$$\mathbf{x}(t) = \mathbf{V}(t)\,\mathbf{x}_r(t) + \mathbf{e}(t),$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{V}}(t)\,\mathbf{x}_r(t) + \mathbf{V}(t)\,\dot{\mathbf{x}}_r(t) + \dot{\mathbf{e}}(t)$$

Petrov-Galerkin condition: $\mathbf{W}(t) \perp \boldsymbol{\epsilon}(t)$

$\mathbf{W}(t)^T \cdot \Big|$

$$\mathbf{E}(t)\,\mathbf{V}(t)\,\dot{\mathbf{x}}_r(t) = \Big(\mathbf{A}(t)\,\mathbf{V}(t) - \mathbf{E}(t)\,\dot{\mathbf{V}}(t)\Big)\,\mathbf{x}_r(t) + \mathbf{B}(t)\,\mathbf{u}(t) + \boldsymbol{\epsilon}(t)$$
$$\mathbf{y}_r = \mathbf{C}(t)\,\mathbf{V}(t)\,\mathbf{x}_r(t)$$

Linear Time-Varying Model Order Reduction: tMOR

Linear time-varying (LTV) system

$$\mathbf{E}(t)\,\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{x}(t) \in \mathbb{R}^n$$
$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t)$$

$r \ll n$ **tMOR**

Approximation of the full state vector:

$$\mathbf{x}(t) = \mathbf{V}(t)\,\mathbf{x}_r(t) + \mathbf{e}(t),$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{V}}(t)\,\mathbf{x}_r(t) + \mathbf{V}(t)\,\dot{\mathbf{x}}_r(t) + \dot{\mathbf{e}}(t)$$

Petrov-Galerkin condition: $\mathbf{W}(t) \perp \boldsymbol{\epsilon}(t)$

$$\overbrace{\mathbf{W}(t)^T\mathbf{E}(t)\mathbf{V}(t)}^{\mathbf{E}_r(t)}\,\dot{\mathbf{x}}_r(t) = \left(\overbrace{\mathbf{W}(t)^T\mathbf{A}(t)\mathbf{V}(t)}^{\mathbf{A}_r(t)} - \mathbf{W}(t)^T\mathbf{E}(t)\dot{\mathbf{V}}(t)\right)\mathbf{x}_r(t) + \overbrace{\mathbf{W}(t)^T\mathbf{B}(t)}^{\mathbf{B}_r(t)}\,\mathbf{u}(t)$$

$$\mathbf{y}_r(t) = \underbrace{\mathbf{C}(t)\mathbf{V}(t)}_{\mathbf{C}_r(t)}\,\mathbf{x}_r(t)$$

Linear Time-Varying Model Order Reduction: tMOR

Linear time-varying (LTV) system

$$\mathbf{E}(t)\,\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{x}(t) \in \mathbb{R}^n$$
$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t)$$

$r \ll n$ tMOR

Approximation of the full state vector:

$$\mathbf{x}(t) = \mathbf{V}(t)\,\mathbf{x}_r(t) + \mathbf{e}(t),$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{V}}(t)\,\mathbf{x}_r(t) + \mathbf{V}(t)\,\dot{\mathbf{x}}_r(t) + \dot{\mathbf{e}}(t)$$

Petrov-Galerkin condition: $\mathbf{W}(t) \perp \boldsymbol{\epsilon}(t)$

Linear time-varying reduced order model

$$\mathbf{E}_r(t)\,\dot{\mathbf{x}}_r(t) = \left(\mathbf{A}_r(t) - \mathbf{W}(t)^T\,\mathbf{E}(t)\,\dot{\mathbf{V}}(t)\right)\mathbf{x}_r(t) + \mathbf{B}_r(t)\,\mathbf{u}(t)$$
$$\mathbf{y}_r(t) = \mathbf{C}_r(t)\,\mathbf{x}_r(t)$$

$$\mathbf{E}_r(t) = \mathbf{W}(t)^T\,\mathbf{E}(t)\,\mathbf{V}(t), \qquad \mathbf{A}_r(t) = \mathbf{W}(t)^T\,\mathbf{A}(t)\,\mathbf{V}(t)$$
$$\mathbf{B}_r(t) = \mathbf{W}(t)^T\,\mathbf{B}(t), \qquad \mathbf{C}_r(t) = \mathbf{C}(t)\,\mathbf{V}(t)$$

Linear Time-Varying Model Order Reduction: tMOR

Linear time-varying (LTV) system

$$\mathbf{E}(t)\,\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{x}(t) \in \mathbb{R}^n$$
$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t)$$

$r \ll n$ **tMOR**

Approximation of the full state vector:

$$\mathbf{x}(t) = \mathbf{V}(t)\,\mathbf{x}_r(t) + \mathbf{e}(t),$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{V}}(t)\,\mathbf{x}_r(t) + \mathbf{V}(t)\,\dot{\mathbf{x}}_r(t) + \dot{\mathbf{e}}(t)$$

Petrov-Galerkin condition: $\mathbf{W}(t) \perp \boldsymbol{\epsilon}(t)$

Linear time-varying reduced order model

$$\mathbf{E}_r(t)\,\dot{\mathbf{x}}_r(t) = \left(\mathbf{A}_r(t) - \mathbf{W}(t)^T\,\mathbf{E}(t)\,\dot{\mathbf{V}}(t)\right)\mathbf{x}_r(t) + \mathbf{B}_r(t)\,\mathbf{u}(t)$$
$$\mathbf{y}_r(t) = \mathbf{C}_r(t)\,\mathbf{x}_r(t)$$
$$\mathbf{E}_r(t) = \mathbf{W}(t)^T\,\mathbf{E}(t)\,\mathbf{V}(t), \qquad \mathbf{A}_r(t) = \mathbf{W}(t)^T\,\mathbf{A}(t)\,\mathbf{V}(t)$$
$$\mathbf{B}_r(t) = \mathbf{W}(t)^T\,\mathbf{B}(t), \qquad \mathbf{C}_r(t) = \mathbf{C}(t)\,\mathbf{V}(t)$$

Problem: How to deal with the additional time-derivative term?

Straightforward approaches for LTV-MOR: Moving Loads

Moving Loads

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{E},\mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{x}(t) \in \mathbb{R}^n$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \qquad \mathbf{B}(t) \in \mathbb{R}^{n\times m},\ \mathbf{C} \in \mathbb{R}^{q\times n} \qquad \mathbf{u}(t) \in \mathbb{R}^m$$

Approach 1: Similar to **Two-step approach** [Stykel/Vasilyev '15]

Main idea:

1. Shift the time-variability of $\mathbf{B}(t)$ to the input variables: $\underset{(n,m)}{\mathbf{B}(t)} = \underset{(n,N)}{\mathbf{B}}\ \underset{(N,m)}{\tilde{\mathbf{B}}(t)}$

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\ \overbrace{\tilde{\mathbf{B}}(t)\,\mathbf{u}(t)}^{\tilde{\mathbf{u}}(t)}$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

N : finite elements

2. Perform model order reduction with the resulting MIMO LTI-system $(\mathbf{E},\mathbf{A},\mathbf{B},\mathbf{C})$:

MIMO LTI-MOR

- BT
- MIMO RK
- MIMO IRKA

$$\overbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}^{\mathbf{E}_r}\ \dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}^{\mathbf{A}_r}\ \mathbf{x}_r(t) + \overbrace{\mathbf{W}^T\mathbf{B}}^{\mathbf{B}_r}\ \overbrace{\tilde{\mathbf{B}}(t)\,\mathbf{u}(t)}^{\tilde{\mathbf{u}}(t)}$$
$$\mathbf{y}_r(t) = \underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r}\ \mathbf{x}_r(t)$$

Straightforward approaches for LTV-MOR: Moving Loads

Moving Loads

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{E},\mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{x}(t) \in \mathbb{R}^{n}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \qquad \mathbf{B}(t) \in \mathbb{R}^{n\times m},\ \mathbf{C} \in \mathbb{R}^{q\times n} \qquad \mathbf{u}(t) \in \mathbb{R}^{m}$$

Approach 2: One-sided projection with **output Krylov subspace** [Cruz/Lohmann]

Main idea:

Since an input Krylov subspace would yield a time-varying projection matrix

$$\mathbf{V}(t) := \left[\mathbf{A}_{s_0}^{-1}\mathbf{B}(t) \quad \mathbf{A}_{s_0}^{-1}\mathbf{E}\mathbf{A}_{s_0}^{-1}\mathbf{B}(t) \quad \ldots \quad (\mathbf{A}_{s_0}^{-1}\mathbf{E})^{r-1}\mathbf{A}_{s_0}^{-1}\mathbf{B}(t)\right],$$

perform a one-sided projection $\mathbf{V} = \mathbf{W}$ with an output Krylov subspace:

$$\mathbf{W} := \left[\mathbf{A}_{s_0}^{-T}\mathbf{C}^{T} \quad \mathbf{A}_{s_0}^{-T}\mathbf{E}^{T}\mathbf{A}_{s_0}^{-T}\mathbf{C}^{T} \quad \ldots \quad (\mathbf{A}_{s_0}^{-T}\mathbf{E}^{T})^{r-1}\mathbf{A}_{s_0}^{-T}\mathbf{C}^{T}\right].$$

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Reduced order model:

$$\overbrace{\mathbf{W}^T\mathbf{E}\mathbf{W}}^{\mathbf{E}_r}\,\dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T\mathbf{A}\mathbf{W}}^{\mathbf{A}_r}\,\mathbf{x}_r(t) + \overbrace{\mathbf{W}^T\mathbf{B}(t)}^{\mathbf{B}_r(t)}\,\mathbf{u}(t)$$

$$\mathbf{y}_r(t) = \underbrace{\mathbf{C}\mathbf{W}}_{\mathbf{C}_r}\,\mathbf{x}_r(t)$$

Straightforward approaches for LTV-MOR: Moving Sensors

Moving Sensors

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \qquad \mathbf{E},\mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{x}(t) \in \mathbb{R}^{n}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) \qquad \mathbf{B} \in \mathbb{R}^{n\times m},\ \mathbf{C}(t) \in \mathbb{R}^{q\times n} \qquad \mathbf{y}(t) \in \mathbb{R}^{q}$$

Approach 1: Similar to **Two-step approach** [Stykel/Vasilyev '15]

Main idea:

1. Shift the time-variability of $\mathbf{C}(t)$ to the output variables: $\underset{(q,n)}{\mathbf{C}(t)} = \underset{(q,N)}{\tilde{\mathbf{C}}(t)}\ \underset{(N,n)}{\mathbf{C}}$

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}(t)\,\underbrace{\mathbf{C}\,\mathbf{x}(t)}_{\tilde{\mathbf{y}}(t)}$$

N : finite elements

2. Perform model order reduction with the resulting MIMO LTI-system $(\mathbf{E},\mathbf{A},\mathbf{B},\mathbf{C})$:

MIMO LTI-MOR

- BT
- MIMO RK
- MIMO IRKA

$$\overbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}^{\mathbf{E}_r}\,\dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}^{\mathbf{A}_r}\,\mathbf{x}_r(t) + \overbrace{\mathbf{W}^T\mathbf{B}}^{\mathbf{B}_r}\,\mathbf{u}(t)$$

$$\mathbf{y}_r(t) = \tilde{\mathbf{C}}(t)\,\underbrace{\underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r}\,\mathbf{x}_r(t)}_{\tilde{\mathbf{y}}_r(t)}$$

Straightforward approaches for LTV-MOR: Moving Sensors

Moving Sensors

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \qquad \mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{x}(t) \in \mathbb{R}^{n}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) \qquad \mathbf{B} \in \mathbb{R}^{n\times m},\ \mathbf{C}(t) \in \mathbb{R}^{q\times n} \qquad \mathbf{y}(t) \in \mathbb{R}^{q}$$

Approach 2: One-sided projection with **input Krylov subspace** [Cruz/Lohmann]

Main idea:

Since an output Krylov subspace would yield a time-varying projection matrix

$$\mathbf{W}(t) := \left[\mathbf{A}_{s_0}^{-T}\mathbf{C}(t)^T \quad \mathbf{A}_{s_0}^{-T}\mathbf{E}^T\mathbf{A}_{s_0}^{-T}\mathbf{C}(t)^T \quad \dots \quad (\mathbf{A}_{s_0}^{-T}\mathbf{E}^T)^{r-1}\mathbf{A}_{s_0}^{-T}\mathbf{C}(t)^T\right],$$

perform a one-sided projection $\mathbf{W} = \mathbf{V}$ with an input Krylov subspace:

$$\mathbf{V} := \left[\mathbf{A}_{s_0}^{-1}\mathbf{B} \quad \mathbf{A}_{s_0}^{-1}\mathbf{E}\mathbf{A}_{s_0}^{-1}\mathbf{B} \quad \dots \quad (\mathbf{A}_{s_0}^{-1}\mathbf{E})^{r-1}\mathbf{A}_{s_0}^{-1}\mathbf{B}\right].$$

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Reduced order model:

$$\overbrace{\mathbf{V}^T\mathbf{E}\mathbf{V}}^{\mathbf{E}_r}\,\dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{V}^T\mathbf{A}\mathbf{V}}^{\mathbf{A}_r}\,\mathbf{x}_r(t) + \overbrace{\mathbf{V}^T\mathbf{B}}^{\mathbf{B}_r}\,\mathbf{u}(t)$$

$$\mathbf{y}_r(t) = \underbrace{\mathbf{C}(t)\mathbf{V}}_{\mathbf{C}_r(t)}\,\mathbf{x}_r(t)$$

Straightforward approaches for LTV-MOR: Combined case

Moving Loads + Sensors

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{E},\mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{x}(t) \in \mathbb{R}^{n}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) \qquad \mathbf{B}(t) \in \mathbb{R}^{n\times m},\ \mathbf{C}(t) \in \mathbb{R}^{q\times n} \qquad \mathbf{u}(t) \in \mathbb{R}^{m}, \mathbf{y}(t) \in \mathbb{R}^{q}$$

Approach 1: Two-step approach can be pursued without problems

Main idea:

1. Shift time-variability of $\mathbf{B}(t)$ and $\mathbf{C}(t)$:

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\overbrace{\tilde{\mathbf{B}}(t)\,\mathbf{u}(t)}^{\tilde{\mathbf{u}}(t)}$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}(t)\,\underbrace{\mathbf{C}\,\mathbf{x}(t)}_{\tilde{\mathbf{y}}(t)}$$

$$\underset{(n,m)}{\mathbf{B}(t)} = \underset{(n,N)}{\mathbf{B}}\ \underset{(N,m)}{\tilde{\mathbf{B}}(t)}$$

$$\underset{(q,n)}{\mathbf{C}(t)} = \underset{(q,N)}{\tilde{\mathbf{C}}(t)}\ \underset{(N,n)}{\mathbf{C}}$$

2. Perform model order reduction with the resulting MIMO LTI-system $(\mathbf{E},\mathbf{A},\mathbf{B},\mathbf{C})$:

MIMO LTI-MOR

- BT
- MIMO RK
- MIMO IRKA

$$\overbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}^{\mathbf{E}_r}\,\dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}^{\mathbf{A}_r}\,\mathbf{x}_r(t) + \overbrace{\mathbf{W}^T\mathbf{B}}^{\mathbf{B}_r}\,\overbrace{\tilde{\mathbf{B}}(t)\,\mathbf{u}(t)}^{\tilde{\mathbf{u}}(t)}$$

$$\mathbf{y}_r(t) = \tilde{\mathbf{C}}(t)\,\underbrace{\underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r}\,\mathbf{x}_r(t)}_{\tilde{\mathbf{y}}_r(t)}$$

Straightforward approaches for LTV-MOR: Combined case

Moving Loads + Sensors

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \qquad \mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n} \qquad \mathbf{x}(t) \in \mathbb{R}^{n}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) \qquad \mathbf{B}(t) \in \mathbb{R}^{n\times m},\ \mathbf{C}(t) \in \mathbb{R}^{q\times n} \qquad \mathbf{u}(t) \in \mathbb{R}^{m}, \mathbf{y}(t) \in \mathbb{R}^{q}$$

Approach 2: One-sided projection would yield time-varying projection matrices!

Main idea:

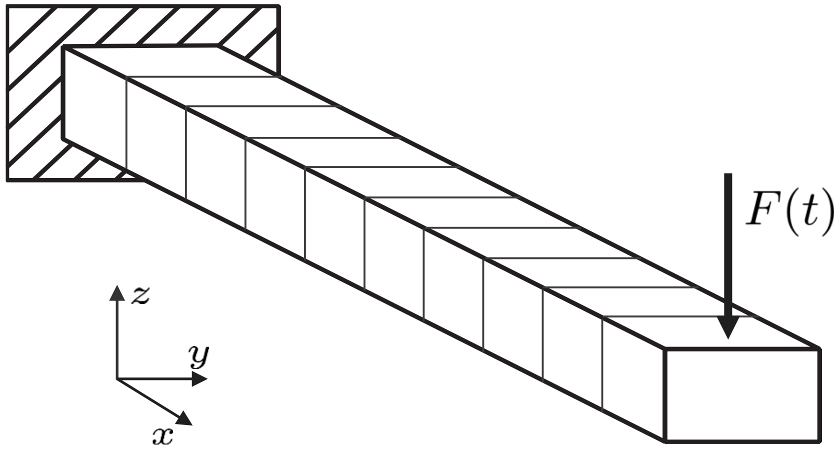
Input Krylov subspace would yield a time-varying projection matrix:

$$\mathbf{V}(t) := \left[\mathbf{A}_{s_0}^{-1}\mathbf{B}(t) \quad \mathbf{A}_{s_0}^{-1}\mathbf{E}\mathbf{A}_{s_0}^{-1}\mathbf{B}(t) \quad \dots \quad (\mathbf{A}_{s_0}^{-1}\mathbf{E})^{r-1}\mathbf{A}_{s_0}^{-1}\mathbf{B}(t)\right].$$

Output Krylov subspace would yield a time-varying projection matrix:

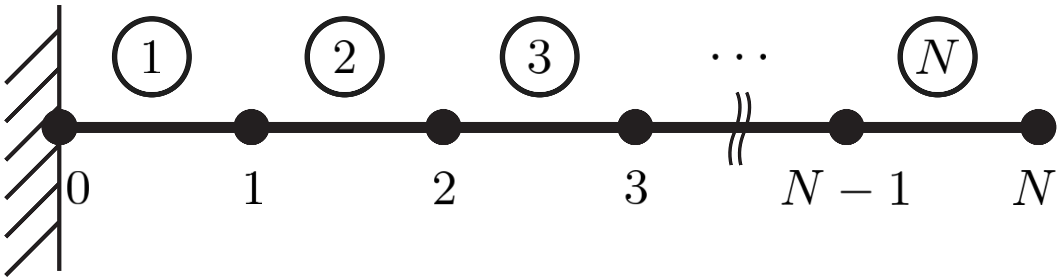
$$\mathbf{W}(t) := \left[\mathbf{A}_{s_0}^{-T}\mathbf{C}(t)^T \quad \mathbf{A}_{s_0}^{-T}\mathbf{E}^T\mathbf{A}_{s_0}^{-T}\mathbf{C}(t)^T \quad \dots \quad (\mathbf{A}_{s_0}^{-T}\mathbf{E}^T)^{r-1}\mathbf{A}_{s_0}^{-T}\mathbf{C}(t)^T\right].$$

Numerical example: Timoshenko beam with moving load



Parameters of the beam
Length: L
Height: h
Thickness: t
Density of material: ρ
Mass: m
Young's modulus: E
Poisson's ratio: ν
Shear modulus: G

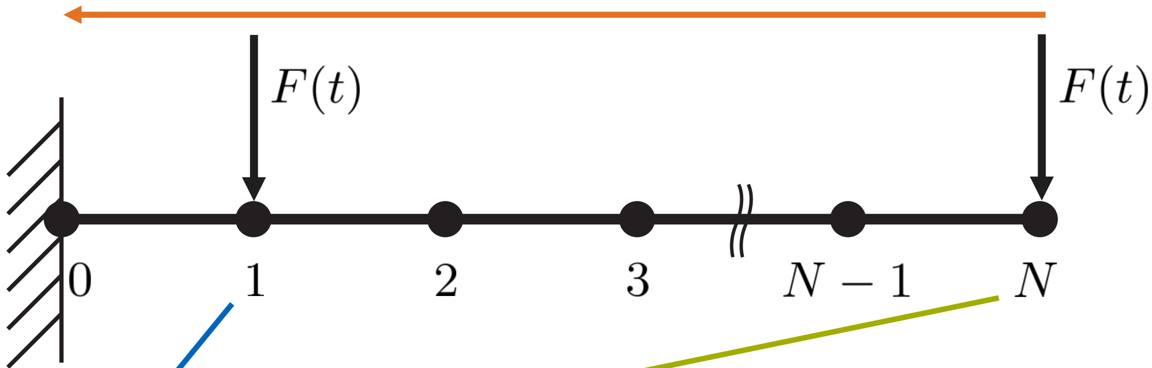
- Moving load position causes time-varying behaviour
- Spatial discretization with finite element method (FEM)



N finite elements with length $l = \frac{L}{N}$

Numerical example: Timoshenko beam with moving load

[Panzer et al. '09]



N : finite elements

$$-\hat{\mathbf{b}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dots, -\hat{\mathbf{b}}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Interpolation of the input vector:

$$\hat{\mathbf{b}}(t) = \sum_{i=1}^{N} \omega_i(t) \hat{\mathbf{b}}_i$$

LTV first-order model:

$$\overbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}^{\mathbf{E}} \overbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}^{\dot{\mathbf{x}}}(t) = \overbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}^{\mathbf{x}}(t) + \overbrace{\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{b}}(t) \end{bmatrix}}^{\mathbf{b}(t)} F(t)$$

$$y(t) = \underbrace{\begin{bmatrix} \hat{\mathbf{c}}^T & \mathbf{0}^T \end{bmatrix}}_{\mathbf{c}^T} \underbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\mathbf{x}}(t)$$

with:

$$\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{6N \times 6N}, \quad \hat{\mathbf{c}}^T = -\hat{\mathbf{b}}_N^T \in \mathbb{R}^{1 \times 6N}$$

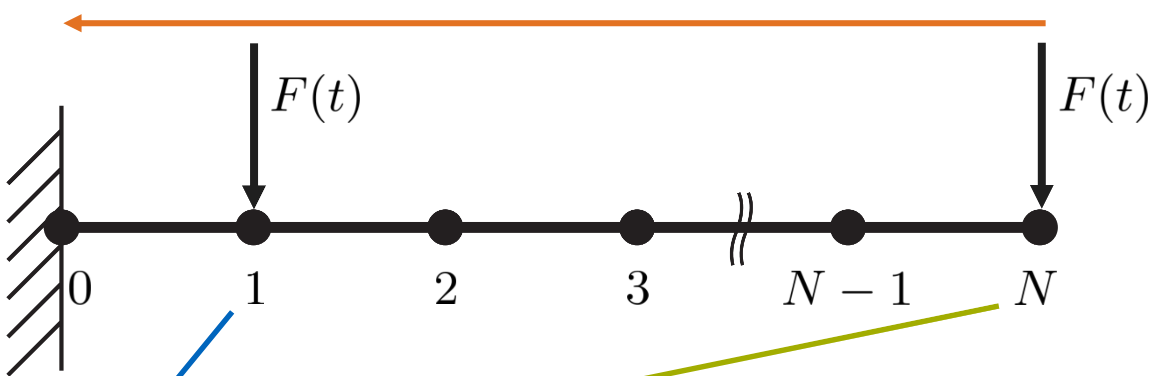
$\mathbf{F} = \mathbf{I}$ chosen

$$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{2 \cdot 6N \times 2 \cdot 6N}, \quad \mathbf{c}^T \in \mathbb{R}^{1 \times 2 \cdot 6N}$$

Original order: $2 \cdot 6N$

Numerical example: Timoshenko beam with moving load

[Panzer et al. '09]



N : finite elements

$$-\hat{\mathbf{b}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, -\hat{\mathbf{b}}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot intput matrix:

$$\mathbf{B} = [-\hat{\mathbf{b}}_1, -\hat{\mathbf{b}}_2, \ldots, -\hat{\mathbf{b}}_N]$$

LTV first-order model:

$$\overbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}^{\mathbf{E}} \overbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}^{\dot{\mathbf{x}}}(t) = \overbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}^{\mathbf{x}}(t) + \overbrace{\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{b}}(t) \end{bmatrix}}^{\mathbf{b}(t)} F(t)$$

$$y(t) = \underbrace{\begin{bmatrix} \hat{\mathbf{c}}^T & \mathbf{0}^T \end{bmatrix}}_{\mathbf{c}^T} \underbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\mathbf{x}}(t)$$

Approach 1:
Two-step approach

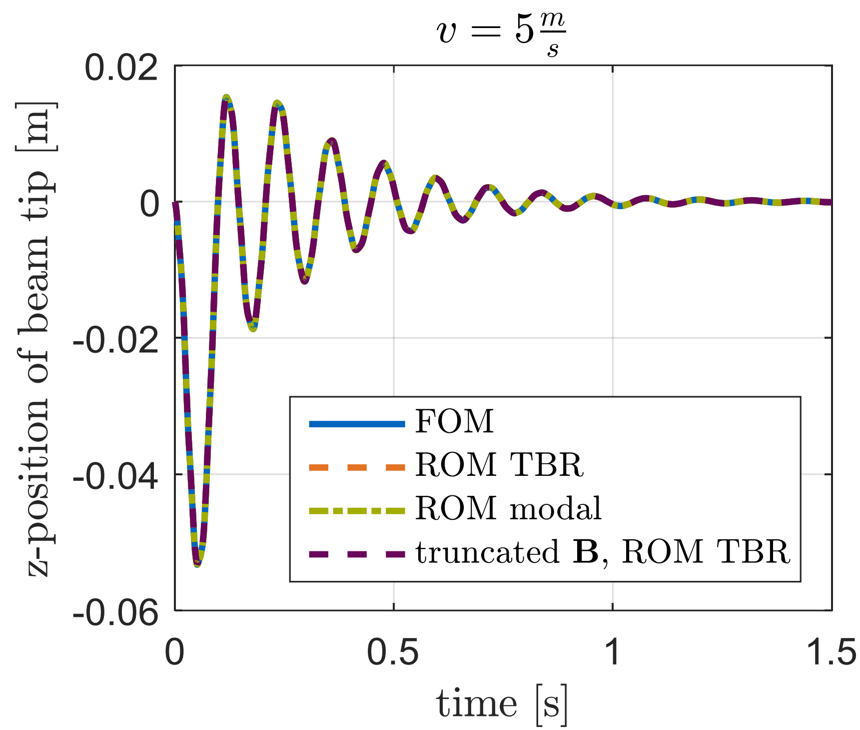
$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\overbrace{\tilde{\mathbf{b}}(t)\,u(t)}^{\tilde{\mathbf{u}}(t)}$$

$$y(t) = \mathbf{c}^T\,\mathbf{x}(t)$$

Timoshenko beam with moving load: Approach 1

Reduction using Balanced Truncation

Length of the beam	L = 1 m
Load amplitude	F(t) = 20 N
Velocity of the moving load	v = 5 m/s
Number of finite elements	N = 151
Original order	n = 1812
Reduced order	r = 10
Simulation with "lsim"	dt = 0.001 s



- Reduced order model obtained with two-step approach and Balanced Truncation yields good results in this case.
- Truncation of **B** and subsequent MOR can still yield good results.

Error TBR: $||y - y_{\text{TBR}}||^2_{\mathcal{L}_2} = 4.53 \cdot 10^{-5}$

Error Modal: $||y - y_{\text{modal}}||^2_{\mathcal{L}_2} = 2.78 \cdot 10^{-4}$

Truncation of $\mathbf{B}$: $\mathbf{B}_{\text{trunc}} \in \mathbb{R}^{1812 \times 76}$

Error: $||y - y_{\text{truncTBR}}||^2_{\mathcal{L}_2} = 1.86 \cdot 10^{-3}$

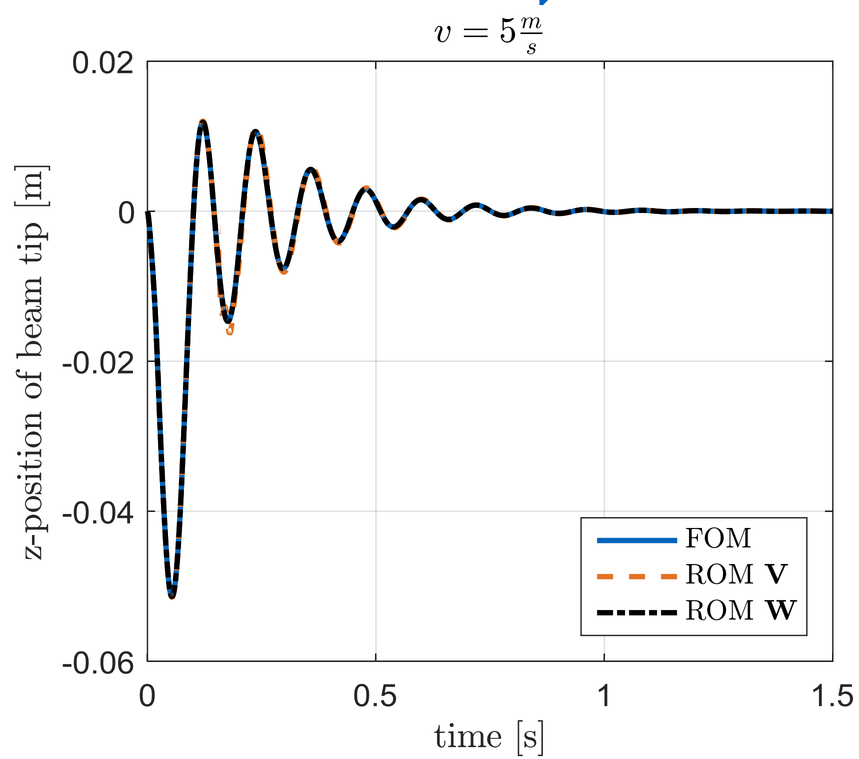
Timoshenko beam with moving load: Approach 2

Reduction with output Krylov subspace

$$\mathbf{W} := \left[\mathbf{A}_{s_0}^{-T}\mathbf{c}^T \quad \mathbf{A}_{s_0}^{-T}\mathbf{E}^T\mathbf{A}_{s_0}^{-T}\mathbf{c}^T \quad \ldots \quad (\mathbf{A}_{s_0}^{-T}\mathbf{E}^T)^{r-1}\mathbf{A}_{s_0}^{-T}\mathbf{c}^T\right]$$

$\mathbf{V} = \mathbf{W}$ ⇨ Time-independent projection matrices ⇨ $\dot{\mathbf{V}} = \mathbf{0}$

Length of the beam	L = 1 m
Load amplitude	F(t) = 20 N
Velocity of the moving load	v = 5 m/s
Number of finite elements	N = 151
Original order	n = 1812
Reduced order	r = 10
Expansion points	$s_0 = 0$
Simulation with "lsim"	dt = 0.001 s

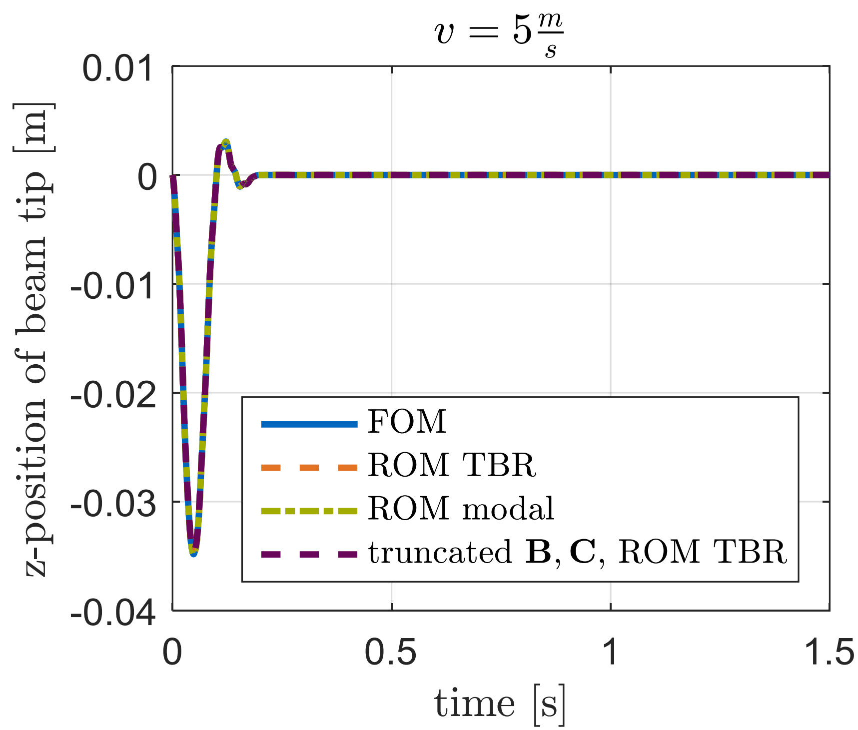


Reduced order model obtained with output Krylov subspace (ROM **W**) yields good results for the case of moving loads

Timoshenko beam with moving load + sensor: Approach 1

Reduction using Balanced Truncation

Length of the beam	L = 1 m
Load amplitude	F(t) = 20 N
Velocity of moving load & sensor	v = 5 m/s
Number of finite elements	N = 151
Original order	n = 1812
Reduced order	r = 10
Simulation with “lsim”	dt = 0.001 s



- Reduced order model obtained with two-step approach and Balanced Truncation yields good results in this case.
- Truncation of **B, C** and subsequent MOR can still yield good results.

Error TBR: $||y - y_{\text{TBR}}||^2_{\mathcal{L}_2} = 4.63 \cdot 10^{-5}$

Error Modal: $||y - y_{\text{modal}}||^2_{\mathcal{L}_2} = 2.34 \cdot 10^{-4}$

Truncation of $\mathbf{B}$ and $\mathbf{C}$:

Error: $||y - y_{\text{truncTBR}}||^2_{\mathcal{L}_2} = 4.63 \cdot 10^{-5}$

Summary and Outlook

Summary:

- **Goal:** Reduction of high dimensional LTV systems (e.g. systems with moving load)
- Projection-based tMOR for the reduction of LTV systems
- Some straightforward approaches for special cases
  - Two-step approach
  - Reduction with input/output Krylov subspace for moving sensor/moving load
- Application of straightforward approaches to Timoshenko beam with moving load and/or sensor

Outlook:

- Application of two-step approach with other MIMO LTI-MOR techniques (e.g. MIMO-RK, MIMO-IRKA)
- Further development of the matrix interpolation for the reduction of LPV systems and investigation of the influence of the additional time-derivative terms

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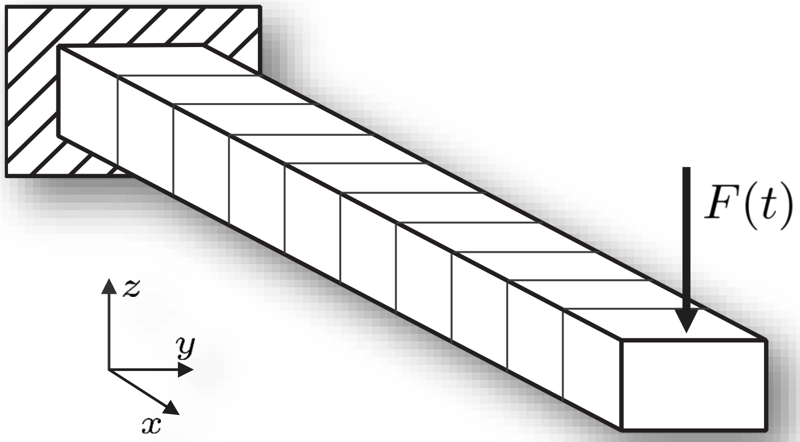
Chair of Automatic Control

Technische Universität München

Model Order Reduction of Linear Time-Varying Systems: Some straightforward approaches



MOR 4 MEMS 2015

Karlsruhe, 18th November 2015

Thank you for your attention

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Backup

Part II

Parametric Model Order Reduction

pMOR by Matrix Interpolation

Properties:

- Local pMOR approach
- Analytical expression of the parameter-dependency in general not available
- Model only available at certain parameter sample points

Main idea:

1. Individual reduction of each local model
2. Transformation of the local reduced models
3. Interpolation of the reduced matrices

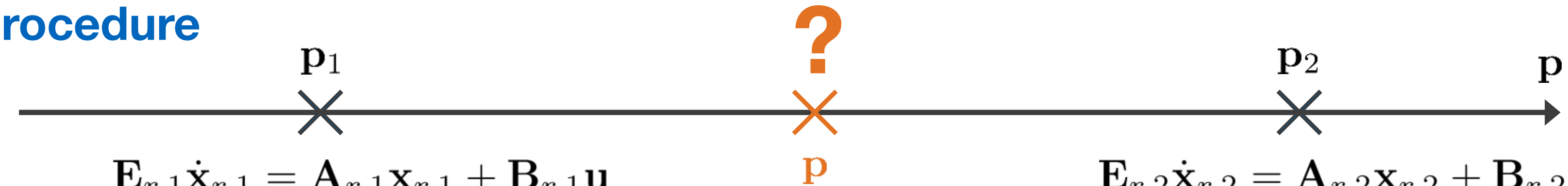
Advantages
• No analytically expressed parameter-dependency required • Any desired MOR technique applicable for the local reduction • Offline/Online decomposition • Reduced order independent of the number of local models

Drawbacks
• Choice of degrees of freedom – Parameter sample points – Interpolation method • Stability preservation • Error bounds

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure



$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1,\dots,k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{M}_i^T \cdot \Big|$$

$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$

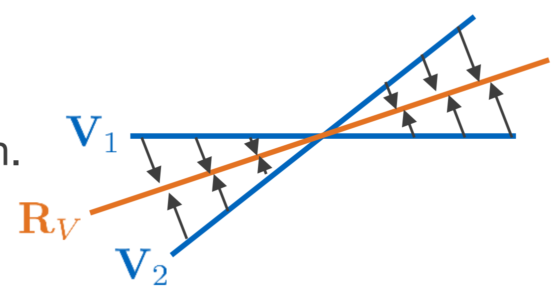
$$\mathbf{E}_{r,i}\mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{T}_i\hat{\mathbf{x}}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{T}_i\hat{\mathbf{x}}_{r,i}(t)$$

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$

$$\mathbf{V}_{all} = [\mathbf{V}_1, \dots, \mathbf{V}_k]$$
$$\mathbf{V}_{all} \overset{\text{SVD}}{=} \mathbf{USN}^T$$
$$\mathbf{R}_V = \mathbf{U}(:, 1:r)$$

How do we choose $\mathbf{T}_i$?

Goal: Adjustment of the local bases $\mathbf{V}_i$ to $\hat{\mathbf{V}}_i = \mathbf{V}_i\mathbf{T}_i$, in order to make the gen. coordinates $\hat{\mathbf{x}}_{r,i}$ compatible w.r.t. a reference subspace $\mathbf{R}_V$.



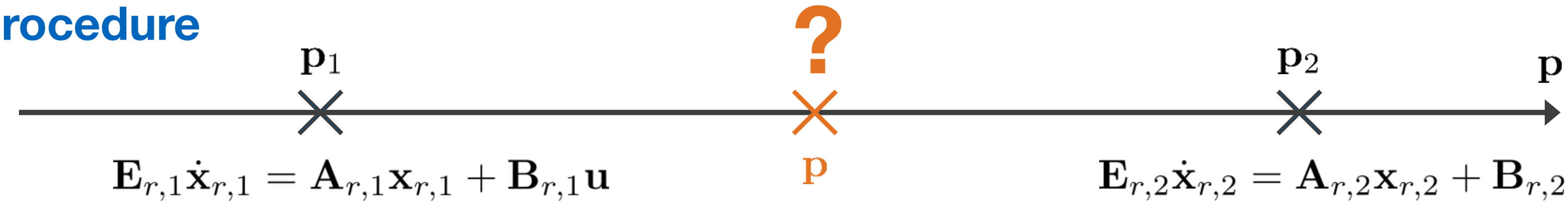
High correlation $\hat{\mathbf{V}}_i \leftrightarrow \mathbf{R}_V$:

$$\mathbf{T}_i^T\mathbf{V}_i^T\mathbf{R}_V \overset{!}{=} \mathbf{I}$$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure



$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u} \qquad \mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u} \qquad \mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t)$$

$$\mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i, \quad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \ldots, k$

$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$

$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

2.) Transformation to generalized coordinates

$\mathbf{M}_i^T \cdot \Big|$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t)$$

$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t)$$

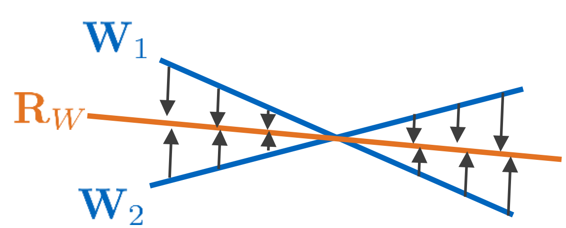
$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1} \qquad \mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}$$

$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$

Analogous to $\mathbf{R}_V$ or $\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$

How do we choose $\mathbf{M}_i$?

Goal: Adjustment of the local bases $\mathbf{W}_i$ to $\hat{\mathbf{W}}_i = \mathbf{W}_i\mathbf{M}_i$, in order to describe the local reduced models w.r.t. the same reference basis $\mathbf{R}_W$.

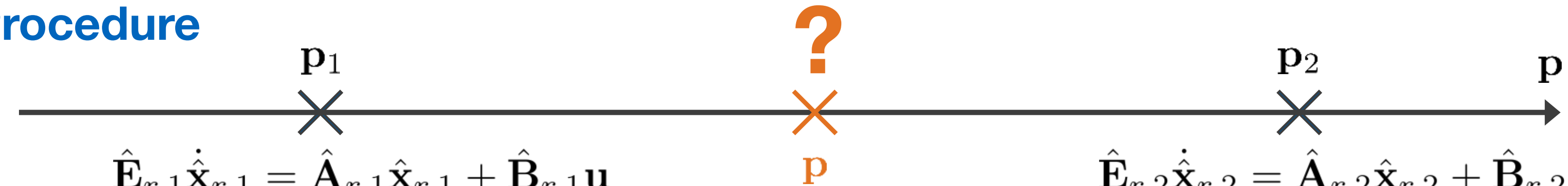


High correlation $\hat{\mathbf{W}}_i \leftrightarrow \mathbf{R}_W$: $\mathbf{M}_i^T\mathbf{W}_i^T\mathbf{R}_W \overset{!}{=} \mathbf{I}$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure



$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1}$$

$$\hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \ldots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{M}_i^T \cdot \Big|$$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t)$$

$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}$$
$$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$$

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$

$$\mathbf{V}_{all} = [\mathbf{V}_1, \ldots, \mathbf{V}_k]$$
$$\mathbf{V}_{all} \overset{\text{SVD}}{=} \mathbf{USN}^T$$
$$\mathbf{R}_V = \mathbf{U}(:, 1:r)$$

3.) Interpolation

$$\hat{\mathbf{E}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{E}}_{r,i}, \quad \hat{\mathbf{A}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{A}}_{r,i}$$
$$\hat{\mathbf{B}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{B}}_{r,i}, \quad \hat{\mathbf{C}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{C}}_{r,i}$$

$$\sum_{i=1}^k \omega_i(\mathbf{p}) = 1$$

Part III

Time-varying Parametric Model Order Reduction

Reduction of Systems with Moving Loads

Balanced Truncation for LTV systems

Linear time-varying system:

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

Solution of two Lyapunov-DE (LDE):

$$\mathbf{A}(t)\mathbf{P}(t)\mathbf{E}(t)^T + \mathbf{E}(t)\mathbf{P}(t)\mathbf{A}(t)^T + \mathbf{B}(t)\mathbf{B}(t)^T = \dot{\mathbf{P}}(t)$$
$$\mathbf{P}(t_0) = \mathbf{0}$$
$$\mathbf{A}(t)^T\mathbf{Q}(t)\mathbf{E}(t) + \mathbf{E}(t)^T\mathbf{Q}(t)\mathbf{A}(t) + \mathbf{C}(t)^T\mathbf{C}(t) = \dot{\mathbf{Q}}(t)$$
$$\mathbf{Q}(t_e) = \mathbf{0}$$

Switched Linear System + BT

Switched linear system:

$$\mathbf{E}_\alpha\dot{\mathbf{x}}(t) = \mathbf{A}_\alpha\mathbf{x}(t) + \mathbf{B}_\alpha\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_\alpha\mathbf{x}(t)$$

BT for each subsystem:

$$\left.\begin{array}{l}\mathbf{A}_\alpha\mathbf{P}_\alpha\mathbf{E}_\alpha^T + \mathbf{E}_\alpha\mathbf{P}_\alpha\mathbf{A}_\alpha^T + \mathbf{B}_\alpha\mathbf{B}_\alpha^T = \mathbf{0}\\ \mathbf{A}_\alpha^T\mathbf{Q}_\alpha\mathbf{E}_\alpha + \mathbf{E}_\alpha^T\mathbf{Q}_\alpha\mathbf{A}_\alpha + \mathbf{C}_\alpha^T\mathbf{C}_\alpha = \mathbf{0}\end{array}\right. \Rightarrow \mathbf{V}_\alpha, \mathbf{W}_\alpha$$

Model reduction: $\mathbf{E}_{r,\alpha},\ \mathbf{A}_{r,\alpha},\ \mathbf{B}_{r,\alpha},\ \mathbf{C}_{r,\alpha}$

Two-step approach

I) Low-rank approximation: $\mathbf{B}(t) \approx \hat{\mathbf{B}}\mathbf{\Psi}(t)$

$\mathbf{u}(t) \in \mathbb{R}^m$ $\quad\xrightarrow{m \gg \hat{m}}\quad$ $\hat{\mathbf{u}}(t) = \mathbf{\Psi}(t)\mathbf{u}(t) \in \mathbb{R}^{\hat{m}}$

$\mathbf{B}(t) \in \mathbb{R}^{n\times m}$ $\qquad$ $\hat{\mathbf{B}} \in \mathbb{R}^{n\times\hat{m}}$

II) LTI-MOR: Reduction of the resulting LTI system with Rational Krylov, IRKA, BT, …

Parametric LTI system + pMOR

Global IRKA: $\mathbf{E}_i,\ \mathbf{A}_i,\ \mathbf{B}_i,\ \mathbf{C}_i$

$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i),\ \mathbf{W}_i := \mathbf{W}(\mathbf{p}_i) \qquad \mathbf{p}_i,\ i = 1,\dots,k$$
$$\mathbf{V} = \left[\mathbf{V}_1,\dots,\mathbf{V}_k\right],\ \mathbf{W} = \left[\mathbf{W}_1,\dots,\mathbf{W}_k\right]$$

Matrix Interpolation:

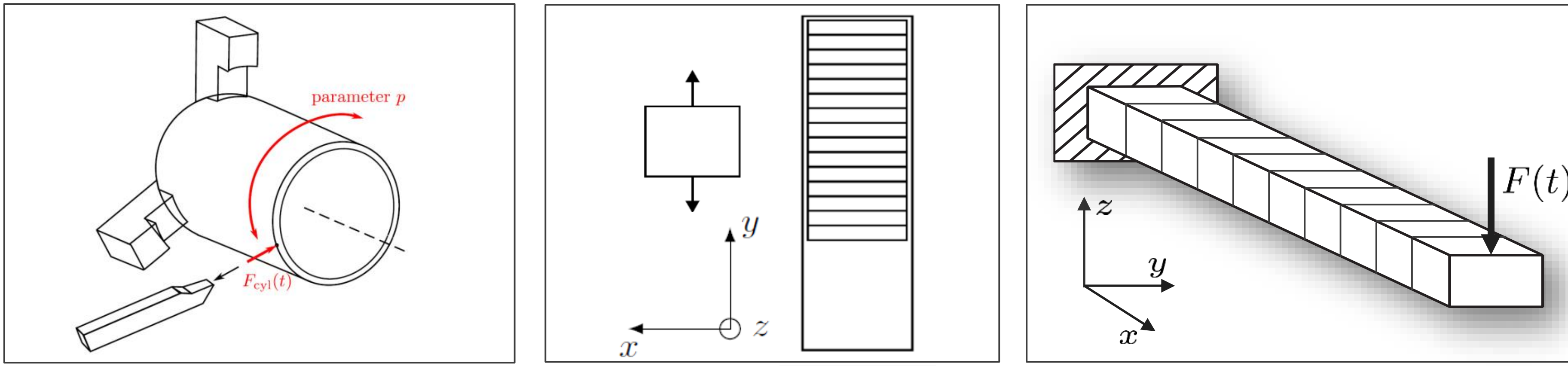
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i),\ \mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$
$$\mathbf{T}_i,\ \mathbf{M}_i$$

Interpolation of reduced system matrices

Reduction of Moving Loads by Matrix Interpolation

Systems with Moving Loads:

- Location of the load varies with time
- Moving load is considered as time-dependent parameter



thin-walled cylinder | thermo-elastic machine stand | Timoshenko beam

Linear parameter-varying (LPV) system:

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t) \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) \qquad \mathbf{x}(t) \in \mathbb{R}^n$$

- System matrices explicitly depend on time-varying parameters
- Special class of linear time-varying (LTV) or nonlinear systems

Goal: Reduction of high dimensional LPV systems by matrix interpolation

Time-Varying Parametric Model Order Reduction: p(t)MOR

Linear parameter-varying (LPV) system

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p}(t))\mathbf{x} + \mathbf{B}(\mathbf{p}(t))\mathbf{u} \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$

$$\mathbf{y} = \mathbf{C}(\mathbf{p}(t))\mathbf{x} \qquad \mathbf{x} \in \mathbb{R}^n$$

$r \ll n$ **p(t)MOR**

Approximation of the full state vector:

$$\mathbf{x} = \mathbf{V}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{e},$$

$$\dot{\mathbf{x}} = \dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r + \dot{\mathbf{e}}$$

Petrov-Galerkin condition: $\mathbf{W}(\mathbf{p}(t)) \perp \boldsymbol{\epsilon}$

$\mathbf{W}(\mathbf{p}(t))^T \cdot \Big|$

$$\mathbf{E}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r = \Big(\mathbf{A}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t)) - \mathbf{E}(\mathbf{p}(t))\dot{\mathbf{V}}(\mathbf{p}(t))\Big)\mathbf{x}_r + \mathbf{B}(\mathbf{p}(t))\mathbf{u} + \boldsymbol{\epsilon}$$

$$\mathbf{y}_r = \mathbf{C}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))\mathbf{x}_r$$

Time-Varying Parametric Model Order Reduction: p(t)MOR

Linear parameter-varying (LPV) system

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p}(t))\mathbf{x} + \mathbf{B}(\mathbf{p}(t))\mathbf{u} \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$

$$\mathbf{y} = \mathbf{C}(\mathbf{p}(t))\mathbf{x} \qquad \mathbf{x} \in \mathbb{R}^n$$

$r \ll n$ **p(t)MOR**

Approximation of the full state vector:

$$\mathbf{x} = \mathbf{V}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{e},$$

$$\dot{\mathbf{x}} = \dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r + \dot{\mathbf{e}}$$

Petrov-Galerkin condition: $\mathbf{W}(\mathbf{p}(t)) \perp \boldsymbol{\epsilon}$

$$\overbrace{\mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))}^{\mathbf{E}_r(\mathbf{p}(t))}\dot{\mathbf{x}}_r = \left(\overbrace{\mathbf{W}(\mathbf{p}(t))^T\mathbf{A}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))}^{\mathbf{A}_r(\mathbf{p}(t))} - \mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_r + \overbrace{\mathbf{W}(\mathbf{p}(t))^T\mathbf{B}(\mathbf{p}(t))}^{\mathbf{B}_r(\mathbf{p}(t))}\mathbf{u}$$

$$\mathbf{y}_r = \underbrace{\mathbf{C}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))}_{\mathbf{C}_r(\mathbf{p}(t))}\mathbf{x}_r$$

Time-Varying Parametric Model Order Reduction: p(t)MOR

Linear parameter-varying (LPV) system

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p}(t))\mathbf{x} + \mathbf{B}(\mathbf{p}(t))\mathbf{u} \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$

$$\mathbf{y} = \mathbf{C}(\mathbf{p}(t))\mathbf{x} \qquad \mathbf{x} \in \mathbb{R}^n$$

$r \ll n$ **p(t)MOR**

Approximation of the full state vector:

$$\mathbf{x} = \mathbf{V}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{e},$$

$$\dot{\mathbf{x}} = \dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r + \dot{\mathbf{e}}$$

Petrov-Galerkin condition: $\mathbf{W}(\mathbf{p}(t)) \perp \boldsymbol{\epsilon}$

Parameter-varying reduced order model

$$\mathbf{E}_r(\mathbf{p}(t))\dot{\mathbf{x}}_r = \left(\mathbf{A}_r(\mathbf{p}(t)) - \mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_r + \mathbf{B}_r(\mathbf{p}(t))\mathbf{u}$$

$$\mathbf{y}_r = \mathbf{C}_r(\mathbf{p}(t))\mathbf{x}_r$$

$$\mathbf{E}_r(\mathbf{p}(t)) = \mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t)), \qquad \mathbf{A}_r(\mathbf{p}(t)) = \mathbf{W}(\mathbf{p}(t))^T\mathbf{A}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))$$

$$\mathbf{B}_r(\mathbf{p}(t)) = \mathbf{W}(\mathbf{p}(t))^T\mathbf{B}(\mathbf{p}(t)), \qquad \mathbf{C}_r(\mathbf{p}(t)) = \mathbf{C}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))$$