Elastic waves propagation in 1D fractional non-local continuum *

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Abstract

Aim of this paper is the study of waves propagation in a fractional, non-local 1D elastic continuum. The non-local effects are modeled introducing long-range central body interactions applied to the centroids of the infinitesimal volume elements of the continuum. These non-local interactions are proportional to a proper attenuation function and to the relative displacements between non-adjacent elements. It is shown that, assuming a power-law attenuation function, the governing equation of the elastic waves in the *unbounded domain*, is ruled by a Marchaud-type fractional differential equation. Wave propagation in *bounded domain* instead involves only the integral part of the Marchaud fractional derivative. The dispersion of elastic waves, as well as waves propagation in unbounded and bounded domains are discussed in detail.

1 Introduction

Elastic waves propagation in solid mechanics has been widely studied in the context of physical and engineering problems since the end of the nineteenth century. Propagation of elastic waves in 1D domains are usually analyzed with the well celebrated D'Alembert or Riemann solutions (see e.g. [1]). The main feature of waves propagation in unbounded elastic solids is related to the constant speed of the traveling disturbances that depends on the mass density as well as on some mechanical parameters, whereas it is independent of the wavelength. Such independence has not been observed in experimental measures of elastic waves speed in real materials. As in fact, it is well known that, as soon as the wavelength decreases, a lower propagation speed is usually detected and this effect is known as *dispersion* of elastic waves. Such a phenomenon is the main responsible of traveling disturbances shape changes in 1D elastic waveguides and it is commonly detected in lattice-type materials. Similar effects may be also detected in waves scattering across periodic or, nearly-periodic elastic waveguides ([31]).

Detailed discussion of waves propagation in crystal lattices, faced in the context of lattice mechanics may be found in basic book (see [6] and [5] among others). At the borders of the

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Brillouin zone, as soon as the wavelength become comparable to the lattice distances, there are large deviations between the speed of traveling waves, predicted with continuum mechanics, and that evaluated by means of the lattice dynamics theory. This phenomenon is due to the lack of a specific parameter related to the structure of the solid in the continuum mechanics model ([34]).

In this context, several studies involving structured elastic continuum are available from the beginning of the sixties ([28], [17], [33], [29]) by means of suitable extensions of the theory of deformable directors ([16]), up to more recent studies ([17], [25]) introducing various forms of the theory of micro-morphic continuum. Alternatively, a different approach accounting for the discrete nature of the solids has been proposed at the end of the sixties ([24]) and later on applied to several fields of physical and engineering context ([18], [19], [20], [3], [2]). In those latter approaches it has been assumed that the stress-strain relation is defined in an Hilbert space with assigned metric depending on the relative distance of the state variables ([22]). Very recently a general framework for non-local 1D solid with Eringen integral model appeared in scientific literature [4].

The governing equations of the problem, studied in the context of non-local elasticity, yield dispersion relations for unbounded domains but as far as bounded elastic solids are analyzed the frequency-wavenumber ($\omega - \kappa$) relations involve *escape frequencies* corresponding to non-propagating waves ([9]). This aspect cannot be explained in the 1D model of elastic waves propagation since the presence of escape frequencies in classical mechanics is due to the presence of elastic external restraints along the waveguide, not included in the model.

A different, physically-based, approach to non-local mechanics has been recently introduced in the context of fractional calculus ([14]) and after that, successively investigated either in variational and thermodynamic context ([11], [13]).

The non-local effects have been accounted for introducing, in the equilibrium equation of a generic volume element an additional contribution representing the resultant of the long-range forces exchanged between the considered element and the surrounding non-adjacent volume elements of the solid. The mathematical model of the long-range interactions has been assumed dependent on the product of the interacting volumes, on their relative displacements, as well as on a proper, monotonically decreasing function. This latter function, dubbed *attenuation function*, or *distance decaying function*, contains additional parameters related to the inner microstructure of the material. Under the assumption of power-law decay a fractional differential equation of Marchaud-type has been obtained in unbounded domain ([14]). Similar fractional equation with a different fractional order may be also obtained, in unbounded domains, by the integral model of non-local elasticity with a proper, fractionally decaying attenuation function as recently proposed by [26]. Non-local approaches with long-range elastic forces have been conducted in static setting in order to capture the mechanical behavior of materials exhibited during load tests. In other studies ([35]) dynamic analysis of elastic continuum with long-range interactions have been performed in the context of exponential decay of the long-range effects. Yet, if we assume that similarity properties of the long-range interactions keep at smaller observation scales then the fractional power-law is, in the authors' opinion, the best choice for the decaying function, because it is typical of self-similar transformations ([7], [8]).

In this paper, the propagation of elastic waves in 1D case is faced in the context of fractional power-law long-range interactions. The propagating phenomenon is analyzed in unbounded continuum domain providing the frequency-wavenumber curves assessing the presence of the elastic waves dispersion. Standing waves analysis in bounded domain as well as disturbances propagation in bounded 1D elastic waveguides are also assessed. The formulation of the field equations are here derived from the weak form of motions equations as in Hamiltonian mechanics.

2 The 1D Elastodynamics in Presence of Fractional Long-Range Interactions

Fundamentals of the elastodynamics problem within the physically-based approach to non-local mechanics ([14]) in presence of long-range central interactions will be shortly outlined in this section for clarity's sake as well as to introduce appropriate notation.

Let us assume to deal with a 1D homogeneous elastic domain with Young elastic modulus E, uniform mass density ρ, in the region $[a,b] \in \mathbb{R}$ of an Euclidean space as in Fig.(1) and let us denote the generic volume element as $dV(x) = Adx$, with A the cross section area of the 1D solid. It is assumed that the axial external body force field $f(x,t)\, dV(x)$ with $f(x,t)$, where t is the time, is equilibrated by the contact forces coming from adjacent volumes, $N(x,t)$ and $N(x+dx,t)$ that play the same role of lattice interactions between particles involved in *next nearest* (NN) models in [5]. Other contribution to the equilibrium of the external load field $f(x,t)$ in such a physically-based approach to non-local mechanics is represented by the long-range body forces between volume $dV(x)$ and the non-adjacent volume elements $dV(\xi)$, as like as in *next to nearest neighborhood* (NNN) interaction lattice models ([30]).

Such internal body forces, denoted as $Q(x,t)$ are the resultant of the long-range interactions $q(x,\xi,t)$ that each volume $dV(\xi)$ with $\xi \in [a,b]$ exerts on volume $dV(x)$. In particular it is assumed that $q(x,\xi,t)$ is proportional to the product of interacting volumes, and, under the assumption of linear elastic state of the solid, to the product of the relative displacement, indicated as $\eta(x,\xi,t) = u(x,t) - u(\xi,t)$. Since all physical interactions decay with the inter-distance, the long-range interaction forces $q(x,\xi,t)$ will be also proportional to a distance-decaying function $g(x,\xi)$ depending of the material at hands.

The choice of the distance decaying function is a crucial step in the mechanics of the non-local continuum with long-range interactions. Selection of the function $g(x,\xi)$ in the class of Helmoltz or bi-Helmoltz type decaying functions yields, with the physically-based model, an integro-differential formulation of the governing equations ([35]).

In the authors' opinion, the evidence of long-range effects is manifested at micro or meso-scale, and only an accurate description of the material micro-structure might suggest the correct form and the parameters of the function $g(x,\xi)$. Some recent attempt to model materials micro-structure has been formulated in the context of fractal geometry ([15]), that, in the authors' opinion corresponds to a model in which self-similar interactions are introduced. This leads towards particles interactions with fractional power-law decaying function. Based on such considerations, in the following, the function $g(x,\xi)$ will be assumed as:

$$g(x,\xi) = \frac{c_\alpha E \beta_2 \alpha}{\Gamma(1-\alpha)} \frac{1}{|x-\xi|^{1+\alpha}}, \qquad 0 < \alpha \leq 1 \tag{1}$$

where $\Gamma(\cdot)$ is the Euler gamma function, β_2 is a coefficient accounting for the percentage of the non-local interactions with $0 \leq \beta_2 \leq 1$, c_α is a dimensional parameter, $[c_\alpha] = L^{\alpha-4}$ that rules the intensity of the long-range interactions. The real coefficient α is related to the fractal dimension of the inner micro-structure and it may be selected after proper experimental set-up, that is underway in the authors' research activities. The governing equation of the proposed model is obtained by a variational approach as:

$$\begin{aligned} \mathcal{H}(\dot{u},u,\varepsilon,\eta) = &\int_{t_1}^{t_2} \left\{ \frac{A\rho}{2} \int_a^b \dot{u}(x,t)^2 dx - \frac{A\bar{E}}{2} \int_a^b \varepsilon(x,t)^2 dx \right\} dt + \\ &+ \int_{t_1}^{t_2} \left\{ \frac{A^2}{4} \int_a^b \int_a^b g(x,\xi)\, \eta(x,\xi,t)^2 d\xi dx + A \int_a^b P(u,x,t)\, dx \right\} dt \end{aligned} \tag{2}$$

where $\mathcal{H}(\dot{u},u,\varepsilon,\eta)$ is the extended Hamiltonian (see e.g. [35]), $P(u,x,t)$ is the potential energy of the external force field and $\bar{E} = \beta_1 E$, with $\beta_1 = 1 - \beta_2$ is the reduced elastic modulus that accounts for long-range effects.

The field equations of the elastodynamics in conjunction with the natural boundary conditions may be obtained by variations with respect to the state variables of the elastic problem, namely $\dot{u}(x), u(x), \epsilon(x)$ and $\eta(x,\xi)$ yielding:

$$\rho u_{tt} - \bar{E} u_{xx} + \beta_2 c_a E \left[\left({}_x\hat{D}^{\alpha}_{a+} u \right)(x,t) + \left({}_x\hat{D}^{\alpha}_{b-} u \right)(x,t) \right] = f(x,t) \tag{3}$$

where we denoted $[\cdot]_{tt} = \partial^2/\partial t^2$ and $[\cdot]_{xx} = \partial^2/\partial x^2$. The natural boundary conditions associated to the proposed Hamiltonian reads:

$$\bar{E}\, A\varepsilon(a,t) = A\; \sigma^{(\ell)}(a,t) = -F_a(t) \tag{4a}$$

$$\bar{E}\, A\varepsilon(b,t) = A\; \sigma^{(\ell)}(b,t) = F_b(t) \tag{4b}$$

where $\sigma^{(\ell)}(x,t) = \bar{E}\varepsilon(x,t)$ is the well-known, local Cauchy stress and, at the limit it corresponds to the contact interactions between adjacent volumes of the NN lattice models. The very remarkable result exploited in Eq.(4a,4b) has been discussed in detail on rigorous basis in previous papers (see [11], [13]) but it may be also easily explained by considering that non-local interactions have been modeled as body forces and then they do not affect the mechanical boundary conditions involving only contact forces. The integral operators $\left({}_x\hat{D}^{\alpha}_{a+} u \right)(x,t)$ and $\left({}_x\hat{D}^{\alpha}_{b-} u \right)(x,t)$ in eq.(3) represent the integral parts of the Marchaud fractional derivatives $\left({}_x D^{\alpha}_{a+} u \right)(x,t)$ and $\left({}_x D^{\alpha}_{b-} u \right)(x,t)$ on the finite domain that are written as:

$$\begin{aligned} \left({}_x D^{\alpha}_{a+} u \right)(x,t) &= \frac{u(x,t)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{u(x,t) - u(\xi,t)}{(x-\xi)^{1+\alpha}} d\xi = \\ &= \frac{u(x,t)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \left({}_x\hat{D}^{\alpha}_{a+} u \right)(x,t) \end{aligned} \tag{5a}$$

$$\begin{aligned} \left({}_x D^{\alpha}_{b-} u \right)(x,t) &= \frac{u(x,t)}{\Gamma(1-\alpha)(b-x)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{u(x,t) - u(\xi,t)}{(\xi-x)^{1+\alpha}} d\xi = \\ &= \frac{u(x,t)}{\Gamma(1-\alpha)(b-a)^{\alpha}} + \left({}_x\hat{D}^{\alpha}_{b-} u \right)(x,t) \end{aligned} \tag{5b}$$

For unbounded waveguides the field equation of the elastic waves propagation is obtained for $a \to -\infty$, $b \to \infty$ in eq.(3). In this case, the Marchaud fractional derivatives will involve only the integral terms in eqs.(5a,5b) and they will be denoted as $\left({}_x D^{\alpha}_{+} u \right)(x,t)$ and $\left({}_x D^{\alpha}_{-} u \right)(x,t)$. In this setting the governing equation of the waves propagation reverts to

$$\rho u_{tt}(x,t) - \bar{E} u_{xx}(x,t) + \beta_2 c_{\alpha} E \left[\left({}_x D^{\alpha}_{+} u \right)(x,t) + \left({}_x D^{\alpha}_{-} u \right)(x,t) \right] = f(x,t) \tag{6}$$

that is a fractional differential equation with constant coefficients that will be extensively used in the next section to study dispersion and disturbance propagation of elastic waves in unbounded domains. Other generalization of the classical wave equation that involves fractional differentiation in time can be found in literature in ([27]) and ([23]).

The integro-differential problem represented in eq.(3) possesses an equivalent mechanical model that may be used to better clarify the physics of the proposed model of non-local elasticity. As in fact, the fractional differential equation may be reverted into a discrete version introducing the difference operator $d/dx \cong \Delta/\Delta x$ so that the following system of ordinary differential equations is obtained as:

$$\begin{aligned} & M\ddot{u}_j + K^{(\ell)}(u_{j+1} + 2u_j - u_{j-1}) + \\ & + \sum_{h=1}^{j-1} K^{(n\ell)}_{hj}(u_j - u_h) - \sum_{h=j+1}^{n} K^{(n\ell)}_{hj}(u_h - u_j) = F_j(t) \end{aligned} \tag{7}$$

with $(j = 1, ..., n)$ where $F_j(t) = f(x,t)\,\Delta V$, $M = \rho A \Delta x$ is the mass of volume $\Delta V = A\Delta x$, and $K^{(\ell)} = \bar{E}A/\Delta x$, $K_{hj}^{(n\ell)} = \Delta V^2 \alpha c_\alpha E \beta_2 / \left(\Gamma(1-\alpha)\,|x_j - x_h|^{1+\alpha}\right)^{-1}$ are, respectively, the local and non-local stiffness, and $F_j(t) = f(x,t)\,\Delta V$.

Eq.(7) corresponds exactly to the equilibrium equations of the point-spring model depicted in Fig.(2) where only four points have been reported for clarity. The dynamic equilibrium equations of such a model may be cast in matrix form as:

$$\mathbf{M}\ddot{\mathbf{u}} + \left(\mathbf{K}^{(\ell)} + \mathbf{K}^{(n\ell)}\right)\mathbf{u} = \mathbf{F} \tag{8}$$

with, $\mathbf{M}$ a diagonal mass matrix, and:

$$\mathbf{K}^{(\ell)} = \begin{bmatrix} K^{(\ell)} & -K^{(\ell)} & ... & 0 \\ & 2K^{(\ell)} & -K^{(\ell)} & \\ & \text{(sym)} & ... & ... \\ & & & K^{(\ell)} \end{bmatrix} \tag{9}$$

$$\mathbf{K}^{(n\ell)} = \begin{bmatrix} K_{11}^{(n\ell)} & -K_{12}^{(n\ell)} & ... & -K_{1n}^{(n\ell)} \\ ... & K_{22}^{(n\ell)} & ... & -K_{2n}^{(n\ell)} \\ ... & ... & ... & ... \\ -K_{n1}^{(n\ell)} & ... & ... & K_{nn}^{(n\ell)} \end{bmatrix} \tag{10}$$

are the stiffness matrices of the local and long-range (non-local) interactions where the diagonal terms read: $K_{jj}^{(n\ell)} = \sum_{\substack{h=1 \\ h\neq j}}^{n} K_{jh}^{(n\ell)}$.

Because of the equivalence between the fractional differential equation and its discretized version as $\Delta x \to 0$ reported in eq.(7), the proposed model has been dubbed as *physically-based model* of non-local mechanics. It must be remarked that eq.(7) coincides with the NN Born-Von Karman monoatomic lattice model assuming $\Delta x = d$, being d the lattice constant whereas if also the springs connecting non-adjacent elements are present then it may be thought as an NNN Born-Von Karman lattice model. The equivalent mechanical representation of the local elastic model here proposed will be used in the next section to provide waves propagations in bounded waveguides.

3 Analysis of waves propagation in 1D solid with fractionally decaying long-range interactions

The elastic model studied in the context of fractional decay of long-range interactions is applied, in this section to unbounded waveguides, aiming to provide the effects of the long-range interactions on traveling disturbances as well as on the speed of traveling waves. In the first part of this section the analysis of a disturbance traveling along the waveguide will be analyzed and the dispersion of elastic waves will be examined, in terms of the frequency-wavenumber relations. In the second part of this section the analysis of elastic waves in bounded waveguides will be expoloited to highlight the presence of boundary effects that are a crucial aspect of non-local elasticity.

3.1 Dispersion of elastic waves in 1D unbounded waveguide

Analysis of traveling disturbances in unbounded waveguides is ruled by the fractional differential equation reported in eq.(6). Let us assume that, at time instant $t = 0$, the initial displacement and velocity of the solid reads: $u(x,0) = \bar{u}(x)$ and $u_t(x,0) = \bar{v}(x)$. Let us denote as $\hat{u}(\kappa,t)$ the spatial Fourier transform as $\hat{u}(\kappa,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\kappa x} dx$ where $i = \sqrt{-1}$ is the imaginary unit, κ is the wavenumber. Then, by assuming $f(x,t) = 0$, and making the spatial Fourier transform of Eq.(6) the following differential equation is readily obtained:

$$\hat{u}_{tt}(\kappa,t) + c_l^2 h(\kappa)\, \hat{u}(\kappa,t) = 0 \tag{11}$$

where the wavenumber function $h(\kappa)$ is related to the fractional form of the decaying function as:

$$h(\kappa) = \beta_1 \kappa^2 + 2c_\alpha \cos(\alpha\pi/2)\, |\kappa|^\alpha\, \beta_2; \qquad (0 < \alpha \leq 1) \tag{12}$$

In the former equation, the second term is proportional to the Fourier transform of the spatial Riesz fractional derivatives. It may be observed that, in the latter equation the case $\alpha = 1$ may be recovered observing that for this value of α the sum of left and right fractional derivatives vanishes and the local case $\omega = \beta_1 c_l \kappa$ is recovered. The solution of the second-order, linear differential equation reported in Eq.(11) is provided by the linear combination:

$$\hat{u}(\kappa,t) = d_1 \cos\left(c_l \sqrt{h(\kappa)} t\right) + d_2 \sin\left(c_l \sqrt{h(\kappa)} t\right) \tag{13}$$

where $c_l = \sqrt{\bar{E}/\rho}$ is the speed of elastic waves in the context of classical mechanics and d_1 and d_2 are related to the initial condition as:

$$\hat{u}(\kappa,0) = \int_{-\infty}^{\infty} \bar{u}(x)\, e^{-i\kappa x} dx$$
$$\dot{\hat{u}}(\kappa,0) = \int_{-\infty}^{\infty} \bar{v}(x)\, e^{-i\kappa x} dx$$

It has to be remarked that as $\beta_2 = 0$, that is for a classical local continuum, the function $\hat{u}(\kappa,t)$ in eq.(13) coincides with the classical solution of waves propagation on 1D Cauchy solid. A numerical solution of a traveling Gaussian type disturbance along the waveguide has been reported in Fig.(3) for different values of the fractional differentiation order α. The initial conditions of the traveling disturbance has been assumed as: $\bar{u}(x) = 1/\left(\sqrt{2\pi s}\right) \exp\left[-x^2/s^2\right]$ and $\bar{v}(x) = 0$ where s is a scale parameter ruling the spread of the Gaussian initial disturbance. The parameters chosen for the simulation are: $\rho = 5.1 \times 10^{-9} kg/mm^3$, $L = 1000mm$, $E = 72kN/mm^2$, $A = 100mm^2$, $s = 20$, $c_\alpha = 1mm^{\alpha-4}$ and $\beta_1 = 0.9$.

It is well-known that disturbance propagation in unbounded waveguide, faced in the framework of classical continuum mechanics ($\beta_2 = 0$ or $\alpha = 1$) involves the presence of a couple of identical disturbances traveling in opposite directions with constant speed c_l as from D'Alambert solution of the wave equation. A different scenario appears for the non-local model of elastic continuum investigated in the paper. In fact in Fig.(3) it may be seen that as soon as the differentiation index α takes on real numerical values the shapes of the traveling disturbance along the waveguide is sharply modified.

Observation of the propagating disturbance shows that as long as the central core of the disturbance propagates away from the perturbed zone this location does not came at rest but an alternating sequence of positive and negative displacement concentrations are involved in the central zone of the waveguide. It may be observed that the time interval between two positive (or negative) peaks of the perturbed pulse is related to the fractional order of differentiation α. Values of the parameter α close to one leads to waves scattering similar to the local case (see Fig.3) that is exactly achieved for $\alpha \to 1$.

The shape modification of traveling disturbance involves different propagation speed of the elastic waves with different wavelength in the $(\omega - \kappa)$ spectrum. Such a phenomenon is characteristic of crystal lattice models of real materials and it is detected from the deviation of the $(\omega - \kappa)$ relations from the straight line $\omega = c_l \kappa$ at the border of the Brillouin zone. In dispersive continuum this phenomenon is known as *dispersion* of elastic waves. This aspect is captured by the proposed model of non-local elasticity and the proper $(\omega-\kappa)$ relations is obtained introducing a double, space-time, Fourier transform of Eq.(3) in the unbounded case, defined as: $U(\kappa,\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,t)\,e^{-i(\omega t+\kappa x)}dxdt$ that yields, after some straightforward manipulations an $(\omega-\kappa)$ relation of the form:

$$\omega(\kappa)=c_l\left(\beta_1\kappa^2+2\beta_2 c_\alpha\cos(\alpha\pi/2)\,|\kappa|^{\alpha}\right)^{1/2};\qquad(0<\alpha\leq 1)\tag{14}$$

The particular case $\alpha=1$ yields the well-known $(\omega-\kappa)$ relation of the classical continuum mechanics $\omega=\beta_1 c_l\kappa$. For non-integer values of the fractional differentiation index α the non-linear $(\omega-\kappa)$ relation involves different propagation speed $c(\kappa)=d\omega(\kappa)/d\kappa$ at different wavelenghts $\lambda=1/\kappa$, i.e. as like as the typical behavior of dispersive media.

Various dispersion curves for different values of the fractional differentiation order α have been reported in Fig.(4) without contact forces (e.g. $\beta_1\to 0$) since they are inessential to discuss the non-linear dispersion relation in Eq.(14). In Fig.(4a) the dispersion curves of the proposed model show that as $\kappa\to 0$, $d\omega(\kappa)/d\kappa|_{\kappa=0}\to\infty$ that is, a vertical slope $\kappa=0$. This behavior is strictly related with the features of traveling disturbances along the waveguide reported in Fig.(3a) because it implies that infinitely long elastic waves behaves statically. This consideration means that the initial disturbance is transfered to all the long-range elastic bonds between volume elements composing the waveguide and as soon as the disturbance moves away from the initially perturbed core the same location is still loaded by all the long-range forces. Such an effect is the main reason for the presence of compressive and tensile disturbances in the initially perturbed core of the elastic waveguide as observed in Fig.(3).

A similar scenario does not appear with a smoother class of the distance-decaying function $g(x,\xi)=g(x-\xi)$ yielding a dispersion relation in the form ([35]):

$$\omega(\kappa)=c_l\left(\beta_1\kappa^2+\beta_2 A\int_{-\infty}^{\infty}g(x,\xi)\left(\cos(\kappa x)-\cos(\kappa\xi)\right)d\xi\right)^{1/2}\tag{15}$$

that has been reported with dotted and dashed lines in Fig.(4b) for Helmoltz and Bi-Helmoltz decaying functions, respectively defined as:

$$g(x,\xi)=\left(\beta_2 E/Al^3\right)\mathrm{e}^{-|x-\xi|/l}\tag{16a}$$

$$g(x,\xi)=(\beta_2 E/A)\left\{\mathrm{e}^{-|x-\xi|/l_1}/l_1^3-\mathrm{e}^{-|x-\xi|/l_2}/l_2^3\right\}\tag{16b}$$

where l, l_1 and l_2 are internal lengths dependent on the microstructure. The dispersion curves have been contrasted with the $(\omega-\kappa)$ well-known relation of the Born-Von Karman model $\omega/c_l=sin(\pi\kappa/d)$, where d is the lattice distance, to show the degree of accuracy in the description of the dispersive phenomenon.

It must be remarked that the dispersion curves obtained in Eq.(16) are totally in agreement with the dispersion curves obtained by the integral model of non-local elasticity (see [18], [21], [9]). This is not surprising since it has been widely shown that the proposed model of non-local elasticity, returns the integral model of non-local elasticity *in unbounded domains* for some specific class of distance decaying function as fractional-type ([14]) or Helmoltz and bi-Helmoltz decay ([35]). The main difference between the introduction of the hypersingular decay reported in Eq.(1) and smoother version of the long-range distance-decaying is related to the behavior of the function $g(x,\xi)$ as $x\to\xi$. As in fact the presence of a non-essential singularity as $x\to\xi$ involves a supplementary restraint between adjacent volumes as $u(x)\to u(\xi)$ that regularizes the displacement function in presence of concentrated loads as it has been shown in previous studies (see [12]).

3.2 Standing wave analysis in bounded waveguides

Analysis of elastic waves in bounded waveguides is presented in this section to show the physical effects of the boundaries involved in the proposed model of non-local elasticity. This is an important issue of the paper since the presence of boundaries is hardly accounted for in the existent non-local elasticity theories yet used by several authors in waves scattering analysis.

In this setting let us assume that the considered waveguide is located at abscissas $a = 0, b = L$ and that no external body force field is applied to the waveguide so that $f(x,t) = 0$, without loss of generality.

Some physical insights in waves propagation in elastic bounded waveguides may be shown considering an elastic waveguide in the interval $[0, L]$ fully restrained at $x = 0$ with prescribed, assigned displacement at $x = L$.

In this case the integro-differential boundary value problem is written as:

$$\left\{ \begin{array}{l} u_{tt} - \beta_1 c_l^2 u_{xx} + c_\alpha \beta_2 c_l^2 \left[\left({}_x\hat{D}^\alpha_{a+} u \right)(x,t) + \left({}_x\hat{D}^\alpha_{b-} u \right)(x,t) \right] = 0 \\ u(0,t) = 0; \qquad u(L,t) = U_L \sin(\Omega t) \end{array} \right. \tag{17}$$

where U_L is the amplitude and Ω the forcing frequency .

The solution of the elastic problem may be obtained considering the fractional operator $\hat{D}^\alpha[s](x) = \left({}_x\hat{D}^\alpha_{a+} s \right)(x) + \left({}_x\hat{D}^\alpha_{b-} s \right)(x)$ that may be discretized by means of the fractional finite difference operator ([32]), yet used in static context ([14]). Following [32] the discretized form of the integral part of the Marchaud fractional derivative may be written as: $\hat{D}^\alpha[s](x_j) = \Delta^\alpha_{x_j} s(x) + O(\Delta x)$ where $\Delta^\alpha_{x_j} s(x)$ is the fractional difference operator defined as:

$$\begin{array}{l} \Delta^\alpha_{x_j} s(x) = \frac{\alpha^{-1}}{\Gamma(1-\alpha)} \times \\ \times \left\{ \sum\limits_{h=1}^{j-1} \left[(x_{j-h+1})^{-\alpha} - (x_{j-h})^{-\alpha} \right] s(x_h) + \sum\limits_{r=j+1}^{n} \left[(x_{j-r})^{-\alpha} - (x_{j+1-r})^{-\alpha} \right] s(x_r) \right\} \end{array} \tag{18}$$

leading to a discretized set of linear differential equation in the unknown functions $u_j(t)$ with $j = 1, 2, ..., n$. It has previously shown ([14]) that as $\Delta x \to 0$ Eq.(18) coalesces with the non-local terms reported in eq.(7). Propagation of elastic waves due to the presence of a sinusoidally varying displacements of the rightmost edge has been reported in Fig.(5) and (6), respectively for the local case ($\alpha \to 1$) and for the non-local fractional model with $\alpha = 0.5$. The parameters used for computation are: $L = 200$, $n = 1201$, $E = 72\, kN/mm^2$, $\beta_1 = 0.76$, $A = 100\, mm^2$, whereas the parameter $c_\alpha = 0.21 mm^{\alpha-4}$ and $\rho = 2.5 \times 10^{-6}\, kg/mm^3$. The observation of the waves propagation shows that in presence of fractional long-range interactions a marked scattering of the waves propagating away from the perturbed edge is experienced as it is stressed by the absence of a marked wavefront separating the perturbed space from the unperturbed zone of the waveguide.

The edge effects in elastic waves scattering are also shown in Figs.(7a,c) representing the axial displacement of a double restrained waveguide with an initial sinusoidally varying shape as $u(x,0) = \bar{u}(x) = sin(\bar{\kappa} x)$ and $v(x,0) = \bar{v}(x) = 0$ with $\bar{\kappa} = 1/mm$ for different values of the fractional strength c_α in Eq.(17). It may be observed that the presence of the two borders modify the initial sinusoidal shape as time varies in presence of long-range interactions. The extent of such a modification is influenced by the values of the coefficient c_α.

The asymptotic condition (as $t \to \infty$) is a stationary, non-propagating, elastic wave that may be obtained assuming a steady-state motion of the elastic waveguide in the form

$$u(x,t) = \psi_j(x) e^{i\omega_j t} \tag{19}$$

where we denoted ω_j and $\psi_j(x)$ the $j-th$ natural frequency and eigenmode, respectively of the waveguide that are obtained replacing Eq.(19) into Eq.(17) yielding the boundary value problem:

$$\text{(20)} \qquad \begin{cases} \omega_j^2\psi_j\left(x\right)+\beta_1 c_l^2\frac{d^2\psi_j(x)}{dx^2}-c_\alpha\beta_2 c_l^2\left[\left(\hat{D}\psi_j\right)(x)\right]=0 \\ \psi_j\left(0\right)=0; \qquad \psi_j\left(L\right)=0 \end{cases}$$

Such a problem may be solved by discretization with the aid of fractional finite differences $\Delta^{\alpha}_{x_k}\psi\left(x\right)$, $1\leq k\leq n$ previously outlined. The boundary value problem is therefore reverted to a discrete eigenvalue problem with eigenvalues ω_j obtained as solution of the secular equation:

$$\text{(21)} \qquad Det\left[\omega_j^2\mathbf{M}+\mathbf{K}\right]=0$$

with $\mathbf{K}=\mathbf{K}^{(\ell)}+\mathbf{K}^{(\mathbf{n}\ell)}$.

In order to make a feasible comparison between the local and the non-local discrete model for different values of α, firstly, for each value of the parameter α the coefficient c_α has been selected to have the coincidence between the first natural circular frequency. In other words, for every value of α the system' stiffness is rescaled by proper choice of c_α to fit the first local natural frequency, that for the parameters selected in this example, is $413.13 rad/s$. Then, the first fifteen natural frequencies for different values of the parameter $\alpha=0; 0.3; 0.5$ and 0.9 are contrasted in Fig.(8).

It can be observed that with $\alpha=1$ the effect of long-range forces vanishes and the model coincides with the 1D elastic bar. For values of α in the proximity of the unity, i.e. $\alpha=0.9$ in Fig.(8), the non-local natural frequencies are *lower* than the local's one as reported in other works presented in literature. As soon as the value of α decreases, the bar stiffness increases and consequently the lower frequencies are *higher* in the non-local model than in the local one.

Similar considerations may be ensued from the observation of the corresponding standing waves $\psi_j\left(x\right)$ contrasted with the local eigenfunctions in Figs.(9) and (10). In particular, in Figs.(9) the first four eigenfunctions are plotted. From this picture it can be noted that both $\alpha=1$ and $\alpha=0$ are eigenfunctions of local type, with the difference that the latter is stiffer than the former, which conversely exactly coincides with the local bar.

It may be observed that, as far as the non-local effects have been accounted for, significant differences both in the natural frequencies and in the standing wave shape functions are experienced. As like as α wander off from zero and one (local behavior) the effect of the border in the eigenmodes becomes relevant.

For higher eigenfunctions, as the eighth eigenfunction reported in Fig.(10), the stiffer character of the non-local model with respect to the local one is more evident, because of the presence of the long-range interactions. Fig.(10) shows in fact that the wavelength for $\alpha=0.3$ and $\alpha=0.9$ (dashed lines) are bigger than the local (continuous line).

Finally, we stress that it has been proved in ([10]) that for very particular cases of boundary conditions the proposed long-range interaction model reverts to the integral model of non-local elasticity. Based on this consideration we conclude that the proposed model of non-local elasticity is a generalization, on physical grounds, of the integral models of non-local elasticity theory, proposed by Eringen and co-workers.

4 Conclusions

In this paper waves propagation in 1D elastic waveguides has been framed in the context of fractional-type non-local continuum. The non-local effects have been accounted for in the model, introducing long-range interactions between non-adjacent volumes, that are monotonically decreasing with the interaction distance. Moreover, they are proportional to the product of interacting volumes as well as to their relative axial displacements. Such an interaction model is totally equivalent to a 1D point-spring network with linear springs connecting adjacent and non-adjacent volume elements. These latter springs possess distance-decaying stiffness and they are physically equivalent to the long-range effects evidenced in NNN lattice models of material. The model has been investigated either in unbounded domains to deal with dispersion of elastic waves as well as in bounded domains to highlights edge effects involved in the model. Waves propagation in unbounded domain has been found to be ruled by a fractional differential equation of Marchaud-type. Different situation arises in the analysis of bounded domain, since the governing equation proves to be an integrodifferential one with hyper-singular kernel representing only the integral parts of the Marchaud fractional derivatives. It has been shown that under the assumption of hypersingular, fractional decay of the long-range interactions a sequence of tensile and compressive pulses is spread away from an initial elongation of the central core of the elastic waveguides showing evident differences in the propagation speed at different wavelengths. This behavior has been confirmed by the observation of the frequency-wavenumber relations showing marked non-linearity for non-integer values of the differentiation index. Some numerical examples have been reported also for the case of bounded domain introducing propagation of sinusoidal displacements as well as the traveling of a sinusoidal shape along a double bounded waveguide.

5 Acknowledgments

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FIGURES

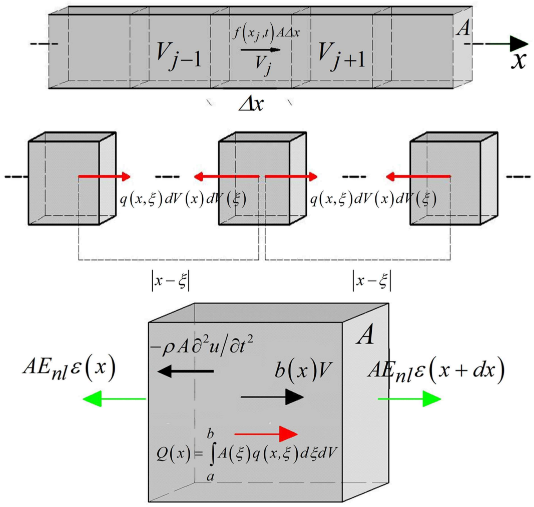

Figure 1: Equilibrium of a solid element with long-range interactions

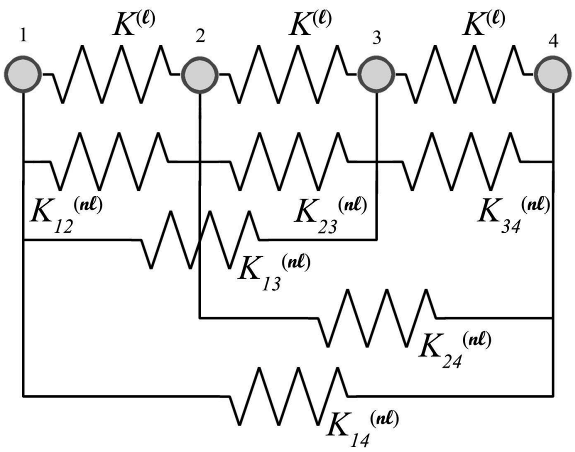

Figure 2: Spring-mass model with long range interactions

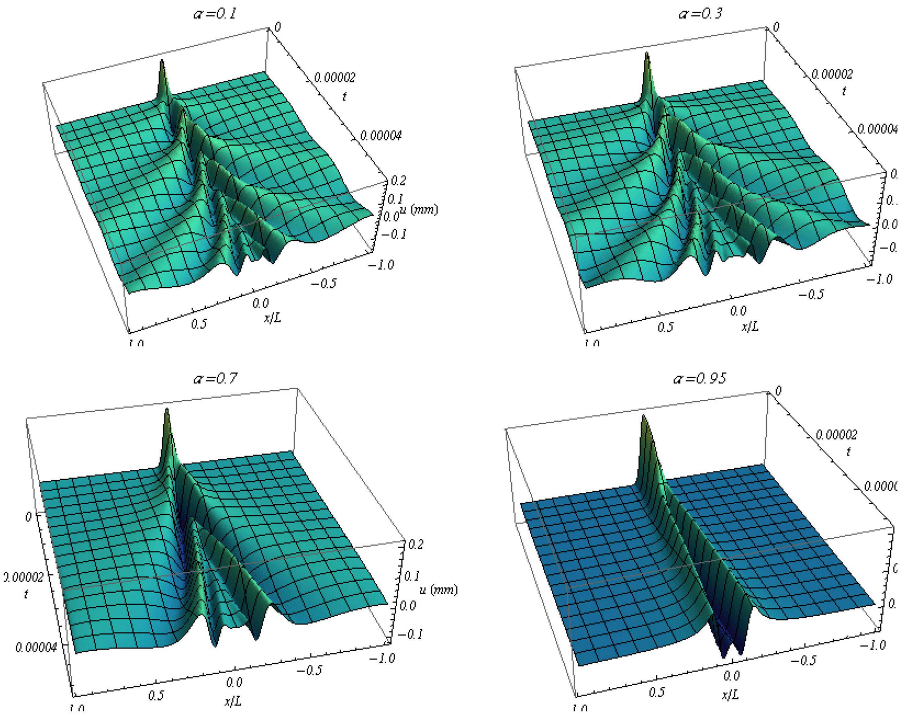

Figure 3: Disturbance propagations for different α

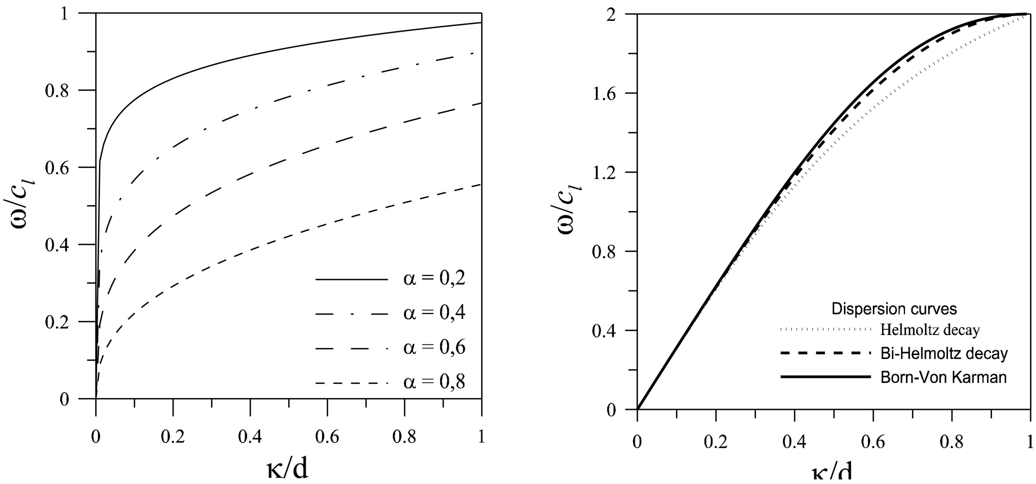

Figure 4: Dispersion curves for: a) power-law fractional decaying long-range interactions; b) exponential type decaying long-range interactions with $l = 0.38$, $l_1 = 0.32$, $l_2 = 0.22$

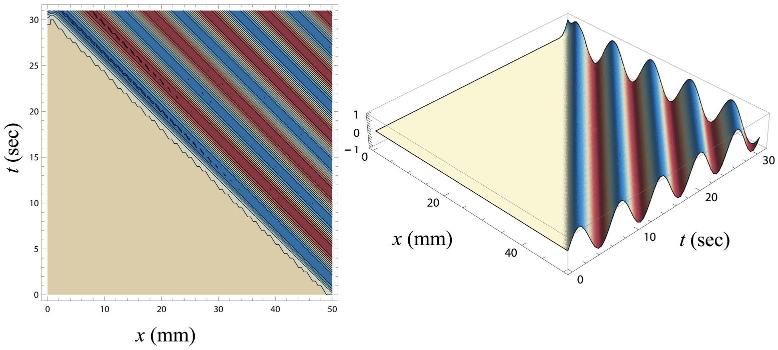

Figure 5: Classical local wave propagation

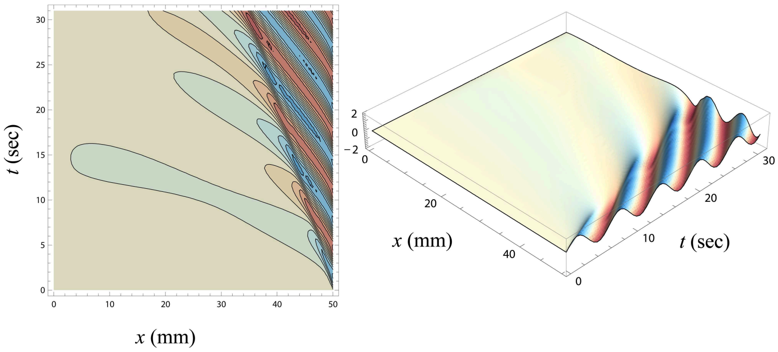

Figure 6: Non-local wave propagation

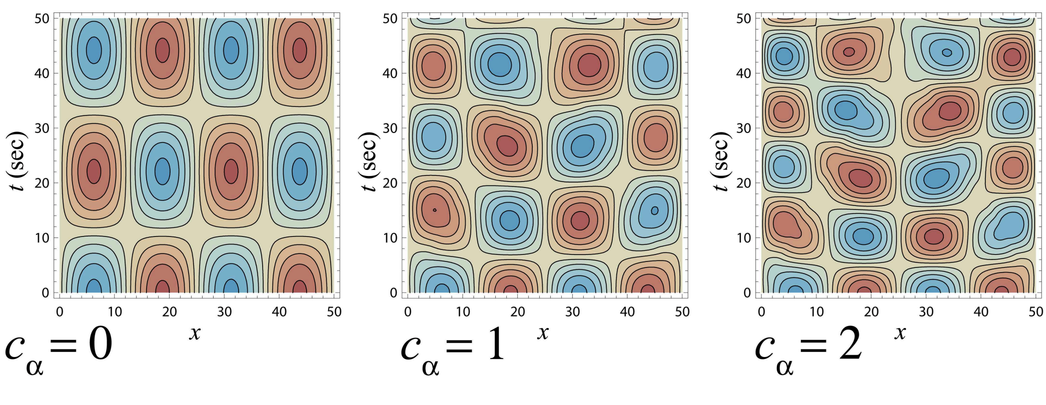

Figure 7: Propagation of a sinusoidal initial deformation varying c_α, with $L = 50\,mm$, $\alpha = 0.5$

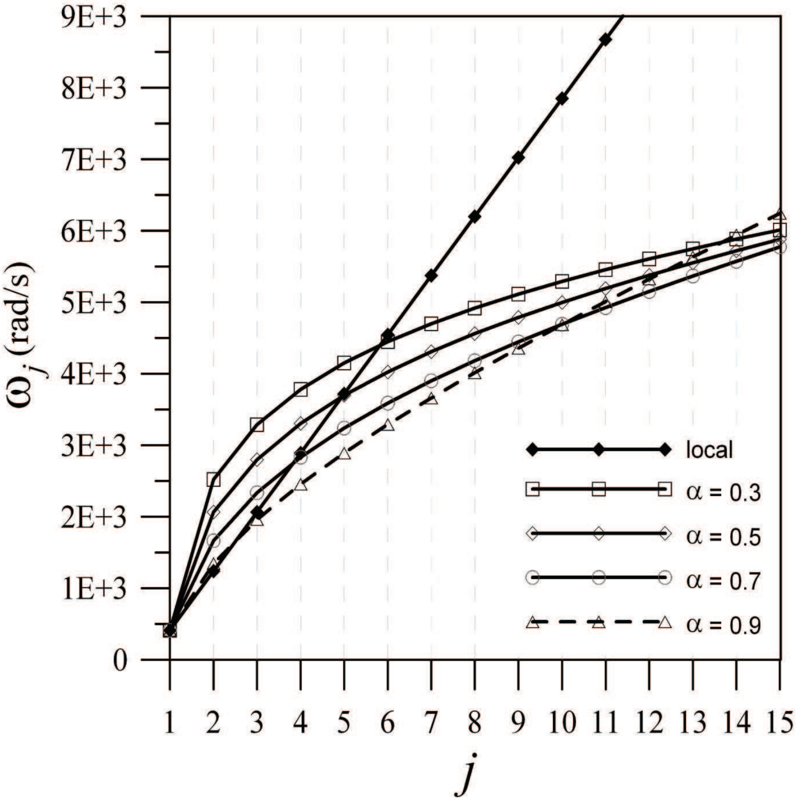

Figure 8: Comparison between natural circular frequencies varying the non-local parameter α

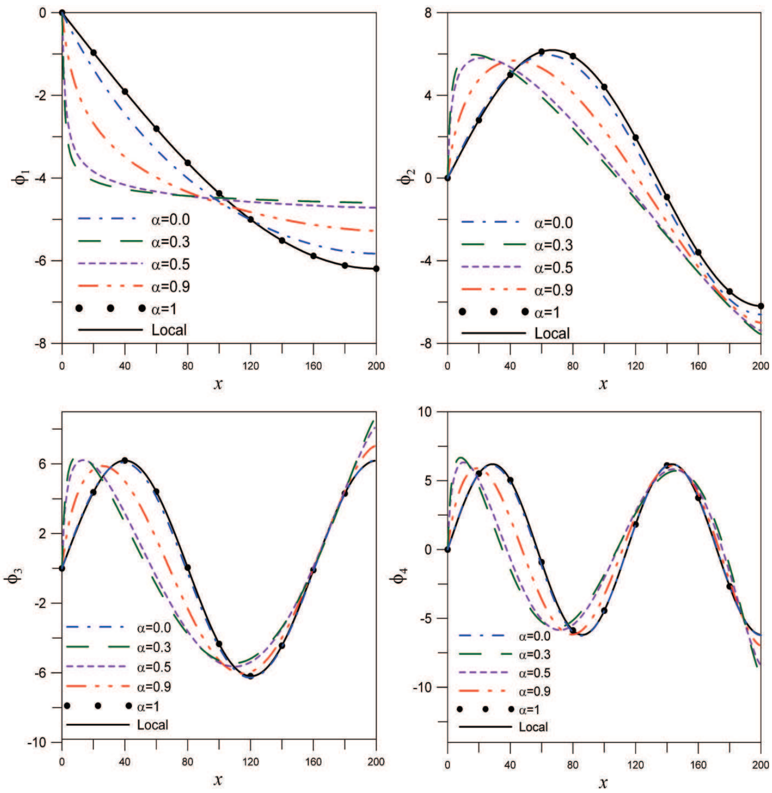

Figure 9: First four eigenmodes varying α

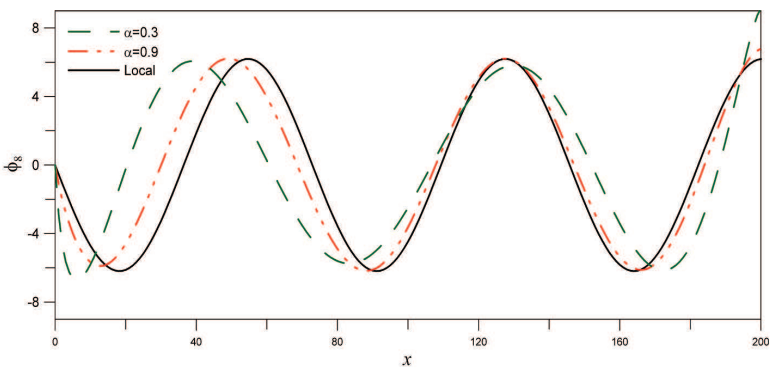

Figure 10: Eighth eigenmode varying α