Reduced equations of motion for a wheeled inverted pendulum

Sergio Delgado * **, Sneha Gajbhiye** ** **, Ravi N. Banavar** **

* *Technische Universität München, Boltzmannstr. 15, D-85748, Garching (Tel: +49-89-289 15679; e-mail: s.delgado@tum.de).*
** *Indian Institute of Technology Bombay, India (e-mail: {banavar,sneha}@sc.iitb.ac.in)*

Abstract: This paper develops the equations of motion in the reduced space for the wheeled inverted pendulum, which is an underactuated mechanical system subject to nonholonomic constraints. The equations are derived from the Lagrange-d'Alembert principle using variations consistent with the constraints. The equations are first derived in the shape space, and then, a coordinate transformation is performed to get the equations of motion in more suitable coordinates for the purpose of control.

Keywords: Underactuated mechanical systems, nonholonomic constraints.

1. INTRODUCTION

The Wheeled Inverted Pendulum (WIP) - and its commercial version, the Segway - has gained interest in the past several years due to its maneuverability and simple construction (see e.g. Grasser et al. [2002], Segway [2015, Jan]). Other robotic systems based on the WIP are becoming popular as well in the robotic community for human assistance or transportation as can be seen in the works of Li et al. [2012], Nasrallah et al. [2007], Baloh and Parent [2003]. A WIP consists of a vertical body with two coaxial driven wheels.

The stabilization and tracking control for the WIP is challenging: the system belongs to the class of underactuated mechanical systems, since the control inputs are less than the number of configuration variables: There are a total of two control variables τ_1 and τ_2 which are the torques applied to rotate the wheels, and six configuration variables, namely, the x- and y- position of the WIP on the horizontal plane, the relative rotation angle of each of the wheels with respect to the body ϕ_1 and ϕ_2, the orientation angle θ, and the tilting angle α. In addition, the system is restricted by nonholonomic (nonintegrable) constraints and is thus not smoothly stabilizable at a point as proven by Brockett [1983]. These constraints do not restrict the state space on which the dynamics evolve, but the motion direction at a given point: The rolling constraint impedes a sideways motion, and the forward velocity of the WIP and its yaw rate are directly given by the angular velocity of the wheels. Wheeled robots have largely been considered as purely kinematic systems, due to the simplification in the motion and controllability analysis. The WIP, however, needs to be stabilized by dynamic effects, such that the complete dynamics need to be taken into account. In mechanical systems with nonholonomic constraints the configuration space Q is a finite dimensional smooth manifold, TQ is the tangent bundle - the velocity phase space - and a smooth (non-integrable) distribution $\mathcal{D} \subset TQ$ defines the constraints [1]. While traditional approaches like the Lagrange-d'Alembert equations lead to the equations of motion of nonholonomic mechanical systems (see, e. g., Pathak et al. [2005]), geometric approaches help to understand the structure and the intrinsic properties of the system. There is a lot of work regarding the modeling of nonholonomic systems, see for example Bloch [2003], Ostrowski [1999], Bloch et al. [1996], Bloch et al. [2009] and the references therein. These geometric tools help understand the mechanism of locomotion, i. e., the way motion is generated by changing the shape of the mechanical system.

Symmetries can be exploited to develop dynamical models in a reduced space. Roughly speaking, the Lagrangian L exhibits a symmetry if it does not depend on one configuration variable, lets say, q_j. The variable q_j is called cyclic. The Lagrangian is thus invariant under transformations in cyclic coordinates. Lie group action and symmetry reduction has been successfully applied to model other types of nonholonomic mechanical systems in the differential geometric framework. See for example the works by Bloch et al. [1996], Ostrowski [1999], Gajbhiye and Banavar [2012]. As shown, e. g., by Ostrowski [1999], the resulting equations can be put in a simplified form containing apart from the reduced equations of motion, also the momentum and reconstruction equation, which describe the dynamics of the system along the group directions. That is, how the system translates and rotates in space due to the change in the shape variables. Bloch et al. [2009] further show the advantage of using the Hamel equations to obtain the reduced nonholonomic equations of motion: The momentum equation is in this case given in a body frame which appears to be more natural than in a spatial frame, for the latter is rarely conserved for systems with nonholonomic constraints. The derivation of the reduced nonholomonic equations can be done as well using the constrained Lagrangian and a so-called *Ehresmann connection* which relates motion along the shape directions with the motion

[1] The distribution $\mathcal{D}$ defines the admissible velocities

along the group directions. The approach is based on taking admissible virtual displacements from the Lagrange d'Alembert principle. Admissible means, that the variations satisfy the constraints (given by the connection). This paper follows this modeling tool. Note that we are not imposing the constraints before taking variations, we are taking variations according to the constraints.

Several control laws have been applied to the WIP, mostly using linearized models as can be seen in Li et al. [2012]. There is still the need to exploit the nonlinear geometric structure of the WIP to stabilize and control the system using coordinate-free control laws. Nasrallah et al. [2007] develop a model based on the Euler-Rodrigues parameters and analyze the controllability of the WIP moving on an inclined plane. Pathak et al. [2005] develop a model using the Lagrange-d'Alembert equations and check the strong accessibility condition. The aim of this note is to explore the motion of the system in the reduced (shape) space which lead to some net displacement of the mobile robot (motion in the group space) independently from the starting point. Additionally, we present the equations of motion in more suitable coordinates [2] for control or trajectory planning purposes: Since the shape space of the WIP is not fully actuated, the control task becomes difficult in these coordinates. The choice of the model can be done depending on which better suits the task.

Notation: Contrary to most of the literature, we use here the matrix/vector representation instead of the index convention. Readers are encouraged to read the referred literature for the wide-used index convention. Further, we use the following simplified notation for the transposed Jacobian: $\partial_x^T = \left(\frac{\partial}{\partial x}\right)^T$.

2. EQUATIONS OF MOTION IN SHAPE SPACE

Consider the configuration space $Q = G \times S$, where S denotes the *shape* space and G denotes the *group* space: Q is a trivial principal bundle with fibers G over a base manifold S. The shape space, as the name suggests, denotes the space of the possible shapes of the system. As stated by Ostrowski [1999], this division is natural in mechanisms that locomote, like mobile robots, where position changes are generated by (mostly cyclic) changes in the shape. See for example the oscillations of a snake-board which create the forward motion, or the rotation of the wheels of a mobile platform resulting in a platforms displacement due to the rolling-without-slipping interaction with the environment. The internal shapes of the WIP are solely defined by the relative angles of the wheels with respect to the body. And since the gravity acts on the WIP depending on the tilting angle (the gravity breaks the symmetry), and it is crucial for the stability of the system, the tilting angle is also considered as a shape variable (more on that later). Note that the net motion resulting from a change in shape is independent from the initial position (we assume, that the WIP is moving on a horizontal plane). Mathematically speaking is this nothing but an invariance (symmetry) of the Lagrangian under a change in position (group) coordinates. We are therefore interested in the reduced equations of motion in shape space variables.

[2] The same equations of motion can be found in Pathak et al. [2005]

On the configuration space $Q = G \times S$, the Lagrangian is a function $L : TG \times TS \to \mathbb{R}$ and the distribution characterizing the nonholonomic constraints is given by $\mathcal{D} \subset TQ$. A curve $q(t)$ on Q is said to satisfy the constraints if $\dot{q}(t) \in \mathcal{D}_q, \forall t$. This nonholonomic restriction can also be given in local coordinates as

$$\dot{g} + \mathbb{A}^T \dot{s} = 0, \tag{1}$$

where $g \in G$ and $s \in S$, and the matrix $\mathbb{A}$ describes how $\dot{g}$ and $\dot{s}$ are related to each other due to the constraints. Recall that the equations governing the dynamics of the system satisfy the Lagrange-d'Alembert Principle (F_{ext} denote the external forces)

$$\delta \int L(q, \dot{q}) dt + \int F_{ext}^T \, \delta q \, dt = 0, \tag{2}$$

which is equivalent to

$$\int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \left(\frac{\partial L}{\partial q} \right) - F_{ext}^T \right] \delta q \, dt = 0. \tag{3}$$

Independent from the Lie-group structure, we can intrinsically eliminate the Lagrange-multipliers which arise from the constraint forces and write the reduced equations of motion using the Ehresmann connection (1), which is nothing but a way to split the tangent space into a horizontal (tangent to the shape space) and a vertical (tangent to the group space) part [3]. The curves $q(t)$ solving the equations of motion need to satisfy the constraints. Thus, the variations $\delta q = (\delta s, \delta g)$ are of the form $\delta g + \mathbb{A}^T \delta s = 0$ (see Bloch et al. [1996]). We assume, that the external forces are input torques τ and only act on the shape variables, i. e., $F_{ext}^T \, \delta q = \tau^T \, \delta s$. This assumption is valid, since we will consider group space motion only as a result of a change in the shape variables, and we neglect friction forces. Equation (3) takes the following form

$$\begin{aligned} &\int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) - \left(\frac{\partial L}{\partial s} \right) - \tau^T \right] \delta s \, dt \\ &- \int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{g}} \right) - \left(\frac{\partial L}{\partial g} \right) \right] \mathbb{A}^T \delta s \, dt = 0. \end{aligned} \tag{4}$$

To eliminate the group velocities $\dot{g}$, define the constrained Lagrangian

$$L_c(s, g, \dot{s}) = L(s, g, \dot{s}, -\mathbb{A}^T \dot{s}). \tag{5}$$

The following relationships hold

$$\frac{\partial L_c}{\partial \dot{s}} = \frac{\partial L}{\partial \dot{s}} + \frac{\partial L}{\partial \dot{g}} \frac{\partial \dot{g}}{\partial \dot{s}} = \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial \dot{g}} \mathbb{A}^T \tag{6}$$

$$\frac{\partial L_c}{\partial s} = \frac{\partial L}{\partial s} + \frac{\partial L}{\partial \dot{g}} \frac{\partial \dot{g}}{\partial s} = \frac{\partial L}{\partial s} - \frac{\partial L}{\partial \dot{g}} \frac{\partial (\mathbb{A}^T \dot{s})}{\partial s} \tag{7}$$

$$\frac{\partial L_c}{\partial g} = \frac{\partial L}{\partial g} + \frac{\partial L}{\partial \dot{g}} \frac{\partial \dot{g}}{\partial g} = \frac{\partial L}{\partial g} - \frac{\partial L}{\partial \dot{g}} \frac{\partial (\mathbb{A}^T \dot{s})}{\partial g}. \tag{8}$$

According to (4), and using the mentioned relationships (6) - (8), the equations of motion in terms of the constrained Lagrangian L_c are given by

$$\frac{d}{dt} \left(\partial_{\dot{s}}^T L_c \right) - \partial_s^T L_c + \mathbb{A} \, \partial_g^T L_c = \tau - \mathbb{B} \, \partial_{\dot{g}}^T L, \tag{9}$$

where

$$\begin{aligned} \mathbb{B} \, \partial_{\dot{g}}^T L &= \frac{d}{dt} \left(\mathbb{A} \, \partial_{\dot{g}}^T L \right) - \mathbb{A} \frac{d}{dt} \left(\partial_{\dot{g}}^T L \right) \\ &\quad + \left(\mathbb{A} \, \partial_g^T (\mathbb{A}^T \dot{s}) - \partial_s^T (\mathbb{A}^T \dot{s}) \right) \partial_{\dot{g}}^T L \\ &\Rightarrow \mathbb{B} = \dot{\mathbb{A}} - \partial_s^T (\mathbb{A}^T \dot{s}) + \mathbb{A} \, \partial_g^T (\mathbb{A}^T \dot{s}). \end{aligned} \tag{10}$$

[3] The reader is referred to the references for detailed information regarding Ehresmann connections

The equations of motion can be further simplified if there additionally exists a symmetry with respect to the group variables, meaning that the constrained Lagrangian is independent from g and thus $\partial_g^T L_c = 0$. In that case the equations of motion are

$$\frac{d}{dt}\left(\partial_{\dot{s}}^T L_c\right) - \partial_s^T L_c = \tau - \mathbb{B}\,\partial_{\dot{g}}^T L. \tag{11}$$

Note that these are not the Euler-Lagrange equations for the constrained Lagrangian $L_c(s, \dot{s})$ - that would have been the case for holonomic (integrable) constraints. The term $\mathbb{B}\,\partial_{\dot{g}}^T L$ on the right hand side of (11) would be missing if we had imposed the constraints before taking the variations as shown in Bloch [2003]. This "forcing" term is written in terms of the curvature of the Ehresmann connection (1) as stated in Bloch et al. [1996], and can be interpreted as additional gyroscopic forces.

3. THE WHEELED INVERTED PENDULUM

Figure 1 shows a simple scheme of the WIP. It is basically a body of mass m_B (center of mass at a distance b from the wheels rotation axes) mounted on two wheels of radius r. The distance between the wheels is $2d$ and their mass is denoted by m_W. The wheels are directly attached to the body and can rotate independently. Since they are actuated by motors sitting on the body itself, a tilting motion will automatically rotate the wheels by the tilting angle if the wheels are blocked. The body needs to be stabilized in the upper position through a back and forth motion of the system similar to the inverted pendulum on a cart. In this section we introduce the configuration variables and move on to defining the velocities, the Lagrangian and the constraints.

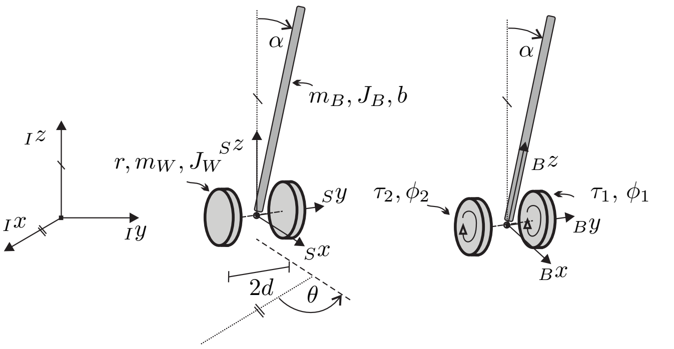

Fig. 1. The Wheeled Inverted Pendulum

We will mainly make use of three different coordinate systems: The inertial I-System, the shaft S-System, which has been rotated around the $_Iz$-axis by the yaw angle θ and will be used only for visualizing purposes, and the body fixed B-System, which is attached to the pendulum's body. For completeness, we also introduce the W_j- frames (for $j = 1, 2$), which are fixed to the wheels. The following notation has been adopted. $_I(*)$ - inertial frame, $_S(*)$ - shaft frame, $_B(*)$ - body (pendulum) frame, $_{W_j}(*)$ - wheel j frame. The triples $({}_i\hat{e}_1, {}_i\hat{e}_2, {}_i\hat{e}_3)$ for $i = \{I, S, B, W_j\}$ denote the unit vectors of the respective coordinate system. The set of generalized coordinates describing the WIP consists of

(1) Coordinates of the origin of the body-fixed coordinate system in the horizontal plane ($x, y \in \mathbb{R}^2$)
(2) Heading angle around the $_Iz$-axis ($\theta \in \mathbb{S}^1$)
(3) Tilting angle around the $_Sy$-axis ($\alpha \in \mathbb{S}^1$)
(4) Relative rotation angle of each of the wheels with respect to the body around the $_{W_j}y$-axis, which coincides with the $_By$-axis ($\phi_1 \in \mathbb{S}^1$ and $\phi_2 \in \mathbb{S}^1$)

The configuration space $Q = G \times S$ of the system is thus $(\mathbb{R}^2 \times \mathbb{S}^1) \times (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)$. Let $\mathbf{X}$ be a point on the pendulum, then the relationship between the coordinates of $\mathbf{X}$ in different frames is given by

$${}_I\mathbf{X} = R_{IS}(\theta)R_{SB}(\alpha)\,{}_B\mathbf{X}; \quad {}_S\mathbf{X} = R_{SB}(\alpha)\,{}_B\mathbf{X},$$

where $R_{IS}(\theta)$ is the orientation of the shaft with respect to the inertial frame, and $R_{SB}(\alpha)$ is the orientation of the pendulum with respect to the shaft given by

$$R_{IS}(\theta) = R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

$$R_{SB}(\alpha) = R_\alpha = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}. \tag{13}$$

Analogously, let $\mathbf{X}$ be a point on the left wheel - wheel 1 (the right wheel case works the same way). The orientation of the wheel with respect to the pendulum is given by

$$R_{W_1}(\phi_1) = R_{\phi_1} = \begin{bmatrix} \cos\phi_1 & 0 & \sin\phi_1 \\ 0 & 1 & 0 \\ -\sin\phi_1 & 0 & \cos\phi_1 \end{bmatrix}, \tag{14}$$

such that the coordinate representation of $\mathbf{X}$ in the inertial and the shaft frame reads

$${}_I\mathbf{X} = R_\theta R_\alpha R_{\phi_1}\,{}_1\mathbf{X}, \quad {}_S\mathbf{X} = R_\alpha R_{\phi_1}\,{}_1\mathbf{X}.$$

Vectors represented in one of the systems (sys) can be transformed into a different coordinate system by the following rule:

$$I\text{-sys} \xrightarrow[z\text{-axis}]{\theta} S\text{-sys} \xrightarrow[y\text{-axis}]{\alpha} B\text{-sys} \xrightarrow[y\text{-axis}]{\phi_{1/2}} W_{1/2}\text{-sys}.$$

Note that ϕ_1 and ϕ_2 are the relative angles of the left and right wheel with respect to the body. This definition seems natural: Since the wheels sit on the body, we can measure the *relative* angle of rotation with respect to the body. The absolute wheel rotation angles are given by $\varphi_1 = \alpha + \phi_1$ and $\varphi_2 = \alpha + \phi_2$. Now, let $\mathbf{r}_S$ be the origin of the B-coordinate frame[4] (position of the shaft) given as

$$\mathbf{r}_S = x\,{}_I\hat{e}_1 + y\,{}_I\hat{e}_2 + r\,{}_I\hat{e}_3, \tag{15}$$

where r is the wheel radius. A point $\mathbf{X}$ on the pendulum (body/bar) will hence be given in inertial coordinates as

$$(x)^b = R_\theta R_\alpha\,\mathbf{X} + \mathbf{r}_S. \tag{16}$$

For $\mathbf{X}$ being a point on the left (right) wheel expressed in body fixed coordinates, its inertial position is given as

$$(x)^{w_1} = R_\theta R_\alpha (R_{\phi_1}\mathbf{X} + d\,{}_B\hat{e}_2) + \mathbf{r}_S, \tag{17}$$

$$(x)^{w_2} = R_\theta R_\alpha (R_{\phi_2}\mathbf{X} - d\,{}_B\hat{e}_2) + \mathbf{r}_S. \tag{18}$$

3.1 Velocities

Differentiating (15) we get the translational velocity of the origin of the B-frame

$${}_I\dot{\mathbf{r}}_S = \dot{x}_I\hat{e}_1 + \dot{y}_I\hat{e}_2. \tag{19}$$

[4] Note that the origin of both, the S- and the B-coordinate frame coincide

Now, we differentiate (16), (17), and (18) to calculate the inertial velocity of a point on the pendulum, and on the wheels, which is given as

$$(\dot{x})^b = (\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,\mathbf{X} + \dot{\mathbf{r}}_S \quad (20)$$

$$(\dot{x})^{w_1} = (\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_1}\mathbf{X} + d\,{}_B\hat{e}_2) + R_\theta R_\alpha \dot{R}_{\phi_1}\mathbf{X} + \dot{\mathbf{r}}_S, \quad (21)$$

$$(\dot{x})^{w_2} = (\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_2}\mathbf{X} - d\,{}_B\hat{e}_2) + R_\theta R_\alpha \dot{R}_{\phi_2}\mathbf{X} + \dot{\mathbf{r}}_S. \quad (22)$$

Since all rotation matrices satisfy $R(t)^T R(t) = I,\ \forall t$, by differentiating with respect to time we get the relation $R^T\dot{R} + \dot{R}^T R = 0$, meaning that $R^T\dot{R}$ is skew symmetric. The matrix $\hat{\omega} = R^T\dot{R}$, defined as

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad (23)$$

denotes the relative angular velocity of the body with respect to its body fixed coordinate frame and expressed in the body frame. The body velocities can therefore be given as

$$(\dot{x})^b = R_\theta R_\alpha (R_\alpha^T(\widehat{\omega}_\theta)R_\alpha + \widehat{\omega}_\alpha)\,\mathbf{X} + \dot{\mathbf{r}}_S, \quad (24)$$

$$(\dot{x})^{w_1} = R_\theta R_\alpha (R_\alpha^T(\widehat{\omega}_\theta)R_\alpha + \widehat{\omega}_\alpha)(R_{\phi_1}\mathbf{X} + d\,{}_B\hat{e}_2) + R_\theta R_\alpha R_{\phi_1}(\widehat{\omega}_{\phi_1})\,\mathbf{X} + \dot{\mathbf{r}}_S, \quad (25)$$

$$(\dot{x})^{w_2} = R_\theta R_\alpha (R_\alpha^T(\widehat{\omega}_\theta)R_\alpha + \widehat{\omega}_\alpha)(R_{\phi_2}\mathbf{X} - d\,{}_B\hat{e}_2) + R_\theta R_\alpha R_{\phi_2}(\widehat{\omega}_{\phi_2})\,\mathbf{X} + \dot{\mathbf{r}}_S. \quad (26)$$

3.2 Lagrangian

The kinetic energy T is the sum of the kinetic energy terms of each of the bodies (pendulum and wheels)

$$T = \frac{1}{2}\int_{\mathcal{B}} \|(\dot{x})^b\|^2 \rho(\mathbf{X})dV + \frac{1}{2}\int_{\mathcal{W}_1} \|(\dot{x})^{w_1}\|^2 \rho(\mathbf{X})dV + \frac{1}{2}\int_{\mathcal{W}_2} \|(\dot{x})^{w_2}\|^2 \rho(\mathbf{X})dV. \quad (27)$$

The potential energy due to the gravity is given by

$$V = m_b g b\,({}_I\hat{e}_3{}^T R_\theta R_\alpha\,{}_B\hat{e}_3) = m_b g b\,\langle {}_I\hat{e}_3,\, R_\theta R_\alpha\,{}_B\hat{e}_3\rangle, \quad (28)$$

where $b\,{}_B\hat{e}_3$ denotes the vector from the origin of the body frame to the body's center of mass, expressed in body fixed coordinates, and g the gravity constant. The Lagrangian is defined as $L = T - V$. Using (20), (21), (22), and (28), L can be written as

$$\begin{aligned} L = &\frac{1}{2}\int_{\mathcal{B}} \|(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,\mathbf{X} + \dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X})dV \\ &+ \frac{1}{2}\int_{\mathcal{W}_1} \|(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_1}\mathbf{X} + d\,{}_B\hat{e}_2) \\ &+ R_\theta R_\alpha \dot{R}_{\phi_1}\mathbf{X} + \dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X})dV \\ &+ \frac{1}{2}\int_{\mathcal{W}_2} \|(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_2}\mathbf{X} - d\,{}_B\hat{e}_2) \\ &+ R_\theta R_\alpha \dot{R}_{\phi_2}\mathbf{X} + \dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X})dV \\ &- m_b g b\,\langle {}_I\hat{e}_3, R_\theta R_\alpha\,{}_B\hat{e}_3\rangle. \end{aligned} \quad (29)$$

Assume that the moment of inertia of body and the wheels have the following diagonal form (in body-fixed coordinates)

$$J_B = \begin{bmatrix} J_{B_{xx}} & 0 & 0 \\ 0 & J_{B_{yy}} & 0 \\ 0 & 0 & J_{B_{zz}} \end{bmatrix}, \quad J_W = \begin{bmatrix} J_{W_{xx}} & 0 & 0 \\ 0 & J_{W_{yy}} & 0 \\ 0 & 0 & J_{W_{xx}} \end{bmatrix}. \quad (30)$$

The kinetic energy of the body is given as

$$T_B = \frac{1}{2}m_B\|\dot{\mathbf{r}}_S + b(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)_B\hat{e}_3\|^2 + \frac{1}{2}\omega_b^T J_B \omega_b, \quad (31)$$

where $\omega_b = {}_B\omega_b$ is the absolute rotation of the body in body coordinates

$$\widehat{\omega}_b = R_\alpha^T(\widehat{\omega}_\theta)R_\alpha + \widehat{\omega}_\alpha.$$

Let $\omega_{w_j} = \omega_b + \omega_{\phi_j}$ be the absolute rotation of the wheel j in its wheel fixed frame W_j. From (21) and (22), and with the fact, that $R_{\alpha B}\hat{e}_2 = {}_S\hat{e}_2$, and $\dot{R}_{\alpha B}\hat{e}_2 = 0$, the kinetic energy of the wheels takes the form

$$T_{W_1} = \frac{1}{2}m_W\|\dot{\mathbf{r}}_S + d\dot{R}_\theta\,{}_S\hat{e}_2\|^2 + \frac{1}{2}\omega_{w_1}^T J_W \omega_{w_1} \quad (32)$$

$$T_{W_2} = \frac{1}{2}m_W\|\dot{\mathbf{r}}_S - d\dot{R}_\theta\,{}_S\hat{e}_2\|^2 + \frac{1}{2}\omega_{w_2}^T J_W \omega_{w_2}. \quad (33)$$

The potential energy is given by $V = m_B g b \cos\alpha$. The Lagrangian is simply $L = T_B + T_{W_1} + T_{W_2} - V$.

3.3 Constraints

Since the wheels roll without slipping on the plane, the velocity of the center points of the wheels

$$(\mathbf{r}_c)_1 = \mathbf{r}_S + dR_\theta R_\alpha\,{}_B\hat{e}_2 \quad (34)$$

$$(\mathbf{r}_c)_2 = \mathbf{r}_S - dR_\theta R_\alpha\,{}_B\hat{e}_2 \quad (35)$$

is solely given by their (absolute) rotation

$$(\dot{\mathbf{r}}_c)_1 = rR_\theta R_\alpha(\widehat{\omega}_{w_1})R_\alpha^T R_\theta^T\,{}_I\hat{e}_3 \quad (36)$$

$$(\dot{\mathbf{r}}_c)_2 = rR_\theta R_\alpha(\widehat{\omega}_{w_2})R_\alpha^T R_\theta^T\,{}_I\hat{e}_3. \quad (37)$$

Differentiating (34) and (35), and comparing them to the equations above yields the rolling constraints as a subset $\mathcal{D}$ of TQ

$$\dot{\mathbf{r}}_S + d(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,{}_B\hat{e}_2 = rR_\theta R_\alpha(\widehat{\omega}_{w_1})R_\alpha^T R_\theta^T\,{}_I\hat{e}_3 \quad (38)$$

$$\dot{\mathbf{r}}_S - d(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,{}_B\hat{e}_2 = rR_\theta R_\alpha(\widehat{\omega}_{w_2})R_\alpha^T R_\theta^T\,{}_I\hat{e}_3. \quad (39)$$

4. GROUP ACTION, INVARIANCE OF THE LAGRANGIAN AND DISTRIBUTION, AND EQUATIONS OF MOTION

The left (or right) action of a Lie group G on a smooth manifold M is a mapping $\Phi : G \times M \to M$. Assuming the action of G is free and proper (Φ is simple, and therefore M/G is a smooth manifold and the mapping $\pi : M \to M/G$ is a submersion), the Lagrangian L is said to be invariant under the group action if L remains invariant under the induced action of G on TM. For more details on this topic see, e.,g., Marsden and Ratiu [1994], or Holm et al. [2009]. Consider the configuration space Q as a submanifold of the space $\tilde{Q} = (\mathbb{R}^3 \times \mathbb{S}^1) \times (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) = \tilde{G} \times S$, where S denotes the *shape* space and consists of the tilt angle α, and the relative wheel angles ϕ_1 and ϕ_2. On this extended space, the Lagrangian is a function $\tilde{L} : T\tilde{Q} \to \mathbb{R}$ and the distribution characterizing the constraints is given by $\tilde{\mathcal{D}} \subset T\tilde{Q}$. Note that $\tilde{Q}$ is a trivial principal bundle with fibers $\tilde{G}$ over a base manifold S. If the Lagrangian $\tilde{L}$ and the Distribution $\tilde{\mathcal{D}}$ are invariant under the action of $\tilde{G}$, the dynamics can be reduced to the quotient space $\tilde{Q}/\tilde{G}$ (the set of orbits), which is diffeomorphic to $\tilde{\mathfrak{g}} \times S$. The left action of the Lie group[5]

$$\tilde{G} = \{(\mathbf{s}, \bar{R}) \in \mathbb{R}^3 \times \mathbb{S}^1 \,|\bar{R}\hat{e}_3 = \hat{e}_3\}. \tag{40}$$

on the manifold $\tilde{Q}$ is given by

$$\begin{aligned}\Phi_{(\bar{R},\mathbf{s})} :&((\mathbf{r}_S, R_\theta), (R_\alpha, R_{\phi_1}, R_{\phi_2}))\\ &\to ((\bar{R}\mathbf{r}_S + \mathbf{s}, \bar{R}R_\theta), (R_\alpha, R_{\phi_1}, R_{\phi_2})).\end{aligned} \tag{41}$$

The left action on the tangent-lifted coordinates of the manifold $\tilde{Q}$ is

$$\begin{aligned}T\Phi_{(\bar{R},\mathbf{s})} :&((\dot{\mathbf{r}}_S, \dot{R}_\theta), (\dot{R}_\alpha, \dot{R}_{\phi_1}, \dot{R}_{\phi_2}))\\ &\to ((\bar{R}\dot{\mathbf{r}}_S, \bar{R}\dot{R}_\theta), (\dot{R}_\alpha, \dot{R}_{\phi_1}, \dot{R}_{\phi_2})).\end{aligned} \tag{42}$$

Claim: The Lagrangian $L = \tilde{L}|_Q$ and the distribution $\mathcal{D}$ are invariant under the action of the group

$$\tilde{G} = \{(\mathbf{s}, \bar{R}) \in \mathbb{R}^3 \times \mathbb{S}^1 \,|\bar{R}\hat{e}_3 = \hat{e}_3\}. \tag{43}$$

Proof. Under the left action of $\tilde{G}$, the Lagrangian (29) is given by

$$\begin{aligned}L = &\frac{1}{2}\int_{\mathcal{B}} \|\bar{R}(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,\mathbf{X} + \bar{R}\dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X}) dV\\ &+ \frac{1}{2}\int_{\mathcal{W}_1} \|\bar{R}(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_1}\mathbf{X} + d\,{}_B\hat{e}_2)\\ &+ \bar{R}R_\theta R_\alpha \dot{R}_{\phi_1}\mathbf{X} + \bar{R}\dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X}) dV\\ &+ \frac{1}{2}\int_{\mathcal{W}_2} \|\bar{R}(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_2}\mathbf{X} - d\,{}_B\hat{e}_2)\\ &+ \bar{R}R_\theta R_\alpha \dot{R}_{\phi_2}\mathbf{X} + \bar{R}\dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X}) dV\\ &- m_b g b\,\langle {}_I\hat{e}_3, \bar{R}R_\theta R_\alpha\,{}_B\hat{e}_3\rangle \end{aligned} \tag{44}$$

$$\begin{aligned}= &\frac{1}{2}\int_{\mathcal{B}} \|(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,\mathbf{X} + \dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X}) dV\\ &+ \frac{1}{2}\int_{\mathcal{W}_1} \|(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_1}\mathbf{X} + d\,{}_B\hat{e}_2)\\ &+ R_\theta R_\alpha \dot{R}_{\phi_1}\mathbf{X} + \dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X}) dV\\ &+ \frac{1}{2}\int_{\mathcal{W}_2} \|(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)(R_{\phi_2}\mathbf{X} - d\,{}_B\hat{e}_2)\\ &+ R_\theta R_\alpha \dot{R}_{\phi_2}\mathbf{X} + \dot{\mathbf{r}}_S\|^2 \rho(\mathbf{X}) dV\\ &- m_b g b\,\langle {}_I\hat{e}_3, \bar{R}R_\theta R_\alpha\,{}_B\hat{e}_3\rangle, \end{aligned} \tag{45}$$

where it has been used the fact that $\bar{R}^T\bar{R} = I$. The Lagrangian coincides with (29), since $\bar{R}^T{}_I\hat{e}_3 = {}_I\hat{e}_3$.

The distribution $\mathcal{D} = \tilde{\mathcal{D}}|_Q$ is described by the equations (38) and (39). Since $\mathcal{D}_q \subset T_qQ$, under left group action of $\tilde{G}$ the distribution $\mathcal{D}$ becomes

$$\begin{aligned}\bar{R}\dot{\mathbf{r}}_S = &\mp d\bar{R}(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,{}_B\hat{e}_2\\ &+ r\bar{R}R_\theta R_\alpha(R_\alpha^T \bar{R}^T \bar{R}\dot{R}_\alpha + R_{\phi_{1/2}}^T \dot{R}_{\phi_{1/2}})R_\alpha^T R_\theta^T \bar{R}^T {}_I\hat{e}_3\end{aligned} \tag{46}$$

$$\begin{aligned}\Rightarrow \dot{\mathbf{r}}_S = &\mp d(\dot{R}_\theta R_\alpha + R_\theta \dot{R}_\alpha)\,{}_B\hat{e}_2\\ &+ rR_\theta R_\alpha(R_\alpha^T \dot{R}_\alpha + R_{\phi_{1/2}}^T \dot{R}_{\phi_{1/2}})R_\alpha^T R_\theta^T {}_I\hat{e}_3. \qquad \blacksquare\end{aligned}$$

4.1 Constrained equations of motion of the WIP

Since a curve $q(t)$ satisfies the constraints if $\dot{q}(t) \in \mathcal{D}_q, \forall t$, we can express the distribution given by (38) and (39) as $\mathcal{D} = \{(s, g, \dot{s}, \dot{g}) \in TQ\,|\,\dot{g} + \mathbb{A}^T\dot{s} = 0\}$, for the group variables $g = (x, y, \theta)^T$, and shape variables $s = (\alpha, \phi_1, \phi_2)^T$. The constraints are satisfied in that specific set of local coordinates for

$$\mathbb{A} = \frac{1}{2}\begin{bmatrix} -2\,r\cos\theta & -2\,r\sin\theta & 0\\ -r\cos\theta & -r\sin\theta & r/d\\ -r\cos\theta & -r\sin\theta & -r/d \end{bmatrix}. \tag{47}$$

We can get the constrained Lagrangian L_c by replacing

$$\dot{g} = -\mathbb{A}^T\dot{s}$$

in (29) according to (47). Due to the symmetry of the Lagrangian and the distribution, $L_c = L_c(s, \dot{s})$ is additionally independent from g. The governing equations describing the dynamics of the system are thus given by (11). The input is simply the torque applied to the wheels

$$\tau = (0 \;\; \tau_1 \;\; \tau_2)^T. \tag{48}$$

The mass matrix for this system is defined as $M_c = \frac{\partial^2 L_c}{\partial\dot{s}\partial\dot{s}}$, which explicitly results in

$$M_c = \begin{bmatrix} M_{c_{11}} & M_{c_{12}} & M_{c_{13}}\\ M_{c_{12}} & M_{c_{22}} & M_{c_{23}}\\ M_{c_{13}} & M_{c_{23}} & M_{c_{33}} \end{bmatrix}, \tag{49}$$

where

$$\begin{aligned} M_{c_{11}} &= c_3 + r^2 c_1 + 2rc_2\cos\alpha\\ M_{c_{12}} &= M_{c_{13}} = \frac{r}{2}c_2\cos\alpha + \frac{r^2}{2}c_1\\ M_{c_{23}} &= \frac{r^2}{4}c_1 - \frac{r^2}{4d^2}(c_4\sin^2\alpha + c_5)\\ M_{c_{22}} &= M_{c_{33}} = \frac{r^2}{4}c_1 + \frac{r^2}{4d^2}(c_4\sin^2\alpha + c_5), \end{aligned}$$

and with

$$\begin{aligned} c_1 &= m_B + 2m_W + 2\frac{1}{r^2}J_{W_{yy}}\\ c_2 &= m_B b, \quad c_3 = m_B b^2 + J_{B_{yy}}\\ c_4 &= m_B b^2 + J_{B_{xx}} - J_{B_{zz}}\\ c_5 &= 2J_{W_{xx}} + J_{B_{zz}} + 2m_W d^2 + 2\frac{d^2}{r^2}J_{W_{yy}}. \end{aligned}$$

The matrix of the Coriolis and centrifugal forces C_c can be derived from the Christoffel symbols (Bloch [2003]) or by using the following relations:

$$C_c\dot{s} = \dot{M}_c\dot{s} - \frac{1}{2}\partial_s^T(\dot{s}^T M_c\dot{s}) \tag{50}$$

$$\dot{M}_c = C_c^T + C_c. \tag{51}$$

It can be explicitly written as

$$C_c = \begin{bmatrix} -r\,c_2\,\dot{\alpha}\sin\alpha & -\delta\,\dot{\phi}_{1\text{-}2} & \delta\,\dot{\phi}_{1\text{-}2}\\ -r/2\;c_2\,\dot{\alpha}\sin\alpha + \delta\,\dot{\phi}_{1\text{-}2} & \delta\,\dot{\alpha} & -\delta\,\dot{\alpha}\\ -r/2\;c_2\,\dot{\alpha}\sin\alpha - \delta\,\dot{\phi}_{1\text{-}2} & -\delta\,\dot{\alpha} & \delta\,\dot{\alpha} \end{bmatrix},$$

where $\delta = \frac{r^2}{4d^2}\,c_4\sin\alpha\cos\alpha$, $\dot{\phi}_{1\text{-}2} = \dot{\phi}_1 - \dot{\phi}_2$. The forcing term on the right hand side is

$$-\mathbb{B}\,\partial_{\dot{g}}^T L_{|_{\dot{g}=-\mathbb{A}\dot{s}}} = J_c\dot{s},$$

$$\text{where}\quad J_c = \begin{bmatrix} 0 & -\beta & \beta\\ \beta & 0 & \beta\\ -\beta & -\beta & 0 \end{bmatrix}, \quad \beta = \frac{r^3}{4d^2}c_2(\dot{\phi}_2 - \dot{\phi}_1)\sin\alpha.$$

The term corresponding to the gravitational forces is simply the gradient of the potential

$$\nabla_s V = \begin{bmatrix} -c_2 g\sin\alpha\\ 0\\ 0 \end{bmatrix}. \tag{52}$$

[5] Translation, and rotation about the z-axis

The equations of motion can be now written as the equations of motion in the reduced (shape) space and the reconstruction equation [6]

$$M_c\ddot{s} + C_c\dot{s} + \nabla_s V = \tau + J_c\dot{s} \tag{53}$$

$$\dot{g} = -\mathbb{A}^T\dot{s}. \tag{54}$$

4.2 Change of coordinates

Since the shape variables are not fully actuated, separating the equations of motion into dynamics of the shape variables, and reconstruction equations does not simplify the control problem, nor any controllability analysis or trajectory planing. Thus, we want to get a functional relationship between the inputs and the group variables instead. We also need to consider the tilting angle α, for this variable is critical for the stability of the WIP. Let us consider the relation

$$v = \frac{r}{2}\left(\dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\alpha}\right), \qquad \dot{\theta} = \frac{r}{2d}\left(\dot{\phi}_2 - \dot{\phi}_1\right)$$

to get the equations of motion in terms of the forward acceleration $\dot{v}$ of the WIP and the yawing rotation $\dot{\theta}$. Let us introduce the velocities $\dot{\xi} = (v, \dot{\alpha}, \dot{\theta})^T$, such that $\dot{\xi} = T^{-1}\dot{s}$, for the constant matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1/r & -1 & -d/r \\ 1/r & -1 & d/r \end{bmatrix}. \tag{55}$$

In the new coordinates ξ, the equations of motion are

$$M\ddot{\xi} + C\dot{\xi} + \nabla_\xi V = \tilde{\tau} + J\dot{\xi} \tag{56}$$

$$\dot{g} = -\mathbb{A}^T T\dot{\xi}. \tag{57}$$

The Mass matrix considerably simplifies to

$$M = T^T M_c T = \begin{bmatrix} c_1 & c_2\cos\alpha & 0 \\ c_2\cos\alpha & c_3 & 0 \\ 0 & 0 & c_4\sin^2\alpha + c_5 \end{bmatrix}, \tag{58}$$

such that the matrix of the Coriolis and centrifugal forces becomes

$$C = \begin{bmatrix} 0 & -c_2\,\dot{\alpha}\sin\alpha & 0 \\ 0 & 0 & -c_4\,\dot{\theta}\sin\alpha\cos\alpha \\ 0 & c_4\,\dot{\theta}\sin\alpha\cos\alpha & c_4\,\dot{\alpha}\sin\alpha\cos\alpha \end{bmatrix}, \tag{59}$$

the term corresponding to the gravitational forces is

$$\nabla_\xi V = \begin{bmatrix} 0 \\ -c_2\, g\sin\alpha \\ 0 \end{bmatrix}, \tag{60}$$

and the forcing term on the right hand side is

$$J\dot{\xi} = \begin{bmatrix} 0 & 0 & c_2\,\dot{\theta}\sin\alpha \\ 0 & 0 & 0 \\ -c_2\,\dot{\theta}\sin\alpha & 0 & 0 \end{bmatrix}\dot{\xi}. \tag{61}$$

In the new coordinates ξ, the systems input is given by

$$\tilde{\tau} = T^T\tau = \begin{bmatrix} 1/r\,(\tau_1+\tau_2) \\ -(\tau_1+\tau_2) \\ d/r\,(\tau_2-\tau_1) \end{bmatrix}, \tag{62}$$

and the reconstruction equation $\dot{g} = -\mathbb{A}^T T\dot{\xi}$ for the position is simply

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta.$$

Note that the new coordinates ξ consist of the length of the path, and the tilting and yaw angles, α and θ, respectively. The equations of motion (56) are the same derived by Pathak et al. [2005] using the Lagrange-d'Alembert equations, since the assumptions made for developing the model are identical.

[6] There is no momentum equation for this system.

5. CONCLUSION

In this paper we have derived the reduced equations of motion for the WIP from the Lagrange-d'Alembert Principle and identifying inherent symmetries in the system. Our immediate goal is to identify the Lie-Poisson structure for this nonholonomic system and employ this feature in arriving at a suitable control law for the objective of stabilization and, later on, trajectory tracking.

ACKNOWLEDGEMENTS

A part of this work was performed during the first author's visit to the Systems and Control group, IIT Bombay. The first author thanks the TUM Graduate School for supporting this work.

REFERENCES

Baloh, M. and Parent, M. (2003). Modeling and model verification of an intelligent self-balancing two-wheeled vehicle for an autonomous urban transportation system. In *Proc. Conf. Comp. Intellig., Robot. Autonom. Syst.*

Bloch, A.M. (2003). *Nonholonomic mechanics and control.* Interdisciplinary applied mathematics: Systems and control. Springer.

Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., and Murray, R.M. (1996). Nonholonomic mechanical systems with symmetry. *Archive for Rational Mechanics and Analysis*, 136(1), 21–99.

Bloch, A.M., Marsden, J.E., and Zenkov, D.V. (2009). Quasivelocities and symmetries in non-holonomic systems. *Dynamical Systems*, 24(2), 187–222.

Brockett, R.W. (1983). *Asymptotic Stability and Feedback Stabilization.* Defense Technical Information Center.

Gajbhiye, S. and Banavar, R.N. (2012). The Euler-Poincaré equations for a spherical robot actuated by a pendulum. In *Proc. 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control.*

Grasser, F., D'Arrigo, A., Colombi, S., and Rufer, A.C. (2002). Joe: a mobile, inverted pendulum. *IEEE Transactions on Industrial Electronics*, 49(1), 107–114.

Holm, D.D., Schmah, T., and Stoica, C. (2009). *Geometric Mechanics and Symmetry.* Oxford University Press Inc., New York.

Li, Z., Yang, C., and Fan, L. (2012). *Advanced Control of Wheeled Inverted Pendulum Systems.* Springer.

Marsden, J.E. and Ratiu, T.S. (1994). *Introduction to Mechanics and Symmetry.* Springer-Verlag, New York.

Nasrallah, D., Michalska, H., and Angeles, J. (2007). Controllability and posture control of a wheeled pendulum moving on an inclined plane. *Robotics, IEEE Transactions on*, 23(3), 564–577.

Ostrowski, J.P. (1999). Computing reduced equations for robotic systems with constraints and symmetries. *IEEE T. Robotics and Automation*, 15(1), 111–123.

Pathak, K., Franch, J., and Agrawal, S. (2005). Velocity and position control of a wheeled inverted pendulum by partial feedback linearization. *Robotics, IEEE Transactions on*, 21(3), 505–513.

Segway (2015, Jan). [online]. http://www.segway.com.