Nonlinear acoustic modeling of an industrial premixed combustor

A. Di Vita*, G. Mori

Ansaldo Energia, PDE/ISV/CEN, corso Perrone 118, Genoa, Italy
* Corresponding author: Andrea.DiVita@aen.ansaldo.it

We describe the spontaneous onset of pressure oscillations in a burner for lean, subsonic, premix combustion as the result of a feedback between perturbations of heat release at the flame and of chemical composition at the fuel inlet. A Feigenbaum-like limit cycle describes oscillations near the onset. Markstein's kinematic equation describes the evolution of the flame surface in time. A suitably rewritten Rayleigh's criterion of stability in thermo-acoustics dictates the shape of the flame at marginal stability. We write down analytical formulae for both amplitude and frequency. These formulae involve few, dimensionless, geometry-dependent constant quantities. The amplitude of pressure oscillations increases with increasing Mach number and decreasing fuel content.

1 The problem

A simplified description of the pressure oscillations in a burner for lean premix combustion burner is presented. Starting from first principles, we derive analytical formulas for both frequency and amplitude of the oscillations. We solve the equations of motion assuming a thin, axisymmetric, lean premixed flame. We deal with a mixture of just two components, air and fuel, with Mach number $Ma \ll 1$. Combustion occurs at the flame only. The total unperturbed pressure has the same value everywhere; we take also a given value for the pressure of the fuel at the inlet. We assume no fixed, flat flame; we allow the flame to move in agreement with the equations of motion, and take into account its motion explicitly. We focus our attention on the onset of oscillations. A suitably reformulated version of Rayleigh's criterion of stability [1] allows the unperturbed, oscillation-free flame to be stable. A perturbation of the amount of heat released by combustion induces an acoustic oscillation which leaves the flame and arrives at the inlet of the air-fuel mixture. Since the pressure of the fuel at the inlet is a constant, the acoustic perturbation at the inlet affects the air pressure only. The stoichiometry of the mixture is therefore also affected. The resulting perturbation of stoichiometry is carried by the fluid towards the flame, and affects the value of the flame velocity there. In turn, this change of flame velocity affects the heat released by combustion. A feedback mechanism is established. When this feedback is destabilized, oscillations occur [3]. We include the time evolution of the flame surface with the help of Markstein equation [7], in order to take into account the non-negligible flame curvature in Ansaldo burners. We describe the oscillation with the help of a limit cycle [4]. Here we investigate no perturbations of vorticity, no acoustic interactions among many flames and no longitudinal jump of velocity (in contrast with [11]). We do not discuss the experimentally observed discontinuities of the frequency as the air mass flow varies. Furthermore, steady, oversimplified flow patterns between the fuel inlet and the flame are assigned as a starting guess, so we cannot deal with acoustic excitation of Kelvin-Helmholz instability at the inlet. Our results stand as a proof-of concept only. However, we invoke no ad hoc transfer function and 'describing function'. It turns out that lean combustion at high power is the most unstable. A flame becomes more unstable with the increase of the Mach number – see [10].

2 The basic equations

We assume that no net mass source exists, so that the balance of mass reads

$$\frac{d\rho}{dt} + \rho \nabla . \mathbf{v} = 0 \tag{1}$$

everywhere. We neglect viscosity, so that the balance of momentum reads

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = 0. \tag{2}$$

We neglect radiation and heat conduction, so that the energy balance reads

$$\rho T \frac{ds}{dt} = \rho \frac{du}{dt} + p \nabla . \mathbf{v} \tag{3}$$

Energy and entropy are additive quantities, i.e. both are the sum of a contribution of the flame and a contribution of the fluid outside the flame: $\rho du/dt = (\rho du/dt)_{fluid} + (\rho du/dt)_{flame}$, $\rho T ds/dt = (\rho T ds/dt)_{fluid} + (\rho T ds/dt)_{flame}$:

$$\left(\rho T \frac{ds}{dt}\right)_{flame} = \left(\rho \frac{du}{dt}\right)_{flame} + q^*; \qquad \left(\rho T \frac{ds}{dt}\right)_{flame} = q_{comb} \tag{4}$$

$$\left(\rho T \frac{ds}{dt}\right)_{fluid} = \left(\rho \frac{du}{dt}\right)_{fluid} + p \nabla . \mathbf{v} - q^*; \qquad \left(\rho T \frac{ds}{dt}\right)_{fluid} = 0. \tag{5}$$

Here q_{comb}, $\mathbf{v}$, T, s, $u = p/(\rho(\gamma - 1))$, p, ρ, γ, and q^* are the combustion power density (> 0 inside the flame, $= 0$ elsewhere), the velocity, the temperature (supposed to be the same for all species), the entropy per unit mass, the internal energy per unit mass, the total pressure, the total mass density, the specific heat ratio (supposed to be the same for all species), and the density of mechanical power supplied by the flame to the fluid. In the following, we write $a(t) = a_0 + \epsilon a_1(t)$, $\epsilon \ll 1$ for the generic quantity $a(t)$.

3 The model

Linearisation of eqns (1), (2) and (5) gives [2]

$$\left(\frac{D^2}{Dt^2} - c_{s0}^2 \Delta\right) p_1 = (\gamma - 1) \frac{Dq_1^*}{Dt}, \tag{6}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{v_0}.\nabla$ and $c_{s0} \equiv (\gamma p_0/\rho_0)^{1/2}$ is the speed of sound; here $\nabla p_0 = 0$, and both $\nabla \mathbf{v_0}$ and ∇c_{s0} have been neglected for simplicity (as for $\nabla \mathbf{v_0}$, this assumption is dropped below). If the flame is a thin region of area $A_f \equiv \int_{flame} d^2x$ and constant thickness $d_0 = d(p_0)$, the solution of eqn (6) – which we obtain with the method of Green's function for an assigned burner geometry – gives the perturbation of the total pressure at the fuel inlet

$$p_{in\,1}\Big|_{t+t_{ac}} = \zeta_N (\gamma - 1) \frac{Q_{exch\,1}}{4(c_{s0} - v_0) A_f}\Bigg|_t, \tag{7}$$

where $v_0 = Ma\, c_{s0}$ and $Q_{exch} \equiv \int_{flame} q^* d\mathbf{x}$; moreover ζ_N and t_{ac} are a geometry-dependent, dimensionless quantity and a time delay required by acoustic perturbations to reach the fuel inlet starting from the flame (by 'geometry' we mean here both the geometry of the burner and the flame shape described below). Below, we compute both Q_{exch} and A_f explicitly. To this purpose, we perform integration of eqn (4) on the flame volume:

$$Q_{exch}\Big|_t = Q_{comb}\Big|_t - \frac{p_0 d_0}{\gamma - 1}\frac{dA_f}{dt}\Big|_t, \tag{8}$$

where $Q_{comb} \equiv \int_{flame} q_{comb} d\mathbf{x}$ is the power produced by combustion, we neglect relative changes of pressure in comparison with relative changes in A_f (this choice will be justified below) and write $d[\int a d\mathbf{x}]/dt = \int \rho[d(a/\rho)/dt]d\mathbf{x}$. In turn, the latter relationship follows from eqn (1) and the identity $[\int_\Omega d\mathbf{x}\, a]/dt = \int_\Omega d\mathbf{x}\, [\partial a/\partial t + \nabla.(a\mathbf{v})]$ [9]. As for steady states, $(dA_f/dt)_0 = 0$ – i.e. $dA_f/dt = (dA_f/dt)_1$ – and $Q_{exch\,0} = Q_{comb\,0}$. Generally speaking, $Q_{exch} = Q_{comb}$ at all times for fixed flames only ($dA_f/dt = 0$). Below, we provide an explicit expression for each term on the R.H.S. of eqn (8). As for Q_{comb}, we assume that combustion involves two species: the fuel (with molar density n_{fuel}, partial pressure $p_{fuel} = n_{fuel}RT$, R perfect gas constant, and molar mass m_{fuel}) and the air (with molar density n_{air}, partial pressure $p_{air} = n_{air}RT$ and molar mass m_{air}). We introduce two dimensionless quantities: the air molar fraction $z \equiv n_{air}/(n_{fuel} + n_{air})$ ($0 \le z \le 1$) and the mass fraction of fuel $Y \equiv n_{fuel}m_{fuel}/\rho$. No chemical reaction occurs between the flame and the fuel inlet, hence Dalton's law of partial pressures gives $p = p_{fuel} + p_{air}$; moreover, $\rho = n_{fuel}m_{fuel} + n_{air}m_{air}$ and the fact that air and fuel have the same (input) temperature T_{in} at the inlet gives

$$\rho Y = \frac{m_{fuel}\, p_{fuel\,in}}{RT_{in}}, \tag{9}$$

where $p_{fuel\,in}$ is the (input) fuel partial pressure at the fuel inlet. In lean combustion $z \approx 1$ and all the injected fuel is burnt at the flame. We write [7] $Q_{comb} = \int_{flame} d^2x\, \mathbf{n}.\mathbf{v_f}\, H\rho Y$, where H, $\mathbf{n}$ and $\mathbf{v_f} = \mathbf{n}v_f$ are the heat produced per unit mass of fuel, the unit normal vector to the flame and the flame velocity respectively. Eqn (9) links $H\rho Y$, $p_{fuel\,in}$ and T_{in}, and v_f may depend on T and z. We assume that $z = z_{flame}$ and $T = T_{flame}$ everywhere on the flame, so that

$$Q_{comb}\Big|_t = A_f\, v_f\, H\rho Y\Big|_t. \tag{10}$$

Landau's jump conditions at the flame [6] link $T_{flame\,0}$ and T_{in}. As for A_f and dA_f/dt, equation $g(\mathbf{x}, t) = 0$ describes a thin flame. Markstein's equation [7] $\partial g/\partial t + \mathbf{v}.\nabla g - \mathbf{v_f}.g = 0$ describes its evolution. For an axisymmetric flame, $g(X, Z, t) = Z - f(X, t)$, where Z and X are the axial and the radial coordinate respectively ($0 \le X \le R_{max}$) in a cylindrical coordinate system, and the flame at time t is generated by the rotation of the curve $f(X, t)$ (the 'flame shape') in the plane $\{X, Z\}$ around the $X = 0$ axis. Then, the definition of A_f and Markstein's equation give

$$A_f\Big|_t = 2\pi \int dX\, X \sqrt{1 + \left(\frac{df}{dX}\right)^2}\Bigg|_t \tag{11}$$

$$\left.\begin{aligned} \left(\frac{dA_f}{dt}\right)_1\Bigg|_t &= z_{flame\,1}\frac{dv_f}{dz_{flame}}\, I\Big|_t \\ I\Big|_t &\equiv 2\pi \int dX\, X \frac{1}{1 + \left(\frac{df}{dX}\right)^2}\left(\frac{df}{dX}\right)^2 \frac{d^2 f}{dX^2}\Big|_t \end{aligned}\right\} \tag{12}$$

respectively. According to eqn (12), $|dv_f/dz_{flame}|$ may be large even if $|z_{flame\,1}| \ll 1$, as $z \approx 1$: this justifies our previous choice of allowing large relative variations of A_f. Moreover, flat flames

$(df/dX = 0)$ are also fixed $(dA_f/dt = 0)$. Finally, if we increase the relative content of fuel $(z_{flame\,1} < 0)$ we obtain $z_{flame\,1}(dv_f/dz_{flame}) > 0$ as $dv_f/dz_{flame} < 0$ in lean combustion, i.e. we increase v_f. In convex flames $(d^2f/dX^2 < 0)$, therefore, eqn (12) leads to $dA_f/dt < 0$, i.e. to a decrease in A_f. Intuitively, the flame as a whole approaches therefore the fuel inlet, where p is likely to have a node; Rayleigh's criterion of stability [8] predicts therefore stability, in agreement with the well-known, experimentally observed flame stabilisation through fuel enrichment. In the unperturbed state the computations of the integrals in eqns (11-12) requires explicit knowledge of the unperturbed flame shape $f_0(X)$. We obtain the latter invoking $Ma \ll 1$ between the flame and the fuel inlet (so that we may introduce the stream function ψ for the incompressible flow in this region) and with the help of the variational principle discussed in [1]

$$\int K \frac{df_0}{dX} d\psi = max. \text{ with fixed } A_{f0} = 2\pi \int dX\, X \sqrt{1 + \left(\frac{df_0}{dX}\right)^2} \tag{13}$$

where $K \equiv \dfrac{d^2f_0/dX^2}{\left(1 + (df_0/dX)^2\right)^{3/2}}$, incompressibility dictates the value of A_{f0}:

$$v_{f0}\, A_{f0} = \pi R_{max}^2 v_0, \tag{14}$$

and a suitable guess is to be provided for $\psi(X, Z)$. Physically, eqn (13) selects possible stable steady flame shapes in agreement with Rayleigh's criterion; since we are interested in the onset of oscillations i.e. on the loss of stability of steady states, it is only natural to invoke eqn (13) to describing the physical state we start from in our analysis. The proof of [1] invokes the well-known jump conditions of [6] at the flame, so no longitudinal jump of velocity is considered. In contrast with [1], here we investigate no quasi-flat flames. We allow our guess for ψ to include vortices; admittedly, this is in contrast with the assumption of negligible ∇v_0 invoked in the proof of eqn (6), but leads to more realistic results. Numerical solutions of eqn (13) show that our guess vortices are associated to convex flame shapes, in agreement with both CFD and our discussion above. This point deserves further discussion, which is definitely outside the limit of the present work. Here we look for solutions of eqn (13) in a simple convex shape, i.e. the parabola $f_0(X) = c - a(X - b)^2$. Here eqn (13) dictates the dependence on v_{f0} of the quantities a, b, c, A_{f0}, I_0, $[dA_f/d(v_f/v_0)]_0$ and the distance $\Delta X \equiv 2\sqrt{c/a}$ between the two intersections of the parabola with the $Z = 0$ axis.

As for the flame velocity v_f, we do not know it exactly, as only laminar flame models are available. However v_f has its maximum value v_{fM} when the reaction is stoichiometric, i.e. $z_{flame} = z_{sto}$ $(= 10/11$ for methane in air). If there is either no air $(z_{flame} = 0)$ or no fuel $(z_{flame} = 1)$ then $Q_{comb} = 0$. Then, we may reasonably write:

$$v_f\Big|_t = v_{fM}\, h(z_{flame}\Big|_t), \qquad \text{where } 0 \leq h \leq 1, \quad h(0) = h(1) = 0, \quad \text{and } h(z_{sto}) = 1. \tag{15}$$

If there is just one maximum of h, then for $z_{flame} \approx 1$ (lean combustion) v_f increases with decreasing z_{flame}, i.e. increasing relative abundance of fuel. We write h in the form $h = c_3\, z_{flame}^{1+c_1}\,(1 - z_{flame})^{1+c_2}$ where c_1, c_2 and c_3 are dimensionless constants to be determined. Relationships $h(z_{sto}) = 1$ and $(dh/dz)_{z_{sto}} = 0$ allow us to compute v_f unambiguously provided that two quantities – say v_{fM} and c_2 – are known.

Accordingly, $Q_{comb}\Big|_t$ in eqn (10) is known provided that $z_{flame}\Big|_t$ is known. Moreover, the absence of chemical reactions between the fuel inlet and the flame implies:

$$z_{flame}\Big|_t = z_{in}\Big|_{t - t_{conv}}, \tag{16}$$

where t_{conv} is the time delay required by perturbations of chemical composition – convected by the fluid – to reach the flame starting from the fuel inlet. Finally, $p_{fuel\,in} = constant$ and Dalton's law of partial pressures give:

$$p_{in\,1}\Big|_{t-t_{conv}} = \frac{p_{fuel\,in}}{(1-z_{in})_1}\Bigg|_{t-t_{conv}}. \tag{17}$$

Formally, eqns (7-17) link $p_{in\,1}\Big|_{t+t_{ac}}$ and $p_{in\,1}\Big|_{t-t_{conv}}$, or equivalently, the couple $\{z_{flame}, A_f\}\Big|_{t+\tau}$ and the couple $\{z_{flame}, A_f\}\Big|_{t}$, where $\tau \equiv t_{ac} + t_{conv}$.

4 The limit cycle

Our line of thought is as follows. In the particular case of flat flames, A_f is constant and the evolution of the system is described by a 1-dimensional differentiable mapping ('return map') $z_n \to z_{n+1}$ were $z_n \equiv z_{flame}(t)$ and $z_{n+1} \equiv z_{flame}(t+\tau)$. Since $0 \leq z_{flame} \leq 1$ and both $z_n = 0$ and $z_n = 1$ are mapped to $z_{n+1} = 0$ regardless of the actual value of τ, as no combustion occurs if either air or fuel is absent, the mapping has at least one maximum in the range $0 \leq z_{flame} \leq 1$. Then the results of [4] apply, and a transition ('bifurcation') from a stable steady state to a stably oscillating state ('limit cycle') may occur; z_{flame} oscillates in the latter state between an upper bound z_{max} and a lower bound z_{min}. We derive a formula for the amplitude $z_{max} - z_{min}$ of the oscillation near the bifurcation, where $z_{max} - z_{min} \propto |z_{flame\,1}| \propto |(dA_f/dt)_1| \propto O(\epsilon)$. The introduction of time-dependent A_f requires cumbersome algebra in the general case. However, we take advantage of the stability of the limit cycle against small perturbations. An example of such small perturbation is just the contribution of dA_f/dt near the bifurcation, and a perturbative approach in dealing with dA_f/dt is therefore justified. All the same, it is precisely this term which allows us to compute the (so far unspecified) oscillation frequency $\approx |(dA_f/dt)_1/A_{f1}|$. In detail, eqns (7), (10), (15) and (17) give:

$$q_{exch\,1}\Big|_{t-\tau} = \frac{1}{(1-z_{in})_1}\Bigg|_{t-t_{conv}}, \tag{18}$$

where we have defined $\alpha \equiv \rho Y H\, v_{fM}\, A_f$, $\xi \equiv \alpha\zeta/p_{fuel\,in}$, $\zeta = \zeta_N(\gamma-1)/[4A_f(c_{s0}-v_0)]$, $q_{exch} \equiv \xi\, Q_{exch}/\alpha$ and we have neglected terms $\propto O(\epsilon^2)$. Let us write $da = a_1$ for the generic quantity a. The boundary condition $q_{exch} = 0$ for $z_{in} = 0$ (i.e. no combustion without air) allows straightforward integration of eqn (18):

$$q_{exch}\Big|_{t-\tau} = -1 + \frac{1}{(1-z_{in})_1}\Bigg|_{t-t_{conv}} \tag{19}$$

(no such condition applies for $z_{in} = 1$, as $lim_{z_{in}\to 1} Y = 0$ and $q_{exch} \propto 1/Y \to \infty$). Equations (16) and (19) give

$$z_{n+1} = 1 - \frac{1}{1+q_{exch\,n}} \tag{20}$$

where $a_n \equiv a(t)$, $a_{n+1} \equiv a(t+\tau)$ and we dropped the pedix 'flame'. As for $q_{exch\,n}$, eqns (8), (10) and (15) give:

$$q_{exch\,n} = \xi\, h(z_n) - \frac{\zeta_N\, p_0 d_0}{4p_{fuel\,in}(c_{s0}-v_0)}\frac{dA_f/dt}{A_f}. \tag{21}$$

In turn, equations (11), (12) and (15) give straightforwardly

$$\frac{dA_f/dt}{A_f} = -4\frac{z_{n+1}-z_n}{\Delta X}\, v_{fM}\, \frac{dh}{dz_n}, \tag{22}$$

for a parabolic flame shape in the realistic limit of long, thin flames ($ac \to \infty$). Here we have assumed $z_{n+1} - z_n \approx z_{n1}$. This is justified for small amplitude $|z_{n+1} - z_n| \approx |z_{max} - z_{min}|$. Equations (20-22) give

$$z_{n+1} = 1 - \frac{1}{1 + \xi\, h(z_n) + (dh/dz_n)(z_{n+1} - z_n)\,\delta}, \tag{23}$$

where $\delta \equiv \zeta_N\, p_0 d_0\, v_{fM}/\,[p_{fuel\,in}\, \Delta X\, (c_{s0} - v_0)] \ll 1$ for thin flames, as $\delta \propto d_0$. Equation (23) is the fundamental result of our work. Note that it contains neither t_{ac} nor t_{conv}. Firstly, we discuss the flat flame limit $\delta \to 0$, then we Taylor-expand the R.H.S. of eqn (23) in the small parameter $(z_{n+1} - z_n)\delta$.

5 Fixed, flat flame

If $\delta \to 0$ then eqn (23) reduces to

$$z_{n+1} = 1 - \frac{1}{1 + \xi\, h(z_n)}, \tag{24}$$

which is just what we would have derived from eqns (18-22), had we postulated $dA_f/dt \equiv 0$ from the beginning. Equation (24) is the return map we referred to above. It involves ξ and c_2 (through h). In lean combustion $z \approx 1$, hence $dz_{n+1}/dz_n < 0$. Bifurcation occurs [4] at the solutions $z_{cr}(c_2, \delta = 0)$, $\xi_{cr}(c_2, \delta = 0)$ of

$$z_{n+1} = z_n; \qquad\qquad F(\xi,\, z_n,\, \delta = 0) = 0, \tag{25}$$

where $F \equiv 1 + dz_{n+1}/dz_n$. Oscillations are triggered when $\xi > \xi_{cr,\delta=0} = 0$, i.e. when Ma becomes too large. We compute the amplitude of the oscillation as follows. Our choice for h allows us to show that the function $z_{n+2}(z_n) - z_n$ behaves like a cubic function of z_n near the bifurcation. It has just one zero for $\xi < \xi_{cr,\delta=0} = 0$ and three zeros, namely z_{min}, $z_{cr,\delta=0}$ and z_{max} for $\xi > \xi_{cr,\delta=0} = 0$. The computation reduces therefore to an investigation of these zeros. It turns out that:

$$|z_{max} - z_{min}| = 0 \quad \text{for } \xi < \xi_{cr,\delta=0}; \qquad |z_{max} - z_{min}| = k_4\sqrt{\xi - \xi_{cr,\delta=0}} \quad \text{for } \xi > \xi_{cr,\delta=0}; \tag{26}$$

where k_4 is a dimensionless function of the derivatives of the return map (see below). In spite of eqn (26), the flat flame limit is not sufficient for practical purposes, for three reasons. Firstly, no information concerning c_2 is obtained. Secondly, eqn (24) provides us with no information about oscillation frequency, as dA_f/dt is irrelevant for $\delta \to 0$. Finally, $z_{cr,\delta=0}$ is an output of eqn (25); in contrast, in real life we want to know at which value of Ma oscillations start for a given chemical composition z_{in}. In order to overcome these obstacles, we come back to eqn (23).

6 Moving, curved flame

For thin flames, δ is small. Then, we make a small error if we take δ constant in the Taylor expansion after solving eqn (13) for $\Delta X = \Delta X(v_{f0}(z_{in}))$, etc. Physically, this means that the bifurcation we are investigating occurs at the chemical composition assigned by the user, and that just before the onset of oscillation the flame is stable according to Rayleigh's criterion. Neglecting higher-order terms, eqn (23) generalises eqn (24):

$$z_{n+1} = 1 - \frac{1}{1 + \xi\, h(z_n)} + \delta \frac{dh}{dz_n}\left[\frac{1 - z_n}{(1 + \xi h(z_n))^2} - \frac{1}{(1 + \xi h(z_n))^3}\right]. \tag{27}$$

Again, $F \equiv 1 + dz_{n+1}/dz_n$, and bifurcation occurs at $z_n = z_{in}$ and $\xi_{cr}(c_2, \delta)$ such that

$$z_{n+1} = z_n; \qquad F(\xi,\, z_n,\, \delta) = 0. \tag{28}$$

Comparison of eqns (25) and (28) gives two relationships between the three unknown quantities ξ_{cr}, δ and c_2:

$$z_{in} = 1 - \frac{1}{1+\xi_{cr} h(z_{in})} + \delta \left(\frac{dh}{dz_n}\right)_{z_{in}} \left[\frac{1 - z_{in}}{(1+\xi_{cr} h(z_{in}))^2} - \frac{1}{(1+\xi_{cr} h(z_{in}))^3}\right] \tag{29}$$

$$(\partial F/\partial \delta)_* \, \delta + \left(\frac{\partial F}{\partial z_n}\right)_* (z_{in} - z_{cr}(c_2, \delta = 0)) + \left(\frac{\partial F}{\partial \xi}\right)_* (\xi_{cr} - \xi_{cr}(c_2, \delta = 0)) = 0, \tag{30}$$

where the pedix '$*$' means '$\delta = 0,\ \xi = \xi_{cr,\delta=0},\ z_n = z_{cr,\delta=0}$'. We are left with the problem of evaluating c_2. To this purpose, we note that in order to trigger the oscillation, the power delivered by the oscillating flame to the fluid should at least compensate the power carried away by the fluid. Then, marginal stability ($\xi = \xi_{cr},\ z_n = z_{in}$) reads [5]:

$$H(\rho v_f)_0 K_J = \frac{1}{2} \rho_0 c_{s0} \omega^2 R_{max}^2 \sin^2\left(\frac{\omega L}{c_{s0}}\right), \tag{31}$$

where $K_J = K_J(z_{in}, T_{flame}(T_{in})) \approx \mathit{constant}$ as far as $RT_{in} \ll m_{fuel} H$; moreover, L is a typical axial length and $\omega = c_{s0}\sqrt{(\pi/L)^2 + (x_0/R_{max})^2}$ is a typical frequency, $x_0 = 2.405$ being the first zero of a Bessel function of zero-th order. The R.H.S. of eqn (31) is larger than the L.H.S. in stable flames. First instability occurs (Rayleigh criterion) at the first occurrence of $\sin(\omega L/c_{s0}) = 1$. In the limit of large c_{s0} (i.e. $Ma \ll 1$), eqn (31) gives the looked-for, geometry-dependent, third relationship between ξ_{cr}, δ and c_2 for given z_{in} and T_{in}:

$$8\, \xi_{cr}\, K_J\, h(z_{in}) = \gamma(\gamma - 1)\, \zeta_N\, x_0^2. \tag{32}$$

Once ξ_{cr} and c_2 are known from eqns (29), (30) and (32), eqn (14) gives v_{fM} with $v_{f0} = v_{fM}\, h(z_{in})$; eqn (26) gives the amplitude, as ξ_{cr} replaces $\xi_{cr,\delta=0}$, and eqn (12) gives the frequency $|(dA_f/dt)_1/A_{f1}|$.

7 Final results: the flame speed

We apply our discussion to a system with total mass flow $M \equiv \pi R^2 \rho v_0$, pressure p_0, unperturbed air mass fraction z_{in} and unperturbed temperature T_{in} at the inlet. We introduce the quantities

$$k_1 \equiv \frac{\pi R_{max}^2}{A_f}; \qquad k_2 \equiv \frac{m_{fuel} H(\gamma - 1)}{RT_{in}}; \qquad k_3 \equiv \frac{\zeta_N}{4\xi_{cr}};$$

$$k_4 \equiv 4 \left| \frac{3}{2} \frac{\dfrac{\partial\left(\partial z_{n+1}/\partial z_n\right)_*^2}{\partial \xi}}{2\left(\dfrac{\partial^3 z_{n+1}}{\partial z_n^3}\right)_* + 3\left(\dfrac{\partial^2 z_{n+1}}{\partial z_n^2}\right)_*} \right|^{1/2}; \qquad k_5 \equiv 1 + h(z_{in})\, k_1 \left(\frac{d(\ln A_f)}{d(v_f/v_0)}\right)_0.$$

Since the values of k_1, etc. are taken just at the bifurcation for a flame shape which solves eqn (13), the results will be valid just near the onset of the oscillation. The definition of ξ, eqns (9) and (14) give

$$v_0 = \frac{c_{s0}}{1 + k_1\, k_2\, k_3\, \xi_{cr} / \left(\xi\, h(z_{in})\right)} = v_0(\xi),$$

and, in the $Ma \ll 1$ limit, the maximum allowable value for the flame speed:

$$v_{fM} = \frac{c_{s0}}{k_2\, k_3} \frac{\xi}{\xi_{cr}}. \tag{33}$$

Relationships $h(z_{sto}) = 1$ and $(dh/dz)_{z_{sto}} = 0$ give c_1 and c_3, so that the flame speed is

$$\begin{aligned} &v_f(z_{flame}) = v_{fM}\, h(z_{flame}); \qquad c_1 = \frac{1+c_2}{1/z_{sto}-1}; \\ &h(z_{flame}) = \left[\frac{z_{flame}}{(1+c_1)/(2+c_1+c_2)}\right]^{1+c_1} \left[\frac{(1-z_{flame})}{1-(1+c_1)/(2+c_1+c_2)}\right]^{1+c_2}. \end{aligned} \tag{34}$$

8 Final results: pressure oscillation amplitude

Equations (8) and (10) show that the fuel-related ($\propto Y$) contribution to the source $Q_{exch\,1}$ of acoustic oscillation is $\ll$ the term $\propto dA_f/dt$ for lean combustion ($Y \to 0$, i.e. $n_{air} \gg n_{fuel}$). The latter term is the only remaining term in $Q_{exch\,1}$ for $Y = 0$, and is basically due to air for $Y \ll 1$. No measurement of $p_{in\,1}$ may distinguish between signals produced by particles of air and particles of fuel; hence the sound produced by each particle of the flame is the same regardless of its chemical nature. Accordingly, we write

$$\frac{Q_{comb\,1}}{n_{fuel}} \approx \frac{p_0 d_0}{n_{air}\,(\gamma-1)} \left(-\frac{dA_f}{dt}\right)_1. \tag{35}$$

Here eqns (9-10) give

$$Q_{comb\,1} = (A_f\, v_f)_1\, H\, \rho Y = (A_f\, v_f)_1\, H \frac{m_{fuel}}{RT_{in}}\, p_0\, (1-z_{in}). \tag{36}$$

Equations (35-36) give

$$\frac{p_0 d_0}{(\gamma-1)} \left(-\frac{dA_f}{dt}\right)_1 \approx (A_f\, v_f)_1\, H \frac{m_{fuel}}{RT_{in}}\, p_0\, z_{in}. \tag{37}$$

Equations (8) and (37) give the source of the acoustic signal:

$$Q_{exch\,1} = (A_f\, v_f)_1\, H \frac{m_{fuel}}{RT_{in}}\, p_0. \tag{38}$$

Let us compute $(A_f\, v_f)_1$. It does not vanish, as eqn (14) holds in steady state only. Moreover, we may write $v_{f1} = v_{fM}(dh/dz_n)_{z_{in}} |z_{max} - z_{min}|$ and $A_{f1} = (dA_f/dv_f)_0 v_{f1}$; thus, eqns (7), (26) and (38) give the required formula for the amplitude (from the negative to the positive peak of p_1) of the oscillation of pressure at the inlet. The amplitude Δp from zero to the positive peak of p_1 is half this value, i.e.

$$\begin{aligned} \Delta p &= 0, && Ma \le Ma_{cr} \equiv 1/\left[1 + k_1 k_2 k_3/h(z_{in})\right], \\ \Delta p &= \frac{1}{2} p_0\, k_4\, k_5\, \xi_{cr}^{3/2} \left|\frac{dh}{dz}\right|_{z\,in} \left|\frac{1/Ma_{cr}-1}{1/Ma-1}\right|^{1/2}, && Ma > Ma_{cr}. \end{aligned} \tag{39}$$

9 Final results: frequency

The oscillation of the flame area is $|A_{f\,max} - A_{f\,min}| \approx A_{f1} = (dA_f/dv_f)_0\, v_{f1}$. In a neighbourhood of the bifurcation, eqn (15) leads to $(dA_f/dv_f)_0 = [dA_f/d(v_f/v_0)]_0 / v_0(\xi_{cr})$, where eqns (14) and (33) give $v_0(\xi_{cr}) = c_{s0}/|1/Ma_{cr} - 1|$. Finally, eqn (12) gives $(dA_f/dt)_1 = z_{flame\,1}\, (dv_f/dz_{flame})\; I \approx v_{f1}\, I_0$. Accordingly, we write for the frequency of the oscillations near the bifurcation (if any such oscillation occurs, i.e. for $Ma > Ma_{cr}$):

$$\left|\frac{(dA_f/dt)_1}{A_{f1}}\right| = \frac{c_{s0}I_0}{|1/Ma_{cr} - 1| \left(dA_f/d\left(v_f/v_0\right)\right)_0}. \tag{40}$$

10 Conclusions

Equations (34), (39) and (40) are our final results. They describe spontaneous oscillations in a premixed lean burner which is not too far from a guess configuration which satisfies Rayleigh's criterion of stability. We have applied these equations to an Ansaldo burner. (This task required a suitable choice of the stream function of the flow sustaining the flame in the combustion chamber, as well as an estimate of the Green function for the acoustic equation in the Ansaldo burner geometry; these points are not discussed here). According to eqn (34), the flame speed is always larger than the laminar value: this is in agreement with the common practice of Ansaldo burners. According to eqn (39), a lower threshold Ma_{cr} on Mach number Ma exists for the occurrence of pressure oscillations. The leaner the combustion, the lower Ma_{cr}. Above threshold, the relative amplitude of the oscillation increases with increasing Ma and with decreasing fuel content. This result agrees with Ansaldo experience and – qualitatively at least – with [10]. Equation (40) links the oscillation frequency and the sound speed. We encompass the relative amplitude $\Delta p/p_0$ of eqn (39) and the frequency of eqn (40) in a single plot with the help of the Strouhal number defined as $St = R_{max} frequency/v_0$ (Fig.1). Real-life applications show that many different configurations satisfy Rayleigh's criterion: our treatment deals with the perturbations of just one of them. Investigation of such different configurations, as well as of the interactions among different flames will be the subject of future work.

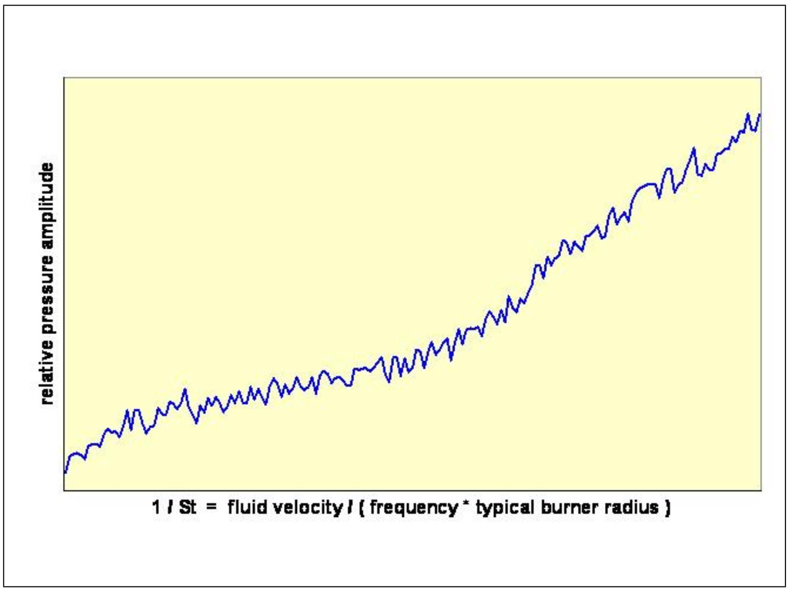

Figure 1: The relative amplitude $\Delta p/p_0$ vs. St according to eqns (39) and (40).

References

[1] A. Di Vita and Mori G. On Rayleigh's criterion and the stability of premixed flames. In *this conference*, 2010.

[2] A.P. Dowling. Modeling and control of acoustic oscillations. In *ASME Turbo Expo 2005: Power for Land, Sea and Air GT2008*, Reno, USA, 2005.

[3] A.P. Dowling and S.R. Stow. A time-domain network model for nonlinear thermoacoustic oscillations. In *ASME Turbo Expo 2008: Power for Land, Sea and Air GT2008*, Berlin, Germany, 2008.

[4] M.J. Feigenbaum. *Universal Behavior In Non-Linear Systems.* P. Universality And Chaos. Adam Hilger, Bristol, 1980.

[5] H. Jones. The generation of sound by flames. *Proc. Roy. Soc A*, 367:291–309, 1979.

[6] L.D. Landau and E.M. Lifshits. *Mécanique des Fluides.* Mir, Moscou, 1971. see §120.

[7] T.C. Lieuwen. Premixed combustion-acoustic wave interactions in combustion instabilities in gas turbine engines: Operational experience, fundamental mechanisms, and modelling. In *Progress In Astronautics And Aeronautics, AIAA*, 2005. see equations (12.1) and (12.7). Note that equations (12.2) has the same physical of (12.1) as far as the flame thickness is constant.

[8] W. Rayleigh. The explanation of certain acoustical phenomena. *Nature*, 18:319–321, 1978.

[9] V.I. Smirnov. *A course in higher mathematics.* MIR, Moscou, 1977.

[10] O.Yu. Travnikov, M.A. Liberman, and V.V. Bychkov. Stability of a planar flame front in a compressible flow. *Phys. Fluid*, 9(12):3935, 1997.

[11] X. Wu, M. Wang, and P. Moin. Combustion instability due to the nonlinear interaction between sound and flame. In *Center for Turbulence Research, Annual Research Briefs 2001*, 2001.