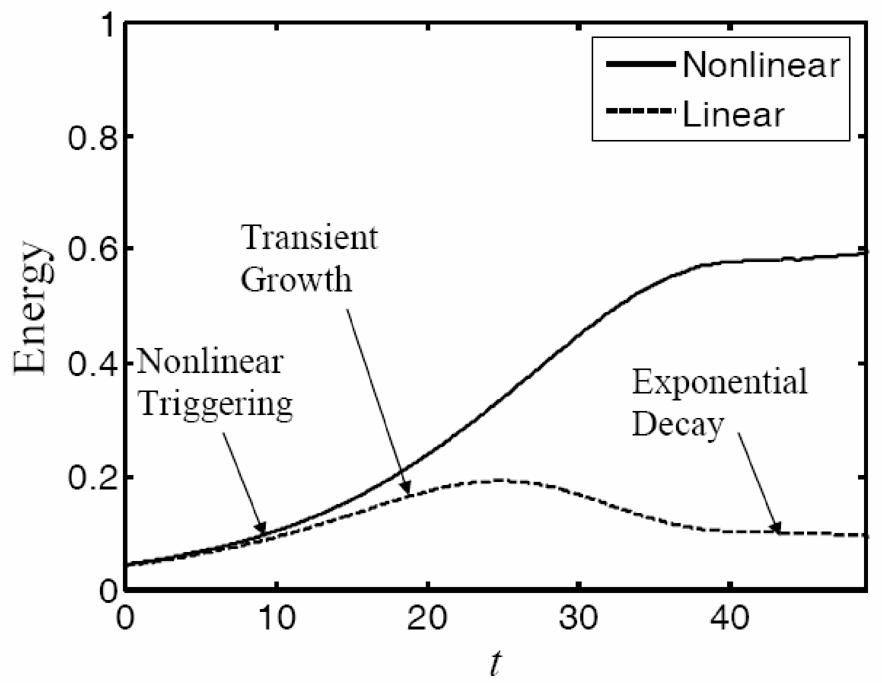
An Overview of Non-normality & Nonlinearity in Thermoacoustic Interactions



R. I. Sujith
Dept. of Aerospace Engineering
IIT Madras

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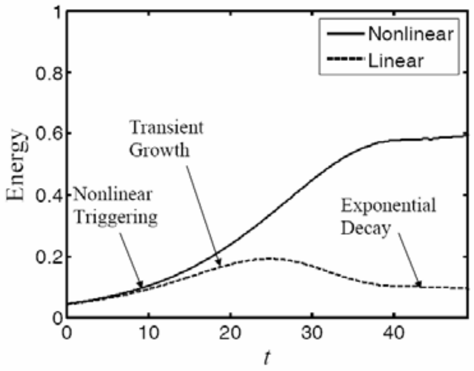
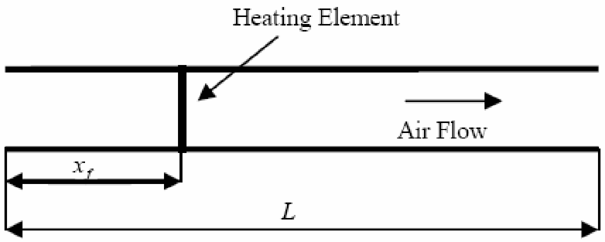
We clarify the role of non-normality & nonlinearity in thermoacoustics

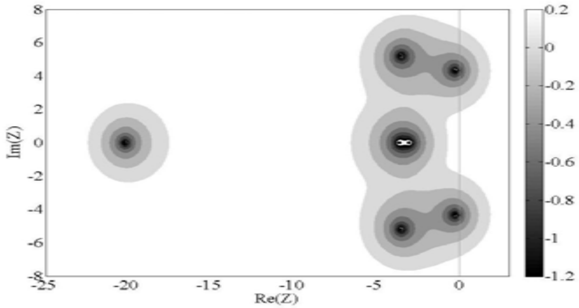
Non-normality & its role in subcritical transition to instability



Rijke tube



How to analyse non-normal systems?

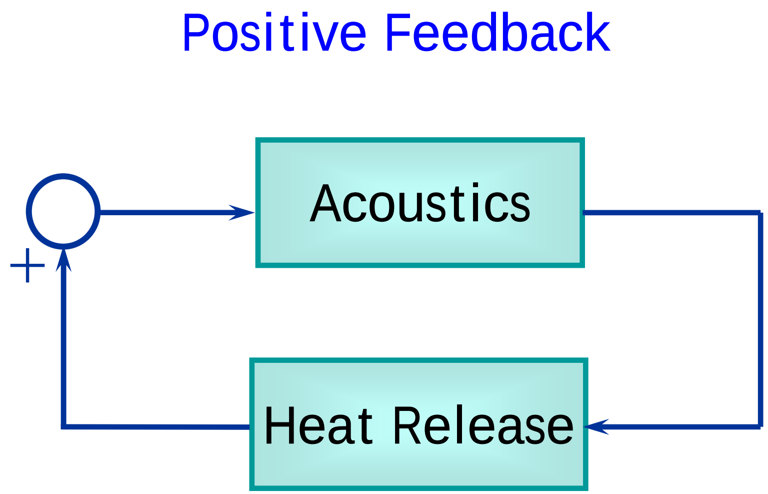


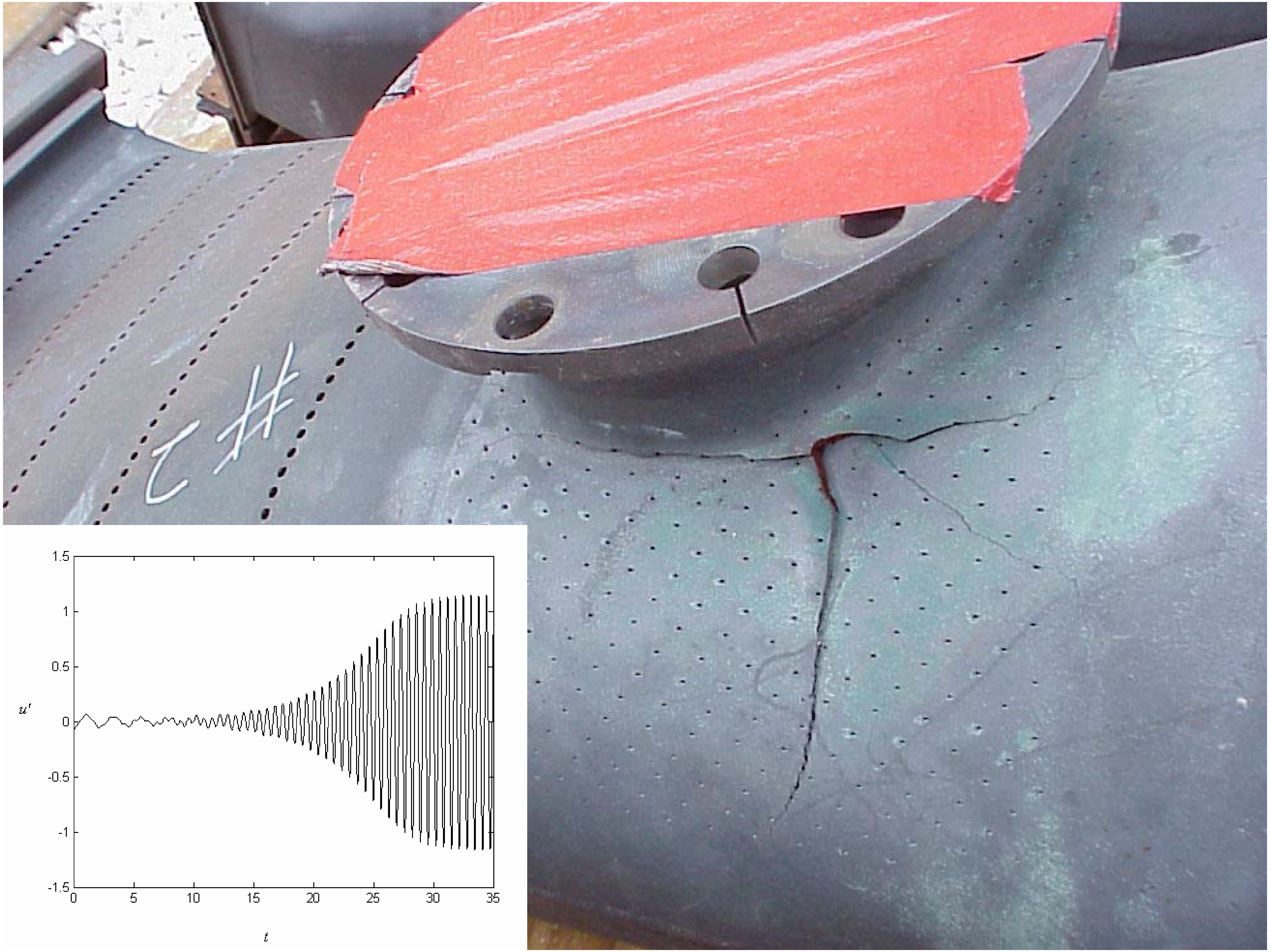
Other thermoacoustic Systems

Combustion instability is a plaguing problem in aero engines & power plants leading to failure

Photograph 2. Intact Nozzle Component

Combustion instability is caused by the feedback between acoustics & combustion





1.5
1
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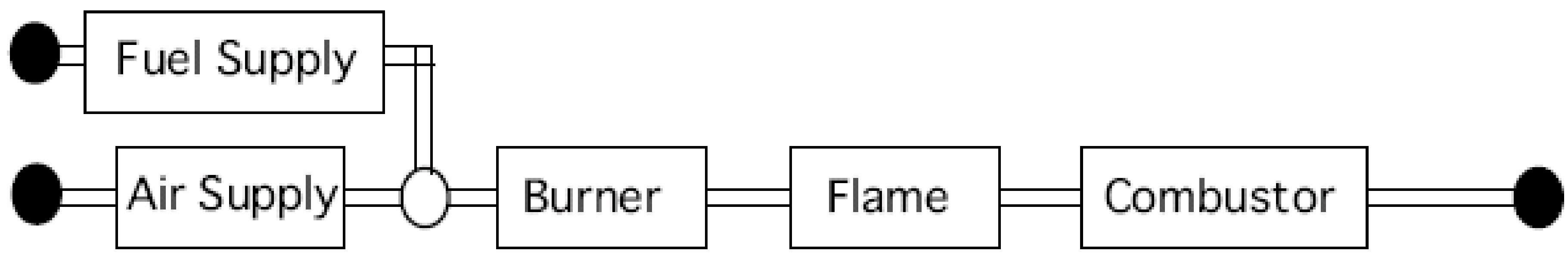
Combustion instability is investigated by testing for unstable eigenvalues of the linearized system



Polifke (2004)

Network model using ***normal modes***

Reasoning:

At low amplitudes, linearised eqns are sufficient to model the evolution

Imaginary part of the eigenvalue is growth rate

$$\omega = 2\pi f + i\,\alpha$$

Complex eigenvalue → ω

frequency → $2\pi f$

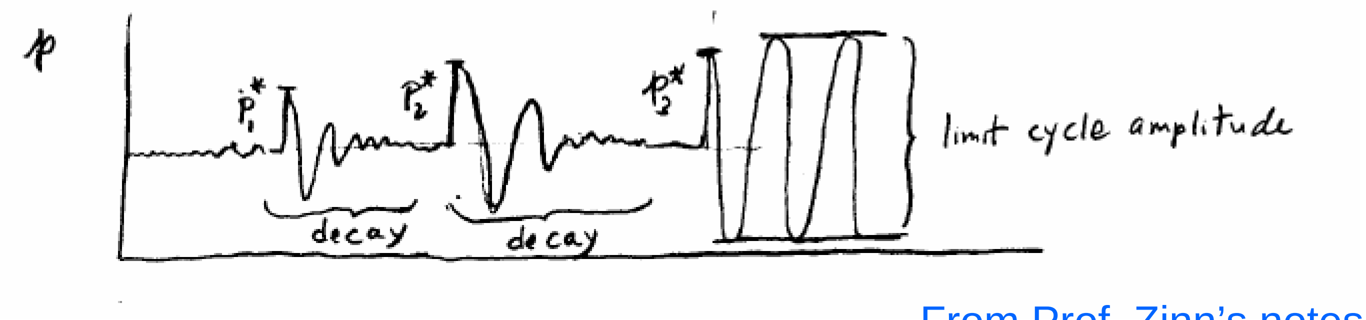
Growth rate → α

$$p' = \hat{p}e^{-i\omega t} = \hat{p}e^{-i(2\pi f + i\alpha)t} = \hat{p}\, e^{-i2\pi f t} e^{\alpha t}$$

Periodic → $e^{-i2\pi f t}$

Exponential growth/decay → $e^{\alpha t}$

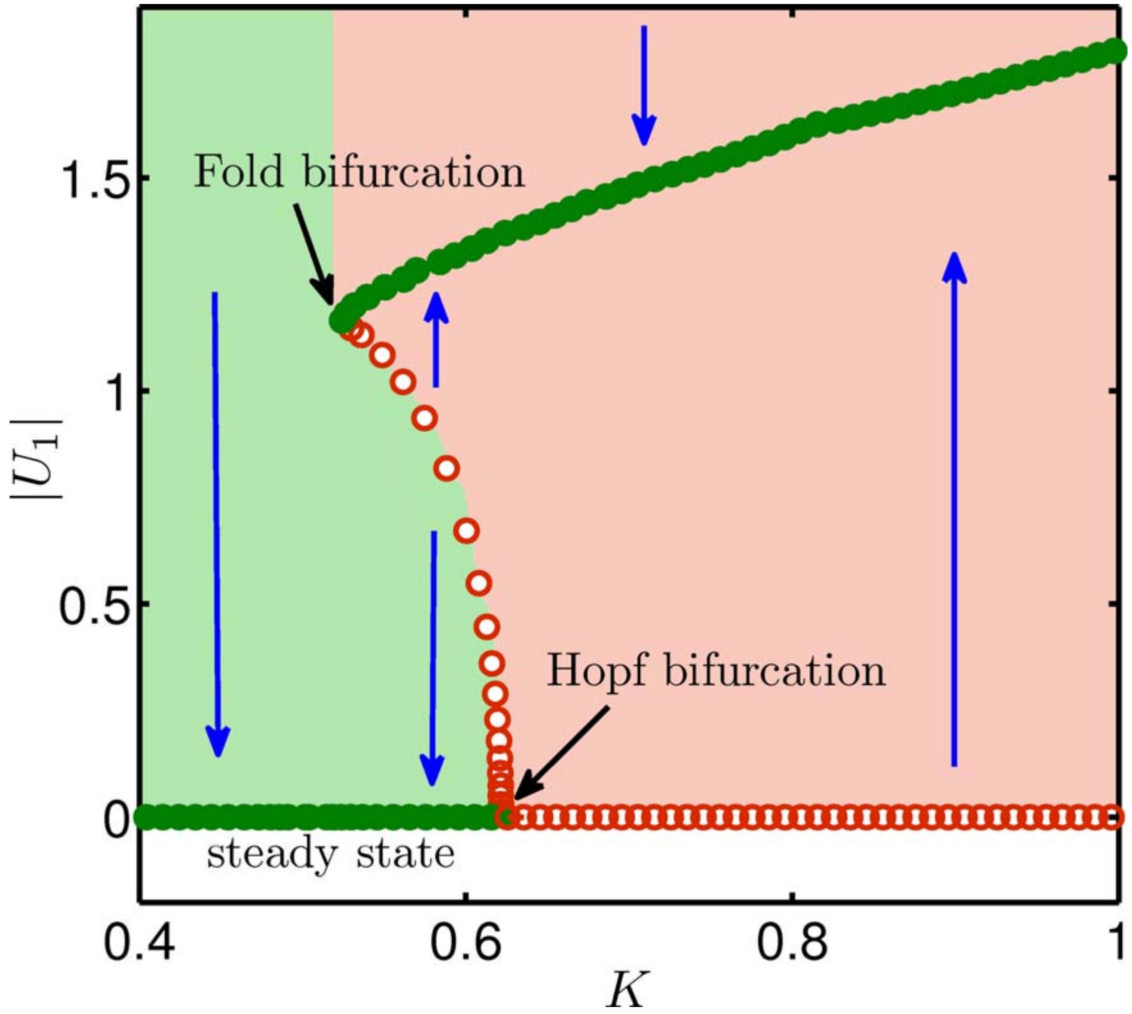
A system is nonlinearly unstable if some finite amplitude disturbance grows with time



From Prof. Zinn's notes

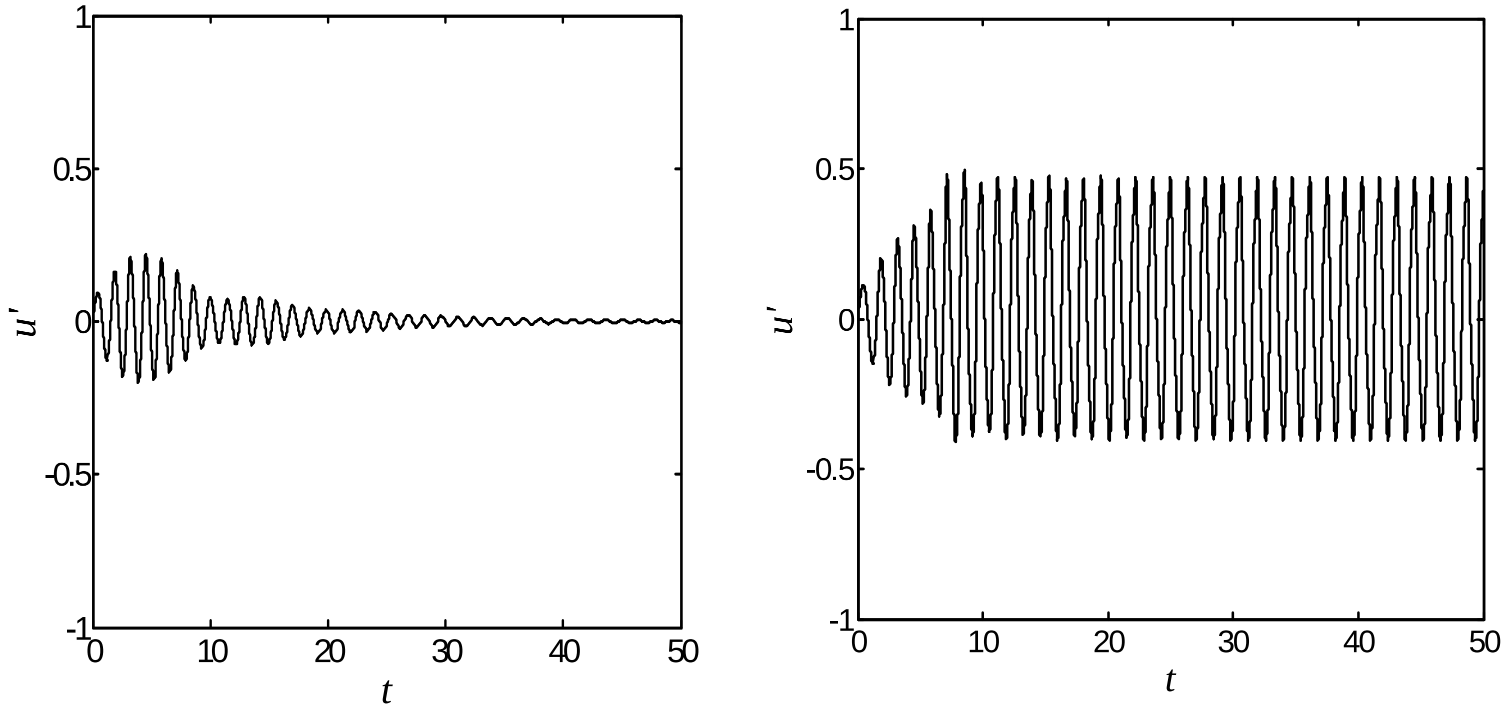
For trigerring instability, the initial amplitude should be greater than a “threshold amplitude”

Increase in power destabilizes the system through a sub-critical Hopf bifurcation



Subramanian et al. (2010, n3l

Can a small, but finite amplitude disturbance cause triggering?



Initial excitation at **small** identical amplitudes.

Let us linearize the acoustics and the heat release.

Momentum

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

Energy

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = (\gamma - 1)\frac{L_a}{c_0}\frac{\dot{Q}'}{\rho_0 c_0^2}$$

Linearized
Heat release

$$\dot{Q}' = R(x, \varepsilon_i)\gamma M u' + S(x, \mu_i)\, p'$$

A system is non-normal when its evolution operator does not commute with its adjoint

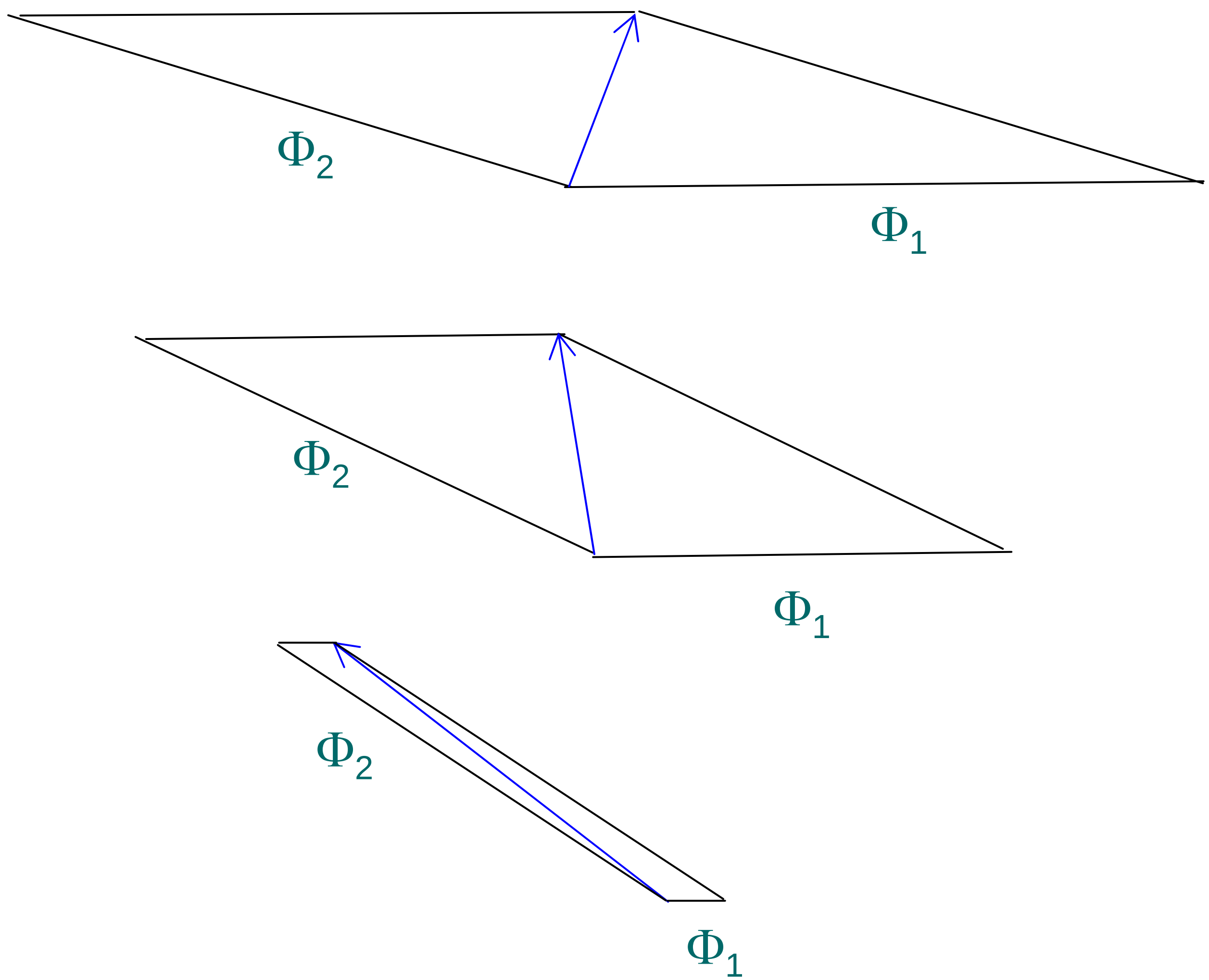
$$\frac{\partial}{\partial t}\begin{bmatrix} \gamma M u' \\ p' \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial x} \\ -\dfrac{\partial}{\partial x} + \dfrac{R L_a(\gamma - 1)}{\rho_0 c_0^3} & \dfrac{S L_a(\gamma - 1)}{\rho_0 c_0^3} \end{bmatrix}\begin{bmatrix} \gamma M u' \\ p' \end{bmatrix}$$

Thermoacoustic interaction is non-normal

Balasubramanian & Sujith: JFM (2008), POF (2008); Nicoud et al. (AIAA J 2007)

“Non-normality”

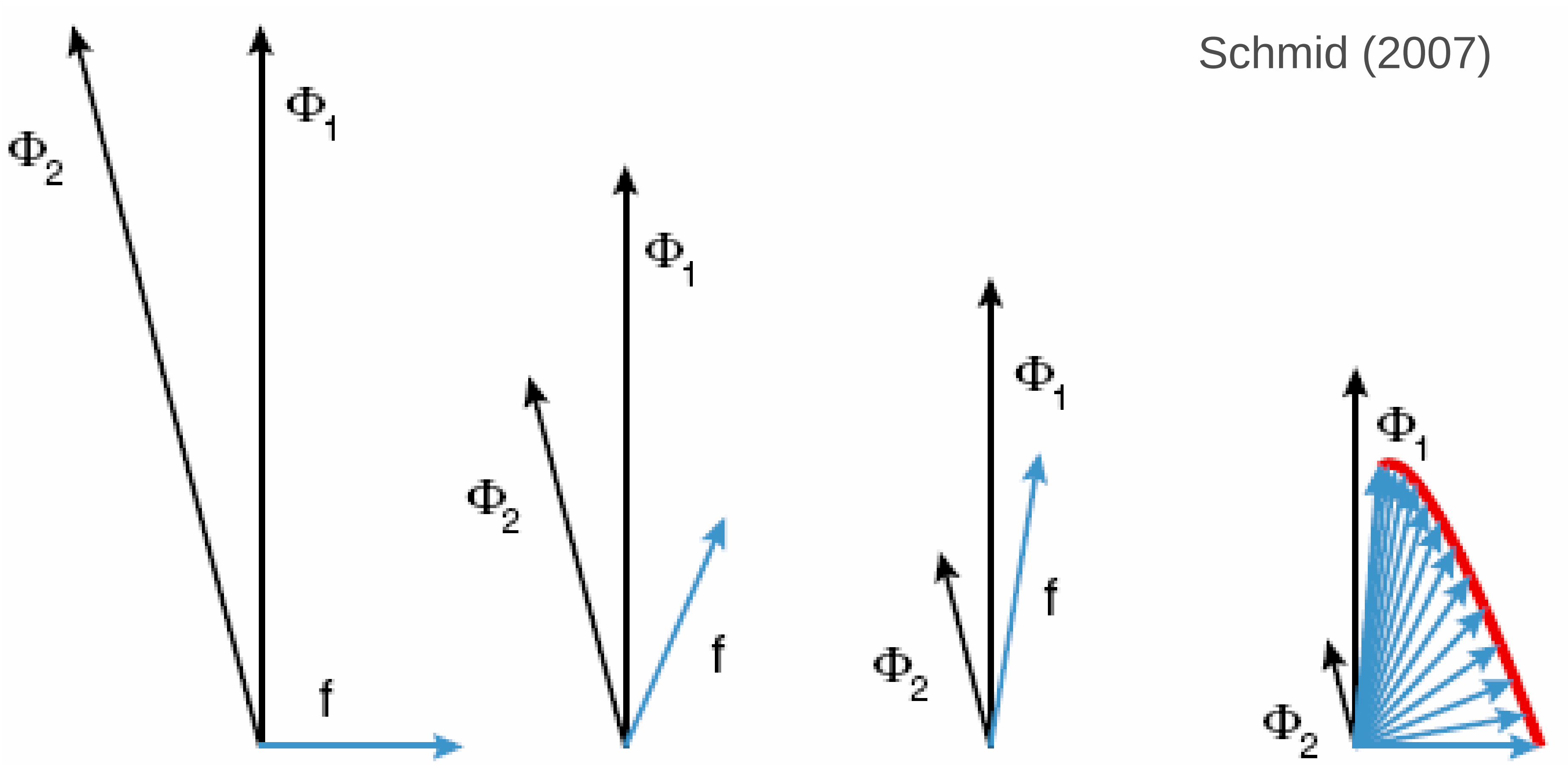
A non-normal system can have transient growth even if the individual modes decay



$$LL^{+} \neq L^{+}L$$

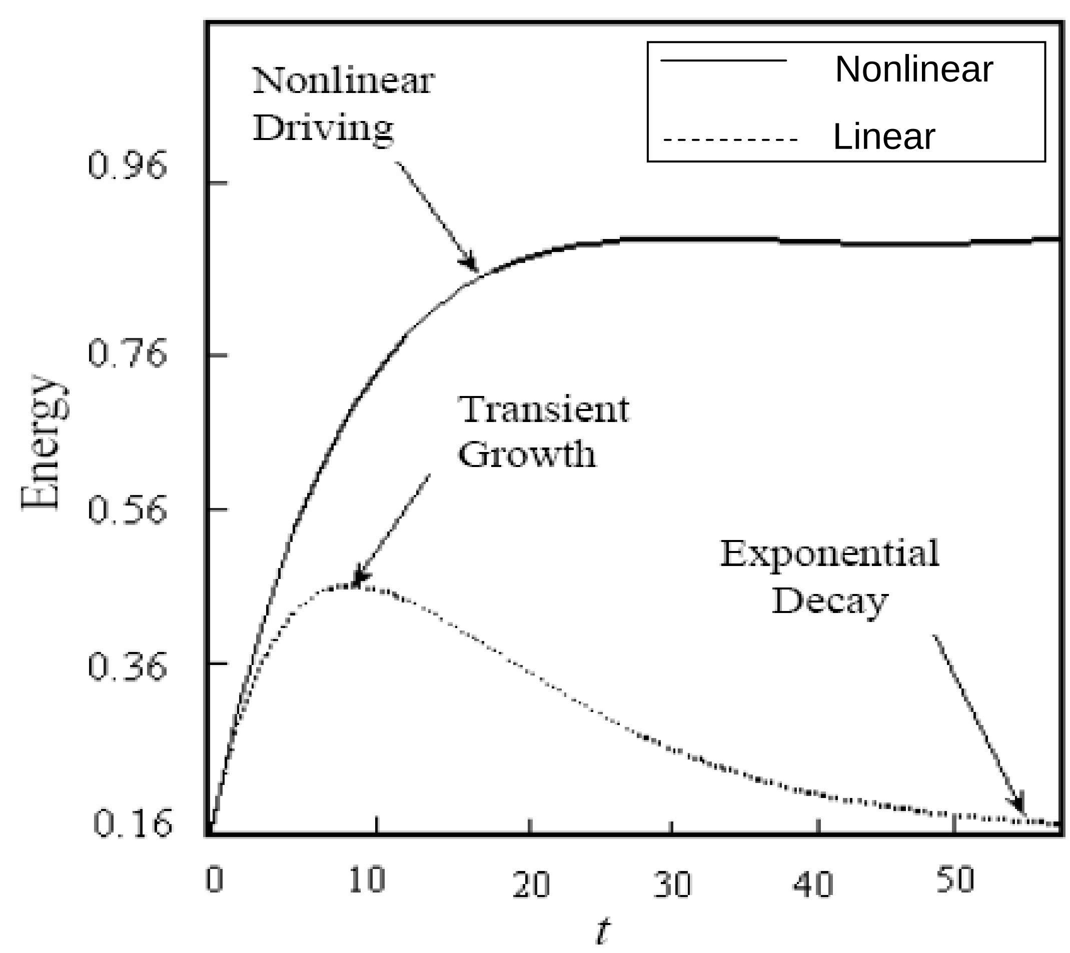
Individual Eigenvalues: Wrong tool to analyse a non-normal system

Superposition of decaying eigenvectors can produce growth in short term



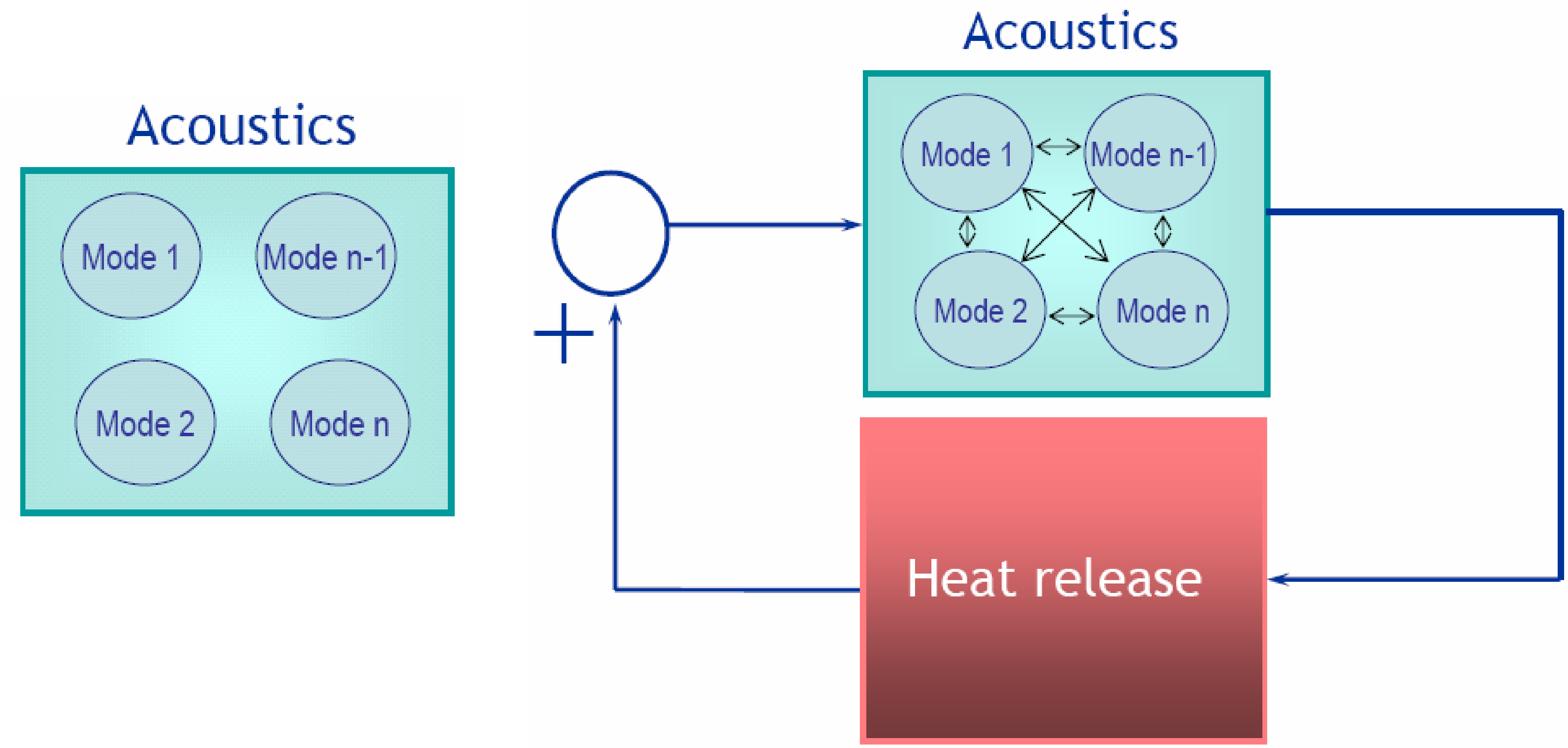
Non-normality: not considered before in thermoacoustic instability

Transient growth can trigger nonlinearities when the amplitude reaches high enough values



Triggerring: Balasubramanian & Sujith: JFM (2008), POF (2008)

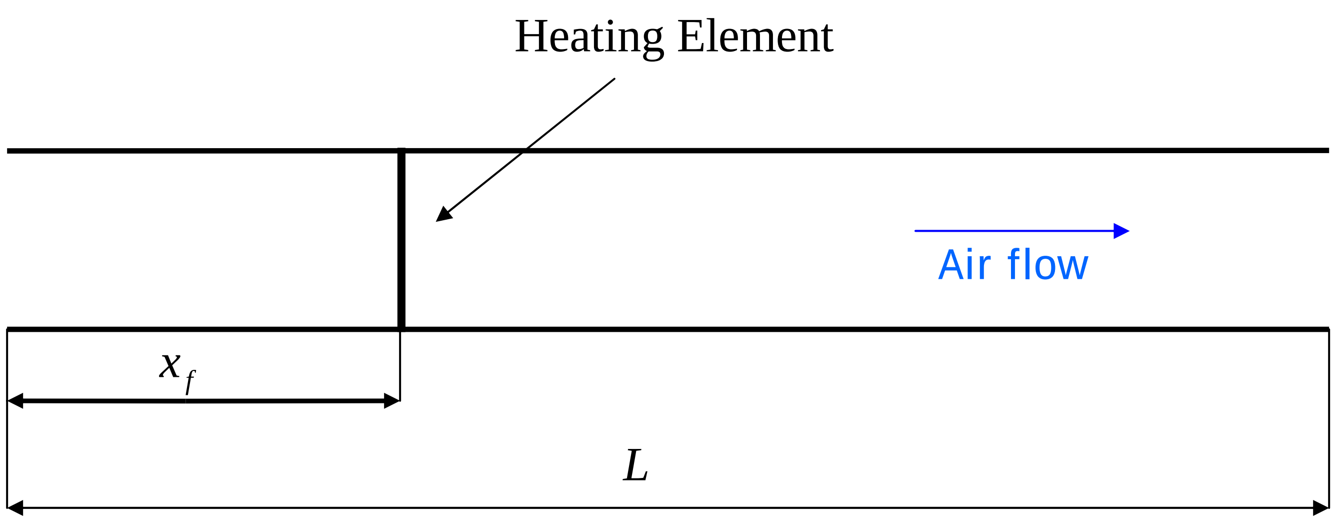
Heat release makes the acoustic modes non-normal



Rijke tube

POF (2008)
AIAA Paper 2007-3428

Horizontal Rijke tube is modeled



We sidestep the effects of natural convection on the mean flow

The acoustic field is solved in the time domain using the Galerkin technique

Momentum: $\gamma M \dfrac{\partial u'}{\partial t} + \dfrac{\partial p'}{\partial x} = 0$

Energy: $\dfrac{\partial p'}{\partial t} + \gamma M \dfrac{\partial u'}{\partial x} = (\gamma - 1)\dfrac{L_a}{c_0}\dfrac{\dot{\tilde{Q}}_{av}}{\rho_0 c_0^2}\dfrac{\delta\left(x - x_f\right)}{L_a}$

Compact Heat Source

Modal expansion:

$$u' = \sum_{j=1}^{N} \eta_j \cos(j\pi x) \quad \text{and} \quad p' = \sum_{j=1}^{N} \frac{\gamma M}{j\pi} \dot{\eta}_j \sin(j\pi x)$$

Heckl's modified form of the King's law is used to model the heat release

$$\dot{Q}' = \dot{Q}'\left(u'_f\left(t-\tau\right)\right)$$

$$\dot{Q}' = \frac{2L_w(T_w - \bar{T})}{S\sqrt{3}}\sqrt{\pi\lambda C_v\bar{\rho}\frac{d_w}{2}}\left[\sqrt{\left|\frac{\bar{u}}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]$$

King's law predicts nonlinearity only for velocity perturbations greater than the mean velocity

Evolution equations obtained from the Galerkin projection are non-normal & nonlinear

$$A\frac{d\chi}{dt}+B_{NN}\chi+B_{NL}(\chi)=0$$

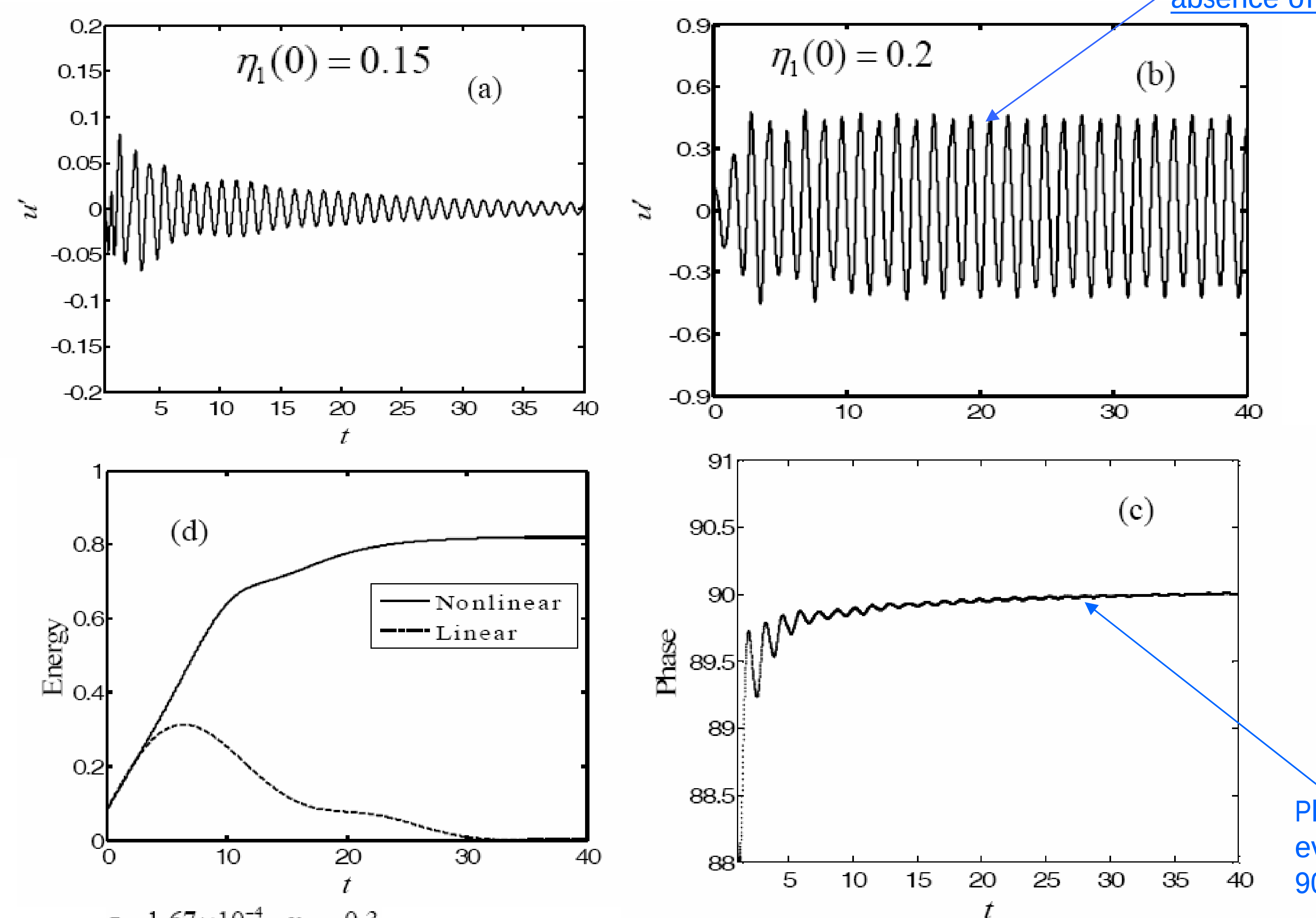
B_{NN} is non-normal

B_{NL} is nonlinear

No damping

Subcritical transition to instability is observed

Limit cycle in the absence of damping



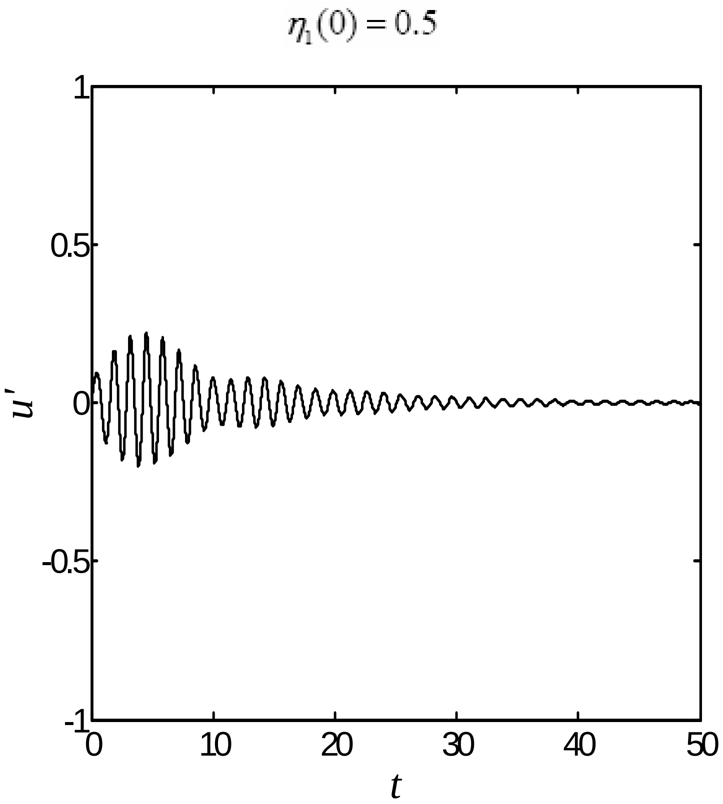
Phase evolves to 90°

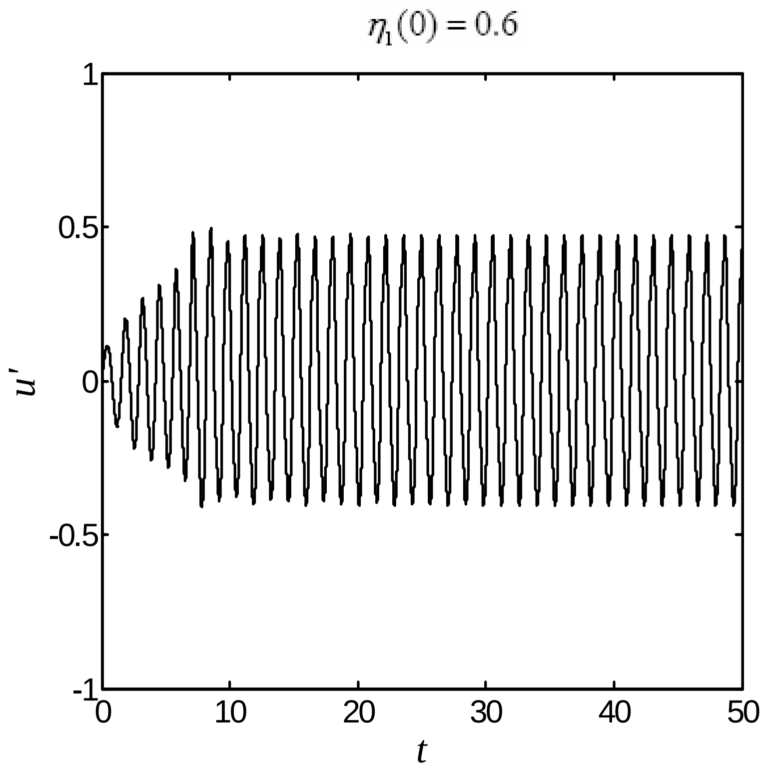
$\tau = 1.67\times10^{-4}$, $x_f = 0.3$

$\bar{u} = 0.5m/s$, $\bar{\rho} = 1.205kg/m^3$, $T_w = 1000K$, $C_v = 719J/kgK$, $\lambda = 0.0328W/mK$ $x_f = 0.29$, $c_0 = 399.6m/s$, $L_w = 3.6m$

Include damping

"Triggering" occurs with damping as well

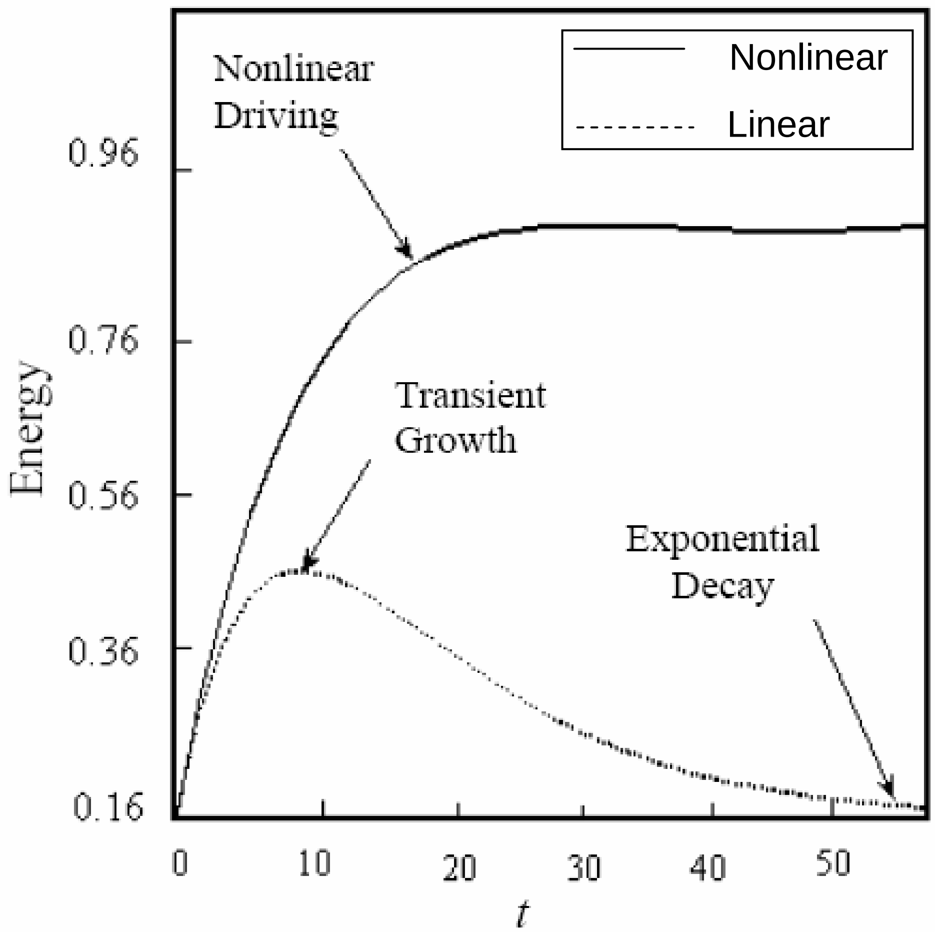




$$\xi_j = \frac{1}{2\pi}\left[c_1\, \omega_j/\omega_1 + c_2\sqrt{\omega_1/\omega_j}\right] \quad c_1 = 0.05,\ c_2 = 0.005$$

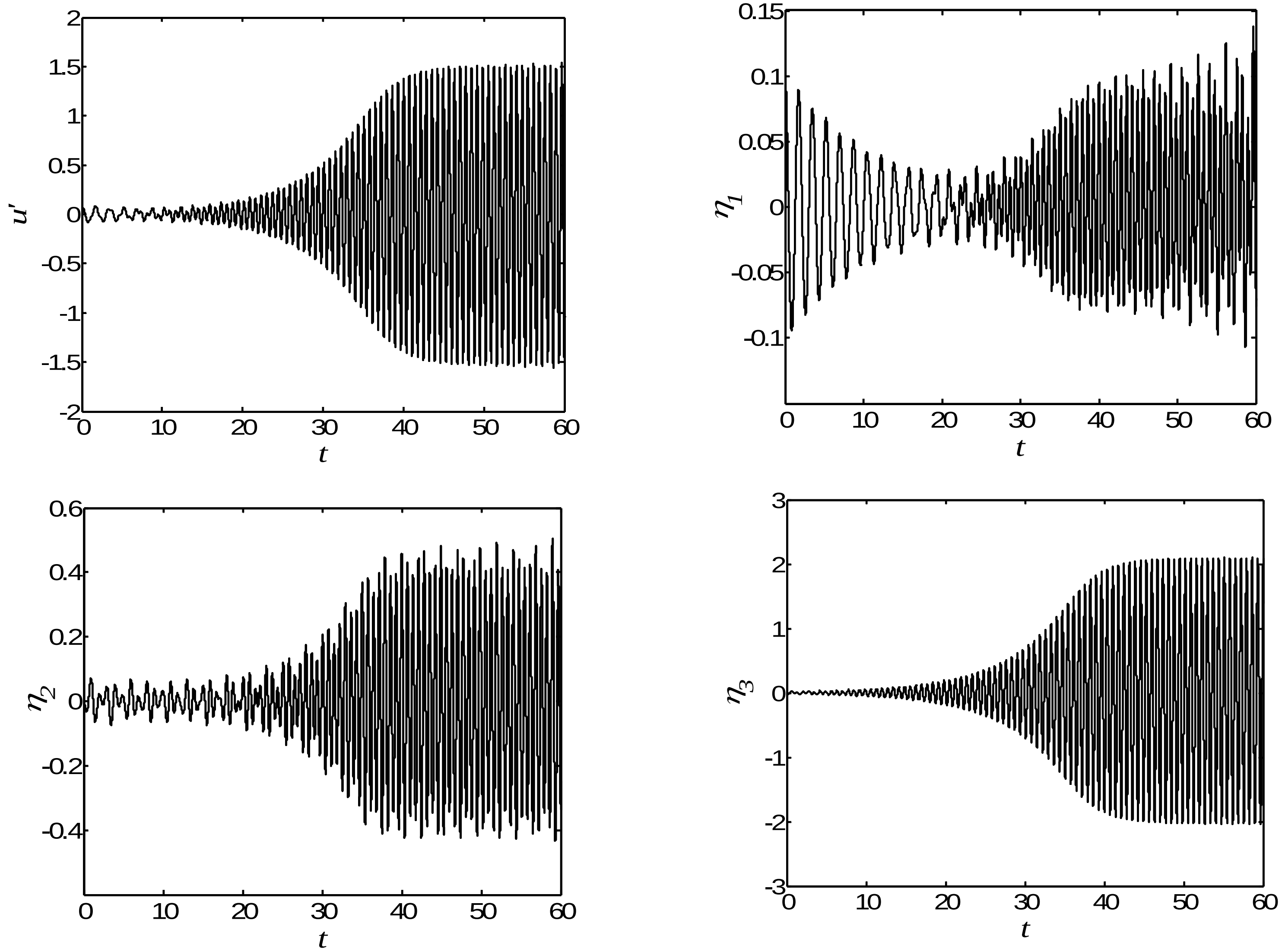
$L_w = 2.0m$, $\lambda = 0.0328W/mK$, $C_v = 719kJ/kgK$, $\bar{\rho} = 1.205\,kg/m^3$, $\bar{u} = 0.6\,m/s$, $T_w = 900K$.
$\tau = 1.67\times10^{-4}$, $x_f = 0.3$

Identical linear & nonlinear simulations at low amplitudes.



Nonlinearity "picks up" after sufficient transient growth

Bootstrapping in an initially decaying system that grows later



$\eta_1(0) = 0.1, \eta_{i \neq 1}(0) = 0, \dot{\eta}_i(0) = 0$ $L_w = 2\text{m}$ and $T_w = 900\text{K}$, $x_f = 0.3$

How can we quantify energy growth?

Nagaraja, Kedia & Sujith (Montreal Symposium, 2008)

2-norm of a vector is its length in Eucledian space

Vector $x = (x_1, x_2, \ldots, x_p)$

Its norm $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_p^2}$

For our vector space with Galerkin modes

$$\chi = [\eta_1 \quad \frac{\dot{\eta}_1}{k_1} \quad \eta_2 \quad \frac{\dot{\eta}_2}{k_2} \quad \cdots\cdots\cdots \quad \eta_N \quad \frac{\dot{\eta}_N}{k_N}]^T$$

$$\|\chi(t)\|^2 = \sum_{i=1}^{N} \left(\eta_i^2 + \frac{\dot{\eta}_i^2}{k_i^2} \right)$$

2-norm of our state vector represents the acoustic energy

$$E(t) = \int_0^1 \left(\frac{1}{2} p'(x,t)^2 + \frac{1}{2} (\gamma M u'(x,t))^2 \right) dx$$

$$E(t) = \left(\frac{\gamma M}{2} \right)^2 \left\| \chi(t) \right\|^2$$

Greatest possible energy growth at time *t*, maximized over all possible initial perturbations is given by

$$G(t) = \max_{\chi(0)} \frac{\left\|\chi(t)\right\|^2}{\left\|\chi(0)\right\|^2} = \left\|\exp(-Lt)\right\|^2$$

We maximise it over all time to get Gmax

We need the 2-norm of $\exp(-Lt)$

$$\frac{d\chi}{dt} + L\chi = 0$$

$$\chi(t) = \exp(-Lt)\chi(0)$$

The 2-norm of any matrix is its principal singularvalue

We define the SVD of a matrix *A* as

$$A = U\Sigma V^T$$

U is a unitary matrix,

Σ is a matrix with non-negative numbers on its diagonal and zeros off the diagonal

V^T is the transpose of V, which is a unitary matrix.

Let us do SVD of our evolution operator

$$\frac{d\chi}{dt} + L\chi = 0$$

$$\chi = [\eta_1 \quad \frac{\dot{\eta}_1}{k_1} \quad \eta_2 \quad \frac{\dot{\eta}_2}{k_2} \quad \cdots\cdots\cdots \quad \eta_N \quad \frac{\dot{\eta}_N}{k_N}]^T$$

$$\chi(t) = \exp(-Lt)\chi(0)$$

Let us do SVD of our evolution operator

$$e^{-Lt} = U\Sigma V^T$$

$$\chi(t) = U\Sigma V^T \chi(0)$$

$$\chi(t) = \exp(-Lt)\chi(0)$$

$$\chi(t) = \exp(-Lt)\chi(0)$$

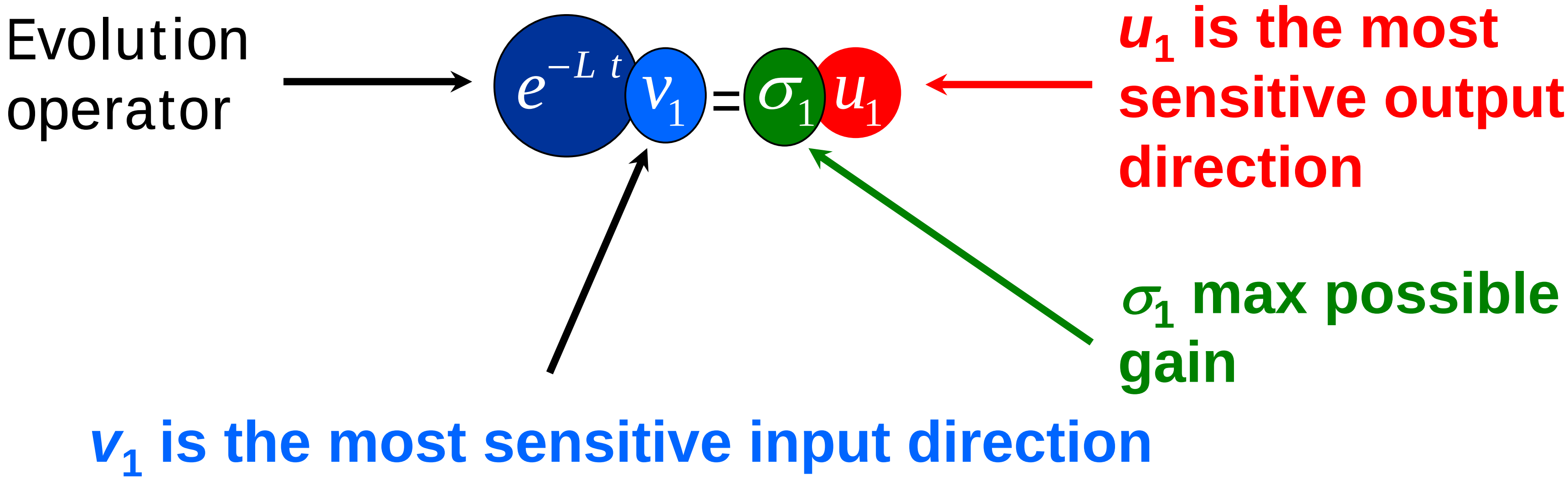
Resolves the initial condition vector into an orthonormal basis of input vectors

$$\chi(t) = U\Sigma V^{T}\chi(0)$$

Represents the output vector as a linear superposition of components along the orthonormal basis formed by the output vectors

$$e^{-Lt} = U \Sigma V^T$$

$$e^{-Lt} V = U \Sigma V^T V = U \Sigma$$

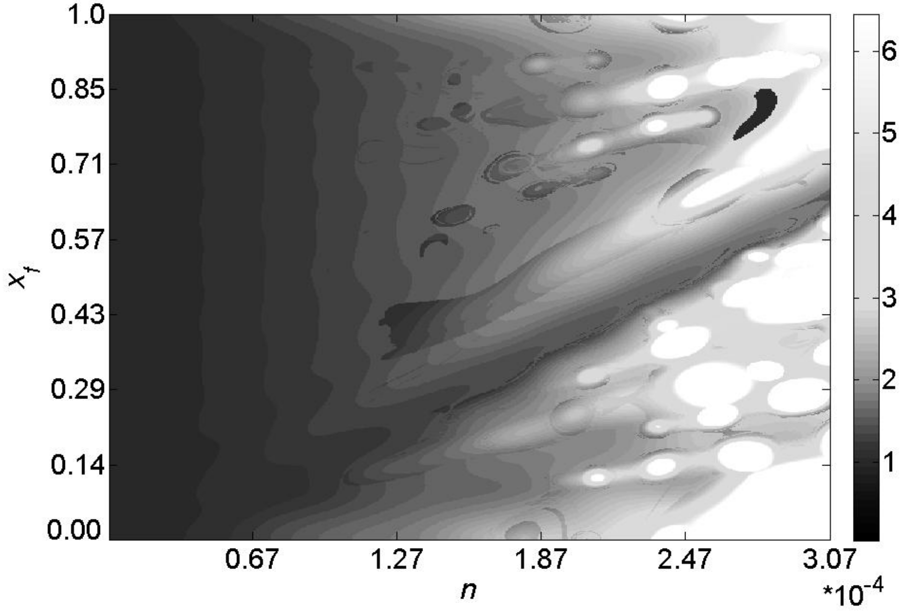


Principal singularvalue - max energy amplification.

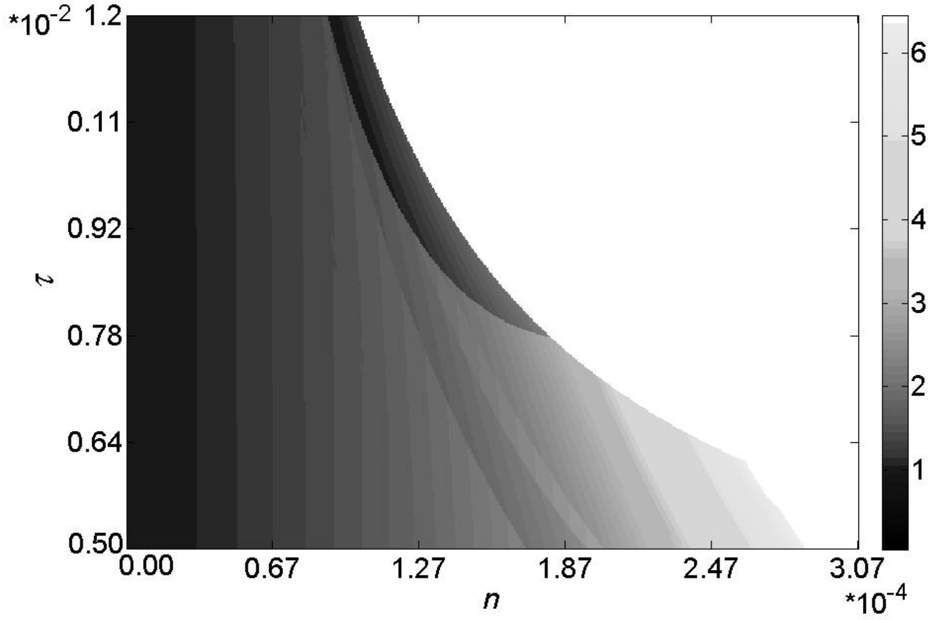
Corresponding
right singularvector - most sensitive initial condition

G_{max} = Infinity	Classical linear instability
G_{max} = 1	Classical linear stability
G_{max} > 1, finite	Transient growth

n - τ model of Crocco captures transient dynamics



$\tau = 0.001$, $c_1 = 0.8$, $M = 1.7 \times 10^{-4}$



$x_f = 0.3$, $c_1 = 0.8$ $M = 1.7 \times 10^{-4}$

There are regions of stability, instability and

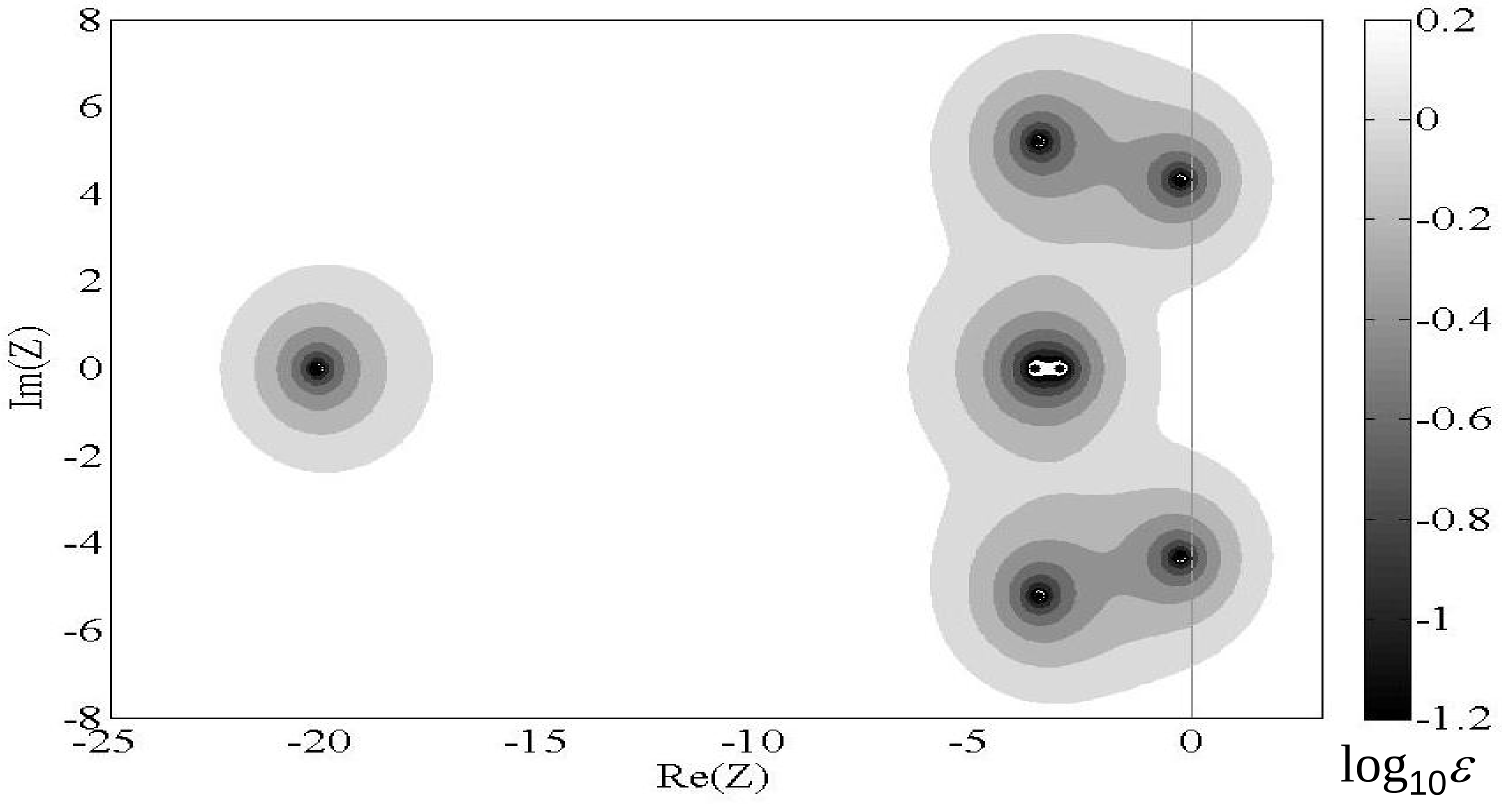
"<u>potential" instability</u>

Pseudospectra for studying non-normal systems

Z is an ε - pseudoeigenvalue of A if it satisfies $||(ZI\text{-}A)^{-1}|| \geq \varepsilon^{-1}$

Set of all points on the complex plane whose minimum value of the singular value of (ZI-A) is less than ε

Contours should protrude into the right half plane for the system to exhibit transient growth



"Necessary condition" for transient growth

Predicting transient growth using Rayleigh criteria requires precise knowledge of initial conditions

Lord Rayleigh

$$\frac{\partial}{\partial t}\int_V \langle \frac{p^2}{2\varrho_0 c^2} + \frac{\varrho u^2}{2} \rangle \, dV + \oint_S (pu) \, dS = \frac{\gamma - 1}{c^2} \int_V \langle pQ \rangle \, dV$$

Ambiguity of initial conditions due to noise makes the identification of transient growth using Rayleigh Criteria difficult.

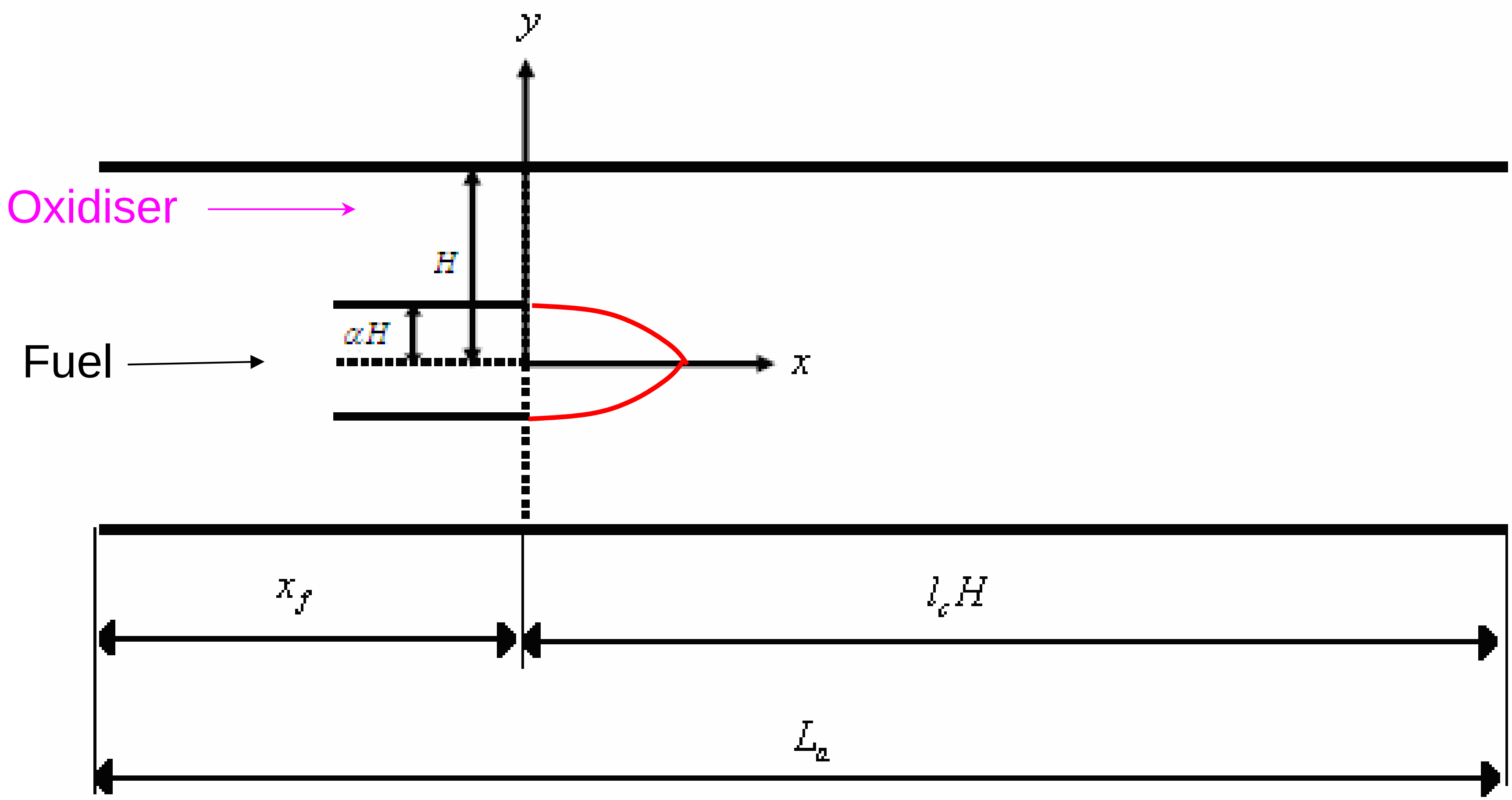
The conditions are now on the evolution operator; hence do not depend on initial conditions

Thermoacoustics involves multiphysics & multiscales

Ducted Burke-Schumann Flame

JFM (2008)
AIAA Paper 2007- 0567

We model 2D co-flow non-premixed combustion



Infinite rate chemistry model is used to model the unsteady diffusion flame

Compact heat source

$$\frac{\partial Z}{\partial t} + u(t)\frac{\partial Z}{\partial x} = \frac{1}{Pe}\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right)$$

Boundary Conditions

i) At $y = \pm 1$, $\frac{\partial Z}{\partial y} = 0$, $\quad 0 \le x < l_c$

ii) At $x = 0$

$Z = -Y_i \quad -\alpha \le y \le \alpha$

$Z = X_i$, $\quad -1 \le y \le -\alpha$ and $\alpha \le y \le 1$

iii) At $x = l_c$, $\frac{\partial Z}{\partial x} = 0$

Galerkin technique is used to solve the unsteady Burke Schumann problem

Superposition of mode functions

$$Z = \sum_m \sum_n A_n \cos(n\pi y)\sin((m+1/2)\frac{\pi x}{l_c})G_m^n(t) + Z_{st}$$

Evolution equations

$$\dot{G}_m^{(n)} + u(t)\sum_k W_{mk} G_k^{(n)} = -\frac{(m+1/2)^2\pi^2}{l_c^2 Pe} G_m^{(n)} - \frac{n^2\pi^2}{Pe} G_m^{(n)} + \left[u(t)-1\right] C_m^{(n)}$$

$$W_{mk} = \int_0^{l_c} \sin\left[(m+1/2)\pi x / l_c\right]\cos\left[(k+1/2)\pi x / l_c\right] dx$$

W **does not commute with its adjoint.**

The heat release is calculated using the thermodynamic relations

Burke Schumann temperature field

$$T_{bs} = T_i + X_i\left(Y_i + Z\right)/\left(X_i + Y_i\right) \qquad Z \leq 0$$

$$T_{bs} = T_i + Y_i\left(X_i - Z\right)/\left(X_i + Y_i\right) \qquad Z \geq 0$$

Heat release

$$\dot{Q}_c = \int_V \left(\frac{dT_{bs}}{dt} + T_{bs}\vec{\nabla}.\vec{u}\right) dV = \int_V \left(\frac{\partial T_{bs}}{\partial t} + \vec{\nabla}.(T_{bs}\vec{u})\right) dV$$

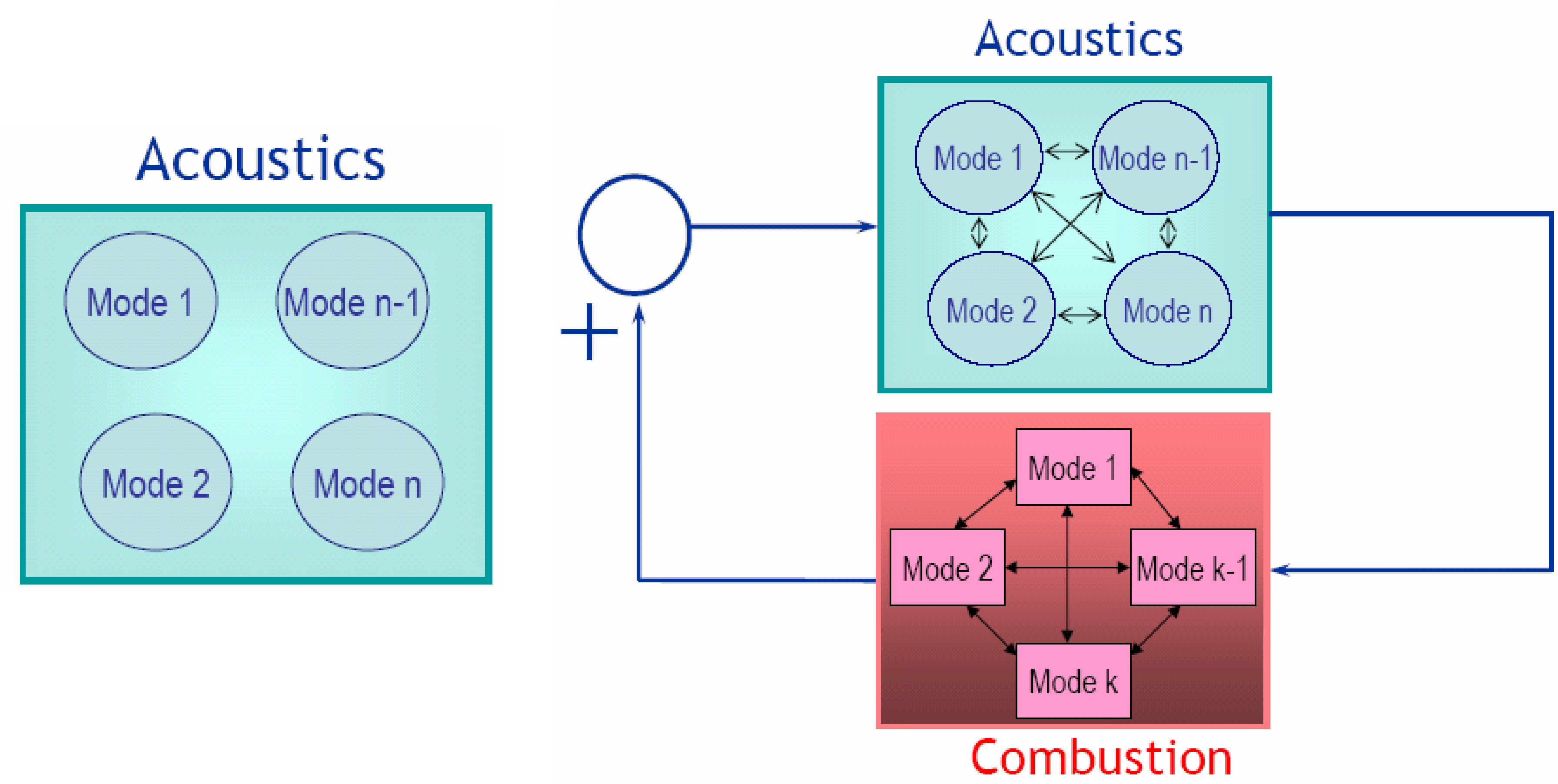
Evolution equations for both acoustic and combustion modes are non-normal & nonlinear

$$A\frac{d\chi}{dt} + B_{NN}\chi + B_{NL}(\chi) = 0$$

B_{NN} is non-normal

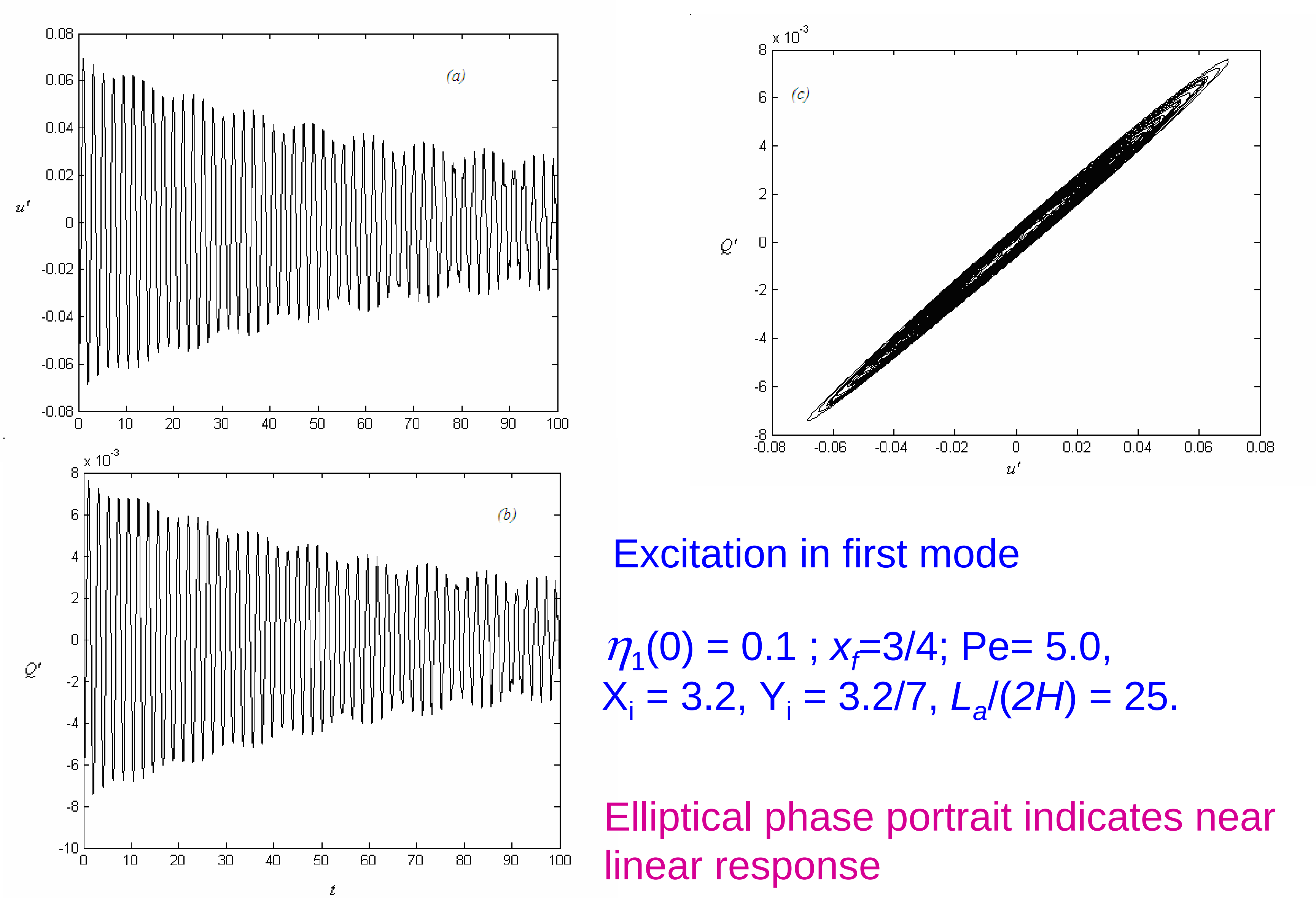
B_{NL} is nonlinear

Combustion makes the acoustic modes non-normal



Combustion modes are non-normal even in the absence of acoustic feedback

The oscillations decay, though there are several periods of short time growth

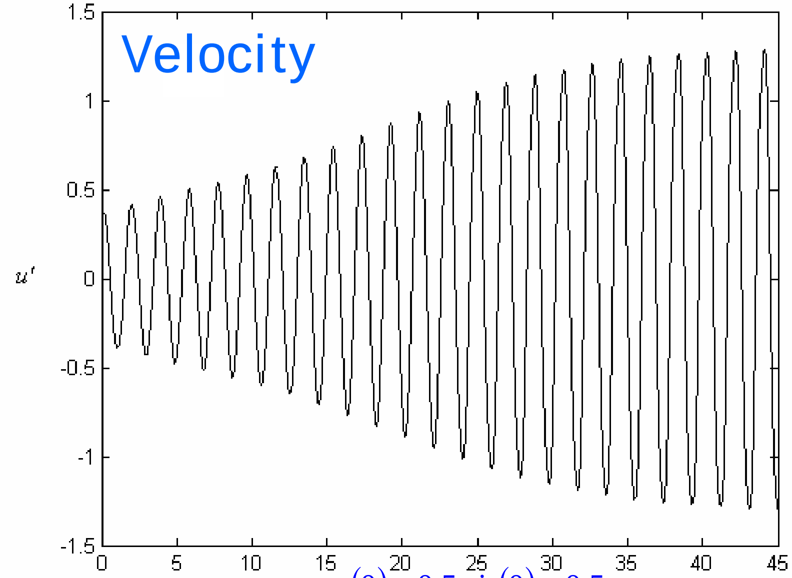


Excitation in first mode

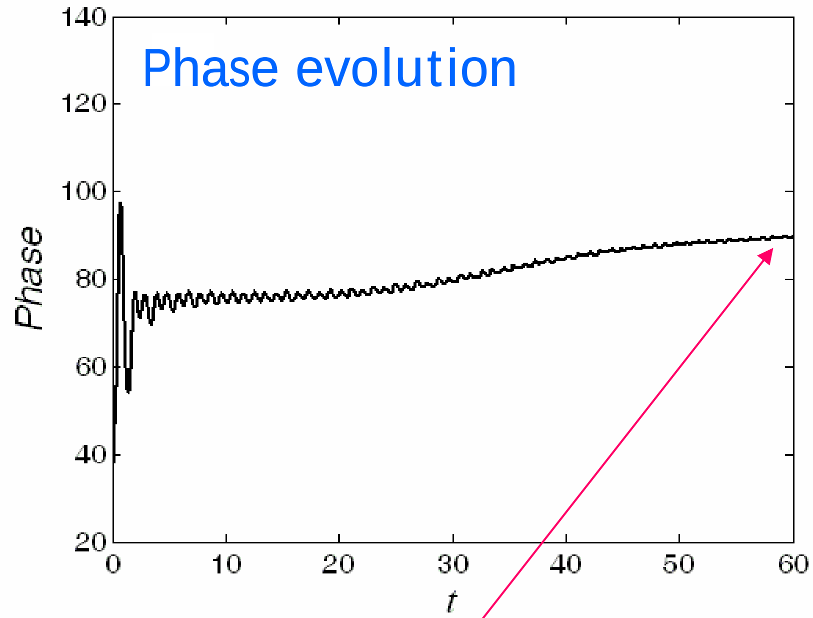
$\eta_1(0) = 0.1$; x_f=3/4; Pe= 5.0,
X_i = 3.2, Y_i = 3.2/7, $L_a/(2H)$ = 25.

Elliptical phase portrait indicates near linear response

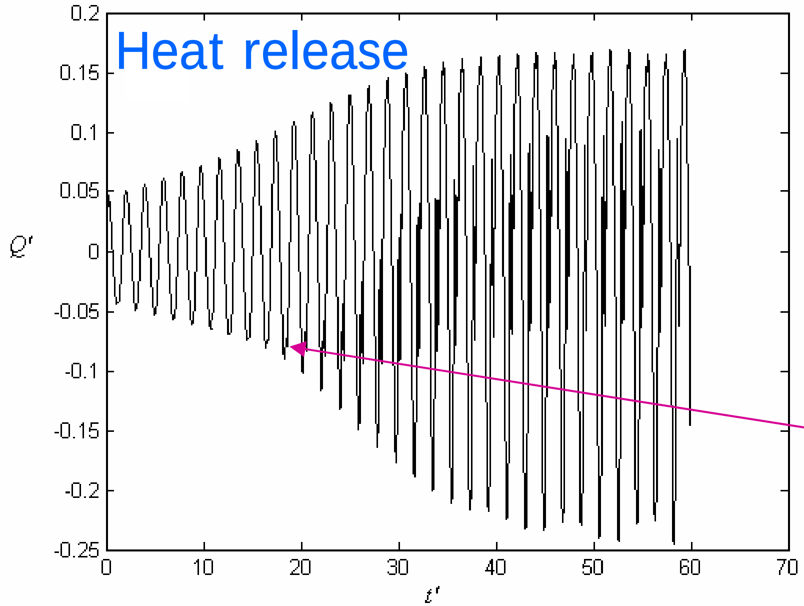
For a different initial condition, transient growth is large enough to trigger nonlinearities.



$\eta_1(0) = 0.5; \dot{\eta}_1(0) = 0.5$

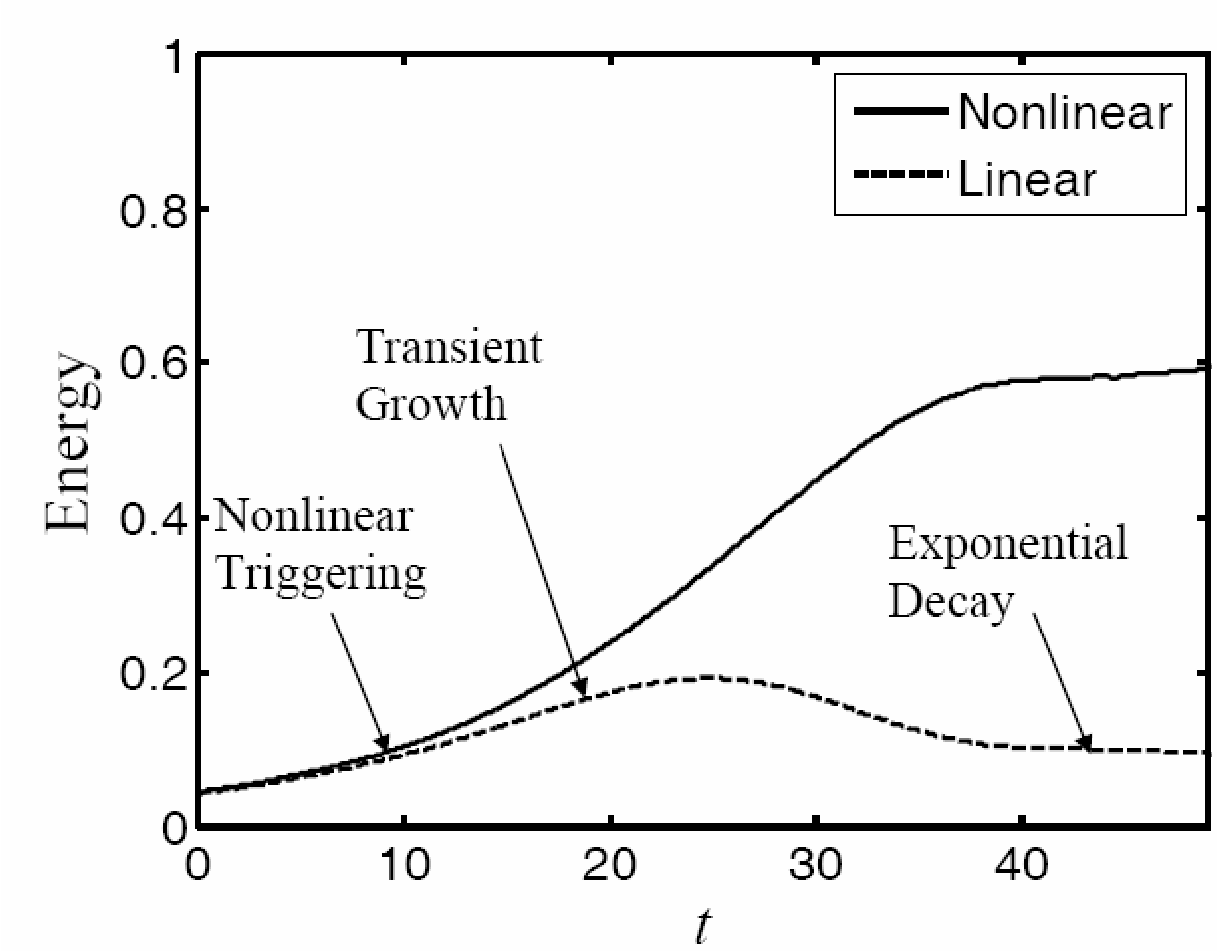


Phase evolves to 90º



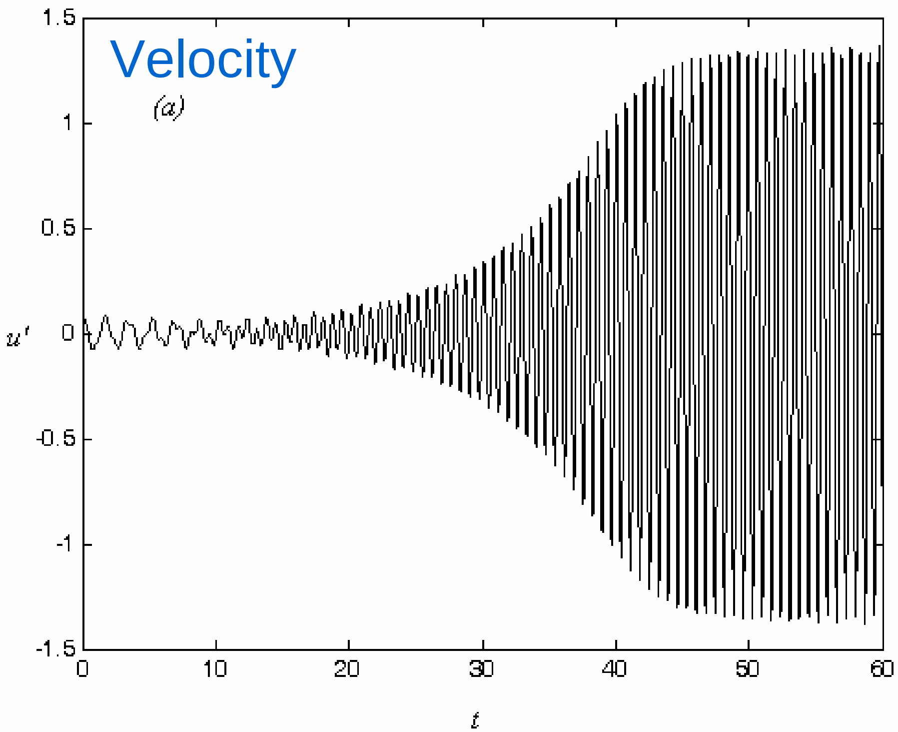
Mean value of heat release changes indicating nonlinear behavior

Linear & nonlinear simulations are identical at low amplitudes.

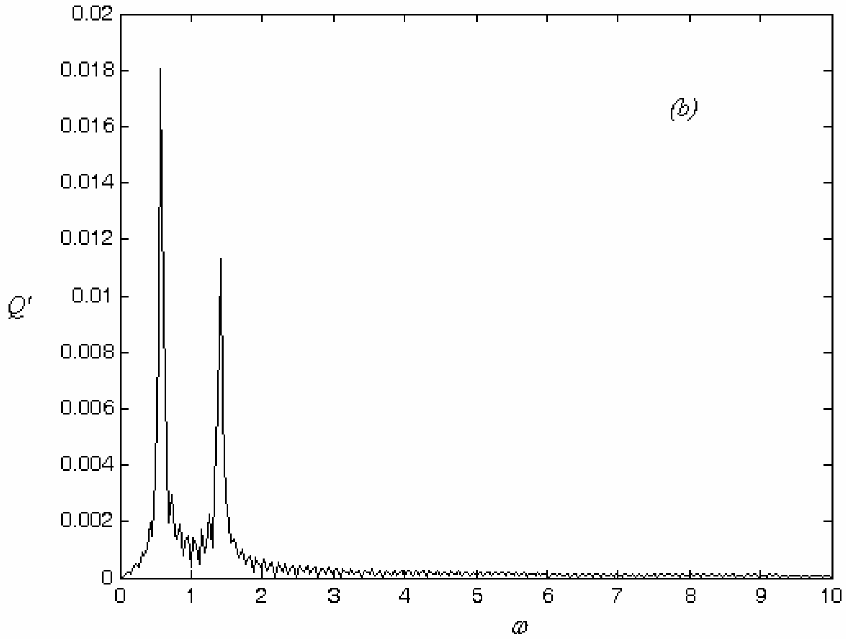


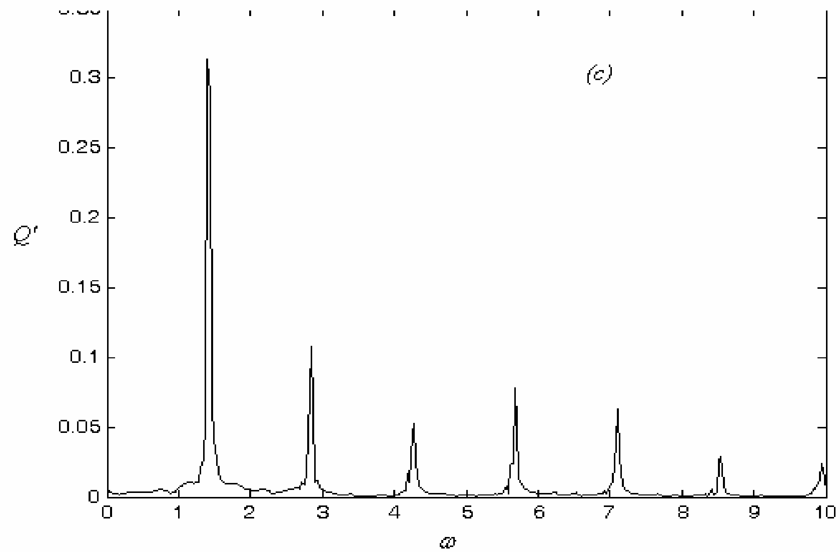
Nonlinearity “picks up” after sufficient transient growth

Combustion Instabilities occur at frequencies far from natural acoustic frequencies



FFT of heat release





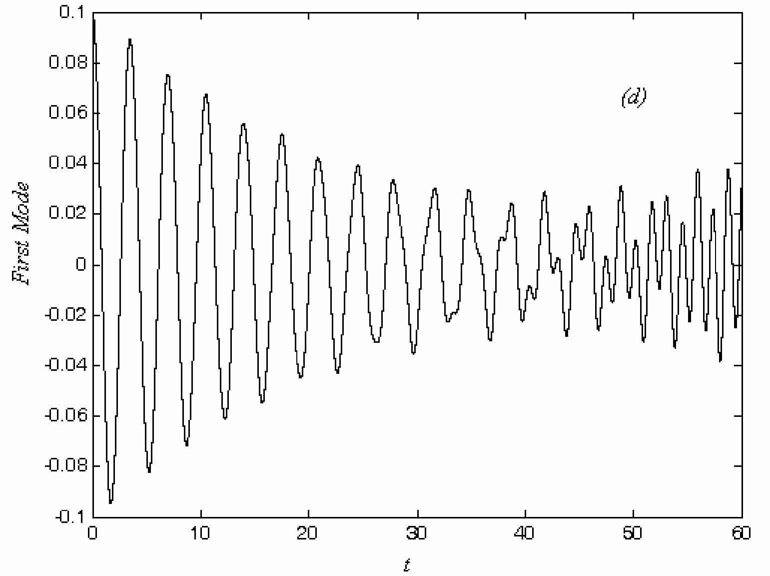
x_f=3/4; Pe= 5.0, X_i = 3.2, Y_i = 3.2/7, $L_a/(2H)$ = 25.

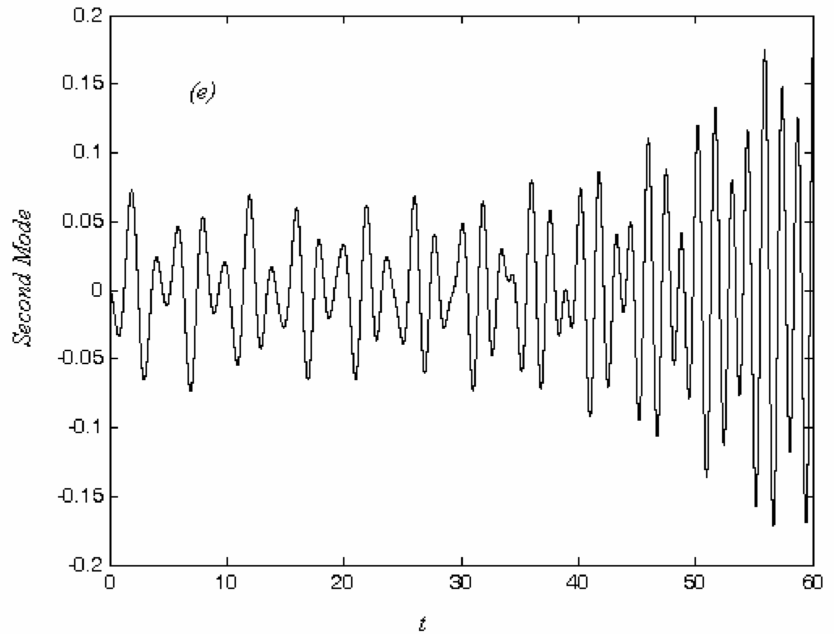
$$\eta_1(0) = 0.1; \dot{\eta}_1(0) = 0.1$$

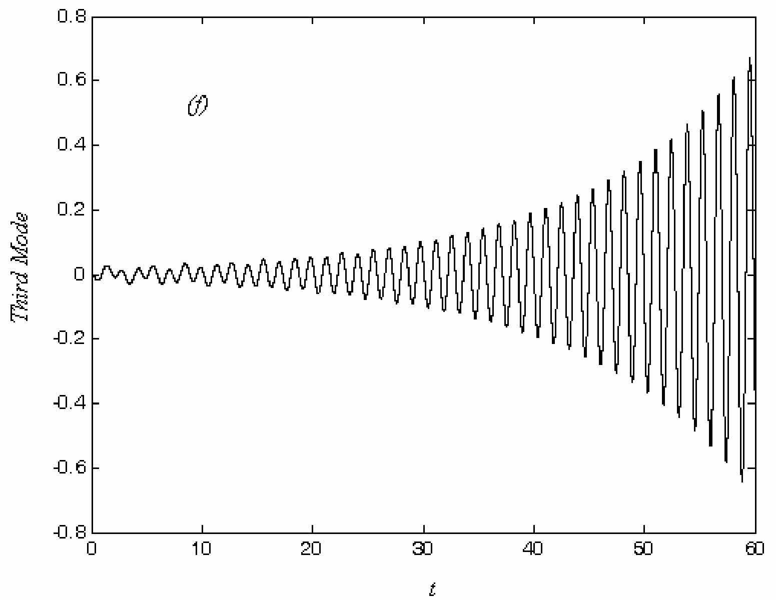
Matveev & Culick (2003);

Schadow & Gutmark (1989)

Bootstrapping in an initially decaying oscillations that grow later





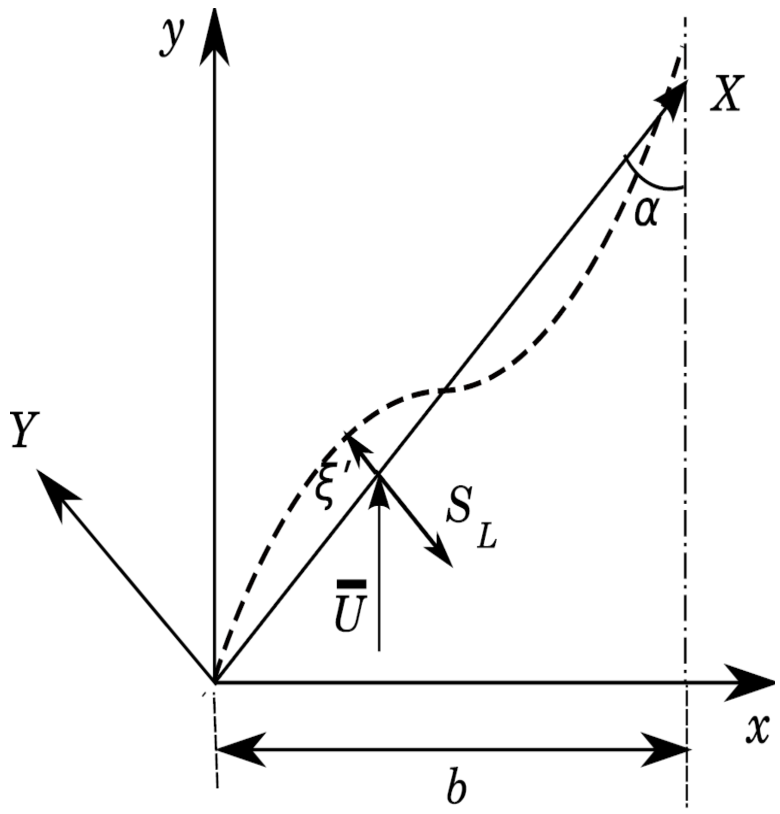


Initially stable 1st mode projects energy to 3rd mode, which later projects energy back to 1st mode.

Is premixed flame-acoustic interaction non-normal?

Yes, indeed!

Premixed flame is modelled using the *G*-equation



$$\frac{\partial \xi'}{\partial t} + (\overline{u} + u')\cos\alpha \frac{\partial \xi'}{\partial X} - (\overline{u} + u')\sin\alpha = -S_L\sqrt{1 + (\frac{\partial \xi'}{\partial X})^2}$$

Heat release rate is correlated to surface area

Schuller (2003)

$$\frac{Q'}{\overline{Q}} = \frac{A'}{\overline{A}}$$

Linearised system of equations for the self evolving system are non-normal

$$\frac{d\chi}{dt} = L\chi$$

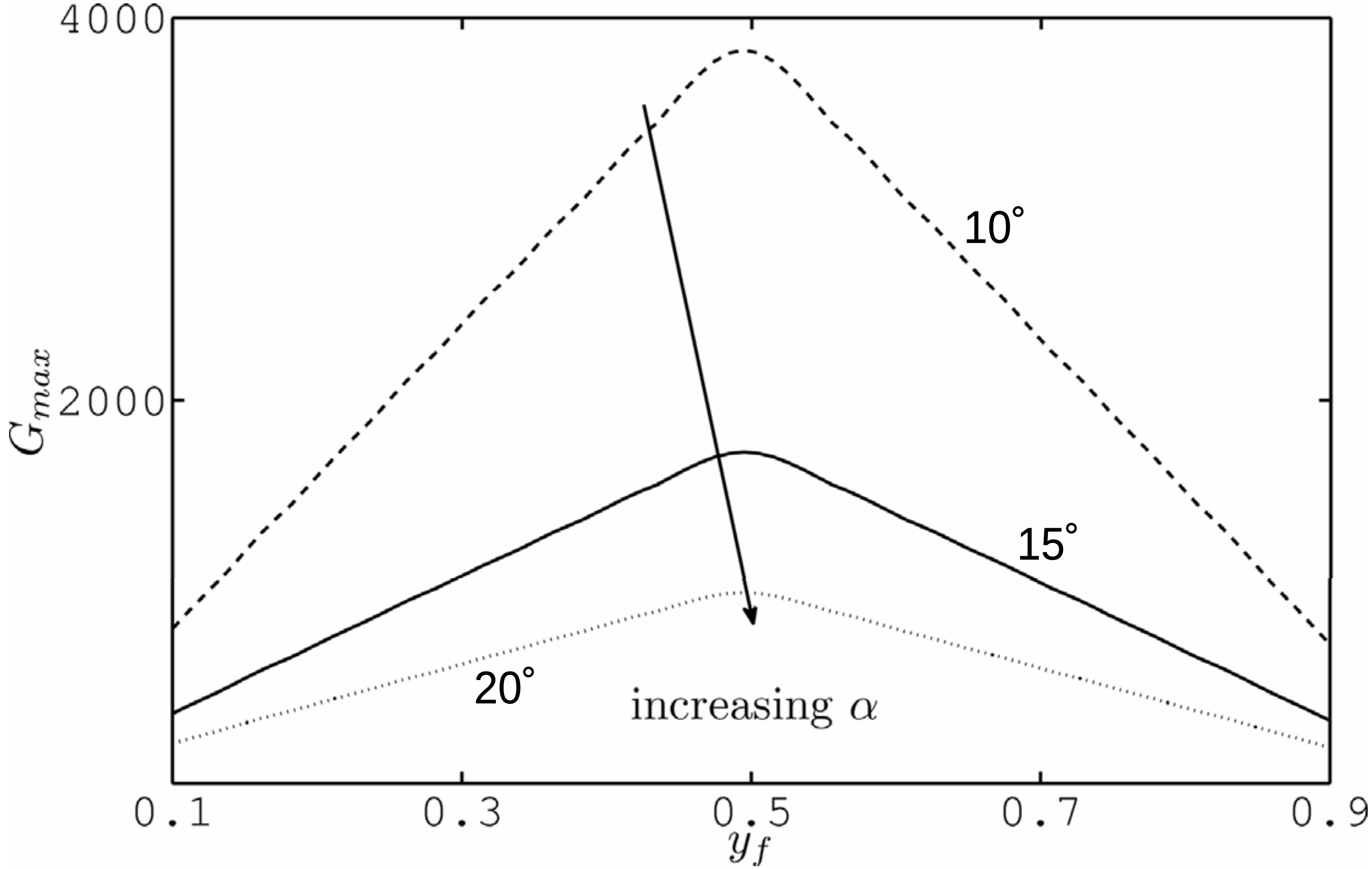
Flame front variables

$$\chi = \left(\eta_1 \quad \dot{\eta}_1/\pi \quad . \quad . \quad . \quad . \quad \eta_N \quad \dot{\eta}_N/N\pi \quad S_1 \quad . \quad . \quad . \quad . \quad S_P\right)^T_{1\times(2N+P)}$$

Acoustic variables

$$L = \begin{pmatrix} C_{2N\times 2N} & D_{2N\times P} \\ E_{P\times 2N} & F_{P\times P} \end{pmatrix}_{(2N+P)\times(2N+P)}$$

Growth factor maximized over all initial conditions and all time is called G_{max}

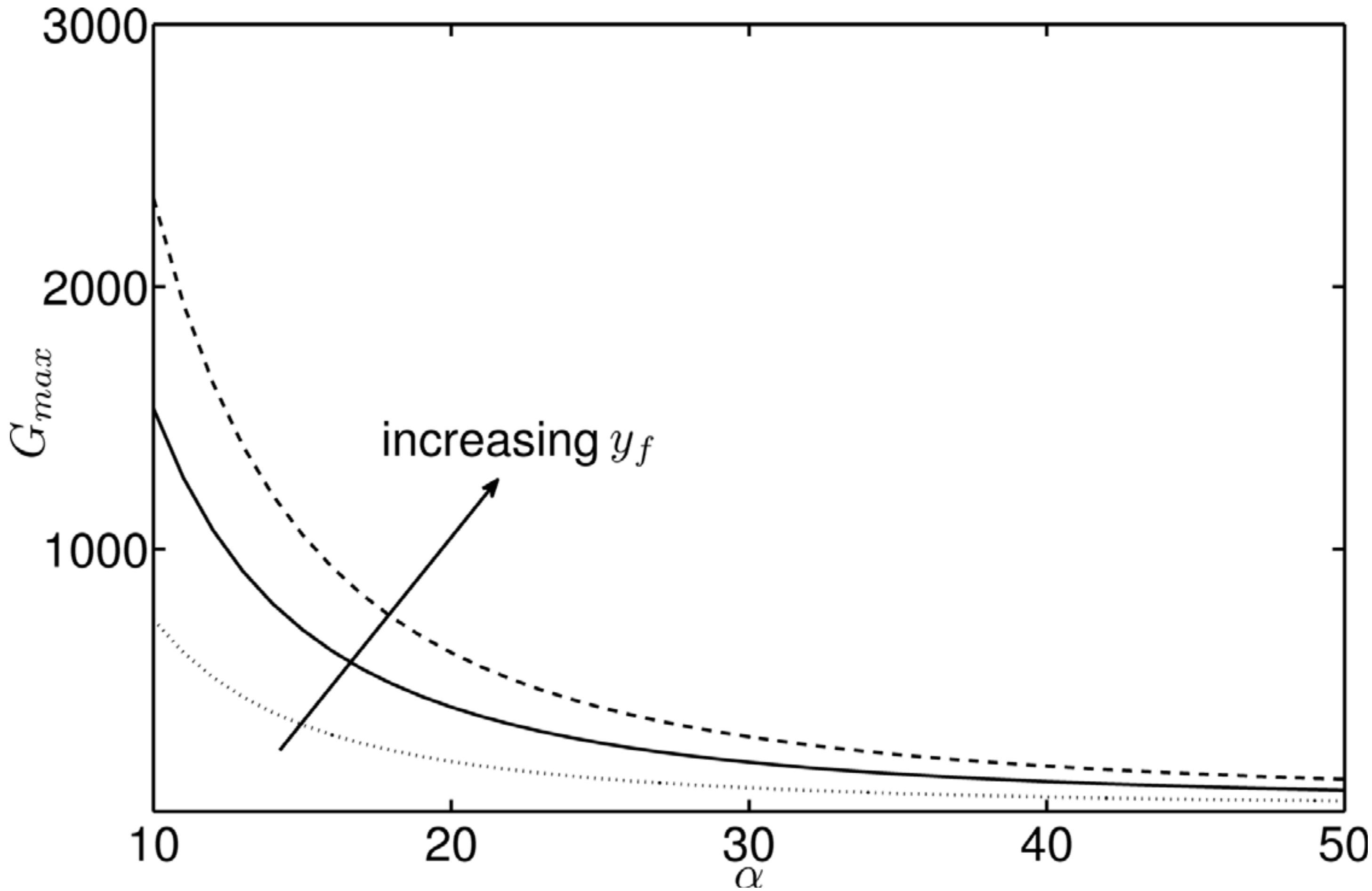


G_{max} increases with flame location till the half duct length

$\phi = 0.6,\ S_L = 0.1231\ m/s,$

$\Delta q_r = 1688500$ J/Kg, $c_1 = 6\times10^{-3},\ c_2 = 6\times10^{-4}$

Transient growth is more for sharper flames



$\phi = 0.6,\ S_L = 0.1231\ m/s,$

$\Delta q_r = 1688500$ J/Kg, $c_1 = 6\times10^{-3},\ c_2 = 6\times10^{-4}$

Thermoacoustic system has more degrees of freedom than the number of acoustic modes

Flame front variables

$$\chi = \left(\eta_1 \quad \dot{\eta}_1 / \pi \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \eta_N \quad \dot{\eta}_N / N\pi \quad S_1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad S_P \right)^T_{1\times(2N+P)}$$

Acoustic variables

$$L = \begin{pmatrix} C_{2N\times 2N} & D_{2N\times P} \\ E_{P\times 2N} & F_{P\times P} \end{pmatrix}_{(2N+P)\times(2N+P)}$$

interaction between acoustic modes

Internal degrees of freedom of the flame front also contribute to the dynamics

Flame front variables

$$\chi = \left(\eta_1 \quad \dot{\eta}_1/\pi \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \eta_N \quad \dot{\eta}_N/N\pi \quad S_1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad S_P \right)^T_{1\times(2N+P)}$$

Acoustic variables

$$L = \begin{pmatrix} C_{2N\times 2N} & D_{2N\times P} \\ E_{P\times 2N} & F_{P\times P} \end{pmatrix}_{(2N+P)\times(2N+P)}$$

interaction between flame elements

Internal degrees of freedom of the flame front also contribute to the dynamics

Flame front variables

$$\chi = \left(\eta_1 \quad \dot{\eta}_1/\pi \quad . \quad . \quad . \quad . \quad \eta_N \quad \dot{\eta}_N/N\pi \quad S_1 \quad . \quad . \quad . \quad . \quad S_P \right)^T_{1\times(2N+P)}$$

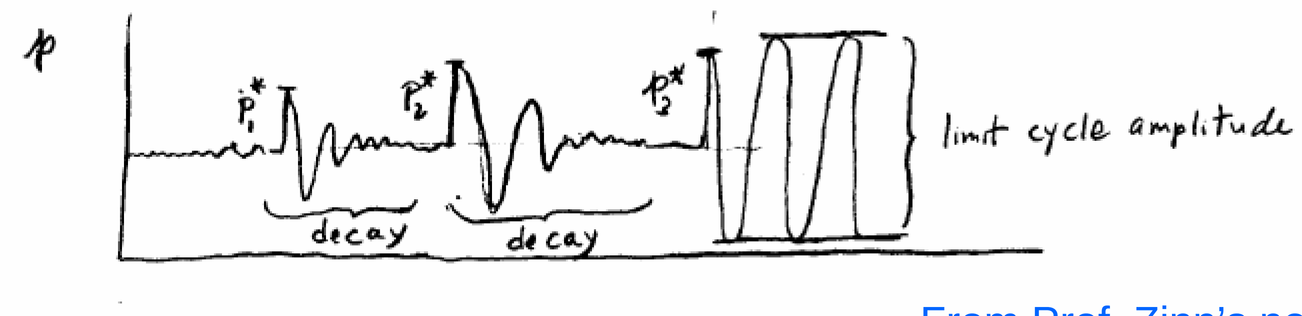
Acoustic variables

$$L = \begin{pmatrix} C_{2N\times 2N} & D_{2N\times P} \\ E_{P\times 2N} & F_{P\times P} \end{pmatrix}_{(2N+P)\times(2N+P)}$$

coupling between acoustics and flame front

Let us redo the analysis of an SRM

A system is nonlinearly unstable if some finite amplitude disturbance grows with time



From Prof. Zinn's notes

For trigerring instability, the initial amplitude should be greater than a "threshold amplitude"

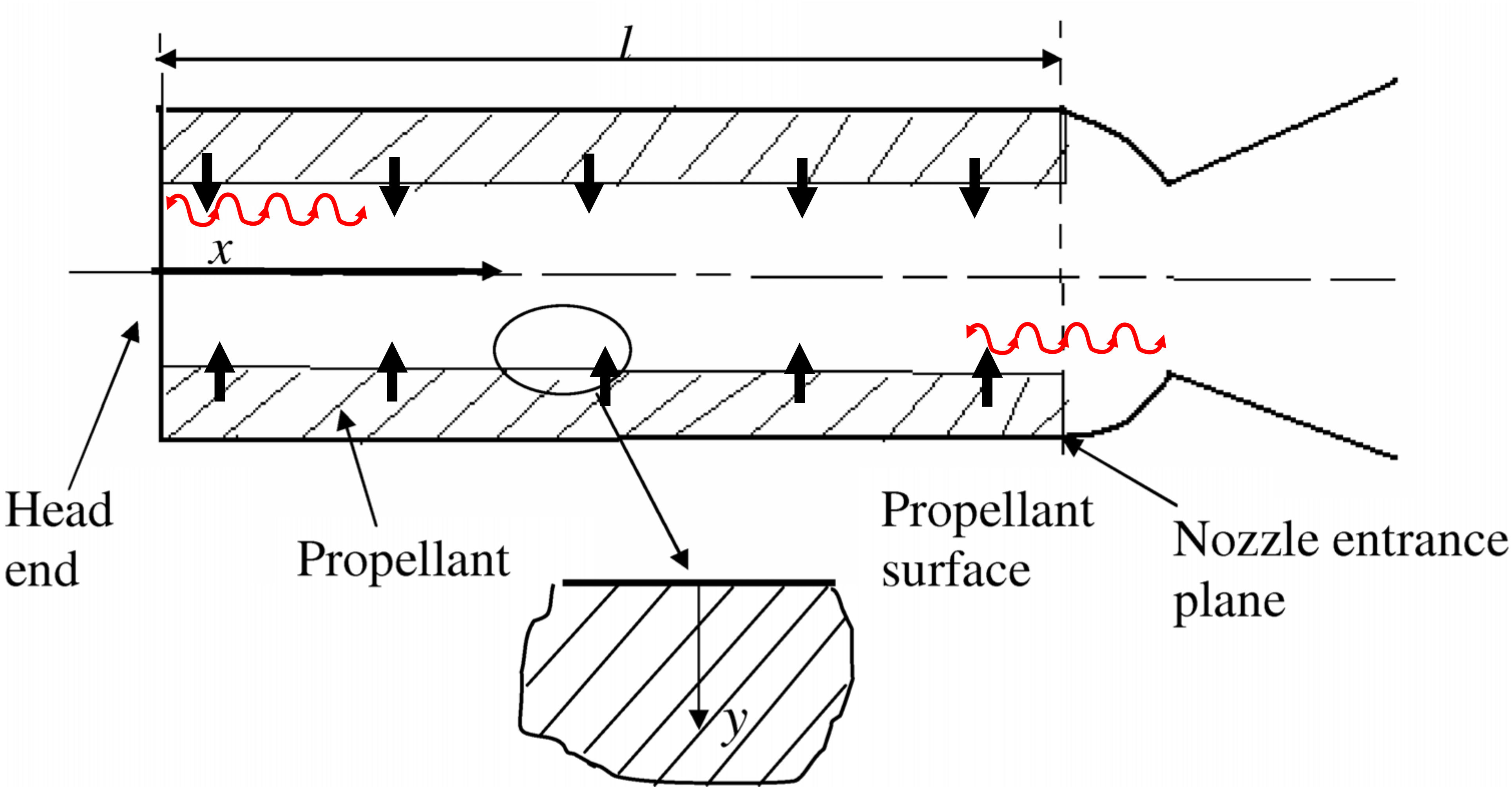
Include non-orthogonality of eigenmodes which plays an important role in the short term dynamics

Use a physics based model for the burn rate response

Include all the nonlinear processes involved

Can we obtain pulsed instability from a small amplitude initial pulse (compared to limit cycle amplitude)?

We consider an SRM here with a prismatic circular combustion chamber



Non-dimensionalised acoustic equations with 2nd order nonlinearity are given by

Momentum

$$\frac{\partial u}{\partial t}+\frac{1}{\gamma M}\frac{\partial p}{\partial x}+M\left[\bar{U}\frac{\partial u}{\partial x}+\frac{dU}{dx}u\right]=k_m[R\bar{U}+u]+\left\{k_m Ru-Mu\frac{\partial u}{\partial x}+\frac{p}{\gamma M}\frac{\partial p}{\partial x}\right\}$$

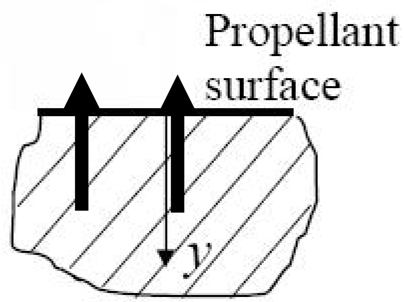
Unsteady Burn rate

Energy

$$\frac{\partial p}{\partial t}+(\xi+\alpha_{NO})\,p+\gamma M\frac{\partial u}{\partial x}+\left[\bar{U}\frac{\partial p}{\partial x}+\gamma\frac{d\bar{U}}{dx}p\right]=k_e R-\left\{Mu\frac{\partial p}{\partial x}+\gamma Mp\frac{\partial u}{\partial x}\right\}$$

$$k_m=-\frac{\bar{m}u_m l}{\bar{p}\gamma M},\; k_e=\frac{\bar{R}\rho_p S_l l}{\bar{\rho}S_c 2},\; u_m=\frac{\bar{R}\rho_p S_l l}{\bar{\rho}S_c 2},\; \bar{m}=\frac{\bar{R}\rho_p S_l}{S_c},\; \bar{U}=\frac{\bar{u}}{u_m}=2x$$

Unsteady burn rate drives the acoustic oscillations



$$\frac{\partial T}{\partial \tau} - (1+R)\frac{\partial T}{\partial y} - \frac{\partial^2 T}{\partial y^2} = 0 \qquad 0 \le y \le \infty, 0 \le \tau \le \infty$$

burn rate

Krier. *et al* 1968

$$R = T_S^m - 1, \qquad T_S(\tau) = T(y=0,\tau)$$

BC

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = -\frac{(1+p)^{2n}\left((1+p)^{n/m} - H\right)}{1+R} - H(1+R)$$

acoustic forcing

$$T(y \to \infty, \tau) = 0 \qquad \frac{\tau}{t} = \frac{lR_0^2}{a\alpha} = F$$

timescale ratio

The acoustic equations are solved by Galerkin technique

$$u = \sum_{m=1}^{N} U_m \sin(\omega_m x),\ p = \gamma M \sum_{m=1}^{N} P_m \cos(\omega_m x),\ R = \sum_{m=1}^{N} \left[R_m^c \cos(\omega_m x) + R_m^s \sin(\omega_m x) \right]$$

Momentum

$$\dot{U}_n + 2\sum_{m=1}^{N} \left(U_m I_{n,m}^1 + P_m I_{n,m}^2 + R_m^c I_{n,m}^3 + R_m^s I_{n,m}^4 \right) = 2\left\{ k_m N_n^1 - M N_n^2 - \gamma N_n^3 \right\}$$

Energy

$$\dot{P}_n - (\xi_n + \alpha_{NO}) P_n + \frac{2}{\gamma M}\left(\sum_{m=1}^{N} \left[U_m I_{n,m}^5 + P_m I_{n,m}^6 + R_m^c I_{n,m}^7 \right] \right) = \frac{2}{\gamma M}\left\{ \gamma M^2 N_n^4 - (\gamma M)^2 N_n^5 \right\}$$

$\{\ \}$ Non linear terms

Linearized governing equations are used investigate the non-normality behavior

$$\frac{d\chi}{dt} = L\chi$$

$$\chi = \left(\Omega \quad \Psi_1^c \, \Psi_2^c \quad . \quad . \quad \Psi_N^c \quad \Psi_1^s \, \Psi_2^s \, . \quad . \quad . \quad \Psi_N^s\right)^T_{1\times 2NM_g}$$

$$\Omega = \left(U_1 \quad P_1 \quad . \quad . \quad . \quad U_N \quad P_N\right)^T_{1\times 2N}$$

Acoustic variables

$$\Psi_n^c = \left(\beta_1 T_{n(1)}^c \quad \beta_2 T_{n(2)}^c \quad . \quad . \quad . \quad . \quad \beta_{M_g-1} T_{n(M_g-1)}^c\right)^T_{1\times(M_g-1)}$$

$$\Psi_n^s = \left(\beta_1 T_{n(1)}^s \quad \beta_2 T_{n(2)}^s \quad . \quad . \quad . \quad . \quad \beta_{M_g-1} T_{n(M_g-1)}^s\right)^T_{1\times(M_g-1)}$$

Unsteady burn rate variables

$$L = \begin{pmatrix} A_{2N\times 2N} & B_{2N\times(2N(M_g-1))} \\ C_{(2N(M_g-1))\times 2N} & D_{2N(M_g-1)\times 2N(M_g-1)} \end{pmatrix}_{(2NM_g)\times(2NM_g)}$$

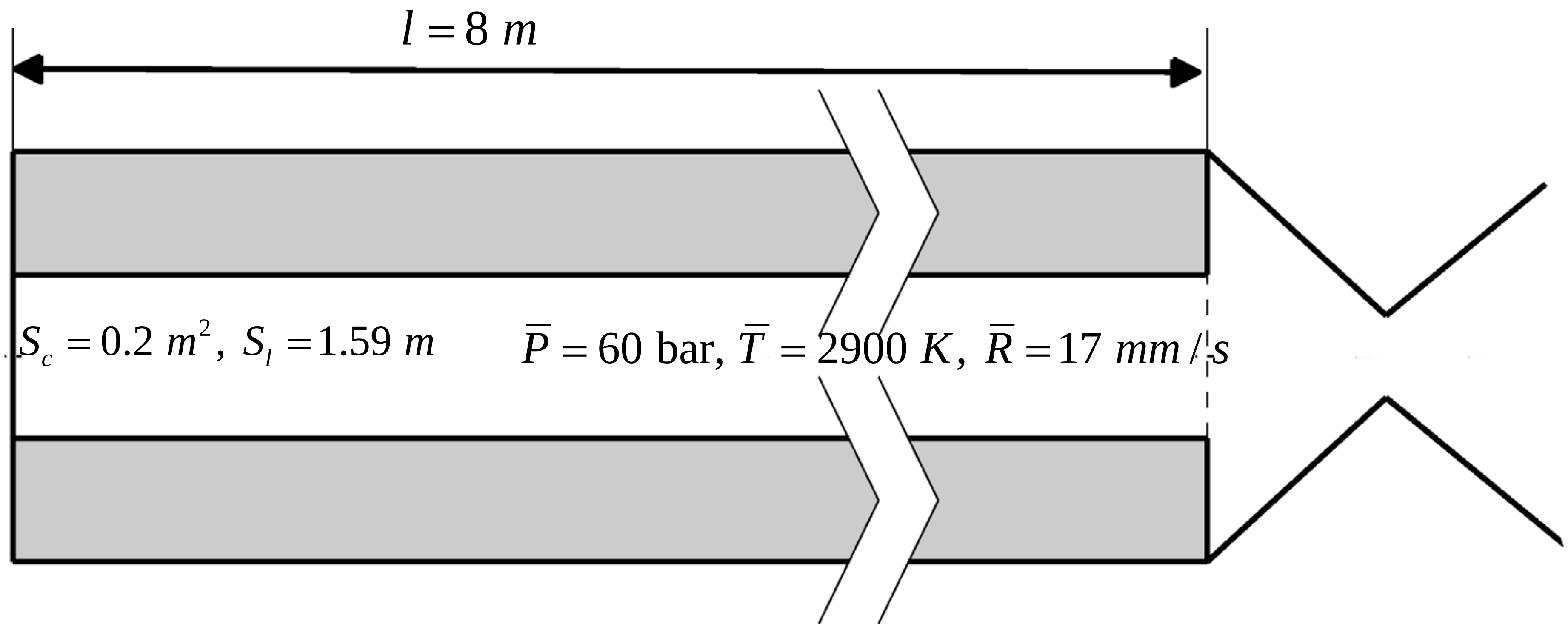
acoustics

burn rate to acoustic feedback

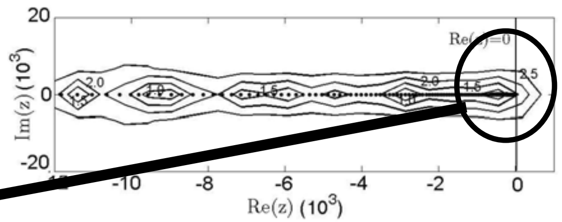
acoustic to burn rate feedback

burn rate

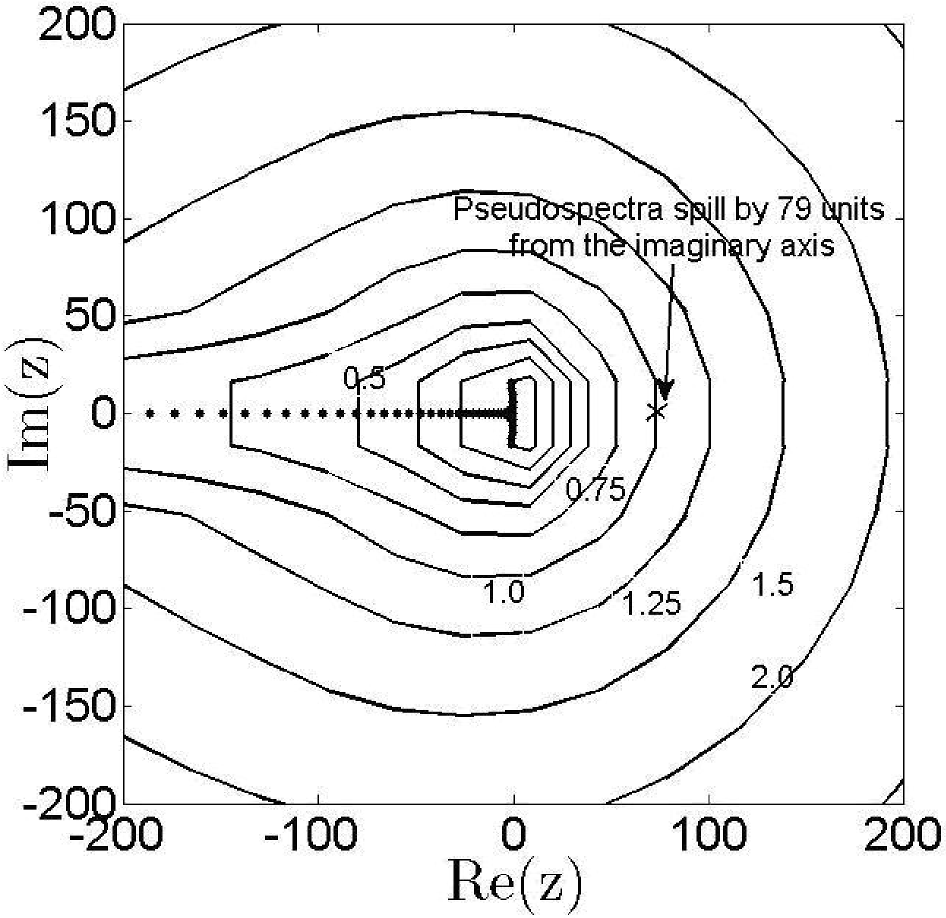
Numerical simulations have been performed for the following value of the parameters.



Non-normality can be studied by pseudospectra



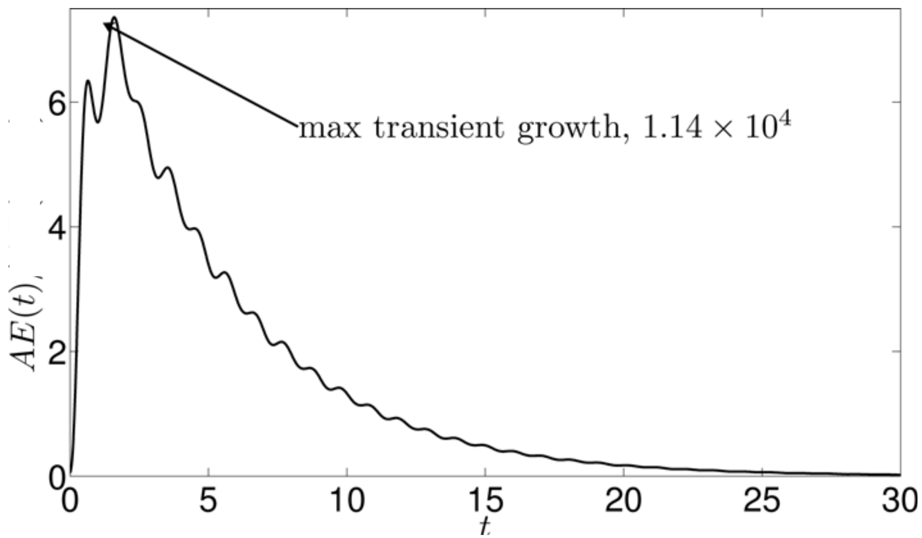
Pseudospectra



$$c_1 = 0.03, c_2 = 0.02,$$

$$N = 5, M_g = 150, \|L\| = 1.88 \times 10^4,$$

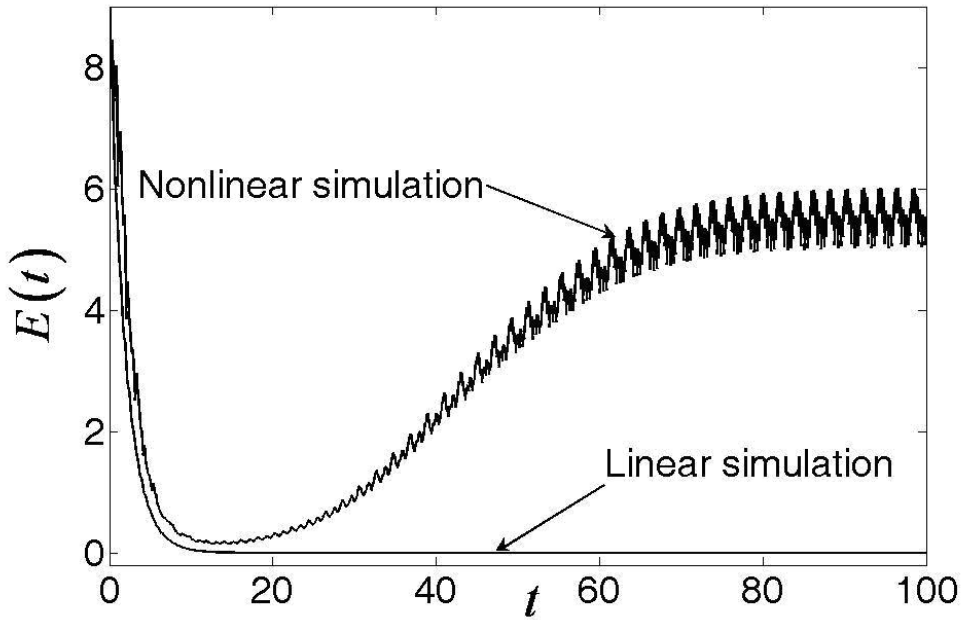
$$\varepsilon_{\max} = 1 \times 10^{2.5}, \frac{\varepsilon_{\max}}{\|L\|_2} << 1$$



Max transient growth

Non-normality plays an important role in pulsed instability

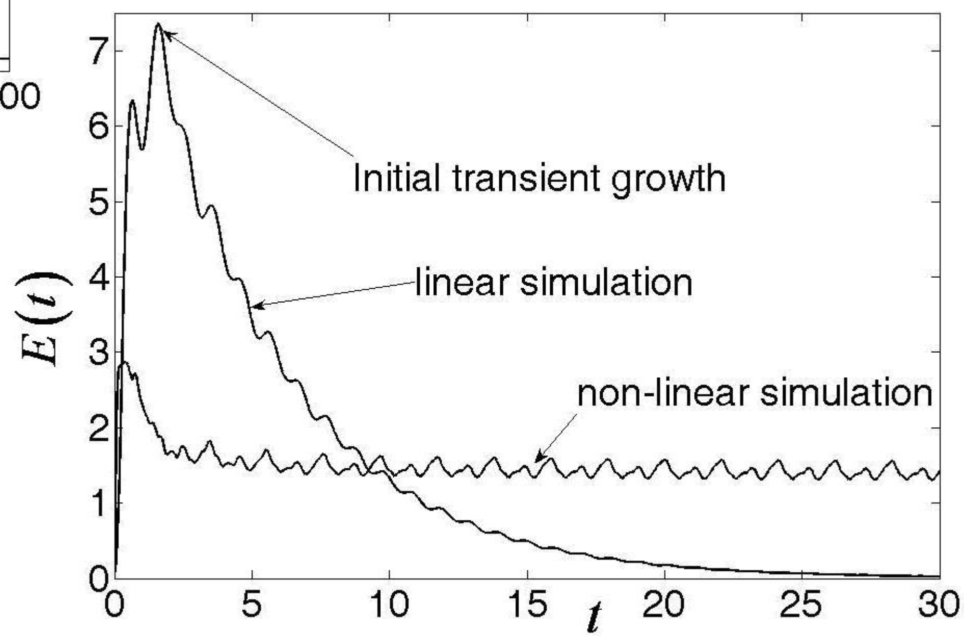
$$U_2(0) = 3,\ P_2(0) = 3,\ P_{m\neq 2}(0) = U_{m\neq 2}(0) = 0,$$
$$MT_p(\eta = 1, 0) = 0.03,\ MT_p(\eta \neq 1, 0) = 0,$$
$$C_1 = 0.02,\ C_2 = 0.02,\ E(t=0) = 8.43$$



Triggering with large initial condition

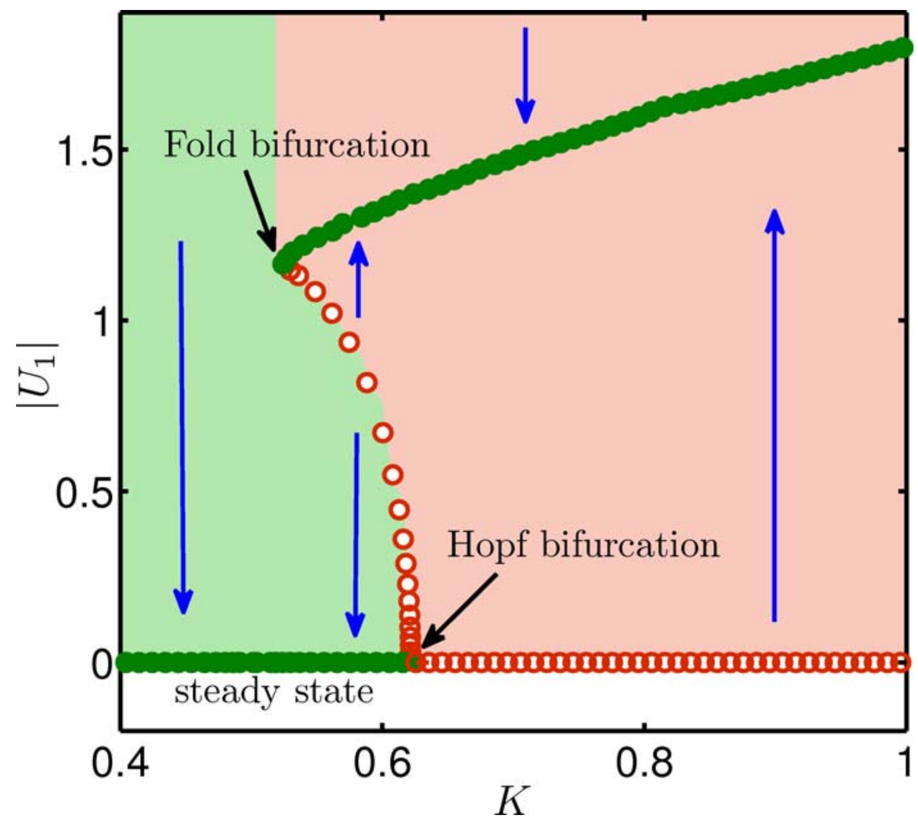
Triggering with transient growth

$$\chi(0) = V_{opt},\ C_1 = 0.03,\ C_2 = 0.02,$$
$$E(t=0) = 6.4\times 10^{-4}$$



How can we get the bifurcation diagram?

Continuation methods can capture bifurcations

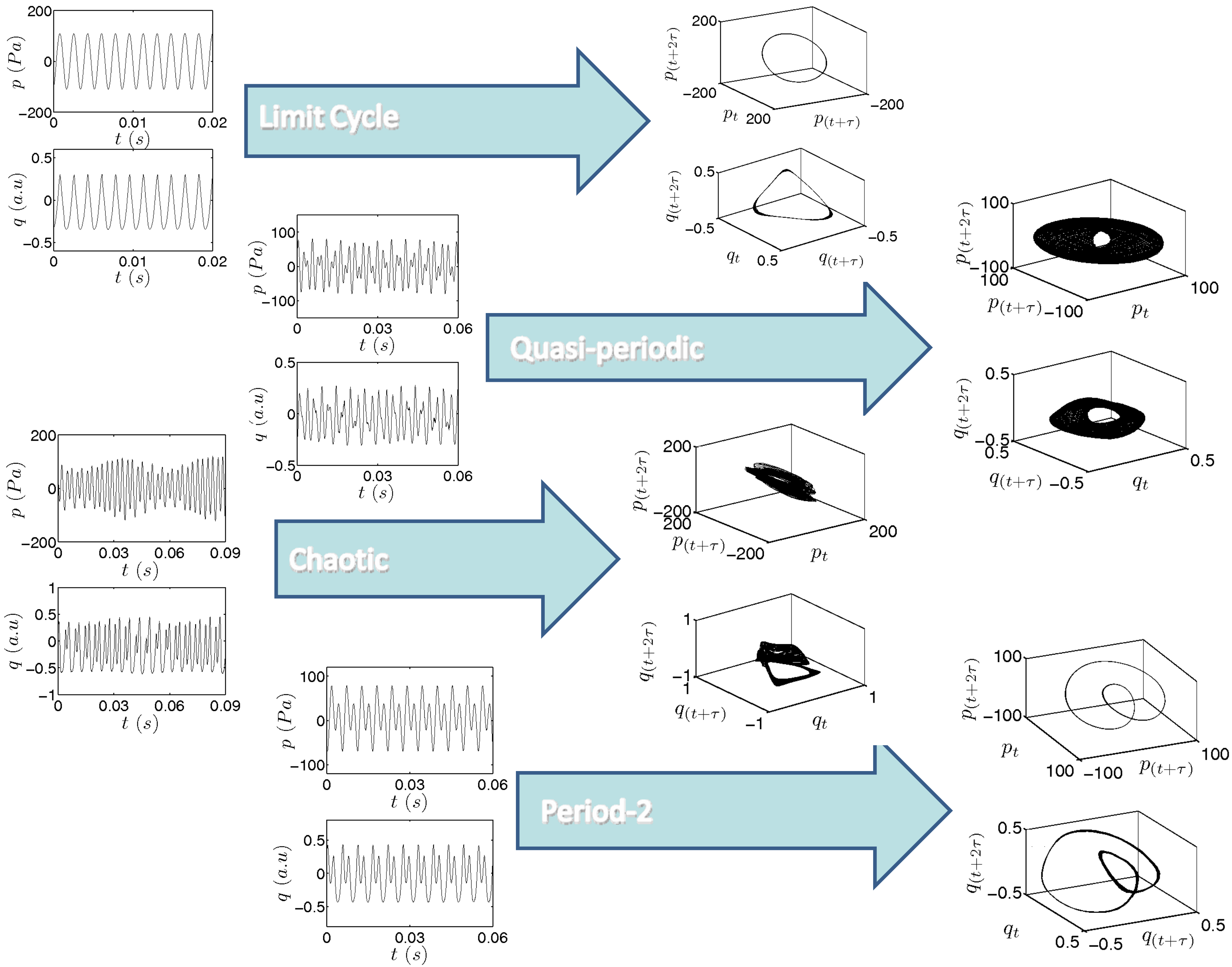


What are the asymptotic states of a thermoacoustic system?

Fixed point

Limit Cycle

What else?......



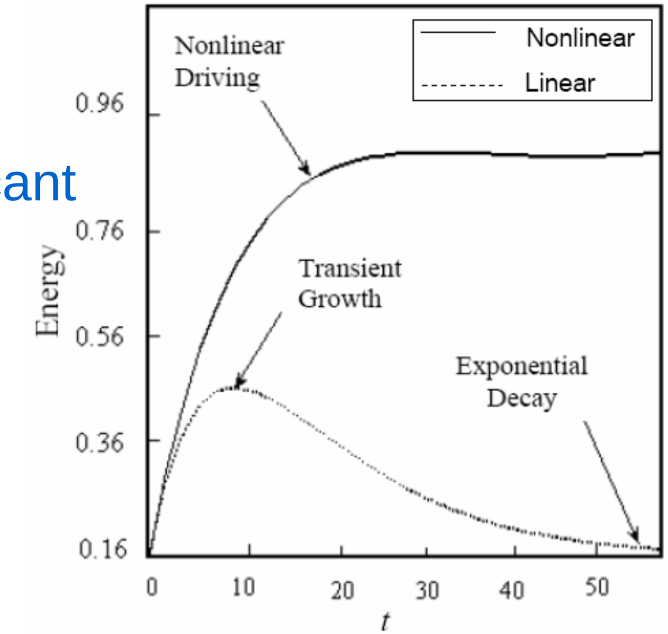
Limit Cycle
Quasi-periodic
Chaotic
Period-2

Industrial/aero combustors have turbulent flow

A lot more work & excitement awaits us!

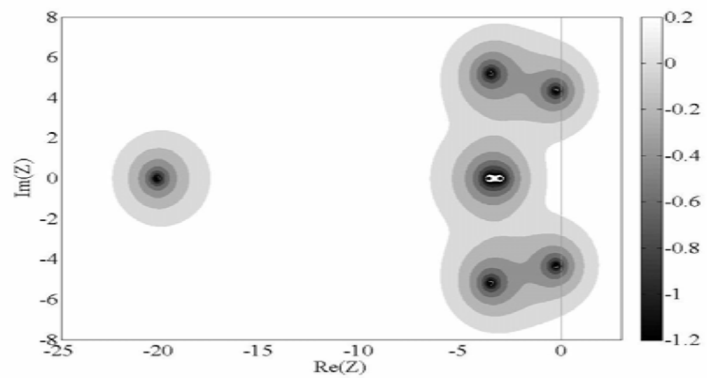
In summary, thermoacoustic interaction is non-normal and nonlinear

Transient growth can lead to high enough amplitudes where nonlinearities become significant



Non-normality & nonlinearity leads to subcritical transition to instability

Individual eigenvalues: wrong tools for analyzing a non-normal system.



Need to adopt tools such as SVD and ε pseudospectra