Elementary non-linear processes in aero-acoustics of internal flows

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Outline

1. Wave distortion 1-D frictionless

    (shock waves)

2. Driven resonance in closed pipes

    (sub-harmonic resonance, turbulence, thermal effects)

3. Driven resonance in open pipes

    (vortex shedding)

4. Self-sustained oscillation

    (Rijke tube)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Mass conservation

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right) = -\nabla p + \nabla \cdot \vec{\vec{\sigma}} + \vec{f}$$

Newton

Local thermodynamic equilibrium

$$p = p(\rho, s) \Rightarrow \quad p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$$

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Frictionless, Isentropic and One-dimensional

1-D:

$$\vec{v} = \left(u(x,t),0,0\right); \quad p = p(x,t);$$

$$\rho = \rho(x,t); \; c = c(x,t)$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

Equations of motion

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x}$$

Isentropic flow

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = c^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right)$$

Characteristic form: step 1

$$\frac{c^2}{\rho c}[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}] + c\frac{\partial u}{\partial x} = 0$$

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{c}{\rho c}\frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = c^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right)$$

Eliminate density

Characteristic form

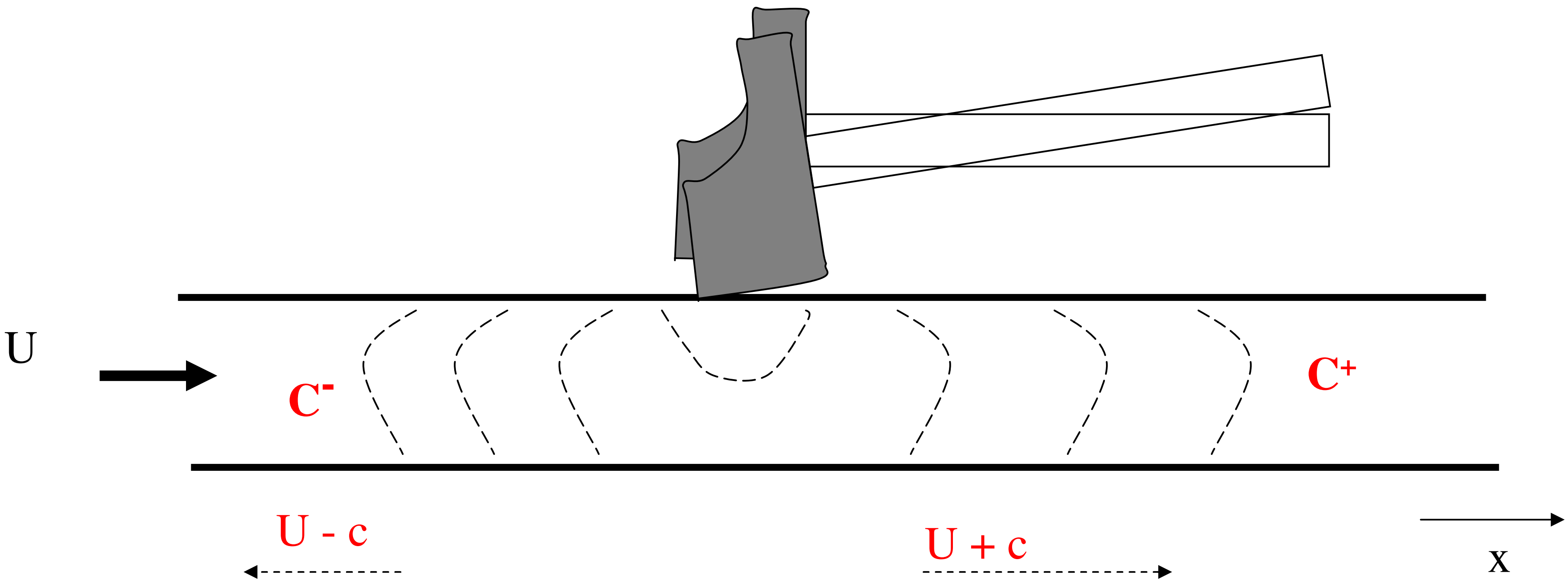
$$\frac{1}{\rho c}\left[\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right]+c\frac{\partial u}{\partial x}=0$$

Adding or subtracting

$$\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{c}{\rho c}\frac{\partial p}{\partial x}$$

$$\left(\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial x}\right)\left[u\pm\int\frac{dp}{\rho c}\right]=0$$

Sound waves



Characteristics

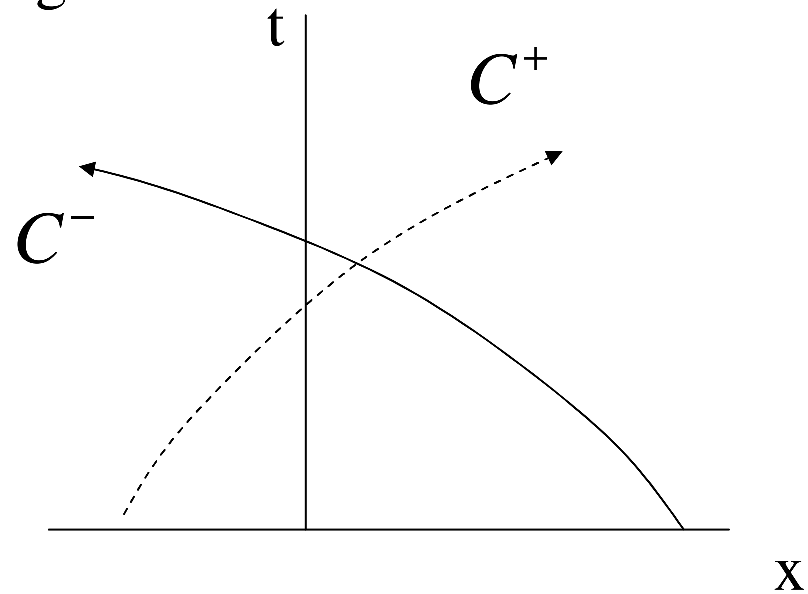
$$C^+ : \quad \frac{dx}{dt} = u + c$$

$$\Theta^+ = u + \int \frac{dp}{\rho c}$$

Along the lines in the (x,t) diagram

$$C^- : \quad \frac{dx}{dt} = u - c$$

$$\Theta^- = u - \int \frac{dp}{\rho c}$$

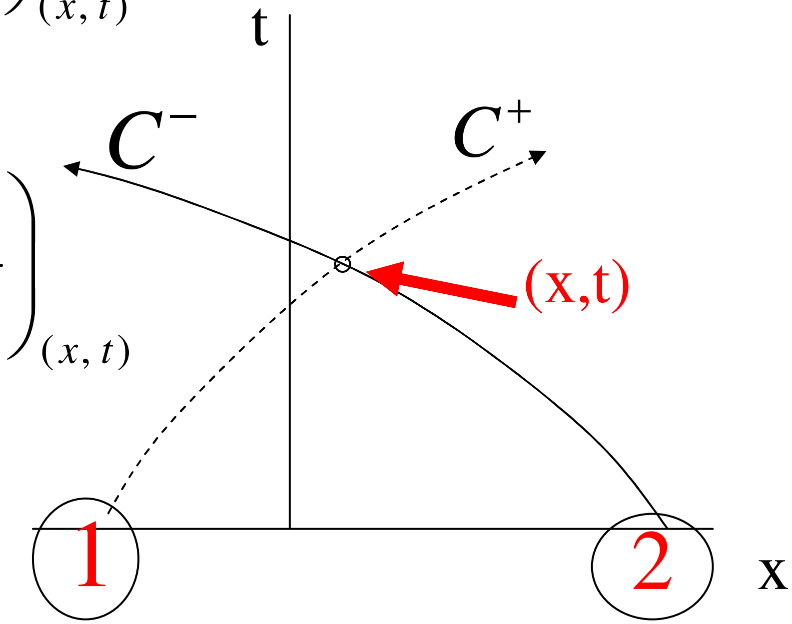


We follow either a C^+ line or a C^- line.

Initial value problem

$$\Theta^{+} = \left(u + \int \frac{dp}{\rho c}\right)_{(x_1,0)} = \left(u + \int \frac{dp}{\rho c}\right)_{(x,t)}$$

$$\Theta^{-} = \left(u - \int \frac{dp}{\rho c}\right)_{(x_2,0)} = \left(u - \int \frac{dp}{\rho c}\right)_{(x,t)}$$



We find u(x,t) and $\int dp/\rho c$ at the intersection of the two characteristics C^+ and C^-.

Simple wave: travelling into a uniform region

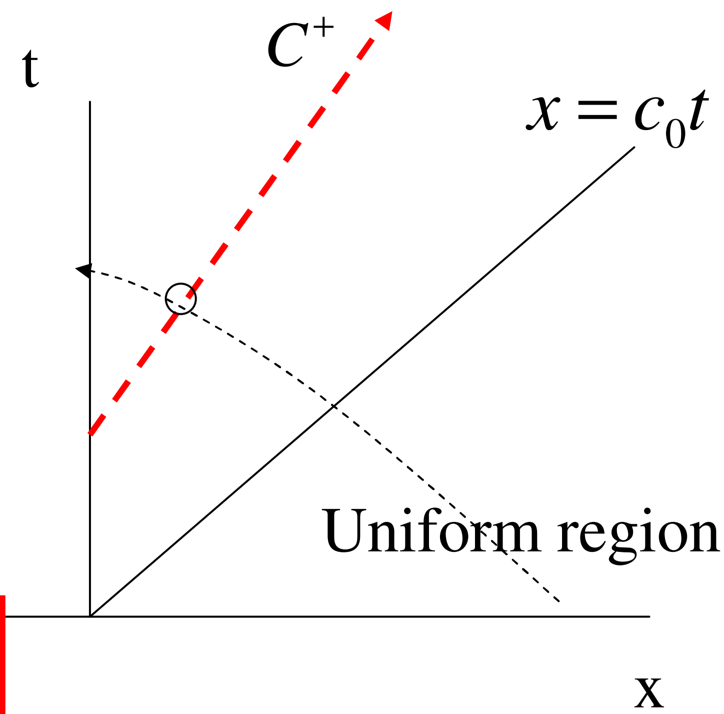
$$C^+ : u + \int \frac{dp}{\rho c} = \Theta^+$$

$$C^- : u - \int \frac{dp}{\rho c} = -\left(\int \frac{dp}{\rho c} \right)_0$$

$$u = \frac{1}{2}\left(\Theta^+ - \left(\int \frac{dp}{\rho c} \right)_0 \right)$$

$$\Rightarrow \boxed{u + c = const}$$

$$\int \frac{dp}{\rho c} = \frac{1}{2}\left(\Theta^+ + \left(\int \frac{dp}{\rho c} \right)_0 \right)$$



C^+ straight line

Fundamental derivative

$$\frac{dp}{\rho\, c} = \frac{1}{\rho\, c}\left(\frac{\partial p}{\partial c^2}\right)_s 2c\,dc = \frac{dc}{\Gamma - 1}$$

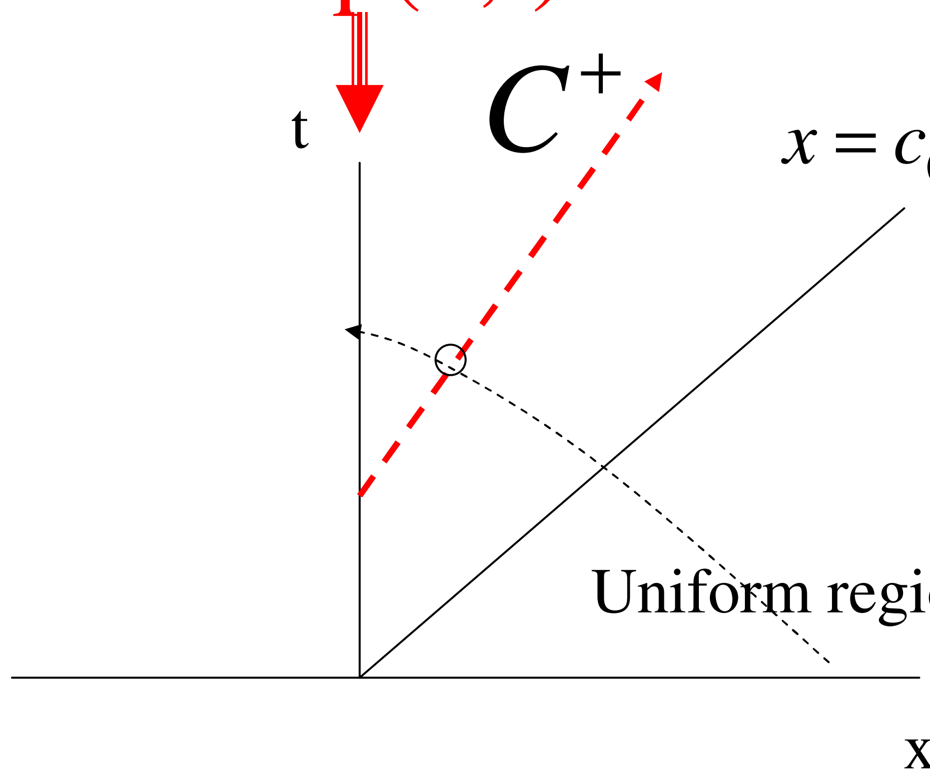
$$\Gamma = 1 + \frac{\rho}{2}\left(\frac{\partial c^2}{\partial p}\right)_s$$

Simple wave in calorically perfect gas: $\Gamma = \frac{\gamma+1}{2}$

Boundary condition p(0,t)

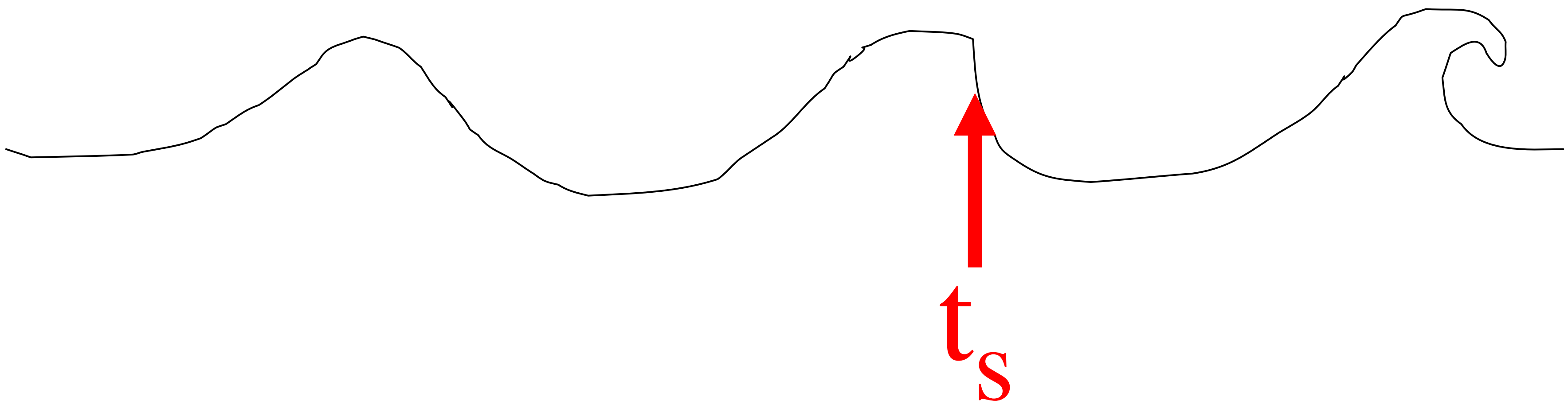
$$c(0,t) = c_0 \left(\frac{p(0,t)}{p_0} \right)^{\frac{\gamma-1}{2\gamma}} ;$$

$$u(0,t) = \frac{2c_0}{\gamma-1} \left(\frac{c(0,t)}{c_0} - 1 \right)$$



Both u and c increase with increasing pressure because $\gamma > 1$

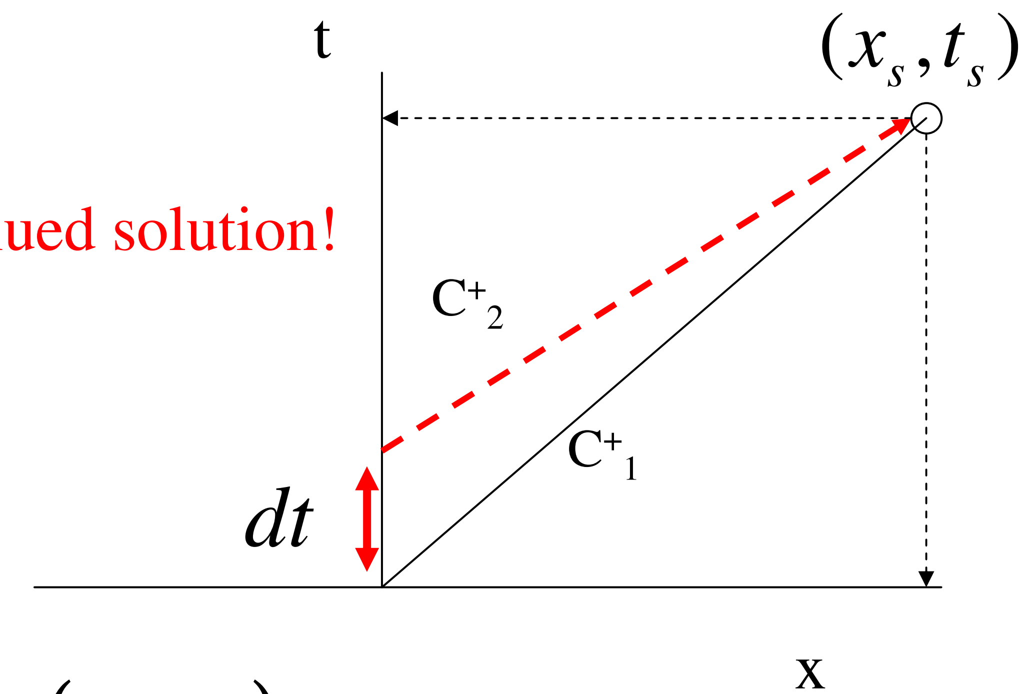
Wave propagation non-linearity



Single valued solution fails for $t>t_s$

Compression wave: Boundary condition p(0,t)

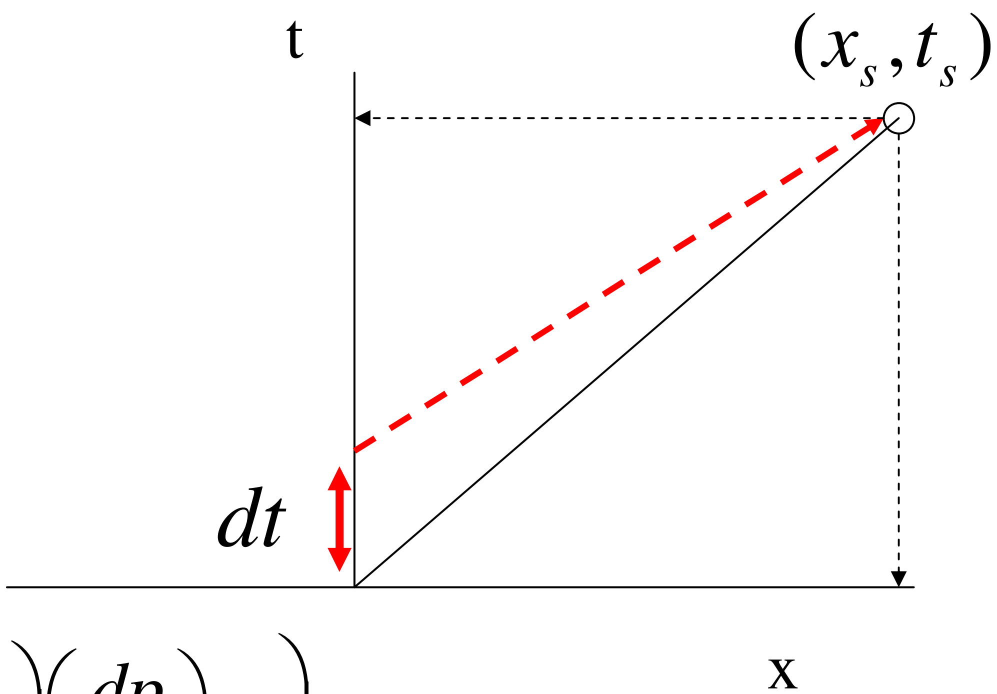
For $t > t_s$ we have a multiple valued solution!



$$\frac{x_s}{t_s} = c_0$$

$$\frac{x_s}{t_s - dt} = u(0, dt) + c(0, dt)$$

Simple wave:
Boundary condition p(0,t)

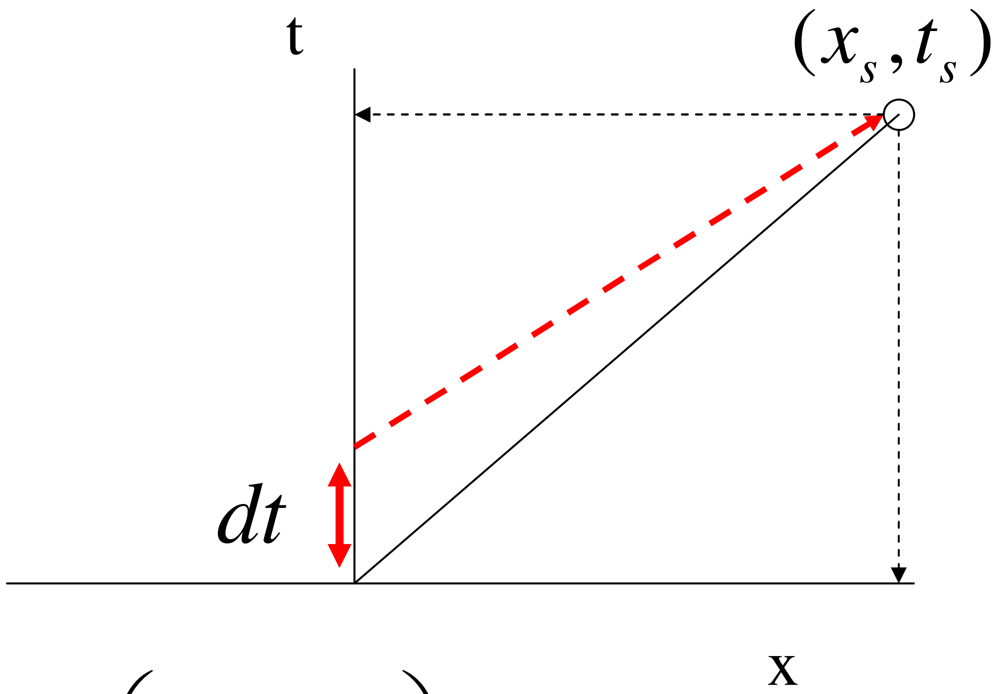


$$\frac{x_s}{t_s} = c_0$$

$$\frac{x_s}{t_s - dt} = u(0, dt) + c(0, dt)$$

$$\frac{x_s}{t_s} = (1 - \frac{dt}{t_s})\left(c_0 + \left(\frac{d(u+c)}{dp} \right)\left(\frac{dp}{dt} \right) dt \right)$$

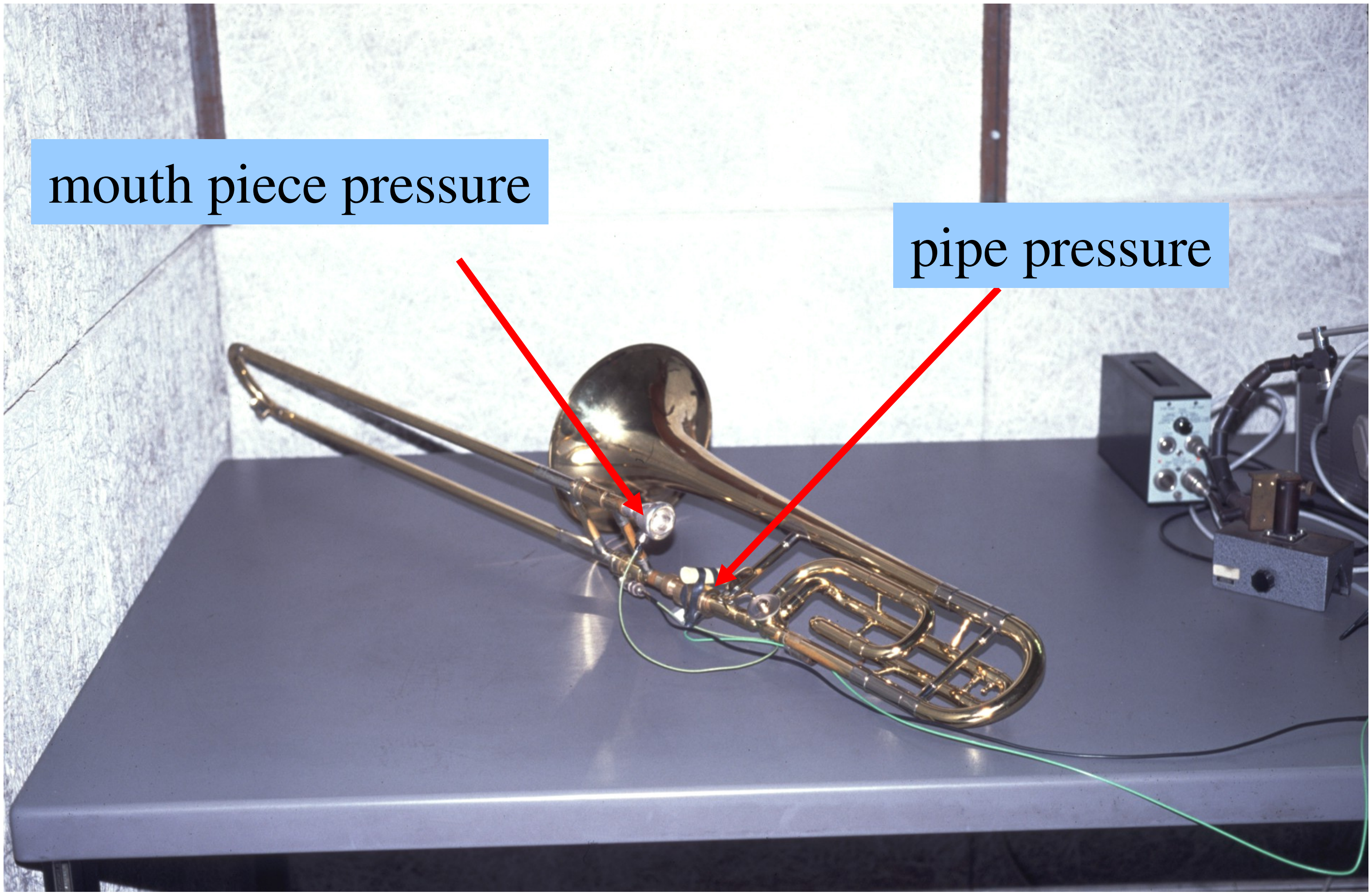
Simple wave:
Boundary condition p(0,t)



$$t_s = \frac{\rho_0 c_0^2}{\Gamma (dp/dt)_0} \approx \frac{2\gamma}{\gamma+1}\left(\frac{p}{dp/dt}\right)_0$$

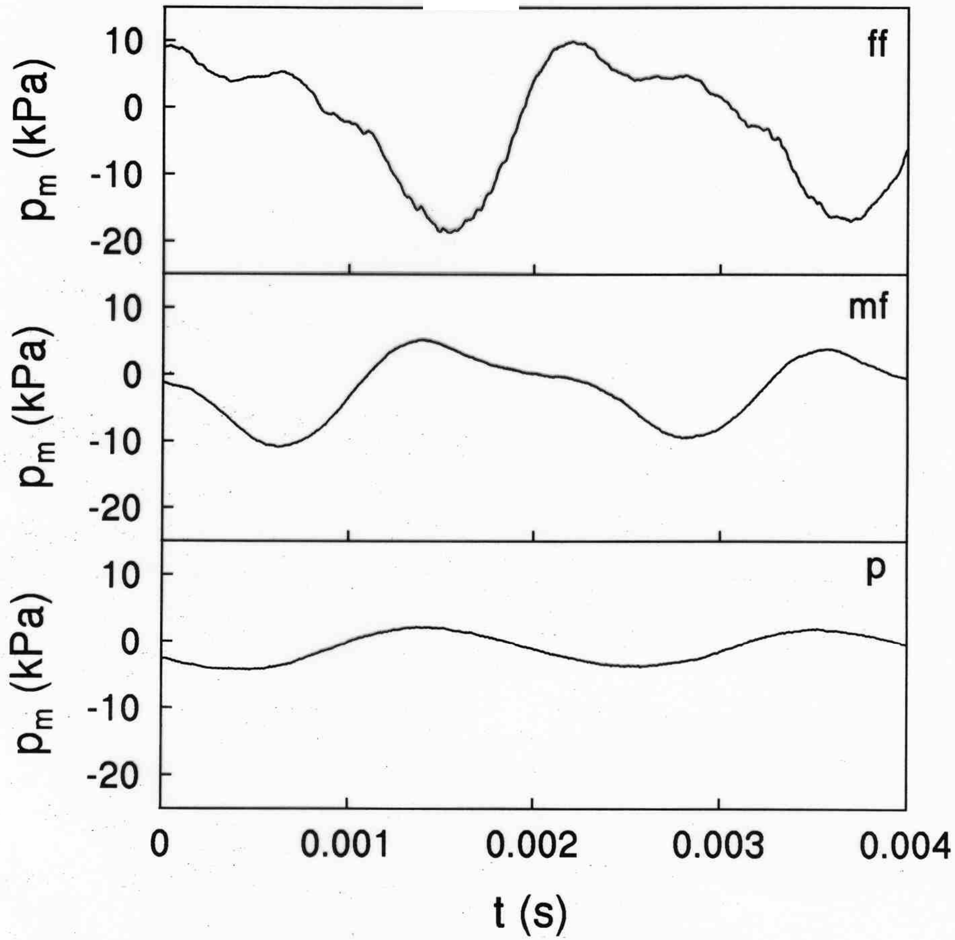
Brassy sound

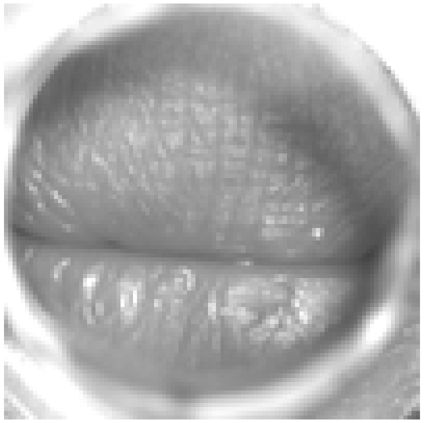
(Beauchamp, Hirschberg, Msallam)



Lips of Murray Campbell

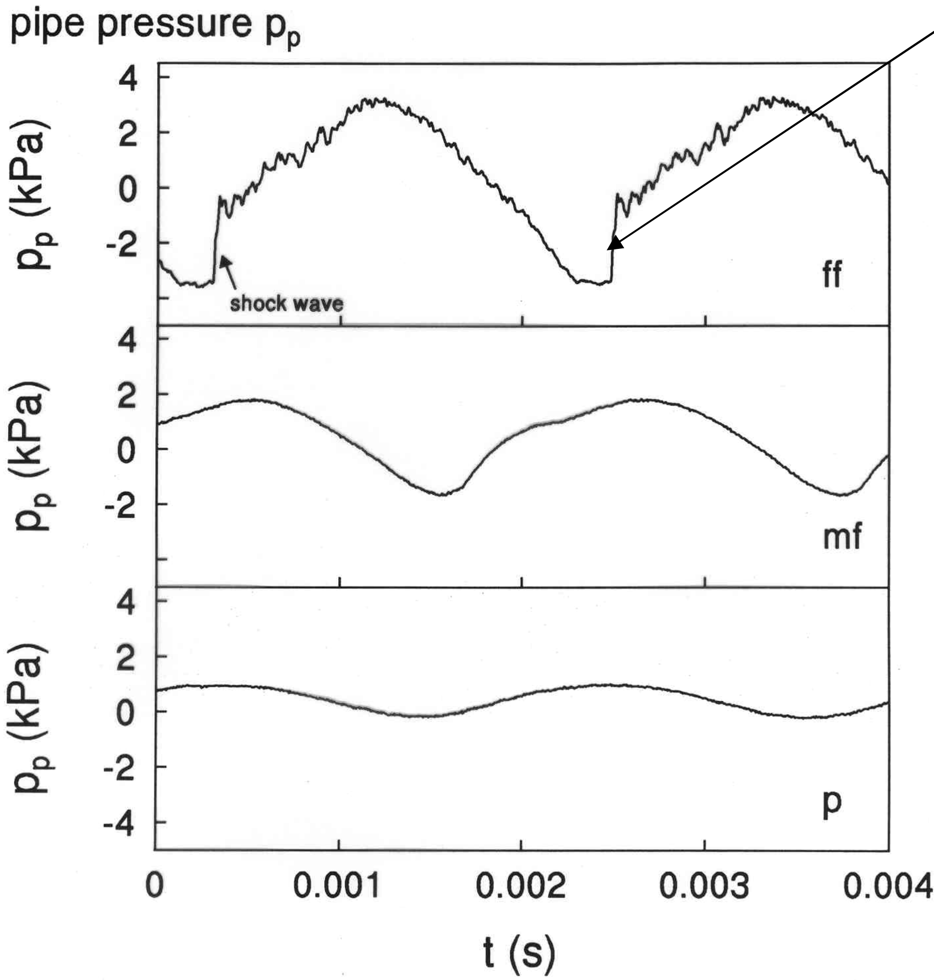
Pressure in mouth piece

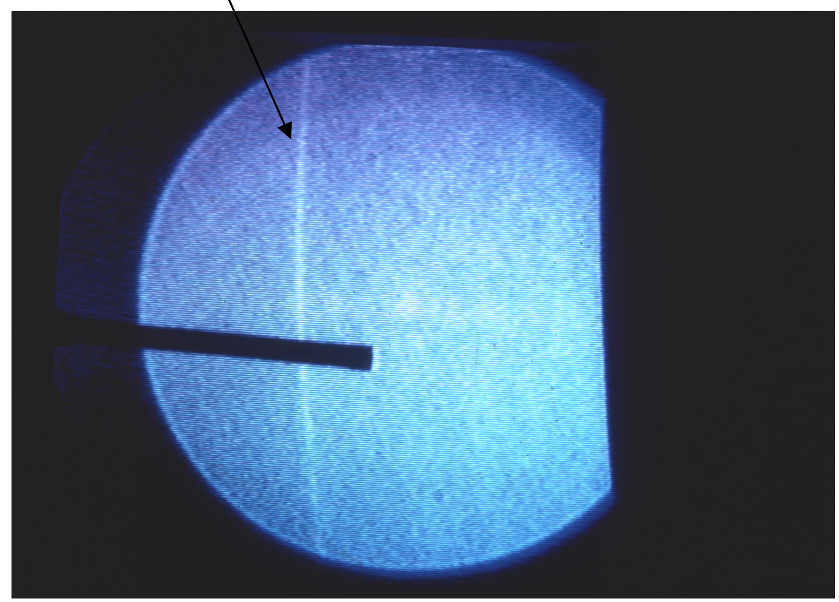




Pressure in trombone

Shock wave



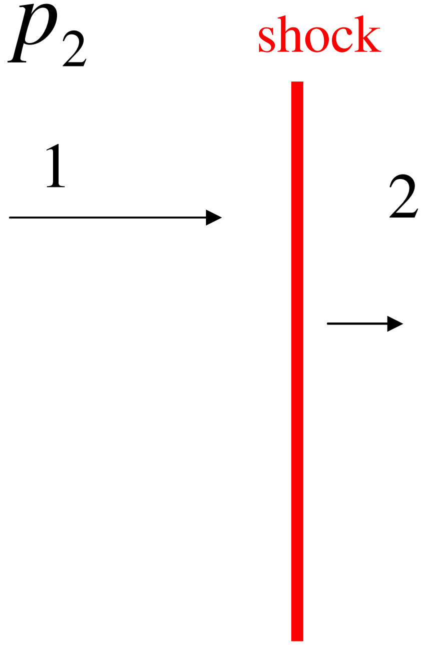


Integral conservation laws across a shock moving with the shock.

$$\rho_1(u_1 - u_s) = \rho_2(u_2 - u_s) = \phi_m$$

$$\rho_1(u_1 - u_s)^2 + p_1 = \rho_2(u_2 - u_s)^2 + p_2$$

$$h_1 + \frac{(u_1 - u_s)^2}{2} = h_2 + \frac{(u_2 - u_s)^2}{2}$$

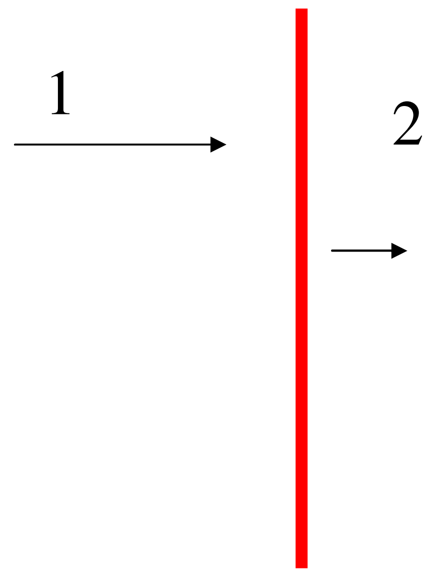


Rankine Hugoniot (RH)

- Eliminate the velocity

$$\phi_m{}^2 = -\frac{p_2 - p_1}{\dfrac{1}{\rho_2} - \dfrac{1}{\rho_1}} = -2\frac{h_2 - h_1}{\dfrac{1}{\rho_2{}^2} - \dfrac{1}{\rho_1{}^2}}$$

$$\Rightarrow \boxed{(p_2 - p_1)(\frac{1}{\rho_2} + \frac{1}{\rho_1}) = h_2 - h_1}$$

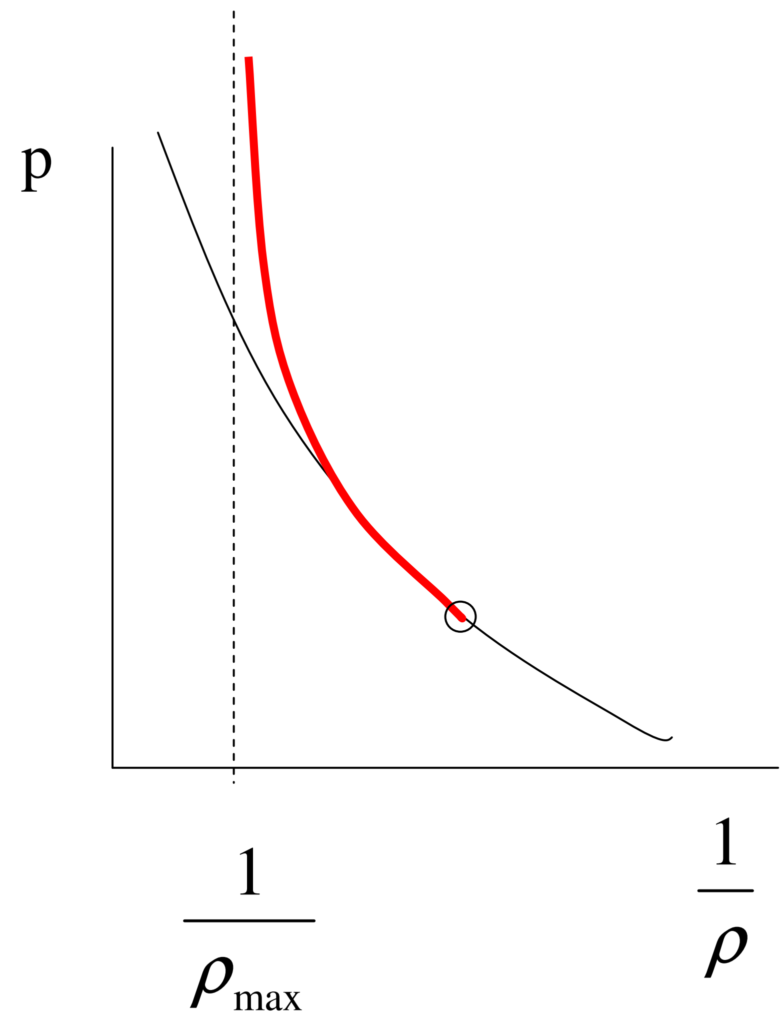


Comparison of RH with isentrope

$$(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) = h_2 - h_1$$

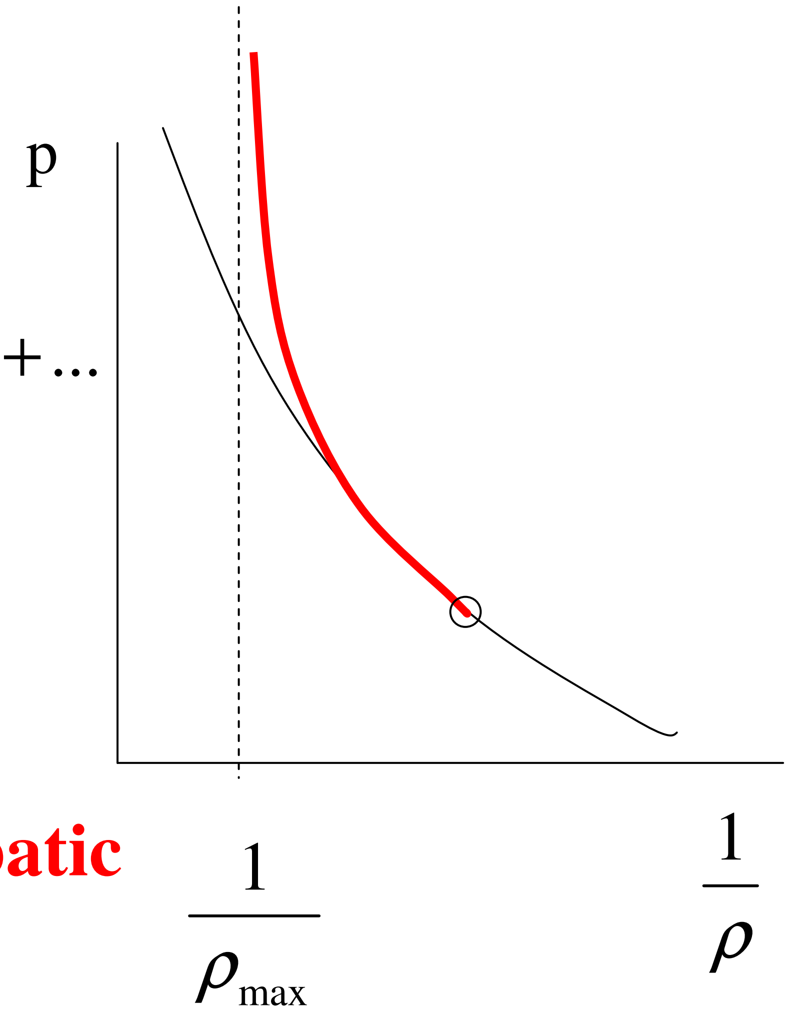
$$h = c_p T = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \exp\left(\frac{s - s_0}{c_v}\right)$$



Weak shock

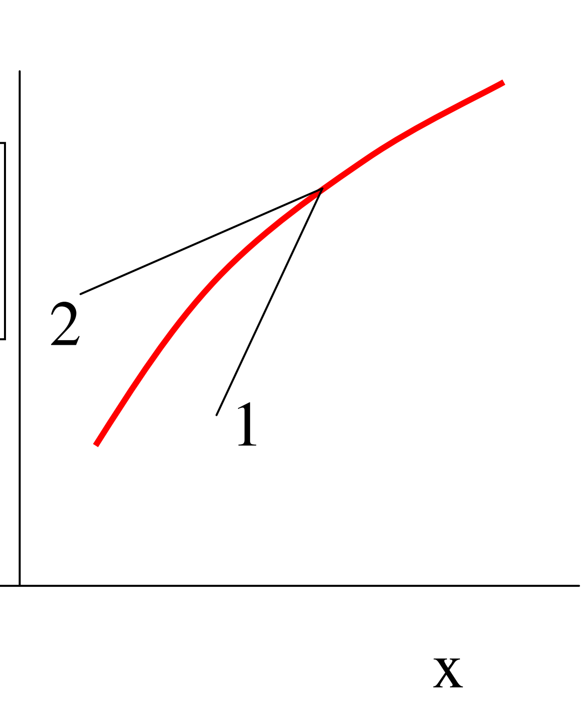
$$\frac{s_2 - s_1}{c_v} = \frac{1}{12 c_v T_1} \left(\frac{\partial^2 (1/\rho)}{\partial p^2} \right)_s (p_2 - p_1)^3 + \ldots$$



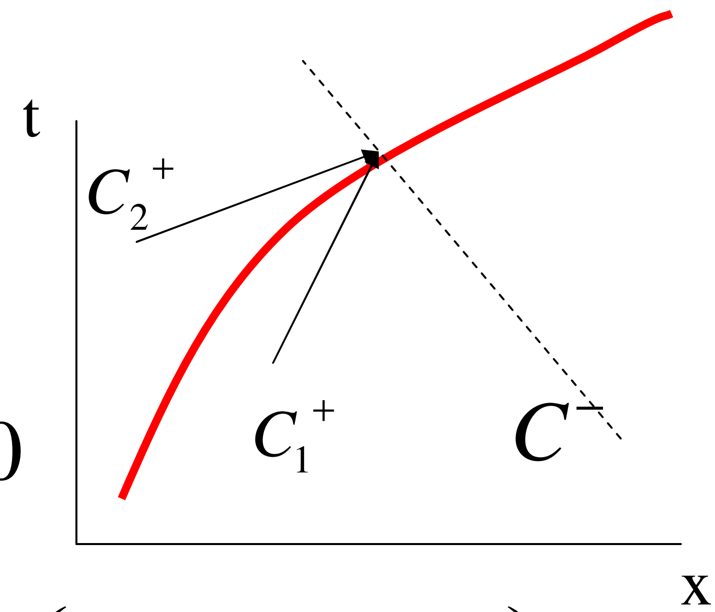
Weak shocks are almost adiabatic

Speed of weak shock wave

$$u_s \approx \frac{1}{2}[(u_1 + c_1) + (u_2 + c_2)]$$



Weak shock:
change in (u+c)

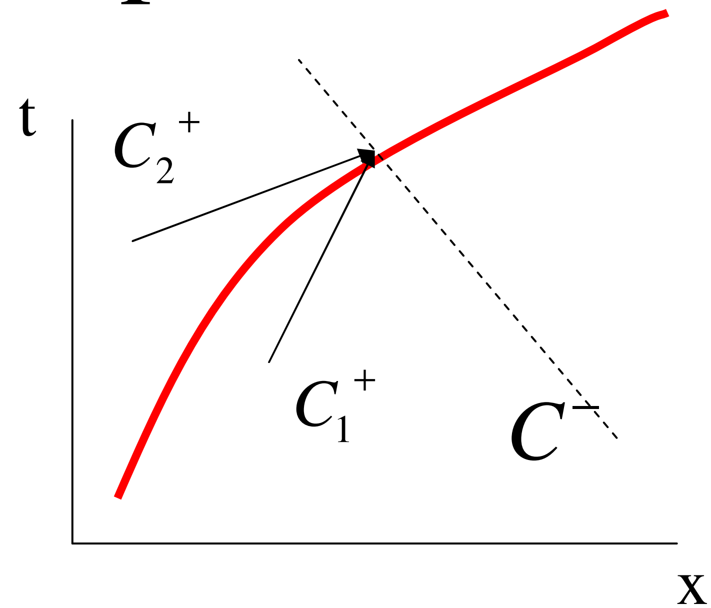


$$C^- : \quad d\left(u - \int \frac{dp}{\rho c}\right) = du - \frac{dp}{\rho c} = 0$$

$$d(u+c) = \frac{dp}{\rho c} + \left(\frac{\partial c^2}{\partial p}\right)_s \frac{dp}{2c} = \frac{c}{\rho}\left(1 + \frac{\rho}{2}\left(\frac{\partial c^2}{\partial p}\right)_s\right) d\rho$$

$$\boxed{d(u+c) = \frac{c\Gamma}{\rho} d\rho}$$

Weak shock speed



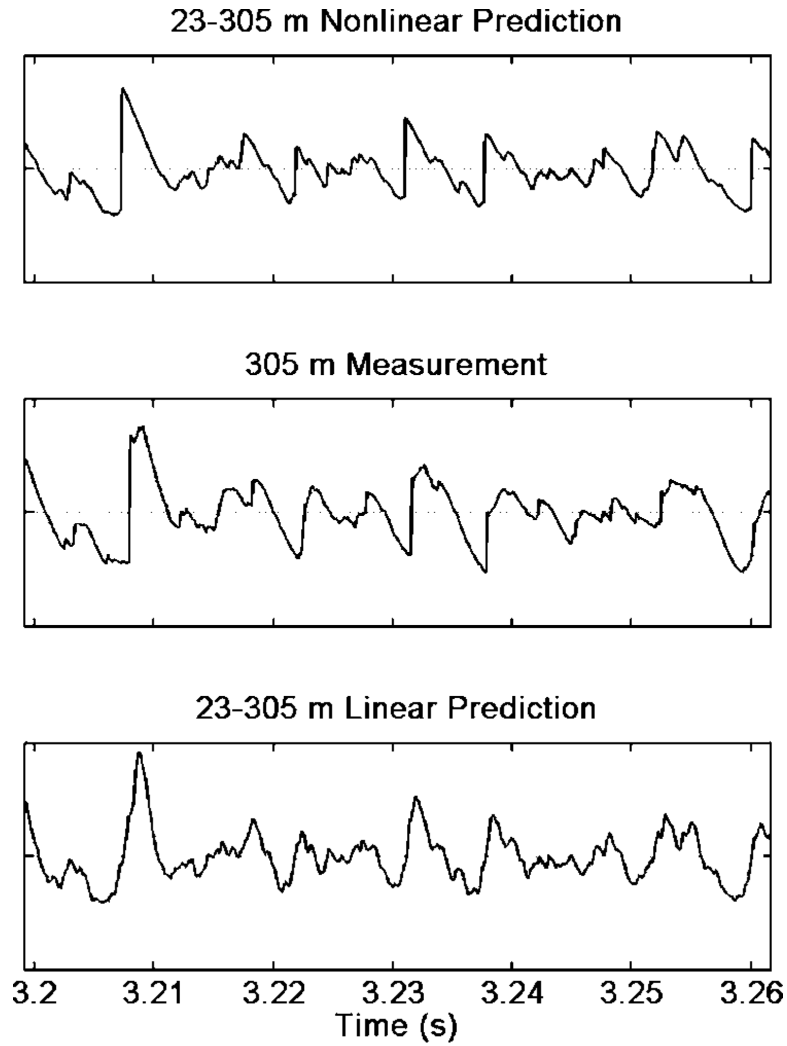
$$d(u+c) = \frac{c\Gamma}{\rho} d\rho$$

$$u_2 + c_2 = u_1 + c_1 + \frac{c\Gamma}{\rho}\Delta\rho + \ldots$$

$$u_1 = 0$$

$$u_s = \frac{1}{2}(u_2 + c_2 + u_1 + c_1) = c_1\left(1 + \frac{\Gamma}{2}\frac{\Delta\rho}{\rho}\right) + \ldots$$

Shock waves in aircraft noise



K.L. Gee et al (JASA 2008)

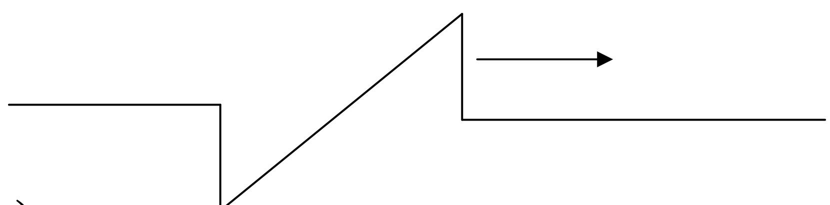
Weak shock theory

(Beyer 1997, Pierce 1981…)

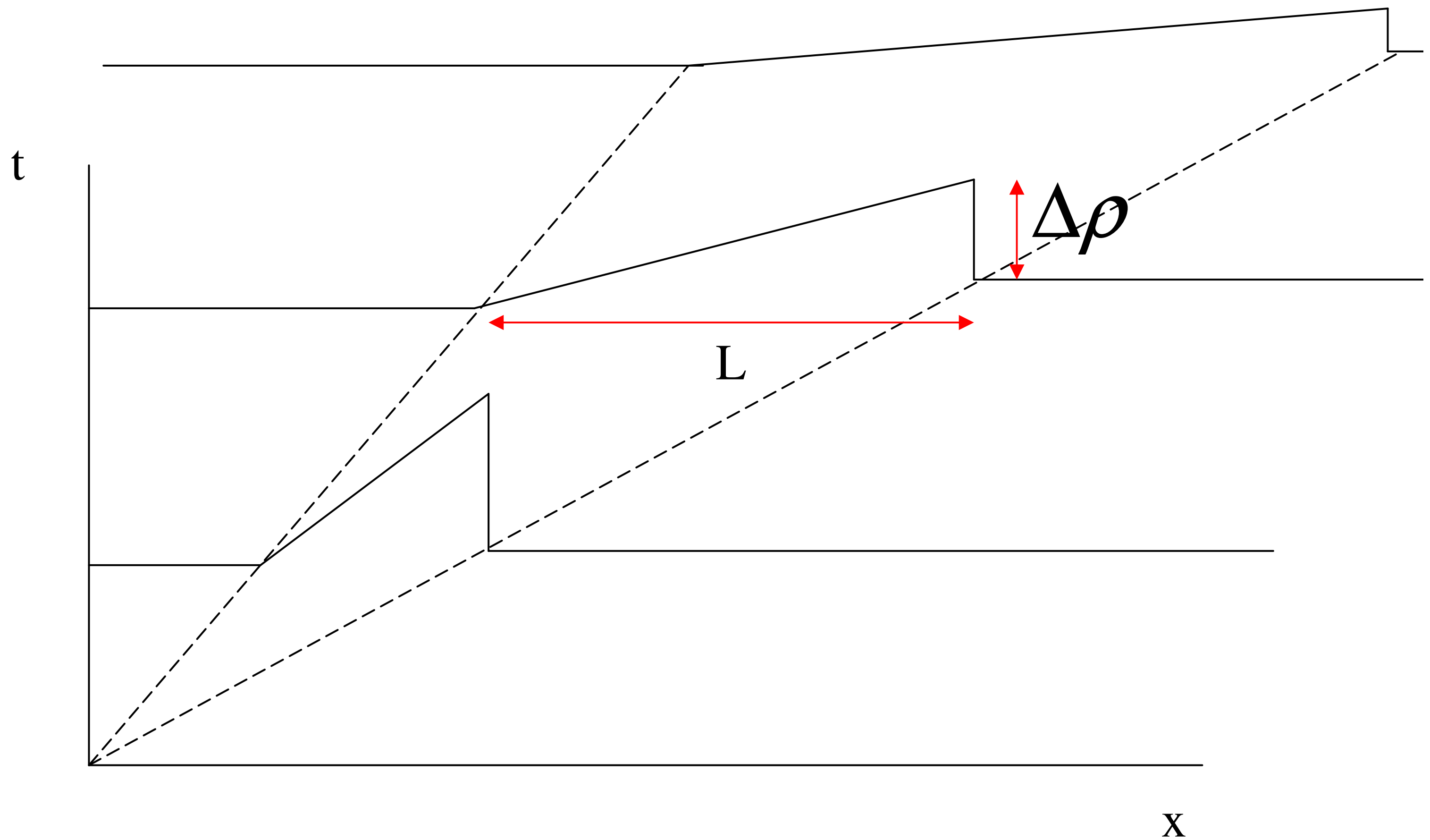
- Neglecting the effect of entropy change on the wave propagation

- Predicting shock attenuation due to friction and heat transfer in the shock wave

- Sawtooth solution
- N-waves
- Aircraft noise (Perception)

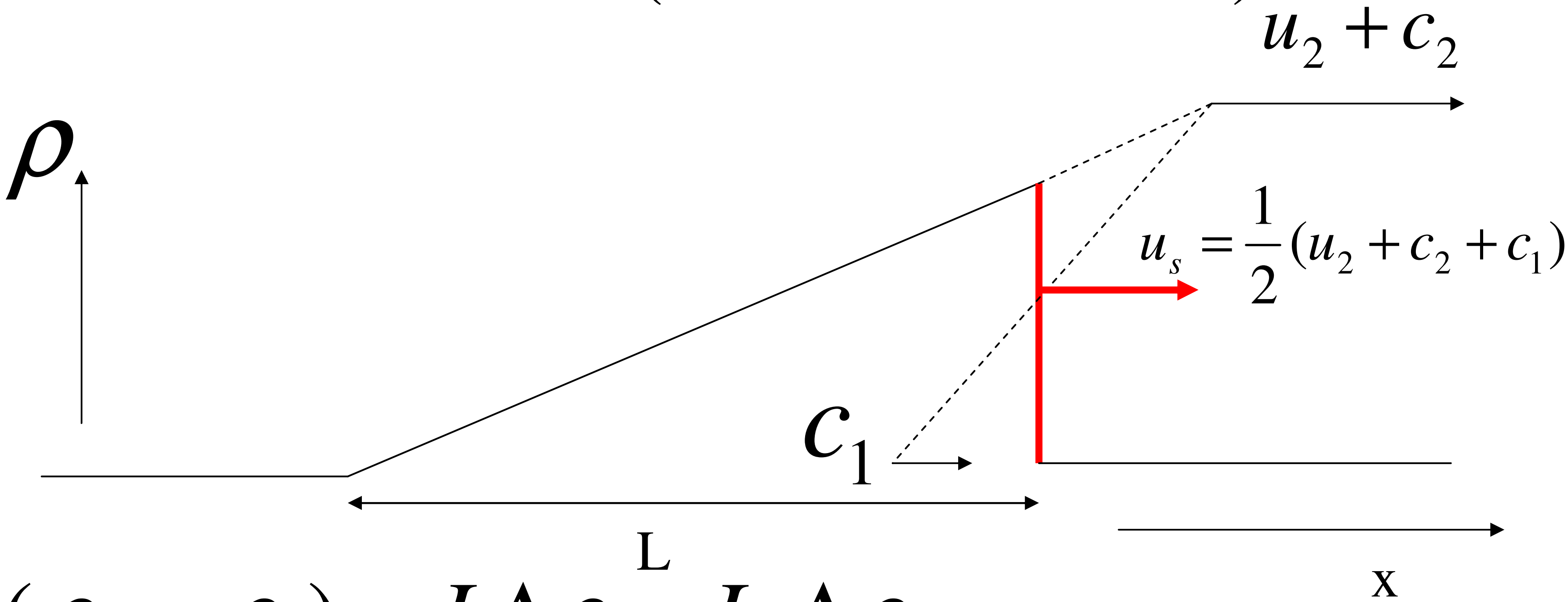


Limit of wave shape (Sawtooth)



Mass $L\,\Delta\rho = L_0 \Delta\rho_0$ (we neglect entropy changes across the shock)

Weak shock:
area rule (Landau 1942-1945)



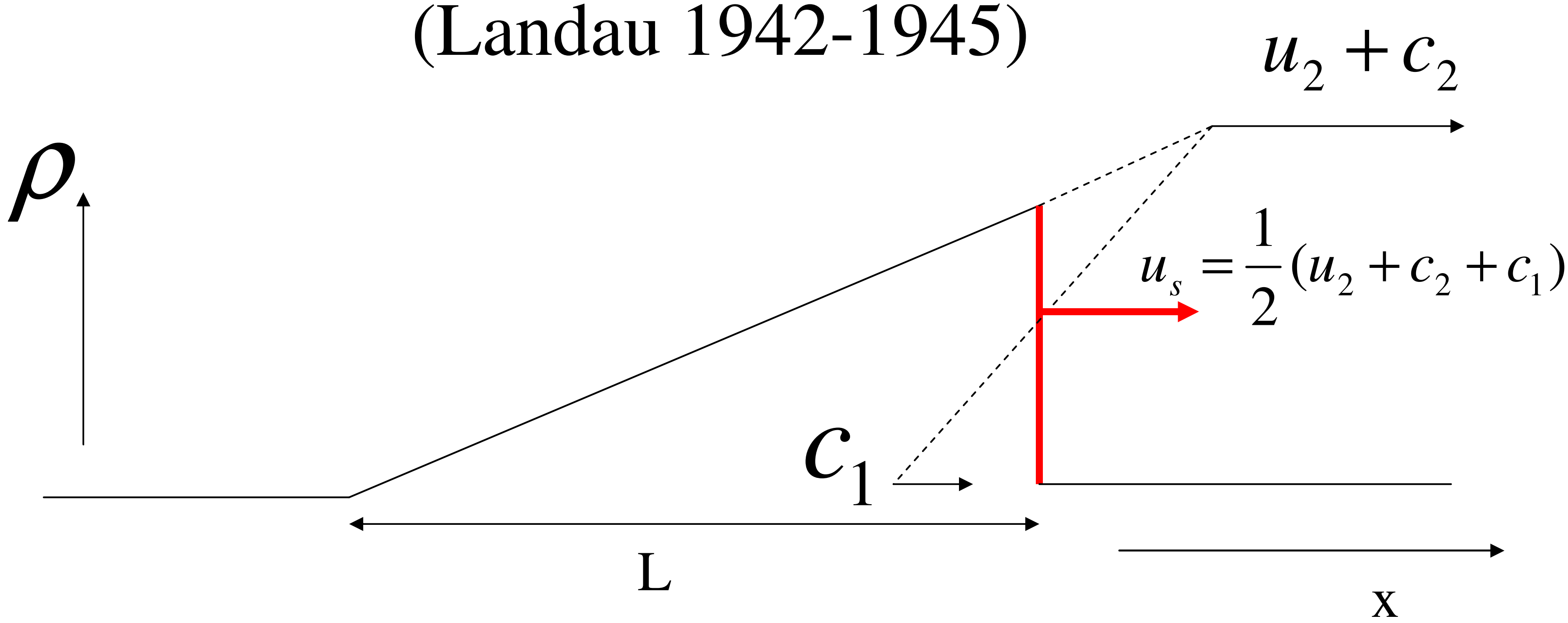
$$L(\rho_2 - \rho_1) = L\Delta\rho = L_0\Delta\rho_0$$

$$\boxed{\Delta\rho = \frac{L_0\Delta\rho_0}{L}}$$

Mass conservation

Weak shock: attenuation

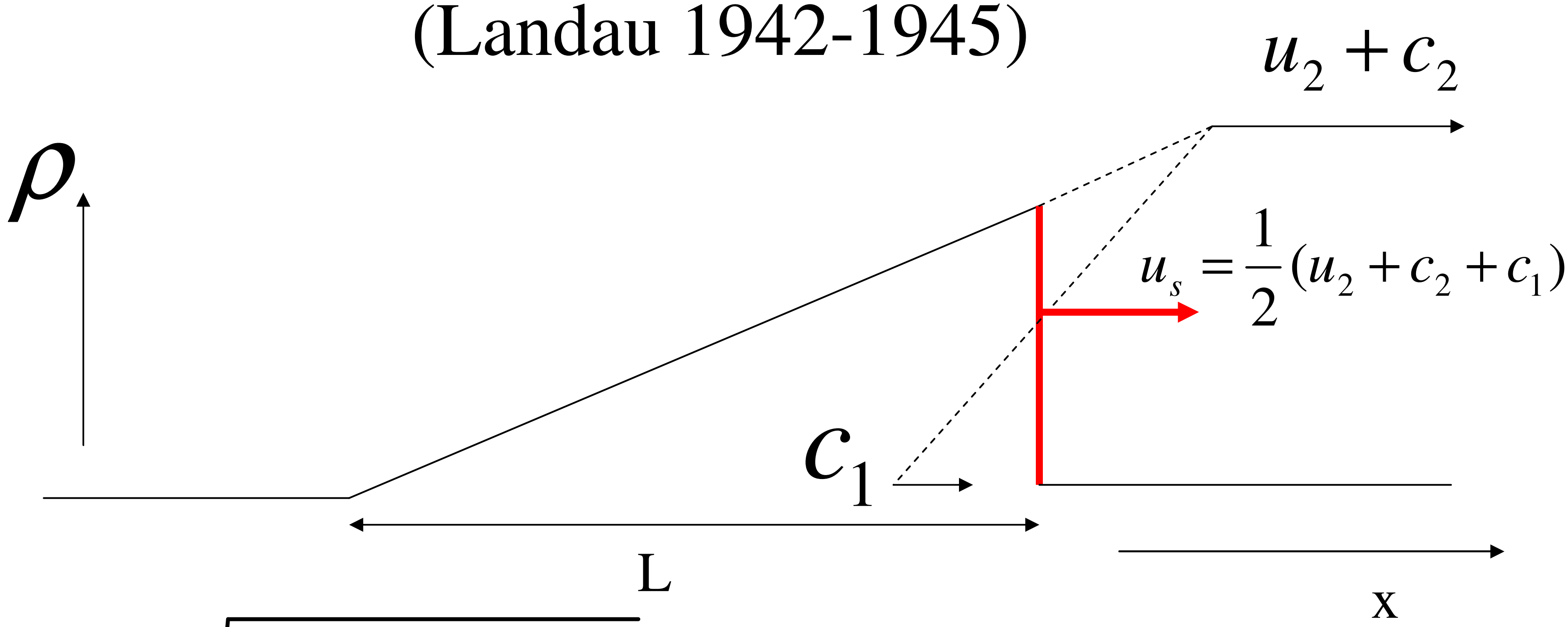
(Landau 1942-1945)



$$\frac{dL}{dt} = (u_s - c_1) = c_1 \frac{\Gamma}{2} \frac{\Delta\rho}{\rho} = c_1 \frac{\Gamma}{2\rho} \frac{L_0 \Delta\rho_0}{L}$$

Weak shock: attenuation

(Landau 1942-1945)

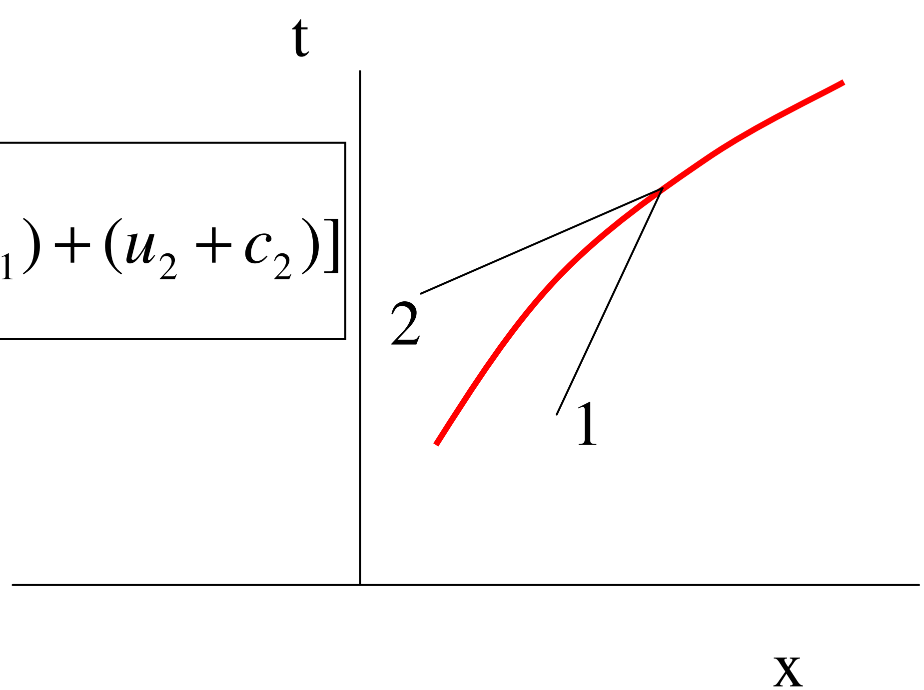


$$L = L_0 \sqrt{1 + \frac{\Gamma \Delta\rho_0}{2\rho} \frac{c_1 t}{L_0}}$$

$$\Delta\rho = \frac{L_0}{L} \Delta\rho_0 \propto \frac{1}{\sqrt{t}}$$

Summary weak shock wave

$$u_s \approx \frac{1}{2}[(u_1 + c_1) + (u_2 + c_2)]$$



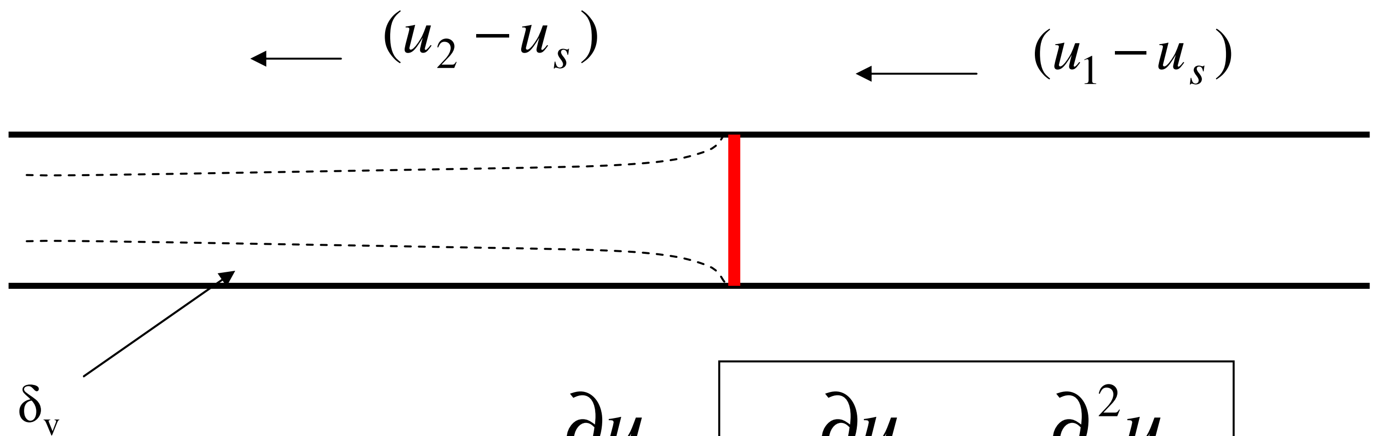
Equal area rule

Shock structure and viscous damping (Lighthill 1956)

$$\left(\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial x}\right)\left[u\pm\int\frac{dp}{\rho c}\right]=\delta\frac{\partial^2 u}{\partial x^2}$$

Burgers equation!

Viscous damping (Chester 1964)



$$(u_2 - u_s)\frac{\partial u}{\partial x} \approx \boxed{c_0 \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2}}$$

$$u(x,\infty) = u_2 - u_s; \quad u(x,0) = -u_s$$

Linear theory for growth viscous boundary layer, allows estimate of viscous force.

Viscous force

For stepwise increase of velocity :

$$\tau_w = -\mu \frac{\partial u}{\partial y} |_w = -\mu u_2 \sqrt{\frac{c_0}{\pi x \nu}} H(c_0 t - x)$$

$$f_x = \frac{\Pi \tau_w}{S_d} = \frac{2\tau_w}{R} \Longleftarrow$$ Integration over pipe section!

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} \approx -\frac{2}{R} u_2 \sqrt{\frac{c_0 \nu}{\pi x}} H(c_0 t - x)$$

Convolution

$$u_2(\xi, t) = \frac{\partial u}{\partial x} \Big|_{(x, t - \frac{\xi}{c_0})} d\xi$$

$$\frac{f_x}{\rho} = \frac{2\tau_w}{R\rho} = -\frac{2}{R}\sqrt{\frac{c_0 \nu}{\pi}} \int_0^{\infty} \frac{1}{\sqrt{\xi}} \left(\frac{\partial u}{\partial x} \Big|_{(x, t - \frac{\xi}{c_0})} \right) d\xi$$

Factor $(1 + \frac{\gamma - 1}{\sqrt{\Pr}})$ to include effect of heat transfer.

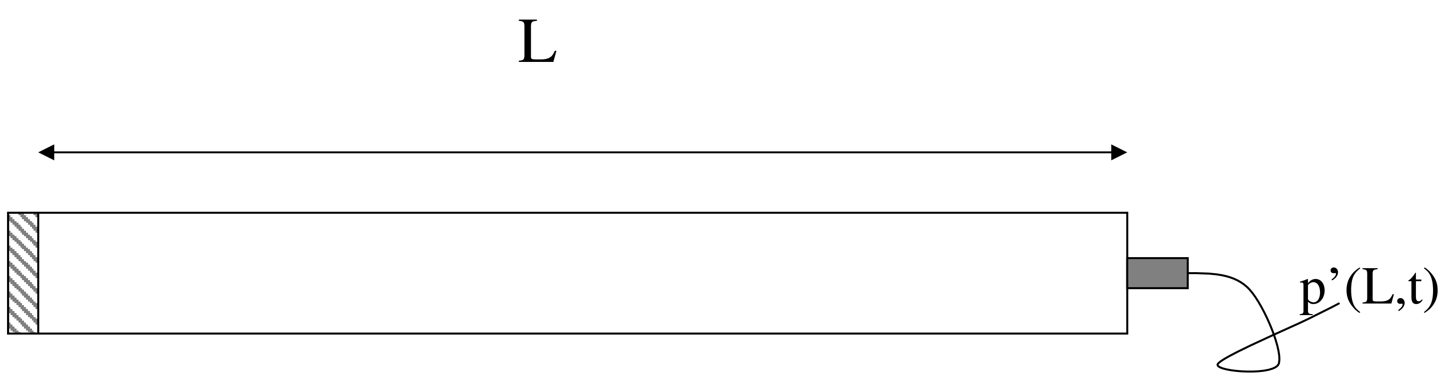
Chester (1964)

- Neglect change in average entropy
- Use non-linear simple wave theory
- Estimate friction for **thin boundary layers** from linear theory

Thin boundary layer approach fails
for Sondhaus tube (Rott 1986)

Resonance of a closed pipe driven by a piston

- Acoustics: Amplitude limited at resonance by friction



$$u'(x,t) = \mathrm{Re}\big[[p^+ \exp(-ikx) - p^- \exp(ikx)]\exp(i\omega t)\big]$$

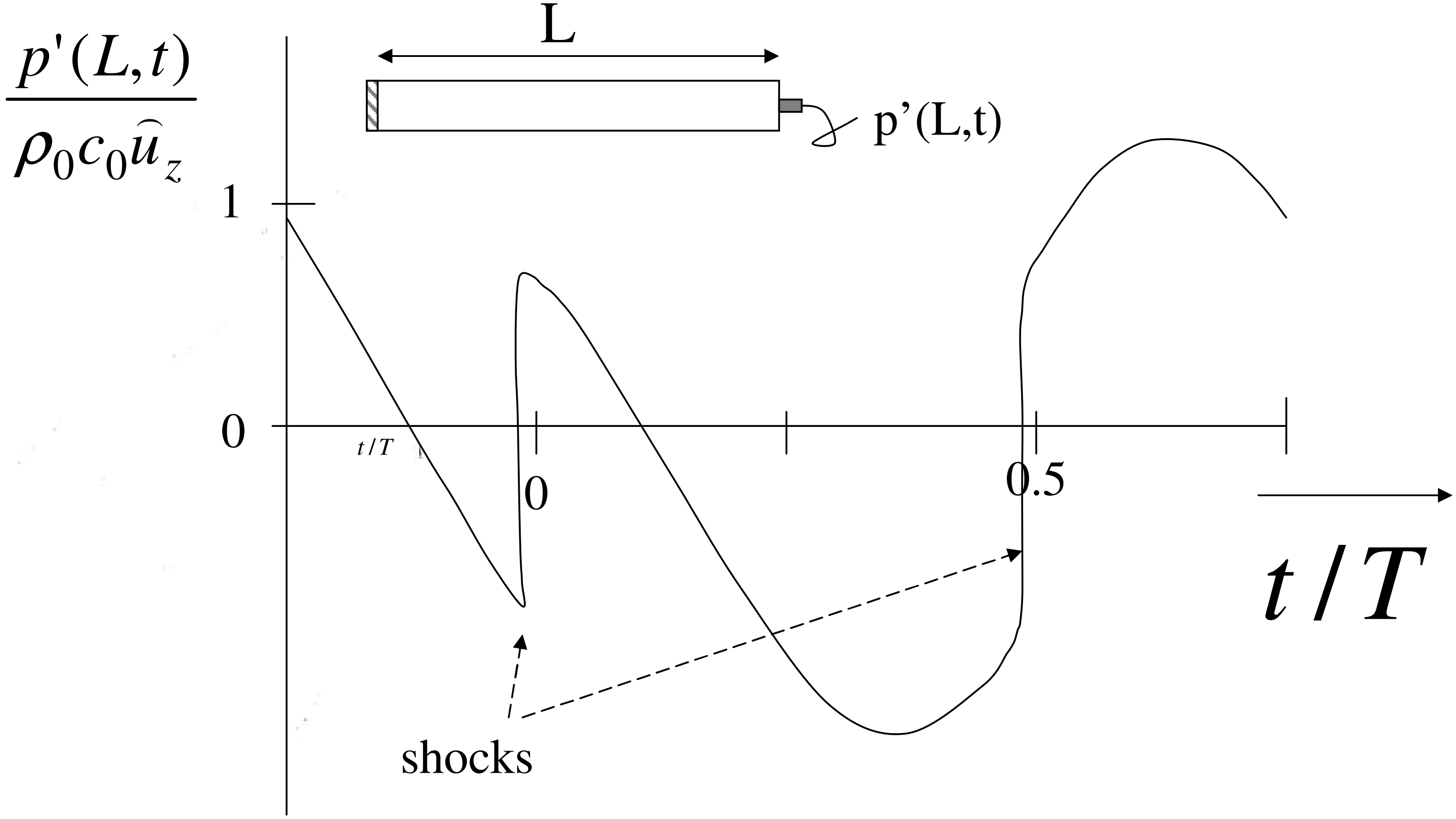
$$u'(0,t) = \mathrm{Re}[\hat{u}_z \exp(i\omega t)]$$

$$u'(L,t) = 0$$

$$p^+ = p^- \exp(-2ikL) = \frac{\hat{u}_z}{1-\exp(-2ikL)}$$

Sub-harmonic excitation

(Keller 1975; Althaus and Thomann 1987)



Due to reflection at close wall, we have long wave propagation distances!

Sub-harmonic excitation
(Keller 1975; Althaus and Thomann 1987)

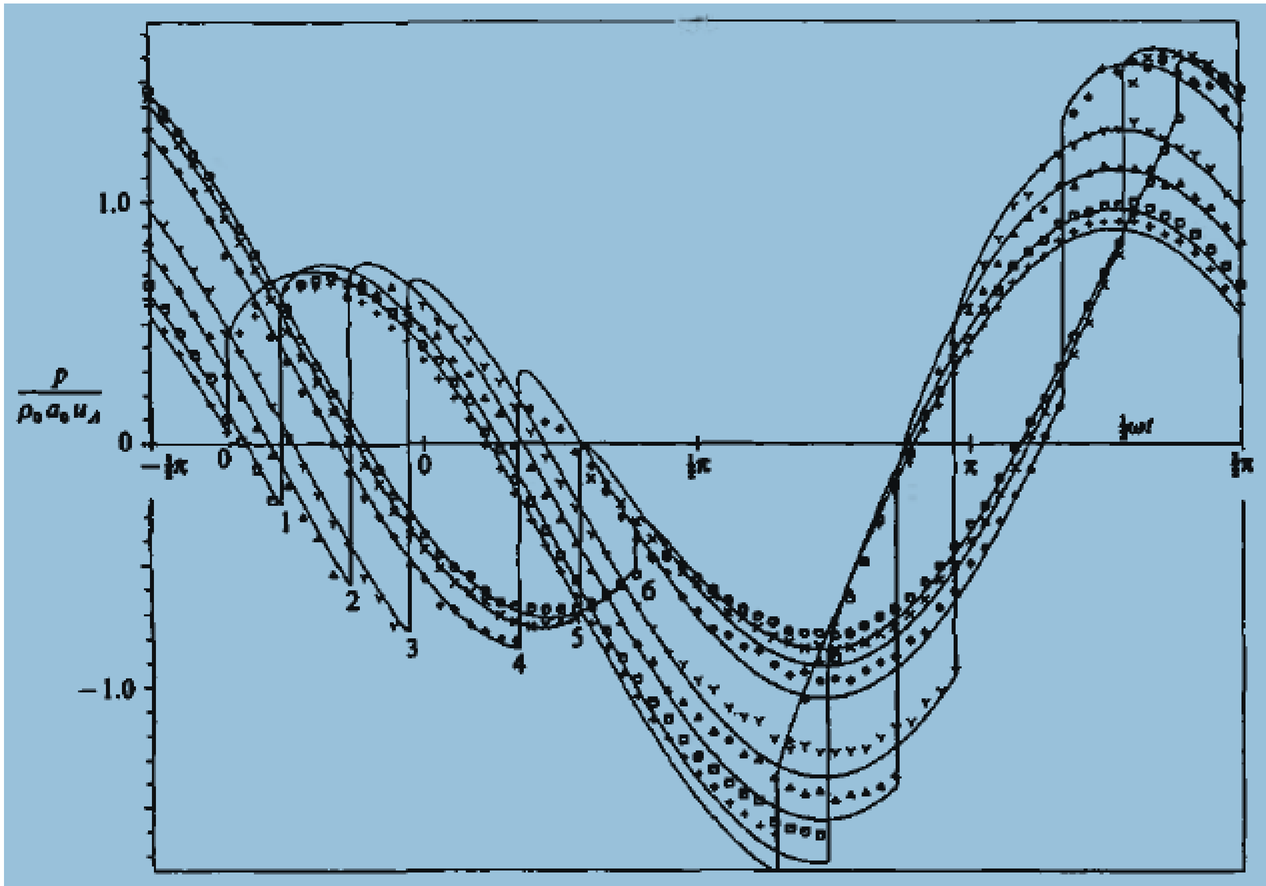


FIGURE 9. Pressure at the closed end for Freon RC-318, experiment V8. 0: $r = -1.076$, $s = 0.348$; 1: $r = -0.940$, $s = 0.347$; 2: $r = -0.596$, $s = 0.344$; 3: $r = -0.271$, $s = 0.342$; 4: $r = 0.360$, $s = 0.335$; 5: $r = 0.636$, $s = 0.334$; 6: $r = 0.810$, $s = 0.332$.

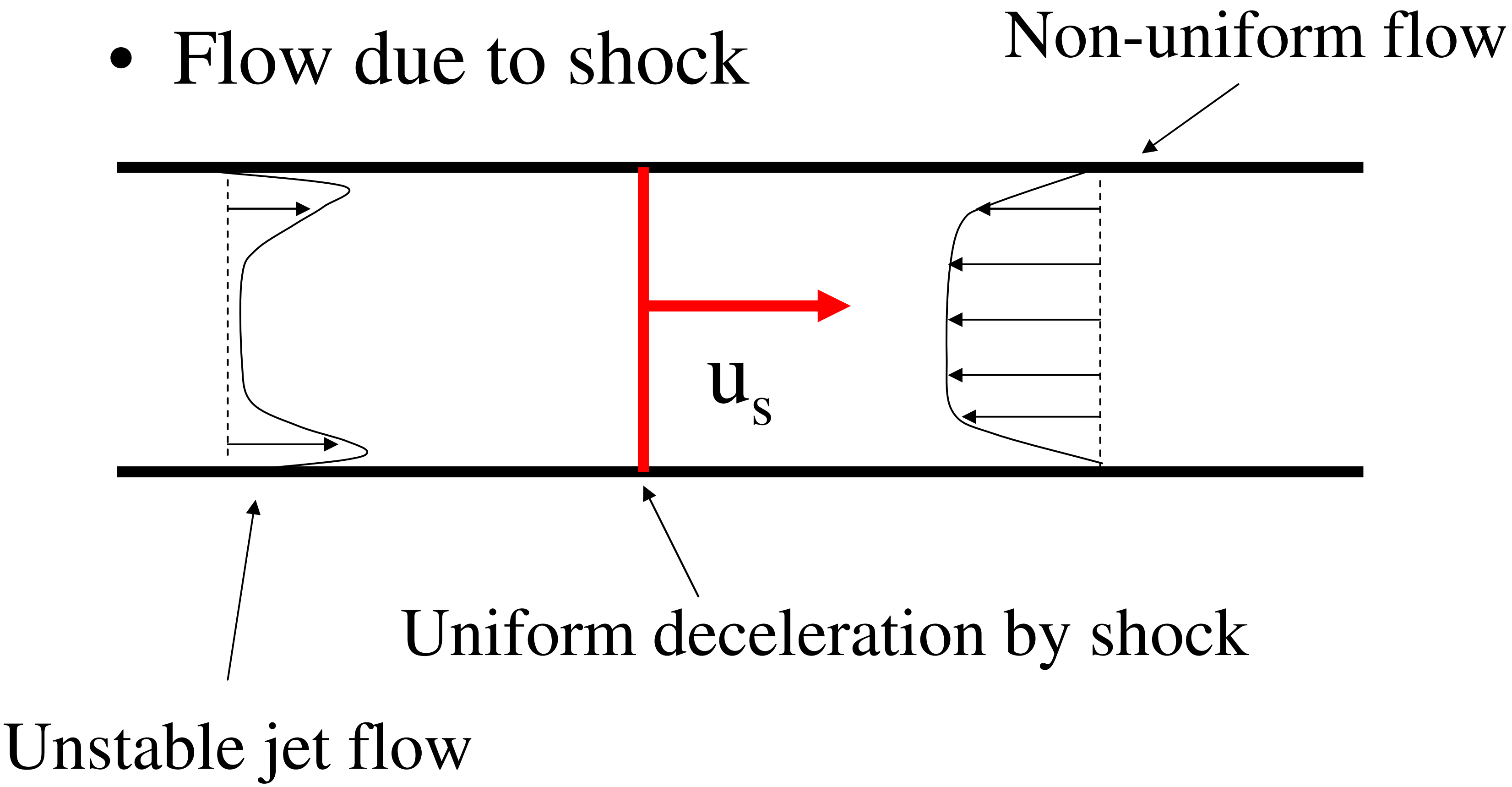
Thermal effects
(Rott 1974-1980, Swift 1992, Bailliet 2001)

- Change of average entropy due to dissipation and heat conduction.

- Thermo-acoustical devices.

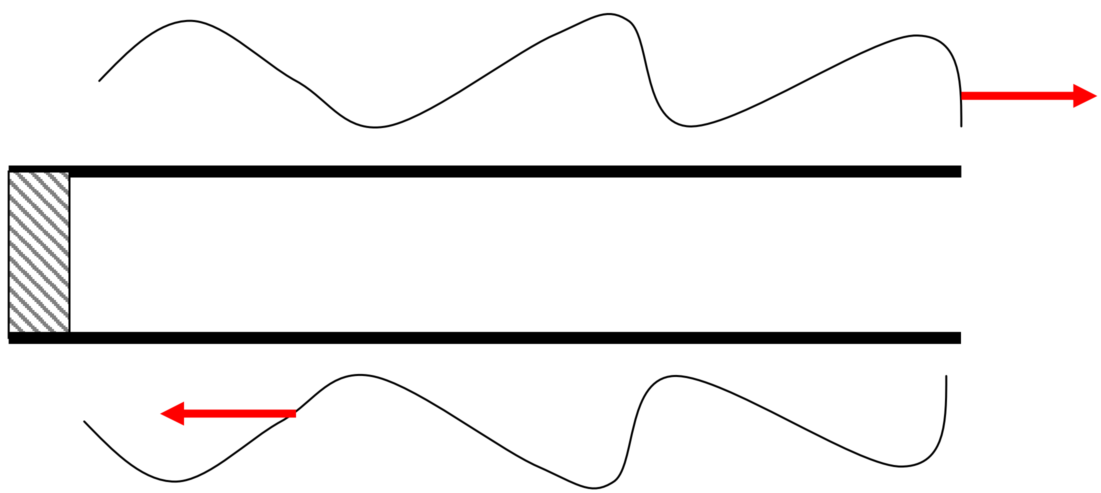
Turbulence in standing wave
(Merkli and Thomann 1975)

- Flow due to shock

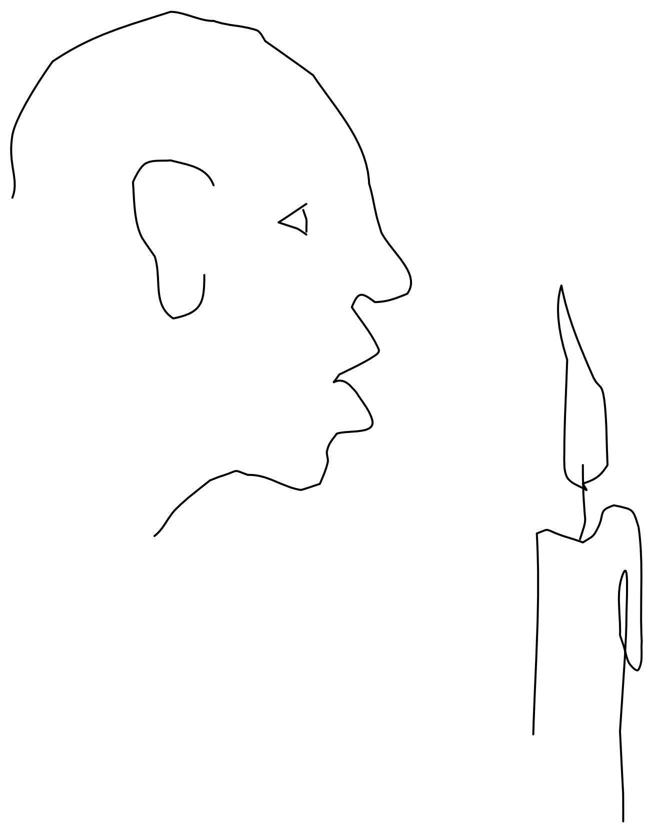


Open tube resonance
(Disselhorst and van Wijngaarden 1980)

- Reflection at open pipe termination results in an inversion of acoustic wave
- Wave steepening from piston to open end is compensated on the way back

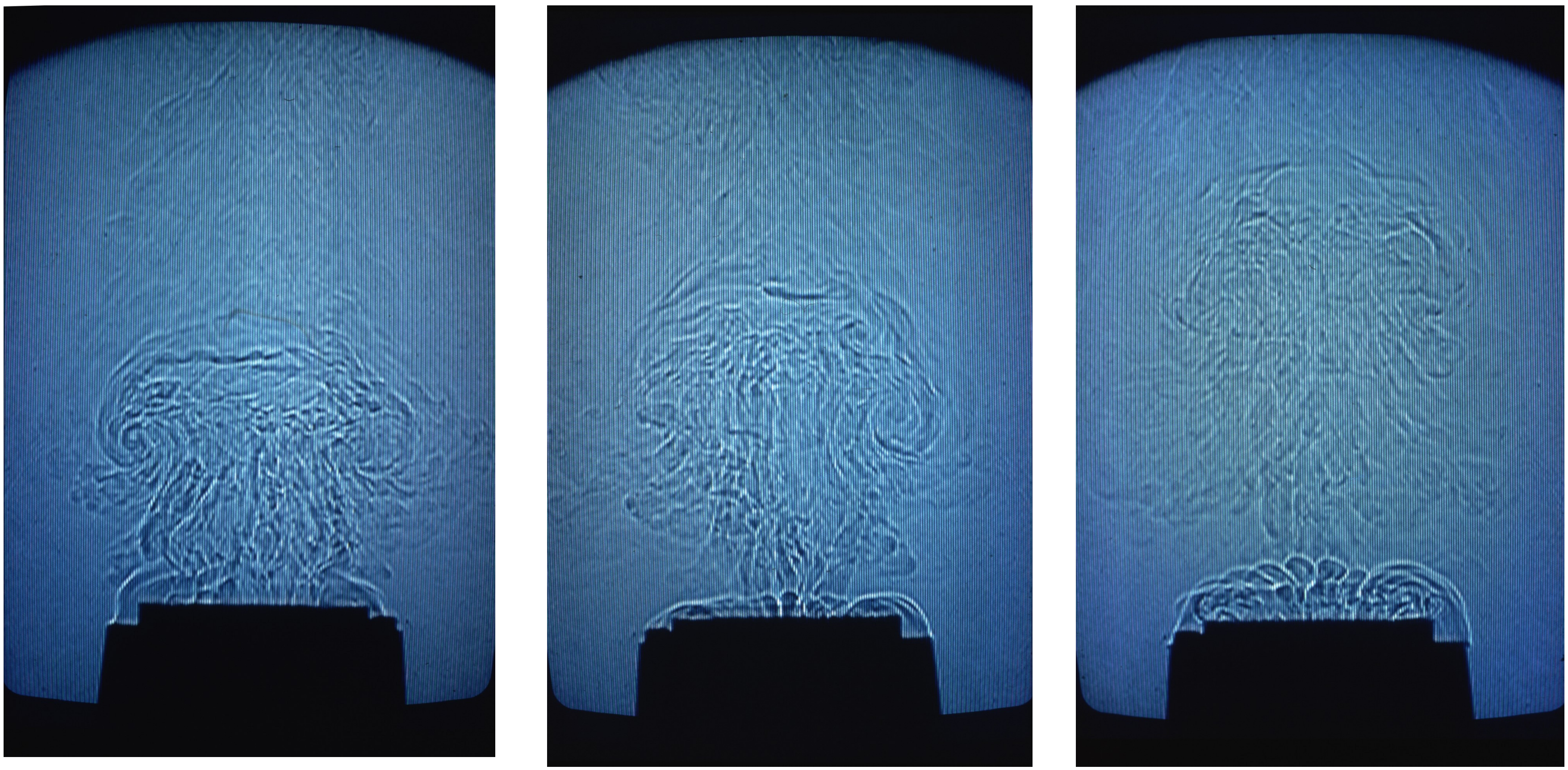


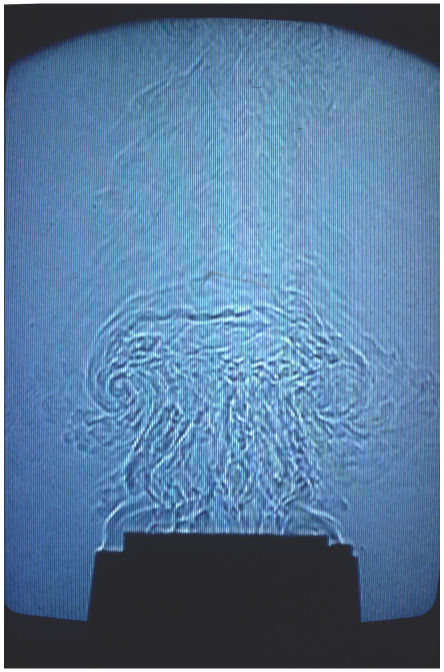
Convective non-linearity



Pipe termination

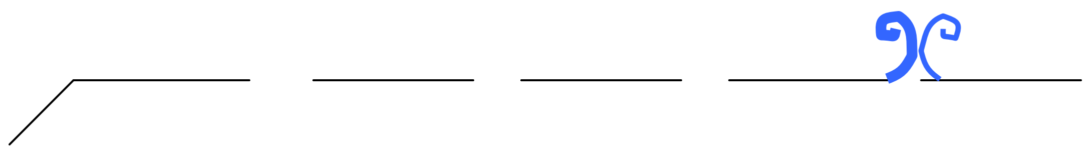
u_{ac} ↑ $\quad$ u_{ac} ↓





Vortex shedding

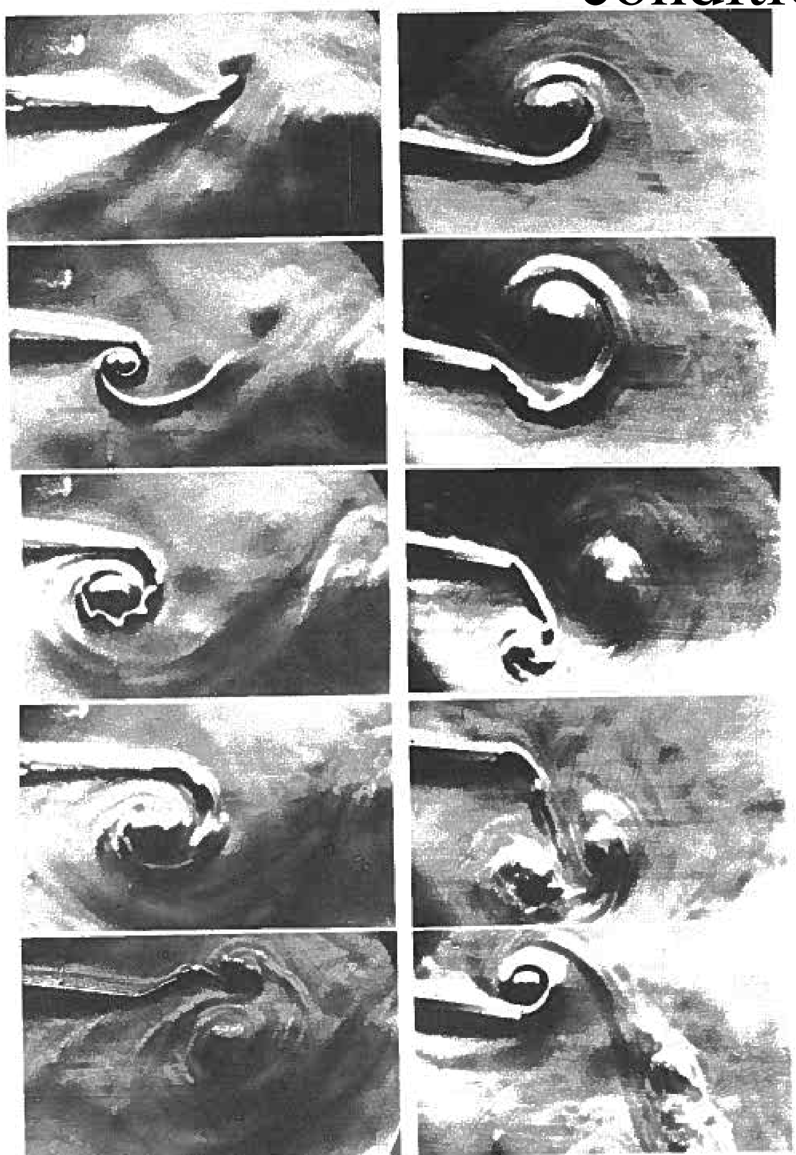
play



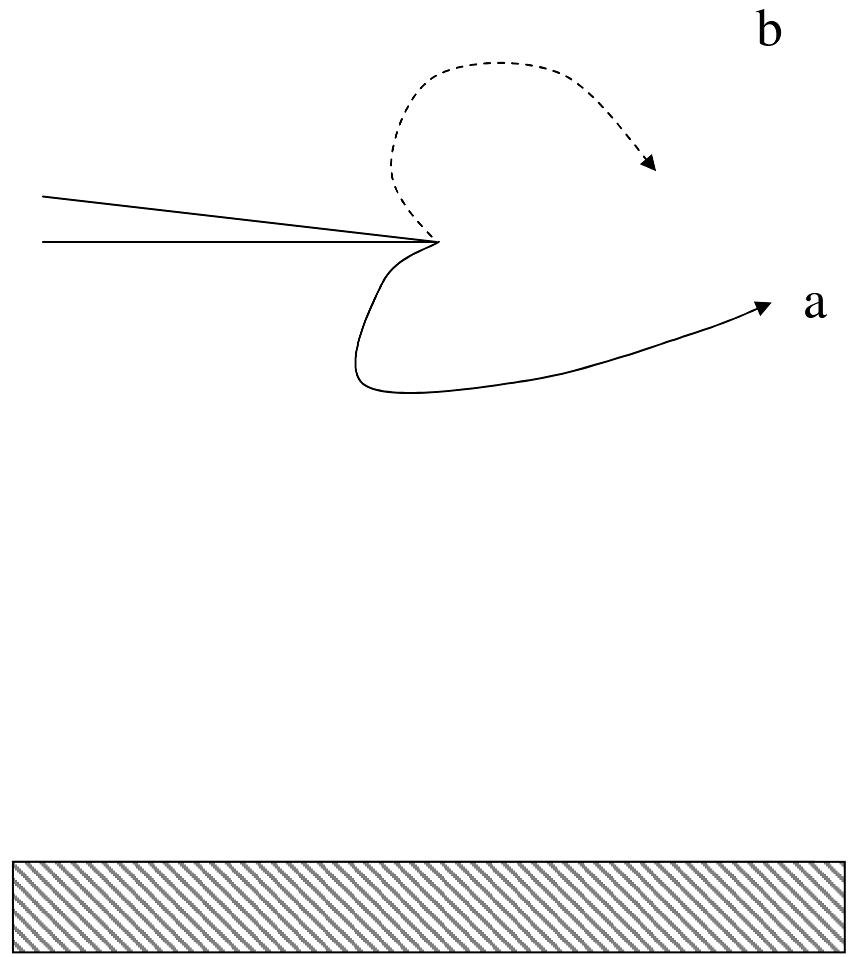
No play

Thin walled clarinet has narrower tone holes
(Keefe 1983, Atig, Dalmont and Gilbert 2004)

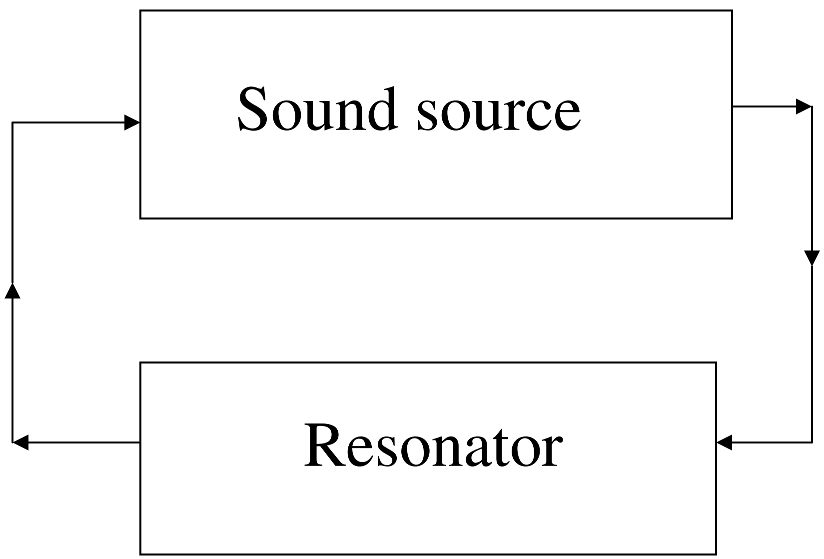
Vortex shedding modes depending on initial conditions! (Disselhorst 1978)



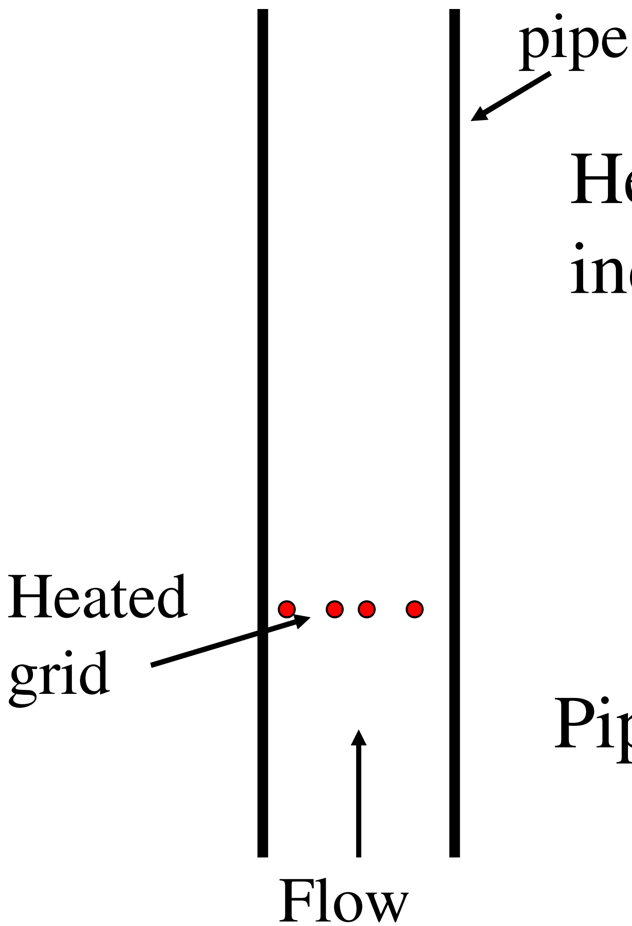
a b



Self-sustained oscillation



Rijke tube

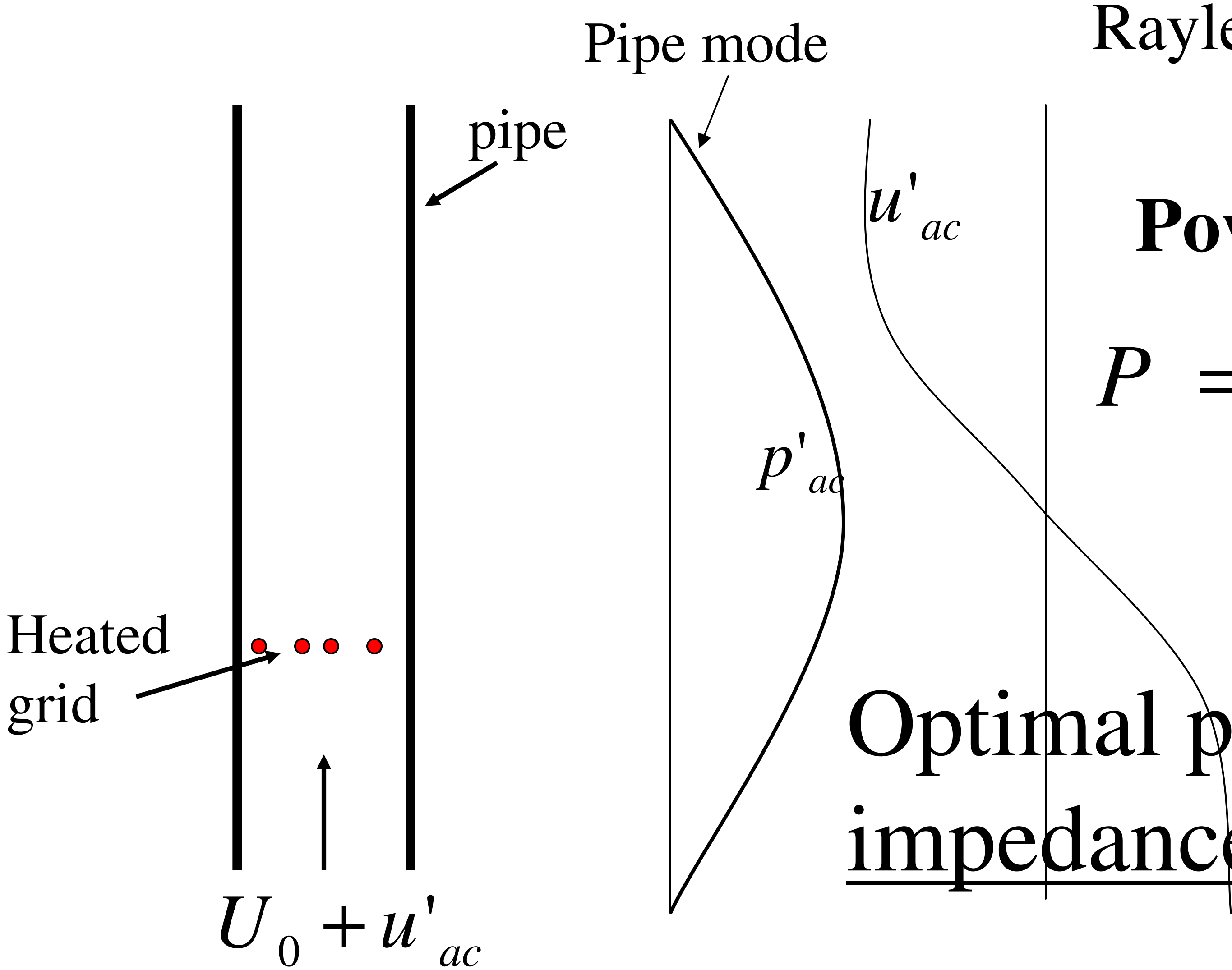


Heat transfer q'_w induces dilatation →

$$\frac{dV}{dt} \propto q'_w$$

Pipe is acoustical mass-spring system

Rijke tube



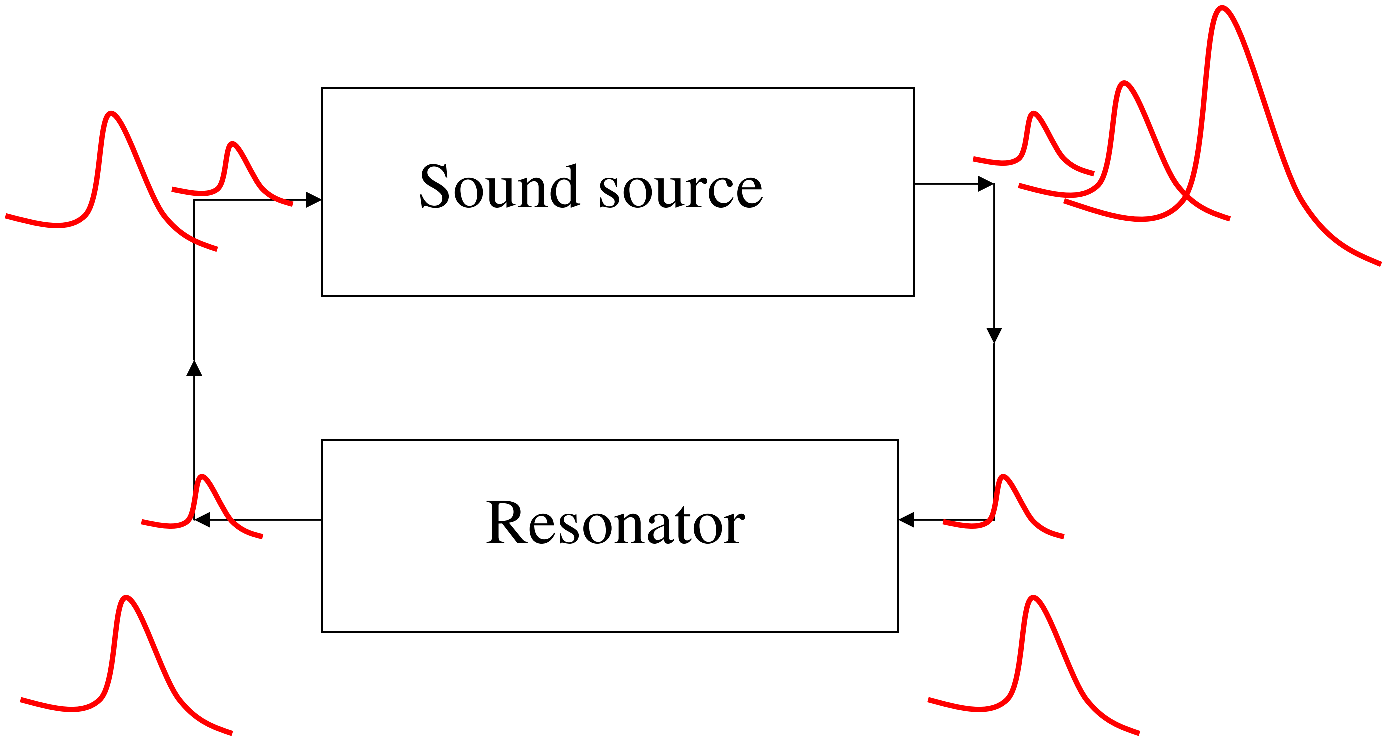
Rayleigh $\frac{dV}{dt} \propto u'_{ac}$

Power:

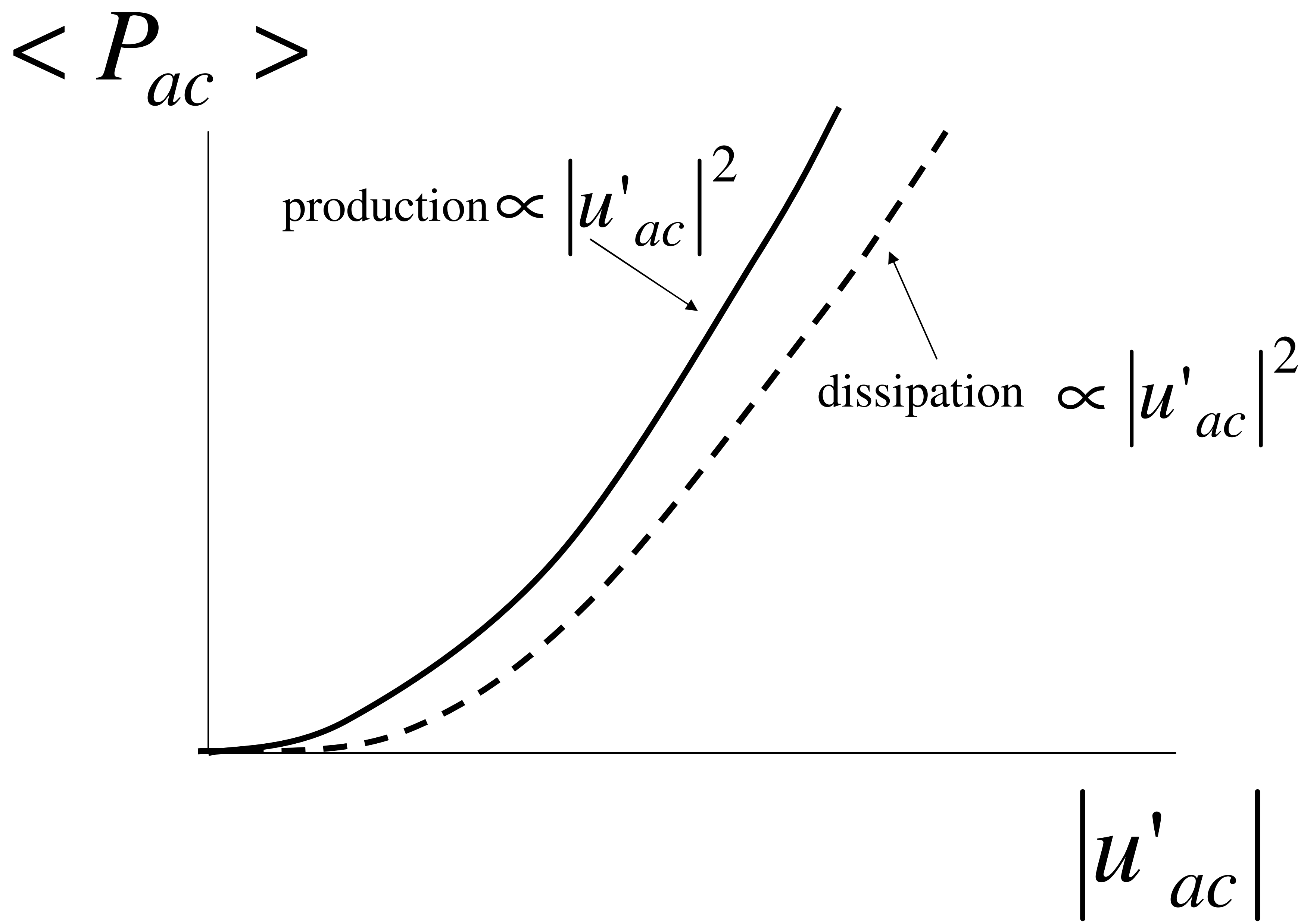
$$P = p'_{ac} \frac{dV}{dt}$$

Optimal position:
<u>impedance matching</u>

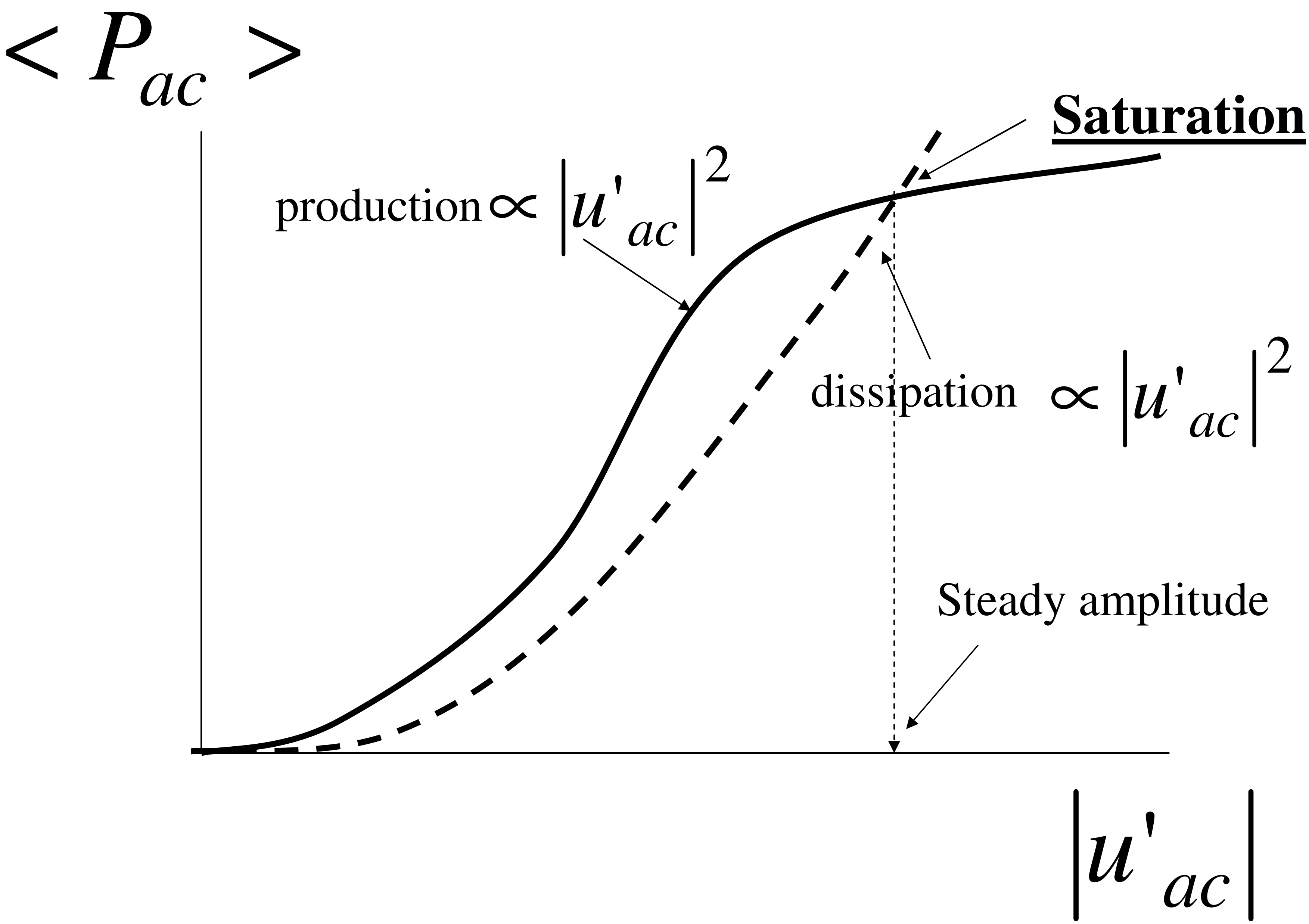
Self-sustained oscillation



Prediction of linear theory

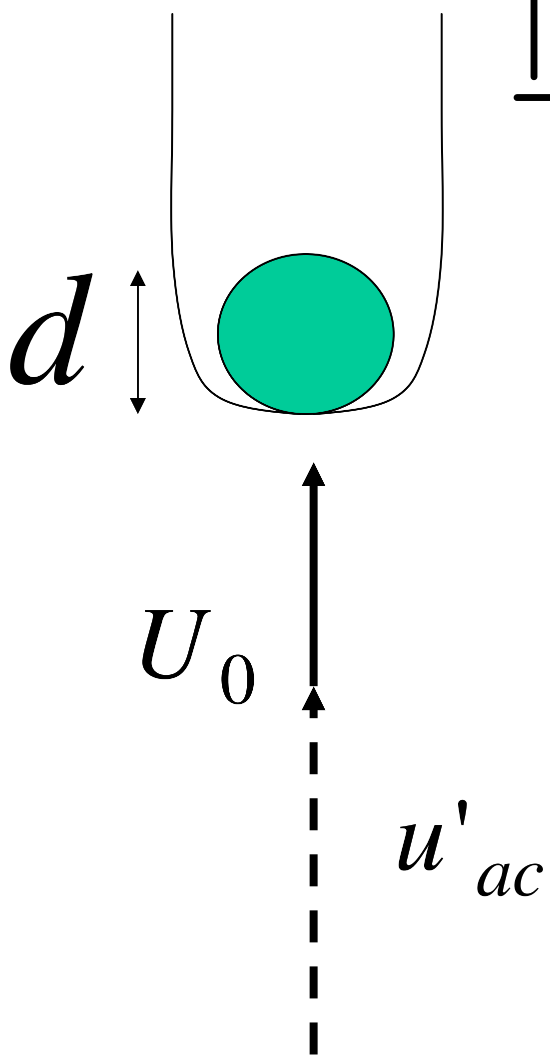


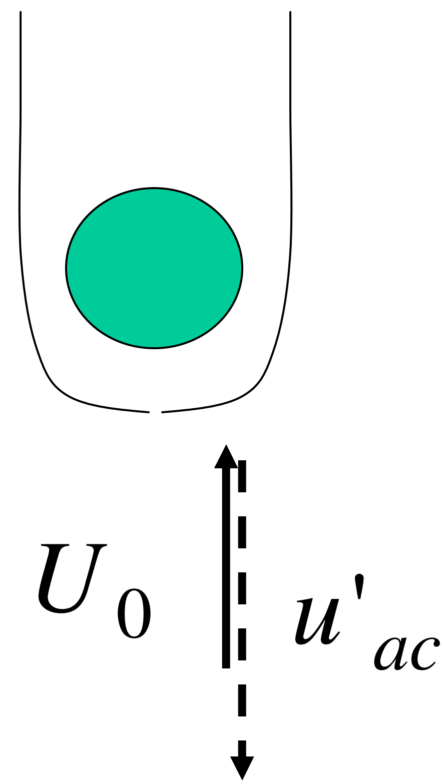
Saturation due to non-linearity



Saturation due to backflow

$$\frac{|u'_{ac}|}{\omega} > d$$





Conclusions

- “Complex problems have simple, easy to understand wrong answers”
  (N.Rott, 1986)

Some references

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Self-similar solution viscous boundary layer

$$u / c_0 = f(\eta)$$

$$\eta = y\sqrt{\frac{c_0}{\nu x}}$$

$$f'' = -\frac{1}{2}\eta f'$$

$$f = \frac{u_2}{c_0\sqrt{\pi}} \int_0^{y\sqrt{\frac{c_0}{\nu x}}} \exp(-\frac{\eta^2}{4}) d\eta - 1$$