KPMG Center of Excellence
in Risk Management

Titel: Discretisation of FBSDEs driven by CÀDLÀG martingales

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Abstract. We consider a backward stochastic differential equation with jumps (BSDEJ) which is driven by a Brownian motion and a Poisson random measure. We present two candidate-approximations to this BSDEJ and we prove that the solution of each candidate-approximation converges to the solution of the original BSDEJ in a space which we specify. We use this result to investigate in further detail the consequences of the choice of the model to (partial) hedging in incomplete markets in finance. As an application, we consider models in which the small variations in the price dynamics are modeled with a Poisson random measure with infinite activity and models in which these small variations are modeled with a Brownian motion. Using the convergence results on BSDEJs, we show that quadratic hedging strategies are robust towards the choice of the model and we derive an estimation of the model risk.

1. Introduction

In the present paper we aim at studying the time-discretisation of forward backward stochastic differential equations (FBSDEs) driven by *càdlàg martingales*. That is FBSDEs of the form

(1.1)
$$\begin{cases} S(t) = S(0) + \int_0^t S(s)a(s)\,\mathrm{d}s + \int_0^t S(s-)\,\mathrm{d}M(s)\,, \\ V(t) = h(S(T)) + \int_t^T \varphi(s, S(s), V(s), \Upsilon(s))\,\mathrm{d}\langle M\rangle_s - \int_t^T \Upsilon(s)\,\mathrm{d}M(s) - L(T) + L(t)\,, \end{cases}$$

where M is a càdlàg martingale, $\langle M\rangle$ is the predictable compensator of the quadratic variation of M, and L is a *martingale orthogonal* to M. h, φ, and a have to fulfill certain conditions that we specify later in the paper. A solution to the backward equation in (1.1) is a triplet (V, Υ, L). We refer to Carbone et al. [6] for the study of the existence and uniqueness of this solution.

Such equations were first derived by Chitashvili [7] as a stochastic version of the Bellman equation in an optimal control problem. They naturally appear for example in quadratic hedging problems (see e.g. Jeanblanc et al. [10] and Di Nunno et al. [8]). They can also characterize solutions of optimal portfolio problems based on utility functions (see e.g. Mania et al. [14], Mania and Tevzada [16], and Mania and Schweizer [15]).

Bouchard and Touzi [5] studied the problem of discretisation of classical BSDEs driven by a Brownian motion using an Euler type scheme. Bouchard and Elie [4] extended this

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approach to classical BSDEs with jumps and studied the induced L^2-error. They showed that the L^2-norm of the error is of the order of the time step.

In the present paper, we first specify the martingale M in (1.1) to be driven by a Brownian motion and by jumps with finite activity. In this case the discretisation of the forward equation S is well studied in the literature and we refer to Platen [19] for a good overview. For the backward equation, we obtain an Euler backward scheme $(\hat{V}, \hat{\Upsilon}, \hat{L})$ which is presented as a solution to a time-discrete BSDE. Then we study the induced L^2-error.

In this context we mention a paper by Lejay et al. [12] in which the authors presented a numerical scheme for solving classical BSDEs with jumps. They consider a time-continuous BSDE driven by a Brownian motion and by jumps. Then they construct a time-discrete BSDE driven by a complete system of three orthogonal discrete time-space martingales, the first being a random walk converging to a Brownian motion; the second being another random walk, independent of the first one, converging to a Poisson process. The third martingale is added to ensure the existence of the solution to the time-discrete BSDE. Thus, this martingale converges to zero for a small time step. In our case $\hat{L}$ is a martingale that represents both a time-discretisation of the orthogonal continuous-time martingale L and an additional term necessary for the existence of the solution to our time-discrete BSDE (see (3.3)). Note that the time-discrete backward scheme in Section 2.2 in Bouchard and Elie [4] does not contain an orthogonal martingale since it is not presented as a solution to a time-discrete BSDE.

Our approach allows us to approximate the process L in (1.1) by a discrete-time process $\hat{L}$. This is important in applications in finance. For example in the case of the study of quadratic hedging strategies, the process L represents the remaining risk in the hedging strategy and hence it is important to study the approximation and the simulation of this process L. Finally, we mention that the study of the approximation in the paper by Lejay et al. [12] was considered in the weak sense. In the present paper, we study the L^2-convergence.

For the study of the convergence, we first investigate a relation of the backward equation in (1.1) to classical BSDEs by applying an *Itô representation* to the martingale L. Then we exploit the results by Bouchard and Elie [4]. This allows us to prove that under certain conditions imposed on the parameters of the forward equation, the Euler scheme we consider converges to the continuous BSDE (1.1) in the L^2-sense. Moreover, we show that the L^2-norm of the error is of the order of the time step.

To complete our study we consider in a second step martingales M driven by a Brownian motion and by jumps with *infinite activity*. We approximate the small jumps by a Brownian motion scaled with the standard deviation of the small jumps. Then we study the discretisation of the approximation and by exploiting the results in Di Nunno et al. [8], we prove that the L^2-norm of the error is of the order of the time step plus the variance of the small jumps. Same type of results hold when ignoring the small jumps in the approximation before performing a discretisation.

The paper is organised as follows. In Section 2 we present the continuous-time FBSDE and in Section 3 its time-discretisation.The convergence of the discrete-time scheme to the continuous-time BSDE is studied in Section 4. We also compute the convergence rate. In Section 5 we complete the analysis by including jumps with infinite activity in the study of the discretisation. Section 6 concludes.

2. Continuous-time model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Fix $T > 0$. Let S be a locally square integrable semimartingale under $\mathbb{P}$ adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the *usual conditions* of right continuity and completeness (see Chapter I in Protter [19]). We denote by $[X, Y]$ the *quadratic covariation* of two given semimartingales X and Y and by $\langle X, Y \rangle$ the compensator of the quadratic covariation also called *predictable compensator*. Let $W = W(t)$ and $B = B(t)$, $t \in [0, T]$, be two independent standard Wiener processes and $\tilde{N} = \tilde{N}(\mathrm{d}t, \mathrm{d}z)$, $(t, z) \in [0, T] \times \mathbb{R}$ be a centered Poisson random measure, i.e. $\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \ell(\mathrm{d}z)\,\mathrm{d}t$, where $\ell(\mathrm{d}z)$ is the Lévy measure and $N(\mathrm{d}t, \mathrm{d}z)$ is the Poisson random measure such that $\mathbb{E}[N(\mathrm{d}t, \mathrm{d}z)] = \ell(\mathrm{d}z)\mathrm{d}t$. Define $\mathcal{B}(\mathbb{R})$ as the σ-algebra generated by the Borel sets $\bar{U} \subset \mathbb{R}$. We assume that the Lévy measure has a finite mass in the tail, i.e. $\ell(|z| \le 1) < \infty$ and that the jump measure has a finite second moment, i.e. $\int_{\mathbb{R}} z^2\, \ell(\mathrm{d}z) < \infty$. We specify the $\mathbb{P}$-augmented filtrations $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$, respectively by

$$\mathcal{F}_t = \sigma \left\{ W(s), \int_0^s \int_A \tilde{N}(\mathrm{d}u, \mathrm{d}z), \quad s \le t, \quad A \in \mathcal{B}(\mathbb{R}) \right\} \vee \mathcal{N},$$

$$\mathcal{G}_t = \sigma \left\{ W(s),\ B(s),\ \int_0^s \int_A \tilde{N}(\mathrm{d}u, \mathrm{d}z), \quad s \le t, \quad A \in \mathcal{B}(\mathbb{R}) \right\} \vee \mathcal{N},$$

where $\mathcal{N}$ represents the set of $\mathbb{P}$-null events in $\mathcal{F}$. We introduce the notation $\mathbb{H} = (\mathcal{H}_t)_{0 \le t \le T}$ such that $\mathcal{H}_t$ will be given by the σ algebra $\mathcal{F}_t$ or $\mathcal{G}_t$ depending on our analysis later. Let $[M] := [M, M]$ and $\langle M \rangle := \langle M, M \rangle$ be respectively the quadratic variation and the predictable compensator associated with a given square integrable martingale M. We define the following spaces;

- L^2_T: the space of all $\mathcal{H}_T$-measurable random variables $X : \Omega \to \mathbb{R}$ such that
$$\|X\|^2 = \mathbb{E}[X^2] < \infty.$$
- $H^2_{[0,T]}$: the space of all $\mathbb{H}$-predictable processes $\phi : \Omega \times [0, T] \to \mathbb{R}$, such that
$$\|\phi\|^2_{H^2_{[0,T]}} = \mathbb{E}\left[\int_0^T |\phi(t)|^2\, \mathrm{d}t \right] < \infty.$$
- $S^2_{[0,T]}$: the space of all $\mathbb{H}$-adapted, càdlàg processes $\gamma : \Omega \times [0, T] \to \mathbb{R}$ such that
$$\|\gamma\|^2_{S^2_{[0,T]}} = \mathbb{E}\left[\sup_{0 \le t \le T} |\gamma^2(t)| \right] < \infty.$$

- $\hat{H}^2_{[0,T]}$: the space of all $\mathbb{H}$-predictable mappings $\theta : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, such that

$$\|\theta\|^2_{\hat{H}^2_{[0,T]}} = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\theta(t,z)|^2 \, \ell(\mathrm{d}z)\, \mathrm{d}t\right] < \infty.$$

- $\tilde{H}^2_{[0,T]}$: the space of all $\mathbb{H}$-adapted processes $\phi : \Omega \times [0,T] \to \mathbb{R}$, such that

$$\|\phi\|^2_{\tilde{H}^2_{[0,T]}} = \mathbb{E}\left[\int_0^T |\phi(t)|^2 \, \mathrm{d}\langle M\rangle_t\right] < \infty.$$

- $\hat{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \ell)$: the space of all $\mathcal{B}(\mathbb{R})$-measurable mappings $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\|\psi\|^2_{\hat{L}^2(\mathbb{R},\mathcal{B}(\mathbb{R}),\ell)} = \int_{\mathbb{R}} |\psi(z)|^2 \, \ell(\mathrm{d}z) < \infty\,.$$

- $\mathcal{L}^2_{[0,T]}$: the space of L^2-bounded martingales M such that

$$\mathbb{E}\left[\int_0^T \mathrm{d}[M]_t\right] < \infty\,.$$

Consider three functions f, φ, and h to which we impose the following assumptions:

Assumptions 1.
(A) $f : [0,T] \times \mathbb{R}^3 \times \hat{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \ell) \to \mathbb{R}$ *is such that*

- $f(\cdot, 0, 0, 0, 0) \in H^2_{[0,T]}$,
- f *satisfies a uniform Lipschitz condition in* $(\cdot, u, v, w, \Gamma)$, *i.e. there exists a constant* C *such that for all* $(u_i, v_i, w_i, \Gamma_i) \in \mathbb{R}^3 \times \hat{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \ell)$, $i = 1, 2$ *we have*

$$\begin{aligned}&|f(t, u_1, v_1, w_1, \Gamma_1) - f(t, u_2, v_2, w_2, \Gamma_2)| \\ &\qquad \le C\Big(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2| + \|\Gamma_1 - \Gamma_2\|\Big), \quad \textit{for all } t \in [0,T]\,.\end{aligned}$$

(B) $\varphi : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ *is such that*

- $\varphi(\cdot, 0, 0, 0) \in \tilde{H}^2_{[0,T]}$,
- φ *satisfies a Lipschitz condition in* (t, u, v, w), *i.e. there exists a constant* C *such that for all* $(t_i, u_i, v_i, w_i) \in [0,T] \times \mathbb{R}^3$, $i = 1, 2$ *we have*

$$\begin{aligned}&|\varphi(t_1, u_1, v_1, w_1) - \varphi(t_2, u_2, v_2, w_2)| \\ &\qquad \le C\Big(|t_1 - t_2| + |u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|\Big)\,.\end{aligned}$$

(C) $h : \mathbb{R} \to \mathbb{R}$ *is Lipschitz and* $h(0) = 0$.

In Sections 2 - 4, we consider $\mathbb{H} = \mathbb{F}$. We define a process M as follows

$$M(t) := \int_0^t b(s)\,\mathrm{d}W(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(\mathrm{d}s, \mathrm{d}z)\,, \tag{2.1}$$

where $b(t), \gamma(t,z) \in \mathbb{R}$, for $t \geq 0,\ z \in \mathbb{R}$. Moreover, we assume that

$$\gamma(t,z) = g(z)\widetilde{\gamma}(t). \tag{2.2}$$

In order to ensure that the process M has a finite second moment, we impose for some $\varepsilon > 0$,

$$G^2(\varepsilon) := \int_{|z|\leq\varepsilon} g^2(z)\,\ell(dz) < \infty\,. \tag{2.3}$$

We consider the following classical stochastic differential equation

$$S(t) = S(0) + \int_0^t S(s)a(s)\,\mathrm{d}s + \int_0^t S(s-)\,\mathrm{d}M(s)\,, \tag{2.4}$$

where $a(t) \in \mathbb{R}$ and $S(0)$ is a positive constant. For S to be positive, we assume $\gamma(t,z) > -1$ for $(t\,,z\,) \in [0,T] \times \mathbb{R}$.

We then consider the associated backward stochastic differential equation

$$V(t) = h(S(T)) + \int_t^T \varphi(s, S(s), V(s), \Upsilon(s))\,\mathrm{d}\langle M\rangle_s - \int_t^T \Upsilon(s)\,\mathrm{d}M(s) - L(T) + L(t)\,, \tag{2.5}$$

where φ and h satisfy respectively Assumptions 1 (B) and (C). A solution to the BSDE with jumps (BSDEJ) of type (2.5) is a triplet $(V, \Upsilon, L) \in \tilde{H}^2_{[0,T]} \times \tilde{H}^2_{[0,T]} \times \mathcal{L}^2_{[0,T]}$ such that Υ is predictable and $[M, L]$ is a local $\mathbb{P}$-martingale. The existence and uniqueness of the solution to (2.5) are well studied in Carbone et al. [6] for general square integrable martingales M not necessarily of the form we impose. We mention that their study does not include the dependence on S in the functions h and φ. However its generalisation to the FBSDE (2.4)-(2.5) is straightforward.

Since $L(T)$ is an $\mathcal{F}_T$-measurable square integrable random variable, then applying the representation theorem (see Kunita and Watanabe [11]) and the fact that $\mathbb{E}[L(T)] = \mathbb{E}[L(0)] = 0$, we get the following representation for the process L

$$L(t) = \int_0^t P(s)\,\mathrm{d}W(s) + \int_0^t \int_{\mathbb{R}} Q(s,z)\,\tilde{N}(\mathrm{d}s, \mathrm{d}z)\,,$$

where $P \in H^2_{[0,T]}$ and $Q \in \hat{H}^2_{[0,T]}$ (see Section 3 in Di Nunno et al. [8] for more details).

In the sequel we aim at rewriting the BSDEJ (2.5) as a classical BSDEJ. Thus we consider first the following classical BSDEJ

(2.6)
$$\begin{cases} -\mathrm{d}U(t) = & f(t, S(t), U(t), Y(t), Z(t,\cdot))\,\mathrm{d}t - Y(t)\,\mathrm{d}W(t) - \int\limits_{\mathbb{R}} Z(t-,z)\,\tilde{N}(\mathrm{d}t,\mathrm{d}z)\,, \\ \quad U(T) = & h(S(T))\,, \end{cases}$$

where f and h fulfill Assumptions 1 (A) and (C). A solution to (2.6) is a triplet $(U,Y,Z) \in S^2_{[0,T]} \times H^2_{[0,T]} \times \hat{H}^2_{[0,T]}$. From Tang and Li [20], we know that this solution exists and is unique.

In the following proposition we rewrite (2.5) as a BSDEJ of type (2.6). We do not present the proof since it follows similar lines as the proof of Lemma 4.1 in Di Nunno et al. [8].

Proposition 2.1. *Define*

$$\kappa(t) := b^2(t) + \int\limits_{\mathbb{R}} \gamma^2(t,z)\,\ell(\mathrm{d}z)\,, \qquad t \in [0,T]\,. \tag{2.7}$$

Assume that

$$C_1 \leq \sqrt{\kappa(t)} \leq C_2\,, \qquad \textit{for all } t \in [0,T]\,, \tag{2.8}$$

where C_1 and C_2 are positive constants. Let V be given by (2.5). *Recall the processes P and Q in the decomposition of L* (5.2). *Then V satisfies a BSDEJ of type* (2.6)*, where*

$$\begin{aligned} Y(t) &= \Upsilon(t)b(t) + P(t)\,, \\ Z(t,z) &= \Upsilon(t)\gamma(t,z) + Q(t,z)\,, \\ f(t,u,v,w,\Gamma(\cdot)) &= \varphi\left(t,u,v,\phi\left(t,w,\Gamma\left(\cdot\right)\right)\right)\kappa(t)\,, \end{aligned} \tag{2.9}$$

with $\phi : [0,T] \times \mathbb{R} \times \hat{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \ell) \to \mathbb{R}$ is such that

$$\phi(t,w,\Gamma(\cdot)) = \frac{1}{\kappa(t)}\left(b(t)w + \int\limits_{\mathbb{R}} \Gamma(z)\gamma(t,z)\,\ell(\mathrm{d}z)\right)\,. \tag{2.10}$$

Let C^1_b denote the space of continuously differentiable functions with bounded derivatives and π_n be the regular time grid

$$\pi_n := \{0 = t_0, \ldots, t_i, \ldots, t_n = T\}\,, \qquad t_i = \frac{iT}{n}\,, \qquad n \in \mathbb{N}\,.$$

We introduce the following estimate which we need for our convergence study later.

Theorem 2.2. *Let V and Υ be as in* (2.5)*. Assume* (2.8) *holds, a, b, and $\tilde{\gamma}$ defined in* (2.2) *are Lipschitz, and the functions h in* (2.5) *and f in* (2.9) *are C_b^1. Then*

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |V(t)-V(t_i)|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\Upsilon(s)-\Upsilon(t_i)|^2 \,\mathrm{d}s\right] \leq \frac{C}{n}\,,$$

where C is a positive constant.

Proof. Introduce the short hand notation

$$Z^\gamma(t) = \int_{\mathbb{R}} Z(t,z)\gamma(t,z)\,\ell(dz)\,.$$

The estimate for V follows immediately from Theorem 2.1 in Bouchard and Elie [4] and Proposition 2.1 in the present paper. For the estimate for Υ, we deduce from (2.10) that

$$\begin{aligned}
|\Upsilon(s)-\Upsilon(t_i)| &= \left|\frac{1}{\kappa(s)}\left(b(s)Y(s)+Z^\gamma(s)\right) - \frac{1}{\kappa(t_i)}\left(b(t_i)Y(t_i)+Z^\gamma(t_i)\right)\right| \\
&\leq \left|\frac{b(s)}{\kappa(s)}Y(s) - \frac{b(t_i)}{\kappa(t_i)}Y(t_i)\right| + \left|\frac{1}{\kappa(s)}Z^\gamma(s) - \frac{1}{\kappa(t_i)}Z^\gamma(t_i)\right|\,. \qquad (2.11)
\end{aligned}$$

We estimate the first term in the last equation to arrive at

$$\begin{aligned}
&\left|\frac{b(s)}{\kappa(s)}Y(s) - \frac{b(t_i)}{\kappa(t_i)}Y(t_i)\right| \\
&\quad\leq \left|\frac{b(s)}{\kappa(s)}Y(s) - \frac{b(t_i)}{\kappa(t_i)}Y(s)\right| + \left|\frac{b(t_i)}{\kappa(t_i)}Y(s) - \frac{b(t_i)}{\kappa(t_i)}Y(t_i)\right| \\
&\quad\leq \left|\frac{b(t_i)}{\kappa(t_i)}\right| |Y(s)-Y(t_i)| + |Y(s)| \left|\frac{b(s)\kappa(t_i)-b(t_i)\kappa(s)}{\kappa(s)\kappa(t_i)}\right| \\
&\quad\leq C\,|Y(s)-Y(t_i)| + C'\,|Y(s)|\,|b(s)-b(t_i)| + \tilde{C}\,|Y(s)|\,|\kappa(s)-\kappa(t_i)|\,,
\end{aligned}$$

where in the latter we used the fact that $\kappa(t)$ is bounded below and above uniformly in t by a constant and that $b(t)$ is bounded above uniformly in t by a constant. Using the fact that b and $\tilde{\gamma}$ are Lipschitz and bounded above uniformly in t, which implies that κ is also Lipschitz, we further get

$$\begin{aligned}
&\left|\frac{b(s)}{\kappa(s)}Y(s) - \frac{b(t_i)}{\kappa(t_i)}Y(t_i)\right|^2 \\
&\quad\leq C\,|Y(s)-Y(t_i)|^2 + C'\,|Y(s)|^2\,|b(s)-b(t_i)|^2 + \tilde{C}\,|Y(s)|^2\,|\kappa(s)-\kappa(t_i)|^2 \\
&\quad\leq C\,|Y(s)-Y(t_i)|^2 + \frac{K}{n^2}\,|Y(s)|^2\,.
\end{aligned}$$

Integrating, summing up, and taking the expectation, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|\frac{b(s)}{\kappa(s)}Y(s)-\frac{b(t_i)}{\kappa(t_i)}Y(t_i)\right|^2 \mathrm{d}s\right] \\
&\leq C\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Y(s)-Y(t_i)|^2 \mathrm{d}s\right]+\frac{K}{n^2}\mathbb{E}\left[\int_0^T |Y(s)|^2 \mathrm{d}s\right] \qquad (2.12)\\
&\leq \frac{C}{n}\,.
\end{aligned}
$$

In the last step we invoked Proposition 4.5 and Remark 2.7 in Bouchard and Elie [4], and we relied on the boundedness by a positive constant of the expectation in the second term in (2.12) (for a proof we refer to Lemma 3.3 in Di Nunno et al. [8]). For the second term in the last inequality of (2.11), we find using the boundedness below and above of κ uniformly in t

$$
\begin{aligned}
&\left|\frac{1}{\kappa(s)}Z^\gamma(s)-\frac{1}{\kappa(t_i)}Z^\gamma(t_i)\right|^2 \\
&\quad\leq C\left(|\kappa(s)-\kappa(t_i)|^2\,|Z^\gamma(s)|^2+|Z^\gamma(s)-Z^\gamma(t_i)|^2\right) \\
&\quad\leq CG(\infty)\left(|\tilde{\gamma}(s)|^2\,|\kappa(s)-\kappa(t_i)|^2+|\tilde{\gamma}(s)-\tilde{\gamma}(t_i)|^2\right)\int_{\mathbb{R}}|Z(s,z)|^2\,\ell(\mathrm{d}z) \\
&\qquad\quad + CG(\infty)|\tilde{\gamma}(t_i)|^2\int_{\mathbb{R}}|Z(s,z)-Z(t_i,z)|^2\,|\,\ell(\mathrm{d}z)\,,
\end{aligned}
$$

where $G(\infty)=\int_{\mathbb{R}} g^2(z)\,\ell(\mathrm{d}z)$ and the function g is as in (2.2). By the Lipschitz property of $\tilde{\gamma}$ and of κ and the boundedness of $\tilde{\gamma}$, we get

$$
\begin{aligned}
&\left|\frac{1}{\kappa(s)}Z^\gamma(s)-\frac{1}{\kappa(t_i)}Z^\gamma(t_i)\right|^2 \\
&\quad\leq \frac{K}{n^2}\int_{\mathbb{R}}|Z(s,z)|^2\,\ell(\mathrm{d}z)+C\int_{\mathbb{R}}|Z(s,z)-Z(t_i,z)|^2\,|\,\ell(\mathrm{d}z)\,.
\end{aligned}
$$

The statement of the theorem follows using Lemma 3.3 in Di Nunno et al. [8] and Corollary 4.1 and Remark 2.7 in Bouchard and Elie [4]. □

3. Discrete-time model

We consider the set up (2.4), (2.5) but now in discrete time. Throughout this paper we shall use the notation

$$
\Delta H(t_{i+1})=H(t_{i+1})-H(t_i)\,,\quad i=0,\dots,n-1\,,
$$

for any process H.

Time-discretisation of the forward equation. The discrete-time version of the process S is denoted by $\hat{S}$ and defined as

$$\hat{S}(t_{i+1}) := \hat{S}(t_i) + \hat{S}(t_i)a(t_i)\,\Delta t_i + \hat{S}(t_i)\,\Delta\hat{M}(t_i) \quad i = 0,\dots,n-1\,, \quad \hat{S}(0) = S(0)\,, \tag{3.1}$$

where

$$\Delta\hat{M}(t_{i+1}) := \int_{t_i}^{t_{i+1}} b(t_i)\,\mathrm{d}W(t) + \int_{t_i}^{t_{i+1}}\int_{\mathbb{R}} g(z)\tilde{\gamma}(t_i)\,\tilde{N}(\mathrm{d}t,\mathrm{d}z)\,, \tag{3.2}$$

for a, b, and $\tilde{\gamma}$ as in (2.1)-(2.4)

Time-discretisation of the backward equation. A discrete-time version of the process V in (2.5) is given by

$$\begin{cases} \hat{V}(t_i) = \hat{V}(t_{i+1}) + \varphi(t_i, \hat{S}(t_i), \hat{V}(t_i), \hat{\Upsilon}(t_i))\,\Delta\langle\hat{M}\rangle_{t_{i+1}} - \hat{\Upsilon}(t_i)\,\Delta\hat{M}(t_{i+1}) - \Delta\hat{L}(t_{i+1})\,, \\ \hat{V}(T) = h(\hat{S}(T))\,, \end{cases} \tag{3.3}$$

where φ and h verify respectively Assumptions 1 (B) and (C). The solution to the latter BSDEJ is a triplet of discrete processes $(\hat{V}(t_i), \hat{\Upsilon}(t_i), \hat{L}(t_i)) \in \tilde{H}^2_{[0,T]} \times \tilde{H}^2_{[0,T]} \times \mathcal{L}^2_{[0,T]}$, $i = 0, ..., n-1$, such that $(\hat{\Upsilon}(t_i))_{\{0\le i\le n\}}$ is predictable, $[\hat{M}, \hat{L}]$ is a local $\mathbb{P}$-martingale, and the mentioned spaces are meant in a discrete setting. Existence and uniqueness of the solution to (3.3) follows from Carbone et al. [6]. Notice that the process $\hat{L}$ in (3.3) is necessary for the existence of the solution since the predictable representation property does not hold in the discrete case (see e.g. Chapter 4 in Protter [18]). Moreover we cannot write this BSDEJ as a time-discrete BSDEJ driven only by a Brownian motion and jumps as we did in the continuous case in Proposition (2.1).

To derive an algorithm from the Euler scheme (3.3), we first take the expectation conditionally on $\mathcal{F}_{t_i}$ on both sides in (3.3) to arrive at the expression for $\hat{V}(t_i)$ (second equation in (3.4)). Then we multiply both sides in (3.3) by $\Delta\hat{M}(t_{i+1})$, take conditional expectation with respect to $\mathcal{F}_{t_i}$, and solve for $\hat{\Upsilon}(t_i)$ using (3.3) and the fact that $[\hat{M}, \hat{L}]$ is a local $\mathbb{P}$-martingale, to find the system for $i = n-1, \dots, 0$

$$\begin{cases} \hat{\Upsilon}(t_i) = \dfrac{n}{T\kappa(t_i)}\mathbb{E}\left[\hat{V}(t_{i+1})\left(b(t_i)\,\Delta W(t_{i+1}) + \displaystyle\int_{\mathbb{R}} \gamma(t_i, z)\tilde{N}\left((t_i, t_{i+1}], \mathrm{d}z\right)\right) \mid \mathcal{F}_{t_i}\right]\,, \\ \hat{V}(t_i) = \mathbb{E}[\hat{V}(t_{i+1})|\mathcal{F}_{t_i}] + \Delta t_{i+1}\varphi(t_i, \hat{S}(t_i), \hat{V}(t_i), \hat{\Upsilon}(t_i))\kappa(t_i)\,, \end{cases} \tag{3.4}$$

where κ is as in (2.7). As for $\hat{L}$, being zero at zero, we have from (3.3)

$$\hat{L}(T) = \hat{V}(T) - \sum_{i=0}^{n-1} \hat{\Upsilon}(t_i)\,\Delta\hat{M}(t_{i+1}) + \sum_{i=0}^{n-1} \varphi\left(t_i, \hat{S}(t_i), \hat{V}(t_i), \hat{\Upsilon}(t_i)\right)\kappa(t_i)\,\Delta t_{i+1} - \hat{V}(0). \tag{3.5}$$

Observe that this algorithm involves the computation of conditional expectations. For a numerical computation or approximation we refer e.g. to Daveloose et al. [9] and Longstaff and Schwartz [13].

4. L^2-convergence of the discretisation scheme

Convergence of the forward equation. In the following theorem we state the discretisation error of the approximation of (2.4) by (3.1). We refer to Platen [17] for a proof.

Theorem 4.1. *Recall the dynamics of S and $\hat{S}$ as in* (2.4) *and* (3.1) *respectively. Assume that the adapted processes a, b, and $\tilde{\gamma}$ are Lipschitz continuous in t. Then we have*

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|S(t)-\hat{S}(t_i)\right|^2\right] \leq \frac{C}{n}\,,$$

for a positive constant C independent of the number of steps.

Convergence of the backward equation. To study the convergence of the time-discrete scheme, we consider a continuous-time version of the process $\hat{M}$ (3.2) as follows

$$M_1(t) = M_1(t_i) + \int_{t_i}^{t} b(t_i)\mathrm{d}W(s) + \int_{t_i}^{t}\int_{\mathbb{R}} \gamma(t_i,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,, \qquad M(0)=0\,.$$

Now consider the $\mathcal{F}_{t_{i+1}}$-measurable random variable

$$\xi(t_{i+1}) := \hat{V}(t_{i+1}) + \int_{t_i}^{t_{i+1}} \varphi(t_i,\hat{S}(t_i),\hat{V}(t_i),\hat{\Upsilon}(t_i))\kappa(t_i)\,\mathrm{d}s\,.$$

We know from the GKW decomposition (see, e.g., Ansel and Stricker [1]) that there exists a predictable process $\Upsilon_1 \in \tilde{H}^2_{[t_i,t_{i+1}]}$ such that

$$\xi(t_{i+1}) = \mathbb{E}\left[\xi(t_{i+1})\,|\mathcal{F}_{t_i}\right] + \int_{t_i}^{t_{i+1}} \Upsilon_1(s)\,\mathrm{d}M_1(s) + \Delta L_1(t_{i+1})\,,$$

where L_1 is a square integrable $\mathbb{P}$-martingale such that $[M_1,L_1]$ is a local $\mathbb{P}$-martingale. From the latter equation and the second equality in (3.4), we deduce

(4.1)

$$\hat{V}(t_{i+1}) = \hat{V}(t_i) - \int_{t_i}^{t_{i+1}} \varphi(t_i,\hat{S}(t_i),\hat{V}(t_i),\hat{\Upsilon}(t_i))\kappa(t_i)\,\mathrm{d}s + \int_{t_i}^{t_{i+1}} \Upsilon_1(s)\,\mathrm{d}M_1(s) + \Delta L_1(t_{i+1})\,.$$

We define a continuous version of $\hat{V}$ as follows

$$\hat{V}(t) := \hat{V}(t_{i+1}) + \int_{t}^{t_{i+1}} \varphi(t_i,\hat{S}(t_i),\hat{V}(t_i),\hat{\Upsilon}(t_i))\kappa(t_i)\,\mathrm{d}s - \int_{t}^{t_{i+1}} \Upsilon_1(s)\,\mathrm{d}M_1(s)$$

$$-L_1(t_{i+1}) + L_1(t)\,. \tag{4.2}$$

The latter is an "intermediate" time-continuous BSDEJ which is needed for the convergence study later on. Since we are in a time-continuous setting, we can apply the classical martingale representation to L_1 to find

$$L_1(t_{i+1}) = L_1(t) + \int_t^{t_{i+1}} P_1(s)\,\mathrm{d}W(s) + \int_t^{t_{i+1}}\int_{\mathbb{R}} Q_1(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,,$$

where $P_1 \in H^2_{[t_i,t_{i+1}]}$ and $Q_1 \in \hat{H}^2_{[t_i,t_{i+1}]}$. Substitution the latter in (4.2) for the continuous version of $\hat{V}$ leads to

$$\begin{aligned}\hat{V}(t) :=\ & \hat{V}(t_{i+1}) + \int_t^{t_{i+1}} \varphi(t_i,\hat{S}(t_i),\hat{V}(t_i),\hat{\Upsilon}(t_i))\kappa(t_i)\,\mathrm{d}s - \int_t^{t_{i+1}} Y_1(t_i,s)\,\mathrm{d}W(s)\\ & - \int_t^{t_{i+1}}\int_{\mathbb{R}} Z_1(t_i,s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned} Y_1(t_i,s) =\ & \Upsilon_1(s)b(t_i) + P_1(s)\,,\\ Z_1(t_i,s,z) =\ & \Upsilon_1(s)\gamma(t_i,z) + Q_1(s,z)\,.\end{aligned} \tag{4.4}$$

Using the fact that $[M_1, L_1]$ is a local $\mathbb{P}$-martingale, we deduce

$$\Upsilon_1(s) = \frac{1}{\kappa(t_i)}\left(b(t_i)Y_1(t_i,s) + \int_{\mathbb{R}} Z_1(t_i,s,z)\gamma(t_i,z)\,\ell(\mathrm{d}z)\right)\,,\ t_i \le s \le t_{i+1}\,, i = 0,...,n-1\,. \tag{4.5}$$

Multiplying by $\Delta M_1(t_{i+1})$ in both sides in (4.1) and taking conditional expectation with respect to $\mathcal{F}_{t_i}$, we obtain

$$\mathbb{E}\left[\hat{V}(t_{i+1})\Delta M_1(t_{i+1})\,|\mathcal{F}_{t_i}\right] = \kappa(t_i)\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Upsilon_1(s)\,\mathrm{d}s\,|\mathcal{F}_{t_i}\right]\,,$$

where κ is as in (2.7). Comparing the latter to the first equality in (3.4), we get

$$\hat{\Upsilon}(t_i) = \frac{n}{T}\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Upsilon_1(s)\,\mathrm{d}s\,|\mathcal{F}_{t_i}\right]\,.$$

In the following two propositions, we compute estimates which we use later in the proofs of the convergence results.

Proposition 4.2. *Let Υ and Υ_1 be respectively as in* (2.5) *and* (4.1)*. Assume* (2.8) *holds and a, b, and $\tilde{\gamma}$ are Lipschitz. Introduce the notation*

$$\delta Y(t) = Y(t) - Y_1(t_i, t)\,, \qquad \delta Z(t,z) = Z(t,z) - Z_1(t_i, t, z)\,, \tag{4.6}$$

where Y, Z are as in (2.6) *and Y_1, Z_1 are as in* (4.4)*. Then*

$$\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon_1(s)|^2\right] \mathrm{d}s \le \frac{K}{n^2} + C \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\delta Y(s)|^2\right] \mathrm{d}s + C \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} \mathbb{E}\left[|\delta Z(s,z)|^2\right] \ell(\mathrm{d}z)\,\mathrm{d}s\,,$$

where K and C are positive constants.

Proof. Introduce the short hand notation

$$Z_1^\gamma(t_i, t) = \int_{\mathbb{R}} Z_1(t_i, t, z)\gamma(t_i, z)\,\ell(\mathrm{d}z)\,.$$

From (2.10) and (4.5), we have

$$\begin{aligned}
&|\Upsilon(s) - \Upsilon_1(s)| \\
&\qquad = \left| \frac{1}{\kappa(s)} \left(b(s)Y(s) + Z^\gamma(s)\right) - \frac{1}{\kappa(t_i)} \left(b(t_i)Y_1(t_i, s) + Z_1^\gamma(t_i, s)\right) \right|\,.
\end{aligned}$$

Then proceeding as in the proof of Theorem 2.2, we get

$$\begin{aligned}
\left| \frac{b(s)}{\kappa(s)} Y(s) - \frac{b(t_i)}{\kappa(t_i)} Y_1(t_i, s) \right|^2 &\le C\,|Y(s) - Y_1(t_i, s)|^2 + \frac{K}{n^2}\,|Y(s)|^2\,, \\
\left| \frac{1}{\kappa(s)} Z^\gamma(s) - \frac{1}{\kappa(t_i)} Z_1^\gamma(t_i, s) \right|^2 &\le C \int_{\mathbb{R}} |Z(s,z) - Z(t_i, s, z)|^2 \,|\, \ell(\mathrm{d}z) + \frac{K}{n^2} \int_{\mathbb{R}} |Z(s,z)|^2\,\ell(\mathrm{d}z)\,,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon_1(s)|^2\right] \mathrm{d}s \le{}& C \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\delta Y(s)|^2\right] \mathrm{d}s + \tilde{C} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} \mathbb{E}\left[|\delta Z(s,z)|^2\right]\,\ell(\mathrm{d}z)\,\mathrm{d}s \\
&+ \frac{K}{n^2} \left(\int_0^T \mathbb{E}\left[|Y(s)|^2\right] \mathrm{d}s + \int_0^T \int_{\mathbb{R}} \mathbb{E}\left[|Z(s,z)|^2\right]\,\ell(\mathrm{d}z)\,\mathrm{d}s \right)\,.
\end{aligned}$$

The statement follows from the boundedness by a positive constant of the integrals in the third and fourth term in the right hand side of the latter inequality (for a proof we refer to Lemma 3.3 in Di Nunno et al. [8]). □

Proposition 4.3. *Let Υ and $\hat{\Upsilon}$ be respectively as in* (2.5) *and* (3.3)*. Assume* (2.8) *holds and a, b, and $\tilde{\gamma}$ are Lipschitz. Then we have*

$$\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\hat{\Upsilon}(t_i) - \Upsilon(s)|^2\right] \mathrm{d}s \le \frac{K}{n^2} + C\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\delta Y(s)|^2\right] \mathrm{d}s + C\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}} \mathbb{E}\left[|\delta Z(s,z)|^2\right] \ell(\mathrm{d}z)\,\mathrm{d}s$$
$$+ C\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon(t_i)|^2\right] \mathrm{d}s\,,$$

where K and C are positive constants and δY and δZ are as in (4.6)*.*

Proof. We introduce $\Upsilon^\star$ as follows

$$\Upsilon^\star(t_i) := \frac{n}{T}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Upsilon(s)\,\mathrm{d}s\,|\mathcal{F}_{t_i}\right],$$

where Υ is as in (2.5). Then we have

$$\begin{aligned}&\mathbb{E}\left[|\hat{\Upsilon}(t_i) - \Upsilon(s)|^2\right]\\ &\quad\le 3\left(\mathbb{E}\left[|\hat{\Upsilon}(t_i) - \Upsilon^\star(t_i)|^2\right] + \mathbb{E}\left[|\Upsilon^\star(t_i) - \Upsilon(t_i)|^2\right] + \mathbb{E}\left[|\Upsilon(t_i) - \Upsilon(s)|^2\right]\right).\end{aligned}$$

As for the first and second terms in the latter equation, we get using Jensen's and Cauchy-Schwartz inequalities

$$\mathbb{E}\left[|\hat{\Upsilon}(t_i) - \Upsilon^\star(t_i)|^2\right] \le \frac{n}{T}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon_1(s) - \Upsilon(s)|^2\right] \mathrm{d}s$$

$$\mathbb{E}\left[|\Upsilon^\star(t_i) - \Upsilon(t_i)|^2\right] \le \frac{n}{T}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon(t_i)|^2\right] \mathrm{d}s\,.$$

Thus

$$\begin{aligned}&\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\hat{\Upsilon}(t_i) - \Upsilon(s)|^2\right] \mathrm{d}s\\ &\quad\le \frac{nC}{T}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon_1(s) - \Upsilon(s)|^2\right] \mathrm{d}s\,\mathrm{d}r + \frac{nC}{T}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon(t_i)|^2\right] \mathrm{d}s\,\mathrm{d}r\\ &\qquad + C\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon(t_i)|^2\right] \mathrm{d}s\end{aligned}$$

$$= C\left(\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon_1(s)-\Upsilon(s)|^2\right]\mathrm{d}s + \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s)-\Upsilon(t_i)|^2\right]\mathrm{d}s\right).$$

Applying Proposition 4.2, the statement follows. ☐

In the following theorem we compute the rate of convergence of the solution of (4.3) to the solution of (2.5) in a space we specify.

Theorem 4.4. *Assume the conditions of Theorem 2.2 hold and $\varphi(0,0,0,0)=0$. Let the triplets (V,Y,Z) and $(\hat V, Y_1, Z_1)$ be respectively the solutions of (2.6) and (4.3). It holds*

$$\max_{i<n}\mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]}\left|V(t)-\hat V(t)\right|^2\right] + \sum_{i=0}^{n-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Y(s)-Y_1(t_i,s)|^2\,\mathrm{d}s\right]$$

$$+\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}} |Z(s,z)-Z(t_i,s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right] \le \frac{C}{n}, \tag{4.7}$$

where C is a positive constant.

Proof. Set $\delta V(t) = V(t)-\hat V(t)$, $\delta\kappa(t) = \kappa(t)-\kappa(t_i)$, $\delta\varphi(s) = \varphi(s,S(s),V(s),\Upsilon(s)) - \varphi(t_i,\hat S(t_i),\hat V(t_i),\hat\Upsilon(t_i))$. Recall the notations δY and δZ in (4.6). In the sequel, C denotes a positive constant independent of i and n and may take different values from line to line. Applying Itô's Lemma, we get

(4.8)

$$A(t) := \mathbb{E}[|\delta V(t)|^2] - \mathbb{E}[|\delta V(t_{i+1})|^2] + \mathbb{E}\left[\int_t^{t_{i+1}} |\delta Y(s)|^2\,\mathrm{d}s\right] + \mathbb{E}\left[\int_t^{t_{i+1}}\int_{\mathbb{R}} |\delta Z(s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right]$$

$$= \mathbb{E}\left[\int_t^{t_{i+1}} 2\delta V(s)\left(\varphi(s,S(s),V(s),\Upsilon(s))\kappa(s) - \varphi(t_i,\hat S(t_i),\hat V(t_i),\hat\Upsilon(t_i))\kappa(t_i)\right)\mathrm{d}s\right]$$

$$\le \mathbb{E}\left[\int_t^{t_{i+1}} 2\delta V(s)\delta\varphi(s)\kappa(t_i)\,\mathrm{d}s\right] + \mathbb{E}\left[\int_t^{t_{i+1}} 2\delta V(s)\varphi\left(s,S(s),V(s),Y(s)\right)\delta\kappa(s)\,\mathrm{d}s\right].$$

Using $2ab \le \alpha a^2 + b^2/\alpha$, for some $\alpha>0$, Assumptions 1 (B), the fact that κ is bounded above uniformly in t, and the fact that b and $\tilde\gamma$ are Lipschitz, we get

$$A(t) \le \alpha\mathbb{E}\left[\int_t^{t_{i+1}} |\delta V(s)|^2\,\mathrm{d}s\right] + \frac{CT}{n}\mathbb{E}\left[\int_t^{t_{i+1}} 2|\delta V(s)\varphi\left(s,S(s),V(s),Y(s)\right)|\,\mathrm{d}s\right]$$

$$+\frac{C}{\alpha}\mathbb{E}\left[\int_t^{t_{i+1}}\left(|t_i-s|^2+|S(s)-\hat{S}(t_i)|^2+|V(s)-\hat{V}(t_i)|^2+|\Upsilon(s)-\hat{\Upsilon}(t_i)|^2\right)\mathrm{d}s\right]$$

$$\le 2\alpha\mathbb{E}\left[\int_t^{t_{i+1}}|\delta V(s)|^2\,\mathrm{d}s\right]+\frac{C}{\alpha n^2}\mathbb{E}\left[\int_t^{t_{i+1}}|\varphi(s,S(s),V(s),Y(s))|^2\,\mathrm{d}s\right]$$

$$+\frac{C}{\alpha}\mathbb{E}\left[\int_t^{t_{i+1}}\left(\left(\frac{T}{n}\right)^2+|S(s)-\hat{S}(t_i)|^2+|V(s)-\hat{V}(t_i)|^2+|\Upsilon(s)-\hat{\Upsilon}(t_i)|^2\right)\mathrm{d}s\right]$$

(4.9)

$$\le 2\alpha\mathbb{E}\left[\int_t^{t_{i+1}}|\delta V(s)|^2\,\mathrm{d}s\right]+\frac{C}{\alpha n^2}\mathbb{E}\left[\int_0^{T}\left(|T|^2+|S(s)|^2+|V(s)|^2+|\Upsilon(s)|^2\right)\mathrm{d}s\right]$$

$$+\frac{C}{\alpha}\mathbb{E}\left[\int_t^{t_{i+1}}\left(\left(\frac{T}{n}\right)^2+|S(s)-\hat{S}(t_i)|^2+|V(s)-\hat{V}(t_i)|^2+|\Upsilon(s)-\hat{\Upsilon}(t_i)|^2\right)\mathrm{d}s\right].$$

Theorem 2.2 yields

$$\begin{aligned}\mathbb{E}\left[|V(s)-\hat{V}(t_i)|^2\right]&\le 2\left(\mathbb{E}\left[|V(s)-V(t_i)|^2\right]+\mathbb{E}\left[|V(t_i)-\hat{V}(t_i)|^2\right]\right)\\ &\le C\left(\frac{1}{n}+\mathbb{E}[|\delta V(t_i)|^2]\right).\end{aligned}\tag{4.10}$$

Hence, using the boundedness by a constant of the expectation in the second term in (4.9), Theorem 4.1, and (4.10) we get

(4.11)

$$A(t)\le C\alpha\mathbb{E}\left[\int_t^{t_{i+1}}|\delta V(s)|^2\,\mathrm{d}s\right]+\frac{C}{n^2}+\frac{C}{\alpha}\int_t^{t_{i+1}}\left(\frac{1}{n}+\mathbb{E}\left[|\delta V(t_i)|^2\right]+\mathbb{E}\left[|\Upsilon(s)-\hat{\Upsilon}(t_i)|^2\right]\right)\mathrm{d}s\,.$$

Recall the expression of A in (4.8). We deduce from (4.11)

$$\begin{aligned}\mathbb{E}[|\delta V(t)|^2]&\le\mathbb{E}[|\delta V(t)|^2]+\mathbb{E}\left[\int_t^{t_{i+1}}|\delta Y(s)|^2\,\mathrm{d}s\right]+\mathbb{E}\left[\int_t^{t_{i+1}}\int_{\mathbb{R}}|\delta Z(s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right]\\ &\le C\alpha\int_t^{t_{i+1}}\mathbb{E}\left[|\delta V(s)|^2\right]\mathrm{d}s+B_i\,,\end{aligned}\tag{4.12}$$

where

$$B_i = \frac{C}{n^2} + \mathbb{E}\left[|\delta V(t_{i+1})|^2\right] + \frac{C}{\alpha}\left(\frac{1}{n^2} + \frac{1}{n}\mathbb{E}\left[|\delta V(t_i)|^2\right] + \int_t^{t_{i+1}} \mathbb{E}\left[\left|\Upsilon(s) - \hat{\Upsilon}(t_i)\right|^2\right] \mathrm{d}s\right).$$

Thus applying Gronwall's lemma to (4.12), we get

$$\mathbb{E}\left[|\delta V(t)|^2\right] \le B_i \exp\left\{\frac{C\alpha}{n}\right\}, \qquad t_i \le t < t_{i+1}\,,\; i = 0, ..., n-1\,,$$

which plugged in (4.12), implies

(4.13)

$$\mathbb{E}[|\delta V(t)|^2] + \int_t^{t_{i+1}} \mathbb{E}\left[|\delta Y(s)|^2\right] \mathrm{d}s + \int_t^{t_{i+1}} \int_{\mathbb{R}} \mathbb{E}\left[|\delta Z(s,z)|^2\right] \ell(\mathrm{d}z)\, \mathrm{d}s \le B_i\left(1 + \alpha\frac{C}{n}\right).$$

Taking $t = t_i$ and applying Proposition 4.3, we get

$$\begin{aligned}
&\mathbb{E}[|\delta V(t_i)|^2] + \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\delta Y(s)|^2\, \mathrm{d}s\right] + \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\delta Z(s,z)|^2\, \ell(\mathrm{d}z)\, \mathrm{d}s\right] \\
&\le \left(1 + \alpha\frac{C}{n}\right)\left(\frac{C}{n^2} + \mathbb{E}\left[|\delta V(t_{i+1})|^2\right] + \frac{C}{n\alpha}\mathbb{E}\left[|\delta V(t_i)|^2\right] + \frac{C}{\alpha}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon(t_i)|^2\right] \mathrm{d}s\right. \\
&\quad \left. + \frac{C}{\alpha}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\delta Y(s)|^2\right] \mathrm{d}s + \frac{C}{\alpha}\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} \mathbb{E}\left[|\delta Z(s,z)|^2\right] \ell(\mathrm{d}z)\, \mathrm{d}s\right).
\end{aligned}$$

For α sufficiently larger than C, we deduce

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}[|\delta V(t_i)|^2] + \frac{1}{2}\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\delta Y(s)|^2\, \mathrm{d}s\right] + \frac{1}{2}\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\delta Z(s,z)|^2\, \ell(\mathrm{d}z)\, \mathrm{d}s\right] \\
&\le \left(1 + \frac{C}{n}\right)\left(\frac{C}{n^2} + \mathbb{E}\left[|\delta V(t_{i+1})|^2\right] + C\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \Upsilon(t_i)|^2\right] \mathrm{d}s\right).
\end{aligned}$$

Iterating the last inequality we get

$$\frac{1}{2}\,\mathbb{E}[|\delta V(t_i)|^2] + \frac{1}{2}\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\delta Y(s)|^2\, \mathrm{d}s\right] + \frac{1}{2}\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\delta Z(s,z)|^2\, \ell(\mathrm{d}z)\, \mathrm{d}s\right]$$

$$
\leq C\left(1+\frac{C}{n}\right)^n\left(\frac{1}{n}+\mathbb{E}\left[|\delta V(T)|^2\right]+\sum_{j=i}^{n-1}\int\limits_{t_j}^{t_{j+1}}\mathbb{E}\left[|\Upsilon(s)-\Upsilon(t_i)|^2\right]\mathrm{d}s\right). \tag{4.14}
$$

Using the estimates in Theorems 2.2 and 4.1 together with the Lipschitz property of h leads to

$$
\mathbb{E}\left[|\delta V(t_i)|^2\right]+\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}|\delta Y(s)|^2\,\mathrm{d}s\right]+\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}\int\limits_{\mathbb{R}}|\delta Z(s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right]\leq\frac{C}{n}. \tag{4.15}
$$

Taking $t=t_i$ in (4.13), summing up, and using Proposition 4.3, we arrive at

$$
\begin{aligned}
&\sum_{i=0}^{n-1}\left(\mathbb{E}[|\delta V(t_i)|^2]+\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}|\delta Y(s)|^2\,\mathrm{d}s\right]+\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}\int\limits_{\mathbb{R}}|\delta Z(s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right]\right)\\
&\quad\leq\sum_{i=0}^{n-1}\left[\left(1+\alpha\frac{C}{n}\right)\left(\frac{C}{n^2}+\mathbb{E}\left[\delta|V(t_{i+1})|^2\right]+\frac{C}{\alpha n}\mathbb{E}\left[|\delta V(t_i)|^2\right]\right.\right.\\
&\qquad+\frac{C}{\alpha}\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}|\delta Y(s)|^2\,\mathrm{d}s\right]+\frac{C}{\alpha}\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}\int\limits_{\mathbb{R}}|\delta Z(s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right]\\
&\qquad\left.\left.+\frac{C}{\alpha}\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}|\Upsilon(s)-\Upsilon(t_i)|^2\,\mathrm{d}s\right]\right)\right],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left[1-\frac{C}{\alpha}\left(1+\frac{\alpha C}{n}\right)\right]\sum_{i=0}^{n-1}\left(\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}|\delta Y(s)|^2\,\mathrm{d}s\right]+\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}\int\limits_{\mathbb{R}}|\delta Z(s,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right]\right)\\
&\quad\leq\left(1+\alpha\frac{C}{n}\right)\frac{C}{n}+C\left(1+\alpha\frac{C}{n}\right)\mathbb{E}\left[\delta V(T)|^2\right]+\left[\left(1+\alpha\frac{C}{n}\right)\frac{C}{\alpha n}-1\right]\mathbb{E}\left[\delta V(t_0)|^2\right]\\
&\qquad+\left[\left(1+\alpha\frac{C}{n}\right)\left(1+\frac{C}{\alpha n}\right)-1\right]\sum_{i=1}^{n-1}\mathbb{E}\left[\delta V(t_i)|^2\right]\\
&\qquad+C\left(1+\alpha\frac{C}{n}\right)\frac{C}{\alpha}\sum_{i=0}^{n-1}\mathbb{E}\left[\int\limits_{t_i}^{t_{i+1}}|\Upsilon(s)-\Upsilon(t_i)|^2\,\mathrm{d}s\right].
\end{aligned}
$$

Using (4.15) and again the Lipschitz property of h, the latter implies that for α sufficiently larger than C, we obtain

$$\begin{aligned} &\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\delta Y(s)|^2 \, \mathrm{d}s\right] + \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\delta Z(s,z)|^2 \, \ell(\mathrm{d}z) \, \mathrm{d}s\right] \\ &\leq C \left[\frac{1}{n} + \frac{1}{n} \sum_{i=1}^{n-1} \mathbb{E}\left[\delta V(t_i)|^2\right] + \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\Upsilon(s) - \Upsilon(t_i)|^2 \, \mathrm{d}s\right]\right) \end{aligned} \tag{4.16}$$

and the statement for the last two terms in (4.7) follows using (4.15) and Theorem 2.2. Finally, observe that

$$\begin{aligned} &\mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} |V(t) - \hat{V}(t)|^2\right] \\ &\leq K \Bigg(\mathbb{E}\left[|V(t_{i+1}) - \hat{V}(t_{i+1})|^2\right] \\ &\quad + \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \left|\varphi(s, S(s), V(s), \Upsilon(s))\kappa(s) - \varphi(t_i, \hat{S}(t_i), \hat{V}(t_i), \hat{\Upsilon}(t_i))\kappa(t_i)\right|^2 \mathrm{d}s\right] \\ &\quad + \mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} \left|\int_t^{t_{i+1}} \delta Y(s) \, \mathrm{d}W(s)\right|^2\right] + \mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} \left|\int_t^{t_{i+1}} \int_{\mathbb{R}} |\delta Z(s,z)|^2 \, \tilde{N}(\mathrm{d}s, \mathrm{d}z)\right|^2\right] \Bigg) . \end{aligned}$$

Then using Burkholder's inequality, the Lipschitz property of φ, and iterating as we did to get (4.14), we deduce the result applying (4.16). □

In the following theorem we compute a rate for the L^2-convergence of $\hat{\Upsilon}$ to Υ.

Theorem 4.5. *Assume the conditions of Theorem 2.2 hold and* $\varphi(0,0,0,0) = 0$. *Let* Υ *and* $\hat{\Upsilon}$ *be respectively as in* (2.5) *and* (3.3). *Then*

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon(s) - \hat{\Upsilon}(t_i)|^2\right] \mathrm{d}s \leq \frac{C}{n} .$$

Proof. Summing up both sides of the inequality in Proposition 4.3, the statement follows by invoking Theorem 2.2 and Theorem 4.4. □

We state a rate of convergence for the process $\hat{L}$ to L in the following theorem.

Theorem 4.6. *Assume the conditions of Theorem 2.2 hold and* $\varphi(0,0,0,0) = 0$. *Let the processes* L *and* $\hat{L}$ *be defined as in* (2.5) *and* (3.3). *Then we have for all* $0 \leq i \leq n-1$,

$$\mathbb{E}\left[\left|L(t_{i+1}) - \hat{L}(t_{i+1})\right|^2\right] \leq \frac{C}{n} ,$$

where C is a positive constant.

Proof. Applying the recursion in (3.3), we get

$$\hat{L}(t_{i+1}) = \hat{V}(t_{i+1}) - \hat{V}(0) + \sum_{k=0}^{i} \varphi(t_k, \hat{S}(t_k), \hat{V}(t_k), \hat{\Upsilon}(t_k))\kappa(t_k)\,\Delta t_{k+1}$$
$$- \sum_{k=0}^{i} \hat{\Upsilon}(t_k)b(t_k)\Delta W(t_{k+1}) - \sum_{k=0}^{i} \int_{\mathbb{R}} \hat{\Upsilon}(t_k)\gamma(t_k,z)\,\tilde{N}\big((t_k,t_{k+1}],\mathrm{d}z\big)\,,$$

while differentiation and integration over $[0,t]$ in (2.5) gives

$$L(t) = V(t) - V(0) - \int_0^t \varphi(s, S(s), V(s), \Upsilon(s))\kappa(s)\,\mathrm{d}s - \int_0^t \Upsilon(s)b(s)\,\mathrm{d}W(s)$$
$$- \int_0^t \int_{\mathbb{R}} \Upsilon(s)\gamma(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,.$$

Hence, we find

$$\mathbb{E}[|L(t_{i+1}) - \hat{L}(t_{i+1})|^2]$$
$$\leq C\mathbb{E}\left[\left|V(t_{i+1}) - \hat{V}(t_{i+1})\right|^2\right] + C\mathbb{E}\left[\left|V(0) - \hat{V}(0)\right|^2\right]$$

(4.17)

$$+ C\mathbb{E}\left[\left|\int_0^{t_{i+1}} \varphi(s, S(s), V(s), \Upsilon(s))\kappa(s)\,\mathrm{d}s - \sum_{k=0}^{i} \varphi(t_k, \hat{S}(t_k), \hat{V}(t_k), \hat{\Upsilon}(t_k))\kappa(t_k)\,\Delta t_{k+1}\right|^2\right]$$

(4.18)

$$+ C\mathbb{E}\left[\left|\int_0^{t_{i+1}} \Upsilon(s)b(s)\,\mathrm{d}W(s) - \sum_{k=0}^{i} \hat{\Upsilon}(t_k)b(t_k)\,\Delta W(t_{k+1})\right|^2\right]$$

(4.19)

$$+ C\mathbb{E}\left[\left|\int_0^{t_{i+1}} \int_{\mathbb{R}} \Upsilon(s)\gamma(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z) - \sum_{k=0}^{i} \int_{\mathbb{R}} \hat{\Upsilon}(t_k)\gamma(t_k,z)\,\tilde{N}\big((t_k,t_{k+1}],\mathrm{d}z\big)\right|^2\right]\,.$$

Define

$$\hat{\Upsilon}(s) = \sum_{k=0}^{n-1} \hat{\Upsilon}(t_k)\mathbf{1}_{[t_k,t_{k+1}]}(s)\,, \quad \hat{b}(s) = \sum_{k=0}^{n-1} b(t_k)\mathbf{1}_{[t_k,t_{k+1}]}(s)\,.$$

After applying the Itô isometry, we estimate the expectation in (4.18)

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^{t_{i+1}} \Upsilon(s)b(s)\,\mathrm{d}W(s) - \sum_{k=0}^{i} \hat{\Upsilon}(t_k)b(t_k)\,\Delta W(t_{k+1})\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_0^{t_{i+1}} \left(\Upsilon(s)b(s) - \hat{\Upsilon}(s)\hat{b}(s)\right)\mathrm{d}W(s)\right|^2\right] = \mathbb{E}\left[\int_0^{t_{i+1}} \left(\Upsilon(s)b(s) - \hat{\Upsilon}(s)\hat{b}(s)\right)^2 \mathrm{d}s\right] \\
&\leq 2\mathbb{E}\left[\int_0^{t_{i+1}} \Upsilon^2(s)\left(b(s) - \hat{b}(s)\right)^2 \mathrm{d}s\right] + 2\mathbb{E}\left[\int_0^{t_{i+1}} \left(\Upsilon(s) - \hat{\Upsilon}(s)\right)^2 \hat{b}^2(s)\,\mathrm{d}s\right] \\
&= 2\mathbb{E}\left[\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} \Upsilon^2(s)\left(b(s) - b(t_k)\right)^2 \mathrm{d}s\right] + 2\mathbb{E}\left[\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} \left(\Upsilon(s) - \hat{\Upsilon}(t_k)\right)^2 b^2(t_k)\,\mathrm{d}s\right].
\end{aligned}$$

The Lipschitzianity and boundedness of b leads to

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^{t_{i+1}} \Upsilon(s)b(s)\,\mathrm{d}W(s) - \sum_{k=0}^{i} \hat{\Upsilon}(t_k)b(t_k)\,\Delta W(t_{k+1})\right|^2\right] \\
&\qquad \leq C\left(\frac{T}{n}\right)^2 \mathbb{E}\left[\int_0^T |\Upsilon(s)|^2\,\mathrm{d}s\right] + C\sum_{k=0}^{i-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}} |\Upsilon(s) - \hat{\Upsilon}(t_k)|^2\,\mathrm{d}s\right] \\
&\qquad \leq \frac{C}{n},
\end{aligned}$$

where in the last step we applied Theorem 4.5 and the boundedness of the expectation in the first term. Using similar arguments, we can prove that the expectations in (4.17) and (4.19) are bounded above by C/n. Finally, the result follows invoking Theorem 4.4. □

5. A note on the infinite activity of the jumps

In the previous section we imposed that the jumps have finite mass in the tail. In this section we aim at completing this analysis by considering jumps with infinite activity. Therefore before discretising, we introduce an approximating FBSDJ where we replace the small jumps by an independent Brownian motion appropriately scaled. Then we introduce a discretisation of the approximating FBSEDJ and we study the convergence of the time-discrete equation to the original one in an L^2-sense.

The idea of approximating the small jumps by a properly scaled independent Brownian motion is motivated by a paper of Asmussen and Rosinski [2]. Another choice would be to truncate the small jumps or rescale the Brownian motion W such that the original process and the approximating one would have the same variance. All these suggested approximations contain a Brownian motion and a compound Poisson process which are both easy to simulate. In the present paper we only consider the first mentioned approximation. However we claim that similar results hold for the other types of approximations. We refer to Benth et al. [3] for the influence of the different approximations on the forward SDE (2.4) and to Di Nunno et al. [8] for the study of the influence of the different approximations to the BSDE (2.5).

5.1. **Approximating continuous-time model.** We consider M as in (2.1), but now where we allow for infinite activity of the jumps. That is we allow the Poisson random measure N to have an infinite mass in the tail, i.e. $\ell(|z| \leq 1) = \infty$. We approximate the small jumps in the martingale M by a Brownian motion B which we scale with the standard deviation of the small jumps. We obtain

$$M_\varepsilon(t) = \int_0^t b(s)\,\mathrm{d}W(s) + \int_0^t G(\varepsilon)\tilde{\gamma}(s)\,\mathrm{d}B(s) + \int_0^t \int_{|z|>\varepsilon} \gamma(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,.$$

From now on we enlarge the filtration $\mathbb{F}$ with the information of the Brownian motion B and thus we consider $\mathbb{H} = \mathbb{G}$. We define the process S_ε as follows

$$S_\varepsilon(t) = S_\varepsilon(0) + \int_0^t S_\varepsilon(s)a(s)\,\mathrm{d}s + \int_0^t S_\varepsilon(s-)\mathrm{d}M_\varepsilon(s)\,,$$

where $S_\varepsilon(0) = S(0)$. Notice that by scaling the Brownian motion with the standard deviation of the small jumps, both processes S and S_ε have the same variance for ε tending to 0. The associated BSDEJ is given by

(5.1)

$$V_\varepsilon(t) = h(S_\varepsilon(T)) + \int_t^T \varphi(s, S_\varepsilon(s), V_\varepsilon(s), \Upsilon_\varepsilon(s))\,\mathrm{d}\langle M_\varepsilon\rangle_s - \int_t^T \Upsilon_\varepsilon(s)\,\mathrm{d}M_\varepsilon(s) - L_\varepsilon(T) + L_\varepsilon(t)\,,$$

where $(V_\varepsilon, \Upsilon_\varepsilon, L_\varepsilon) \in \tilde{H}^2_{[0,T]} \times \tilde{H}^2_{[0,T]} \times \mathcal{L}^2_{[0,T]}$ such that Υ_ε is $\mathbb{G}$-predictable and $[M_\varepsilon, L_\varepsilon]$ is a local $\mathbb{P}$-martingale. From Carbone et al. [6], we know that the solution to (5.1) exists and is unique. Notice that $L_\varepsilon(T)$ is an $\mathcal{G}_T$-measurable square integrable random variable. Then applying the representation theorem (see Kunita and Watanabe [11]) to the process L_ε leads to

$$L_\varepsilon(t) = \int_0^t P_\varepsilon(s)\,\mathrm{d}W(s) + \int_0^t R_\varepsilon(s)\,\mathrm{d}B(s) + \int_0^t \int_{\mathbb{R}} Q_\varepsilon(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,, \tag{5.2}$$

where $P_\varepsilon, R_\varepsilon \in H^2_{[0,T]}$ and $Q_\varepsilon \in \hat{H}^2_{[0,T]}$ (see Section 3 in Di Nunno et al. [8] for more details). Let f_ε be a function that fulfills the following assumption

Assumption 1.
$f_\varepsilon : [0,T] \times \mathbb{R}^3 \times \hat{L}^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \ell) \times \mathbb{R} \to \mathbb{R}$ *is such that for all* $\varepsilon \in [0,1]$,

- $f_\varepsilon(\cdot, 0, 0, 0, 0, 0) \in H^2_{[0,T]}$,

- f_ε *satisfies a uniform Lipschitz condition in* $(\cdot, u, v, w, \Gamma, \zeta)$.

We define the classical BSDEJ

$$
\tag{5.3}
\begin{cases}
-\mathrm{d}U_\varepsilon(t) = & f_\varepsilon(t, S_\varepsilon(t), U_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t,\cdot), \zeta_\varepsilon(t))\,\mathrm{d}t - Y_\varepsilon(t)\,\mathrm{d}W(t) \\
& \qquad - \int_{\mathbb{R}} Z_\varepsilon(t,z)\,\tilde{N}(\mathrm{d}t,\mathrm{d}z) - \zeta_\varepsilon(t)\,\mathrm{d}B(t)\,, \\
U_\varepsilon(T) = & h(S_\varepsilon(T))\,.
\end{cases}
$$

Under Assumption 1 imposed on f_ε, we know from Tang and Li [20] that the solution to (5.3) given by $(U_\varepsilon, Y_\varepsilon, Z_\varepsilon, \zeta_\varepsilon) \in S^2_{[0,T]} \times H^2_{[0,T]} \times \hat{H}^2_{[0,T]} \times H^2_{[0,T]}$ exists and is unique. In the following proposition we study the relation of (5.1) to classical BSDEJ's. We do not present the proof since it follows similar arguments as the proof of Lemma 4.1 in Di Nunno et al. [8].

Proposition 5.1. *Assume that* (2.8) *holds. Let* $\tilde{V}_\varepsilon$ *be given by* (5.1)*. Then* $\tilde{V}_\varepsilon$ *satisfies a BSDEJ of type* (5.3)*, where*

$$
\begin{aligned}
Y_\varepsilon(t) &= \Upsilon_\varepsilon(t) b(t) + P_\varepsilon(t)\,, \quad \zeta_\varepsilon(t) = \Upsilon_\varepsilon(t) G(\varepsilon)\tilde{\gamma}(t) + R_\varepsilon(t)\,, \\
Z_\varepsilon(t,z) &= \Upsilon_\varepsilon(t)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z) + Q_\varepsilon(t,z)\,,
\end{aligned}
$$

$$
\tag{5.4}
f_\varepsilon(t,u,v,w,\Gamma(\cdot),\zeta) = \varphi\left(t,u,v,\phi_\varepsilon\left(t,w,\Gamma(\cdot),\zeta\right)\right)\kappa(t)\,,
$$

with $\phi_\varepsilon : [0,T] \times \mathbb{R} \times \hat{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \ell) \times \mathbb{R} \to \mathbb{R}$ *is such that*

$$
\phi_\varepsilon\left(t,w,\Gamma(\cdot),\zeta\right) = \frac{1}{\kappa(t)}\left(b(t)w + G(\varepsilon)\tilde{\gamma}(t)\zeta + \int_{\mathbb{R}} \Gamma(z)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)\,\ell(\mathrm{d}z)\right).
$$

The following theorem is an adaptation of Theorem 2.1 in Bouchard and Elie [4] to the BSDEJ (5.3). The proof follows similar lines as in their paper where we first use a Malliavin derivative with respect to the Brownian motion W to find an estimate for the integrand of the Brownian motion W and then a Malliavin derivative with respect to the Brownian motion B to find an estimate for the integrand of the Brownian motion B.

Theorem 5.2. *Assume* (2.8) *holds,* a*,* b*, and* $\tilde{\gamma}$ *defined in* (2.2) *are Lipschitz, and the functions* h *and* f_ε *in* (5.3) *are* C^1_b*. Let* $(U_\varepsilon, Y_\varepsilon, Z_\varepsilon, \zeta_\varepsilon)$ *be the solution of* (5.3)*. It holds*

$$
\begin{aligned}
&\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |U_\varepsilon(t) - U_\varepsilon(t_i)|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Y_\varepsilon(s) - Y_\varepsilon(t_i)|^2\,\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |\zeta_\varepsilon(s) - \zeta_\varepsilon(t_i)|^2\,\mathrm{d}s\right] + \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}} |Z_\varepsilon(s,z) - Z_\varepsilon(t_i,z)|^2\,\ell(\mathrm{d}z)\,\mathrm{d}s\right] \le \frac{C}{n}\,,
\end{aligned}
$$

where C *is a positive constant.*

Using the latter theorem, Proposition 5.1, and following similar steps as in Theorem 2.2, we deduce the next result.

Theorem 5.3. *Assume* (2.8) *holds,* a*,* b*, and* $\tilde{\gamma}$ *defined in* (2.2) *are Lipschitz, and the functions* h *in* (5.1) *and* f_ε *in* (5.4) *are* C_b^1*. Then for* V_ε *and* Υ_ε *in* (5.1) *it holds that*

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |V_\varepsilon(t) - V_\varepsilon(t_i)|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\Upsilon_\varepsilon(s) - \Upsilon_\varepsilon(t_i)|^2 \,\mathrm{d}s\right] \le \frac{C}{n},$$

where C *is a positive constant.*

5.2. **Approximating discrete-time model.** The discrete-time version of the process S_ε is denoted by $\hat{S}_\varepsilon$ and defined by

(5.5)
$$\hat{S}_\varepsilon(t_{i+1}) := \hat{S}_\varepsilon(t_i) + \hat{S}_\varepsilon(t_i)a(t_i)\,\Delta t_i + \hat{S}_\varepsilon(t_i)\Delta\,\hat{M}_\varepsilon(t_i) \quad i = 0,\dots,n-1, \quad \hat{S}_\varepsilon(0) = S(0),$$

where

$$\Delta\hat{M}_\varepsilon(t_{i+1}) := \int_{t_i}^{t_{i+1}} b(t_i)\,\mathrm{d}W(t) + \int_{t_i}^{t_{i+1}} G(\varepsilon)\tilde{\gamma}(t_i)\,\mathrm{d}B(t) + \int_{t_i}^{t_{i+1}} \int_{|z|>\varepsilon} g(z)\tilde{\gamma}(t_i)\,\tilde{N}(\mathrm{d}t,\mathrm{d}z),$$

for a, b, and $\tilde{\gamma}$ as in (2.1)-(2.4).

A discrete-time version of the process V_ε in (5.1) is given by

$$\begin{cases} \hat{V}_\varepsilon(t_i) = & \hat{V}_\varepsilon(t_{i+1}) + \varphi(t_i, \hat{S}_\varepsilon(t_i), \hat{V}_\varepsilon(t_i), \hat{\Upsilon}_\varepsilon(t_i))\,\Delta\langle\hat{M}_\varepsilon\rangle_{t_{i+1}} - \hat{\Upsilon}_\varepsilon(t_i)\,\Delta\hat{M}_\varepsilon(t_{i+1}) \\ & \qquad -\Delta\hat{L}_\varepsilon(t_{i+1}), \\ \hat{V}_\varepsilon(T) = & h(\hat{S}_\varepsilon(T)), \end{cases} \tag{5.6}$$

where $(\hat{\Upsilon}_\varepsilon(t_i))_{\{0\le i\le n\}}$ is predictable and $[\hat{M}_\varepsilon, \hat{L}_\varepsilon]$ is a local $\mathbb{P}$-martingale. We proceed as in Section 3 to obtain a backward scheme for (5.6) as follows

$$\begin{cases} \hat{\Upsilon}_\varepsilon(t_i) = & \dfrac{n}{T\kappa(t_i)}\mathbb{E}\left[\hat{V}_\varepsilon(t_{i+1})\,\Delta\hat{M}_\varepsilon(t_{i+1})\mid\mathcal{G}_{t_i}\right], \\ \hat{V}_\varepsilon(t_i) = & \mathbb{E}[\hat{V}_\varepsilon(t_{i+1})|\mathcal{G}_{t_i}] + \Delta t_{i+1}\varphi\left(t_i, \hat{S}_\varepsilon(t_i), \hat{V}_\varepsilon(t_i), \hat{\Upsilon}_\varepsilon(t_i)\right)\kappa(t_i), \\ \hat{L}_\varepsilon(T) = & \hat{V}_\varepsilon(T) - \displaystyle\sum_{i=0}^{n-1}\hat{\Upsilon}_\varepsilon(t_i)\,\Delta\hat{M}_\varepsilon(t_{i+1}) + \sum_{i=0}^{n-1}\varphi\left(t_i, \hat{S}_\varepsilon(t_i), \hat{V}_\varepsilon(t_i), \hat{\Upsilon}_\varepsilon(t_i)\right)\kappa(t_i)\,\Delta t_{i+1} \\ & -\hat{V}_\varepsilon(0). \end{cases}$$

where κ is as in (2.7).

5.3. **Convergence of the approximating discrete-time model.** We first study the convergence of the approximating forward equation as we proceeded in Section 4. Then we study the convergence of the backward scheme.
Convergence of the approximating forward equation. In the following theorem we state the error of the discrete approximating process (5.5) to (2.4).

Theorem 5.4. *Recall the dynamics of S and $\hat{S}_\varepsilon$ as in* (2.4) *and* (5.5) *respectively. Assume that the adapted processes a, b, and $\tilde{\gamma}$ are Lipschitz continuous in t. Then we have*

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|S(t) - \hat{S}_\varepsilon(t_i)\right|^2\right] \le \frac{C}{n} + \tilde{C}G^2(\varepsilon)\,,$$

for positive constants C and $\tilde{C}$ independent of the number of steps.

Proof. We have

$$\begin{aligned}
&\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|\hat{S}_\varepsilon(t_i) - S(t)\right|^2\right] \\
&\quad \le \max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|\hat{S}_\varepsilon(t_i) - S_\varepsilon(t)\right|^2\right] + \max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|S_\varepsilon(t_i) - S(t)\right|^2\right].
\end{aligned}$$

Thus the statement follows using an adaptation of Theorem 4.1 to the approximating processes S_ε and $\hat{S}_\varepsilon$ and Proposition 3.3 in Benth et al. [3]. □

Convergence of the approximating backward scheme. Following similar steps as in Section 4, we define a continuous-time version of $\hat{V}_\varepsilon$ as follows

$$\begin{aligned}
\hat{V}_\varepsilon(t) := \hat{V}_\varepsilon(t_{i+1}) + \int_t^{t_{i+1}} \varphi(t_i, \hat{S}_\varepsilon(t_i), \hat{V}_\varepsilon(t_i), \hat{\Upsilon}_\varepsilon(t_i))\kappa(t_i)\,\mathrm{d}s - \int_t^{t_{i+1}} \Upsilon_{1,\varepsilon}(s)\,\mathrm{d}M_{1,\varepsilon}(s) \\
- L_{1,\varepsilon}(t_{i+1}) + L_{1,\varepsilon}(t)\,,
\end{aligned}$$

where $\Upsilon_{1,\varepsilon} \in \tilde{H}^2_{[t_i,t_{i+1}]}$ is predictable, $[M_{1,\varepsilon}, L_{1,\varepsilon}]$ is a local $\mathbb{P}$-martingale, and

$$M_{1,\varepsilon}(t) = M_{1,\varepsilon}(t_i) + \int_{t_i}^{t} b(t_i)\,\mathrm{d}W(s) + \int_{t_i}^{t} G(\varepsilon)\tilde{\gamma}(t_i)\,\mathrm{d}B(s) + \int_{t_i}^{t}\int_{|z|>\varepsilon} \gamma(t_i,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,,$$

with $M_{1,\varepsilon}(0) = 0$. Notice that both the existence of $\Upsilon_{1,\varepsilon}$ and of $L_{1,\varepsilon}$ follow from the GKW decomposition. The martingale representation of $L_{1,\varepsilon}$ yields

$$L_{1,\varepsilon}(t_{i+1}) = L_{1,\varepsilon}(t) + \int_t^{t_{i+1}} P_{1,\varepsilon}(s)\,\mathrm{d}W(s) + \int_t^{t_{i+1}} R_{1,\varepsilon}(s)\,\mathrm{d}B(s) + \int_t^{t_{i+1}}\int_{\mathbb{R}} Q_{1,\varepsilon}(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\,,$$

where $P_{1,\varepsilon}, R_{1,\varepsilon} \in H^2_{[t_i,t_{i+1}]}$ and $Q_{1,\varepsilon} \in \hat{H}^2_{[t_i,t_{i+1}]}$. Using the latter equation we can rewrite $\hat{V}_\varepsilon$ as follows

$$\hat{V}_\varepsilon(t) := \hat{V}_\varepsilon(t_{i+1}) + \int_t^{t_{i+1}} \varphi(t_i, \hat{S}_\varepsilon(t_i), \hat{V}_\varepsilon(t_i), \hat{\Upsilon}_\varepsilon(t_i))\kappa(t_i)\,\mathrm{d}s - \int_t^{t_{i+1}} Y_{1,\varepsilon}(t_i,s)\,\mathrm{d}W(s)$$

$$- \int_t^{t_{i+1}} \int_{\mathbb{R}} Z_{1,\varepsilon}(t_i,s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z) - \int_t^{t_{i+1}} \zeta_{1,\varepsilon}(t_i,s)\,\mathrm{d}B(s)\,, \tag{5.7}$$

where

$$Y_{1,\varepsilon}(t_i,s) = \Upsilon_{1,\varepsilon}(s)b(t_i) + P_{1,\varepsilon}(s)\,,\quad \zeta_{1,\varepsilon}(t_i,s) = \Upsilon_{1,\varepsilon}(s)\tilde{\gamma}(t_i)G(\varepsilon) + R_{1,\varepsilon}(s)\,,$$
$$Z_{1,\varepsilon}(t_i,s,z) = \Upsilon_{1,\varepsilon}(s)\gamma(t_i,z)\mathbf{1}_{\{|z|>\varepsilon\}} + Q_{1,\varepsilon}(s,z)\,. \tag{5.8}$$

Using the fact that $[M_{1,\varepsilon}, L_{1,\varepsilon}]$ is a local $\mathbb{P}$-martingale, we deduce for $t_i \le s \le t_{i+1}$, $i = 0,...,n-1$,

(5.9)

$$\Upsilon_{1,\varepsilon}(s) = \frac{1}{\kappa(t_i)}\left(b(t_i)Y_{1,\varepsilon}(t_i,s) + G(\varepsilon)\tilde{\gamma}(t_i)\zeta_{1,\varepsilon}(t_i,s) + \int_{|z|>\varepsilon} Z_{1,\varepsilon}(t_i,s,z)\gamma(t_i,z)\,\ell(\mathrm{d}z)\right)\,.$$

Thus following similar steps as in Section 4, we find

$$\hat{\Upsilon}_\varepsilon(t_i) = \frac{n}{T}\,\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Upsilon_{1,\varepsilon}(s)\,\mathrm{d}s \,|\, \mathcal{G}_{t_i}\right]\,. \tag{5.10}$$

Using the continuous-time version (5.7)-(5.8) of the process $\hat{V}_\varepsilon$ and also the processes $\Upsilon_{1,\varepsilon}$ and $\hat{\Upsilon}_\varepsilon$ as defined respectively in (5.9) and (5.10), we can follow similar steps as in Section 4 to prove the convergence of the approximating discrete-time scheme to the approximating continuous-time model. In the following theorem we state the obtained convergence rates. These results are the analogons of those in Theorem 4.4, Theorem 4.5, and Theorem 4.6. The proofs are hence skipped.

Theorem 5.5. *Assume the conditions of Theorem 5.3 hold and $\varphi(0,0,0,0) = 0$. Let the triplets $(V_\varepsilon, \Upsilon_\varepsilon, L_\varepsilon)$ and $(\hat{V}_\varepsilon, \hat{\Upsilon}_\varepsilon, \hat{L}_\varepsilon)$ be respectively the solutions of (5.1) and (5.6). It holds*

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|V_\varepsilon(t) - \hat{V}_\varepsilon(t)\right|^2\right] + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\Upsilon_\varepsilon(s) - \hat{\Upsilon}_\varepsilon(t_i)|^2\right]\,\mathrm{d}s$$
$$+ \max_{i<n} \mathbb{E}\left[\left|L_\varepsilon(t_{i+1}) - \hat{L}_\varepsilon(t_{i+1})\right|^2\right] \le \frac{C}{n}\,,$$

where C is a positive constant.

Finally we present the convergence of the approximating discrete-time model to the continuous-time model.

Theorem 5.6. *Assume the conditions of Theorem 5.3 hold and* $\varphi(0,0,0,0)=0$*. Let the triplets* (V,Υ,L) *and* $(\hat{V}_\varepsilon,\hat{\Upsilon}_\varepsilon,\hat{L}_\varepsilon)$ *be respectively the solutions of* (2.5) *and* (5.6)*. It holds*

$$\begin{aligned}&\max_{i<n}\mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]}\left|V(t)-\hat{V}_\varepsilon(t)\right|^2\right]+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[|\Upsilon(s)-\hat{\Upsilon}_\varepsilon(t_i)|^2\right]\mathrm{d}s\\ &+\max_{i<n}\mathbb{E}\left[\left|L(t_{i+1})-\hat{L}_\varepsilon(t_{i+1})\right|^2\right]\le\frac{C}{n}+\tilde{C}G^2(\varepsilon),\end{aligned}\tag{5.11}$$

where $G(\varepsilon)$ *is as in* (2.3) *and* C *and* $\tilde{C}$ *are positive constants.*

Proof. Recall the functions f and f_ε respectively in (2.9) and (5.4). Observe that

$$\begin{aligned}&|f(t,u_1,v_1,w_1,\Gamma_1)-f_\varepsilon(t,u_2,v_2,w_2,\Gamma_2,\zeta)|\\ &\le C\left(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|+\|\Gamma_1-\Gamma_2\|+|\zeta|+\tilde{G}(\varepsilon)\|\Gamma_1\|\right).\end{aligned}$$

Thus using Theorem 5.5 in the present paper, Theorem 4.2 in Di Nunno et al. [8], the Lipschitz property of h, and Proposition 3.3 in Benth et al. [3], we deduce

$$\begin{aligned}&\max_{i<n}\mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]}\left|\hat{V}_\varepsilon(t)-V(t)\right|^2\right]\\ &\le 2\max_{i<n}\mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]}\left|\hat{V}_\varepsilon(t)-V_\varepsilon(t)\right|^2\right]+2\max_{i<n}\mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]}|V_\varepsilon(t)-V(t)|^2\right]\\ &\le\frac{\hat{C}}{n}+\tilde{C}G^2(\varepsilon).\end{aligned}$$

The estimates for the second term and for the third term in the left hand side of equation (5.11) follow using Theorem 4.4 and Theorem 4.5 in Di Nunno et al. [8],Theorem 5.5 in the present paper, the Lipschitz property of h, and Proposition 3.3 in Benth et al. [3]. □

6. Conclusion

We studied in this paper time-discretisation of (2.4)-(2.5), where we specified M to be driven by a Brownian motion and jumps. We considered jumps with finite activity in a first step. Then we included jumps with infinite activity. Using the GKW decomposition, we derived the "intermediate" time-continuous FBSDEJ (4.2). Exploiting this latter equation, the results in Bouchard and Elie [4] and in Di Nunno et al. [8], we proved the convergence of the time-discrete scheme to the time-continuous equation. Moreover, we showed that the L^2-error is of the order of the time step.

As far as further investigations are concerned, we consider in another paper applications of our study to the problem of hedging in finance. Thus we consider quadratic hedging strategies in incomplete markets for which we derive the related FBSDE. Then we study discretisation and simulation of these strategies and we present numerical examples.

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