Effect of Large Disturbances on the Local Behavior of Nonlinear Physically Interconnected Systems

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Abstract: Systems that are physically interconnected, e.g. electric power systems, evolve in the regime of physical consistency laws, such as balance of power flows. This complicates the analysis and control of the global system behavior drastically, because mathematically these balance laws induce nonlocal relations among all dynamical variables, thus leading to a system of differential algebraic equations. While a stability theory for local analysis is available, i.e. for the small perturbations evolving about a steady-state power flow solution as operational set-point, little is known about dynamical behavior under large disturbances that induce changes in the steady-state operation point. Recently, this problem has been listed in numerous publications as key challenge to overcome in the restructuring process of electric power systems thus preserving electric infrastructures from black-outs. Here we present a novel method that bridges this gap between local, linear stability analysis, and nonlinear changes in steady-state. We provide an analytical framework suited to quantify the maximum change of a specific eigenvalue, given a large, external disturbance affecting the system at specific locations. Our method bases on a Lagrangian method for constrained optimization and the use of Gâteaux differentials to compute first variations. We formulate a Lagrangian with eigenvalue variation as objective function and set-point dependent eigenvalue problem as equality constraint. The application of necessary optimality conditions provides sensitivity fields w.r.t. set-point changes succeeding the given, external disturbance. The application of the approach in the electric power system is discussed.

Keywords: Physics of Circuits, Lagrangian methods, Linear Stability, Electric Power Systems

1. INTRODUCTION

Approaches to the mathematical modeling and operation of large-scale systems are usually modular due to the complexity of the systems to be analyzed. For the class of physically networked systems a decoupling into a set of subsystems is particularly difficult, because the interconnection of subsystems imposes algebraic constraints on the dynamic variables of the system, see for example Seiler (2010). These constraints naturally lead to differential algebraic equations (DAEs) for the system model, where the algebraic part represents physical consistency relations, see for instance Seiler (2010), Chapter 1, and Ascher and Petzold (1998), Chapter 1 and 9. Depending on the physical context, consistency means conservation of mass, energy, or momentum, or balance and continuity of power flows, e.g. validity of Kirchhoff laws in electric networks. While from the physical point of view it is clear that algebraic conditions are not additional restrictions on the solution space - because the physical behavior automatically satisfies physical consistency by its nature - from the mathematical point of view their appearance complicates the analysis and design of control strategies for large-scale systems significantly. This is due to the fact that the validity of physical balance (or conservation) laws mathematically results in integrability conditions which lead to existence criteria for solutions on some manifold, see Ascher and Petzold (1998), Chapter 9, and Seiler (2010), Chapter 1. In general this implies a nonlinear space of physically consistent system configurations. Classical analysis and control synthesis approaches might be only locally valid, i.e. in a (compact) neighborhood of some attractive steady-state system configuration. Another complicating issue is that the actual network structure of physically interconnected systems is often in the algebraic part of the DAEs. This renders the application of known multi-agent and graph-theoretical approaches difficult. These two issues are two main challenges to be met in the restructuring process of electric power systems, and they have recently lead to a series of large power system black-outs, e.g. in 2003 and 2006 in the USA and Europe, see Bialek (2007) and UCTE (2007). Traditional steady-state decentralized operation turns out to be insufficient to account for overall system stability in situations where the transmission network has become a platform for the exchange of large amounts of electric power between far distant locations in the network. This leads to changes in steady-state power flow at time-scales that interact with traditional steady-state real-time operation. Thus, the open problem is to analyze system behavior under large disturbances, i.e. the effect of changes in power balance on the (local) dynamic behavior.

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Little is known about the large signal behavior of electric power systems, as already stated in Dobson and Lu (1992). The relationship between steady-state dynamic models for electric power systems and the power flow equations is analyzed for example in Sauer and Pai (1990). It turns out that the power flow Jacobian enters the linearized dynamics for small disturbances evolving about a steady-state; singularity of the Jacobian matrix implies singularity of the linearized system dynamics. The work of Cao and Hill (2010) clarifies this result for linear DAE models with inputs; singularity will affect the system even when "perfectly" controlled, where "perfect" control means that control inputs are adjusted to meet the power flow requirements. However, in reality this assumption of "perfect" control can not be met, as the cited black-outs impressively demonstrate. Conversely, in Wang et al. (2001) it is shown that the power flow Jacobian becomes ill-conditioned for highly loaded networks, and the question arises, how a change in steady-state power flow - represented by the ill-conditioned power flow Jacobian - interacts with the local (linearized) behavior of controlled small perturbations. Thus, of practical interest are those situations where the power flow Jacobian is not singular, but may induce low-damped oscillatory behavior as a result of uncoodinated decentralized controls that do not meet the power flow requirements, as opposed to the case of Cao and Hill (2010). It is to note that classical stability theory for DAEs, see for instance Hill and Mareels (1990), may not apply in these situations, because the technical condition that the operating region is contained in a compact positively invariant set may be violated when a disturbance is large. Then, the steady-state changes and the succeeding operation point may leave the compact set. This condition allows to represent the DAE system locally in ODE form on the basis of conditions for the existence of solutions as first integrals, see Hill and Mareels (1990) and Dobson and Lu (1992).

This paper aims at bridging the methodological gap between static methods to solve steady-state set-point equations for the physical network and methods for stability analysis of (controlled) small perturbation dynamics. We derive a formula to quantify the effect of the non-singular Jacobian matrix - related to nonlinear changes in the steady-state operating configuration - on the behavior of small disturbances that evolve about a fixed steady-state set-point solution. The quantification is accomplished by means of the variation of a specifically chosen eigenvalue in the complex domain. Our novel approach is based on a Lagrangian method for constrained optimization, where the nonlinearity stemming from variations in the algebraic balance conditions is handled using a sensitivity approach within the formalism of Gâteaux differentiation. Our framework is motivated by sensitivity studies in physical systems with infinite dimensional state, see for instance Qadri and Juniper (2012) and Meliga et al. (2010), where we make use of the fact that the discretized and linearized local dynamics are described by a finite dimensional DAE system. In contrast to classical eigenvalue sensitivity frameworks, as for instance derived in Verghese et al. (1982), here we go one step further an compute variations of an eigenvalue. Moreover, in our framework we are able to retain physical meaning of network structure in that we are able to relate the locational structure and magnitude of a large external disturbance as network input to the variation of the output, i.e. the chosen eigenvalue.

The paper is structured as follows: In Section 2 we introduce the class of (large-scale) physically networked systems, approaches for their control, and give a problem formulation. In Section 3 we put the problem in the setting of constrained optimization, and develop our analytical framework. In Section 4 we discuss our results on the example of electric power system problems, before we conclude.

Notation: We use $\langle \cdot, \cdot \rangle$ to denote the inner product, and $||\cdot||$ denotes the 2-norm. The formal adjoint is denoted by $(\cdot)^{\dagger}$. The notation $\partial \boldsymbol{f} / \partial \boldsymbol{x}$ denotes differentiation w.r.t. vector elements of $\boldsymbol{x}$, and $\mathrm{grad}_{\boldsymbol{x}} \boldsymbol{f}$ is the gradient of a vector-valued function $\boldsymbol{f}$ in the direction of the vector $\boldsymbol{x}$.

2. MODEL SETUP & STABILITY PROBLEM

Motivated by electric power systems in this work we consider physically networked systems. The problem under investigation is to quantify changes in a specifically chosen eigenvalue when external disturbances are large enough to induce changes in the nonlinear steady-state solution of the DAE system that serves as operational set-point for small disturbance dynamics.

2.1 Large-Scale Physically Networked Systems Modeling

Large-scale physically networked systems are formally represented in modularized form as tuple $\Sigma = (\{\Sigma_i\}, \mathscr{G})$, where a set of subsystems Σ_i is interconnected according to a graph $\mathscr{G}$. The graph is a tuple $(\mathcal{V}, \mathcal{E})$ comprising the set of vertices, denoted as $\mathcal{V}$, and the set of edges, denoted as $\mathcal{E}$, which connect vertices pairwise. Each subsystem is a controlled dynamical system described by a vector-valued ODE in explicit form, i.e. $\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i),\ i \in \mathcal{V}$, where $\boldsymbol{x}_i$ is the state of a subsystem, and $\boldsymbol{u}_i$ the control vector. With N denoting the number of subsystems, the overall system state vector is $\boldsymbol{x}^T = (\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_N^T)$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{u}^T = (\boldsymbol{u}_1^T, \ldots, \boldsymbol{u}_N^T)$, $\boldsymbol{u} \in \mathbb{R}^r$ is the stacked vector of system controls. The graph represents the physical interconnection, where edges are constituted as terminals, in the sense of Willems (2010).

Remark 1. The systems notion of a terminal and physical interconnection is well explained in the circuits related work of Willems (2010) (which does also apply to dynamical systems and state space models, see for instance Anderson and Vongpanitlerd (2006)): "interconnection leads to terminals that share their potential and current, (up to a sign) while energy transfer occurs through ports. Terminal connection is a local operation, while energy involves several terminals (that satisfy Kirchhoff's laws) simultaneously, and is therefore action at a distance". By that physically consistent stability studies require nonlocal methods, because energy as vehicle that drives physical motion is not a local quantity.

That is, balance equations of algebraic type are valid globally, whereat the interconnection structure of $\mathscr{G}$, contained in $\mathcal{E}$, determines those variables of the subsystems that enter the algebraic equations, and thus, from a physical viewpoint, contribute to balance of power flows. The action of such consistency conditions imposes nonlocal, instantaneous relations among all dynamical variables that constrain the overall system behavior to some restricted (lower dimensional) space, see Seiler (2010), Chapter 1, and Ascher and Petzold (1998), Chapter 9. The system behavior therefore is represented by DAEs of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}), \tag{1a}$$

$$\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}), \tag{1b}$$

where $\boldsymbol{y} \in \mathbb{R}^q$ denotes algebraic variables, see the following remark 2 for a mathematical argument for their emergence. The algebraic part (1b) represents so-called network equations, which refer to an invariant of the reduced ODE system (1a), see Ascher and Petzold (1998), Chapter 9, i.e. its trajectories evolve on the (smooth) manifold $\mathcal{M} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}\}$. Physically it represents instantaneous balance of power flows within $\mathscr{G}$, similar to the familiar nodal equations in electric circuits, see Ilic and Zaborszky (2000), Chapter 5, and the following remark 3.

Remark 2. The algebraic variables are obtained as a result of the implicit functions theorem. A dynamical system (here without inputs) with state denoted by $\boldsymbol{z}$ is generally represented by an implicit vector equation $\boldsymbol{F}(\dot{\boldsymbol{z}}, \boldsymbol{z}) = \boldsymbol{0}$. When the Jacobian $\partial \boldsymbol{F}/\partial \dot{\boldsymbol{z}}$ is singular, then the state can be partitioned as $\boldsymbol{z}^T = (\boldsymbol{x}^T, \boldsymbol{y}^T)$ leading to the generic form (1) (without controls in the autonomous case). Thereby, the Jacobian of the reduced ODE system $\partial \boldsymbol{f}/\partial \dot{\boldsymbol{x}}$ has full rank, and the same rank as $\partial \boldsymbol{F}/\partial \dot{\boldsymbol{z}}$. Thus, the dimension of the vector $\boldsymbol{y}$ containing the algebraic variables is determined by the rank deficiency of $\partial \boldsymbol{F}/\partial \dot{\boldsymbol{z}}$. This argument is central to the understanding of why classical systems tools may not be valid, i.e. physically networked systems are singular, and cannot readily be transformed into a system of ODEs, unless one has knowledge about the n-dimensional manifold $\mathcal{M}$ on which the reduced n-dimensional ODE system evolves.

Remark 3. In electric power systems (1a) describes the set of dynamic machines generating electric power. The vector $\boldsymbol{y}$ contains mainly variables for voltage phasors at network buses, see Cao and Hill (2010). The network buses are usually separated into a slack bus, serving as reference, PV-buses, i.e. generator nodes injecting generated real power P at given voltage magnitude V, and PQ-buses, i.e. nodes representing loads via active and reactive powers P and Q, see Sauer and Pai (1998), Chapter 7. The graph structure is explicitly visible in the Jacobian of the algebraic equations $\boldsymbol{J}_{\mathrm{ae}} := \partial \boldsymbol{g}/\partial \boldsymbol{y}$ for the load-buses, i.e. in the load flow equations. That is, we have $J_{\mathrm{ae},ik} \neq 0$, whenever nodes i and k are interconnected, i.e. $ik \in \mathcal{E}$, else $J_{\mathrm{ae},ik} = 0$, see Barret et al. (1997), Chapter 2.

The goal for real-time control is steady-state stability of an operational set-point, where set-points are obtained as equilibrium point of (1), denoted by $\boldsymbol{z}_{\mathrm{opt}}$, with $\boldsymbol{z}^T = (\boldsymbol{x}^T, \boldsymbol{y}^T)$, $\boldsymbol{z} \in \mathbb{R}^{n+q}$, see also remark 2, i.e. $\boldsymbol{z}_{\mathrm{opt}}$ satisfies

$$\begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}_{\mathrm{opt}}, \boldsymbol{y}_{\mathrm{opt}}, \boldsymbol{0}) \\ \boldsymbol{g}(\boldsymbol{x}_{\mathrm{opt}}, \boldsymbol{y}_{\mathrm{opt}}) \end{bmatrix} = \boldsymbol{0}. \tag{2}$$

This equilibrium point is computed as solution of a static optimization problem, for instance via Newton iterations. Defining the resolvent (or residual) operator

$$\boldsymbol{\mathcal{R}}(\boldsymbol{z}) := \begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{0}) \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{bmatrix}, \tag{3}$$

a steady-state operation condition is for example obtained from optimization as

$$\boldsymbol{z}_{\mathrm{opt}} = \arg\min ||\boldsymbol{\mathcal{R}}(\boldsymbol{z})||. \tag{4}$$

In (2) controls are neglected because they are zero in equilibrium, i.e. when no deviations from the steady-state $\boldsymbol{z}_{\mathrm{opt}}$ occur. In contrast to the homogeneous case (2), inhomogeneity due to a large, external disturbance $\boldsymbol{d} \in \mathbb{R}^{n+q}$ affects the system (1) as external forcing of the operation state via the relation

$$\boldsymbol{\mathcal{R}}(\boldsymbol{z}_{\mathrm{opt}}^+) + \boldsymbol{d} = \boldsymbol{0}, \tag{5}$$

where $\boldsymbol{z}_{\mathrm{opt}}^+$ denotes the post-disturbance operation point, in contrast to the pre-disturbance operation point $\boldsymbol{z}_{\mathrm{opt}}$. Therefore the forced algebraic balance equations (5) are the constitutive equations defining $\boldsymbol{z}_{\mathrm{opt}}^+$. In the following we consider disturbances $\boldsymbol{d}$ such that a solution $\boldsymbol{z}_{\mathrm{opt}}^+$ exists, and the Jacobian $\partial \boldsymbol{\mathcal{R}}/\partial \boldsymbol{z}|_{\boldsymbol{z}_{\mathrm{opt}}}$ is non-singular. The disturbance enters in additive form, because it is external, i.e. a forcing of the nonlinear balance laws, and physically it induces the internal re-balancing of steady-state power flows.

When an equilibrium point is given, the equations for small perturbations, denoted by $\Delta \boldsymbol{z}^T = (\Delta \boldsymbol{x}^T, \Delta \boldsymbol{y}^T)$, are obtained via linearization of (1) without inputs about $\boldsymbol{z}_{\mathrm{opt}}$, so that

$$\Delta \dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{z})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{z}_{\mathrm{opt}}} \Delta \boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}(\boldsymbol{z})}{\partial \boldsymbol{y}}\right|_{\boldsymbol{z}_{\mathrm{opt}}} \Delta \boldsymbol{y}, \tag{6a}$$

$$\boldsymbol{0} = \left.\frac{\partial \boldsymbol{g}(\boldsymbol{z})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{z}_{\mathrm{opt}}} \Delta \boldsymbol{x} + \left.\frac{\partial \boldsymbol{g}(\boldsymbol{z})}{\partial \boldsymbol{y}}\right|_{\boldsymbol{z}_{\mathrm{opt}}} \Delta \boldsymbol{y}. \tag{6b}$$

We write the linear DAE system (6) in matrix notation as

$$\begin{pmatrix} \Delta \dot{\boldsymbol{x}} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}}) \begin{pmatrix} \Delta \boldsymbol{x} \\ \Delta \boldsymbol{y} \end{pmatrix}, \tag{7a}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{J}_{\mathrm{ae}} \end{bmatrix} \triangleq \left.\frac{\partial \boldsymbol{\mathcal{R}}(\boldsymbol{z})}{\partial \boldsymbol{z}}\right|_{\boldsymbol{z}_{\mathrm{opt}}} = \begin{bmatrix} \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} & \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} \\ \dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} & \dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}} \end{bmatrix}_{\boldsymbol{z}_{\mathrm{opt}}}, \tag{7b}$$

where $\boldsymbol{J}_{\mathrm{ae}}$ denotes the Jacobian of the network equations, see also remark 3.

In practice, real-time control of large-scale physically networked systems like electric power systems is accomplished locally, i.e. spatially located at vertices $i \in \mathcal{V}$ using proportional feedback controllers under the assumption of linearity, and without a global system control strategy, see for example Sauer (2005). While controllers are acting decentralized, their actions are physically coupled via the balance of power flows, which leads to dynamical interference of control actions that is not taken into account.

Remark 4. The differential algebraic nature of electric power systems leads to a separation of models and problems at different time scales. For long time scales, power system planning is accomplished on grounds of purely static methods, e.g. in terms of snapshots of steady-state load flow solutions at load buses. Real-time control of small perturbations however is accomplished on grounds of localized measures of full linear dynamics (7) at generator buses, given a steady-state operation condition. Thus, tools to analyze dynamics are separated from tools for static balance considerations, which leads to significant problems in the operation of modern electric power systems, see Barret et al. (1997), Chapter 11, and Bialek (2007).

Remark 5. In view of electric power system applications the work of Cao and Hill (2010) introduces function mappings to describe the procedure of obtaining an equilibrium $\boldsymbol{z}_{\mathrm{opt}}$ without resorting to the full resolvent as in (2). There, an operating point is obtained on the basis of the load flow (network) equations, being related to the Jacobian $\boldsymbol{J}_{\mathrm{ae}}$ of the algebraic variables, see (7b). This is of importance because only $\boldsymbol{J}_{\mathrm{ae}}$ carries information about the physical network structure, and it is the major tool to couple dynamic and static methods, see remark 4,

2.2 Network Structure and Linear Stability under Changes of Operating Point

Traditionally, an ODE system is obtained from (7a) via elimination of algebraic variables. Therefore one solves (6b) for $\Delta\boldsymbol{y}$ and substitutes into (6a), which leads to the equivalent ODE system

$$\Delta\dot{\boldsymbol{x}} = [\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{J}_{\mathrm{ae}}^{-1}\boldsymbol{A}_{21}]\Delta\boldsymbol{x}. \tag{8}$$

The physical interconnection structure, which is originally contained in $\boldsymbol{J}_{\mathrm{ae}}$, is then no more present as sparsity pattern in the system matrix of (8); it gets lost through the inversion and matrix multiplications. Thus, the sparsity pattern of a local ODE representation of a physically networked system carries no physical meaning in the sense of transmitting energy between coupled states. This is a consequence of Kirchhoff balance laws, and contrasts classical multi-agent models where subsystems interact through a communication network whose graph can be interpreted as transportation structure for some information quantity.

Therefore, we stay within the setting of the DAE system (7). Then, linear (in)stability is derived from modal analysis in terms of the generalized eigenvalue problem

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{B}\boldsymbol{v}_i, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \tag{9}$$

where $(\lambda_i, \boldsymbol{v}_i)$ denotes an eigenpair. The linear behavior is characterized by the eigenvalues $\lambda_i = \sigma_i + j\omega_i$, where σ_i is the growth rate, and ω_i the frequency of the eigenmode corresponding to the i-th eigenvector $\boldsymbol{v}_i$. Denote by $(\lambda_i^\dagger, \boldsymbol{w}_i)$ the associated adjoint eigenpair satisfying the adjoint generalized eigenvalue problem

$$\left(\lambda_i^\dagger \boldsymbol{B} - \boldsymbol{A}^\dagger\right)\boldsymbol{w}_i = \boldsymbol{0}. \tag{10}$$

Then the local behavior of (1) is described by the evolution of small amplitude perturbations, starting in $\Delta\boldsymbol{z}(t=0)$, at fixed operating point, i.e. by the linear combination of the operational state $\boldsymbol{z}_{\mathrm{opt}}$ and superpositioned eigenmodes

$$\boldsymbol{z}(t) = \boldsymbol{z}_{\mathrm{opt}} + \Delta\boldsymbol{z}(t) = \boldsymbol{z}_{\mathrm{opt}} + \sum_i \langle \boldsymbol{w}_i, \Delta\boldsymbol{z}(0)\rangle\, e^{\lambda_i t}\boldsymbol{v}_i. \tag{11}$$

When all eigenvalues are stable, i.e. they lie in the open left-half-plane of the complex plane $\Re(\lambda_i) < 0$, for all i, then the eigenmode i with the growth rate closest to zero is the dominant eigenmode. The dominant eigenmode is the most critical for stability. In the following we choose the dominant eigenvalue as output measure and denote it by λ.

Problem statement: The problem we consider is to find an expression for the change of the dominant eigenvalue $\delta\lambda = \lambda^+ - \lambda = \lambda(\boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}}^+(\boldsymbol{d}))) - \lambda(\boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}}))$, when a large enough, external disturbance vector $\boldsymbol{d}$, with structure such that the component d_i enters at component z_i, forces the system, and leads to a new operational steady-state $\boldsymbol{z}_{\mathrm{opt}}^+$. A change $\delta\boldsymbol{z}_{\mathrm{opt}} = \boldsymbol{z}_{\mathrm{opt}}^+ - \boldsymbol{z}_{\mathrm{opt}}$ of steady-state, induced by $\boldsymbol{d}$, affects the local behavior via the newly parameterized system matrix $\boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}^+})$ with associated dominant eigenvalue λ^+. That is, we consider the dominant eigenvalue λ as function of the operating point $\boldsymbol{z}_{\mathrm{opt}}$, with the dependencies (9) and (7). The operating point in turn is a function of the algebraic balance equations in equilibrium according to (2) and (4), and of the external disturbance $\boldsymbol{d}$ that acts as forcing on the internally balanced equilibrium configuration, when present.

Note that the algebraic balancing constraint (5) imposes a generally nonlinear relation, which has to be respected in the derivation process for a quantitative expression of $\delta\lambda$. In the following we use the symbol δ preceding a variable as notation for the first variation of the variable, and equivalently as notation for the difference between the respective post- and pre-disturbance values. Further we assume λ having algebraic multiplicity one, which assures continuous dependence on system parameters.

Remark 6. Making equivalent use of δ for first variations and differences we obtain only estimates of respective variable changes. These become more exact, and related to physical reality, the "smaller" the nonlinearity is, i.e. the better the first variation is approximated by a difference within a Taylor series expansion to first order.

3. EFFECTS OF THE BALANCE EQUATIONS ON THE LOCAL BEHAVIOR

The problem under investigation is reformulated in a variational setting. By that, the first variation of the dominant eigenvalue as nonlinear function of an external disturbance can be computed via a linear expression involving sensitivity vectors. First order optimality conditions lead to analytical expressions for these sensitivity vectors, and it is seen, that the Jacobian of the algebraic balance equations for set-points acts in terms of a gain on the sensitivity vector of a specific eigenvalue w.r.t. steady-state changes.

3.1 Variational Formulation Using a Lagrangian Method and Gâteaux Differentials

The variation of the function λ at steady-state condition $\boldsymbol{z}_{\mathrm{opt}}$, $\delta\lambda$, with respect to the directional input $\boldsymbol{d}$ is mathematically described by the Gâteaux differential, i.e. the first variation of the dominant eigenvalue. Denoting the variation in the steady-state operating condition by $\delta\boldsymbol{z}_{\mathrm{opt}}(\boldsymbol{d}) = \boldsymbol{z}_{\mathrm{opt}}^+(\boldsymbol{d}) - \boldsymbol{z}_{\mathrm{opt}}(\boldsymbol{d}=\boldsymbol{0})$, the first variation is defined as

$$\delta\lambda(\boldsymbol{z}_{\mathrm{opt}}; \boldsymbol{d}) \triangleq \lim_{\tau\to 0} \frac{\lambda^+(\boldsymbol{z}_{\mathrm{opt}} + \tau\delta\boldsymbol{z}_{\mathrm{opt}}(\boldsymbol{d})) - \lambda(\boldsymbol{z}_{\mathrm{opt}})}{\tau}, \tag{12}$$

under the normalization $||\delta\boldsymbol{z}_{\mathrm{opt}}|| = 1$. This functional is a nonlinear mapping between pre- and post-disturbance situations, because $\boldsymbol{z}_{\mathrm{opt}}^+ = \boldsymbol{z}_{\mathrm{opt}} + \delta\boldsymbol{z}_{\mathrm{opt}}(\boldsymbol{d})$, the new operating point, is computed such that it satisfies (5), and $\boldsymbol{\mathcal{R}}(\cdot)$ is a system of nonlinear algebraic equations.

The first variation (12) has a linear expression in terms of the Gâteaux derivative $\mathrm{grad}_{\boldsymbol{z}}\lambda =: \mathcal{S}_{\boldsymbol{z}}(\lambda)$ at point $\boldsymbol{z}_{\mathrm{opt}}$

in the direction of the vector $\delta \boldsymbol{z}_{\text{opt}}$, so that (12) can be equivalently stated as

$$\delta\lambda(\boldsymbol{z}_{\text{opt}};\boldsymbol{d}) \equiv \delta\lambda(\boldsymbol{z}_{\text{opt}};\delta\boldsymbol{z}_{\text{opt}}(\boldsymbol{d})) \triangleq \langle \boldsymbol{\mathcal{S}}_z(\lambda), \tau\delta\boldsymbol{z}_{\text{opt}}\rangle . \quad (13)$$

The Gâteaux differential is continuous in the input argument, so that we assume continuity of $\delta\lambda$ in $\delta\boldsymbol{z}_{\text{opt}}(\boldsymbol{d})$ and $\boldsymbol{d}$, respectively. Conceptually, the first variation $\delta\lambda$ with the external disturbance as input can formally be expressed as

$$\delta\lambda(\boldsymbol{z}_{\text{opt}};\boldsymbol{d}) \triangleq \tau \langle \boldsymbol{\mathcal{S}}_d(\lambda), \boldsymbol{d}\rangle . \quad (14)$$

where $\boldsymbol{\mathcal{S}}_d(\lambda) := \text{grad}_{\boldsymbol{d}}\lambda$ is the sensitivity of the dominant eigenvalue in the direction of the disturbance vector $\boldsymbol{d}$. Here, the normalization $||\boldsymbol{d}|| = 1$ formally applies. Thus, in a variational setting the dominant eigenvalue deviation $\delta\lambda$ w.r.t. nonlinear changes in the operating point induced by an external disturbance $\boldsymbol{d}$ has a linear expression. This motivates the variational approach using the framework of Gâteaux differentials.

Remark 7. In (14), the linear scaling with the parameter τ follows from the fact that the first variation is linear in the input argument for the inner product, and by that homogenous (or scale invariant), i.e.

$$\delta\lambda(\boldsymbol{z}_{\text{opt}};\tau\boldsymbol{d}) = \tau\delta\lambda(\boldsymbol{z}_{\text{opt}};\boldsymbol{d}). \quad (15)$$

For the purpose of deriving estimates of changes $\delta\lambda$ induced by unnormalized external disturbances, $||\boldsymbol{d}|| \neq 1$, we use scale invariance together with the factor τ to recast an estimate for the effect of the original magnitude input. In that we write $||\boldsymbol{d}|| \neq 1$ normalized and scaled as $\tau\boldsymbol{d}, ||\boldsymbol{d}|| = 1$.

The problem of finding the first variation of the dominant eigenvalue under large disturbances can be solved by finding a formula for $\boldsymbol{\mathcal{S}}_z(\lambda)$ and an expression that relates it to $\boldsymbol{\mathcal{S}}_d(\lambda)$. For this purpose we resort to a Lagrangian framework for (un-)constrained optimization, as in the work of Meliga et al. (2010) or Qadri and Juniper (2012).

Every dominant eigenvalue $\lambda(\boldsymbol{A}(\boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d}))$ is required to satisfy the respective eigenvalue problem (9). By definition, when $\boldsymbol{d} \neq \boldsymbol{0}$ and large, then there will be a nonzero difference $\delta\lambda$ which is a first variation. That is, the sensitivity $\boldsymbol{\mathcal{S}}_{\boldsymbol{d}}(\lambda)$ is a complex valued (steepest descent) gradient direction along which the effect of $\boldsymbol{d}$ on the output measure $||\lambda^+ - \lambda||$ is greatest, where the distance between the two complex variables is measured by identifying $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$ equipped with the norm $||\cdot||$. The two requirements which are set by definition - satisfying the eigenvalue problem and being a first variation - impose an extremum condition on the associated Lagrangian function given by

$$\begin{aligned}\mathcal{L}(\lambda, \boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d})) &= ||\lambda(\boldsymbol{A}(\boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d}))) - \lambda(\boldsymbol{A}(\boldsymbol{z}_{\text{opt}}))|| \\ &- \left\langle \boldsymbol{\mu}, \left[\lambda(\boldsymbol{A}(\boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d})))\boldsymbol{B} - \boldsymbol{A}(\boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d}))\right] \boldsymbol{v}\right\rangle \\ &- \langle \boldsymbol{\mu}, [\lambda(\boldsymbol{A}(\boldsymbol{z}_{\text{opt}}))\boldsymbol{B} - \boldsymbol{A}(\boldsymbol{z}_{\text{opt}})]\, \boldsymbol{v}\rangle,\end{aligned} \quad (16)$$

where the vector $\boldsymbol{\mu}$ is a suitable vector of Lagrange multipliers. The constraint optimization problem is as follows: Given $(\lambda, \boldsymbol{v})$ and an input $\boldsymbol{d}$, the goal is find a sensitivity such that $||\lambda^+ - \lambda||$ is maximized under the constraint of the post-disturbance eigenvalue problem.

Remark 8. Since we are interested in maximal deviations of an eigenvalue in the complex plane we can equivalently write (16) as

$$\mathcal{L} = \lambda - \left\langle \boldsymbol{\mu}, [\lambda^+\boldsymbol{B} - \boldsymbol{A}(\boldsymbol{z}_{\text{opt}^+})]\boldsymbol{v}\right\rangle , \quad (17)$$

because any change in the cost is unit proportional to the variation magnitude, i.e. $||\lambda^+ - \lambda|| = 1 \cdot (\Re(\delta\lambda) + \Im(\delta\lambda))$, for small variations, thus having the same form as in the work of Meliga et al. (2010).

Referring to the maximum principle, the extremum condition corresponds to the stationarity of the first variation $\delta\mathcal{L}$, i.e.

$$0 \overset{!}{=} \delta\mathcal{L} = \delta\lambda(\boldsymbol{z}_{\text{opt}};\boldsymbol{d}) - \delta\left[\left\langle \boldsymbol{\mu}, \left[\lambda^+\boldsymbol{B} - \boldsymbol{A}(\boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d}))\right] \boldsymbol{v}\right\rangle\right] . \quad (18)$$

Stationarity of the Lagrangian corresponds to the validity of first order necessary optimality conditions and yields an optimality system of equations from where optimal sensitivities can be found, see Gunzburger (2003), Chapter 2. Using the Lagrangian (16), the unconstrained optimization problem is as follows: Given $(\lambda, \boldsymbol{v})$ and a vector of external disturbances $\boldsymbol{d}$, find the sensitivity vector $\boldsymbol{\mathcal{S}}_{\boldsymbol{d}}(\mathcal{L})$ and the vector of Lagrange multipliers $\boldsymbol{\mu}$, such that the functional $\mathcal{L}(\lambda, \boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{z}^+_{\text{opt}}(\boldsymbol{d}))$ is rendered stationary.

Remark 9. Because λ is a complex number, the sensitivity vector $\boldsymbol{\mathcal{S}}_d(\lambda)$ is a complex vector. By that, for given disturbance $\boldsymbol{d}$, the real part $\Re(\boldsymbol{\mathcal{S}}_d(\lambda))$ is the sensitivity of growth rate changes, and $\Im(\boldsymbol{\mathcal{S}}_d(\lambda))$ is the sensitivity of frequency changes. A solution to the unconstrained optimization problem yields the complex vector $\boldsymbol{\mathcal{S}}_d(\lambda)$, and from that the effect of $\boldsymbol{d}$ on other objectives than maximizing $||\delta\lambda||$ can be derived by choosing other combinations of real and imaginary parts of $\boldsymbol{\mathcal{S}}_d(\lambda)$ that are of interest.

3.2 Estimating Eigenvalue Displacements and Main Result

The relation between the sensitivity $\boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda)$ and the sensitivity $\boldsymbol{\mathcal{S}}_{\boldsymbol{d}}(\lambda)$ is as follows.

Lemma 1. Consider a system with steady-state dynamics according to the DAE representation (7) and a large, external disturbance $\tau\boldsymbol{d}$, $||\boldsymbol{d}|| = 1, \tau > 0$, such that the post-disturbace operational point satisfies (5). Assume the sensitivity vector given by $\boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda)$ satisfies the first variation equation (13). Then, this relation for the first variation of the dominant eigenvalue of system (7) satisfies the equation

$$\delta\lambda = \tau \langle \boldsymbol{\mathcal{S}}_{\boldsymbol{d}}(\lambda), \boldsymbol{d}\rangle , \quad (19)$$

with

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{d}}(\lambda) = -\left(\boldsymbol{A}^{-1}\right)^\dagger \boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda), \quad (20)$$

where $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{z}_{\text{opt}})$ is the system matrix.

Proof 1. Assuming $\tau\boldsymbol{d}$ to be small, see remark 6 with remark 7, the relation between $\delta\boldsymbol{z}_{\text{opt}}$ and $\tau\boldsymbol{d}$ computes from (5) by taking differentials, i.e. a first order approximation at the pre-disturbance operating point $\boldsymbol{z}_{\text{opt}}$, so that

$$\left[\frac{\partial\boldsymbol{\mathcal{R}}(\boldsymbol{z})}{\partial\boldsymbol{z}}\right]_{\boldsymbol{z}_{\text{opt}}} \delta\boldsymbol{z}_{\text{opt}} + \tau\boldsymbol{d} = \boldsymbol{0} \quad (21a)$$

$$\Rightarrow \delta\boldsymbol{z}_{\text{opt}} = -\left[\frac{\partial\boldsymbol{\mathcal{R}}(\boldsymbol{z})}{\partial\boldsymbol{z}}\right]^{-1}_{\boldsymbol{z}_{\text{opt}}} \tau\boldsymbol{d}. \quad (21b)$$

With $\partial\boldsymbol{\mathcal{R}}/\partial\boldsymbol{z} \equiv \boldsymbol{A}$ from (7b), substitution of (21b) into (13) yields

$$\delta\lambda = \left\langle \boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda), -\boldsymbol{A}^{-1}\tau\boldsymbol{d}\right\rangle = \left\langle -\left(\boldsymbol{A}^{-1}\right)^\dagger \boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda), \tau\boldsymbol{d}\right\rangle . \quad (22)$$

Then, according to (15), scale invariance of the functional $\delta\lambda$ leads to (19) with (20).

■

In the main result that follows we give the relation between $\tau \boldsymbol{d}, ||\boldsymbol{d}|| = 1$ and the first variation $\delta\lambda$.

Theorem 1. Consider a system with steady-state dynamics according to the DAE representation (7) and a large, external disturbance $\tau \boldsymbol{d}, \quad ||\boldsymbol{d}|| = 1, \tau > 0$, such that the post-disturbance operation point satisfies (5). Then, the effect of the external disturbance on the small-signal behavior described by the dominant eigenvalue is captured by the first variation

$$\delta\lambda = \tau \left\langle \left(\boldsymbol{A}^{-1}\right)^{\dagger} \left[\frac{\partial[\boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}})\boldsymbol{v}]}{\partial \boldsymbol{z}}\right]^{\dagger} \boldsymbol{w}, \boldsymbol{d} \right\rangle, \tag{23}$$

with the normalization $\langle \boldsymbol{w}, \boldsymbol{B}\boldsymbol{v}\rangle = 1$ applied.

Proof 2. Using lemma 1, it remains to show that

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda) = -\left[\frac{\partial[\boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}})\boldsymbol{v}]}{\partial \boldsymbol{z}}\right]^{\dagger} \boldsymbol{w}, \tag{24}$$

The first variation $\delta\lambda = \lambda^{+} - \lambda$ results from the optimality system determined by first order optimality conditions that render the Lagrangian $\mathcal{L}$ given in (16) stationary. Using the chain rule the extremum condition (**??**) can be written as

$$0 = \delta\mathcal{L} =$$
$$\underbrace{\left\langle \frac{\partial\mathcal{L}}{\partial \boldsymbol{z}}, \delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle}_{\text{(i)}} + \underbrace{\left\langle \frac{\partial\mathcal{L}}{\partial \boldsymbol{\mu}}, \delta\boldsymbol{\mu} \right\rangle}_{\text{(ii)}} + \underbrace{\left[\left\langle \frac{\partial\mathcal{L}}{\partial \lambda}, \delta\lambda \right\rangle + \left\langle \frac{\partial\mathcal{L}}{\partial \boldsymbol{v}}, \delta\boldsymbol{v} \right\rangle\right]}_{\text{(iii)}}. \tag{25}$$

For the first variation of the Lagrangian to vanish it is sufficient for all terms (i)-(iii) to be zero. Evaluating the second term in (iii) we obtain

$$\frac{\partial\mathcal{L}}{\partial \boldsymbol{v}} = \langle \boldsymbol{\mu}, \lambda\boldsymbol{B} - \boldsymbol{A}\rangle = \left\langle \left(\lambda^{\dagger}\boldsymbol{B} - \boldsymbol{A}^{\dagger}\right)\boldsymbol{\mu}, \boldsymbol{1} \right\rangle, \tag{26}$$

and hence,

$$\left\langle \frac{\partial\mathcal{L}}{\partial \boldsymbol{v}}, \delta\boldsymbol{v} \right\rangle = \left\langle \left(\lambda^{\dagger}\boldsymbol{B} - \boldsymbol{A}^{\dagger}\right)\boldsymbol{\mu}, \delta\boldsymbol{v} \right\rangle \tag{27}$$

follows. Thus, whenever the vector of Lagrange multipliers corresponds to the adjoint eigenvector associated to $\boldsymbol{v}$ this term vanishes, because it satisfies the adjoint eigenvalue problem (10).

Evaluating the first term in (iii), which has to vanish whenever the second term vanishes, we obtain

$$\frac{\partial\mathcal{L}}{\partial \lambda} = 1 - \langle \boldsymbol{\mu}, \boldsymbol{B}\boldsymbol{v}\rangle. \tag{28}$$

Thus, with $\boldsymbol{\mu} \equiv \boldsymbol{w}$, the requirement of (28) to vanish is fulfilled if and only if the normalization condition

$$\langle \boldsymbol{w}, \boldsymbol{B}\boldsymbol{v}\rangle = 1 \tag{29}$$

is true. Then, the term $\left\langle \frac{\partial\mathcal{L}}{\partial\lambda} \right\rangle$ vanishes, and (iii) becomes zero. From here on we replace $\boldsymbol{\mu}$ by $\boldsymbol{w}$.

The term (ii) vanishes, too, with the choice $\boldsymbol{\mu} = \boldsymbol{w}$, because then

$$\frac{\partial\mathcal{L}}{\partial \boldsymbol{w}} = \langle \boldsymbol{1}, (\lambda\boldsymbol{B} - \boldsymbol{A})\,\boldsymbol{v}\rangle \tag{30}$$

which is equivalently zero, whenever $(\lambda, \boldsymbol{v})$ is an eigenpair. This is the case by definition of the problem setting.

Thus, given an eigenpair $(\lambda, \boldsymbol{v})$, if the vector of Lagrange multipliers $\boldsymbol{w}$ satisfies the normalization condition (29) and the adjoint eigenvalue problem (10), then

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\mathcal{L}) = \mathrm{grad}_{\boldsymbol{z}}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial \boldsymbol{z}}. \tag{31}$$

Consequently, the extremum condition (25) becomes

$$0 = \delta\mathcal{L} = \left\langle \frac{\partial\mathcal{L}}{\partial \boldsymbol{z}}, \delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle \tag{32a}$$

$$= \left\langle \frac{\partial\lambda}{\partial \boldsymbol{z}}, \delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle - \left\langle \boldsymbol{w}, \frac{\partial\left[(\lambda\boldsymbol{B} - \boldsymbol{A})\boldsymbol{v}\right]}{\partial \boldsymbol{z}}\delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle \tag{32b}$$

$$= \langle \boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda), \delta\boldsymbol{z}_{\mathrm{opt}}\rangle - \left\langle \boldsymbol{w}, \frac{\partial\left[(\lambda\boldsymbol{B} - \boldsymbol{A})\boldsymbol{v}\right]}{\partial \boldsymbol{z}}\delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle. \tag{32c}$$

From expression (32b) together with (32a) and (32c) it follows that

$$\langle \boldsymbol{S}_{\boldsymbol{z}}(\lambda), \delta\boldsymbol{z}_{\mathrm{opt}}\rangle = \left\langle \boldsymbol{w}, \frac{\partial\left[(\lambda\boldsymbol{B} - \boldsymbol{A})\boldsymbol{v}\right]}{\partial \boldsymbol{z}}\delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle \tag{33}$$

$$= \left\langle \left[\frac{\partial\left[(\lambda\boldsymbol{B} - \boldsymbol{A})\boldsymbol{v}\right]}{\partial \boldsymbol{z}}\right]^{\dagger}\boldsymbol{w}, \delta\boldsymbol{z}_{\mathrm{opt}} \right\rangle. \tag{34}$$

This implies

$$\boldsymbol{S}_{\boldsymbol{z}}(\lambda) = \left[\frac{\partial\left[(\lambda\boldsymbol{B} - \boldsymbol{A})\boldsymbol{v}\right]}{\partial \boldsymbol{z}}\right]^{\dagger}\boldsymbol{w}, \tag{35}$$

and since $\lambda\boldsymbol{B}\boldsymbol{v} = \textit{const.}$, because λ is fixed at the point $\boldsymbol{z}_{\mathrm{opt}}$ about which we expand, but the system matrix is parameterized by the operating point, we obtain the result

$$\boldsymbol{S}_{\boldsymbol{z}}(\lambda) = -\left[\frac{\partial\left[\boldsymbol{A}(\boldsymbol{z}_{\mathrm{opt}})\boldsymbol{v}\right]}{\partial \boldsymbol{z}}\right]^{\dagger}\boldsymbol{w}. \tag{36}$$

■

In words, theorem 1 states that an external, sufficiently large, and spatially localized power disturbance directly affects the dynamical behavior via the first variation of a specific eigenvalue. Then, the causal chain of the effect is as follows: a disturbance enters the system, it forces internal re-balancing of the algebraic variables in the network described by the Jacobian $\boldsymbol{J}_{\mathrm{ae}}$, and leads to transitioning to a new operation point according to $\partial\boldsymbol{\mathcal{R}}/\partial\boldsymbol{z}$, see remark 5. These shifts in network state due to algebraic balance laws interact with the local behavior in terms of a gain matrix, that amplifies or weakens the complex, small signal sensitivity of an eigenvalue $\boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda)$.

Remark 10. In our sensitivity analysis to a steady forcing the definition of the sensitivities depends on the choice of inner product (they are coordinate dependent) through computation of adjoint quantities, however, the variation of the dominant eigenvalue does not, see Meliga et al. (2010). Thus our result is independent of the specific choice of coordinates and relates to a generic system property.

Remark 11. Using classical results from eigenvalue sensitivity analysis, see for instance Chatelin (1983), Chapter 1, one can estimate variations of a specific eigenvalue λ according to the formula

$$\lambda^{+} - \lambda = \boldsymbol{w}^{\dagger}\boldsymbol{P}\boldsymbol{v} + O(\tau^{2}), \tag{37}$$

where $\boldsymbol{P}$ is a perturbation matrix, such that λ^{+} is the eigenvalue associated to $\boldsymbol{A}^{+} := \boldsymbol{A} + \boldsymbol{P}$, with $\tau = ||\boldsymbol{P}||$ being suitably small. In contrast to this classical result, our approach includes nonlinear dependencies of $\delta\lambda$ resulting from the algebraic balance equations, and disturbing with $\boldsymbol{d}$ other than with $\boldsymbol{P}$ has a physical interpretation in terms of the components z_i at which the system is forced.

It is not a priori clear what structure and perturbation norm an equivalent matrix perturbation $\boldsymbol{P}$ should have.

4. DISCUSSION FOR ELECTRIC POWER SYSTEMS

The main result in theorem 1 is used to explain a mechanism for a subcritical Hopf-bifurcation, and highly sensitive dynamics in high-loading situations. These phenomena are typical in general large-scale physical systems but rarely observed at hand of classical power grid models. This gap is bridged using our main result, and it is shown that the combination of methods for static power flow computations with methods for analysis of dynamics are of importance in accounting for current changes in the electricity industry.

Electric Power System Equations: A structure-preserving classical model consists of swing dynamics for generators and the (lossless) network equations, as given in Sauer and Pai (1998), Chapter 7, where

$$\dot{\theta}_i = \omega_i - \omega_{\mathrm{sync}}, \tag{38a}$$

$$\frac{2H_i}{\omega_{\mathrm{sync}}}\dot{\omega}_i = T_{Mi} - \sum_{k=1}^{N+M} V_i V_k \sin(\theta_i - \theta_k), \tag{38b}$$

$$0 = P_{Li}(V_i) - \sum_{k=1}^{N+M} V_i V_j B_{ik} \sin(\theta_i - \theta_k), \tag{38c}$$

$$0 = Q_{Li}(V_i) + \sum_{k=1}^{N+M} V_i V_j B_{ik} \cos(\theta_i - \theta_k). \tag{38d}$$

Here, θ_i is the generator and voltage angle, ω_i the frequency, V_i a voltage magnitude, H_i is the mechanical inertia, T_{Mi} the constant mechanical torque input, which is usually controlled, and B_{ik} is the interconnection susceptance between bus i and k. Equations (38c) and (38d), are the algebraic network equations, referring to (1b). The vector of dynamical variables contains the states of the N generators, i.e. $\boldsymbol{x}^T = (\theta_1, \omega_1, \ldots, \theta_N, \omega_N)$, and the vector of algebraic variables contains the voltage phasors at M load-buses, i.e. $\boldsymbol{y}^T = (\theta_{N+1}, V_{N+1}, \ldots, \theta_{N+M}, V_{N+M})$, see also remark 3. In (38), the dynamic variables are coupled with the static network equations via active power flow (38c).

High-Loading Situations and Subcritical Hopf-Bifurcation: When the power grid becomes highly loaded, it is observed that the inherent tolerance to local, small disturbances reduces and the power system moves from being elastic to brittle, see IEA (2008). Despite this fact, load flow studies are methodologically separated from local stability and local sensitivity studies of dynamics, see remark 4. Theorem 1 shows that small changes in the balanced load flows (38c) and (38d), see remark 3, have a direct impact on the dynamical ebhavior via first variations $\delta\lambda$. Moreover, the load flow Jacobian $\boldsymbol{J}_{\mathrm{ae}}$ - and by that the power flow Jacobian $\partial\boldsymbol{\mathcal{R}}/\partial\boldsymbol{z}$, see remark 5 - acts as gain on the sensitivity direction $\boldsymbol{\mathcal{S}}_{\boldsymbol{z}}(\lambda)$ of the local dominant behavior. In the following we argue that the gain results in an amplification of local sensitivity rather than a damping. When control is not "perfect", a subcritical Hopf-bifurcation may be induced by the increased sensitivity, i.e. a pair of complex eigenvalues crosses the imaginary axis while a steady-state power flow solution still exists, see Sauer and Pai (1998), Chapter 8.

In a subcritical but highly loaded regime the load flow Jacobian is not singular, but ill-conditioned, see Wang et al. (2001). Ill-conditionedness means, that the matrix $\boldsymbol{J}_{\mathrm{ae}}$ has a large condition number κ, which is defined as

$$\kappa = ||\boldsymbol{J}_{\mathrm{ae}}|| \cdot ||\boldsymbol{J}_{\mathrm{ae}}^{-1}|| = \frac{\sigma_{\max}(\boldsymbol{J}_{\mathrm{ae}})}{\sigma_{\min}(\boldsymbol{J}_{\mathrm{ae}})}, \tag{39}$$

where $\sigma_{\mathrm{max/min}}$ represents the maximal/minimal singular value. For electric power systems in stressed situations the value κ is of the order 10^3 as a result of $\sigma_{\min} \to 0$, see Wang et al. (2001). Writing (38c) and (38d) for small power disturbances ΔP_{Li} and ΔQ_{Li} yields to first order (and after re-ordering the vector $\boldsymbol{y}$)

$$\begin{pmatrix}\Delta\boldsymbol{P}_L\\ \Delta\boldsymbol{Q}_L\end{pmatrix} = \boldsymbol{J}_{\mathrm{ae}}\begin{pmatrix}\Delta\boldsymbol{\theta}\\ \Delta\boldsymbol{V},\end{pmatrix} \Leftrightarrow \begin{pmatrix}\Delta\boldsymbol{\theta}\\ \Delta\boldsymbol{V},\end{pmatrix} = \boldsymbol{J}_{\mathrm{ae}}^{-1}\begin{pmatrix}\Delta\boldsymbol{P}_L\\ \Delta\boldsymbol{Q}_L\end{pmatrix}. \tag{40}$$

From diagonal dominance and positivity of $\boldsymbol{J}_{\mathrm{ae}}^{-1}$, see Barret et al. (1997), Chapter 2, it follows that the inverse of the Jacobian acts as amplifier on small power injections, with a worst case gain factor in the order of the natural norm induced on the vector of power disturbances, i.e. $1/\sigma_{\min}$, which approaches infinity the closer the system is operated to singularity (i.e. the higher the system is loaded). A subcritical Hopf-bifurcation may occur in situations of "non-perfect" control, because then the system exhibits complex eigenvalues, in contrast to purely real ones in the case of open-loop, or perfectly controlled small disturbance dynamics, see Sauer and Pai (1998), Chapter 8, and Cao and Hill (2010). Real eigenvalues imply symmetric, non-interacting local dynamics and no drift or ill-conditioning due to transport of power.

Transport Phenomena and Trading of Large Amounts of Electricity: Theorem 1 shows that transfer of power within the power grid directly affects the small-signal dynamics, when transport phenomena evolve on a time-scale similar to that of real-time operation (seconds to minutes). The previous discussion of high-loading situations motivates high eigenvalue sensitivity in linearized models of full power system dynamics, and in physically networked systems in general. In fact, it is seen that in numerous large-scale physical systems, which are represented by partial differential equations (PDEs), the linearized system operator is highly ill-conditioned, exhibiting high eigenvalue sensitivity, see Trefethen and Embree (2005). It has been shown, that convective transport, i.e. drift terms that arise naturally in PDE settings, is a source of ill-conditionedness of physical linear dynamics, see Chomaz (2005), and Mangesius and Polifke (2011) for a network model illustration. However, the system matrix used for the modal analysis of systems of type (38) usually has condition number (close to) one, because the system matrix is (close to) symmetric, see Sauer and Pai (1998), Chapter 8. Hence, eigenvalue sensitivity is low, despite the fact that it fits the class of large, physically networked systems. An explanation using theorem 1 could be as follows: the load flow Jacobian, which represents power transport within the system, introduces large gains, and with that a high sensitivity of dominant eigenvalue positions in the complex plane, w.r.t. small disturbances entering the system.

This discussion on including power flow considerations into analysis of linear behavior is of immediate importance in practice. Today's electricity infrastructure is not suited to support large shifts of power, see UCTE (2007). However, liberalization tendencies and integration of renewable energy sources have lead to an economic motif force, that drives transfer of ever growing power volumes all across the continent (of Europe) at ever shorter time intervals.

5. CONCLUSION AND FUTURE WORK

In this paper we present an analytical framework that allows to quantify the effect of an external, large disturbance on the local (linearized) dynamics of physically networked systems. Our methodology is particularly useful for the analysis of electric power systems. In specific, it bridges a methodological gap between different types of models that are used in the analysis of different dynamic phenomena: we integrate models for the study of long-term dynamics, which base on snapshots of operating points obtain from nonlinear static optimization, with dynamic stability methods, that classically base on constant steady-state solutions and modal analysis. Our result exposes the relation between load flow optimization problems and dynamical behavior, where the load flow Jacobian takes the role of a gain. In high-loading situations nonlinear mechanisms, and highly sensitive, nonlocal coupling determine the behavior. The recent trends of liberalization, with ever growing trading of large power volumes across countries, and integration of variable generation make our result relevant, for instance for novel design and analysis tools supporting innovative operation schemes, beyond traditional decentralized, and hierarchically separated approaches.

Work is in progress, where our framework is applied to define and specify technical flexibility within a power system. The distributed computation of sensitivity vectors, and the use of the latter for real-time distributed control, both combined in a novel control architecture seems to be a promising approach for research in the direction of restructured electric power systems.

REFERENCES

Anderson, B.D. and Vongpanitlerd, S. (2006). *Network Analysis and Synthesis - A Modern Systems Theory Approach.* Dover Publications, Inc.

Ascher, U.M. and Petzold, L.R. (1998). *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations.* SIAM.

Barret, J.P., Bornard, P., and Meyer, B. (1997). *Power System Simulation.* Chapman & Hall.

Bialek, J. (2007). Why has it happened again? comparison between the ucte blackout in 2006 and the blackouts of 2003. In *Power Tech, 2007 IEEE Lausanne*, 51–56. doi: 10.1109/PCT.2007.4538291.

Cao, G.Y. and Hill, D. (2010). Power system voltage small-disturbance stability studies based on the power flow equation. *Generation, Transmission Distribution, IET*, 4(7), 873–882. doi:10.1049/iet-gtd.2010.0016.

Chatelin, F. (1983). *Spectral Approximation of Linear Operators.* SIAM.

Chomaz, J.M. (2005). Global instabilities in spatially developing flows: Non-normality and nonlinearity. *Annual Review of Fluid Mechanics*, 37(1), 357–392.

Dobson, I. and Lu, L. (1992). Computing an optimum direction in control space to avoid saddle node bifurcation and voltage collapse in electric power systems. *IEEE Transactions on Automatic Control*, 37, 1616–1620.

Gunzburger, M.D. (2003). *Perspectives in Flow Control and Optimization*, chapter Three Approaches to Optimal Control and Optimization, 11–61. SIAM - Advances in Design and Control.

Hill, D. and Mareels, I.M.Y. (1990). Stability theory for differential/algebraic systems with application to power systems. *Circuits and Systems, IEEE Transactions on*, 37(11), 1416–1423. doi:10.1109/31.62415.

IEA (2008). Empowering variable renewables - options for flexible electricity systems. Technical report, Inernational Energy Agency.

Ilic, M. and Zaborszky, J. (2000). *Dynamics and Control of Large Electric Power Systems.* John Wiley & Sons.

Mangesius, H. and Polifke, W. (2011). A discrete-time, state-space approach for the investigation of non-normal effects in thermoacoustic systems. *International Journal of spray and combustion dynamics*, 3, 331–350.

Meliga, P., Sipp, D., and Chomaz, J.M. (2010). Open-loop control of compressible afterbody flows using adjoint methods. *Physics of Fluids*, 22, 053109.

Qadri, U. and Juniper, M. (2012). A theoretical approach to the passive control of spiral vortex breakdown. In *2012 UKACC International Conference on Control*, 656–661. doi:10.1109/CONTROL.2012.6334707.

Sauer, P.W. (2005). Reactive power and voltage control issues in electric power systems. In J. Chow, F. Wu, and J. Momoh (eds.), *Applied Mathematics for Restructured Electric Power Systems*, Power Electronics and Power Systems, 11–24. Springer US.

Sauer, P.W. and Pai, M.A. (1998). *Power System Dynamics and Stability.* Prentice Hall.

Sauer, P. and Pai, M. (1990). Power system steady-state stability and the load-flow jacobian. *IEEE Transactions on Power Systems*, 5(4), 1374–1384.

Seiler, W. (2010). *Involution - The Formal Theory of Differential Equations and its Applications in Computer Algebra*, volume 24 of *Algorithms and Computation in Mathematics.* Springer.

Trefethen, L.N. and Embree, M. (2005). *Spectra and Pseudospectra - The behavior of Nonnormal Matrices and Operators.* Princeton University Press.

UCTE (2007). Final report - system disturbance on 4 november 2006. Technical report, Union for the Coordination of Transmission of Electricity.

Verghese, G.C., Perez-Arriaga, I.J., and Schweppe, F. (1982). Selective modal analysis with applications to electric power systems, part 1 & 2. *IEEE Transactions on Power Apparatus and Systems*, PAS-101(9), 3117–3134.

Wang, Y., Silva, L.C.D., Xu, W., and Zhang, Y. (2001). Analysis of ill-conditioned power-flow problems using voltage stability methodology. *Generation, Transmission and Distribution, IEE Proceedings-*, 148(5), 384–390. doi:10.1049/ip-gtd:20010424.

Willems, J.C. (2010). Terminals and ports. *IEEE Circuits and Systems*, 10(4), 8–26.