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Value of Information Analysis with Structural Reliability Methods

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Abstract

When designing monitoring systems and planning inspections, engineers must assess the benefits of the additional information that can be obtained and weigh them against the cost of these measures. The Value of Information (VoI) concept of the Bayesian statistical decision analysis provides a formal framework to quantify these benefits. This paper presents the determination of the VoI when information is collected to increase the reliability of engineering systems. It is demonstrated how structural reliability methods can be used to effectively model the VoI and an efficient algorithm for its computation is proposed. The theory and the algorithm are demonstrated by an illustrative application to monitoring of a structural system subjected to fatigue deterioration.

Keywords

Inspection; structural health monitoring; reliability; Bayesian analysis; updating; decision analysis; value of information

1 Introduction

When it is required to make decisions under uncertainty and risk, one often has the possibility to gather further information prior to making the decision. Such information reduces the uncertainty and thus facilitates improved decision making. This explains the success of structural health monitoring (SHM), advanced inspection methods, remote sensing and other monitoring techniques for civil infrastructures, to which I will refer collectively as monitoring systems.

As experienced engineers are well aware, collecting the information comes at a price that is not always justified by its benefit. Unfortunately, this is often discovered only after the installation of a monitoring system. A mathematical framework exists for quantitatively assessing the benefit of a monitoring system prior to installing it: the value of information (VoI) analysis from Bayesian statistical decision theory [1-3] that has been considered by civil and structural engineers since the early 1970s [4]. The late Prof. Wilson Tang was one of the first to notice the potential of Bayesian methods and VoI concepts to optimize engineering decisions [5-7]. In his paper published in 1973 [5], he described Bayesian updating of probabilistic models of flaws with inspection results, which preceded the optimization of inspections in aircraft and offshore structures subject to fatigue deterioration in the 1970s and 80s [8-12]. These works were among the first applications of Bayesian decision analysis for optimizing the collection of information in an industrial context. Similar efforts were made in the field of transportation infrastructure management, based on Markovian deterioration models [13]. In recent years, the optimization of monitoring systems through explicit computation of the VoI has found increased interest in various fields of civil and infrastructure engineering. Explicit computation of the VoI for optimizing inspections and structural health monitoring in deteriorating structures was proposed in [14-18]. Optimization of sensor placement based on VoI has been studied in [19]. In geotechnical engineering, which has always been strongly relying on monitoring, the effect of information quality has been investigated [20]; an explicit quantification of the VoI for head monitoring of levees is described in [21]. In the field of natural hazards, the VoI concept has been applied for prioritizing post-earthquake inspections of bridges [22] and for quantifying the value of improved climate models when designing offshore structures against extreme wave loads [23]. VoI analysis is and has been applied in many other fields of engineering and science, including oil exploration [24] and environmental health risk management [25].

Determining the VoI requires significant modeling and computational efforts. Computationally efficient evaluations of the VoI was considered mainly in the field of machine learning and artificial intelligence [19, 26-28]. In these areas, prediction models used for the VoI computations are typically based on known probabilistic dependences among a potentially large number of random variables. In contrast, in infrastructure and civil engineering, prediction models are often based on advanced physically-based models, which describe the monitored phenomena. As an example, when planning the monitoring of a bridge, one can make use of detailed mechanical models of the structure. Furthermore, the monitoring is often installed not to guide the every-day operation of the system, but for early detection of deterioration or damages that may impair the safety of the system. These applications motivate the combination of the VoI concept with structural reliability methods, which were developed to efficiently compute the probability of system failure via advanced physically-based models.

This paper presents the modeling and computation of VoI based on structural reliability methods. A modeling framework is proposed, which is especially suitable when probabilistic physically-based models of the monitored systems and processes are available, e.g. in structural engineering applications. On this basis, a computationally efficient algorithm is developed for estimating the VoI. The framework and the algorithm are illustrated through an application to monitoring of a structure subject to fatigue deterioration, which demonstrates the effectiveness and efficiency of the proposed approach. The paper closes with a discussion on the difficulties encountered in determining the VoI in realistic engineering problems.

2 Value of information analysis

2.1 Decision-theoretic framework

As a premise, I assume that all consequences (costs of monitoring, mitigation actions as well as failure consequences) can be expressed either in monetary values or in a common measure of utility U. I adopt the classical expected utility framework [29] according to which an optimal decision under uncertainty is the one maximizing the expected utility $\mathrm{E}[U]$. For simplicity, I further restrict the presentation to situations in which all consequences can be expressed as monetary costs C and in which utility is proportional to $-C$, corresponding to a risk-neutral decision maker. The optimal decision is thus the one that minimizes the expected cost $\mathrm{E}[C]$. It is straightforward to adapt the methods presented in this paper to the case of a risk-averse decision maker or to situations with non-monetary consequences, if preferences of the decision maker can be expressed through utility functions.

Following the classical structural reliability modeling framework [30], the uncertainty associated with the phenomena under consideration is characterized by a vector $\mathbf{X}$ of random variables. The relation between $\mathbf{X}$ and the events of interest is a deterministic one, e.g. the failure event is described through the limit state function $g_F(\mathbf{X})$ as $F = \{g_F(\mathbf{X}) \leq 0\}$. In this framework, model uncertainties are included through additional random variables in $\mathbf{X}$.

In a classical decision analysis under uncertainty, the goal is to identify the actions a that minimize $\mathrm{E}[C]$, e.g. the maintenance and repair actions a that ensure an optimal balance between the cost of a and the risk associated with failure. Additionally, information can be collected prior to making the action decision a. Therefore, a so-called test decision e is made on what information to collect (e stands for experiments). This is, e.g., a decision on the design of a monitoring system or a decision on the inspection schedule. The extended decision problem is to find the combination of monitoring decision e and action decision a that minimizes $\mathrm{E}[C]$. This problem is known in the literature as *preposterior decision analysis* [4]. These problems can be graphically modeled through decision trees and decision graphs (also called influence diagrams), Figure 1. The decision tree explicitly depicts all possible states of random variables and decisions. In contrast, the decision graph provides a more concise representation, which additionally reflects the causal relations between the random variables and the decisions. Implementations of the decision graph for computing the VoI can be found in [16, 31].

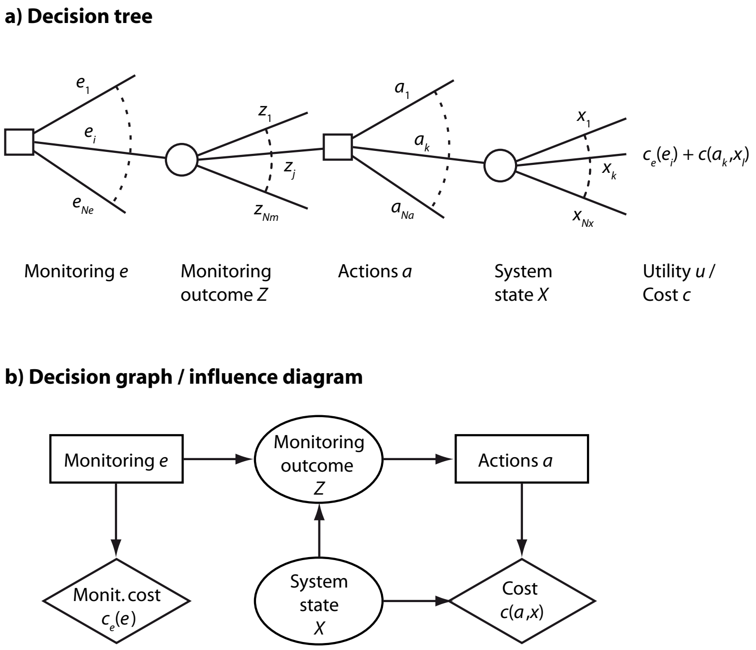

Figure 1.The basic decision problem when planning monitoring and inspection measures: (a) decision tree and (b) corresponding decision graph. Here it is assumed that the cost of monitoring $c_e(e)$ and the cost of the action and system state $c(a, x)$ are additive.

This paper focuses on the computation of the value of information (VoI) of a given monitoring system. The optimization of the monitoring system (the test decision e) is not explicitly considered. However, the VoI is the total expected net benefit of a given monitoring system and is thus the central part of any preposterior decision analysis. The optimal monitoring system is the one maximizing the VoI minus the cost of monitoring.

In the following, the optimization of the decision a is presented prior to considering the monitoring results. This follows the logic that monitoring results enable improved action decisions and that their benefit can thus only be quantified when explicitly modeling the action decision.

2.2 Prior decision optimization

Before applying monitoring, the optimization of the decision a must be based on the prior knowledge, characterized through the prior probability distribution of $\mathbf{X}$. The prior optimization problem is:

$$\begin{aligned} a_{opt} &= \arg\min_a \mathrm{E}_{\mathbf{X}}[c(a, \mathbf{X})] \\ &= \arg\min_a \int_{\mathbf{X}} c(a, \mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \,. \end{aligned} \tag{1}$$

$c(a, \mathbf{x})$ is the cost associated with a given set of actions a and realization $\mathbf{x}$, and $\mathrm{E}_{\mathbf{X}}$ denotes the expectation with respect to $\mathbf{X}$. Throughout the paper I use the notation $\int_{\mathbf{X}} \mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathrm{d}x_1 \dots \mathrm{d}x_n$.

In engineering decision problems involving reliability, the consequences typically depend on discrete events describing the system state, such as failure F or a set of damage levels (e.g., in performance-based earthquake engineering). In the structural reliability framework, these events correspond to domains in the outcome space of $\mathbf{X}$. Let $E_1, E_2, \dots, E_m$ denote the mutually exclusive, collectively exhaustive system states in the general case. (If only failure F is of interest, it is $E_1 = F$ and $E_2 = \bar{F}$.) The optimization problem can then be written as

$$a_{opt} = \arg\min_a \sum_{i=1}^{m} c_{E_i}(a) \Pr(E_i) \,. \tag{2}$$

Here, $c_{E_i}(a)$ is the cost associated with event E_i and decision a. Let C_{prior} denote the expected cost associated with this optimal decision a_{opt}, i.e.

$$C_{prior} = \min_a \sum_{i=1}^{m} c_{E_i}(a) \Pr(E_i) = \sum_{i=1}^{m} c_{E_i}(a_{opt}) \Pr(E_i)\,. \tag{3}$$

The probability of E_i is computed using structural reliability methods as

$$\begin{aligned} \Pr(E_i) &= \int_{\mathbf{X}} I\big(\mathbf{x} \in \Omega_{E_i}\big) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \int_{\mathbf{x}\in\Omega_{E_i}} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x}\,. \end{aligned} \tag{4}$$

$I(\cdot)$ is the indicator function and Ω_{E_i} is the domain in the outcome space of $\mathbf{X}$ corresponding to event E_i. In structural reliability, Ω_{E_i} is defined in terms of limit state functions $g_i(\mathbf{x})$. In case of a single limit state function, it is $\Omega_{E_i} = \{g_i(\mathbf{x}) \le 0\}$. More generally, Ω_{E_i} is defined through unions and intersections of multiple $\{g_i(\mathbf{x}) \le 0\}$, [32].

To make these abstract concepts more apprehensible, consider the simple example of a mechanical system that either functions during its entire service life or fails at some point in time. For simplicity, the time of the failure is considered irrelevant. The failure event is described as $F = \{g_F(\mathbf{X}) \le 0\}$, where $\mathbf{X}$ includes parameters describing deterioration of the system. It is possible to perform maintenance actions during the service life. Let a_0 (do nothing) and a_m (maintenance) denote the two decision alternatives. If maintenance is performed, instead of F one needs to consider F_m, failure of the maintained system. This event is described through a corresponding limit state function $g_{Fm}(\cdot)$ as $F_m = \{g_{Fm}(\mathbf{X}) \le 0\}$. Therefore, there are four distinct events (system states): $E_1 = \bar{F} \cap \bar{F}_m$, $E_2 = F \cap \bar{F}_m$, $E_3 = \bar{F} \cap F_m$, and $E_4 = F \cap F_m$.

The cost of maintenance is c_m. The cost of failure is c_F. The expected cost of the two action alternatives are:

$$\begin{aligned} \mathrm{E}[C|a_0] = \mathrm{E}_{\mathbf{X}}[c(a_0, \mathbf{X})] &= \int_{\mathbf{X}} c(a_0, \mathbf{X}) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \int_{\mathbf{X}} \{c_F\, I[g_F(\mathbf{X}) \le 0]\} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= c_F \int_{\mathbf{X}} I[g_F(\mathbf{X}) \le 0] f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= c_F \Pr(F), \end{aligned} \tag{5}$$

$$
\begin{aligned}
\mathrm{E}[C|a_m] = \mathrm{E}_{\mathbf{X}}[c(a_m,\mathbf{X})] &= \int_{\mathbf{X}} c(a_m,\mathbf{X}) f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{X}} \{c_m + c_F\, I[g_{Fm}(\mathbf{X}) \leq 0]\}\, f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= c_m + c_F \int_{\mathbf{X}} I[g_{Fm}(\mathbf{X}) \leq 0]\, f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= c_m + c_F \Pr(F_m).
\end{aligned}
\tag{6}
$$

The integrals in Eqs. (5) and (6) are classical structural reliability problems and can be solved e.g. with FORM/SORM or sampling-based methods.

The results in Eq. (5) and (6) are rather trivial, and the decision optimization $a_{opt} = \arg\min_a(\mathrm{E}[C|a_0], \mathrm{E}[C|a_m])$ is straightforward. Nevertheless, it will later prove beneficial to explicitly model the mutually exclusive events E_1, E_2, E_3, E_4. The costs associated with these events and decisions a_0 or a_m are summarized in Table 1.

Table 1. Costs associated with the mutually exclusive events $E_1 = \bar{F} \cap \bar{F}_m$, $E_2 = F \cap \bar{F}_m$, $E_3 = \bar{F} \cap F_m$, and $E_4 = F \cap F_m$.

$c_{E_1}(a_0) = 0$,	$c_{E_2}(a_0) = c_F$,	$c_{E_3}(a_0) = 0$,	$c_{E_4}(a_0) = c_F$,
$c_{E_1}(a_m) = c_m$,	$c_{E_2}(a_m) = c_m$,	$c_{E_3}(a_m) = c_m + c_F$,	$c_{E_4}(a_m) = c_m + c_F$.

Since it is $F = E_2 \cup E_4$ and $F_m = E_3 \cup E_4$, it should be clear that the optimization following Eq. (2) gives the same result as the above solution through Eqs. (5) and (6).

Note: For many applications, it is initially more intuitive to model the effect of a decision through changes of the probability distribution of $\mathbf{X}$, i.e. by replacing $f_{\mathbf{X}}(\mathbf{x})$ with $f_{\mathbf{X}|a}(\mathbf{x}|a)$ in Eq. (1). In the above example, this would signify not making a distinction between F and F_m. The effect of the maintenance would instead be modeled through the difference between $f_{\mathbf{X}|a}(\mathbf{x}|a_0)$ and $f_{\mathbf{X}|a}(\mathbf{x}|a_m)$, resulting in $\Pr(F|a_0)$ and $\Pr(F|a_m)$. However, such a modeling approach is less rigorous, as the model parameters before and after the maintenance are in fact different random variables. For this reason, the intuitive approach has several disadvantages over the proposed approach: (a) The random variables representing the same model parameter before and after an action are often dependent. This cannot be modeled with the intuitive approach, where the parameter is represented with only one random variable that is defined through two or more conditional distributions. With the proposed approach, modeling the dependence is straightforward since the parameter is represented through multiple random variables, which are included jointly in $\mathbf{X}$ and can thus be correlated. (b) The intuitive approach hinders the standardization of the computations. With the proposed approach, the optimization of the action decision in Eq. (2) is completely general. It is only necessary to identify the correct limit state functions and the corresponding domains Ω_{E_i}. (c) The proposed structural reliability based modeling facilitates the computations of the VoI, as presented later.

2.3 Perfect information

Perfect information corresponds to the hypothetical situation in which there is no uncertainty on $\mathbf{X}$. In this case, the decision maker is always able to select the best action a. In the outcome space of $\mathbf{X}$, one can identify domains in which one action is optimal, Figure 2. These domains are described by one or multiple limit state functions and are equal to the domains Ω_{E_i} or unions thereof. The case shown in Figure 2 corresponds to the simple example provided earlier. The decision a_m (maintenance) is optimal only under $E_2 = \{g_F(\mathbf{X}) \leq 0\} \cap \{g_{Fm}(\mathbf{X}) > 0\}$, i.e. when the component fails without maintenance F but does not fail with maintenance $\overline{F_m}$. For all other events, i.e. for $E_1 \cup E_3 \cup E_4$, a_0 (do nothing) is optimal. This follows directly from Table 1 (given that the cost of maintenance is smaller than the cost of failure, $c_m < c_F$). Note that perfect information means here that the failure is known to occur (or not) at the time of making the decision, i.e. *before* it actually does occur (or not).

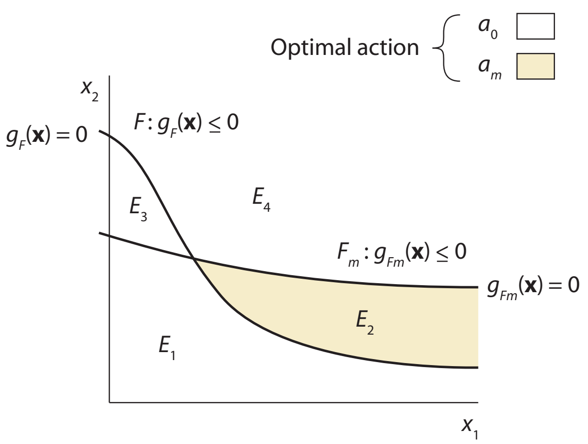

Figure 2. The optimal action corresponding to each outcome $\mathbf{x} = [x_1; x_2]$ can be described through limit state functions $g(\mathbf{x})$. The situation shown here corresponds to the basic example of section 2.2 with two action alternatives. Action a_m (maintenance) is optimal only in the event $E_2 = \{g_F(\mathbf{X}) \leq 0\} \cap \{g_{Fm}(\mathbf{X}) > 0\}$, else action a_0 (do nothing) is optimal.

To formalize the concept of decisions under perfect information, let $a^*_{opt}(\mathbf{x})$ denote the optimal decision for given $\mathbf{x}$. It is

$$a_{opt}^{*}(\mathbf{x}) = \arg\min_{a} c(a, \mathbf{x}). \tag{7}$$

When comparing the cost associated with this optimal action $a_{opt}^{*}(\mathbf{x})$ to the one of choosing the prior optimal action a_{opt}, the so-called conditional value of perfect information (CVoPI) is obtained for given system state $\mathbf{X} = \mathbf{x}$:

$$CVoPI(\mathbf{x}) = c\left(a_{opt}, \mathbf{x}\right) - c\left(a_{opt}^{*}(\mathbf{x}), \mathbf{x}\right). \tag{8}$$

Since the costs are uniquely determined by the events E_i, the CVoPI depends only on which event E_i occurs. Therefore, the CVoPI can be expressed for given E_i:

$$CVoPI_{E_i} = c_{E_i}\left(a_{opt}\right) - c_{E_i}\left(a_{opt,i}^{*}\right), \tag{9}$$

where the notation $a_{opt,i}^{*}$ is introduced for the optimal decision under event E_i.

From Eq. (9) it can be observed that the CVoPI is non-zero only if the optimal decision under a known system state E_i, $a_{opt,i}^{*}$, differs from the optimal decision that is taken under prior information, a_{opt}. In case $a_{opt,i}^{*} \neq a_{opt}$, the former will lead to lower cost and therefore the CVoPI cannot be negative.

A-priori, the true value of $\mathbf{X}$ and the true event E_i are not known. Nevertheless, it is possible to compute the hypothetical value of perfect information (VoPI) [1], defined as the expected value of the CVoPI:

$$\begin{aligned} VoPI &= \mathrm{E}_{\mathbf{X}}[CVoPI(\mathbf{X})] \\ &= \int_{\mathbf{X}} c\left(a_{opt}, \mathbf{x}\right) - c\left(a_{opt}^{*}(\mathbf{x}), \mathbf{x}\right) \mathrm{d}\mathbf{x} \\ &= \int_{\mathbf{X}} c\left(a_{opt}, \mathbf{x}\right) \mathrm{d}\mathbf{x} - \int_{\mathbf{X}} c\left(a_{opt}^{*}(\mathbf{x}), \mathbf{x}\right) \mathrm{d}\mathbf{x} \\ &= \min_{a} \int_{\mathbf{X}} c(a, \mathbf{X}) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} - \int_{\mathbf{X}} \min_{a} c(a, \mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= C_{prior} - \int_{\mathbf{X}} \min_{a} c(a, \mathbf{x})\, \mathrm{d}\mathbf{x}\ , \end{aligned} \tag{10}$$

or alternatively as:

$$
\begin{aligned}
VoPI &= \sum_{i=1}^{m} CVoPI_{E_i} \Pr(E_i) \\
&= C_{prior} - \sum_{i=1}^{m} c_{E_i}\left(a_{opt,i}^{*}\right) \Pr(E_i) \, .
\end{aligned} \tag{11}
$$

This value thus corresponds to the difference in expected utility between the situation a-priori and the expected utility under a situation of perfect information. The VoPI is the upper limit of the value *any* monitoring system can have, irrespective of its capabilities. Any monitoring system that is more expensive than the VoPI will not be efficient.

By definition, the VoPI cannot be negative, since $\min_a \int_{\mathbf{X}} c(a, \mathbf{x}) \mathrm{d}\mathbf{x} \geq \int_{\mathbf{X}} \min_a c(a, \mathbf{x}) \, \mathrm{d}\mathbf{x}$. This follows also from the fact that the CVoPI cannot be negative.

Note: Viable action alternatives may exist, which are not optimal under any known $\mathbf{x}$ and therefore would not appear in Figure 2, but which may be optimal under conditions of uncertainty.

2.4 Imperfect information

In real applications, monitoring systems provide only imperfect information on $\mathbf{X}$. Most measurements are subject to random errors or uncertainty. But even in the absence of these, monitoring systems are imperfect because they do not provide direct information on all $\mathbf{X}$. As an example, for a case where perfect knowledge of material parameters is available, if the future loading remains uncertain, the failure event cannot be predicted with certainty. Additionally, most measurements are indirect, in particular for monitoring of existing structures and geotechnical applications [33].

2.4.1 Bayesian updating

Imperfect information can be used to learn about $\mathbf{X}$ and, consequently, about the events $E_1, \dots, E_m$. Bayesian updating is the mathematical framework for learning the probability distribution of $\mathbf{X}$ and the probabilities of $E_1, \dots, E_m$ with new imperfect information [5, 34, 35].

Following Bayes' rule, the conditional distribution of $\mathbf{X}$ given an observation Z, the posterior distribution, is:

$$
f_{\mathbf{X}|Z}(\mathbf{x}) = \frac{L(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})}{\int_{\mathbf{X}} L(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x}} \, . \tag{12}
$$

The likelihood function $L(\mathbf{x})$ describes the relation between the monitoring outcome event Z and the uncertain variables $\mathbf{X}$. It is defined as [36]:

$$L(\mathbf{x}) \propto \Pr(Z|\mathbf{X} = \mathbf{x}). \tag{13}$$

In the structural reliability context, the monitored quantities are modeled as functions q_i of $\mathbf{X}$, and the monitoring outcome event Z can be expressed by means of $q_i(\mathbf{X})$. The most common case is that of measurements $\mathbf{y} = [y_1, \dots, y_m]$ of quantities $q(\mathbf{X}) = [q_1(\mathbf{X}), q_2(\mathbf{X}), \dots, q_m(\mathbf{X})]$. If measurement errors ϵ_i are additive and statistically independent random variables, the relation between measurements y_i and $\mathbf{x}$ is $y_i = q_i(\mathbf{x}) + \epsilon_i$. Thus the event Z is defined as $Z = \{\mathbf{y} = q(\mathbf{X}) + \boldsymbol{\epsilon}\}$, with $\boldsymbol{\epsilon} = [\epsilon_1, \dots, \epsilon_m]$.

It follows that $y_i - q_i(\mathbf{x}) = \epsilon_i$, and the likelihood function of the monitoring outcome $\mathbf{y}$ is

$$L(\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{y}|\mathbf{x}) = \prod_{i=\mathbf{1}}^{m} f_{\boldsymbol{\epsilon}_i}[y_i - q_i(\mathbf{x})]\,. \tag{14}$$

For further details on how to model observations with likelihood functions, the reader is referred to [35, 37].

In structural reliability analysis, the explicit computation of $f_{\mathbf{X}|Z}(\mathbf{x})$ can be circumvented and instead $\Pr(E_i|Z)$ can be obtained directly from the definition of the conditional probability as

$$\Pr(E_i|Z) = \frac{\Pr(E_i \cap Z)}{\Pr(Z)}\,. \tag{15}$$

As shown in [37], structural reliability methods can be used to compute both the numerator and the denominator in Eq. (15). Thereby it is relevant to distinguish between two classes of monitoring outcomes: Those that provide *inequality information* and those that provide *equality information* [37, 38]. Monitoring outcomes of the inequality type can be characterized through a function $h(\mathbf{X})$ as follows:

$$Z = \{h(\mathbf{X}) \leq 0\}\ . \tag{16}$$

Examples include monitoring outcomes such as “deformations are larger than a threshold” or “no defect found”. For inequality information, reliability updating is straightforward, since $h(\mathbf{X})$ can be interpreted as a limit state function describing the event Z and any of the available structural reliability methods can be applied. In this case, it is not necessary to explicitly formulate the likelihood function.

Monitoring outcomes of the equality type are thus called because they can be described by an equality, such as $Z = \{\mathbf{y} = q(\mathbf{X}) + \boldsymbol{\epsilon}\}$ introduced above. In the general case we can write $Z = \{\mathbf{y} = \mathbf{Y}\}$, where $\mathbf{Y}$ is the monitoring outcome as predicted by the model (for the special case it is $\mathbf{Y} = q(\mathbf{X}) + \boldsymbol{\epsilon}$). Examples include measurements of defect sizes, deformations or loads. Most monitoring outcomes are of this form. For equality observations, measurements are best described through the likelihood function $L(\mathbf{x})$, such as Eq. (14). Direct application of Eq. (15) is not straightforward for equality information, because it requires the solution of surface integrals to compute the probabilities [39, 40]. An efficient alternative was proposed by the author in [37], which is based on transforming the likelihood function $L(\mathbf{x})$, into equivalent inequality information, which can be efficiently and effectively combined with existing structural reliability methods to evaluate Eq. (15).

A simple solution to updating with equality information, which is also a special case of the method proposed in [37], is to first update the distribution of $\mathbf{X}$ following Eq. (12) and then compute the conditional probability $\Pr(E_i|Z)$ by performing reliability analysis with the posterior PDF $f_{\mathbf{X}|Z}(\mathbf{x})$. Using a Monte Carlo simulation approach, an estimate of $\Pr(E_i|Z)$ is obtained as:

$$\begin{aligned}
\Pr(E_i|Z) &= \int_{\mathbf{X}} I\big(\mathbf{x} \in \Omega_{E_i}\big) f_{\mathbf{X}|Z}(\mathbf{x}) \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{X}} I\big(\mathbf{x} \in \Omega_{E_i}\big) \frac{L(\mathbf{x}) f_X(\mathbf{x})}{\int_{\mathbf{X}} L(\mathbf{x}) f_X(\mathbf{x}) \mathrm{d}\mathbf{x}} \mathrm{d}\mathbf{x} \\
&= \frac{\int_{\mathbf{X}} I\big(\mathbf{x} \in \Omega_{E_i}\big) L(\mathbf{x}) f_X(\mathbf{x}) \mathrm{d}\mathbf{x}}{\int_{\mathbf{X}} L(\mathbf{x}) f_X(\mathbf{x}) \mathrm{d}\mathbf{x}} \\
&\approx \frac{\sum_{k=1}^{n_{MCS}} I\big(\mathbf{x}_k \in \Omega_{E_i}\big) L(\mathbf{x}_k)}{\sum_{k=1}^{n_{MCS}} L(\mathbf{x}_k)}.
\end{aligned} \tag{17}$$

where $\mathbf{x}_k, k = 1, \dots, n_{MCS}$, are samples from the prior PDF $f_{\mathbf{X}}(\mathbf{x})$. The Monte Carlo procedure is generally inefficient as it requires a large number of samples n_{MCS} to achieve sufficient accuracy. For more efficient methods, the reader is referred to [37].

2.4.2 Conditional value of information

Once an observation Z has been made and the conditional $\Pr(E_i|Z)$, $i = 1, \dots m$, have been computed, decision optimization is in analogy to the prior decision optimization of Eq. (2):

$$a_{opt|Z} = \arg\min_a \sum_{i=1}^{m} c_{E_i}(a) \Pr(E_i|Z) \,. \tag{18}$$

The only difference to Eq. (2) is the replacement of $\Pr(E_i)$ with the conditional $\Pr(E_i|Z)$. The optimization conditional on Z is called *posterior* decision analysis and does not represent a computational issue once Bayesian updating has been performed.

For a given Z, it is furthermore possible to compute the conditional value of information:

$$CVoI_Z = \sum_{i=1}^{m} c_{E_i}(a_{opt}) \Pr(E_i|Z) - \sum_{i=1}^{m} c_{E_i}\left(a_{opt|Z}\right) \Pr(E_i|Z) \,. \tag{19}$$

Note that the CVoI is zero if the posterior optimal decision $a_{opt|Z}$ is the same as the a-priori optimal decision a_{opt}, and positive otherwise. The CVoI in itself is uninteresting. Once the observation Z is made, $\Pr(E_i|Z)$ represents the new state of nature according to which decisions should be made; it is futile to compare $a_{opt|Z}$ to the results of the original prior decision analysis a_{opt}. The true interest is in the value of information (VoI) of the monitoring system before an observation Z is made.

2.4.3 Value of information (VoI)

The VoI is the expected value of the CVoI with respect to all possible measurement outcomes $VoI = \mathrm{E}[CVoI]$. In case of a finite number of mutually exclusive measurement outcome events $Z_1, \ldots Z_l$, (monitoring outcomes of the inequality type), it is

$$\begin{aligned} VoI &= \sum_{j=1}^{l} CVoI_{Z_j} \Pr(Z_j) \\ &= \sum_{j=1}^{l} \Pr(Z_j) \left[\sum_{i=1}^{m} c_{E_i}(a_{opt}) \Pr(E_i|Z_j) - \sum_{i=1}^{m} c_{E_i}\left(a_{opt|Z}\right) \Pr(E_i|Z_j) \right] \\ &= C_{prior} - \left[\sum_{j=1}^{l} \Pr(Z_j) \min_a \sum_{i=1}^{m} c_{E_i}(a) \Pr(E_i|Z_j) \right] . \end{aligned} \tag{20}$$

The first term in the last line follows from the fact that

$$\sum_{j=1}^{l} \sum_{i=1}^{m} c_{E_i}(a_{opt}) \Pr(E_i|Z_j) \Pr(Z_j) = \sum_{i=1}^{m} c_{E_i}(a_{opt}) \sum_{j=1}^{l} \Pr(E_i \cap Z_j) \tag{21}$$

$$= \sum_{i=1}^{m} c_{E_i}(a_{opt}) \Pr(E_i)$$
$$= C_{prior} \, .$$

This shows that the expected cost associated with the prior decision does not depend on the monitoring outcome (as it clearly should not).

In case of continuous measurement outcomes $Z = \{\mathbf{Y} = \mathbf{y}\}$ (monitoring outcomes of the equality type), it is

$$\begin{aligned} VoI &= \int_{\mathbf{Y}} CVoI_Z f_{\mathbf{Y}}(\mathbf{y}) \mathrm{d}\mathbf{y} \\ &= C_{prior} - \left[\int_{\mathbf{Y}} f_{\mathbf{Y}}(\mathbf{y}) \min_a \sum_{i=1}^{m} c_{E_i}(a) \Pr(E_i | \mathbf{Y} = \mathbf{y}) \, \mathrm{d}\mathbf{y} \right] , \end{aligned} \tag{22}$$

wherein $f_{\mathbf{Y}}(\mathbf{y})$ is the joint PDF of the monitoring outcomes $\mathbf{Y}$.

Numerical methods are necessary to compute the integral in Eq. (22). Thereby, the conditional $\Pr(E_i | \mathbf{Y} = \mathbf{y})$ will have to be computed many times. A computationally efficient procedure for computing $\Pr(E_i | \mathbf{Y} = \mathbf{y})$ repetitively is thus crucial for computing the VoI in this case. Such a procedure is proposed in the following section.

3 Computationally efficient VoI analysis with structural reliability methods

The computationally costly part in computing the VoI is the evaluation of the system model as a function of $\mathbf{x}$, which is required to assess the system performance through $I(\mathbf{x} \in \Omega_{E_i})$, and to compute the monitored quantities, $h_i(\mathbf{x})$. Efficiency is thus measured by the number of model evaluations.

For monitoring systems whose outcome is of the inequality information type, efficient algorithms for evaluating $\Pr(E_i | Z)$ are available through structural reliability methods, e.g. based on FORM or advanced simulation combined with Eq. (15) [37, 38, 41]. These methods require computing $I(\mathbf{x} \in \Omega_{E_i})$ and $h_i(\mathbf{x})$ only for a small number of values of $\mathbf{x}$, typically in the order of $10^2 - 10^3$ as long as the dimension of $\mathbf{X}$ is limited. Once $\Pr(E_i | Z_j), i = 1, \dots, m, j = 1, \dots, l$, is computed, the VoI is obtained through Eq. (20).

In the following, I focus on the VoI analysis for monitoring systems whose outcomes are of the equality information type, since this is the more common situation and identifying computationally efficient solutions is less straightforward. An efficient solution to calculating the VoI in Eq. (22) is proposed. The main difficulty lies in the need for integrating over $\mathbf{Y}$. For this task, Monte Carlo methods seem appropriate. The simplest solution is offered by crude Monte Carlo simulation (MCS), as proposed in [16]. With the MCS approach, Eq. (22) is approximated by

$$VoI \approx C_{prior} - \left[\sum_{j=1}^{n_{SY}} \min_a \sum_{i=1}^{m} c_{E_i}(a) \frac{\sum_{k=1}^{n_{MCS}} I\left(\mathbf{x}_k \in \Omega_{E_i}\right) L\left(\mathbf{x}_k | \mathbf{y}_j\right)}{\sum_{k=1}^{n_{MCS}} L\left(\mathbf{x}_k | \mathbf{y}_j\right)} \right]. \quad (23)$$

Here I have employed the MCS approximation of $\Pr(E_i|Z)$ given in Eq. (17). $L\left(\mathbf{x}_k|\mathbf{y}_j\right)$ denotes the likelihood of $\mathbf{x}_k$, where the dependence on the monitoring outcome $\mathbf{y}_j$ is made explicit. The $\mathbf{x}_k, k = 1, \dots, n_{MCS}$, are samples from $f_{\mathbf{X}}(\mathbf{x})$, and the $\mathbf{y}_j, j = 1, \dots, n_{SY}$, are samples from $f_{\mathbf{Y}}(\mathbf{y})$. Following [16], the latter can be obtained by sampling first $\mathbf{x}_k$, then determining the distribution $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}_k)$, which is equal to the likelihood function Eq. (14), and sampling from this distribution. This introduces a correlation between the samples of $\mathbf{X}$ and $\mathbf{Y}$, but this is not critical. The computationally expensive part is the evaluation of the model, which is required for determining $I\left(\mathbf{x}_k \in \Omega_{E_i}\right)$ and $L\left(\mathbf{x}_k|\mathbf{y}_j\right)$. In most problems, one model evaluation will be required to determine $I\left(\mathbf{x}_k \in \Omega_{E_i}\right)$ and $L\left(\mathbf{x}_k|\mathbf{y}_j\right)$ for every $\mathbf{x}_k$. The computational effort is thus approximately proportional to n_{MCS}. Note that sampling from the conditional $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\boldsymbol{x}_k)$ is inexpensive, and it may thus be desirable to choose $n_{SY} > n_{MCS}$. In this case, several samples of measurement outcomes $\mathbf{Y}$ are generated based on the same sample $\mathbf{x}_k$. This introduces a correlation among the samples of $\mathbf{Y}$, which may become relevant when the problem dimension n is large.

The MCS approach is inefficient for problems involving reliability, where relevant monitoring outcomes (i.e. those which trigger mitigation actions) are expected to occur with a small probability only and where the probabilities of relevant events E_i are small. To obtain accurate solutions with MCS, therefore, a large number of model evaluations would be necessary. For this reason, an importance sampling (IS) scheme is proposed in the following.

With IS, n_{IS} weighted samples $\mathbf{x}_k$ of $\mathbf{X}$ are generated according to the IS density $\psi_{\mathbf{X}}(\mathbf{x})$. The IS estimate of $\Pr(E_i|\mathbf{Y} = \mathbf{y}_j)$ is:

$$\Pr\left(E_i \middle| \mathbf{Y} = \mathbf{y}_j\right) \approx \frac{\sum_{k=1}^{n_{IS}} w_X(\mathbf{x}_k) I\left(\mathbf{x}_k \in \Omega_{E_i}\right) L\left(\mathbf{x}_k | \mathbf{y}_j\right)}{\sum_{k=1}^{n_{IS}} w_X(\mathbf{x}_k) L\left(\mathbf{x}_k | \mathbf{y}_j\right)}, \tag{24}$$

with importance sampling weight

$$w_X(\mathbf{x}) = \frac{f_X(\mathbf{x})}{\psi_{\mathbf{X}}(\mathbf{x})}. \tag{25}$$

For the integration over $\mathbf{Y}$, the entire outcome space of $\mathbf{Y}$ is relevant. Furthermore, for fixed $\mathbf{x}_k$, evaluations of the likelihood function are inexpensive. For these reasons, an MCS approach to the integration over $\mathbf{Y}$ would be computationally efficient. Unfortunately, when performing an IS over $\mathbf{X}$, the simple sampling scheme for $\mathbf{Y}$, which is applicable in the MCS approach, is not available. For this reason, IS must also be used for the integration over $\mathbf{Y}$. With an IS approach, the VoI is approximated by:

$$\begin{aligned} VoI &\approx C_{prior} - \frac{1}{n_{ISY}} \sum_{j=1}^{n_{ISY}} w_Y\left(\mathbf{y}_j\right) \min_a \sum_{i=1}^{m} c_{E_i}(a) \Pr\left(E_i \middle| \mathbf{Y} = \mathbf{y}_j\right) \\ &\approx C_{prior} - \frac{1}{n_{ISY}} \sum_{j=1}^{n_{ISY}} w_Y\left(\mathbf{y}_j\right) \min_a \sum_{i=1}^{m} c_{E_i}(a) \frac{\sum_{k=1}^{n_{IS}} w_X(\mathbf{x}_k) I\left(\mathbf{x}_k \in \Omega_{E_i}\right) L\left(\mathbf{x}_k | \mathbf{y}_j\right)}{\sum_{k=1}^{n_{IS}} w_X(\mathbf{x}_k) L\left(\mathbf{x}_k | \mathbf{y}_j\right)}. \end{aligned} \tag{26}$$

The importance sampling weight $w_Y(\mathbf{y})$ is

$$w_Y(\mathbf{y}) = \frac{f_{\mathbf{Y}}(\mathbf{y})}{\psi_{\mathbf{Y}}(\mathbf{y})}. \tag{27}$$

where $\psi_{\mathbf{Y}}(\mathbf{y})$ is the IS density of $\mathbf{Y}$.

The PDF of $\mathbf{Y}$, $f_{\mathbf{Y}}(\mathbf{y})$, is unknown, but it can be estimated as

$$f_{\mathbf{Y}}(\mathbf{y}) \approx \frac{1}{n_{IS}} \sum_{k=1}^{n_{IS}} w_X(\mathbf{x}_k) f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}_j | \mathbf{x}_k). \tag{28}$$

Noting that $L\left(\mathbf{x}_k | \mathbf{y}_j\right) = f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}_j | \mathbf{x}_k)$, it can be seen that this estimator of $f_{\mathbf{Y}}(\mathbf{y})$ is equal to the denominator in Eq. (26) divided by n_{IS}. Combining Eqs. (26) – (28), the final IS estimator of VoI is obatained:

$$VoI \approx C_{prior} - \frac{1}{n_{ISY}} \frac{1}{n_{IS}} \sum_{j=1}^{n_{ISY}} \frac{1}{\psi_{\mathbf{Y}}(\mathbf{y}_j)} \min_a \sum_{i=1}^{m} c_{E_i}(a) \sum_{k=1}^{n_{IS}} w_X(\mathbf{x}_k) I(\mathbf{x}_k \in \Omega_{E_i}) L(\mathbf{x}_k|\mathbf{y}_j) \quad (29)$$

It remains to select efficient IS densities for **X** and **Y**, which is crucial for the efficiency of the IS approach. The key to identifying an IS density $\psi_{\mathbf{X}}(\mathbf{x})$ for **X** is to consider the hypothetical situation of perfect information illustrated in Figure 2. Under perfect information, the regions in the outcome space of **X**, in which a particular decision is optimal, can be identified. Ideally, the IS density is focused on those parts of these regions with highest probability density. In agreement with classical IS approaches in structural reliability [e.g., 42], these correspond to the areas around the so-called most likely failure points (MLFPs), also called design points. When transforming all limit state functions, and hence the domains describing the events E_i, into the space of standard normal random variables **U**, these are the areas closest to the origin. One possibility, which is employed in the application example described later, is to choose a kernel density for $\psi_{\mathbf{X}}(\mathbf{x})$, with kernels centered around the MLFPs and the origin.

For **Y**, the original PDF $f_{\mathbf{Y}}(\mathbf{y})$ would be an effective sampling density, as discussed earlier. Since $f_{\mathbf{Y}}(\mathbf{y})$ is not known, a sampling density $\psi_{\mathbf{Y}}(\mathbf{y})$ that is an approximation to $f_{\mathbf{Y}}(\mathbf{y})$ can be obtained from a few samples of $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}_k)$, where the $\mathbf{x}_k$ are the samples used in (29), thus avoiding additional model runs.

4 Decisions at multiple points in time

In most applications, action decisions a can be made at multiple points in time, at which different amounts of information from the monitoring system are available. A classic example is monitoring and inspection of deteriorating structures [43]. Consider two points in time t_1 and t_2 with $t_1 < t_2$, at which decisions $a(t_1)$ and $a(t_2)$ are made. The monitoring outcome of t_1 is available only for the decision $a(t_1)$, whereas both monitoring outcomes are available for making the decision $a(t_2)$. To compute the VoI, the action decisions must be optimized sequentially, following the sequence of available information. An approximate solution is obtained as follows. Expanding Eq. (18), one gets:

$$a_{opt|Z}(t_1) = \arg\min_{a(t_1)} \left[\min_{a(t_2)\ldots a(t_m)} \sum_{i=1}^{m} c(E_i, a(t_1), \ldots, a(t_m)) \Pr(E_i|Z(t_1)) \right] \quad (30)$$

$$a_{opt|Z}(t_2) = \arg\min_{a(t_2)} \left[\min_{a(t_3)\ldots a(t_m)} \sum_{i=1}^{m} c(E_i, a_{opt|Z}(t_1), a(t_2), \ldots, a(t_m)) \Pr(E_i|Z(t_2)) \right] \quad (31)$$

$$\vdots$$

$$a_{opt|Z}(t_m) = \arg\min_{a(t_m)} \sum_{i=1}^{m} c(E_i, a_{opt|Z}(t_1),, \dots, a_{opt|Z}(t_{m-1}), a(t_m)) \Pr(E_i|Z(t_m)) . \qquad (32)$$

$Z(t_i)$ refers to all monitoring information collected up to time t_i.

The computation of the VoI is then performed following the second line of Eq. (20), where $a_{opt|Z}$ is replaced by the set $\left[a_{opt|Z}(t_1), \dots, a_{opt|Z}(t_m)\right]$. The IS solution of Eq. (29) is equally applicable to this case.

The above procedure is only an approximation, as it does not take into account the possibility that it can be beneficial to delay an action because the decision on the appropriate action will be improved when more information is available later. The procedure will thus underestimate the overall VoI. Nevertheless, for many applications the approximation will be reasonable.

Unfortunately, in the general case the exact computation of the VoI is associated with an exponential increase in computation cost with increasing number of decision times. In risk-based inspection planning, this has motivated the identification of decisions based on simple decision rules, e.g. performing a repair when the measured size of an identified defect exceeds a threshold value [43, 44]. Through such decision rules, the decision at each time step is readily identified, and the VoI computation can again follow the second line of Eq. (20). For the special case that the relevant phenomena can be modeled as discrete Markov processes, algorithms for solving partially observable Markov decision processes [13, 45] can be employed to compute the VoI. These algorithms have a computation cost that increases only linearly with number of time steps; their disadvantage is the limitation to discrete random variables. For any problem with discrete random variables that is modeled through graphical models, also the limit memory decision diagrams (LIMID) may be viable alternative [22, 46].

5 Illustrative application to monitoring of a structure subject to fatigue deterioration

An illustrative application is presented of the theory and the proposed solution strategy to monitoring of a structural component subject to fatigue. For the sake of a clear presentation, I simplify the application, yet it includes most features of a real application.

5.1 Life-cycle model

The structure has a life-time of 20 years and is inspected and maintained every 5 years. During these campaigns, the considered component may be inspected and/or replaced. I consider the following action alternatives:

a_0, a_5, a_{10}, a_{15}: the component is replaced in year 0, 5, 10 or 15, respectively;

a_n: the component is not replaced.

The VoI is computed for two exemplary options:

(1) perform a measurement in year 5;

(2) perform measurements in year 0 and 5.

In case of option (1), alternative a_0 is not relevant, as will be seen from the prior decision analysis. In case of option (2), two action decisions are considered, one in year 0 and one in year 5, following Section 4.

5.2 Deterioration model

Fatigue deterioration of the component is described by a classical simplified model taken from the literature [38]. The crack growth due to the stress ranges ΔS is described by Paris' law as

$$\frac{\mathrm{d}l(n)}{\mathrm{d}n} = C\left[\Delta S\sqrt{\pi l(n)}\right]^m \tag{33}$$

Here, l is the crack depth, n is the number of stress cycles, ΔS is the stress range per cycle (constant stress amplitudes are assumed) and C and m are empirically determined model parameters. In this formulation of Paris' law, the geometry correction factor is one, which in theory corresponds to the case of a crack in a plate with infinite size. With the boundary condition $l(n=0) = l_0$, this differential equation can be solved for the crack depth as a function of time t [38]:

$$l(\mathbf{x}, t) = \left[\left(1 - \frac{m}{2}\right) C\Delta S^m \pi^{\frac{m}{2}} \nu t + l_0^{\left(1-\frac{m}{2}\right)}\right]^{\frac{1}{1-\frac{m}{2}}}. \tag{34}$$

t is the time in years and ν is the annual cycle rate, so that $n = \nu t$. The event of failure is described by the limit state function g_F as a function of $l(\mathbf{x}, t)$ and the critical crack depth l_c:

$$g_F(\mathbf{x}, t) = l(\mathbf{x}, t) - l_c \tag{35}$$

Here, the random variables of the problem are $\mathbf{X} = [l_0; \Delta S]$. The model parameters are summarized in Table 2.

Table 2. Parameters of the crack growth example.

Variable	Distribution	Mean	c.o.v.
l_0 [mm]	exponential	1	1
l_c [mm]	deterministic	50	-
ΔS [Nmm^{-2}]	lognormal	60	0.25
m [$-$]	deterministic	3.5	-
C [N$^{-3.5}$mm$^{\frac{25}{4}}$]	deterministic	$\exp(-33)$	-
ν [yr^{-1}]	deterministic	10^5	-

Based on the limit state function for failure, Eq. (35), the relevant events, which determine the consequences and hence the optimal action decision, are defined as:

- $E_1 = F(5\text{yr}) = \{g_F(\mathbf{X}, 5\text{yr}) \leq 0\}$;
- $E_2 = \overline{F(5\text{yr})} \cap F(10\text{yr}) = \{g_F(\mathbf{X}, 5\text{yr}) > 0\} \cap \{g_F(\mathbf{X}, 10\text{yr}) \leq 0\}$;
- $E_3 = \overline{F(10\text{yr})} \cap F(15\text{yr}) = \{g_F(\mathbf{X}, 10\text{yr}) > 0\} \cap \{g_F(\mathbf{X}, 15\text{yr}) \leq 0\}$;
- $E_4 = \overline{F(15\text{yr})} \cap F(20\text{yr}) = \{g_F(\mathbf{X}, 15\text{yr}) > 0\} \cap \{g_F(\mathbf{X}, 20\text{yr}) \leq 0\}$;
- $E_5 = \overline{F(20\text{yr})} = \{g_F(\mathbf{X}, 20\text{yr}) > 0\}$.

5.3 Measurement model

In the inspection campaign at time t_i, the crack depth can be measured. The measurement y_{t_i} has independent, additive measurement error ϵ_{m,t_i}. The likelihood function describing the measurement is

$$L(\mathbf{x}) = f_\epsilon\left(y_{t_i} - l(\mathbf{x}, t_i)\right). \tag{36}$$

with f_ϵ being a zero-mean normal PDF with standard deviation σ_ϵ. Unless otherwise stated, it is $\sigma_\epsilon = 1\text{mm}$.

For the case of two measurements y_0 and y_5 in years 0 and 5, the likelihood function is (see also Eq. (14)):

$$L(\mathbf{x}) = f_\epsilon(y_0 - l(\mathbf{x}, 0\text{yr}))f_\epsilon(y_5 - l(\mathbf{x}, 5\text{yr})). \tag{37}$$

5.4 Repair and cost model

For illustrative purposes, I consider two cases:

1. Perfect repair: Following a repair, the component will not fail.
2. Imperfect repair: Following a repair, the component is characterized by a new initial crack depth l_0'. The repaired component is again subject to fatigue deterioration. The new crack depth l_0' has the same probability distribution as l_0, but is independent of the latter. The stress range ΔS is the same before and after repair.

The number of random variables thus depends on the modeling of the repair. In case 1, the set of random variables is $\mathbf{X} = [l_0; \Delta S]$; in case 2, it is $\mathbf{X} = [l_0; l_0'; \Delta S]$.

In sections 5.5 to 5.8, the computations and results for case 1 are presented. Because this case includes only two random variables, it facilitates graphical representation. Extension to case 2 is considered in section 5.9.

The following cost model is applied, which considers discounting of costs:

- Cost of repair in years 0, 5, 10, 15, respectively:
  $c_{R0} = 8 \cdot 10^4$, $c_{R5} = 5 \cdot 10^4$, $c_{R10} = 3 \cdot 10^4$, $c_{R15} = 1.6 \cdot 10^4$
- Cost of failure in the periods 0-5, 5-10, 10-15, 15-20 years, respectively:
  $c_{F5} = 1.6 \cdot 10^6$, $c_{F10} = 10^6$, $c_{F15} = 6 \cdot 10^5$, $c_{F20} = 3.6 \cdot 10^5$

5.5 Decision analysis under prior and perfect information

The domains of the optimal actions under perfect information are shown in Figure 3. On the left-hand side, the domains are shown in the outcome space of **X**; on the right-hand side, the domains are shown in the outcome space of standard normal random variables **U**. The latter are computed by a transformation $\mathbf{x} = T(\mathbf{u})$. In the general case, T can be any of the classical transformations applied in structural reliability analysis, e.g. the Rosenblatt transformation [47] or the Nataf transformation [48]. For the considered example, due to the statistical independence of l_0 and ΔS, these reduce to the marginal transformations $l_0 = F_{l_0}^{-1}[\Phi(u_1)]$ and $\Delta S = F_{\Delta S}^{-1}[\Phi(u_2)]$. $\Phi(\cdot)$ is the standard normal cumulative distribution function; $F_{l_0}^{-1}(\cdot)$ and $F_{\Delta S}^{-1}(\cdot)$ are the inverse CDFs of l_0 and ΔS.

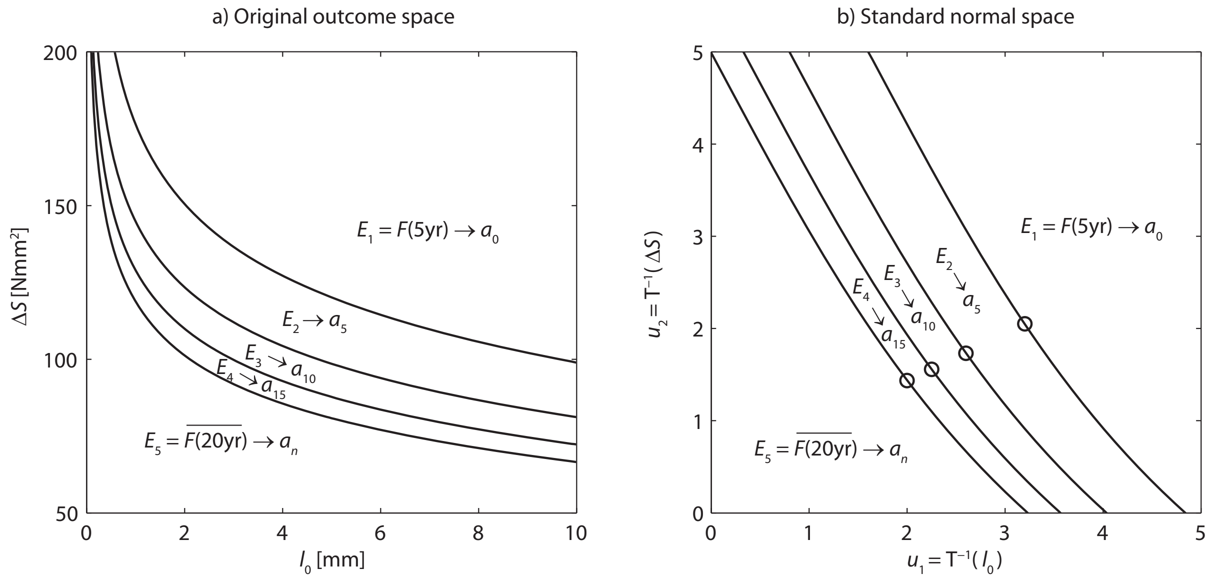

Figure 3. Optimal actions under perfect information. (a) In the outcome space of $\boldsymbol{X}$*; (b) transformed to standard normal space. Circles indicate the most likely failure points (MLFPs) in standard normal space.*

The corresponding a-priori probabilities as obtained with FORM are summarized in Table 3. As evident from the almost linear behavior of the limit state surfaces around the MLFPs visible in Figure 3b, FORM provides accurate results.

Table 3. A-priori probabilities (FORM results).

Event	Probability
E_1: Failure in period 0 - 5 years	$7.2 \cdot 10^{-5}$
E_2: Failure in period 5 - 10 years	$8.1 \cdot 10^{-4}$
E_3: Failure in period 10 - 15 years	$2.2 \cdot 10^{-3}$
E_4: Failure in period 15 - 20 years	$3.8 \cdot 10^{-3}$
E_5: No failure before year 20	0.993

5.5.1 Prior decision analysis

A-priori, the optimal decision a_{opt} is found using Eq. (2). With the cost model and the probabilities reported in Table 3, the optimal decision is a_n: do nothing. The corresponding expected cost is calculated with Eq. (3) as

$$C_{prior} = 1.6 \cdot 10^6 \cdot 7.2 \cdot 10^{-5} + 10^6 \cdot 8.1 \cdot 10^{-4} + 6 \cdot 10^5 \cdot 2.2 \cdot 10^{-3} + 3.6 \cdot 10^5 \cdot 3.8 \cdot 10^{-3} + 0 \cdot 0.993 = \underline{3.62 \cdot 10^3.}$$

5.5.2 Value of perfect information VoPI

The optimal actions under different evidence are: a_0 in case E_1, i.e. the component should be replaced in year 0 if it were to fail prior to year 5; a_5 in case E_2, i.e. the component should be replaced in year 5 if it were to fail in the period from year 5 to year 10; and so on. If the component does not fail during the service life (event E_5), the optimal action is a_n: do nothing. Given E_5, the conditional value of perfect information (CVoPI) is zero, since the optimal action in this case is equal to the one found with the prior decision analysis. Given any of the other events, the CVoPI is positive.

The value of perfect information is determined with Eq. (11) as

$$VoPI = 3.62 \cdot 10^3 - (8 \cdot 10^4 \cdot 7.2 \cdot 10^{-5} + 5 \cdot 10^4 \cdot 8.1 \cdot 10^{-4} + 3 \cdot 10^4 \cdot 2.2 \cdot 10^{-3} + 1.6 \cdot 10^4 \cdot 3.8 \cdot 10^{-3} + 0 \cdot 0.993) = \underline{3.45 \cdot 10^3}.$$

This value is the upper limit of the benefit that can be achieved with any monitoring system.

5.6 Importance sampling

Based on the identified design points (MLFPs) shown in Figure 3b, an IS density $\psi_{\mathbf{X}}$ is is selected. The $\psi_{\mathbf{X}}$ is specified in the space of standard normal random variables, i.e. $\psi_{\mathbf{U}}$ is specified and the samples of $\mathbf{X}$ are obtained by sampling $\mathbf{u}_k$ from $\mathbf{U}$ and transforming the samples, $\mathbf{x}_k = T(\mathbf{u}_k)$, according to section 5.5.

Following Section 3, a kernel density is selected for $\psi_{\mathbf{U}}$ (and consequently for $\psi_{\mathbf{X}}$), with five standard normal distributions as kernels. The first four kernels are centered around the 4 MLFPs shown in Figure 3b, the fifth is centered around the origin. (The origin is the “MLFP” of the no-failure event E_5.) The resulting $\psi_{\mathbf{U}}$ is shown in Figure 4, together with random samples generated from $\psi_{\mathbf{U}}$.

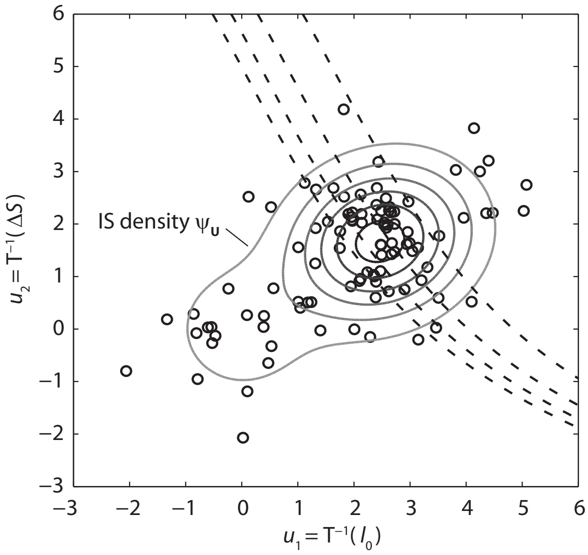

Figure 4. Importance sampling density, shown together with 100 random samples. Dashed lines indicate the borders of the domains corresponding to $E_1, \dots, E_5$ (see Figure 3).

The IS distribution for the measurement results $\psi_{\mathbf{Y}}$ is constructed by evaluating the model for the samples from $\psi_{\mathbf{U}}$, sampling corresponding measurement outcomes using $f_{\mathbf{Y}|\mathbf{X}}$ and then fitting a joint normal distribution to these samples.

5.7 Bayesian updating

Likelihood functions for exemplarily monitoring outcomes are shown in Figure 5. On the left-hand side, likelihood functions are shown for a monitoring outcome $y_5 = 6$mm in year 5. On the right-hand side, likelihood functions are shown for a monitoring outcome $y_0 = 3$mm in year 0 and $y_5 = 6$mm in year 5. The effect of the measurement error on the likelihood function is clearly visible. For the case of two measurements and a monitoring system with small measurement error $\sigma_\epsilon = 0.3$mm, one is able to determine the uncertain parameters with good accuracy, i.e. the posterior variance becomes small in this case. With only one measurement, the resulting posterior variance remains significant even in the case of small measurement error, since different combinations of parameters l_0 and ΔS lead to the same crack size.

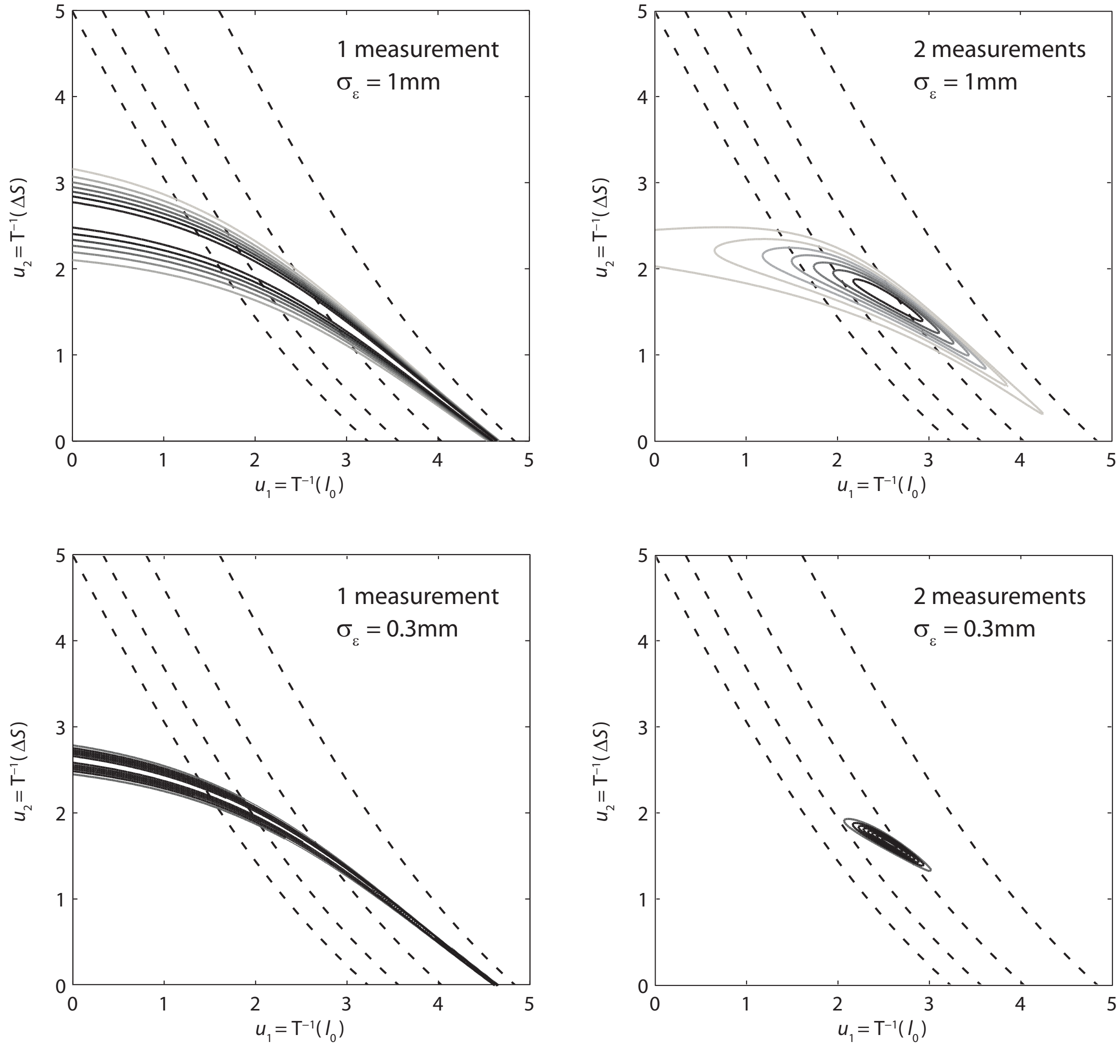

Figure 5. Likelihood functions shown for measurement results $y_0 = 3mm$ and $y_5 = 6mm$ and different measurement accuracy σ_ϵ. The likelihood functions on the left-hand-side are for a measurement at year 5 only, those on the right hand side represent the combined measurement in year 0 and 5. Dashed lines indicate the borders of the domains corresponding to $E_1, \dots, E_5$ (Figure 3).

5.8 Value of information

The value of information (VoI) is computed for the case of one measurement in year 5 and for the case of two measurements in year 0 and 5. In the latter case, the subsequent decision making outlined in Section 4 is applied to identify the optimal decision after the first measurement.

To assess the efficiency and accuracy of the proposed IS approach, computations are performed with different numbers of samples and are compared to solutions obtained with MCS. The results are summarized in Table 4. The VoI of two measurements is only slightly larger than the VoI of one measurement only. This indicates that with $\sigma_\epsilon = 1.0$mm the first measurement in year 0 does not provide much useful additional information.

Table 4. VoI computed with importance sampling (IS) and Monte Carlo simulation (MCS). Mean value and standard deviations were evaluated by repeating the computations 20 times.

	1 measurement		2 measurements	
Computation	Mean	St. dev.	Mean	St. dev.
MCS (10^4 samples)	1447	(264)	1624	(246)
IS (10^3 samples)	1443	(124)	1636	(49)
IS (10^2 samples)	1586	(385)	1721	(404)

The results in Table 4 indicate the effectiveness of the proposed IS solution. Compared to MCS, with IS the number of model evaluations can be reduced by approximately a factor of 50 to achieve the same accuracy.

5.8.1 VoI as a function of measurement accuracy

To obtain further insights into the VoI, the resulting VoI is computed for varying measurement accuracy, i.e. for different values of σ_ϵ. The results are summarized in Figure 6. For comparison, also the value of perfect information VoPI is shown.

For larger measurement uncertainty ($\sigma_\epsilon \geq 1\text{mm}$), two measurements do not provide significantly more information than one measurement. In this case, the second measurement in year 5 is dominating the posterior distribution and the updated probabilities $\Pr(E_i|Z)$. This can be observed graphically by comparing the likelihood functions in Figure 5. For larger measurement uncertainty (upper figures), the differences between one and two measurements are less distinct than for smaller measurement uncertainty (lower figures).

With a single measurement, the VoI is far from the VoPI even for $\sigma_\epsilon = 0$. The reason is that knowing the crack size only at one point in time does not allow one to determine the model parameters $\mathbf{X} = [l_0; \Delta S]$ uniquely, as should be evident from the likelihood function for the case of one measurement and $\sigma_\epsilon = 0.3\text{mm}$ shown in Figure 5. This demonstrates that "perfect information" requires not only zero measurement uncertainty, but also measuring enough relevant quantities. For the case of two measurements shown in Figure 6, the VoI would eventually reach the VoPI as $\sigma_\epsilon = 0$. In this idealized example, two exact measurements at two different points in time are sufficient to exactly determine the two model parameters l_0 and ΔS, thus eliminating all uncertainty.

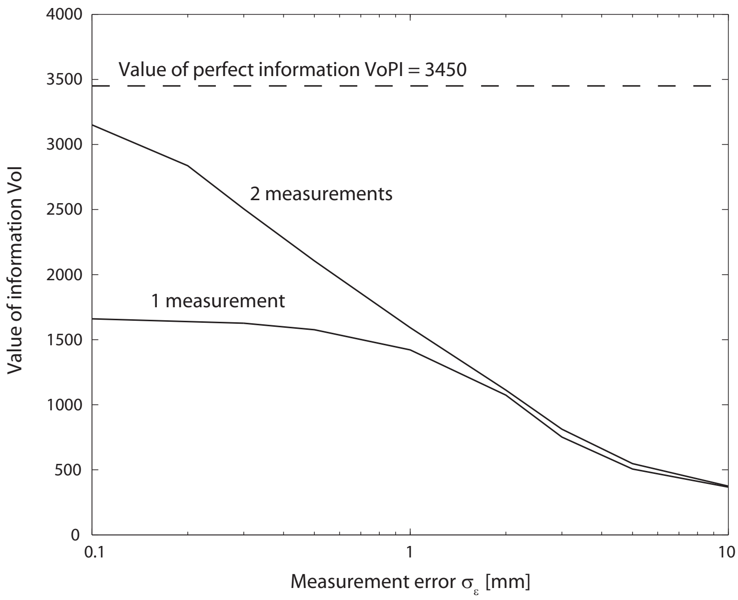

Figure 6. VoI as a function of measurement accuracy. Computations are performed with IS using 10^5 *samples.*

5.9 Imperfect repair

In case of imperfect repair, the problem has three random variables as discussed in section 5.4. After the repair, the performance of the component is described by the same mechanical model, but with new initial crack depth l_0' and different starting time. The limit state function describing a failure τ years after a repair is

$$g_{F_R}(\mathbf{x}, \tau) = l_r(\mathbf{x}, \tau) - l_c \,. \tag{38}$$

The function l_r is equal to l defined in Eq. (34), wherein l_0 is replaced with l_0'. Since a repair is possible already in year 0, the time τ can be up to 20 years. $g_{F_R}(\mathbf{x}, \tau)$ must thus be evaluated for $\tau = 5{,}10{,}15{,}20$ years. A total of 25 events $E_1, \dots, E_{25}$ must now be considered, representing combinations of possible failure times of the original component and of the repaired component. For each combination of events and action a, the corresponding costs can be assigned. These are not reported here for brevity. Exemplarily, event E_{10} is equal to failure of the original structure in the period of $5 - 10$ years and no failure of the repaired structure, i.e. $E_{10} = \{g_F(\mathbf{X}, 5\text{yr}) > 0 \cap g_F(\mathbf{X}, 10\text{yr}) \leq 0 \cap g_{F_R}(\mathbf{X}, 20\text{yr}) > 0\}$. The costs associated with event E_{10} are c_{R0} in case of action a_0, c_{R5} in case of a_5 and c_{F10} otherwise. The VoI computations are then again performed following Eq. (29).

It is found that the consideration of the possible failure following a repair has little impact on the result. The changes in the results compared to the case with perfect repair are within the range of scatter of the IS results; the results reported in Table 4 are thus also valid for the case of imperfect repair. The reason for these small differences in the results? The probability of performing a repair is only in the order of 0.01 – 0.02. Therefore, the additional expected cost due to a potential second failure is only minor. This demonstrates that in this application there is no need to explicitly model the behavior of the repaired component when computing the VoI.

6 Discussion

The aim of this paper is to propose a rigorous modeling framework and efficient computational algorithms for evaluating the value of information VoI of monitoring systems based on structural reliability methods. An importance sampling solution was developed and its efficiency was demonstrated through the presented example application. To facilitate a graphical interpretation of the results, I studied a problem with only two random variables. For problems with many random variables, importance sampling solutions become inefficient. However, in most applications it will be possible to reduce to number of relevant random variables to less than ten by a-priori identifying the most influential variables, thus facilitating the use of the proposed solution strategy.

Arguably the most difficult part in realistic applications is the need to explicitly model the relevant decision processes over the entire service life period of the monitoring system. As pointed out in [49], contrary to the examples of Bayesian decision analysis provided in text books, real situations in which the VoI should be estimated are not simple. A multitude of potential action alternatives a exist, and engineers are not trained to prescribe quantitative rules for which decision to take in which circumstances, except for simple cases. The typical behavior of engineers under situations of uncertainty is to collect some information, and to go from there. Hence the popularity of monitoring systems. Nevertheless, the process of systematically describing the potential monitoring outcomes and the action alternatives is a highly useful process, even when it is not possible to provide exact and complete answers.

In many instances, the decision model can be kept quite simple and only main events and actions must be included explicitly in the model. As an example, it is often not necessary to explicitly model the performance of a system after repair or other mitigation actions [43], as was also found in the example application in this paper. It is also important to point out that an exact assignment of costs or utility values is often not necessary. For decision making purposes it is typically sufficient to provide approximate estimates of costs when computing the VoI.

Finally, to compute the VoI using the proposed methodology, a probabilistic model of the monitored processes and engineering system is required. Although such a model can be at varying degrees of detailing, establishing such a model represents an additional effort if it has not already been prepared for other purposes. It must thus first be decided if the additional modeling effort is justified. This requires one to appraise the value of information of a probabilistic reliability analysis, in other words: estimating the VoI of a VoI analysis. While such an estimate will be based on expert estimates rather than on detailed modeling and quantitative analysis, it follows the same steps as the VoI analysis of the monitoring system, i.e. it must be determined what the possible actions are and how a VoI analysis allows to support identifying the optimal actions. If the monitoring system is inexpensive, installing it without further analysis might be the best option; otherwise, a VoI analysis will likely pay off.

7 Conclusion

Value of information (VoI) is a powerful theory to assess the usefulness of monitoring or any other means of obtaining information. The difficulty in practical applications of VoI lies in (a) proper probabilistic modeling of the monitored process and the monitoring itself, (b) the modeling of the action alternatives following the monitoring results, and (c) the computational efforts in evaluating the VoI. In this paper, it was shown that structural reliability methods can provide an effective framework for understanding, modeling and computing the VoI. The theory was developed, an efficient algorithm for computation was proposed and the example application, considering monitoring of a structure subject to fatigue deterioration, provided an illustration of the theory and its implementation. The example application also showed that a simplified modeling of the potential action alternatives is typically sufficient. This highlights that the formal process of the VoI analysis provides useful insights even when the VoI cannot be determined exactly due to the complexity of the decisions involved.

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