Pricing exotic options under regime switching:
A Fourier transform method

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Abstract
This paper considers the valuation of exotic options (i.e. digital, barrier, and lookback options) in a
Markovian, regime-switching, Black-Scholes model. In Fourier space, analytical expressions for the
option prices are derived via the theory on matrix Wiener-Hopf factorizations. A comparison to
numerical alternatives, i.e. the Brownian bridge algorithm or a finite element scheme, demonstrates
that the given formulas are easy to implement and lead to accurate and unbiased price estimates.

Keywords: regime switching, Markov switching, Wiener-Hopf factorization, option pricing.
Classification: 60G40, 60J22, 60J27, 60J28.

The natural and intuitive idea of regimes in financial time series has become a widely accepted feature.
Following Hamilton [1989]'s seminal work, regime switching model constitute a very simple extension
of the Black-Scholes model to stochastic volatility. A continuous time, finite-state, Markov chain
generates the switches between the model parameters and thus allows to incorporate the impact of
structural changes in the economic conditions on the price dynamics. Typically, two or three regimes
provide a good fit to monthly stock market returns. One reason for their popularity is the fact that
regime switching models are conceptionally very simple (conditional on the regimes, the innovations
are normally distributed) and thus analytically tractable. Nevertheless, they can generate many non-
linear effects like heavy tails or volatility clusters. Furthermore, they allow us to depart from the
unsatisfactory assumption of stationary increments in Lévy models.

The aim of this paper is to price (exotic) options in a regime switching model. The contributions
are twofold: (1) We show how several exotic options (i.e. digital, barrier, and lookback options) can
be priced using the theory on matrix Wiener–Hopf factorizations. (2) We compare the results to
numerical alternatives, i.e. the Brownian bridge algorithm, an analytic approximation by Lo et al.
[2003], Elliott et al. [2014], and a backward finite element scheme.

Among others, Jiang and Pistorius [2008] showed that the Fourier transform of the first-passage time
in a regime switching model can be represented as a function of the matrix Wiener–Hopf factorization
(see also London et al. [1982], Kennedy and Williams [1990], Barlow et al. [1990], Rogers [1994],
Asmussen [1995]). In a regime switching model, the matrix Wiener–Hopf factorization is determined
by a quadratic matrix equation. Generally, the factorization has to be computed numerically (see, for
1 Model description

On the filtered probability space \((\Omega, \mathbb{F}, \mathcal{F}, \mathbb{Q})\), we consider the process \(S = \{S_t\}_{t \geq 0}\) described by

\[
\frac{dS_t}{S_t} = r dt + \sigma Z_t dW_t, \quad S_0 = \exp(x) > 0,
\]

where \(r\) is the risk-less interest rate, \(\sigma\) the regime-dependent volatility, \(Z = \{Z_t\}_{t \geq 0} \in \{1, 2, \ldots, M\}\) a time-homogeneous Markov chain with intensity matrix \(Q_0\), and \(W = \{W_t\}_{t \geq 0}\) an independent Brownian motion. The initial value is \(S_0 = e^x \in \mathbb{R}^+\). The filtration \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is generated by the pair \((W, Z)\), i.e. \(\mathcal{F}_t = \sigma\{W_s, Z_s : 0 \leq s \leq t\}\). The model is fully determined if an initial state (or, more generally, an initial distribution \(\pi_0 := (Q(Z_0 = 1), Q(Z_0 = 2), \ldots, Q(Z_0 = M))\) on the states) is defined.

The meaning of the entries \(q_{ij} := Q_0(i, j)\) of the intensity matrix \(Q_0\) can easily be explained: The time spent in state \(i\) is exponentially distributed with intensity \(\lambda = -q_{ii} > 0\). If a state change from the current state \(i\) occurs, the probability of moving to state \(j \neq i\) is \(-q_{ij}/q_{ii} \geq 0\) (note that, since \(Q_0\) is an intensity matrix, \(-\sum_{i \neq j} q_{ij} = q_{ii}\)).

For a matrix \(Q \in \mathbb{C}^{M \times M}\), we define the matrix exponential \(\exp(Q)\) via the power series

\[
\exp(Q) := \sum_{n=0}^{\infty} \frac{Q^n}{n!}.
\]

\(^1\) (Numerical) solutions to the (matrix) Wiener–Hopf factorization are possible in more general model settings (see, among others, Boyarchenko and Levendorski˘ı [2008], Jiang and Pistorius [2008], Kudryavtsev and Levendorski˘ı [2012], Mijatović and Pistorius [2013]).

\(^2\) An intensity matrix has negative diagonal and non-negative off-diagonal entries. Each row sums up to zero.
Then, the characteristic function \( \varphi_t(u, \pi_0) := \mathbb{E}[\exp(iu(\ln(S_t) - x))] \) of log-returns in a regime switching model is given by (see, e.g., Buffington and Elliott [2002]; Elliott et al. [2005])

\[
\varphi_t(u, \pi_0) = \pi_0 \exp \left( Q_0' t + (i u r - \Sigma^2 u^2/2) t \right) 1 ,
\]

where \( \Sigma := \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_M) \), \( \exp(\cdot) \) denotes the matrix exponential function, \( ' \) transpose, and \( 1 \) a column vector of ones of appropriate size.

The first-passage times on two constant barriers \( e^b < S_0 = e^x < e^a \) are defined as

\[
T_{ab} := \begin{cases} 
\inf \{ t \geq 0 : S_t \notin (e^b, e^a) \}, & \text{if such a } t \text{ exists,} \\
\infty, & \text{if } S_t \text{ never hits the barriers.}
\end{cases}
\]

Here, \( T_{ab} \) is the first time the process \( S_t \) hits one of the two barriers \( e^a \) or \( e^b \). The corresponding first-passage time densities are – for \( t \in (0, \infty) \) – denoted

\[
f_{ab}(t, \pi_0) := \mathbb{Q}(T_{ab} \in dt | \pi_0).
\]

### 2 Option pricing

In the following, we price exotic options in a regime switching model. Section 2.1 deals with the general case of \( M \) regimes: After pricing call options, we recall the Fourier transform of the first-passage time in regime switching models and introduce the matrix Wiener–Hopf factorization. Digital, barrier, and lookback options can then be represented as Fourier integrals (see also Jiang and Pistorius [2008]). In Section 2.2 we exploit that the matrix Wiener–Hopf factorization can be solved analytically in the case of 2 regimes (see, for example, Hieber [2012]).

#### 2.1 \( M \) states

As the characteristic function of log-returns in a regime switching model is known (see Equation (2)), it is possible to price (vanilla) call options via Fourier pricing. If there are multiple regimes, this is more convenient than the \( M \)-fold integrals presented in, for example, Elliott et al. [2005].

**Theorem 1 ((Vanilla) call options)**

Consider the regime switching model as defined in Equation (1). Under \( \mathbb{Q} \), the price of a call option with strike \( K \) and time to maturity \( T \) is recovered as

\[
C(S_0, T, \pi_0) = \frac{e^{-rT}S_0}{\pi} \int_0^\infty \frac{e^{-(\alpha-i\theta)\ln(K/S_0)} \varphi_T(\theta - (1 + \alpha) i, \pi_0)}{\alpha^2 + \alpha - \theta^2 + i(2\alpha + 1)\theta} d\theta ,
\]

where \( \alpha \in [1, 2] \) is an arbitrary constant and \( \varphi_t(u, \pi_0) \) the characteristic function of log-returns in a regime switching model given in (2).

**Proof**

See, for example, Carr and Madan [1999], Raible [2000].

\[ \square \]
The first-passage time densities $f_{ab}(t, \pi_0)$ can be expressed in terms of the matrix Wiener–Hopf factorization (see, for example, Jiang and Pistorius [2008] and the references therein). The one-sided densities $f_{a, -\infty}(t, \pi_0)$ and $f_{\infty, b}(t, \pi_0)$ are recalled in Theorem 2. Note that the class of irreducible $M \times M$ generator matrices (non-negative off-diagonal entries and non-positive row sums) is denoted by $Q_M$.

**Theorem 2 (First-passage time densities)**

Consider the regime switching model as defined in Equation (1) with initial distribution on the states $\pi_0 \in \mathbb{R}^{1 \times M}$. For two constant barriers $e^b < S_0 = e^x < e^a$, we get for $t \in (0, \infty)$$^3$

$$f_{a, -\infty}(t, \pi_0) = \frac{1}{\pi} \int_0^\infty e^{-iut} \left( \pi_0 \exp \left( Q_+(a - x) \right) \mathbf{1} \right) du,$$

$$f_{\infty, b}(t, \pi_0) = \frac{1}{\pi} \int_0^\infty e^{-iut} \left( \pi_0 \exp \left( Q_-(x - b) \right) \mathbf{1} \right) du,$$

where $\exp(\cdot)$ denotes the matrix exponential and $\mathbf{1}$ a column vector of ones of appropriate size. For $u > 0$, the matrices $Q_+, Q_- \in Q_M$ are the matrix Wiener–Hopf factorization of $(S, Z)$ defined via

$$\Xi(-Q_+) = \Xi(Q_-) = 0,$$

$$\Sigma := \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_M), \quad V := \text{diag}(r - \sigma_1^2/2, r - \sigma_2^2/2, \ldots, r - \sigma_M^2/2),$$

and $I_M$ (an $M \times M$ identity matrix).

**Proof**

See, for example, Rogers [1994] and Jiang and Pistorius [2008].

Similar results have been derived for two-sided first-passage times and in the more general case of regime switching exponential jump-diffusion models (see, for example, Jiang and Pistorius [2008]). If we are able to compute the matrix Wiener-Hopf factorization, Theorem 2 demonstrates how to compute the first-passage time densities, a result that then allows us to price many exotic options in a regime switching model.

For 2 and 3 regimes and in the case where $r = 0$, closed-form solutions for the matrix Wiener–Hopf factors $(Q_+, Q_-)$ are available (see, for example, Section 2.2 and Hieber [2012]). In the general case of $M$ states, using diagonalization techniques, Rogers and Shi [1994] present an efficient algorithm to numerically compute the matrix Wiener–Hopf factorization $(Q_+, Q_-)$.

As a first application of Theorem 2 to option pricing, Theorem 3 presents prices for digital options, i.e. options that pay 1 if the barrier $e^b < S_0$ is hit during the lifetime of the option and 0 otherwise.

**Theorem 3 (Digital options)**

Consider the regime switching model as defined in Equation (1) and a barrier $e^b < S_0$. Under $Q$, the price of a digital option with payoff $\mathbbm{1}_{\{T, \pi_0 \}}$ at maturity $T$ is given by

$$D(S_0, T, \pi_0) = \frac{e^{-rT}}{\pi} \int_0^\infty \frac{1 - e^{-iuT}}{iu} \left( \pi_0 \exp \left( Q_-(x - b) \right) \mathbf{1} \right) du,$$

$^3$Note that there might be a positive probability that the barrier is never hit, i.e. $\int_0^\infty f_{a, -\infty}(t, \pi_0) dt \leq 1$. 

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where \( \exp(\cdot) \) denotes the matrix exponential, \( \mathbf{1} \) a column vector of ones of appropriate size, and \( Q_- \) the matrix Wiener–Hopf factor as defined in Theorem 2.

**Proof**

Using Theorem 2 and applying Fubini’s theorem, we find that

\[
D(S_0, T, \pi_0) := e^{-rT} \mathbb{Q}(T_\infty, b \leq T) = e^{-rT} \int_0^T f_{\infty, b}(t, \pi_0) dt
\]

\[
= e^{-rT} \int_0^T \frac{1}{\pi} \int_0^\infty e^{-it} \left( \pi_0 \exp \left( Q_- (x - b) \right) \mathbf{1} \right) du dt
\]

\[
= \frac{e^{-rT}}{\pi} \int_0^\infty \frac{1}{iu} \left( \pi_0 \exp \left( Q_- (x - b) \right) \mathbf{1} \right) du .
\]

Combining the results of digital options in the latter theorem and on call options in Theorem 1, it is possible to price barrier options in a regime switching model, see Theorem 4. Down-and-out barrier options pay \( \max(S_T - K, 0) \) at maturity \( T \) if the barrier \( e^b \leq K \) is not hit during the lifetime of the option and 0 otherwise.

**Theorem 4 (Barrier options)**

Consider the regime switching model as defined in Equation (1). Under \( \mathbb{Q} \), the price of a barrier option with barrier \( e^b < S_0 \) and strike \( K := e^b > 0 \), i.e. with payoff \( \mathbb{1}_{\{T_\infty, b \leq T\}} \max(S_T - K, 0) \) at maturity \( T \), is given by

\[
B(S_0, T, \pi_0) = C(S_0, T, \pi_0) - \frac{e^{-rT}}{\pi} \sum_{j=1}^M \int_0^T \left( \int_0^\infty e^{-it} \left( \pi_0 \exp \left( Q_- (x - b) \right) e_j \right) du \right) C(e^b, T - t, e_j) dt ,
\]

where \( C(S_0, T, \pi_0) \) is the price of a vanilla call option, \( \exp(\cdot) \) denotes the matrix exponential, \( e_j \) the \( j \)-th unit vector, and \( Q_- \) the matrix Wiener–Hopf factor as defined in Theorem 2.

**Proof**

We denote – for \( t \in (0, \infty) \) – the first-passage time density conditional on \( Z_{T_\infty, b} = j \) by \( f_{\infty, b}(t, \pi_0, e_j) \), where \( e_j \) is the \( j \)-th unit vector. Then, the price of the barrier option can be represented as

\[
B(S_0, T, \pi_0) = C(S_0, T, \pi_0) - \sum_{j=1}^M \int_0^T e^{-rt} f_{\infty, b}(t, \pi_0, e_j) C(e^b, T - t, e_j) dt ,
\]

where the series sums over the possible states at the time of the first barrier crossing.

Plugging in Equation (7), one obtains the stated results.
Remark 5 (Implementation of Theorem 4)

To evaluate the barrier option price in Theorem 4 numerically, it might sense to further simplify the integrals. Plugging in Equations (2), (5), we obtain from Theorem 4

\[
\int_0^T e^{-rt} f_{\infty, b}(t, \pi_0, e_j) C(e^b, T - t, e_j) \, dt
\]

\[
= e^{-rT} \int_0^T \left( \frac{1}{\pi} \int_0^\infty e^{-iut} \left( \pi_0 \exp \left( Q_-(x - b) \right) e_j \right) du \right)
\]

\[
\cdot \left( \frac{S_0}{\pi} \int_0^\infty e^{(-\alpha - i\theta)(k-b)} \frac{\varphi_{T-1}(\theta - (1 + \alpha)i, e_j)}{\alpha^2 + \alpha - \theta^2 + i(2\alpha + 1)\theta} \, d\theta \right) \, dt
\]

\[
= \frac{e^{-rT} S_0}{\pi^2} \int_0^\infty \left[ \frac{e^{(-\alpha - i\theta)(k-b)}}{\alpha^2 + \alpha - \theta^2 + i(2\alpha + 1)\theta} \right.
\]

\[
\left. \cdot \int_0^T \int_0^\infty e^{-iut} \varphi_{T-1}(\theta - (1 + \alpha)i, e_j) \, dt \left( \pi_0 \exp \left( Q_-(x - b) \right) e_j \right) du \right] \, d\theta
\]

\[
= \frac{e^{-rT} S_0}{\pi^2} \int_0^\infty \left[ \frac{e^{(-\alpha - i\theta)(k-b)}}{\alpha^2 + \alpha - \theta^2 + i(2\alpha + 1)\theta} \right.
\]

\[
\left. \cdot \int_0^\infty \left( \pi_0 \exp \left( Q_-(x - b) \right) e_j \right) e^{-iuT} \left( e_j \Phi_T(\theta - (1 + \alpha)i) \mathbf{1} \right) du \right] \, d\theta ,
\]

where \( \Phi_T(v) := \left( Q_0' + iu + ivr - \Sigma^2 v^2 / 2 \right)^{-1} \left( \exp(Q_0' t + (iu + ivr - \Sigma^2 v^2 / 2)t) - I_M \right), \alpha \in [1, 2], \) and \( I_M \) denotes the \( M \times M \) identity matrix.

This then results in the numerically more convenient expression

\[
B(S_0, T, \pi_0) = C(S_0, T, \pi_0) - \frac{e^{-rT} S_0}{\pi^2} \int_0^\infty \left[ \frac{e^{(-\alpha - i\theta)(k-b)}}{\alpha^2 + \alpha - \theta^2 + i(2\alpha + 1)\theta} \right.
\]

\[
\left. \cdot \int_0^\infty \left( \pi_0 \exp \left( Q_-(x - b) \right) \right) e^{-iuT} \left( e_j \Phi_T(\theta - (1 + \alpha)i) \mathbf{1} \right) du \right] \, d\theta .
\]

Finally, we turn our attention to lookback options. Therefore, define the maximum of \( S = \{S_t\}_{t \geq 0} \) on the time interval \([0, T]\) by

\[
\mathcal{M}(T) := \max_{t \in [0, T]} S_t .
\]

A lookback strike put option then pays \( \max(\mathcal{M}(T) - S(T), 0) \) at maturity \( T \). Its price in a regime switching model is derived in Theorem 6.
2.1 M states

Theorem 6 (Lookback strike put options)
Consider the regime switching model as defined in Equation (1). At maturity $T$, a lookback put option pays $\max(M(T) - S(T), 0)$. Under $Q$, its price is given by

$$L(S_0, T, \pi_0) = e^{-rT} \pi \int_{x}^{\infty} \int_{0}^{\infty} \frac{1 - e^{-iuT}}{iu} \left( \pi_0 Q_+ \exp \left( Q_+(a - x) \right) 1 \right) du da - S_0.$$ 

where $\exp(\cdot)$ denotes the matrix exponential, $1$ a column vector of ones of appropriate size, and $Q_+$ is the matrix Wiener–Hopf factor as defined in Theorem 2.

Proof
Since $M(T) \geq S(T)$, we can price lookback strike put options as

$$L(S_0, T, \pi_0) = e^{-rT} \mathbb{E}_Q [M(T) - S(T)] = e^{-rT} \mathbb{E}_Q [M(T)] - S_0.$$ 

Using Theorem 2, the expected maximum can then be derived as

$$\mathbb{E}_Q [M(T)] = \int_{x}^{\infty} e^a \int_{0}^{T} \frac{\partial}{\partial a} f_{a, -\infty}(t, \pi_0) dt da$$

$$= \int_{x}^{\infty} e^a \int_{0}^{T} \left( \frac{1}{\pi} \int_{0}^{\infty} e^{-iut} \left( \pi_0 Q_+ \exp \left( Q_+(a - x) \right) 1 \right) du \right) dt da$$

$$= \frac{1}{\pi} \int_{x}^{\infty} \int_{0}^{T} \frac{1 - e^{-iuT}}{iu} \left( \pi_0 Q_+ \exp \left( Q_+(a - x) \right) 1 \right) du da. \quad \square$$

Remark 7 (Implementation of Theorem 6)
If one wants to numerically evaluate the integrals in Theorem 6, one can speed up the implementation by restricting the maximum $M(T)$ to a bounded interval, i.e. choose a suitable $a_{\min}, a_{\max}$ such that $e^x < e^{a_{\min}} \leq M(T) \leq e^{a_{\max}} < \infty$. Then

$$L(S_0, T, \pi_0) \approx \frac{1}{\pi} \int_{x}^{\infty} \int_{0}^{T} \frac{1 - e^{-iuT}}{iu} \left( \pi_0 Q_+ \exp \left( Q_+(a - x) \right) 1 \right) du da - S_0$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-iuT} - 1}{iu} \left( Q_+ + I_M \right)^{-1} Q_+ \left( \exp \left( Q_+(a_{\max} - x) + I_M a_{\max} \right) \right. \left. - \exp \left( Q_+(a_{\min} - x) + I_M a_{\min} \right) \right) 1 du - S_0.$$ 

We represented digital, barrier, and lookback options as functionals of the matrix Wiener–Hopf factorization. Generally, the latter has to be solved numerically. However, in the case of $M = 2$ regimes, analytical solutions of the matrix Wiener–Hopf factors are available. This then gives us closed-form expressions for many exotic option prices.
2.2 2 states

First-passage times and option prices in the 2-state model depend on the roots of the so-called Cramér–Lundberg equation given by

\[ \left( \frac{1}{2} \sigma_1^2 \beta^2 + \mu_1 \beta + q_{11} - u \right) \left( \frac{1}{2} \sigma_2^2 \beta^2 + \mu_2 \beta + q_{22} - u \right) - q_{11} q_{22} = 0. \]  \hspace{1cm} (10)

This quartic equation has 4 unique real roots \(-\infty < \beta_{1,u} < \beta_{2,u} < 0 < \beta_{3,u} < \beta_{4,u} < \infty\) (see Guo [2001b] for a proof). Those roots are available in closed-form, see, for example, Abramowitz and Stegun [1965], p. 17f. Depending on those roots, Theorem 8 presents the matrix Wiener–Hopf factorization \((Q_+, Q_-)\) in the case of \(M = 2\) regimes.

**Theorem 8 (2-state model: Matrix Wiener–Hopf factorization)**

Consider the regime switching model as defined in Equation (1) with \(M = 2\) states and \(q_{11} q_{22} \neq 0\).

(a) The matrix Wiener–Hopf factorization \((Q_+, Q_-)\) is given by

\[ Q_+ = \begin{pmatrix} \frac{-2 \beta_{1,u} \beta_{3,u} + 2 \beta_{2,u} q_{11} + 2 \beta_{1,u} \beta_{4,u} + 2 \beta_{3,u} \beta_{4,u} + 2 q_{11} - u}{\sigma_1^2} & \frac{-2 \beta_{1,u} \beta_{3,u} + 2 \beta_{2,u} q_{11} + 2 \beta_{1,u} \beta_{4,u} + 2 \beta_{3,u} \beta_{4,u} + 2 q_{11} - u}{\sigma_2^2} \\ \frac{-2 \beta_{1,u} \beta_{3,u} + 2 \beta_{2,u} q_{11} + 2 \beta_{1,u} \beta_{4,u} + 2 \beta_{3,u} \beta_{4,u} + 2 q_{11} - u}{\sigma_1^2} & \frac{-2 \beta_{1,u} \beta_{3,u} + 2 \beta_{2,u} q_{11} + 2 \beta_{1,u} \beta_{4,u} + 2 \beta_{3,u} \beta_{4,u} + 2 q_{11} - u}{\sigma_2^2} \end{pmatrix}, \quad Q_- = \begin{pmatrix} \frac{\beta_{1,u} \beta_{2,u} - 2 q_{11}}{\sigma_1^2} & \frac{2 q_{11}}{\sigma_2^2} \\ \frac{\beta_{1,u} \beta_{2,u} - 2 q_{11}}{\sigma_2^2} & \frac{-2 \beta_{1,u} \beta_{3,u} + 2 \beta_{2,u} q_{11} + 2 \beta_{1,u} \beta_{4,u} + 2 \beta_{3,u} \beta_{4,u} + 2 q_{11} - u}{\sigma_1^2} \end{pmatrix}, \]

where \(-\infty < \beta_{1,u} < \beta_{2,u} < 0 < \beta_{3,u} < \beta_{4,u} < \infty\) are the roots of Equation (10).

(b) The matrix exponentials of \((Q_+, Q_-)\) are, for \(m \in \mathbb{R}^+\), given by

\[ \exp(Q_+ m) = \frac{\beta_{3,u} e^{-\beta_{4,u} m} - \beta_{4,u} e^{-\beta_{3,u} m}}{\beta_{3,u} - \beta_{4,u}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{-\beta_{3,u} m} - e^{-\beta_{4,u} m}}{\beta_{3,u} - \beta_{4,u}} Q_+, \]

\[ \exp(Q_- m) = \frac{\beta_{1,u} e^{\beta_{2,u} m} - \beta_{2,u} e^{\beta_{1,u} m}}{\beta_{1,u} - \beta_{2,u}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{\beta_{1,u} m} - e^{\beta_{2,u} m}}{\beta_{1,u} - \beta_{2,u}} Q_. \]

**Proof**

See, for example, Hieber [2012].

If we apply Theorem 1 for \(m = x - b\), we obtain a simplified expression for \(\exp(Q_- (x - b))\) in Theorems 3 and 4. The first-passage time densities can then also be expressed in terms of the roots of Equation (10). For \(\pi_0 = (1, 0)\), we obtain

\[ f_{a,-\infty}(t, (1, 0)) = \frac{1}{\pi} \int_0^\infty \frac{e^{-iu t}}{\beta_{3,u} - \beta_{4,u}} \left( \frac{\beta_{3,u} - \beta_{3,u} \beta_{4,u} + 2 q_{11}}{\beta_{3,u} + \beta_{4,u} + 2 \mu_1 \sigma_1^2} e^{-\beta_{4,u} (a-x)} - \frac{\beta_{4,u} - \beta_{3,u} \beta_{4,u} + 2 q_{11}}{\beta_{3,u} + \beta_{4,u} + 2 \mu_1 \sigma_1^2} e^{-\beta_{3,u} (a-x)} \right) du, \]

\[ f_{\infty,b}(t, (1, 0)) = \frac{1}{\pi} \int_0^\infty \frac{e^{-iu t}}{\beta_{1,u} - \beta_{2,u}} \left( \frac{\beta_{1,u} - \beta_{1,u} \beta_{2,u} + 2 q_{11}}{\beta_{1,u} + \beta_{2,u} + 2 \mu_1 \sigma_1^2} e^{\beta_{2,u} (x-b)} - \frac{\beta_{2,u} - \beta_{1,u} \beta_{2,u} + 2 q_{11}}{\beta_{1,u} + \beta_{2,u} + 2 \mu_1 \sigma_1^2} e^{\beta_{1,u} (x-b)} \right) du, \]

see also Guo [2001a], Hieber [2013]. This result then yields analytical expressions for the prices of digital, barrier, and lookback options in Fourier space (see Theorems 3 – 6).
3 Numerical comparison and applications

Table 1 (Vanilla) call option prices $C(S_0, T, \pi_0)$ in a regime switching model comparing the backward finite elements scheme by Boyle and Draviam [2007] (left, $\Delta t = 0.01$, $S_{min} = 0$, $S_{max} = 200$, $N = 10001$), an unbiased Monte-Carlo simulation ($10^8$ simulation runs), and the matrix Wiener–Hopf factorization (middle). The parameter sets are taken from Boyle and Draviam [2007]: $S_0 = K = 100$, $Z_0 = 1$, $Q(0,1,2) = Q(2,1) = 0.5$, $r = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $T = 1$. For the Monte-Carlo simulation $95\%$ confidence intervals are given. The right column gives the corresponding prices $\mathcal{C}(S_0, T, \pi_0)$ in a Black-Scholes model ($\sigma = \sigma_1$).

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<tr>
<td>Boyle, Draviam [2007]</td>
<td>$C(S_0, T, \pi_0)$</td>
<td>$C(S_0, T, \pi_0)$</td>
<td>$C(S_0, T, \pi_0)$</td>
</tr>
<tr>
<td>$S_0 = 94.0$</td>
<td>5.8579</td>
<td>5.8600 ± 0.0021</td>
<td>5.8615</td>
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<td>$S_0 = 96.0$</td>
<td>6.9178</td>
<td>6.9239 ± 0.0023</td>
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<td>$S_0 = 98.0$</td>
<td>8.0775</td>
<td>8.0840 ± 0.0024</td>
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<tr>
<td>$S_0 = 100.0$</td>
<td>9.3324</td>
<td>9.3387 ± 0.0025</td>
<td>9.3392</td>
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<tr>
<td>$S_0 = 102.0$</td>
<td>10.6769</td>
<td>10.6836 ± 0.0026</td>
<td>10.6840</td>
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<td>$S_0 = 104.0$</td>
<td>12.1045</td>
<td>12.1136 ± 0.0027</td>
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<tr>
<td>$S_0 = 106.0$</td>
<td>13.6082</td>
<td>13.6164 ± 0.0028</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>5.1096</td>
<td>5.1096</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>6.1624</td>
<td>6.1624</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>7.3248</td>
<td>7.3248</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>8.5917</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>9.9563</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>11.4110</td>
<td>11.4110</td>
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<tr>
<td>$\mathcal{C}(S_0, T, \pi_0)$</td>
<td>12.9475</td>
<td>12.9475</td>
<td>12.9475</td>
</tr>
</tbody>
</table>

3 Numerical comparison and applications

We now compare our analytical solutions to a backward finite element scheme (see, for example, Boyle and Draviam [2007]), the Brownian bridge algorithm (see, for example, Hieber and Scherer [2010]), and an analytical approximation of barrier option prices by Lo et al. [2003], Elliott et al. [2007], Boyle and Draviam [2007] Monte-Carlo Fourier techniques.

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In a first step, the Fourier integral in Theorem 1 is implemented to price (vanilla) call options.

\[ C(S_0, T, \pi_0) = S_0 \Phi \left( \frac{\ln(S_0/K) + (r + \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} \right) - K e^{-rT} \Phi \left( \frac{\ln(S_0/K) + (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} \right), \]

\[ \mathcal{B}(S_0, T, \pi_0) = \mathcal{C}(S_0, T, \pi_0) - e^{-\frac{T}{4}} \ln(S_0/K) \mathcal{C}(D^2/S_0, T, \pi_0), \]

\[ \mathcal{D}(S_0, T, \pi_0) = \Phi \left( \frac{\ln(B/S_0) - (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} \right) - e^{-\frac{T}{4}} \ln(S_0/K) \Phi \left( \frac{\ln(B/S_0) + (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} \right), \]

\[ \mathcal{E}(S_0, T, \pi_0) = S_0 e^{-rt} \left( 1 - \frac{\sigma_1^2}{2r} \right) \Phi \left( \frac{(-r + \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} \right) + S_0 \left( 1 + \frac{\sigma_1^2}{2r} \right) \Phi \left( \frac{(-r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} \right), \]

where $B := \exp(b)$. 

\[ \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt. \]
3 Numerical comparison and applications

Regime switching model

<table>
<thead>
<tr>
<th></th>
<th>Brownian bridge</th>
<th>matrix W.–H.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D(S_0, T, \pi_0))</td>
<td>0.7650 ± 0.0031</td>
<td>0.7651</td>
</tr>
<tr>
<td>(S_1)</td>
<td>18.3219 ± 0.0139</td>
<td>18.3155</td>
</tr>
<tr>
<td>(S_2)</td>
<td>49.8258 ± 0.0164</td>
<td>49.8278</td>
</tr>
<tr>
<td>(S_3)</td>
<td>4.7844 ± 0.0077</td>
<td>4.7803</td>
</tr>
<tr>
<td>(S_4)</td>
<td>13.2036 ± 0.0122</td>
<td>13.2080</td>
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<tr>
<td>(S_5)</td>
<td>19.9005 ± 0.0146</td>
<td>19.8916</td>
</tr>
<tr>
<td>(S_6)</td>
<td>0.7651 ± 0.0031</td>
<td>0.7651</td>
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Black–Scholes

<table>
<thead>
<tr>
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<th>matrix W.–H.</th>
</tr>
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<tr>
<td>(D(S_0, T, \pi_0))</td>
<td>0.0416</td>
<td>0.0416</td>
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<tr>
<td>(S_1)</td>
<td>10.9707</td>
<td>10.9707</td>
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<tr>
<td>(S_2)</td>
<td>42.7432</td>
<td>42.7467</td>
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<tr>
<td>(S_3)</td>
<td>1.3909</td>
<td>1.3909</td>
</tr>
<tr>
<td>(S_4)</td>
<td>1.3909</td>
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<tr>
<td>(S_5)</td>
<td>1.3909</td>
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</tr>
<tr>
<td>(S_6)</td>
<td>1.3909</td>
<td>1.3909</td>
</tr>
</tbody>
</table>

Table 2 Digital option prices \(D(S_0, T, \pi_0)\) in a regime switching model comparing the Brownian bridge algorithm (left, \(10^7\) simulation runs) and the matrix Wiener–Hopf factorization (middle). The parameter sets are taken from Hieber and Scherer [2010]: \(S_0 = Z_0 = 1\), \(r = 0.03\), \(K = e^b\), and

\(S_1 - S_3\): \(\sigma_1 = 0.15\), \(\sigma_2 = 0.25\), \(Q_0(1, 2) = 0.8\), \(Q_0(2, 1) = 0.6\); \(K = 0.6\) \((S_1)\), \(K = 0.8\) \((S_2)\), \(K = 0.9\) \((S_3)\).

\(S_4 - S_6\): \(\sigma_1 = 0.10\), \(\sigma_2 = 0.25\), \(K = 0.8\); \(Q_0(1, 2) = 0.2\), \(Q_0(2, 1) = 0.1\) \((S_4)\), \(Q_0(1, 2) = 1.0\), \(Q_0(2, 1) = 0.6\) \((S_5)\), \(Q_0(1, 2) = 3.0\), \(Q_0(2, 1) = 2.0\) \((S_6)\).

For the Brownian bridge algorithm 95% confidence intervals are given. The right column gives the corresponding prices \(\overline{D(S_0, T, \pi_0)}\) in a Black–Scholes model \((\sigma = \sigma_1)\).

Table 1 uses the parameter set in Boyle and Draviam [2007] and compares the Fourier technique to a finite element scheme (see Table 1 in Boyle and Draviam [2007]) and to an unbiased Monte-Carlo simulation. We find that Fourier prices are within the 95% confidence intervals of the Monte-Carlo simulation. The results by Boyle and Draviam [2007] seem to very slightly underestimate prices – an indicator for a discretization bias.

Then, we turn to digital options. This now allows to apply the matrix Wiener–Hopf technique in Theorem 3. Using the dataset of Hieber and Scherer [2010], we compare our results to the Brownian bridge algorithm (see Table 2). Again the prices are within the 95% confidence intervals of the Brownian bridge algorithm. We realize that for those kind of options, regime switching seems to have a significant impact on option prices. Black–Scholes prices \((\sigma = \sigma_1)\) are especially for the parameter sets \(S_1, S_4 - S_6\) significantly lower than the prices in a regime switching environment.

Next, we use the same parameter sets to price barrier options. Prices computed via the matrix Wiener–Hopf technique are given in Theorem 4. Table 3 compares those prices to the Brownian bridge algorithm (see the results in Table 1 in Hieber and Scherer [2010]) and to an analytical approximation by Lo et al. [2003], Elliott et al. [2014]. Again we are well within the Monte-Carlo confidence intervals. The analytical approximation by Lo et al. [2003], Elliott et al. [2014] very slightly underestimates the option prices. For barrier options, the regime switching component does not seem to significantly influence option prices.

Last, we price lookback options. Table 4 compares the prices from Theorem 6 to a backward fi-
### 3 Numerical comparison and applications

<table>
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<tr>
<td></td>
<td>$B(S_0, T, \pi_0)$</td>
</tr>
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<td></td>
<td>$\mathcal{B}(S_0, T, \pi_0)$</td>
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</table>

#### Table 3
Barrier option prices $B(S_0, T, \pi_0)$ in a regime switching model comparing the Brownian bridge algorithm (left, $10^7$ simulation runs) and the matrix Wiener–Hopf factorization (middle). The parameter sets are taken from Hieber and Scherer [2010]: $S_0 = Z_0 = 1$, $r = 0.03$, $K = e^b$, and

$S_1 - S_3$: $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $Q_0(1, 2) = 0.8$, $Q_0(2, 1) = 0.6$; $K = 0.6$ ($S_1$), $K = 0.8$ ($S_2$), $K = 0.9$ ($S_3$).

$S_4 - S_6$: $\sigma_1 = 0.10$, $\sigma_2 = 0.25$, $K = 0.8$; $Q_0(1, 2) = 0.2$, $Q_0(2, 1) = 0.1$ ($S_4$), $Q_0(1, 2) = 1.0$, $Q_0(2, 1) = 0.6$ ($S_5$), $Q_0(1, 2) = 3.0$, $Q_0(2, 1) = 2.0$ ($S_6$).

For the Brownian bridge algorithm 95% confidence intervals are given. The right column gives the corresponding prices $\mathcal{B}(S_0, T, \pi_0)$ in a Black-Scholes model ($\sigma = \sigma_1$).

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<tr>
<td></td>
<td>$\mathcal{L}(S_0, T, \pi_0)$</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{L}(S_0, T, \pi_0)$</td>
</tr>
</tbody>
</table>

#### Table 4
Lookback strike put option prices $\mathcal{L}(S_0, T, \pi_0)$ in a regime switching model comparing the backward finite elements scheme by Boyle and Draviam [2007] (left, $\Delta t = 0.01$, $S_{\text{min}} = 0$, $S_{\text{max}} = 200$, $N = 10001$) and the matrix Wiener–Hopf factorization (middle, $b_{\text{min}} = 1e-05$, $b_{\text{max}} = 1$ in Remark 7). The parameter sets are taken from Boyle and Draviam [2007]: $S_0 = K = 100$, $Z_0 = 1$, $Q_0(1, 2) = Q_0(2, 1) = 0.5$, $r = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $T = 1$. The right column gives the corresponding prices $\mathcal{L}(S_0, T, \pi_0)$ in a Black-Scholes model ($\sigma = \sigma_1$).
nite element scheme (see Table 11 in Boyle and Draviam [2007]). Again note that the prices by Boyle and Draviam [2007] slightly underestimate the analytical expressions – an indicator for a small discretization bias.

In Tables 1 to 4, one observes that the analytical Black-Scholes prices (marked by a $\bar{}$) are closely recovered by Fourier integrals and matrix Wiener–Hopf factorization.

4 Conclusion

This paper demonstrates how one can price exotic options (i.e. digital, barrier, and lookback options) in a Markovian, regime-switching, Black-Scholes model. In Fourier space, analytical expressions for the option prices are given as functionals of the matrix Wiener–Hopf factorization. In the case of 2 or 3 regimes or in the case of a zero interest rate the matrix Wiener–Hopf factorization can be solved analytically (see, for example, Section 3 or Hieber [2012]). This approach turns out to be easy to implement; the resulting option prices are accurate and unbiased. In a numerical case study we confirm that the presence of regime switching in financial data has a significant effect on option prices – especially for digital options.

One can generalize the presented results in several directions – regarding both the considered financial model and the payoff streams of the financial contracts. However, the analytical tractability might be reduced or lost. First, if one adds exponential or phase-type jumps, one can still derive modified matrix Wiener–Hopf factorizations (see, for example, Jiang and Pistorius [2008]). Secondly, the Fourier transform of the two-sided first-passage time can also be represented as a functional of the matrix Wiener–Hopf factorization, a result that allows us to price, for example, double barrier options.

References


References


References


