The mean of Marshall–Olkin dependent exponential random variables

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Abstract

The probability distribution of $S_d := X_1 + \ldots + X_d$, where the vector $(X_1, \ldots, X_d)$ is distributed according to the Marshall–Olkin law, is investigated. Closed-form solutions are derived in the general bivariate case and for $d \in \{2, 3, 4\}$ in the exchangeable subfamily. Our computations can be extended to higher dimensions, which, however, becomes cumbersome due to the large number of involved parameters. For the Marshall–Olkin distributions with conditionally independent and identically distributed components, however, the limiting distribution of S_d/d is identified as d tends to infinity. This result might serve as a convenient approximation in high-dimensional situations. Possible fields of application for the presented results are reliability theory, insurance, and credit-risk modeling.

1 Introduction

The distribution of $S_d := X_1 + \ldots + X_d$ has been treated considerably in the literature. For mathematical tractability, the individual random variables X_k are often considered to be independent, see, e.g., [2], an hypothesis that is hardly never met in real-world applications. Another case where the distribution of the sum is known is when $(X_1, \ldots, X_d)$ has an elliptical distribution, see [10], a stability result that (at least partially) explains the popularity of elliptical distributions. Again, it could be that this distributional assumption does not hold for the application one has in mind. In our study we assume $(X_1, \ldots, X_d)$ to be distributed according to the Marshall–Olkin law; a popular assumption for dependent lifetimes in insurance and credit-risk modelling, see [13], [9]. With this interpretation in mind, S_d/d denotes the average lifetime of dependent exponential random variables. Applications might be the costs of an insurance company in the case of a natural catastrophe ([7]) or maintenance fees that have to be paid as long as some system is working. Related studies on the probability of a

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sum of dependent risks can be found in the literature related to insurance and risk-management, see, e.g., ([1], [22], [25], [5]).

We derive $\mathbb{P}(S_2 > x)$ explicitely in the general case. One property of the Marshall–Olkin law is the large number of parameters, namely $2^d - 1$, in dimension d, rendering the Marshall–Olkin law challenging to work with as d increases. To account for this, in Section 3 we focus on the exchangeable subfamily, which has only d parameters in dimension d. In our case we compute $\mathbb{P}(S_d > x)$ for $d \in \{2, 3, 4\}$. We guide the interested reader to strategies how extensions to higher dimensions might be achieved. Moreover, we study the asymptotic distribution of S_d/d (when $d \to \infty$) in the subfamily of Marshall–Olkin distributions with conditionally i.i.d. (CIID) components. Upper bounds for the sum of exchangeable vectors of CIID variables are already studied, see [6]. In [25] the asymptotic quantile behaviour of a sum of dependent variables, where the dependence structure is given by an Archimedean copula, is analysed. In our case, the limiting case is related to certain exponential functionals of Lévy subordinators which are studied, e.g., in [14, 3, 23, 16].

The paper is organized as follows: In Section 2 the general Marshall–Olkin distribution is introduced and we compute the distribution of S_2. Section 3 considers the exchangeable case and computes $\mathbb{P}(S_d > x)$ for $d \in \{2, 3, 4\}$. In Section 4 we analyse the asymptotic case $d \to \infty$. Section 5 concludes.

2 The Marshall–Olkin law

[21] introduce a d-dimensional exponential distribution by lifting the univariate *lack of memory property* $\mathbb{P}(X > x + y | X > y) = \mathbb{P}(X > x)$, for all $x, y > 0$, to higher dimensions. If X is supported on $[0, \infty)$ and satisfies the univariate *lack of memory property*, then X is exponentially distributed. If $(X_1, \ldots, X_d)$ and all possible subvectors $(X_{i_1}, \ldots, X_{i_k})$, where $1 \leq i_1 < \ldots < i_k \leq d$, satisfy the multidimensional *lack of memory property*,

$$\mathbb{P}(X_{i_1} > x_{i_1} + y, \ldots, X_{i_k} > x_{i_k} + y | X_{i_1} > y, \ldots, X_{i_k} > y) = \mathbb{P}(X_{i_1} > x_{i_1}, \ldots, X_{i_k} > x_{i_k}), \qquad (1)$$

where $x_{i_1}, \ldots, x_{i_k}, y > 0$, it is shown in [21] that the only distribution with support $[0, \infty)^d$ satisfying condition (1) is characterized by the survival function introduced in Definition 1 below.

Definition 1 (Marshall–Olkin distribution)
Let $(X_1, \ldots, X_d)$ represent a system of residual lifetimes with support $[0, \infty)^d$. Assume that the remaining components in this vector have a joint distribution that is independent of the age of the system, i.e. $(X_1, \ldots, X_d)$ satisfies the multidimensional lack of memory property (1). Then

$$\bar{F}(x_1, \ldots, x_d) := \mathbb{P}(X_1 > x_1, \ldots, X_d > x_d) = \exp\left(- \sum_{\emptyset \neq I \subset \{1, \ldots, d\}} \lambda_I \max_{i \in I}\{x_i\} \right), \quad x_1, \ldots, x_d \geq 0, \qquad (2)$$

for certain parameters $\lambda_I \geq 0$, $\emptyset \neq I \subset \{1, \ldots, d\}$, and $\sum_{I: k \in I} \lambda_I > 0$, $k = 1, \ldots, d$. This multivariate probability law is called Marshall–Olkin distribution.

This distribution has key impact in reliability theory [21, 8], credit-risk management [13], and insurance [9]. Interpreting X_k as lifetime of component k, λ_I represents the intensity of the arrival time of a "shock" influencing the lifetime of all components in I. This can be seen from the canonical construction of the Marshall–Olkin distribution which is the following *fatal-shock* model, see [20, 8]. Let E_I, $\emptyset \neq I \subset \{1,\dots,d\}$, be exponentially distributed random variables with parameters $\lambda_I \geq 0$. We assume all E_I to be independent and interpret them as the arrival times of exogenous shocks to the respective components in I and define

$$X_k := \min\{E_I | \emptyset \neq I \subset \{1,\dots,d\}, k \in I\} \in (0,\infty), \quad k = 1,\dots,d, \tag{3}$$

where the variable X_k is the first time a shock hits component[1] k. The random vector $(X_1,\dots,X_d)$ as defined in Equation (3) follows the Marshall–Olkin distribution.

Next, we derive the probability distribution of $S_2 = aX_1 + bX_2$, where a, b are positive constants. Providing an interpretation, with $a = b = 1/2$ the quantity $S_2/2$ is precisely the average lifetime of the two components. To simplify notation we write $\lambda_1, \lambda_2, \lambda_{12}$ instead of $\lambda_{\{1\}}, \lambda_{\{2\}}, \lambda_{\{1,2\}}$ and we write E_1, E_2, E_{12} instead of $E_{\{1\}}, E_{\{2\}}, E_{\{1,2\}}$.

Lemma 1 (The weighted sum of two lifetimes)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let (X_1, X_2) be a random vector constructed as in (3) and a, b positive constants. The survival function of the weighted sum of X_1 and X_2 is computed as

$$\begin{aligned} \mathbb{P}(aX_1 + bX_2 > x) &= \frac{\lambda_1}{\lambda_1 - (\lambda_2 + \lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b})\frac{x}{a+b}}\right) \\ &+ \frac{\lambda_2}{\lambda_2 - (\lambda_1 + \lambda_{12})\frac{b}{a}} e^{-(\lambda_1+\lambda_{12})\frac{x}{a}} \left(1 - e^{-(\lambda_2 - (\lambda_1+\lambda_{12})\frac{b}{a})\frac{x}{a+b}}\right) + e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}. \end{aligned} \tag{4}$$

Proof

$$\begin{aligned} \mathbb{P}(aX_1 + bX_2 > x) &= \mathbb{P}(aX_1 + bX_2 > x, X_1 < X_2) + \mathbb{P}(aX_1 + bX_2 > x, X_2 < X_1) \\ &+ \mathbb{P}(aX_1 + bX_2 > x, X_1 = X_2). \end{aligned}$$

Observe that, $X_1 < X_2 \Leftrightarrow E_1 < X_2, \quad X_2 < X_1 \Leftrightarrow E_2 < X_1, \quad X_1 = X_2 \Leftrightarrow E_{12} < \min\{E_1, E_2\}$,

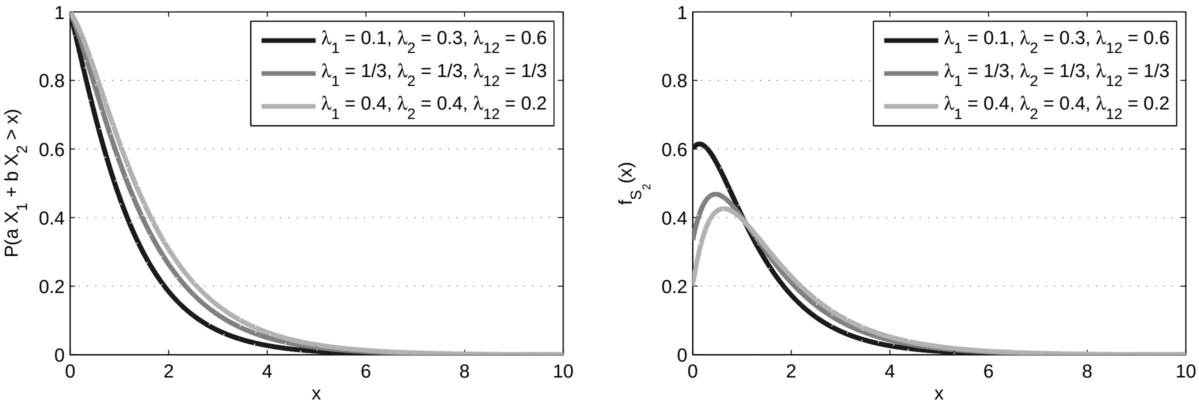
and, $\min\{E_1, E_2\} \sim Exp(\lambda_1 + \lambda_2)$.

Then,

$$\begin{aligned} &\mathbb{P}(aX_1 + bX_2 > x, X_1 < X_2) = \mathbb{P}(aX_1 + bX_2 > x, E_1 < X_2) = \mathbb{P}\left(X_2 > E_1 > \frac{x - bX_2}{a}\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(X_2 > E_1 > \frac{x - bX_2}{a} | E_1\right)\right] = \int_0^\infty \mathbb{P}\left(X_2 > y_1 > \frac{x - bX_2}{a}\right) f_{E_1}(y_1) dy_1 \\ &= \frac{\lambda_1}{\lambda_1 - (\lambda_2 + \lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b})\frac{x}{a+b}}\right) + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_{12}} e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}. \end{aligned}$$

[1] The parameters $\lambda_I \geq 0$ represent the intensities of the exogenous shocks. Some of these can be 0, in which case $E_I \equiv \infty$. We require $\sum_{\emptyset \neq I : k \in I} \lambda_I > 0$, so for each $k = 1,\dots,d$ there is at least one subset $I \subset \{1,\dots,d\}$, containing k, such that $\lambda_I > 0$. Therefore, (3) is well-defined.

$\mathbb{P}(aX_1 + bX_2 > x, X_2 < X_1)$ and $\mathbb{P}(aX_1 + bX_2 > x, X_1 = X_2)$ are computed in the same way. □



(a) Survival function (b) Density function

Figure 1 The survival and density function of $S_2 = aX_1 + bX_2$, where $a = 30\%$ and $b = 70\%$.

Once the survival function of S_2 is known, one can further compute the density and the Laplace transform of S_2.

Corollary 1
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, (X_1, X_2) a random vector constructed as in (3), and a, b positive constants. Then the Laplace transform of $S_2 = aX_1 + bX_2$ is given by

$$\begin{aligned}\psi_{S_2}(t) = \mathbb{E}\left[e^{-tS_2}\right] \quad = \quad & \frac{\lambda_1(\lambda_2+\lambda_{12})b}{(\lambda_1 b - (\lambda_2+\lambda_{12})a)\,(\lambda_2+\lambda_{12}+tb)} + \frac{\lambda_2(\lambda_1+\lambda_{12})a}{(\lambda_2 a - (\lambda_1+\lambda_{12})b)\,(\lambda_1+\lambda_{12}+ta)} \\ & - \left(\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12})a} + \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12})b}\right)\frac{a+b}{\lambda_1+\lambda_2+\lambda_{12}+t(a+b)} \\ & + \frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}+t(a+b)}. \end{aligned} \tag{5}$$

Proof
We first need to compute the probability density function:

$$\begin{aligned} f_{S_2}(x) = \frac{d}{dx}\left(1 - \bar{F}_{S_2}(x)\right) = & \frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12})a} e^{-(\lambda_2+\lambda_{12})x/b}\left(1 - e^{-(\lambda_1-(\lambda_2+\lambda_{12})a/b)\frac{x}{a+b}}\right) \\ & + \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12})b} e^{-(\lambda_1+\lambda_{12})x/a}\left(1 - e^{-(\lambda_2-(\lambda_1+\lambda_{12})b/a)\frac{x}{a+b}}\right) \\ & + \frac{\lambda_{12}}{a+b} e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}. \end{aligned} \tag{6}$$

So, the Laplace transform is computed by evaluating the integral

$$\psi_{S_2}(t) = \int_0^\infty e^{-tx} f_{S_2}(x)dx. \qquad \square$$

Remark 1
Note that when $\lambda_1 - (\lambda_2 + \lambda_{12})a/b = 0$ or $\lambda_2 - (\lambda_1 + \lambda_{12})b/a = 0$ Equations (4), (5), and (6) are not defined. By computing the respective limits (that do exist!) when the parameters approach such a constellation, the functions can be continued continuously.

If one aims at generalizing these results to higher dimensions, one notices that the number of involved shocks and parameters, i.e. $2^d - 1$ in dimension d, renders this problem extremely intractable already for moderate dimensions d. A subclass with fewer parameters is obtained by considering the Marshall–Olkin law with exchangeable components. This yields a parametric family with d parameters in dimension d, allowing us to derive the distribution of S_d in higher dimensions.

3 The exchangeable Marshall–Olkin law

The aim of this section is to compute the survival function of S_d in the exchangeable case. We introduce the subfamily of exchangeable Marshall–Olkin laws in order to deal with the problem of overparameterization. For a deeper background on exchangeable Marshall–Olkin laws see [18], [19] (Chapter 3, Section 3.2). A random vector $(X_1, \ldots, X_d)$ is said to be exchangeable if for all permutations π on $\{1, \ldots, d\}$ it satisfies

$$\mathbb{P}(X_1 > x_1, \ldots, X_d > x_d) = \mathbb{P}(X_1 > x_{\pi(1)}, \ldots, X_d > x_{\pi(d)}), \quad x_1, \ldots, x_d \in \mathbb{R}, \tag{7}$$

or, alternatively in the Marshall–Olkin context, if the exchangeability condition

$$|I| = |J| \Rightarrow \lambda_I = \lambda_J, \tag{8}$$

is met. The proof that (8) is equivalent to $(X_1, \ldots, X_d)$ being exchangeable can be found in [19], page 124. Condition (8) means that two shocks affecting subsets with identical cardinalities have the same intensity λ_I. Hence, in this section we denote by λ_1 the intensity of all shocks affecting precisely one component, by λ_2 all shocks affecting two components, and so on.

Let $(X_1, \ldots, X_d)$ be a random vector following the Marshall–Olkin distribution, defined as in Equation (3). Then the survival function of the exchangeable Marshall–Olkin law is given by

$$\bar{F}(x_1, \ldots, x_d) = \exp\left(- \sum_{k=1}^{d} x_{(d+1-k)} \sum_{i=0}^{d-k} \binom{d-k}{i} \lambda_{i+1} \right), \quad x_1, \ldots, x_d \geq 0, \tag{9}$$

$x_{(1)} \leq \ldots \leq x_{(d)}$ being the ordered list of $x_1, \ldots, x_d$.

Observe that now instead of dealing with $2^d - 1$ parameters λ_I we just have to work with d parameters $\lambda_1, \ldots, \lambda_d$, which simplifies the process of computing the required probabilities.

In the following, we present the survival function of the sum of components of Marshall–Olkin random vectors in low dimensional exchangeable cases (2, 3, and 4-dimensional).

Lemma 2 (The sum of $\mathbf{d} \in \{2,3,4\}$ lifetimes)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$...

i) ... let (X_1, X_2) be a 2-dimensional exchangeable Marshall–Olkin random vector. Then,

$$\mathbb{P}(X_1 + X_2 > x) = \frac{2\lambda_1 e^{-(\lambda_1+\lambda_2)x}}{\lambda_2}\left(e^{\lambda_2 \frac{x}{2}} - 1\right) + e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}, \quad x \geq 0. \tag{10}$$

ii) ... let (X_1, X_2, X_3) be a 3-dimensional exchangeable Marshall–Olkin random vector. Then,

$$\begin{aligned}\mathbb{P}(X_1 + X_2 + X_3 > x) = {} & e^{-(3\lambda_1+3\lambda_2+\lambda_3)\frac{x}{3}} + \frac{6\lambda_1(2\lambda_1 + 3\lambda_2 + \lambda_3)}{(3\lambda_2+\lambda_3)(\lambda_2+\lambda_3)} \\ & e^{-(2\lambda_1+3\lambda_2+\lambda_3)\frac{x}{2}}\left(e^{\left(\frac{3\lambda_2+\lambda_3}{2}\right)\frac{x}{3}} - 1\right) + \frac{3\lambda_2(\lambda_2+\lambda_3) - 6\lambda_1(\lambda_1+\lambda_2)}{(\lambda_2+\lambda_3)(3\lambda_2+2\lambda_3)} \\ & e^{-(\lambda_1+2\lambda_2+\lambda_3)x}\left(e^{(3\lambda_2+2\lambda_3)\frac{x}{3}} - 1\right), \quad x \geq 0.\end{aligned} \tag{11}$$

iii) ... let (X_1, X_2, X_3, X_4) be a 4-dimensional exchangeable Marshall–Olkin random vector. Then,

$$\begin{aligned}\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x) = {} & 24 \cdot P_1 + 12 \cdot P_2 + 12 \cdot P_3 \\ + {} & 12 \cdot P_4 + 4 \cdot P_5 + 4 \cdot P_6 + 6 \cdot P_7 + P_8, \quad x \geq 0,\end{aligned} \tag{12}$$

where,

$$\begin{aligned}
P_1 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2<X_3<X_4)\\
&= \lambda_1(\lambda_1+\lambda_2)f_{11}\Big(\frac{32f_{10}}{f_1f_2f_4f_5}e^{-f_1\frac{x}{4}}-\frac{27f_{10}}{f_2f_3f_7f_8}e^{-f_3\frac{x}{3}}+\frac{4f_{10}}{f_4f_6f_7f_9}e^{-f_9\frac{x}{2}}\\
&- \frac{1}{f_5f_6f_8}e^{-f_{10}x}\Big),\\
P_2 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2<X_3=X_4)\\
&= \lambda_1(\lambda_1+\lambda_2)f_6\Big(\frac{8}{f_1f_2f_4}e^{-f_1\frac{x}{4}}-\frac{9}{f_2f_3f_7}e^{-f_3\frac{x}{3}}+\frac{2}{f_4f_7f_9}e^{-f_9\frac{x}{2}}\Big),\\
P_3 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2=X_3<X_4)\\
&= \lambda_1(\lambda_2+\lambda_3)\Big[\frac{f_{10}}{f_2}\Big(\frac{16}{f_1f_5}e^{-f_1\frac{x}{4}}-\frac{9}{f_3f_8}e^{-f_3\frac{x}{3}}\Big)+\frac{1}{f_5f_8}e^{-f_{10}x}\Big],\\
P_4 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2<X_3<X_4)\\
&= \lambda_2f_{11}\Big[\Big(\frac{2f_{10}}{f_4f_6f_9}-\frac{1}{f_5f_6}+\frac{1}{f_1f_9}\Big)e^{-f_1\frac{x}{4}}-\frac{2f_{10}}{f_4f_6f_9}e^{-f_9\frac{x}{2}}+\frac{1}{f_5f_6}e^{-f_{10}x}\Big],\\
P_5 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2=X_3<X_4)\\
&= \lambda_3\Big(\frac{4f_{10}}{f_1f_5}e^{-f_1\frac{x}{4}}-\frac{1}{f_5}e^{-f_{10}x}\Big),\\
P_6 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2=X_3=X_4)\\
&= \frac{\lambda_1}{f_2}(\lambda_3+\lambda_4)\Big(\frac{4}{f_1}e^{-f_1\frac{x}{4}}-\frac{3}{f_3}e^{-f_3\frac{x}{3}}\Big),\\
P_7 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2<X_3=X_4)\\
&= \frac{\lambda_2f_6}{f_4}\Big(\frac{2}{f_1}e^{-f_1\frac{x}{4}}-\frac{1}{f_9}e^{-f_9\frac{x}{2}}\Big),\\
P_8 &= \mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2=X_3=X_4)=\frac{\lambda_4}{f_1}e^{-f_1\frac{x}{4}},
\end{aligned}$$

and

$$\begin{aligned}
f_1 &= 4\lambda_1+6\lambda_2+4\lambda_3+\lambda_4, & f_5 &= 6\lambda_2+8\lambda_3+3\lambda_4, & f_9 &= 2\lambda_1+5\lambda_2+4\lambda_3+\lambda_4,\\
f_2 &= 6\lambda_2+4\lambda_3+\lambda_4, & f_6 &= \lambda_2+2\lambda_3+\lambda_4, & f_{10} &= \lambda_1+3\lambda_2+3\lambda_3+\lambda_4,\\
f_3 &= 3\lambda_1+6\lambda_2+4\lambda_3+\lambda_4, & f_7 &= 3\lambda_2+4\lambda_3+\lambda_4, & f_{11} &= \lambda_1+2\lambda_2+\lambda_3.\\
f_4 &= 4\lambda_2+4\lambda_3+\lambda_4, & f_8 &= 3\lambda_2+5\lambda_3+2\lambda_4,
\end{aligned}$$

Proof
We prove the case $d=2$, considering that the proofs for $d=3$ and $d=4$ are done in the same way.

$$\begin{aligned}
\mathbb{P}(X_1+X_2>x) &= 2\mathbb{P}(X_1+X_2>x|X_1<X_2)\mathbb{P}(X_1<X_2)\\
&+ \mathbb{P}(X_1+X_2>x|X_1=X_2)\mathbb{P}(X_1=X_2).
\end{aligned}$$

such that $E_1, E_2 \sim Exp(\lambda_1)$ *and* $E_{12} \sim Exp(\lambda_2)$ *and note that since we are working on the exchangeable case,*

$$\mathbb{P}(X_1 + X_2 > x | X_1 > X_2)\mathbb{P}(X_1 > X_2) = \mathbb{P}(X_1 + X_2 > x | X_1 < X_2)\mathbb{P}(X_1 < X_2).$$

Taking into account that, $X_1 < X_2 \Leftrightarrow E_1 < \min\{E_2, E_{12}\}$ *and* $X_1 = X_2 \Leftrightarrow \min\{E_1, E_2\} > E_{12}$*,*

$$\begin{aligned}\mathbb{P}(X_1 + X_2 > x) &= 2\mathbb{P}(E_1 + \min\{E_2, E_{12}\} > x | E_1 < \min\{E_2, E_{12}\}) \\ &\cdot \mathbb{P}(E_1 < \min\{E_2, E_{12}\}) + \mathbb{P}(E_{12} + E_{12} > x | E_{12} < \min\{E_1, E_2\})\mathbb{P}(E_{12} < \min\{E_1, E_2\}) \\ &= 2\mathbb{E}\left[\mathbb{P}(\min\{E_2, E_{12}\} > E_1 > x - \min\{E_2, E_{12}\} | E_1)\right] \\ &+ \mathbb{E}\left[\mathbb{P}(\min\{E_1, E_2\} > E_{12} > \frac{x}{2} | \min\{E_1, E_2\})\right].\end{aligned}$$

Then, from the so-called min-stability of the exponential distribution, $\min\{E_1, E_2\} \sim Exp(2\lambda_1)$ *and* $\min\{E_2, E_{12}\} \sim Exp(\lambda_1 + \lambda_2)$*,*

$$\begin{aligned}\mathbb{E}\left[\mathbb{P}(\min\{E_2, E_{12}\} > E_1 > x - \min\{E_2, E_{12}\} | E_1)\right] &= \frac{\lambda_1}{\lambda_2} e^{-(\lambda_1+\lambda_2)x} \left(e^{\lambda_2 \frac{x}{2}} - 1\right) + \frac{\lambda_1}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}, \\ \mathbb{E}\left[\mathbb{P}(\min\{E_1, E_2\} > E_{12} > \frac{x}{2} | \min\{E_1, E_2\})\right] &= \frac{\lambda_2}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}.\end{aligned}$$

So,

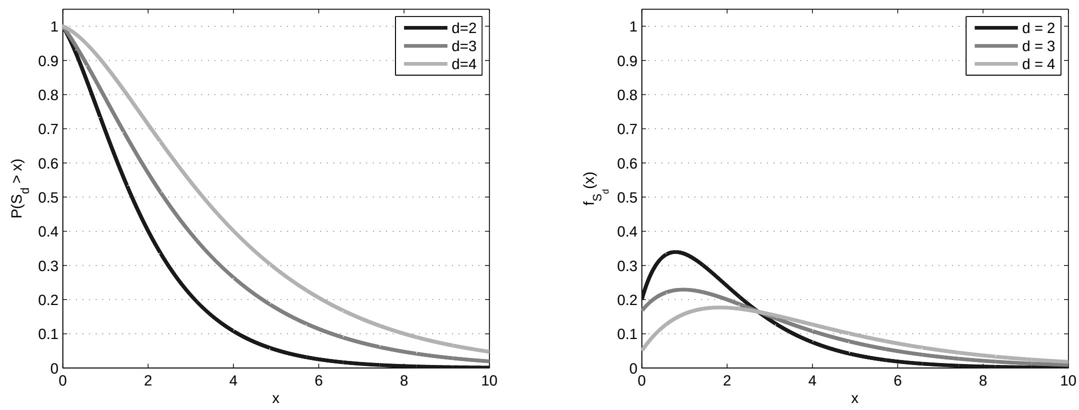
$$\begin{aligned}\mathbb{P}(X_1 + X_2 > x) &= 2\Big(\frac{\lambda_1}{\lambda_2} e^{-(\lambda_1+\lambda_2)x} \left(e^{\lambda_2 \frac{x}{2}} - 1\right) + \frac{\lambda_1}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}\Big) \\ &+ \frac{\lambda_2}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda 2)\frac{x}{2}} = \frac{2\lambda_1 e^{-(\lambda_1+\lambda_2)x}}{\lambda_2} \left(e^{\lambda_2 \frac{x}{2}} - 1\right) + e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}.\end{aligned}$$

Note that (from Remark 2 below) in case $d = 3$

$$\begin{aligned}\mathbb{P}(X_1 + X_2 + X_3 > x) &= 6\mathbb{P}(X_1 + X_2 + X_3 > x | X_1 < X_2 < X_3)\mathbb{P}(X_1 < X_2 < X_3) \\ &+ 3\mathbb{P}(X_1 + X_2 + X_3 > x | X_1 = X_2 < X_3)\mathbb{P}(X_1 = X_2 < X_3) \\ &+ 3\mathbb{P}(X_1 + X_2 + X_3 > x | X_1 < X_2 = X_3)\mathbb{P}(X_1 < X_2 = X_3) \\ &+ \mathbb{P}(X_1 + X_2 + X_3 > x | X_1 = X_2 = X_3)\mathbb{P}(X_1 = X_2 = X_3)\end{aligned}$$

has to be computed and in $d = 4$

$$
\begin{aligned}
\mathbb{P}(X_1+X_2+X_3+X_4>x) = {} & 24\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2<X_3<X_4) \\
& +12\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2<X_3=X_4) \\
& +12\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2=X_3<X_4) \\
& +12\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2<X_3<X_4) \\
& +4\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2=X_3<X_4) \\
& +4\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1<X_2=X_3=X_4) \\
& +6\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2<X_3=X_4) \\
& +\mathbb{P}(X_1+X_2+X_3+X_4>x|X_1=X_2=X_3=X_4). \qquad \square
\end{aligned}
$$



(a) Survival function (b) Density function

Figure 2 Plots of the survival and density function for S_d, $d = 2, 3, 4$, in the exchangeable case. The parameters considered are in the two-dimensional case: $\lambda_1 = 0.6, \lambda_2 = 0.4$, in the three-dimensional case: $\lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.5$, and when $d = 4$: $\lambda_1 = 0.05, \lambda_2 = 0.1, \lambda_3 = 0.15, \lambda_4 = 0.2$.

Remark 2 (Generalizing the results to higher dimensions)
Marshall–Olkin multivariate distributions are not absolutely continuous, i.e. there is a positive probability that several components take the same value, $\mathbb{P}(X_1 = \ldots = X_d) > 0$. It is possible to compute the expression

$$\mathbb{P}(X_1 + \ldots + X_d > x, X_1 = \ldots = X_d), \tag{13}$$

for all dimensions $d \in \mathbb{N}$, by recalling Pascal's triangle.

$$
\begin{aligned}
PM_d^d &:= \mathbb{P}(X_1+\ldots+X_d > x, X_1=\ldots=X_d) \\
&= \frac{\lambda_d}{\sum_{i=0}^d \binom{d}{i}\lambda_i} e^{-\left(\sum_{i=0}^d \binom{d}{i}\lambda_i\right)\frac{x}{d}}, \quad \lambda_0 = 0.
\end{aligned} \tag{14}
$$

However, the generalization to arbitrary singular events is not that obvious. Observe that from a sum of d elements we have to take into account the cases where we have k equalities in the conditions of the conditional probabilities, $k \in \{0, \dots, d-1\}$. The number of cases which have to be taken into account is given by the binomial coefficient $(d-1)$ *choose* (k).

Take for example the case $d = 4$:

i) Number of cases where $k = 0$, i.e. there is no equality in the condition: $\binom{3}{0} = 1$,

$$\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < \ldots < X_{i_4}), \quad \text{where} \quad i_k \neq i_j \in \{1,2,3,4\}.$$

ii) Number of cases where there is one equality ($k = 1$) in the condition: $\binom{3}{1} = 3$,

$$\begin{aligned}
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} = X_{i_2} < X_{i_3} < X_{i_4}), \\
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < X_{i_2} = X_{i_3} < X_{i_4}), \\
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < X_{i_2} < X_{i_3} = X_{i_4}),
\end{aligned}$$

where $i_k \neq i_j \in \{1,2,3,4\}$.

iii) Number of cases where there are 2 equalities ($k = 2$) in the condition: $\binom{3}{2} = 3$,

$$\begin{aligned}
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} = X_{i_2} = X_{i_3} < X_{i_4}), \\
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < X_{i_2} = X_{i_3} = X_{i_4}), \\
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} = X_{i_2} < X_{i_3} = X_{i_4}),
\end{aligned}$$

such that $i_k \neq i_j \in \{1,2,3,4\}$.

Since we are in the exchangeable case, we need to calculate how many times each probability has to be added. For this purpose let us consider the definition of *permutation of multisets*

$$PM_d^{a_1,a_2,\ldots,a_{k-1},a_k} := \frac{d!}{a_1! \cdot a_2! \cdot \ldots \cdot a_{k-1}! \cdot a_k!}, \tag{15}$$

where in our case $a_1, \ldots, a_d$ represent the numbers of elements which are equal and how they are located in each condition. Note that $\sum_{i=1}^{k} a_i = d$. Let us illustrate this relation with the example of $d = 4$:

$$
\begin{aligned}
\mathbb{P}(X_1+\ldots+X_4>x) \;=\; & PM_4^{1,1,1,1}\cdot\mathbb{P}(X_1+\ldots+X_4,\underbrace{X_1}_{1}<\underbrace{X_2}_{1}<\underbrace{X_3}_{1}<\underbrace{X_4}_{1}) \\
+\; & PM_4^{2,1,1}\cdot\mathbb{P}(X_1+\ldots+X_4>x,\underbrace{X_1=X_2}_{2}<\underbrace{X_3}_{1}<\underbrace{X_4}_{1}) \\
+\; & PM_4^{1,2,1}\cdot\mathbb{P}(X_1+\ldots+X_4>x,\underbrace{X_1}_{1}<\underbrace{X_2=X_3}_{2}<\underbrace{X_4}_{1}) \\
+\; & PM_4^{1,1,2}\cdot\mathbb{P}(X_1+\ldots+X_4>x,\underbrace{X_1}_{1}<\underbrace{X_2}_{1}<\underbrace{X_3=X_4}_{2}) \qquad (16) \\
+\; & PM_4^{2,2}\cdot\mathbb{P}(X_1+\ldots+X_4>x,\underbrace{X_1=X_2}_{2}<\underbrace{X_3=X_4}_{2}) \\
+\; & PM_4^{3,1}\cdot\mathbb{P}(X_1+\ldots+X_4>x,\underbrace{X_1=X_2=X_3}_{3}<\underbrace{X_4}_{1}) \\
+\; & PM_4^{1,3}\cdot\mathbb{P}(X_1+\ldots+X_4>x,\underbrace{X_1}_{1}<\underbrace{X_2=X_3=X_4}_{3})+PM_4^{4},
\end{aligned}
$$

the expression for PM_4^4 is given in Equation (14).

Example 1 (Illustrating the effect of different levels of dependence)
In Figure 3, examples for the survival and density function of S_4 for different levels of dependence are visualized.

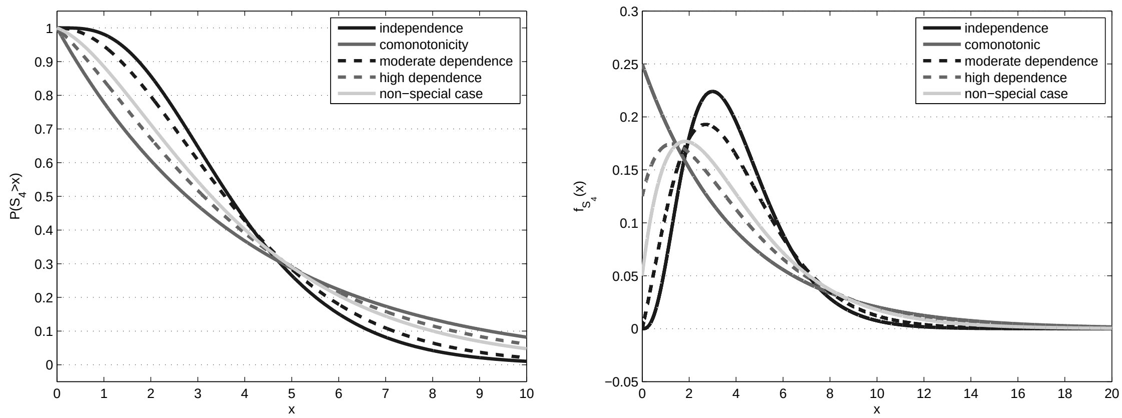
a) *Independence: shocks arriving to just one element are the only ones present in the system, i.e. $\lambda_1>0$ and $\lambda_2=\lambda_3=\lambda_4=0$. In this case the probability distribution of S_d follows the Erlang distribution with rate λ_1 and degrees of freedom 4.*

b) *Comonotonic case: the shock arriving to all components at the same time is the only one influencing the system, i.e. $\lambda_1=\lambda_2=\lambda_3=0$ and $\lambda_4>0$, and the distribution of S_d is exponential with mean $4/\lambda_4$.*

c) *Moderate dependence: in this case the shocks influencing fewer components jointly have the strongest influence, i.e. $\lambda_1>\lambda_2>\lambda_3>\lambda_4>0$.*

d) *High dependence case: shocks arriving to most components jointly have the strongest influence, i.e. $\lambda_4>\lambda_3>\lambda_2>\lambda_1>0$.*

e) *Non-special case: $\lambda_1,\lambda_2,\lambda_3,\lambda_4>0$.*



(a) Survival function (b) Density function

Figure 3 $\mathbb{P}(S_4 > x)$ (left) and $f_{S_4}(x)$ (right) for different assumptions concerning the dependence: a) independence, b) comonotonicity, c) moderate dependence (we consider $\lambda_4 = 0$), d) high dependence (we consider $\lambda_1 = 0$), e) non-special case. In all examples, the marginal laws are considered to be the same, X_i unit exponential random variables, $i = 1, \ldots, 4$.

One can observe in Figure 3 (left) that the intersection of the survival function is around the expected value $\mathbb{E}[S_4] = 4$. When the dependence between the components of the system is strong, the probability of the system to collapse before this intersection is lower than in the cases where the dependence is weak, but once the system survives till this intersection point, in cases with strong dependence the probability that the system will last alive longer is higher than in cases where the dependence is weak. This interpretation can be also seen in the densities (see Figure 3, right). In weak dependence cases, the mass of the probability is concentrated around the expected value, which is translated into having a strong depth in the slope of the survival function (see Figure 3).

4 The extendible Marshall–Olkin law

In this section we show how the probability distribution of S_d/d behaves in the limit when the system grows in dimension, i.e. for $d \to \infty$. For this purpose we work with the extendible subfamily of the Marshall–Olkin law, since we must be able to extend the dimension of the vector $(X_1, \ldots, X_d)$ without destroying its distributional structure. Recall that a random vector is called extendible if there exists an infinite exchangeable sequence $\{\tilde{X}_k\}_{k\in\mathbb{N}}$ such that $(X_1, \ldots, X_d) \stackrel{\mathcal{L}}{=} (\tilde{X}_1, \ldots, \tilde{X}_d)$. De Finetti's Theorem states that this is equivalent to $(\tilde{X}_1, \ldots, \tilde{X}_d)$ being conditionally i.i.d. (see [11]).

For extendible Marshall–Olkin laws there is a canonical construction based on Lévy subordinators, which are non-decreasing Lévy processes, $\{\Lambda_t, t \geq 0\}$, where the Lévy measure $\nu(dx)$ is defined on $\mathcal{B}((0,\infty])$ satisfying $\int(1 \wedge x)\nu(dx) < \infty$:

$$X_k = \inf\{t \geq 0 : \Lambda_t \geq E_k\}, \quad k = 1, \ldots, d. \tag{17}$$

Component X_k is the first-passage time of Λ across E_k and $\{E_k\}_{k\in\mathbb{N}}$ is an i.i.d. sequence of unit exponential random variables. This construction is called the Lévy-frailty construction (for further information on these distributions we refer the reader to [17], [19]) and it defines the subclass of extendible Marshall–Olkin distributions.

Let $\{\Psi(k)\}_{k\in\mathbb{N}}$ be a sequence, derived from evaluating the Laplace exponent Ψ of Λ at the natural numbers. It is shown in [18] that

$$\mathbb{P}(X_1 > x_1, \dots, X_d > x_d) = \exp\left(-\sum_{k=1}^{d} x_{(d-k+1)} \left(\Psi(k) - \Psi(k-1)\right)\right),$$

where $x_{(1)} \leq \ldots \leq x_{(d)}$ is the ordered list of the $x_1, \dots, x_d \geq 0$ (see [17]), is the survival function of $(X_1, \dots, X_d)$ which is completely determined by the sequence $\{\Psi(k)\}_{k\in\mathbb{N}}$. Then, $(X_1, \dots, X_d)$ follows the Marshall–Olkin distribution with parameters

$$\lambda_k = \sum_{i=0}^{k-1} (-1)^i \left(\Psi(d-k+i+1) - \Psi(d-k+i)\right), \quad k = 1, \dots, d.$$

Once we constructed the vector of first-passage times of a *Lévy-subordinator*, $(X_1, \dots, X_d)$, we can prove that when $d \to \infty$, S_d/d and the *exponential functional* of a Lévy-subordinator, $I_\infty = \int_0^\infty e^{-\Lambda_s} ds$, have the same distribution. The exponential functional of a Lévy process, $\{\Lambda_t, t \geq 0\}$, is defined as

$$I_t = \int_0^t e^{-\Lambda_s} ds. \tag{18}$$

Lemma 3
Let $(X_1, \dots, X_d)$ be a random vector following the Marshall–Olkin distribution. Then,

$$\lim_{d\nearrow\infty} \frac{S_d}{d} \overset{\mathcal{L}}{=} I_\infty, \tag{19}$$

where $I_\infty = \int_0^\infty e^{-\Lambda_s} ds$ represents the exponential functional of the Lévy-subordinator $\{\Lambda_t, t \geq 0\}$ at its terminal value. We refer the reader to [3] and [4] for detailed background on exponential functionals of Lévy processes.

Proof
Define $X_k = \inf\{t \geq 0 : \Lambda_t \geq E_k\}$ as in Equation (17). Then

$$\lim_{d\nearrow\infty} \frac{1}{d} \sum_{k=1}^{d} X_k \overset{a.s.}{=} \int_0^\infty e^{-\Lambda_s} ds \Leftrightarrow \mathbb{P}\left(\left|\lim_{d\nearrow\infty} \frac{1}{d} \sum_{k=1}^{d} X_k - \int_0^\infty e^{-\Lambda_s} ds\right| = 0\right) = 1.$$

The strong law of large numbers implies that $\lim_{d\nearrow\infty}\frac{1}{d}\sum_{k=1}^{d}X_k=\mathbb{E}[X_1|\Lambda]$ *holds almost surely.*

Observe that,

$$\begin{aligned}\mathbb{E}[X_1|\Lambda] &= \int_0^\infty x d\mathbb{P}(X_1\le x|\Lambda)=\int_0^\infty x d\mathbb{P}(E_1\le\Lambda_x|\Lambda)=\int_0^\infty x d(1-e^{-\Lambda_x})\\ &= \int_0^\infty -x d(e^{-\Lambda_x})=\Big[-xe^{-\Lambda_x}\Big]_{x=0}^{x=\infty}+\int_0^\infty e^{-\Lambda_x}dx=0+\int_0^\infty e^{-\Lambda_x}dx.\end{aligned}$$

Remark that convergence almost surely implies convergence in distribution. □

Example 2 (The limit of S_d/d in a Poisson-frailty model)
We want to analyse the convergence of $\mathbb{P}\left(S_d/d>x\right)$, $d\ge 2$, $x\ge 0$, *in the limit* $d\to\infty$. *Considering the standard Poisson process as an example,* $N_t=\{N_t\}_{t\ge 0}$ *with intensity* $\beta>0$, *which is a Lévy subordinator. [3] investigates the distribution of the exponential functional of a standard Poisson process,*

$$I_\infty=\int_0^\infty e^{-N_t}dt, \tag{20}$$

using its Laplace transform:

$$\mathbb{E}[e^{\tilde{\lambda}I_\infty}]=\left(\prod_{j=0}^{\infty}(1-\tilde{\lambda}e^{-j})\right)^{-1},\quad \tilde{\lambda}<1. \tag{21}$$

Using the Gaver–Stehfest Laplace inversion technique (see [15], [12],[24]), we numerically compute the survival function of the exponential functional of I_∞ *(Equation* (20)*).*

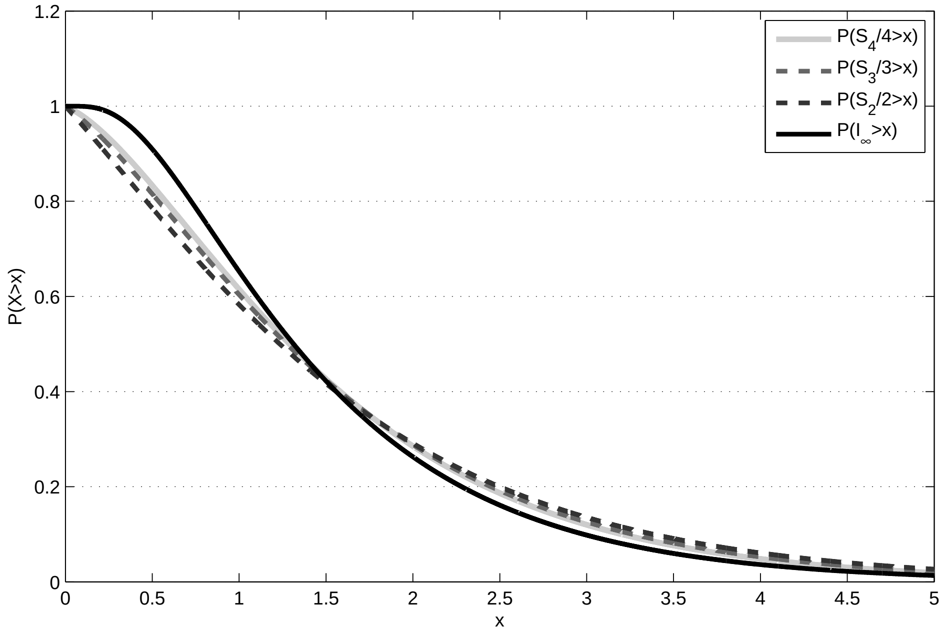


Figure 4 Plot of $\mathbb{P}(S_d/d > x)$, $d = 2, 3, 4$ together with $\mathbb{P}(I_\infty > x)$, $x \geq 0$, where $\beta = 1$.

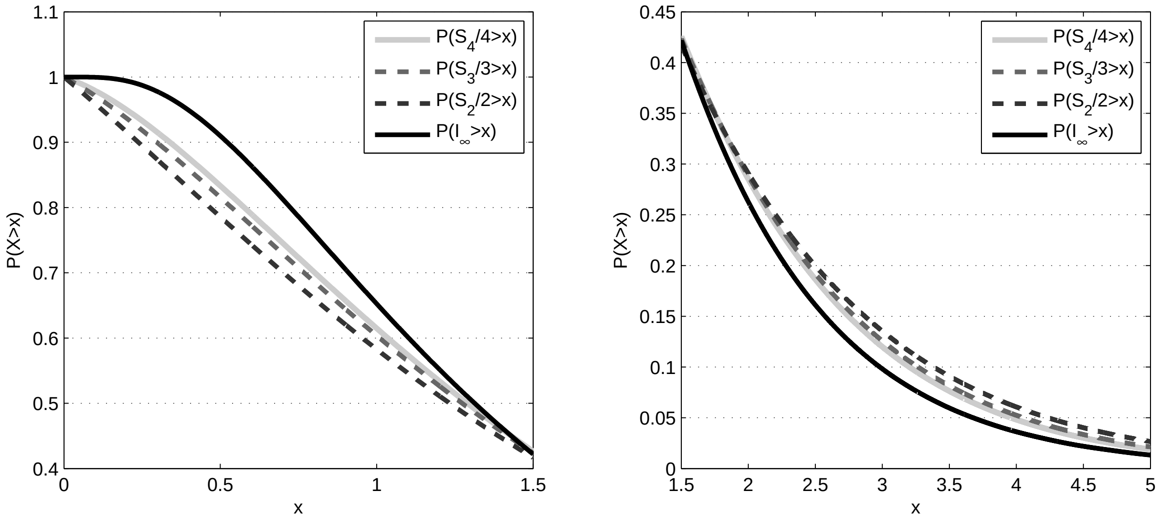


Figure 5 Zoom into Figure 4.

With this example we visualize how $\mathbb{P}(S_d/d > x)$, $d \in \mathbb{N}_0$, converges to $\mathbb{P}(I_\infty > x)$ when $d \to \infty$. In this case the components of the system strongly depend on each other, i.e. $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$.

5 Conclusion

We study the probability distribution of a sum of dependent random variables $S_d = X_1 + \ldots + X_d$ when the dependence structure is given by the Marshall–Olkin distribution. The Marshall–Olkin law possesses interesting properties from a statistical point of view as well as for applications in different fields like financial risk-management or insurance. However, during the construction of this type of dependence structure we encounter the obstacle of overparameterization. In order to deal with this drawback and to make the computations more tractable we work with the exchangeable subfamily, where the amount of parameters is significantly decreased from $2^d - 1$ to d. In low dimensional cases, $d = 2$, $d = 3$, and $d = 4$, we develop the explicit expressions for the distribution of S_d and we give a sketch of how these results can be extended to higher dimensions.

However, note that while the number of factors in the sum increases in one unit the number of cases into consideration for the calculus of the probabilities increases in 2^{d-1}. This is the reason why the problem becomes intractable for $d > 4$ and we focus on analysing the behaviour of S_d/d in the limiting case, $d \to \infty$. For this aim we work with the extendible subfamily, via the Lévy-frailty construction, and we show how the probability distribution of S_d/d is closely related with the probability law of the exponential functional of Lévy-subordinators.

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