Technische Universität München

Department of Mathematics

Master's Thesis

A New Simple Multivariate COGARCH Model for Time Varying Correlations

Thorsten Kud

Supervisor and Advisor: Prof. Dr. Claudia Klüppelberg

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Declaration

I assure the single handed composition of this Master's thesis only supported by declared resources.

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Summary

In this thesis we adapt the discrete-time model for the conditional volatilities and conditional correlations from Brownlees and Engle [1] to a continuous-time model called "MUCO-Diag". Brownlees and Engle introduced an index for systemic risk (SRISK) in financial systems. This index is based on a discrete-time bivariate model for the log returns, where the conditional volatilities and conditional market-firm correlations are separately modelled and then plugged into the model. We adapt the discrete-time model class to a continuous-time analogue: We model the conditional volatilities with an asymmetric GJR COGARCH process and the conditional correlations are modelled continuously in time based on the DCC (Dynamic Conditional Correlation) model of Engle in [2]. We derive several important properties of the "MUCO-Diag" process, such as existence and uniqueness and stationarity. Based on the theoretical results, we present some examples of the modelled volatilities and correlations.

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Contents

1 Introduction

"Financial institutions are systemically important if the failure of the firm to meet its obligations to creditors and customers would have significant adverse consequences for the financial system and the broader economy." [1]
After the financial crisis of 2007-2009 many market participants and academics recognized the importance of systemic risk in financial systems. The above attempt of Federal Reserve Governor Daniel Tarullo to define systemic risk already shows the main idea: If a financial institution faces a capital shortage, this normally doesn't have consequences on the whole financial and real economy, because stronger competitors absorb the capital shortage. But during a financial crisis, such an absorption might be impossible. The consequences are external costs on the real economy [15].
Brownlees and Engle introduced in [1] a systemic risk index SRISK, that measures the expected capital shortfall of a firm during a crisis. The index is based on the SES index by Acharya [15]. The key insight gained by SES was, that capital shortages may have external costs on the real economy, if the whole financial system is in a crisis. SRISK improves an important shortcoming of SES: SRISK doesn't require data from the actual crisis. In order to calculate SRISK, we need to model the conditional volatilities and conditional correlations of the market and firm returns. It all starts with a bivariate model for the market and firm returns:

$$\begin{aligned} r_{m,t} &= \sigma_{m,t}\epsilon_{m,t} \\ r_{i,t} &= \sigma_{i,t}\rho_{i,t}\epsilon_{m,t} + \sigma_{i,t}\sqrt{1-\rho_{i,t}^2}\epsilon_{i,t} \\ (\epsilon_{m,t}, \epsilon_{i,t}) &\sim F \end{aligned}$$

where $(\epsilon_{m,t}, \epsilon_{i,t})$ stand for the shocks, that drive the system. Only the conditional market-firm correlations are considered in this model. We may interpret the conditional market-firm correlation as the accumulated conditional correlations to all other firms from the perspective of each firm.

In this thesis we model the conditional volatilities and conditional correlations continuously in time, based on the above model setting. The motivation comes from the fact, that most financial data is sampled irregularly and therefore needs to be modelled continuously in time. The source of risk $(\epsilon_{m,t}, \epsilon_{i,t})$ is replaced by the jumps of a two-dimensional Levy process. The bivariate model turns into the

[1] "Regulatory Restructuring" Testimony before the Committee on Banking, Housing, and Urban Affairs, U.S. Senate, Washington, D.C., July 23, 2009.

set of stochastic differential equations

$$
\begin{aligned}
dr_{m,t} &= \sigma_{m,t} dL_{m,t} \\
dr_{1,t} &= \sigma_{1,t}\rho_{1,t} dL_{m,t} + \sigma_{1,t}\sqrt{1-\rho_{1,t}^2}\, dL_{1,t}
\end{aligned}
$$

We also decompose the system dynamics and model the conditional volatilities with the corresponding time-continuous GJR COGARCH(1,1) and the conditional correlations are modelled with a new model called "MUCO-Diag", also on the basis of the discrete-time model due to the DCC model, but now continuously in time. We show the existence and uniqueness of the conditional correlation process, give a representation of the unique solution, show the existence of a one-dimensional COGARCH(1,1) bound of the process norm and show that the MUCO-Diag volatility process is a strong Markov process with the weak Feller property. These properties are fundamental for the proof of the stationarity of the solution of the corresponding stochastic differential equations.

The thesis is organized as follows. Chapter 2 gives the definition of the systemic risk index SRISK from [1] and describes the economic approach to model the market and firm returns. Chapter 3 introduces the discrete models for the conditional volatilities and conditional correlations from [1]. Chapter 4.2 contains the model class of multidimensional continuous GARCH models and discusses the most important properties like existence, uniqueness and stationarity of a solution. In chapter 4.3 we create a new simple model for the volatilities by adapting the discrete models of chapter 3 to the model class of multidimensional continuous GARCH models. We also show existence, uniqueness, existence of a one-dimensional COGARCH(1,1) bound of the solution and stationarity of the solution. In chapter 5 we give some examples of the modelled volatilities and correlations. Chapter 6 summarizes our results and gives an outlook. Chapter 7 contains all necessary proofs and definitions.

2 Systemic Risk Index

In the following chapter we introduce the SRISK index from [1], which measures the capital shortfall of an financial institution during a crisis. Brownlees and Engle showed the practical usefulness of this index: "one year and a half before the Lehman bankruptcy, nine companies out of the SRISK top ten turned out to be troubled institutions."(abstract in [1])

2.1 Definition of SRISK

The whole definition is on the basis of [1], where the following systemic risk index was invented. In order to define *systemic risk*, we need the so called *capital buffer* of $i = 1, ..., I$ financial institutions. Capital buffer stands for the working capital of a firm, and is defined as

$$
\begin{aligned}
CB_{i,t} &= W_{i,t} - k(D_{i,t} + W_{i,t}) && (1)\\
&= (1-k)W_{i,t} - kD_{i,t} && (2)
\end{aligned}
$$

where $D_{i,t}$ and $W_{i,t}$ of a firm i denote respectively the book value of debt and the market value of its equity at time t. Many firms maintain a fraction k of its own assets, therefore each firm is only responsible for k percent of the firm debt and may only use $(1-k)$ percent of publicly acquired money. But what is the interpretation of $CB_{i,t}$?
We say, that a firm works properly if $CB_{i,t} > 0$ and experiences a capital shortage, if $CB_{i,t} < 0$. In reality, a capital shortage of a single financial institution normally won't have an influence on the whole financial sector or even the real economy, because there are plenty of other liquid firms to absorb this capital shortage through credits. But if the economy is in distress, a capital shortage might cause further externalities. That means, we are interested in computing the capital shortage of all financial institutions, when the economy is in distress. A distress or systemic event is defined in [1] and also in this work as a drop of the market below a threshold C over a given time period h. $R_{m,t:t+h}$ denotes the simple market return between period t and $t+h$. The systemic event is denoted by $\{R_{m,t:t+h} < C\}$. We then define the *expected capital shortage* as

$$
\begin{aligned}
CS_{i,t:t+h} &= -\mathbb{E}_t(CB_{i,t+h}|R_{m,t:t+h} < C)\\
&= -k\mathbb{E}_t(D_{i,t+h}|R_{m,t:t+h} < C) + (1-k)\mathbb{E}_t(W_{i,t+h}|R_{m,t:t+h} < C)
\end{aligned}
$$

In case of a systemic event, it is challenging to renegotiate debt. Hence we assume that it is impossible, which means $\mathbb{E}_t(D_{i,t+h}|R_{m,t:t+h} < C) = D_{i,t}$. Using this assumption and also the assumption, that $W_{i,t+h} = W_{i,t}R_{i,t:t+h}$, we get:

$$
\begin{aligned}
CS_{i,t:t+h} &= -kD_{i,t} + (1-k)W_{i,t}\mathbb{E}_t(R_{i,t:t+h}|R_{m,t:t+h} < C)\\
&= -kD_{i,t} + (1-k)W_{i,t}MES_{i,t:t+h}(C)
\end{aligned}
$$

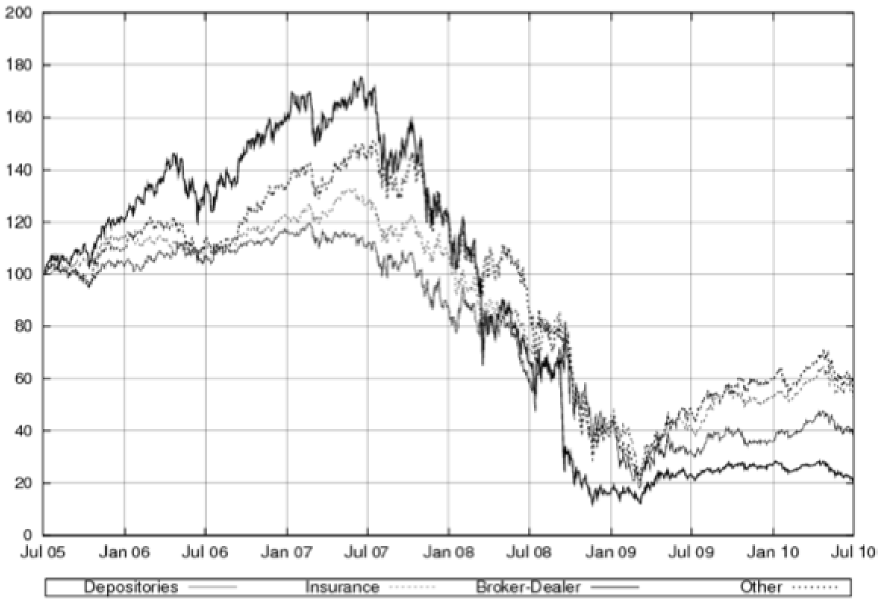

Figure 1: ([1], Fig. 1, page 20) Cumulative average return by industry group between July 2005 and June 2010.

where $MES_{i,t:t+h}(C)$ is the tail expectation of the firm equity returns conditional on the systemic event. We define the systemic risk index of institution i as

$$SRISK_{i,t} = \max(0, CS_{i,t})$$

The total amount of systemic risk in the economy is

$$SRISK_t = \sum_{i=1}^{I} SRISK_{i,t}$$

For a better understanding, we present the example from [1] of calculated aggregate SRISK. They studied a panel of institutions between July 3, 2000 and June, 30, 2010. This panel consists of U.S. financial institutions which have a market capitalization greater than 5 billion US Dollar at the end of June 2007. They extracted daily returns and market capitalization and divided the firms in 4 groups: Depositories (such as Bank of America), Broker-Dealers (such as Lehman Brothers), Insurances (AIG) and Others (non depository institutions, real estate). Figure 1 shows the cumulative average return by each industry group. If we compare figure 2 with figure 3, we see that after July 2007 not only the correlations to the market index go up in each industry group, but also the aggregate SRISK index. Interpretation of aggregate $SRISK_t$: If it comes to a market drop and the government rescues the financial system, $SRISK_t$ gives the expected amount of rescue costs.

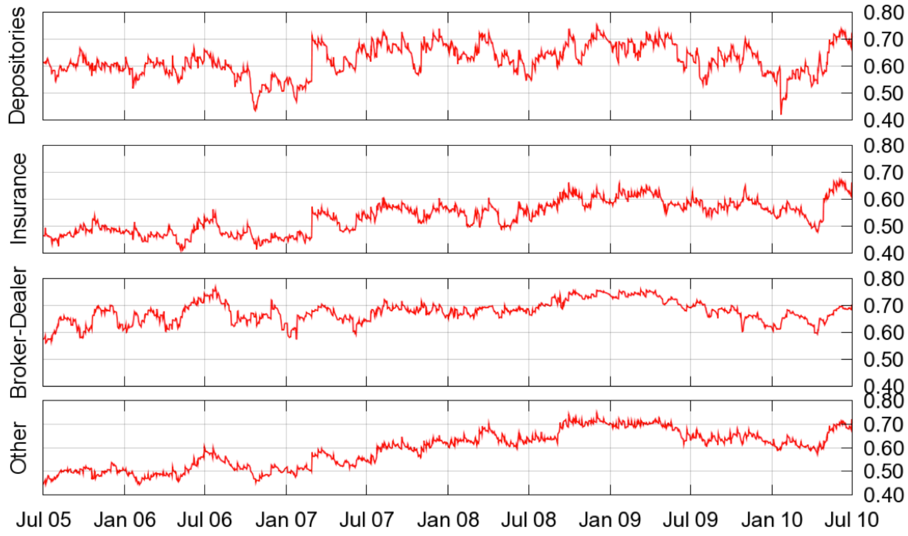

Figure 2: ([1], Fig. 3, page 26) Average in-sample correlations with the market index between July 2005 and June 2010.

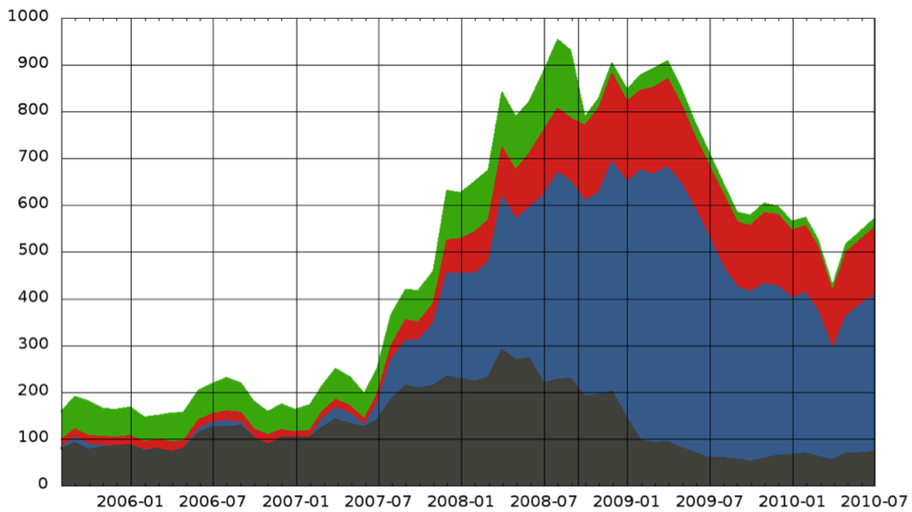

Figure 3: ([1], Fig. 8, page 38) Aggregate SRISK of the biggest U.S. financial firms between July 2005 and June 2010. The industry groups are ordered from top to bottom: Others, Insurance, Depositories and Broker-Dealers

2.2 Economic Approach

Since we are interested in computing $SRISK_{it}$, we need data on the debt, equity and MES of each firm. Debt and equity are publicly available, but MES has to be estimated. We first describe the bivariate dynamic time series model in ([1], chapter 3). Let r_{it} and r_{mt} denote respectively the i^{th} firm's and the market log return on day t. We use a bivariate model for the firm and market log returns, which reads for $t \in \mathbb{N}$ as

$$r_{m,t} = \sigma_{m,t}\epsilon_{m,t} \tag{3}$$

$$r_{i,t} = \sigma_{i,t}\rho_{i,t}\epsilon_{m,t} + \sigma_{i,t}\sqrt{1-\rho_{i,t}^2}\epsilon_{i,t} \tag{4}$$

$$(\epsilon_{m,t}, \epsilon_{i,t}) \sim F \tag{5}$$

where $\sigma_{m,t}$ and $\sigma_{i,t}$ are the conditional standard deviations of the market and firm return, respectively. $\rho_{i,t}$ is the conditional correlation between firm i and the market and $(\epsilon_{m,t}, \epsilon_{i,t})$ are the shocks, the sources for randomness. They are independent and identically distributed *over time* and have zero mean and unit variance. The financial crisis showed, that systemically risky firms are interlinked and therefore highly dependent. So we expect $\epsilon_{m,t}$ and $\epsilon_{i,t}$ at time t to be dependent. To complete the model, we need further descriptions of $\sigma_{i,t}$ and $\rho_{i,t}$,which we introduce in the coming chapters. The conditional volatilities are modelled with a discrete-time (asymmetric) GJR(1,1) model. The correlations are modelled discrete in time with the DCC model, introduced in [2].

3 Discrete-time GARCH Models

AutoRegressive Conditional Heteroscedasticity (ARCH) models were introduced by Engle [19] and then generalised by Bollerslev to the GARCH (generalised ARCH) in [20]. GARCH models are very popular in modelling of time series in finance. In chapter 3.1 we define the one-dimensional GARCH(1,1) model and an asymmetric version, the GJR GARCH model, which also captures the so called "leverage effect". In chapter 3.2 we explain the multivariate DCC model of [2] for time varying correlations.

3.1 One-dimensional Models

We introduce the widely used GARCH model:

Definition 3.1. *([11], page 3)(GARCH(1,1)) Let* $(\epsilon_n)_{n\in\mathbb{N}}$ *be an independent and identically distributed (i.i.d.) non-degenerate series of random variables with* $P\{\epsilon_1 = 0\} = 0$*. Then for* $n \in \mathbb{N}$*:*

$$Y_n = \epsilon_n \sigma_n \tag{6}$$

$$\sigma_n^2 = \beta + \lambda Y_{n-1}^2 + \delta \sigma_{n-1}^2 \tag{7}$$

The parameters β*,* λ *and* δ *satisfy* $\beta > 0$*,* $\lambda \geq 0$ *and* $\delta \geq 0$*.*

The $(Y_n)_{n\in\mathbb{N}}$ process models the observed data, for example log returns, whereas $(\sigma_n)_{n\in\mathbb{N}}$ models the time-varying volatility. The GARCH (Generalized Autoregressive Conditional Heteroscedasticity) model has great importance in the modelling of financial data, such as returns. GARCH models capture the so called "stylized facts" ([12], page 141) of financial times series:

- Returns are not i.i.d. and have low correlation
- Returns are heavy-tailed
- Absolute returns are highly correlated
- Volatility changes (randomly) over time
- Extreme returns appear in clusters

"Autoregressive" means, that past observations and volatilities influence the present volatility and therefore the observation itself. This effect is called "feedback". "Conditional Heteroscedasticity" means, that the volatility is time-varying, conditional and also random. In GARCH models, only the magnitude of past returns determine the future volatility. In practice, we can observe a phenomena called leverage effect: volatilities tend to rise with "bad news" (negative returns) and to fall with "good news" (positive returns). (Nelson, [10]) We model the volatilities of a two factor model of returns using the GJR model, a special Asymmetric Power ARCH model (APARCH). The APARCH model is defined as follows.

Definition 3.2. *([9], chapter 6)(APARCH) Let $(\epsilon_n)_{n\in\mathbb{N}}$ of i.i.d random variables such that $\mathbb{E}(\epsilon_n) = 0$ and $\mathbb{V}ar(\epsilon_n) = 1$. The process $(Y_n)_{n\in\mathbb{N}}$ is called Asymmetric Power ARCH(p,q), and we write APARCH(p,q), if it is satisfying an equation of the following form:*

$$\begin{aligned} Y_n &= \epsilon_n \sigma_n \\ \sigma_n^\delta &= \theta + \sum_{i=1}^{q} \alpha_i h(Y_{n-1}) + \sum_{j=1}^{p} \beta_j \sigma_{n-j}^\delta \end{aligned}$$

where $h(x) = (|x| - \gamma x)^\delta, \theta > 0, \delta > 0, \alpha_i \geq 0, \beta_j \geq 0$ and $|\gamma_i| < 1$.

The APARCH model includes the *GJR* model by choosing $\delta = 2$ ([5], Remark 5.2). Because Brownlees and Engle used in [1] a GJR(1,1) model, we only present the $p = q = 1$ case:

For $0 \leq \gamma < 1$ the GJR model reads

$$\sigma_n^2 = \theta + \hat{\alpha} Y_{n-1}^2 + \beta \sigma_{n-1}^2 + \hat{\gamma} \mathbb{1}_{\{Y_{n-1}<0\}} Y_{n-1}^2$$

with $\hat{\alpha} = \alpha(1-\gamma)^2$ and $\hat{\gamma} = -4\alpha$.

For $-1 < \gamma < 0$ the GJR model reads

$$\sigma_n^2 = \theta + \hat{\alpha} Y_{n-1}^2 + \beta \sigma_{n-1}^2 + \hat{\gamma} \mathbb{1}_{\{Y_{n-1}>0\}} Y_{n-1}^2$$

with $\hat{\alpha} = \alpha(1+\gamma)^2$ and $\hat{\gamma} = -4\alpha\gamma$.

"The main highlight of this specification is its ability to capture the so called leverage effect, that is the tendency of volatility to increase more with negative news rather than positive news."([1], page 16)

In [1] the conditional market volatility $(\sigma_{m,t})_{t\in\mathbb{N}}$ and the conditional firm volatilities $(\sigma_{i,t})_{t\in\mathbb{N}}$ for $i = 1, ..., d$ are each modelled with a GJR GARCH(1,1) model

$$\begin{aligned} \sigma_{m,t}^2 &= \theta_m + \hat{\alpha}_m r_{m,t-1}^2 + \beta_m \sigma_{t-1}^2 + \hat{\gamma}_m \mathbb{1}_{\{r_{m,t-1}>0\}} r_{m,t-1}^2 \\ \sigma_{i,t}^2 &= \theta_i + \hat{\alpha}_i r_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2 + \hat{\gamma}_i \mathbb{1}_{\{r_{i,t-1}>0\}} r_{i,t-1}^2 \end{aligned}$$

with $\hat{\alpha}_k = \alpha_k(1+\gamma_k)^2$, $\hat{\gamma}_k = -4\alpha_k\gamma_k$ for $k \in \{m, 1, ..., d\}$.

3.2 Multivariate Models - The DCC Model

We repeat the discrete-time DCC (Dynamic Conditional Correlation) model for the time varying conditional correlations introduced in [2], a further generalization of Bollerslev's model ([2], page 6).

Let for each $i = 1, ..., d$ and $t \in \mathbb{N}$ be $\epsilon_{im,t} := (\epsilon_{m,t}, \epsilon_{i,t})^T$ an independent and identically distributed sequence of 2-dimensional vectors with mean zero and variance one. The time varying conditional correlation between the returns of the market m and firm i is

$$\rho_{i,t} = \frac{\mathbb{E}_{t-1}[(r_{m,t} - \mathbb{E}[r_{m,t}])(r_{i,t} - \mathbb{E}[r_{i,t}])]}{\sigma_{m,t}\sigma_{i,t}}$$

In order to estimate $\rho_{i,t}$, we model a covariance matrix $Q_{i,t}$

$$Q_{i,t} := \begin{pmatrix} q_{ii,t} & q_{im,t} \\ q_{im,t} & q_{mm,t} \end{pmatrix}$$

through

$$\begin{aligned} Q_{i,t} &= (1 - \alpha_C - \beta_C)S_i + \alpha_C diag(Q_{i,t-1})^{1/2}\epsilon_{im,t-1}\epsilon_{im,t-1}^T diag(Q_{it-1})^{1/2} + \beta_C Q_{it-1} \\ &= (1 - \alpha_C - \beta_C)S_i + \alpha_C \hat{\epsilon}_{im,t-1}\hat{\epsilon}_{im,t-1}^T + \beta_C Q_{i,t-1} \end{aligned}$$

where $t \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta < 1$ and $\hat{\epsilon}_{im,t} = diag(Q_{it})^{1/2}\epsilon_{im,t}$ are the rescaled disturbances. Under the assumption of stationarity the matrix S_i equals the unconditional covariance matrix of the $\hat{\epsilon}_{it}$:

$$S_i = \mathbb{E}[\hat{\epsilon}_{it}\hat{\epsilon}_{it}^T]$$

and may therefore be estimated via $\hat{S}_i = \frac{1}{n}\sum \hat{\epsilon}_{im,t}\hat{\epsilon}_{im,t}^T$. This reduces the number of parameters, which have to be estimated! The remaining parameters are α, β and the initial matrix $Q_{i,0}$.

The estimator $\hat{\rho}_{it}$ of the correlation ρ_{it} is defined by

$$\hat{\rho}_{it} := \frac{q_{im,t}}{\sqrt{q_{ii,t}q_{mm,t}}}$$

In matrix notation this transformation reads as

$$\begin{pmatrix} 1 & \rho_{it} \\ \rho_{it} & 1 \end{pmatrix} = P_{it} = diag(Q_{it})^{-1/2} Q_{it} diag(Q_{it})^{-1/2} \quad (8)$$

We summarize the above model in a definition:

Definition 3.3. *(DCC Model) Let for each $i = 1, ..., d$ and $t \in \mathbb{N}$ be $\epsilon_{im,t} = (\epsilon_{m,t}, \epsilon_{i,t})^T$ an identically distributed sequence of 2-dimensional vectors with mean zero and variance one. Then we call a covariance matrix process $(Q_{it})_{t\in\mathbb{N}}$ modelled by the DCC model, if it satisfies*

$$\begin{aligned} Q_{i,t} &= (1-\alpha_C-\beta_C)S_i + \alpha_C diag(Q_{i,t-1})^{1/2}\epsilon_{im,t-1}\epsilon_{im,t-1}^T diag(Q_{i,t-1})^{1/2} + \beta_C Q_{i,t-1} \\ &= (1-\alpha_C-\beta_C)S_i + \alpha_C \hat{\epsilon}_{im,t-1}\hat{\epsilon}_{im,t-1}^T + \beta_C Q_{i,t-1} \end{aligned}$$

for all $t \in \mathbb{N}$, α_C, $\beta_C \in \mathbb{R}$, $\hat{\epsilon}_{im,t} = diag(Q_{it})^{1/2}\epsilon_{im,t}$ and $\mathbb{S}_i = \mathbb{E}[\hat{\epsilon}_{it}\hat{\epsilon}_{it}^T] \in \mathbb{S}_d^+$, where $\mathbb{S}_d^+$ denotes the set of symmetric and positive semi-definite $d \times d$ matrices. Via the transformation of equation (8), we receive the conditional correlation matrix P_{it}.

4 Continuous-time GARCH Models

4.1 One-dimensional COGARCH

"In practice, for various reasons, including weekend and holiday effects, or in tick-by-tick data, many financial time series are irregularly spaced and this, together with options pricing requirements, in particular, has created a demand for continuous-time models." ([17], page 520) Therefore, we are looking for a continuous model, which also captures the above mentioned "stylized facts", see chapter (3.1). One successful approach by Klüppelberg et al. ([11]) uses only one source of randomness and reads as

Definition 4.1. *([13], page 6)(COGARCH(1,1)) Let $(L_t)_{t\geq 0}$ be a Levy process. Then the COGARCH process $(G_t)_{t\geq 0}$ is defined in terms of its stochastic differential, dG, and the volatility process σ_t satisfies:*

$$dG_t = \sigma_{t-} dL_t, \quad t \geq 0$$
$$d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \Phi\sigma_{t-}^2 d[L,L]_t, \quad t \geq 0$$

for constants $\beta > 0$, $\eta \geq 0$ and $\Phi \geq 0$. $[L,L]_t$ denotes the quadratic variation process of L, defined for $t > 0$ by $[L,L]_t = \sigma_L^2 t + \sum_{0<s\leq t}(\Delta L_s)^2$.

For a short overview of the Levy process notation, see chapter (7.1). We receive the solution of the above stochastic differential equation (SDE) for the squared volatility process with the help of the process $X = (X_t)_{t\geq 0}$ defined by

$$X_t = \eta t - \sum_{0<s\leq t} log(1 + \Phi(\Delta L_s)^2), \quad t \geq 0$$

As in [13], we may write the squared volatility as

$$\sigma_t^2 = e^{-X_t}(\beta \int_0^t e^{X_s} ds + \sigma_0^2), \quad t \geq 0 \tag{9}$$

And we get

$$\sigma_t^2 = \sigma_0^2 + \beta t - \eta \int_0^t \sigma_s^2 ds + \Phi \sum_{0<s\leq t} \sigma_s^2 (\Delta L_s)^2$$

Via the method of Klüppelberg et al. ([11]) we also get a continuous version of the discrete GJR GARCH model introduced in chapter (3.1). For θ, $\alpha > 0$ and γ, $\beta \in (0,1)$ the process $(\sigma_t^2)_{t\geq 0}$ satisfies for $t \geq 0$ ([5], page 69):

$$\sigma_t^2 = \sigma_0^2 + \theta t + log(\beta) \int_0^t \sigma_s^2 ds + \frac{\alpha}{\beta} \sum_{0<s\leq t} \sigma_s^2 h(\Delta L_s)$$

where $h(x) = (|x| - \gamma x)^2$.

4.2 Multidimensional COGARCH

In a multidimensional setting, we need to replace the positive variance by the covariance matrix. Hence, if we try to build a model for the covariance matrix, the stochastic volatility process has to be a stochastic process in the positive semidefinite matrices. The volatility process of a multidimensional GARCH(1,1) model in the BEKK-representation of [8] is given by

$$\Sigma_n = C + A\Sigma_{n-1}^{1/2}\epsilon_{n-1}\epsilon_{n-1}^T\Sigma_{n-1}^{1/2}A^T + B\Sigma_{n-1}B^T$$

with $C \in \mathbb{S}_d^+$, $A, B \in M_d(\mathbb{R})$ and $(\epsilon_n)_{n\in\mathbb{N}}$ being an i.i.d. sequence in $\mathbb{R}^d$. In order to get a time-continuous and multidimensional GARCH(1,1) process, we replace the noise ϵ of a multivariate GARCH(1,1) process by the jumps of a multidimensional Levy process L and the autoregressive structure of the covariance process by a continuous autoregressive (AR) structure.

Definition 4.2. *([3], Definition 3.1)(MUCOGARCH(1,1)) Let L be an $\mathbb{R}^d$-valued Levy process and $A, B \in M_d(\mathbb{R})$, $C \in \mathbb{S}_d^+$. The process $G = (G_t)_{t\in\mathbb{R}^+}$ solving*

$$dG_t = V_{t-}^{1/2}dL_t \tag{10}$$

$$V_t = C + Y_t \tag{11}$$

$$dY_t = (BY_{t-} + Y_{t-}B^T)dt + AV_{t-}^{1/2}d[L,L]_t^d V_{t-}^{1/2}A^T \tag{12}$$

with initial values G_0 in $\mathbb{R}^d$ and Y_0 in $\mathbb{S}_d^+$, is then called a MUCOGARCH(1,1) process. The process $Y = (Y_t)_{t\in\mathbb{R}^+}$ with paths in $\mathbb{S}_d^+$ is referred to as a MUCOGARCH(1,1) volatility process.

The "matrix-integral" $\int_0^t A_{s-}dL_sB_{s-}$ denotes the matrix C_t in $M_{m,s}(\mathbb{R})$ with the ij'th entry $C_{ijt} = \sum_{k=1}^n\sum_{l=1}^r\int_0^t A_{ik,s-}B_{lj,s-}dL_{kl,s}$, where $(A_t)_{t\in\mathbb{R}^+}$ in $M_{m,n}(\mathbb{R})$ and $(B_t)_{t\in\mathbb{R}^+}$ in $M_{r,s}(\mathbb{R})$ are cadlag and adapted processes and $(L_t)_{t\in\mathbb{R}^+}$ in $M_{n,r}(\mathbb{R})$ is a semimartingale, see section 2.1 in [3]. We are interested in the volatility process V_t, which satisfies the SDE

$$dV_t = (B(V_{t-} - C) + (V_{t-} - C)B^T)dt + AV_{t-}^{1/2}d[L,L]_t^d V_{t-}^{1/2}A^T$$

Equivalently we may use the vec operator to write the above SDEs

$$dG_t = V_{t-}^{1/2}dL_t, \qquad V_t = C + Y_t \tag{13}$$

$$dvec(Y_t) = (B\otimes I + I\otimes B)vec(Y_{t-})dt + (A\otimes A)(V_{t-}^{1/2}\otimes V_{t-}^{1/2})dvec([L,L]_t^d) \tag{14}$$

$$dvec(V_t) = (B\otimes I + I\otimes B)(vec(V_{t-}) - vec(C))dt + (A\otimes A)(V_{t-}^{1/2}\otimes V_{t-}^{1/2})dvec([L,L]_t^d) \tag{15}$$

For a proof, see (7.2). We now state the important result of existence and uniqueness of a solution of equation (15):

Theorem 4.1. *([3], Theorem 3.2) Let $A, B \in M_d(\mathbb{R})$, $C \in \mathbb{S}_d^+$ and L be a d-dimensional Levy process. The SDE (15) with initial value $Y_0 \in \mathbb{S}_d^+$ has a unique positive semidefinite solution $(Y_t)_{t\in\mathbb{R}^+}$. The solution $(Y_t)_{t\in\mathbb{R}^+}$ is locally bounded and of finite variation. Moreover, it satisfies $Y_t \geq e^{Bt}Y_0e^{B^Tt}$ for all $t \in \mathbb{R}^+$.*

For a proof see chapter (7.3). Since we proved existence and uniqueness, we want to find a way to represent our solution. We prove now the so called shot noise representation:

Theorem 4.2. *([3], Theorem 3.6) The MUCOGARCH(1,1) volatility process Y satisfies*

$$Y_t = e^{Bt}Y_0e^{B^Tt} + \int_0^t e^{B(t-s)}A(C+Y_{s-})^{1/2}d[L,L]_s^d(C+Y_{s-})^{1/2}A^Te^{B^T(t-s)}$$
$$for \quad all \quad t \in \mathbb{R}^+$$

Proof. The proof is based on ([3], page 100-101). Define $M_t = \int_0^t A(C+Y_{s-})^{1/2}d[L,L]_t^d(C+Y_{s-})^{1/2}A^T$. M is $\mathbb{S}_d^+$-increasing and of finite variation, because all jumps are positive semi-definite and $[L,L]_t^d$ is of finite variation. Y solves the stochastic differential equation

$$dX_t = (BX_{t-} + X_{t-}B^T)dt + dM_t$$

In order to find an explicit representation for Y_t, we refer to Ornstein-Uhlenbeck (OU) processes. Let $(L_t)_{t\in\mathbb{R}^+}$ be a Levy process and consider the stochastic differential equation in one dimension

$$d\sigma_t^2 = -\lambda\sigma_{t-}^2dt + dL_t$$

with some $\lambda \in \mathbb{R}$ and initial value $\sigma_0^2 \in \mathbb{R}$. The solution may be written as

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)}dL_s$$

We identify $M_d(\mathbb{R})$ with $\mathbb{R}^{d^2}$. Let $(L_t)_{t\in\mathbb{R}^+}$ be a Levy process with values in $\mathbb{R}^{d^2}$, $\mathbf{A} : M_d(\mathbb{R}) \longrightarrow M_d(\mathbb{R})$ a linear operator. We call some solution to the SDE

$$dY_t = \mathbf{A}Y_{t-}dt + dL_t$$

a (matrix-valued) process of Ornstein-Uhlenbeck type. As in one dimension, one can show that for some initial value X_0 the solution is unique and given by

$$Y_t = e^{\mathbf{A}t}Y_0 + \int_0^t e^{\mathbf{A}(t-s)}dL_s \tag{16}$$

In our case, the linear operator is $\mathbf{A}(Y) = BY + YB^T = \mathbf{A}_1(Y) + \mathbf{A}_2(Y)$ with $\mathbf{A}_1(Y) = BY$ and $\mathbf{A}_2(Y) = YB^T$. $\mathbf{A}_1$ and $\mathbf{A}_2$ are commutating: $\mathbf{A}_1(\mathbf{A}_2(Y)) = BYB^T = \mathbf{A}_2(\mathbf{A}_1(Y))$ and therefore

$$e^{\mathbf{A}t}Y_0 = e^{(\mathbf{A}_1+\mathbf{A}_2)t}Y_0 = e^{\mathbf{A}_1t}e^{\mathbf{A}_2t}Y_0 = e^{Bt}Y_0e^{B^Tt}$$

Equation (16) and the defintion of M lead to our final representation

$$Y_t = e^{Bt}Y_0 e^{B^T t} + \int_0^t e^{B(t-s)} A(C+Y_{s-})^{1/2} d[L,L]_s^d (C+Y_{s-})^{1/2} A^T e^{B^T(t-s)}$$
$$for \quad all \quad t \in \mathbb{R}^+$$

$\square$

In the following, we will consider a special norm. Let $||\cdot||_2$ denote the operator norm on $M_{d^2}(\mathbb{R})$ associated with the usual euclidian norm. Take a diagonalizable matrix B and let $S \in GL_d(\mathbb{C})$ be a matrix such that $S^{-1}BS$ is diagonal. Define the norm $||\cdot||_{B,S}$ on $M_{d^2}(\mathbb{R})$ by $||X||_{B,S} := ||(S^{-1}\otimes S^{-1})X(S\otimes S)||_2$ for $X \in M_{d^2}(\mathbb{R})$. This norm depends both on B and S and is also an operator norm, associated with the norm $||x||_{B,S} := ||(S^{-1}\otimes S^{-1})x||_2$ on $\mathbb{R}^{d^2}$.

The proof of the stationarity needs some preparation. The first step to show stationarity of our volatility process is the existence of a COGARCH(1,1) process, which bounds the above introduced norm of our MUCOGARCH(1,1) process.

Theorem 4.3. *([3], Theorem 4.1) Let Y be a MUCOGARCH volatility process with initial value $Y_0 \in \mathbb{S}_d^+$ and driven by a Levy process in $\mathbb{R}^d$. Assume, further, that $B \in M_d(\mathbb{R})$ is diagonalizable and let $S \in GL_d(\mathbb{C})$ be such that $S^{-1}BS$ is diagonal. The process solving the SDE,*

$$dy_t = 2\lambda y_{t-}dt + ||S||_2^2||S^{-1}||_2^2 K_{2,B}||A\otimes A||_{B,S}(\frac{||C||_2}{K_{2,B}} + y_{t-})d\hat{L}_t$$
$$y_0 = ||vec(Y_0)||_{B,S}$$

with

$$\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S}\mu_L(ds,dx), \qquad \lambda := max(\Re(\sigma(B)))$$

and

$$K_{2,B} := \max_{X\in\mathbb{S}_d^+, ||X||_2=1}(\frac{||X||_2}{||vec(X)||_{B,S}})$$

is the volatility process of a univariate MUCOGARCH(1,1) process and y satisfies

$$||vec(Y_t)||_{B,S} \leq y_t \qquad for \quad all \quad t\in\mathbb{R}^+ \quad a.s.$$

Moreover, $K_{2,B} \leq \max_{X\in\mathbb{S}_d^+, ||X||_2=1}(\frac{||X||_2}{||vec(X)||_{B,S}}) \leq ||S||_2^2$.

For a proof see chapter (7.3).

From ([3], Theorem 4.4) we know, that the volatility process is a time-homogeneous Markov process with the weak Feller property:

Theorem 4.4. *([3], Theorem 4.4) (Markovian Properties) The MUCOGARCH(1,1) process (G,Y) and its volatility process Y alone are temporally homogeneous strong Markov processes on* $\mathbb{R}^d \times \mathbb{S}_d^+$ *and* $\mathbb{S}_d^+$*, respectively, and they have the weak Feller property.*

This result together with the one-dimensional MUCOGARCH(1,1) bound serve as the fundament for the proof of the stationarity. We use the stationary condition (7.13) to prove the stationarity of the bound, which helps us to show the prerequisites of the Krylov-Bogoliubov Theorem. This theorem, in the end, brings us the desired stationarity of the MUCOGARCH(1,1) volatility process.

With Theorem (4.4) we show the stationarity of the MUCOGARCH volatility process:

Theorem 4.5. *([3], Theorem 4.5) (Stationarity) Let* $B \in M_d(\mathbb{R})$ *be diagonalizable with* $S \in GL_d(\mathbb{C})$ *such that* $S^{-1}BS$ *is diagonal. Furthermore, let L be a d-dimensional Levy process with non zero Levy measure, y be defined as in Theorem 4.3 and* $\alpha_1 := ||S||_2^2||S^{-1}||_2^2 K_{2,B}||A \otimes A||_{B,S}$*. Assume that*

$$\int_{\mathbb{R}^d} log(1 + \alpha_1||vec(yy^T)||_{B,S})\nu_L(dy) < -2\lambda$$

Then there exists a stationary distribution $\mu \in \mathcal{M}_1(\mathbb{S}_d^+)$*, that is, the set of all probability measures on the Borel-*σ*-algebra of* $\mathbb{S}_d^+$*.*

For a proof see chapter (7.5). The above conditions for the stationarity of the MUCOGARCH(1,1) volatility process are similar to the one-dimensional case:

Theorem 4.6. *([11], Theorem 3.1 and 3.2) The squared volatility process* $(\sigma_t^2)_{t\geq 0}$ *as given by equation (9), is a time homogeneous Markov process. Moreover, if we suppose*

$$\int_{\mathbb{R}^d} log(1 + \Phi y^2)\nu_L(dy) < \eta \tag{17}$$

then $\sigma_t^2 \xrightarrow{D} \sigma_\infty^2$ *and if we also assume* $\sigma_0^2 \overset{D}{=} \sigma_0^2$*, independent of* $(L_t)_{t\geq 0}$*, then* $(\sigma_t^2)_{t\geq 0}$ *is strictly stationary.*

4.3 Continuous Modelling of the Correlations

We present a new continuous-time multivariate model for the covariance matrix Q_{it} of the DCC model to model the conditional correlations of each financial institution with the market. So we start with the equations:

$$\begin{aligned} Q_{it} &= (1-\alpha_C-\beta_C)S_i + \alpha_C \hat{\epsilon}_{it-1}\hat{\epsilon}_{it-1}^T + \beta_C Q_{it-1} && (18) \\ &= (1-\alpha_C-\beta_C)S_i + \alpha_C diag(Q_{it-1})^{1/2}\epsilon_{it-1}\epsilon_{it-1}^T diag(Q_{it-1})^{1/2} + \beta_C Q_{it-1} && (19) \end{aligned}$$

where we already used the rescaling of the innovations $\hat{\epsilon}_{it} = diag(Q_{it})^{1/2}\epsilon_{it}$, which ensures that $\{\hat{\epsilon}_{it}, Q_{it}\}$ is a MGARCH process, see [1]. If we compare equation (19) with the BEKK representation of the multidimensional GARCH model

$$\Sigma_n = C + A\Sigma_{n-1}^{1/2}\epsilon_{n-1}\epsilon_{n-1}^T\Sigma_{n-1}^{1/2}A^T + B\Sigma_{n-1}B^T,$$

we recognize a similar structure by choosing the matrices A, B as scalars. Now we proceed like in chapter (4.2): We replace the noise ϵ of equation (19) process by the jumps of a multidimensional Levy process L and the autoregressive structure of the covariance process by a continuous autoregressive (AR) structure:

Definition 4.3. *(MUCO-Diag(1,1)) Let L be an $\mathbb{R}^d$-valued Levy process and $\alpha_C, \beta_C \in \mathbb{R}$, $\alpha_C + \beta_C < 1$, $S \in \mathbb{S}_d^+$. The process $Q_i = (Q_{it})_{t\in\mathbb{R}^+}$ solving*

$$\begin{aligned} Q_{it} &= (1-\alpha_C-\beta_C)S_i + Y_{it} && (20) \\ dY_{it} &= (\beta_C Y_{it-} + Y_{it-}\beta_C)dt + \alpha_C diag(Q_{it-})^{1/2} d[L,L]_t^d diag(Q_{it-})^{1/2} && (21) \end{aligned}$$

with initial value $Q_{i0} \in \mathbb{S}_d^+$, is then called a MUCO-Diag(1,1) volatility process.

Our aim is to show the following properties of the MUCO-Diag(1,1):

- Existence and uniqueness of a solution
- Representation of a solution
- Existence of a COGARCH(1,1) bound for the norm
- Stationarity of the MUCO-Diag(1,1) volatility process

We can directly write the stochastic differential equation for Y_{it}:

$$\begin{aligned} dY_{it} =& (\beta_C Yit- + Y_{it-}\beta_C)dt \\ &+ \alpha_C diag((1-\alpha_C-\beta_C)S + Y_{it})^{1/2} d[L,L]_t^d diag((1-\alpha_C-\beta_C)S + Y_{it})^{1/2} \end{aligned}$$

Analogously to the MUCOGARCH case, see chapter (7.2), we may equivalently use the vec operator to write the above SDEs

$$\begin{aligned} dvec(Y_{it}) =& (\beta_C \otimes I + I \otimes \beta_C) vec(Y_{it-}) dt \\ &+ (\alpha_C \otimes \alpha_C)(diag(C_{\alpha_C,\beta_C} + V_{it-})^{1/2} \otimes diag(C_{\alpha_C,\beta_C} + V_{it-})^{1/2}) dvec([L,L]_t^d) \end{aligned}$$

First of all we want to make sure, that the MUCO-Diag(1,1) is a new process. Therefore we have a deeper look on the structural difference of the MUCOGARCH and MUCO-Diag SDE's. We compare equation (12)

$$dY_t = A^2 (C + Y_{t-})^{1/2} d[L,L]_t^d (C + Y_{t-})^{1/2}$$

with equation (21):

$$dY_{it} = \alpha_C diag(C + Y_{it-})^{1/2} d[L,L]_t^d diag(C + Y_{it-})^{1/2}$$

by setting $B, \beta_C = 0$ and choosing A to be scalar. The main difference is the diagonalization before taking the square root of $(C + Y_{it-})$. The square root of a 2×2 matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is

$$R = \frac{1}{t} \begin{pmatrix} A+s & B \\ C & D+s \end{pmatrix} \tag{22}$$

where $s^2 = det(M) = AD - BC \neq 0$ and $t^2 = A + D + 2s$. If M is diagonal, we just take the square roots of the diagonal entries.

Proposition 4.7. *The MUCOGARCH volatility process and the MUCO-Diag process aren't equal, even if we choose the matrix C and the initial value $Y_{i,0}$ diagonal.*

Proof. We now show, that even if the initial value of the MUCOGARCH volatility process Y_0 and C are diagonal, the volatility process Y_t doesn't have to be diagonal for all $t > 0$, hence the volatility processes aren't equal. We choose $B, \beta_C = 0$, A scalar and the two dimensional Levy process $(L_t)_{t \in \mathbb{R}_+} = (L_1, L_2)_{t \in \mathbb{R}_+}$ to be compound Poisson and we assume, for simplicity, both components to jump at the same time. Let Γ_1 be the first jump time, and we use the shot noise representation of Theorem 4.2 and the definition of the "matrix-integral" to analyse the non-diagonal entries of the 2-dimensional Y_t. Let $t < \Gamma_1$, then we get

$$\begin{aligned} (Y_t)_{12} &= (Y_0 + A^2 \int_0^t (C + Y_{s-})^{1/2} d[L,L]_s^d (C + Y_{s-})^{1/2})_{ij} \\ &= (Y_0)_{ij} + A^2 \sum_{k=1}^2 \sum_{l=1}^2 \int_0^t (C + Y_{s-})_{ik}^{1/2} (C + Y_{s-})_{lj}^{1/2} d[L,L]_{k,l,s}^d \\ &= (Y_0)_{ij} + A^2 \int_0^t (C + Y_{s-})_{11}^{1/2} (C + Y_{s-})_{22}^{1/2} d[L,L]_{1,2,s}^d \\ &= 0 \end{aligned}$$

The double sum vanishes, because C and Y_s stay diagonal (at least until $t < \Gamma_1$). The integral in line three vanishes, because there haven't yet been jumps. For $t = \Gamma_1$ we get

$$\begin{aligned}
(Y_{\Gamma_1})_{12} &= (Y_0)_{12} + A^2 \sum_{k=1}^{2}\sum_{l=1}^{2} \int_0^{\Gamma_1} (C + Y_{s-})_{ik}^{1/2} (C + Y_{s-})_{lj}^{1/2} d[L, L]_{k,l,s}^d \\
&= 0 + A^2 \int_0^{\Gamma_1} (C + Y_{s-})_{11}^{1/2} (C + Y_{s-})_{22}^{1/2} d[L, L]_{1,2,s}^d \\
&= A^2 (C + Y_0)_{11}^{1/2} (C + Y_0)_{22}^{1/2} (\Delta L_1)_{\Gamma_1} (\Delta L_2)_{\Gamma_1} \\
&\neq 0
\end{aligned}$$

If we choose, for example, all diagonal entries to be equal to one. Hence, $(Y_t)_{t \in \mathbb{R}^+}$ is not diagonal and since $(C + Y_{it-})^{1/2} \neq diag(C + Y_{it-})^{1/2}$ for $t > \Gamma_1$, we proved, that both volatility processes can't be equal. $\square$

In order to show existence and uniqueness of a solution, we proceed like in chapter (4.2).

Theorem 4.8. *(Existence and Uniqueness of MUCO-Diag(1,1)) Let L be an $\mathbb{R}^d$-valued Levy process and $\alpha_C, \beta_C \in \mathbb{R}$, $\alpha_C + \beta_C < 1$, $S \in \mathbb{S}_d^+$ and the initial value $Q_{0t} \in \mathbb{S}_d^+$. Then the SDE (21) has a unique and locally bounded solution $Q_t \in \mathbb{S}_d^+$ with finite variation and satisfies the inequality $Q_{it} \geq e^{\beta_C t} Q_{i0} e^{\beta_C^T t}$ for all $t \in \mathbb{R}^+$.*

Proof. Define the mappings $F(vec(y)) = (I_d \otimes \beta + \beta \otimes I_d) vec(y)$ and $G(y) = (\alpha \otimes \alpha)((C + diag(y))^{1/2} \otimes ((C + diag(y))^{1/2}$, where $C := diag((1 - \alpha - \beta)S)$. We receive the SDE in vec notation, using F and G:

$$dvec(Y_{it}) = F(vec(Y_{it}))dt + G(Y_{it-})dvec([L, L]_t^d)$$

Like in Theorem 4.1, the only difference will be to handle the additional function $diag$ in our jump term.

Lemma 4.9. *Consider the mapping $diag : \mathbb{S}_d^+ \longrightarrow \mathbb{S}_d^+, X \mapsto diag(X)$. $diag$ is linear, well-defined and there exists a constant $K > 0$ such that*

$$||diag(A) - diag(B)||_2 \leq K \cdot ||A - B||_2$$

for all $A, B \in M_d(\mathbb{R})$.

Proof. Take an arbitrary $A \in \mathbb{S}_d^+$, then all diagonal entries of A are necessarily non-negative, and hence $diag(A) \in \mathbb{S}_d^+$. We first of all show an inequality, using the 1-matrix-norm instead of the induced euclidean norm:

$$\begin{aligned}
&||diag(A) - diag(B)||_1 = ||diag(A - B)||_1 \\
&= \max_{i=1,\dots,d} \sum_{j=1}^{d} |diag(A - B)_{ij}| = \max_{i=1,\dots,d} |(a_{ii} - b_{ii})| \\
&\leq \max_{i=1,\dots,d} \sum_{j=1}^{d} |a_{ij} - b_{ij}| = ||A - B||_1
\end{aligned}$$

The above inequality holds for all $A, B \in M_d(\mathbb{R})$, which is a finite-dimensional vector space. Therefore all norms on $M_d(\mathbb{R})$ are equivalent and we get our desired constant $K > 0$ such that

$$||diag(A) - diag(B)||_2 \leq K \cdot ||A - B||_2$$

□

We show that G is locally Lipschitz on $U_{C,\epsilon} := \{x = diag(y) \in \mathbb{S}_d^+ : diag(y) > -\epsilon I_d\}$ with $0 < \epsilon < \min \sigma(C_{\alpha,\beta})$ using Lemma 4.8 and the Lemmata from Theorem 4.1:

$$\begin{aligned}
&||G(A) - G(B)||_2 \\
&= ||(\alpha \otimes \alpha)((C_{\alpha,\beta} + diag(A))^{1/2} \otimes ((C_{\alpha,\beta} + diag(A))^{1/2} \\
&- (\alpha \otimes \alpha)((C_{\alpha,\beta} + diag(B))^{1/2} \otimes ((C_{\alpha,\beta} + diag(B))^{1/2}||_2 \\
&\leq 2 \max\{||C_{\alpha,\beta} + diag(A)||_2, ||C_{\alpha,\beta} + diag(B)||_2\} \cdot ||(C_{\alpha,\beta} + diag(A))^{1/2} - (C_{\alpha,\beta} + diag(B))^{1/2}||_2 \\
&= N_{\alpha,\beta}(A, B) \cdot ||(C_{\alpha,\beta} + diag(A))^{1/2} - (C_{\alpha,\beta} + diag(B))^{1/2}||_2 \\
&\leq N_{\alpha,\beta}(A, B) \cdot \frac{1}{2\sqrt{c}} ||(C_{\alpha,\beta} + diag(A)) - (C_{\alpha,\beta} + diag(B))||_2 \\
&\leq N_{\alpha,\beta}(A, B) \cdot \frac{K}{2\sqrt{c}} ||A - B||_2
\end{aligned}$$

where $N_{\alpha,\beta}(A, B) := 2 \max\{||C_{\alpha,\beta} + diag(A)||_2, ||C_{\alpha,\beta} + diag(B)||_2\} < \infty$ if $||A||_2, ||B||_2 < R$ for a constant $R > 0$. Hence, G is locally Lipschitz on $U_{C,\epsilon}$. It remains to show that G has linear growth:

$$\begin{aligned}
||G(A)||_2^2 &= \alpha^4 ||((C_{\alpha,\beta} + diag(A))^{1/2} \otimes ((C_{\alpha,\beta} + diag(A))^{1/2}||_2 = ||C_{\alpha,\beta} + diag(A)||_2^2 \\
&\leq \alpha^4 (||C_{\alpha,\beta}||_2 + ||diag(A)||_2)^2 \leq ||C_{\alpha,\beta}||_2^2 + 2||C_{\alpha,\beta}||_2 K ||A||_2 + K^2 ||A||_2^2
\end{aligned}$$

If $||A||_2 < 1$, we get

$$\begin{aligned}
||G(A)||_2^2 &\leq ||C_{\alpha,\beta}||_2^2 + 2||C_{\alpha,\beta}||_2 K + K^2 ||A||_2^2 \\
&\leq \max\{||C_{\alpha,\beta}||_2^2 + 2||C_{\alpha,\beta}||_2 K, K^2\}(1 + ||A||_2^2)
\end{aligned}$$

If $||A||_2 \geq 1$, we get

$$\begin{aligned}||G(A)||_2^2 &\leq ||C_{\alpha,\beta}||_2^2 + (2||C_{\alpha,\beta}||_2 + K^2)|A||_2^2 \\ &\leq \max\{||C_{\alpha,\beta}||_2^2, 2||C_{\alpha,\beta}||_2 + K^2\}(1 + ||A||_2^2)\end{aligned}$$

So G is locally Lipschitz on $U_{C,\epsilon}$ and has linear growth, hence there exists a unique locally bounded solution $(Y_{it})_{t\in\mathbb{R}^+}$, which also fulfills the claimed inequality, due to the same arguments as in Theorem 4.1. □

Now we know, that a unique solution of the MUCO-Diag volatility SDE's exists and we want to give, like in chapter (4.2), a so called shot noise representation of the MUCO-Diag(1,1) volatility process:

Theorem 4.10. *The MUCO-Diag(1,1) volatility process Y satisfies*

$$\begin{aligned}Y_{it} &= e^{\beta_C t} Y_{i0} e^{\beta_C t} + \int_0^t e^{\beta_C (t-s)} \alpha_C diag(C + Y_{is-})^{1/2} d[L,L]_s^d diag(C + Y_{is-})^{1/2} \alpha_C e^{\beta_C (t-s)} \\ &= e^{2\beta_C t} Y_{i0} + \alpha_C^2 \int_0^t e^{2\beta_C (t-s)} diag(C + Y_{is-})^{1/2} d[L,L]_s^d diag(C + Y_{is-})^{1/2} \\ &\quad for \quad all \quad t \in \mathbb{R}^+\end{aligned}$$

With $M_t = \alpha^2 \int_0^t diag(C + Y_{s-})^{1/2} d[L,L]_t^d diag(C + Y_{s-})^{1/2}$ $\mathbb{S}_d^+$-increasing and of finite variation, Y solves the stochastic differential equation

$$dX_t = (\beta_C X_{t-} + X_{t-}\beta_C)dt + dM_t$$

and the proof is the same as in Theorem 4.2.

Like in the MUCOGARCH case, we prove the stationarity of the volatility process using the following properties: Existence of a one-dimensional MUCOGARCH(1,1) bound, the Markov and the weak Feller property. We start with the bound:

Theorem 4.11. *([3], Theorem 4.1) (Existence of a bound for the norm) Let Y be a MUCO-Diag(1,1) volatility process with $C := (1 - \alpha_C - \beta_C)S$ and initial value $Y_0 \in \mathbb{S}_d^+$ driven by a Levy process in $\mathbb{R}^d$ and K the Lipschitz constant of the diag-function. The process solving the SDE,*

$$\begin{aligned}dy_t &= 2\lambda y_{t-} dt + K\alpha^2 K_{2,B} \left(\frac{||C||_2}{K_{2,B}} + y_{t-}\right) d\hat{L}_t \\ y_0 &= ||vec(Y_0)||_{B,S}\end{aligned}$$

with

$$\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S} \mu_L(ds, dx)$$

and

$$K_{2,B} := \max_{X \in \mathbb{S}_d^+, ||X||_2 = 1} \left(\frac{||X||_2}{||vec(X)||_{B,S}}\right)$$

is the volatility process of a univariate MUCOGARCH(1,1) process and y satisfies

$$||vec(Y_t)||_{B,S} \leq y_t \qquad for \quad all \quad t \in \mathbb{R}^+ \quad a.s.$$

Moreover, $K_{2,B} \leq \max_{X \in \mathbb{S}_d^+, ||X||_2=1} \left(\frac{||X||_2}{||vec(X)||_{B,S}} \right) \leq ||S||_2^2 = 1.$

The proof works as in the MUCOGARCH case: We first show the existence of the bound process for driving compound Poisson processes and transform this result via an approximation argument to the general case with driving Levy processes.

Proof. STEP 1: We show Theorem 4.11 for driving compound Poisson processes.

Lemma 4.12. *([3], Lemma 6.5) Let $(L_t)_{t \in \mathbb{R}^+}$ be a driftless Levy subordinator. Then there exists a Levy process $(\bar{L}_t)_{t \in \mathbb{R}^+}$ in $\mathbb{R}$ such that $L_t = [\bar{L}_t, \bar{L}_t]_t^d$ for all $t \in \mathbb{R}^+$.*

For a proof see chapter (7.4).

$\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S} \mu_L(ds, dx)$ is a driftles subordinator, hence, due to Lemma 4.12 there exists a Levy process $(L_t)_{t \in \mathbb{R}^+}$ such that: $\hat{L}_t = [L_t, L_t]_t^d$.
Let Γ_1 be the time of the first jump of the compound Poisson process. Until Γ_1 the process Y_t follows the deterministic differential equation given by $dY_t = (2\beta_C Y_{t-})dt$ and in vec-notation $dvec(Y_t) = 2\beta_C vec(Y_{t-})dt$ and is uniquely solved by

$$vec(Y_t) = e^{2\beta_C t} vec(Y_0)$$

And we get:

$$||vec(Y_t)||_{B,S} = ||e^{2\beta_C t} vec(Y_0)||_{B,S} = e^{2\beta_C t} ||vec(Y_0)||_{B,S} = e^{2\lambda t} y_0 = y_t$$

Thus our inequality is shown for all $t \in [0, \Gamma_1)$. At time Γ_1 we calculate

$$\begin{aligned}
&||vec(Y_{\Gamma_1})||_{B,S} \\
&\quad = ||vec(Y_{\Gamma_1 -}) + (\alpha_C \otimes \alpha_C)(diag(C + Y_{\Gamma_1})^{1/2} \otimes diag(C + Y_{\Gamma_1})^{1/2}) vec(\Delta L_{\Gamma_1} (\Delta L_{\Gamma_1})^T)||_{B,S} \\
&\quad \leq y_{\Gamma_1} + ||\alpha_C \otimes \alpha_C||_{B,S} ||diag(C + Y_{\Gamma_1})^{1/2} \otimes diag(C + Y_{\Gamma_1})^{1/2}||_{B,S} ||vec(\Delta L_{\Gamma_1} (\Delta L_{\Gamma_1})^T)||_{B,S} \\
&\quad \leq y_{\Gamma_1} + \alpha_C^2 ||diag(C + Y_{\Gamma_1})^{1/2} \otimes diag(C + Y_{\Gamma_1})^{1/2}||_2 \Delta \hat{L}_{\Gamma_1} \\
&\quad \leq y_{\Gamma_1} + K\alpha_C^2 K_{2,B}(||C||_2 + ||Y_{\Gamma_1 -}||_2) \Delta \hat{L}_{\Gamma_1} \\
&\quad \leq y_{\Gamma_1} + K\alpha_C^2 K_{2,B}(||C||_2 + ||vec(Y_{\Gamma_1 -})||_2) \Delta \hat{L}_{\Gamma_1} \\
&\quad = y_{\Gamma_1}
\end{aligned}$$

By iterating the same argument, since compound Poisson is memoryless, gives us the inequality for all $t \in \mathbb{R}^+$.

STEP 2: We want to extend our results above to MUCO-Diag(1,1) processes driven by general Levy processes. The following Proposition shows, that we can approximate MUCO-Diag(1,1) processes by approximating the driving Levy processes with compound Poisson processes.

Proposition 4.13. *Let Y be a MUCO-Diag(1,1) volatility process with $C \in \mathbb{S}_d^{++}$ and $Y_0 \in \mathbb{S}_d^{+}$, driven by a Levy process L in $\mathbb{R}^d$, and let $(\epsilon_n)_{n\in\mathbb{N}}$ be a monotonically decreasing sequence in $\mathbb{R}^+ \setminus \{0\}$ with $\lim_{n\to\infty}\epsilon_n = 0$. Define compound Poisson Levy processes L_n by $L_{n,t} := \int_0^t \int_{\mathbb{R}^d, ||x|| \geq \epsilon_n} x\mu_L(ds,dx)$ for $n \in \mathbb{N}$ and associated MUCO-Diag(1,1) volatility process Y_n by*

$$dY_{n,t} = (2\beta_C Y_{n,t-} + Y_{n,t-}B)dt + Adiag(C + Y_{n,t-})^{1/2} d[L,L]_t^d diag(C + Y_{n,t-})^{1/2} A^T$$
$$Y_{n,0} = Y_0$$

Then $Y_n \to Y$ as $n \to \infty$ almost surely uniformly on compacts.

The proof is analogue to Proposition 7.7: The only difference lies in the choice of the function η in the Gronwall Lemma (Theorem 7.8), where the Lipschitz constant K of the diag operator appears. The inequalities our now:

$$\begin{aligned} ||Y_{n,t}||_2 &\leq (||Y_0||_2 + ||A||_2^2 ||C||_2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_{L_n}(ds,dx)) e^{K||A||_2^2 \int_0^t \int_{\mathbb{R}^d} ||x||_2^2 \mu_{L_n}(ds,dx) + 2||B||_2 t} \\ &\leq (||Y_0||_2 + ||A||_2^2 ||C||_2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds,dx)) e^{K||A||_2^2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds,dx) + 2||B||_2 T} \\ ||Y_t||_2 &\leq (||Y_0||_2 + ||A||_2^2 ||C||_2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds,dx)) e^{K||A||_2^2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds,dx) + 2||B||_2 T} \end{aligned}$$

The rest f the proof works as in the MUCOGARCH(1,1) case.

Proof of Theorem 4.11 for general L: In the last proposition we constructed a sequence $(Y_n)_{n\in\mathbb{N}}$ of compound Poisson driven MUCO-Diag(1,1) processes converging u.c.p. to Y. Let y_n be the univariate $MUCOGARCH(1,1)$ bounds, that means $||vec(Y_{n,t})||_{B,S} \leq y_{n,t}$ for all n and $t \in \mathbb{R}^+$. Due to the defintion of $L_{n,t}$, it is clear, that we add only more jumps through increasing n and therefore

$$y_{n+l,t} \geq y_{n,t}$$

for all $n, l \in \mathbb{N}$ and $t \in \mathbb{R}^+$. Define the process y by:

$$dy_t = 2\lambda y_{t-}dt + K\alpha_C^2 K_{2,B}(\frac{||C||_2}{K_{2,B}} + y_{t-})d\hat{L}_t$$
$$y_0 = ||vec(Y_0)||_{B,S}$$

with $\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S}\mu_L(ds,dx)$ and again $y_{n,t} \leq y_t$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. Passing to the limit $n \to \infty$ in $||vec(Y_{n,t})||_{B,S} \leq y_{n,t} \leq y_t$ shows $||vec(Y_t)||_{B,S} \leq y_t$ for all $t \in \mathbb{R}^+$. □

We now start with the proof for the stationarity of the MUCO-Diag(1,1) volatility process. In the proof for the existence and uniqueness of a solution of equation (21) we already showed, that the coefficients of the SDE

$$\begin{aligned} dvec(Y_{it}) =& (\beta_C \otimes I + I \otimes \beta_C)vec(Y_{it-})dt \\ &+ (\alpha_C \otimes \alpha_C)(diag(C_{\alpha_C,\beta_C} + V_{it-})^{1/2} \otimes diag(C_{\alpha_C,\beta_C} + V_{it-})^{1/2})dvec([L,L]_t^d) \end{aligned}$$

are locally Lipschitz on the open sets $U_{C,\epsilon}$. Hence, all prerequisites for Theorem 7.9 and 7.10 are fulfilled and we can conclude:

Theorem 4.14. *(Markov and weak Feller property) The MUCO-Diag(1,1) volatility process Y is a temporally homogeneous strong Markov process on $\mathbb{S}_d^+$ and has the weak Feller property.*

Using the bound of Theorem 4.11, the strong Markov property and the Krylov-Bogoliubov Theorem of Lemma 7.14, we can also show the stationarity of the MUCO-Diag(1,1) volatility process:

Theorem 4.15. *(Stationarity) Let L be a d-dimensional Levy process with non zero Levy measure, y be defined as in Theorem* 4.11 *and* $\alpha_1 := K\alpha^2 K_{2,B}$. *Assume that*

$$\int_{\mathbb{R}^d} log(1+\alpha_1 ||vec(yy^T)||_{B,S})\nu_L(dy) < -2\beta_C$$

Then there exists a stationary distribution $\mu \in \mathcal{M}_1(\mathbb{S}_d^+)$, that is, the set of all probability measures on the Borel-σ-algebra of $\mathbb{S}_d^+$.

Proof. The proof works like in the MUCOGARCH case, since we also for the MUCO-Diag volatility process showed the existence of a one dimensional bound y. Take λ, $\hat{L}$ and $\hat{L} = [\bar{L}, \bar{L}]$ as in Lemma 4.3. Then we have

$$\int_{\mathbb{R}^d} log(1+\alpha_1 y^2)\nu_{\bar{L}}(dy) = \int_{\mathbb{R}^d} log(1+\alpha_1 ||vec(yy^T)||_{B,S})\nu_L(dy) < -2\lambda$$

and therefore the COGARCH(1,1) bound y from Lemma 4.11 satisfies all prerequisites of Lemma 7.13 and it follows, that our bound converges in distribution to a distribution concentrated on $\mathbb{R}^+$. If we assume that y_0 has this stationary distribution and is independent of $(L_s)_{s\in\mathbb{R}^+}$, $(y_t)_{t\in\mathbb{R}^+}$ is stationary. We want to transform this stationarity result from the bound to our MUCO-Diag(1,1) volatility process. We set $Y_0 = \frac{y_0}{||vec(I_d)||_{B,S}} I_d$, independent of $(L_s)_{s\in\mathbb{R}^+}$ and $||vec(Y_0)||_{B,S} \leq y_0$. Our next step is to show the existence of a stationary solution for the MUCO-Diag(1,1) volatility process Y. The set $\{\mathscr{L}(Y_t) : t \in \mathbb{R}^+\}$ of laws of Y_t forms a tight subset of $\mathcal{M}_1(\mathbb{S}_d^+)$: For all $K > 0$ the set $\{x \in \mathbb{S}_d^+ : ||x|| \leq K\}$ is compact in $\mathbb{S}_d^+$, $\mathbb{P}(||Y_t||_{B,S} \leq K) \geq \mathbb{P}(y_t \leq K)$, because $||Y_t||_{B,S} \leq y_t$ and y is stationary with a stationary distribution concentrated on $\mathbb{R}^+$. $(Y_t)_{t\in\mathbb{R}^+}$ is a weak Feller Markov process (Theorem 4.14) and therefore fulfills all requirements for the Krylov-Bogoliubov Theorem 7.14. We conclude, there exists a stationary distribution $\mu \in \mathcal{M}_1(\mathbb{S}_d^+)$ for the MUCO-Diag(1,1) volatility process Y such that μ is in the closed convex hull of $\{\mathscr{L}(Y_t) : t \in \mathbb{R}^+\}$.

□

5 Simulations

In this chapter we present some examples for the modelled volatilities and correlations. First of all we give a short overview of the discrete and continuous model settings, introduced in chapters 3 and 4. In both cases, discrete and continuous, we have a bivariate model for the returns, where we plug in the distinctly modelled volatilities of one firm and the market and the corresponding market-firm correlation. The driving shocks of the discrete model are replaced by the jumps of a Levy process. The volatilities are modelled wih a GJR GARCH and COGARCH model, where we picked the parameters from [18]. Within the discrete framework the market firm correlation ρ_{1t} is modelled with the DCC approach, introduced in chapter 3.2, where we picked the parameters from [2]. Within the continuous framework, the DCC analogue was introduced in chapter 4.3 with parameters from [3]. In chapter 5.2 we approximate the continuous model by assuming the two dimensional Levy process to be compound Poisson. Due to a high rate λ of the compound Poisson process, we assume the jump times of each component of L to be the same.

Discrete framework

The returns are modelled with a bivariate model

$$\begin{aligned} r_{m,t} &= \sigma_{m,t}\epsilon_{m,t} \\ r_{1,t} &= \sigma_{1,t}\rho_{1,t}\epsilon_{m,t} + \sigma_{1,t}\sqrt{1-\rho_{1,t}^2}\epsilon_{1,t} \\ (\epsilon_{m,t}, \epsilon_{1,t}) &\sim F \end{aligned}$$

where we plug in the distinctly modelled conditional volatilities and conditional correlations! The shocks $(\epsilon_{m,t}, \epsilon_{1,t})$ are normally distributed. The conditional volatilities $(\sigma_{m,t})_{t\in\mathbb{N}}$ and $(\sigma_{1,t})_{t\in\mathbb{N}}$ are each modelled with a GJR GARCH(1,1) model

$$\begin{aligned} \sigma_{m,t}^2 &= \theta_m + \hat{\alpha}_m r_{m,t-1}^2 + \beta_m \sigma_{t-1}^2 + \hat{\gamma}_m \mathbb{1}_{\{r_{m,t-1}>0\}} r_{m,t-1}^2 \\ \sigma_{1,t}^2 &= \theta_1 + \hat{\alpha}_1 r_{1,t-1}^2 + \beta_1 \sigma_{1,t-1}^2 + \hat{\gamma}_1 \mathbb{1}_{\{r_{1,t-1}>0\}} r_{1,t-1}^2 \end{aligned}$$

with $\hat{\alpha}_k = \alpha_k(1+\gamma_k)^2$, $\hat{\gamma}_k = -4\alpha_k\gamma_k$ for $k \in \{m, 1\}$. The DCC approach models the covariance matrix Q_{1t} by

$$Q_{1t} = (1-\alpha_C-\beta_C)S_1 + \alpha_C diag(Q_{1,t-1})^{1/2}\epsilon_{1,t-1}\epsilon_{1,t-1}^T diag(Q_{1,t-1})^{1/2} + \beta_C Q_{1,t-1}$$

with parameters α_C, $\beta_C \in \mathbb{R}$, $\alpha_C + \beta_C < 1$ and $S_1 \in \mathbb{S}_d^+$. This matrix is then mapped in a correlation matrix using the following transformation:

$$\begin{pmatrix} 1 & \rho_{1t} \\ \rho_{1t} & 1 \end{pmatrix} = diag(Q_{1t})^{-1/2} Q_{1t} diag(Q_{1t})^{-1/2}$$

Continuous framework

Let L be an $\mathbb{R}^2$-valued Levy process, then we model for $t \geq 0$ the returns by

$$\begin{aligned} dr_{m,t} &= \sigma_{m,t} dL_{m,t} \\ dr_{1,t} &= \sigma_{1,t}\rho_{1,t} dL_{m,t} + \sigma_{1,t}\sqrt{1-\rho_{1,t}^2}\, dL_{1,t} \end{aligned}$$

The conditional volatility processes $(\sigma_{m,t}^2)_{t\geq 0}$ and $(\sigma_{1,t}^2)_{t\geq 0}$ satisfy

$$\sigma_{k,t}^2 = \sigma_{k,0}^2 + \theta_k t - \eta_k \int_0^t \sigma_s^2 ds + \Phi_k \sum_{0<s\leq t} \sigma_s^2 h(\Delta L_s)$$

where $h(x) = (|x| - \gamma_k x)^2$, $\theta_k > 0$, $\Phi_k \geq 0$ and $|\gamma_k| < 1$ for $k = m, 1$.

As in the discrete case, we first model the covariance matrix Q_{1t}. With $\alpha_C, \beta_C \in \mathbb{R}$, $\alpha_C + \beta_C < 1$ and $S_1 \in \mathbb{S}_d^+$ the model reads as

$$\begin{aligned} Q_{1t} &= (1-\alpha_C-\beta_C)S_1 + Y_{1t} \\ dY_{1t} &= (\beta_C Y_{1t-} + Y_{1t-}\beta_C)dt + \alpha_C diag(Q_{1t-})^{1/2} d[L,L]_t^d diag(Q_{1t-})^{1/2} \end{aligned}$$

And by the transformation

$$P_{1t} = diag(Q_{1t})^{-1/2} Q_{1t} diag(Q_{1t})^{-1/2}$$

we receive the correlation matrix.

5.1 Modelling the GJR-COGARCH Volatilities

The GJR model is a time-continuous model for the volatilities and therefore we need to discretize it. Maller et al. [17] constructed a family of discrete-time processes, which discretized the continuous-time integrated COGARCH process and the volatility process. They showed, that this family of processes converged in the Skorokhod metric to the continuous-time process and it's volatility process. In Behme et al [18], this procedure was also successfully used for the GJR COGARCH. Our programs are based on the programs, introduced in the Master's thesis of Mayr [5]. The parameters for the continuous volatilities are as in [18]: $\theta = 0.0001$, $\eta = -log(0.9) = 0.04576$, $\phi = 1/18 = 0.05556$, $\gamma = 0.2$. Through the parameter transformation, as in [18], the corresponding parameters for the discrete-time GJR GARCH are $\theta = 0.0001$, $\eta = 0.9$, $\beta = 0.05$, $\gamma = 0.2$. In order to receive initial values for the volatilities, we let one simulation run with starting value zero and pick the last value as the initial one for the second simulation.

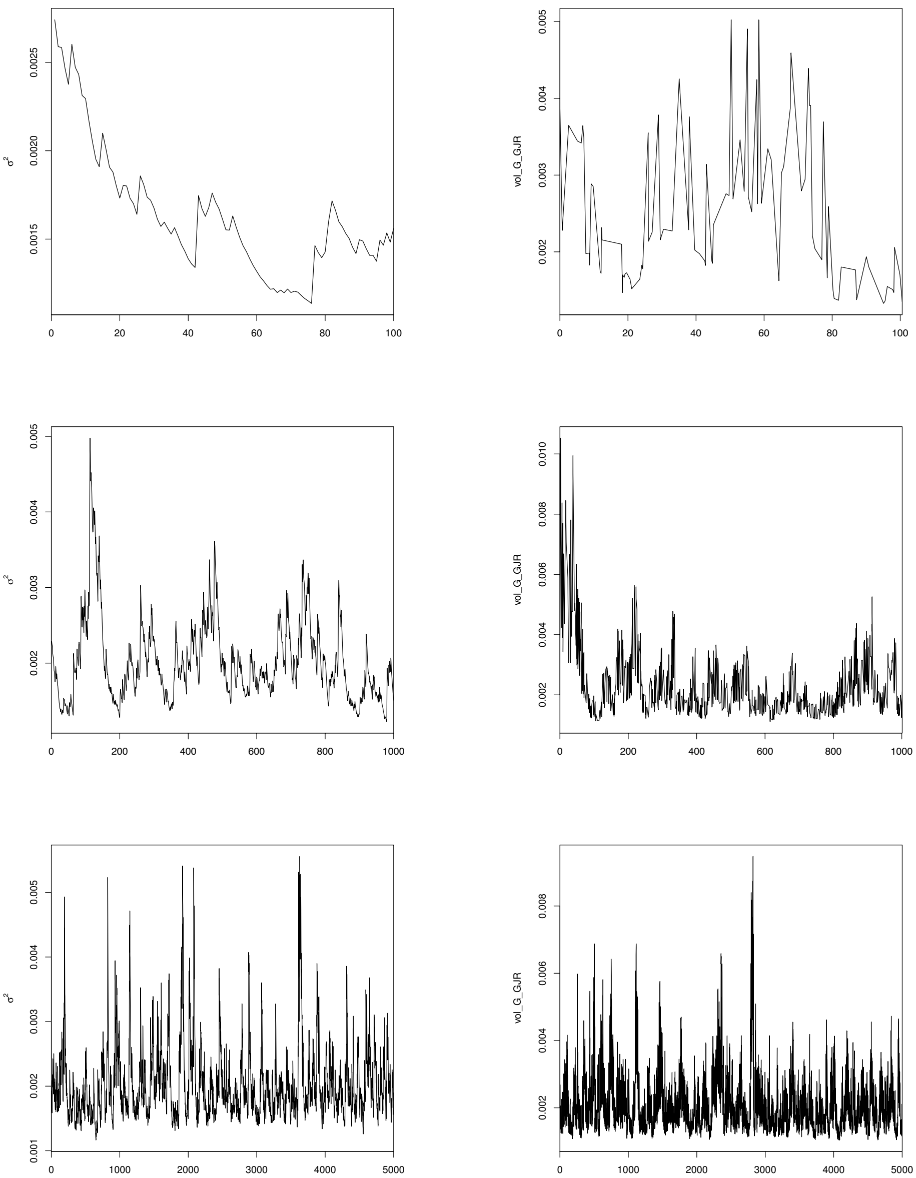

Figure 4: Simulation of GJR GARCH (left column) vs. GJR COGARCH (right column) volatilities. We compare the volatilities for different times $t = 100, 1000, 5000$. Parameters: $\theta = 0.0001$, $\eta = -log(0.9) = 0.04576$, $\phi = 1/18 = 0.05556$, $\gamma = 0.2$

5.2 Approximation of the MUCO-Diag Correlations

In chapter 4.3 we introduced the continuous-time model for the correlation matrix on the basis of the discrete DCC approach. We now discretize the continuous model by assuming the driving Levy processes to be compound Poisson processes. Through the transformation

$$P_{1t} = \begin{pmatrix} 1 & \rho_{1t} \\ \rho_{1t} & 1 \end{pmatrix} = diag(Q_{1t})^{1/2} Q_{1t} diag(Q_{1t})^{1/2} \tag{23}$$

we get the correlation matrix P_{1t} where ρ_{1t} denotes the correlation between the institute and the market. The matrix Q_{1t} is modelled by

$$\begin{aligned} Q_{1t} &= (1 - \alpha_C - \beta_C)S + Y_{1t} \\ dY_{1t} &= (\beta_C Y_{1t-} + Y_{1t-}\beta_C)dt + \alpha_C diag(Q_{1t-})^{1/2} d[L,L]_t^d diag(Q_{1t-})^{1/2} \end{aligned}$$

By integration we get

$$Y_{1t} = 2\beta_C \int_0^t Y_{1t-}dt + \alpha_C \underbrace{\int_0^t diag(Q_{1t-})^{1/2} d[L,L]_t^d diag(Q_{1t-})^{1/2}}_{=:\Omega_{1t}}$$

where Ω_{1t} is a 2×2 matrix. We use the definition of the "matrix-integral" to compute each component of the matrix. Let's start with

$$(\Omega_{1t})_{ij} = \sum_{k=1}^{2} \sum_{l=1}^{2} \int_0^t (diag(Q_{1s-})^{1/2})_{ik} (diag(Q_{1s-})^{1/2})_{lj} d[L,L]_{kl,s}^d$$

which reduces to

$$\begin{aligned} (\Omega_{1t})_{12} &= \int_0^t (diag(Q_{1s-})^{1/2})_{11} (diag(Q_{1s-})^{1/2})_{22} d[L,L]_{12,s}^d \\ (\Omega_{1t})_{ii} &= \int_0^t (diag(Q_{1s-}))_{ii} d[L,L]_{ii,s}^d \quad for \quad i = 1,2. \end{aligned}$$

In order to discretize the continuous model, we approximate a general bivariate Levy process by a two-dimensional compound Poisson process with a high intensity λ. We assume, that both components of L jump at same times. $(L_{mt}, L_{1t})^T = (\sum_{i=1}^{N(t)} X_{m,t_i}, \sum_{i=1}^{N(t)} X_{1,t_i})$ and the jump sizes X are normally distributed with expectation vector μ and correlation matrix Σ. The discontinuous part of the quadratic variation process of the 2-dimensional Levy process $[L,L]_t^d$ equals the matrix

$$[L,L]_t^d = \sum_{0 \le s \le t} (\Delta L_s)(\Delta L_s)^T = \int_0^t \int_{\mathbb{R}} xx^T \mu_L(ds, dx) \tag{24}$$

where μ_L denotes the corresponding jump measure of L. So in the ij'th entry of $[L,L]_t^d$ we sum up all simultaneous jumps of the i'th and j'th component of L_t

until time t. Now we rewrite the expression for the ij'th component of the pseudo correlation matrix and we receive

$$\begin{aligned}(\Omega_{1t})_{12} &= \int_0^t (diag(Q_{1s-})^{1/2})_{11}(diag(Q_{1s-})^{1/2})_{22} d[L,L]^d_{12,s} \\ &= \sum_{k=1}^{N(t)} (diag(Q_{1t_{k-1}})^{1/2})_{11}(diag(Q_{1t_{k-1}})^{1/2})_{22}(\Delta L_1)_{t_k}(\Delta L_2)_{t_k}\end{aligned}$$

and for $i = 1,2$

$$\begin{aligned}(\Omega_{1t})_{ii} &= \int_0^t (diag(Q_{1s-}))_{ii} d[L,L]^d_{ii,s} \\ &= \sum_{k=1}^{N(t)} (diag(Q_{1t_{k-1}}))_{ii}(\Delta L_i)^2_{t_k}\end{aligned}$$

We approximate the first term of Y_{it} by $2\beta_C \sum_{k=1}^{N(t)} (Y_{1t_k})_{ij}(t_k - t_{k-1})$ and if we combine all, we get a recursion formula

$$\begin{aligned}(Y_{1t_{k+1}})_{12} &= 2\beta_C(Y_{1t_k})_{12}(t_{k+1} - t_k) + \alpha_C diag(Q_{1t_k})^{1/2}_{11} diag(Q_{1t_k})^{1/2}_{22}(\Delta L_{t_{k+1}})_1(\Delta L_{t_{k+1}})_2 \\ (Y_{1t_{k+1}})_{ii} &= 2\beta_C(Y_{1t_k})_{ii}(t_{k+1} - t_k) + \alpha_C diag(Q_{1t_k})_{ii}\Delta(L_{t_{k+1}})^2_{ii} \quad for \quad i = 1,2\end{aligned}$$

We plug everything into equation (23) and receive a recursion formula for the correlation ρ_{1t}.

5.3 Modelling the Correlations with the MUCO-Diag Model

In the continuous setting the parameters α_C and β_C are chosen as in [2]: $\alpha_C = 0.06$, $\beta_C = 0.9$ and the shocks are normally distributed like $F \sim \mathcal{N}((0,0),\Sigma)$ with $\Sigma = \begin{pmatrix} 0.03 & 0.001 \\ 0.001 & 0.03 \end{pmatrix}$. The matrix S_1 equals $\begin{pmatrix} 0.03 & 0.001 \\ 0.001 & 0.03 \end{pmatrix}$. We conduct a first simulation with $\rho_{1,0} = 0$ to get an initial value for the correlation.
In the continuous setting, the Levy process is assumed to be compound Poisson with rate $\lambda = 10$ and the jump size distribution is also distributed like F. In order to receive an initial value for the correlation, we conduct a starting simulation with $\rho_{1,0} = 0$ and take the last simulated value of the correlation as the initial value for the second simulation. We choose the parameters α_C, β_C as in [3]: $\alpha_C = 1$, $\beta_C = -1.6$. The matrix S_1 are in both cases equal.

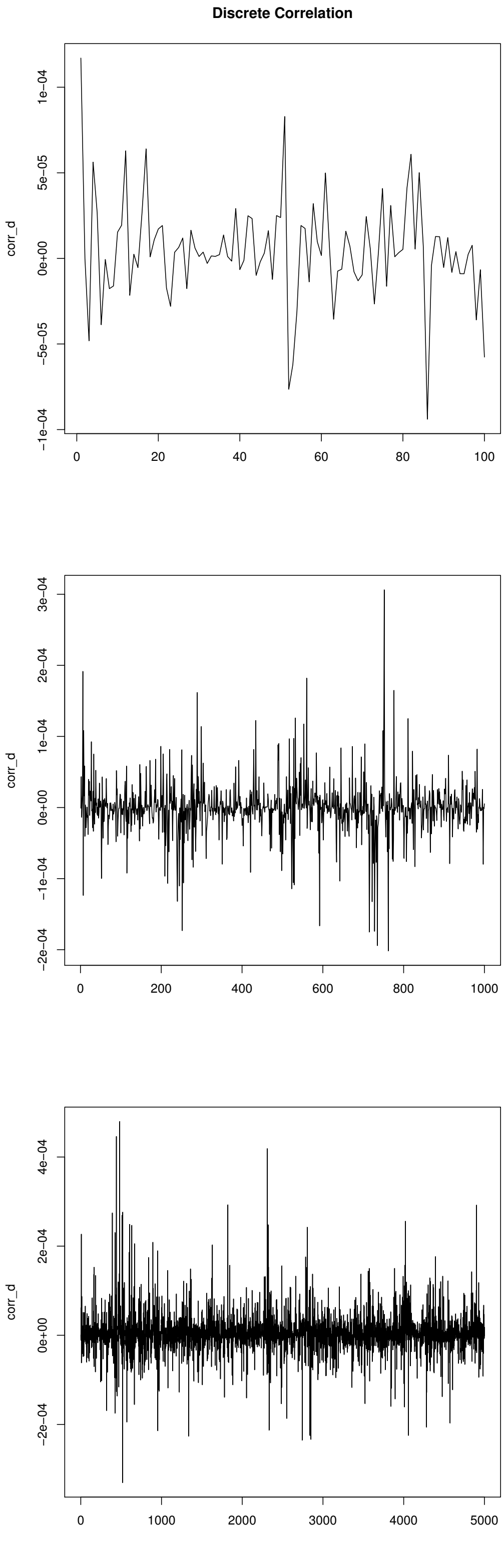

Figure 5: Simulation of Time-Discrete Correlations. We compare the correlations for different times $t = 100, 1000, 5000$. We use the parameters mentioned above.

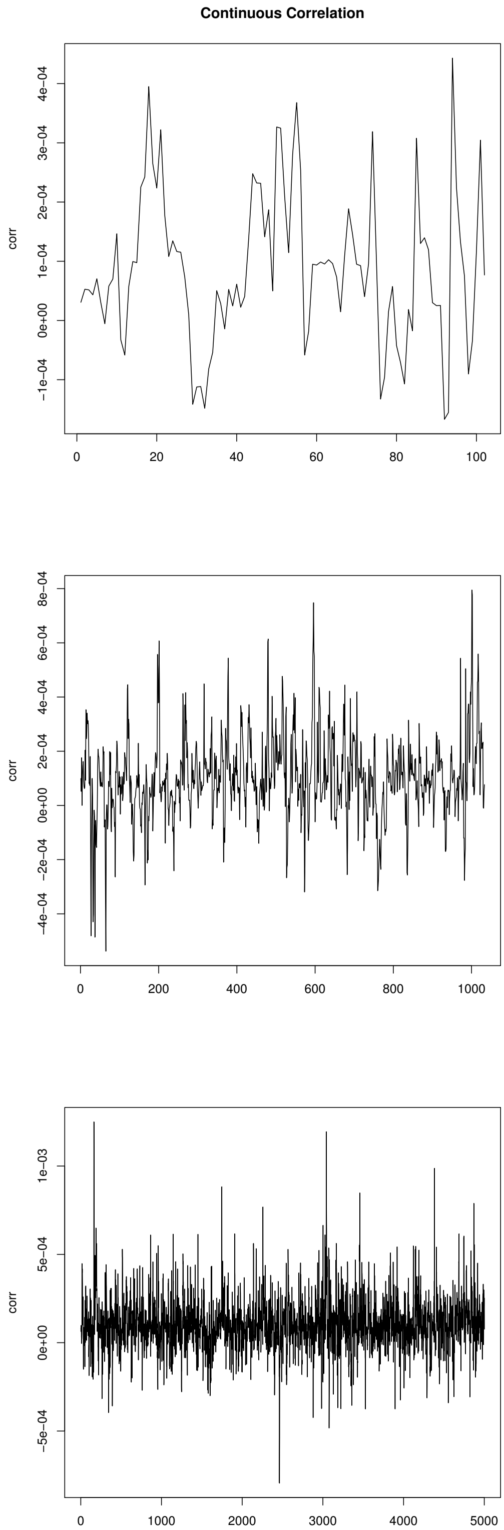

Figure 6: Simulation of Time-Continuous Correlations. We compare the correlations for different times $t = 100, 1000, 5000$. We use the parameters mentioned above.

6 Conclusion and Outlook

We successfully adapted the time-discrete models of Brownlees and Engle in [1] to time-continuous models for the volatilities and the market-firm correlations. The theoretical basis of existence and uniqueness of a solution for the pseudo-correlation process of the SDE for the MUCO-Diag(1,1) process and also the stationarity are proved. The simulations of volatilities and correlations were done with parameters from the literature. The magnitude of both, time-discrete and time-continuous, correlation do not match to correlations in finance, which are typically much higher and non negative. Also a comparison of the time-discrete and time-continuous processes weren't possible, because the exact parameter-transformation between both model classes hasn't yet been available. At this point we are interested in such a parameter transformation and the calculation of some moments in order to start a pseudo-maximum-likelihood-parameter estimation. The parameter estimation is crucial for the usage of the time-continuous model for the SRISK index. For a comprehensive study of the SRISK index with the new model class, we need parameters matching to the financial background.

7 Proofs and Definitions

7.1 Levy Processes

For a comprehensive introduction to Levy processes see [21]. We give the basic notation of Levy processes, based on chapter 2.2 of [3]. Consider a Levy process $L = (L_t)_{t\in\mathbb{R}^+}$ (where $L_0 = 0$ a.s.) in $\mathbb{R}^d$ which is determined by its characteristic function in the Levy-Khintchine form $\mathbb{E}[e^{i<u,L_t>}] = \exp(t\Psi_L(u))$ for $t \in \mathbb{R}^+$ with

$$\Psi_L(u) = i < \sigma_L, u > -\frac{1}{2} < u, \tau_L u > + \int_{\mathbb{R}^d} (e^{i<u,x>} - 1 - i < u, x > \mathbb{1}_{[0,1]}(||x||))\nu_L(dx)$$

where $\sigma_L \in \mathbb{R}^d$, $\tau_L \in \mathbb{S}_d^+$ and the Levy measure ν_L is a measure on $\mathbb{R}^d$, which satisfies $\nu_L(0) = 0$ and $\int_{\mathbb{R}^d}(||x||^2 \wedge 1)\nu_L(dx) < \infty$ and $< \cdot, \cdot >$ denotes the Euclidean scalar product on $\mathbb{R}^d$. We assume L to be cadlag (right continuous with left limits) and denote its jump measure by μ_L, that is, μ_L is the Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}$ given by $\mu_L(B) = \sharp\{s \geq 0 : (s, L_s - L_{s-}) \in B\}$ for any measurable set $B \subset \mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}$. We write $\Delta L_s = L_s - L_{s-}$ if $|L_s - L_{s-}| > 0$.

7.2 The Vec-Representation

Theorem: We may rewrite

$$\begin{aligned} dG_t &= V_{t-}^{1/2} dL_t \\ V_t &= C + Y_t \\ dY_t &= (BY_{t-} + Y_{t-}B^T)dt + AV_{t-}^{1/2} d[L,L]_t^d V_{t-}^{1/2} A^T \end{aligned}$$

into

$$\begin{aligned} dG_t &= V_{t-}^{1/2} dL_t \\ V_t &= C + Y_t \\ dvec(Y_t) &= (B \otimes I + I \otimes B)vec(Y_{t-})dt + (A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2})dvec([L,L]_t^d) \\ dvec(V_t) &= (B \otimes I + I \otimes B)(vec(V_{t-}) - vec(C))dt + (A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2})dvec([L,L]_t^d) \end{aligned}$$

using the vec operator.

Proof.

$$dY_t = \underbrace{(BY_{t-} + Y_{t-}B^T)}_{(1a)} dt + \underbrace{AV_{t-}^{1/2} d[L,L]_t^d V_{t-}^{1/2} A^T}_{(1b)} \tag{25}$$

$$dvec(Y_t) = \underbrace{(B \otimes I + I \otimes B)vec(Y_{t-})}_{(2a)} dt + \underbrace{(A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2})dvec([L,L]_t^d)}_{(2b)} \tag{26}$$

We compare the according parts of the above equations in order to identify the two representations.
Step 1: (1a) and (2a)

The *vec*-operator transforms a $m \otimes n$ matrix A into a column vector by stacking all columns of A on top of one another:

$$vec(A) = [a_{1,1}, ..., a_{m,1}, a_{1,2}, ..., a_{m,2}, ..., a_{1,n}, ..., a_{m,n}]^T \tag{27}$$

Hence, to find the (i, j)'th entry of a $d \otimes d$ matrix A in $vec(A)$, we stack the first $j - 1$ columns on top of another and the remaining i entries of the j'th column. Therefore the (i, j)'th entry of a $d \otimes d$ matrix A equals the $((j - 1) * d + i)$'th entry of $vec(A)$.

$$(BY + YB^T)_{ij} = \sum_{k=1}^{d} b_{ik} y_{kj} + \sum_{k=1}^{d} y_{ik} b_{kj}$$

and accordingly in vec notation

$$\begin{aligned}&(B \otimes I + I \otimes B) vec(Y)_{(i+(j-1)d)} = ((diag_d(b_{j1}) \cdots diag_d(b_{jn})) vec(Y))_i + (B(Y)_j)_i \\ &= b_{j1} y_{i1} + b_{j2} y_{i2} + ... + b_{jn} y_{in} + \sum_{k=1}^{d} b_{ik} y_{kj} = \sum_{k=1}^{d} b_{ik} y_{kj} + \sum_{k=1}^{d} y_{ik} b_{kj}\end{aligned}$$

Step 2: (1b) and (2b)

$$(AV_{t-}^{1/2} d[L, L]_t^d V_{t-}^{1/2} A^T)_{ij} = \sum_{k,l} \int_0^t (AV^{1/2})_{ik,s-} \underbrace{(V^{1/2} A^T)_{lj,s-}}_{((V^{1/2})^T A^T)_{lj} = (AV^{1/2})^T_{lj} = (AV^{1/2})_{jl}} d[L, L]_{kl}^d$$

and accordingly in vec notation

$$(A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2}) dvec([L, L]_t^d)_{(i+(j-1)d)} = (AV_{t-}^{1/2} \otimes AV_{t-}^{1/2}) dvec([L, L]_t^d)_{(i+(j-1)d)}$$

We want to understand the above matrix vector multiplication and therefore we need to analyse the $(i + (j - 1)d)$'th row of the matrix $AV_{t-}^{1/2} \otimes AV_{t-}^{1/2}$.

$$\begin{aligned}&(AV_{t-}^{1/2} \otimes AV_{t-}^{1/2})_{(i+(j-1)d)'th\,row} \\ &= \begin{bmatrix} (AV^{1/2})_{11} AV^{1/2} & \dots & (AV^{1/2})_{1d} AV^{1/2} \\ \vdots & \ddots & \vdots \\ (AV^{1/2})_{d1} AV^{1/2} & \dots & (AV^{1/2})_{dd} AV^{1/2} \end{bmatrix}_{(i+(j-1)d)'th\,row} \\ &= ((AV^{1/2})_{j1} AV^{1/2} \dots (AV^{1/2})_{jd} AV^{1/2})_{i'th\,row} \\ &= ((AV^{1/2})_{j1} (AV^{1/2})_{i'th\,row} \dots (AV^{1/2})_{jk} (AV^{1/2})_{i'th\,row} \dots (AV^{1/2})_{jd} (AV^{1/2})_{i'th\,row})\end{aligned}$$

And we receive our matrix vector multiplication

$$\begin{aligned}
&(A \otimes A)(V_{s-}^{1/2} \otimes V_{s-}^{1/2}) dvec([L,L]^d)_{(i+(j-1)d)} \\
&= \sum_{l=1}^{d} (AV^{1/2})_{jl,s-} \sum_{k=1}^{d} (AV^{1/2})_{ik,s-} \underbrace{dvec([L,L]_t^d)_{((l-1)d+k}}_{=d[L,L]_{lk,t}^d} \\
&= \sum_{k,l} (AV^{1/2})_{jl,s-} (AV^{1/2})_{ik,s-} d[L,L]_{lk}^d
\end{aligned}$$

□

7.3 Existence of the MUCOGARCH(1,1) volatility process

Theorem 4.1: Let $A, B \in M_d(\mathbb{R})$, $C \in \mathbb{S}_d^+$ and L be a d-dimensional Levy process. The SDE (15) with initial value $Y_0 \in \mathbb{S}_d^+$ has a unique positive semidefinite solution $(Y_t)_{t \in \mathbb{R}^+}$. The solution $(Y_t)_{t \in \mathbb{R}^+}$ is locally bounded and of finite variation. Moreover, it satisfies $Y_t \geq e^{Bt} Y_0 e^{B^T t}$ for all $t \in \mathbb{R}^+$.

Proof. The proof is based on ([3], page 98-100). Consider the SDE for Y_t:

$$dY_t = (BY_{t-} + Y_{t-}B)dt + AV_{t-}^{1/2} d[L,L]_t^d V_{t-}^{1/2} A^T$$

This SDE can equivalently be written in vec notation:

$$dvec(Y_t) = (B \otimes I + I \otimes B) vec(Y_{t-}) dt + (A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2}) dvec([L,L]_t^d)$$

In order to show existence and uniqueness of a solution of the above SDE, define the following mappings $F(vec(y)) = (I_d \otimes B + B \otimes I_d) vec(y)$ and $G(y) = (A \otimes A)((C+y)^{1/2} \otimes (C+y)^{1/2})$. Rewriting the SDE gives

$$dvec(Y_t) = F(vec(Y_t))dt + G(Y_{t-}) dvec([L,L]_t^d) \tag{28}$$

We try to show, that G has linear growth and is locally Lipschitz on a well chosen set U. The following lemmata help us to find the right choice of U.

Lemma 7.1. *([4], Problem I.6.11) For all $A, B \in M_d(\mathbb{R})$, we have*

$$||A \otimes A - B \otimes B||_2 \leq 2 \max\{||A||_2, ||B||_2\} ||A - B||_2$$

In particular, the mapping $\otimes : M_d(\mathbb{R}) \longrightarrow M_{d^2}(\mathbb{R}), X \mapsto X \otimes X$ is uniformly Lipschitz on any set of the form $\{x \in M_d(\mathbb{R}) : ||x|| \leq c\}$ with $c > 0$.

Lemma 7.2. *([4], page 305) Let $A, B \in \mathbb{S}_d^+$ and $a > 0$ such that $A, B \geq aI_d$. Then*

$$||A^{1/2} - B^{1/2}||_2 \leq \frac{1}{2\sqrt{a}} ||A - B||_2$$

Hence, the mapping $\otimes : \mathbb{S}_d^+ \longrightarrow \mathbb{S}_d^+, X \mapsto X^{1/2}$ is uniformly Lipschitz on any set of the form $\{x \in \mathbb{S}_d^+ : ||x|| \geq cI_d\} \subset \mathbb{S}_d^{++}$ wth $c > 0$.

Lemma 7.3. *([3], Lemma 6.3) Consider the map $F : \mathbb{S}_d^+ \longrightarrow \mathbb{S}_{d^2}^+, X \mapsto X^{1/2} \otimes X^{1/2}$. F is continuous and uniformly Lipschitz on any set of the form $\{x \in \mathbb{S}_d^+ : ||x|| \geq cI_d, ||x|| \leq \hat{c}\}$ with $c, \hat{c} > 0$. Moreover, we have that $||A^{1/2} \otimes A^{1/2}||_2 = ||A||_2$ for all $A \in \mathbb{S}_d^+$.*

We set $U_{C,\epsilon} := \{x \in \mathbb{S}_d^+ : x > -\epsilon I_d\}$ and choose $0 < \epsilon < \min \sigma(C)$, where $\sigma(C)$ denotes the spectrum of C. Then $U_{C,\epsilon}$ (and $vec(U_{C,\epsilon})$) is open and satisfies for all $x \in U_{C,\epsilon}$:

$$C + x > (\min \sigma(C) - \epsilon)I_d$$

From Lemma (7.3) follows, that G is locally Lipschitz on $U_{C,\epsilon}$ and has linear growth:

$$||G(x)||_2^2 = ||x||_2^2 \leq 1 \cdot (1 + ||x||_2^2)$$

From standard results on stochastic differential equations, like [21], it follows, that the SDE (28) has a unique locally bounded solution $(Y_t)_{t \in \mathbb{R}^+}$ with initial value Y_0 and is also of bounded variation, because t and $[L, L]_t^d$ are of bounded variation. *These results only hold, if we can ensure, that our solution doesn't leave the open set $U_{C,\epsilon}$.* We now prove the inequality $Y_t \geq e^{Bt} Y_0 e^{B^T t}$ for all $t \in \mathbb{R}^+$. Between two jumps Y_t follows the deterministic differential equation $dY_t = (BYt- + Y_{t-}B)dt$, which is uniquely solved by $Y_t = e^{Bt} Y_0 e^{B^T t}$. If $Y_0 \in \mathbb{S}_d^+$, $x^T e^{Bt} Y_0 e^{B^T t} x = (e^{B^T t} x)^T Y_0 e^{B^T t} x \geq 0$ for all $x \in \mathbb{R}^d$ and therefore $Y_t \in \mathbb{S}_d^+$, too. The inequality follows from the fact, that all jumps are positive semi-definite: For all $x \in \mathbb{R}^d$ we have

$$\begin{aligned}
&x^T A V_{t-}^{1/2} \Delta[L, L]_t^d V_{t-}^{1/2} A^T x \\
=&(A^T x)^T V_{t-}^{1/2} \Delta[L, L]_t^d V_{t-}^{1/2} A^T x \\
=&(V_{t-}^{1/2} A^T x)^T \Delta[L, L]_t^d V_{t-}^{1/2} A^T x \\
\geq&0
\end{aligned}$$

since $\Delta[L, L]_t^d$ is a matrix subordinator. An $\mathbb{S}_d$-valued Levy process is said to be a *matrix subordinator*, if $L_t - L_s \in \mathbb{S}_d$ for all $s, t \in \mathbb{R}^+$ with $t > s$, (see [23]). Therefore, $(Y_t)_{t \in \mathbb{R}^+}$ satisfies the claimed inequality $Y_t \geq e^{Bt} Y_0 e^{B^T t}$ and the solution stays as a sum of a positive semi-definite process and positive semi-definite jumps necessarily in $\mathbb{S}_d^+$. □

7.4 The MUCOGARCH(1,1) Bound

Theorem 4.3: Let Y be a MUCOGARCH volatility process with initial value $Y_0 \in \mathbb{S}_d^+$ and driven by a Levy process in $\mathbb{R}^d$. Assume, further, that $B \in M_d(\mathbb{R})$ is diagonalizable and let $S \in GL_d(\mathbb{C})$ be such that $S^{-1}BS$ is diagonal. The process solving the SDE,

$$\begin{aligned}
dy_t &= 2\lambda y_{t-} dt + ||S||_2^2 ||S^{-1}||_2^2 K_{2,B} ||A \otimes A||_{B,S} \left(\frac{||C||_2}{K_{2,B}} + y_{t-}\right) d\hat{L}_t \\
y_0 &= ||vec(Y_0)||_{B,S}
\end{aligned}$$

with

$$\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S}\mu_L(ds, dx), \qquad \lambda := max(\Re(\sigma(B)))$$

and

$$K_{2,B} := \max_{X \in \mathbb{S}_d^+, ||X||_2 = 1} (\frac{||X||_2}{||vec(X)||_{B,S}})$$

is the volatility process of a univariate MUCOGARCH(1,1) process and y satisfies

$$||vec(Y_t)||_{B,S} \le y_t \qquad for \quad all \quad t \in \mathbb{R}^+ \quad a.s.$$

Moreover, $K_{2,B} \le \max_{X \in \mathbb{S}_d^+, ||X||_2=1}(\frac{||X||_2}{||vec(X)||_{B,S}}) \le ||S||_2^2$.

Proof. The proof is based on ([3], page 101-104). We will need the following two lemmata to show the existence of a one-dimensional COGARCH(1,1) bound.

Lemma 7.4. *([3], Lemma 6.5) Let $(L_t)_{t \in \mathbb{R}^+}$ be a driftless Levy subordinator. Then there exists a Levy process $(\bar{L}_t)_{t \in \mathbb{R}^+}$ in $\mathbb{R}$ such that $L_t = [\bar{L}_t, \bar{L}_t]_t^d$ for all $t \in \mathbb{R}^+$.*

Proof. We denote the jump measure of L with μ_L, the Levy measure by ν_L and we may write $L_t = \int_0^t \int_{\mathbb{R}^+} x\mu_L(ds, dx)$, since L_t is driftless. Let

$$\bar{L}_t := \int_0^t \int_{0<x\le 1} \sqrt{x}(\mu_L(ds, dx) - ds\nu_L(dx)) + \int_0^t \int_{x>1} \sqrt{x}\mu_L(ds, dx)$$

the first integral exists, because $\int_{0<x\le 1} \sqrt{x}^2 \nu_L(dx) = \int_{0<x\le 1} x\nu_L(dx) < \infty$, because L is of finite variation. Therefore $(\bar{L}_t)_{t \in \mathbb{R}^+}$ is a Levy measure and also satisfies $L_t = [\bar{L}_t, \bar{L}_t]_t^d$. □

The following elementary properties of the norm$|| \cdot ||_{B,S}$ are important for the upcoming steps.

Lemma 7.5. *([3], Lemma 6.6) It holds that*

$$||S \otimes S||_{B,S} = ||S||_2^2 \quad and \quad ||S^{-1} \otimes S^{-1}||_{B,S} = ||S^{-1}||_2^2$$
$$||x||_{B,S} \le ||S^{-1}||_2^2 ||x||_2 \quad and \quad ||x||_2 \le ||S||_2^2 ||x||_{B,S} \quad for \quad all \quad x \in \mathbb{R}^{d^2}$$
$$||X||_{B,S} \le ||S||_2^2 ||S^{-1}||_2^2 ||X||_2 \quad and \quad ||X||_2 \le ||S||_2^2 ||S^{-1}||_2^2 ||X||_{B,S} \quad for \quad all \quad X \in M_{d^2}(\mathbb{R})$$

STEP 1: We show Theorem 4.3 for driving compound Poisson processes.

$\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S}\mu_L(ds, dx)$ is a driftles subordinator, hence, due to Lemma 7.4 there exists a Levy process $(L_t)_{\mathbb{R}^+}$ such that: $\hat{L}_t = [L_t, L_t]_t^d$.
Let Γ_1 be the time of the first jump of the compound Poisson process. Until Γ_1 the process Y_t follows the deterministic differential equation given by $dY_t = (BYt- + Y_{t-}B^T)dt$ and in vec-notation $dvec(Y_t) = (B \otimes I + I \otimes B)vec(Y_{t-})dt$ and is uniquely solved by

$$vec(Y_t) = e^{(B \otimes I + I \otimes B)t} vec(Y_0)$$

Claim 7.6. $||e^{(B\otimes I+I\otimes B)t}||_{B,S} = e^{2\lambda t}$.

Proof. The upcoming calculations are based on elementary properties of the matrix exponential and the following two calculation rules regarding the Kronecker product: $(A\otimes B)^{-1} = (A^{-1}\otimes B^{-1})$ and $(A\otimes B)(C\otimes D) = (AC\otimes BD)$.

$$\begin{aligned}||e^{(B\otimes I+I\otimes B)t}||_{B,S} &= ||(S^{-1}\otimes S^{-1})e^{(B\otimes I+I\otimes B)t}(S\otimes S)||_2\\ &= ||(S\otimes S)^{-1}e^{(B\otimes I+I\otimes B)t}(S\otimes S)||_2\\ &= ||e^{(S\otimes S)^{-1}(B\otimes I+I\otimes B)(S\otimes S)t}||_2\end{aligned}$$

For the first matrix in the matrix exponential we get

$$\begin{aligned}&(S\otimes S)^{-1}(B\otimes I)(S\otimes S) = (S^{-1}\otimes S^{-1})(B\otimes I)(S\otimes S) = (S^{-1}B\otimes S^{-1}I)(S\otimes S)\\ &= (S^{-1}BS)\otimes(S^{-1}IS) = (S^{-1}BS)\otimes I = \hat{B}\otimes I\end{aligned}$$

where $\hat{B} = S^{-1}BS$ is diagonal. Analogously we receive for the second matrix:

$$(S^{-1}\otimes S^{-1})(I\otimes B)(S\otimes S) = I\otimes(S^{-1}BS) = I\otimes\hat{B}$$

Let $\sigma(B) = \{\lambda_1, \lambda_2, ..., \lambda_d\}$ with $\lambda's \in \mathbb{C}$ be the spectrum of B. The first matrix $\hat{B}\otimes I$ is a $d^2\times d^2$ diagonal matrix, where each eigenvalue appears d times in a row. Whereas the second matrix $I\otimes\hat{B}$ is d times $\hat{B}$. We write the sum of them as D and notice, that D is also diagonal, where each entry represents one possible sum of eigenvalues: $D := diag_{i,j=1,...,d}(\{\lambda_i+\lambda_j\})$. The order of the diagonal entries is not important, because $||\cdot||_2$ is invariant with respect to permutations on diagonal matrices. Define $\lambda := max(\Re(\sigma(B)))$ and we conclude our calculations

$$||e^{(B\otimes I+I\otimes B)t}||_{B,S} = ||e^{Dt}||_2 = e^{2\lambda t}$$

□

And since we proved our claim, we get

$$\begin{aligned}||vec(Y_t)||_{B,S} &= ||e^{(B\otimes I+I\otimes B)t}vec(Y_0)||_{B,S} \le ||e^{(B\otimes I+I\otimes B)t}||_{B,S}||vec(Y_0)||_{B,S}\\ &= e^{2\lambda t}y_0 = y_t\end{aligned}$$

Thus our inequality is shown for all $t\in[0,\Gamma_1)$. At time Γ_1 we calculate

$$\begin{aligned}&||vec(Y_{\Gamma_1})||_{B,S}\\ &= ||vec(Y_{\Gamma_1-}) + (A\otimes A)((C+Y_{\Gamma_1})^{1/2}\otimes(C+Y_{\Gamma_1})^{1/2})vec(\Delta L_{\Gamma_1}(\Delta L_{\Gamma_1})^T)||_{B,S}\\ &\le y_{\Gamma_1} + ||A\otimes A||_{B,S}||(C+Y_{\Gamma_1})^{1/2}\otimes(C+Y_{\Gamma_1})^{1/2}||_{B,S}||vec(\Delta L_{\Gamma_1}(\Delta L_{\Gamma_1})^T)||_{B,S}\\ &\le y_{\Gamma_1} + ||A\otimes A||_{B,S}||S||_2^2||S^{-1}||_2^2||(C+Y_{\Gamma_1})^{1/2}\otimes(C+Y_{\Gamma_1})^{1/2}||_2\Delta\hat{L}_{\Gamma_1}\\ &\le y_{\Gamma_1} + ||A\otimes A||_{B,S}||S||_2^2||S^{-1}||_2^2(||C||_2 + ||Y_{\Gamma_1-}||_2)\Delta\hat{L}_{\Gamma_1}\\ &\le y_{\Gamma_1} + ||A\otimes A||_{B,S}||S||_2^2||S^{-1}||_2^2K_{2,B}(K_{2,B}^{-1}||C||_2 + ||vec(Y_{\Gamma_1-})||_{B,S})\Delta\hat{L}_{\Gamma_1}\\ &= y_{\Gamma_1}\end{aligned}$$

By iterating the same argument, because compound Poisson is memoryless, gives us the inequality for all $t \in \mathbb{R}^+$.

STEP 2: We want to extend our results above to MUCOGARCH(1,1) processes driven by general Levy processes. The following Proposition shows, that we can approximate MUCOGARCH(1,1) processes by approximating the driving Levy processes with compound Poisson processes.

Proposition 7.7. *([3], Proposition 6.7) Let Y be a MUCOGARCH volatility process with $C \in \mathbb{S}_d^{++}$ and $Y_0 \in \mathbb{S}_d^+$, driven by a Levy process L in $\mathbb{R}^d$, and let $(\epsilon_n)_{n\in\mathbb{N}}$ be a monotonically decreasing sequence in $\mathbb{R}^+ \setminus \{0\}$ with $\lim_{n\to\infty} \epsilon_n = 0$. Define compound Poisson processes L_n by $L_{n,t} := \int_0^t \int_{\mathbb{R}^d, ||x|| \geq \epsilon_n} x\mu_L(ds, dx)$ for $n \in \mathbb{N}$ and associated MUCOGARCH volatility process Y_n by*

$$dY_{n,t} = (BY_{n,t-} + Y_{n,t-}B)dt + A(C + Y_{n,t-})^{1/2}d[L, L]_t^d(C + Y_{n,t-})^{1/2}A^T \tag{29}$$

$$Y_{n,0} = Y_0 \tag{30}$$

Then $Y_n \to Y$ as $n \to \infty$ almost surely u.c.p. (uniformly on compacts).

We cite the proof from [3], Proposition 6.7:

Proof. First, observe that $[L_n, L_n]_t^d = \int_0^t \int_{||x|| \geq \epsilon_n} xx^T \mu_L(ds, dx)$ implies that $[L_n, L_n]^d \to [L, L]^d$, as $n \to \infty$ a.s. uniformly on compacts, and that $[L_n, L_n]^d \to [L, L]^d$ is monotonically increasing in n. Since all processes involved are of finite variation, we can prove the claim with a pathwise approach. So, fix $\omega \in \Omega$ and thereby one path. Let $T \in \mathbb{R}^+$ be arbitrary. The Gronwall inequality, Theorem 7.8, shows that

$$\begin{aligned}
||Y_{n,t}||_2 &\leq (||Y_0||_2 + ||A||_2^2||C||_2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_{L_n}(ds, dx))e^{||A||_2^2 \int_0^t \int_{\mathbb{R}^d} ||x||_2^2 \mu_{L_n}(ds,dx) + 2||B||_2 t} \\
&\leq (||Y_0||_2 + ||A||_2^2||C||_2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds, dx))e^{||A||_2^2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds,dx) + 2||B||_2 T} \\
||Y_t||_2 &\leq (||Y_0||_2 + ||A||_2^2||C||_2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds, dx))e^{||A||_2^2 \int_0^T \int_{\mathbb{R}^d} ||x||_2^2 \mu_L(ds,dx) + 2||B||_2 T}
\end{aligned}$$

for all $t \in [0, T]$. Since $Y_t \geq e^{Bt}Y_0e^{B^Tt}$, $Y_{n,t} \geq e^{Bt}Y_0e^{B^Tt}$ and Y_0 is positive semidefinite, $C + Y$ and $(C + Y_n)_{n\in\mathbb{N}}$ all remain in one common compact set in $\mathbb{S}_d^{++}$ on $[0, T]$. Thus, when considering (29) and (12), we can regard the coefficients of these SDE's as being globally Lipschitz with a common Lipschitz coefficient. Thus, [21], Corollary, page 261 after Theorem v.11, implies that $Y_n(\omega) \to Y(\omega)$ uniformly on $[0, T]$. Note that, formally, the result of [21] is applied on the probability space given by the set $\{\omega\}$, the trivial σ-algebra $\{\{\omega\}, \emptyset\}$ (which also gives the filtration) and the Dirac measure with respect to ω. Since $\omega \in \Omega$ and $T \in \mathbb{R}^+$ were arbitrary, this completes the proof. $\square$

We state the Gronwall Lemma, used in the above Proposition:

Theorem 7.8. *(Lemma A.2.35, [21])(Gronwall's Lemma) Let $\Phi : [0, \infty] \to [0, \infty)$ be an increasing function satisfying*

$$\Phi(t) \leq A(t) + \int_0^t \Phi(s)\eta(s)ds \quad t \geq 0,$$

where $\eta : [0, \infty) \to \infty$ is positive and Borel, and $A : [0, \infty) \to [0, \infty)$ is increasing. Then

$$\Phi(t) \leq A(t)exp(\int_0^t \eta(s)ds) \quad t \geq 0.$$

We proceed with the proof of Theorem 4.3 for general L: In the last proposition we constructed a sequence $(Y_n)_{n\in\mathbb{N}}$ of compound Poisson driven MUCOGARCH(1,1) processes converging u.c.p. to Y. Let y_n be the univariate $MUCOGARCH(1,1)$ bounds, that means $||vec(Y_{n,t})||_{B,S} \leq y_{n,t}$ for all n and $t \in \mathbb{R}^+$. Due to the defintion of $L_{n,t}$, it is clear, that we add only more jumps through increasing n and therefore

$$y_{n+l,t} \geq y_{n,t}$$

for all $n, l \in \mathbb{N}$ and $t \in \mathbb{R}^+$. Define the process y by:

$$\begin{aligned} dy_t &= 2\lambda y_{t-}dt + ||S||_2^2||S^{-1}||_2^2 K_{2,B}||A \otimes A||_{B,S}(\frac{||C||_2}{K_{2,B}} + y_{t-})d\hat{L}_t \\ y_0 &= ||vec(Y_0)||_{B,S} \end{aligned}$$

with $\hat{L}_t = \int_0^t \int_{\mathbb{R}^d} ||vec(xx^T)||_{B,S}\mu_L(ds, dx)$ and again $y_{n,t} \leq y_t$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. Passing to the limit $n \to \infty$ in $||vec(Y_{n,t})||_{B,S} \leq y_{n,t} \leq y_t$ shows $||vec(Y_t)||_{B,S} \leq y_t$ for all $t \in \mathbb{R}^+$. □

7.5 Stationarity of the MUCOGARCH(1,1) process

The proof of the stationarity is based on ([3], 104-105) and needs some preparation. First of all we show, that our volatility process is a time-homogeneous Markov process with the weak Feller property. This result together with the one-dimensional MUCOGARCH(1,1) bound serve as the fundament for the proof of the stationarity. We use the stationary condition Lemma (7.13) to prove the stationarity of the bound, which helps us to prove the prerequisites of the Krylov-Bogoliubov Theorem. This theorem, in the end, brings us the desired stationarity of the MUCOGARCH(1,1) volatility process.

Let $U \subseteq \mathbb{R}^d$ be an open set, $f : U \longrightarrow M_{d,m}(\mathbb{R})$ a locally Lipschitz function and $(L_t)_{t\in\mathbb{R}^+}$ a m-dimensional Levy process. We want to show, that the solution of the following stochastic differential equation

$$dX_t = f(X_t)dL_t$$

is Markovian and has the weak Feller property. But before we start, we give some standard notion regarding Markov processes:

Definition 7.1. *([3], Definition 6.7.7) Let $Z = (Z_t)_{t\in\mathbb{R}^+}$ be a process with values in U which is adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$.*
(i) Z is called a Markov process with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$, if

$$E(g(Z_u)|\mathcal{F}_t) = E(g(Z_u)|Z_t)$$

for all $t \in \mathbb{R}^+$, $u \geq t$ and $g : U \longrightarrow \mathbb{R}$ bounded and Borel measurable.
(ii) Let Z be a Markov process and define for all $s, t \in \mathbb{R}^+$, $s \leq t$, the transition functions $P_{s,t}(Z_s, g) = E(g(Z_t)|\mathcal{F})$ with $g : U \longrightarrow \mathbb{R}$ bounded and Borel measurable. If $P_{s,t} = P_{0,t-s} =: P_{t-s}$ for all $s, t \in \mathbb{R}^+$, $s \leq t$, Z is said to be a time homogeneous Markov process.
(iii) A time homogeneous Markov process is called a strong Markov process, if $E(g(Z_{T+s})|\mathcal{F}_T) = P_s(Z_T, g)$ for $g : U \longrightarrow \mathbb{R}$ bounded and Borel measurable and a.s. finite stopping time.

Under the assumptions on U ([3], chapter 6.7.1.2) and the enlargement of the probability space as in ([21], page 293) we are able to show, that the solution $(X_t)_{t\in\mathbb{R}^+}$ is Markovian:

Theorem 7.9. *(Theorem 6.7.8, [22])(Markov process) Under the above assumptions the unique solution $(X_t)_{t\in\mathbb{R}^+}$ to*

$$X_t = X_0 + \int_0^t f(X_{t-})dL_t \tag{31}$$

is a temporally homogeneous strong Markov process on U.

And under the same assumptions we are able to show the weak Feller property of the solution $(X_t)_{t\in\mathbb{R}^+}$, which is defined as

Definition 7.2. *([22], Definition 6.7.11) Let $(P_s)_{s\in\mathbb{R}^+}$ be the transition semi-group of a time homogeneous Markov process Z on U.*
(i) $(P_s)_{s\in\mathbb{R}^+}$ (respectively the associated Markov process Z) is called stochastically continuous, if

$$\lim_{t\to 0, t\geq 0} P_t(x, U(x)) = 1$$

for all $x \in U$ and open neighbourhoods $U(x)$ of x.
(ii) A stochastically continuous semi-group $(P_s)_{s\in\mathbb{R}^+}$ (respectively its associated Markov process Z) is called weakly Feller, if

$$P_s(C_b(U)) \subseteq C_b(U)$$

for all $s \in \mathbb{R}^+$ (iii) A probability measure $\mu \in \mathcal{M}_1(U)$ is said to be an invariant (stationary) measure for the Markovian semi-group $(P_s)_{s\in\mathbb{R}^+}$ (repsectively for its associated Markov process), if $P_s^\star \mu = \mu$ for all $s \in \mathbb{R}^+$.

and we conclude

Theorem 7.10. *(Proposition 6.7.13, [22])(Weak Feller Property) Under the above assumptions the unique solution $(X_t)_{t\in\mathbb{R}^+}$ to*

$$X_t = x + \int_0^t f(X_{t-})dL_t \tag{32}$$

with $x \in U$ is a weak Feller process on U.

Remark. Regarding the definitions of the Markov properties and the weak Feller property, we may replace the open set $U \subseteq \mathbb{R}^d$ by any Polish space.

Stelzer used these general results in ([22], chapter 6.3.3 and 6.7.1.2) to show that the MUCOGARCH and it's volatility process are Markovian and have the weak Feller property:

Theorem 7.11. *([3], Theorem 4.4) (Markovian Properties) The MUCOGARCH(1,1) process (G,Y) and its volatility process Y alone are temporally homogeneous strong Markov processes on $\mathbb{R}^d \times \mathbb{S}_d^+$ and $\mathbb{S}_d^+$, respectively, and they have the weak Feller property.*

We are now ready to prove the stationarity of the volatility process.

Theorem 7.12. *(Theorem 6.7.8, [22])(Stationarity) Let $B \in M_d(\mathbb{R})$ be diagonalizable with $S \in GL_d(\mathbb{C})$ such that $S^{-1}BS$ is diagonal. Furthermore, let L be a d-dimensinal Levy process with non zero Levy measure, y be defined as in Theorem 4.3 and $\alpha_1 := ||S||_2^2||S^{-1}||_2^2K_{2,B}||A \otimes A||_{B,S}$. Assume that*

$$\int_{\mathbb{R}^d} log(1 + \alpha_1||vec(yy^T)||_{B,S})\nu_L(dy) < -2\lambda$$

Then there exists a stationary distribution $\mu \in \mathcal{M}_1(\mathbb{S}_d^+)$, that is, the set of all probability measures on the Borel-σ-algebra of $\mathbb{S}_d^+$.

Proof. The one-dimensional MUCOGARCH(1,1) bound y of Lemma 4.3 is the key for the proof of the stationarity. We first show, that the bound y converges to a stationary distribution and then we shift this property with the help of the Krylov-Bogoliubov Theorem to the MUCOGARCH(1,1) volatility process $(Y_t)_{t\in\mathbb{R}^+}$.

Lemma 7.13. *([7], Definition 2.1, Theorem 3.1) (Stationarity Condition) We define the left-continuous volatility process of the COGARCH(1,1) process by*

$$V_t = \alpha_0 + \alpha_1\mathbf{Y}_{t-} \quad t > 0 \quad V_0 = \alpha_0 + \alpha_1\mathbf{Y}_0$$

where the state process $\mathbf{Y} = (\mathbf{Y}_t)_{t\geq 0}$ is the unique cadlag solution of the stochastic differential equation

$$d\mathbf{Y}_t = -\beta_1\mathbf{Y}_{t-}dt + (\alpha_0 + \alpha_1\mathbf{Y}_{t-})d[L, L]_t^d$$

Let $(\mathbf{Y}_t)_{t\geq 0}$ be the state process of the COGARCH(1,1) process with parameters $-\beta_1, \alpha_1, \alpha_0$. Let L be a Levy process with non-trivial Levy-measure ν_L and suppose that

$$\int_{\mathbb{R}} log(1+\alpha_1 y^2) d\nu_L(y) < \beta_1$$

Then $\mathbf{Y}_t$ converges in distribution to a finite random variable $\mathbf{Y}_\infty$ as $t \to \infty$. It follows that if $\mathbf{Y}_0 = \mathbf{Y}_\infty$ in distribution, then $(\mathbf{Y}_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are strictly stationary.

For a proof see [16], Theorem 3.1. Take λ, $\hat{L}$ and $\hat{L} = [\bar{L}, \bar{L}]$ as in Lemma 4.3. Then we have

$$\int_{\mathbb{R}^d} log(1+\alpha_1 y^2)\nu_{\bar{L}}(dy) = \int_{\mathbb{R}^d} log(1+\alpha_1 ||vec(yy^T)||_{B,S})\nu_L(dy) < -2\lambda$$

and therefore the COGARCH(1,1) bound y from Lemma 4.3 satisfies all prerequisites of Lemma 7.13 and it follows, that our bound converges in distribution to a distribution concentrated on $\mathbb{R}^+$. If we assume that y_0 has this stationary distribution and is independent of $(L_s)_{s\in\mathbb{R}^+}$, $(y_t)_{t\in\mathbb{R}^+}$ is stationary. We want to transform this stationarity result from the bound to our MUCOGARCH volatility process. We set $Y_0 = \frac{y_0}{||vec(I_d)||_{B,S}} I_d$, independent of $(L_s)_{s\in\mathbb{R}^+}$ and $||vec(Y_0)||_{B,S} \leq y_0$. Our next step is to show the existence of a stationary solution for the MUCOGARCH volatility process Y. The set $\{\mathscr{L}(Y_t) : t \in \mathbb{R}^+\}$ of laws of Y_t forms a tight subset of $\mathcal{M}_1(\mathbb{S}_d^+)$: For all $K > 0$ the set $\{x \in \mathbb{S}_d^+ : ||x|| \leq K\}$ is compact in $\mathbb{S}_d^+$, $\mathbb{P}(||Y_t||_{B,S} \leq K) \geq \mathbb{P}(y_t \leq K)$, because $||Y_t||_{B,S} \leq y_t$ and y is stationary with a stationary distribution concentrated on $\mathbb{R}^+$. $(Y_t)_{t\in\mathbb{R}^+}$ is a weak Feller Markov process (Theorem 4.4, [3]) and therefore fulfills all requirements for the Krylov-Bogoliubov Theorem:

Lemma 7.14. *([3], Theorem 6.8) (Krylov-Bogoliubov Existence Theorem) Let E be a Polish space and $(P_s)_{s\in\mathbb{R}^+}$ the transition semigroup of an E-valued weak Feller Markov process. Assume that there is an $\eta \in \mathcal{M}_1(E)$ such that the set $\{P_t^*\eta : t \in \mathbb{R}^+\}$ is tight. Then there exists a $\mu \in \mathcal{M}_1(E)$ such that $P_t^*\mu = \mu$ for all $t \in \mathbb{R}^+$, that is, μ is an invariant measure for $(P_s)_{s\in\mathbb{R}^+}$ or a stationary distribution for the Markov process, respectively, and μ is in the closed (with respect to weak convergence) convex hull of $\{P_t^*\eta : t \in \mathbb{R}^+\}$.*

For a proof see [6], Theorem 4.6. We conclude, there exists a stationary distribution $\mu \in \mathcal{M}_1(\mathbb{S}_d^+)$ for the MUCOGARCH volatility process Y such that μ is in the closed convex hull of $\{\mathscr{L}(Y_t) : t \in \mathbb{R}^+\}$. □

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