An Approach for Stability-Preserving Model Order Reduction for Switched Linear Systems Based on Individual Subspaces

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We present an approach for the stability-preserving model order reduction of switched linear systems with individual subspaces for arbitrary switching signals. First the high-order subsystems of the switched system are reduced by any reduction method and stabilized if necessary. Then the reduced subsystems are made contractive by solving low-dimensional Lyapunov equations. During time simulation the reduced state vector of the active subsystem has to be initialized by the last state vector of the previously active subsystem at the switching points so that the Lyapunov function of the reduced switched system does not grow. This method has no constraints for the structure of the high-order switched systems. The performance of the proposed approach is demonstrated on the basis of a numerical example.

1 Introduction

The rising complexity of engineered systems (e.g. mechanical components or integrated circuits) often leads to large-scale and complex mathematical models. Conventionally, the complexity problem can be tackled by model order reduction (MOR) techniques, whereby a high order system is approximated by a lower order one [1]. However, widely used MOR approaches, like Truncated Balanced Realization [2] (TBR) or Krylov subspace methods [3], are designed for LTI system formulations.
Switched systems are a special class of hybrid and nonlinear systems and are useful because the theory of linear control can be applied [6, 5, 7]. Modeling of nonlinearities by using linear systems and switching between them is applicable for abrupt changes of system properties. This is the case, for example, in vibration or starting procedure simulations of a motor whose damping or stiffness matrices change with the rotational speed as input signal. Consequently, a switching model enables the principle applicability of conventional and well-understood MOR techniques for order reduction of nonlinear systems. However, conventional MOR techniques return a weak performance when directly applied to switched systems [10]. These facts motivated the extension of MOR methods for handling switched linear systems in the last few years. Two main strategies have been investigated so far. First, a holistic reduction of the overall switched system expression is investigated in [11, 12]. This approach has been focused by researchers, since the stability of the reduced switched system can be guaranteed based on a common projection. However, this is at the expense of the obtainable accuracy. Moreover, linear matrix inequalities (LMI) have to be solved whereby the approach becomes progressively computationally inefficient with rising order of the original system. Contrary to that approach, a recently investigated second strategy aims to separately reduce each linear subsystem in order to minimize the error between the dynamical behavior of the reduced and original switched system [13]. The approach achieves accurate results for any kind of switching signal (time-based and state-based switching) and every projection-based MOR method can be used. However, a proof of stability still has to be brought for this performance-oriented approach.

This drawback is tackled within this article. We extend the mentioned method from [13] for arbitrary time-based switching such that the stability is preserved without any constraint for the structure of high-order switched linear systems. The proof of stability is based on the theory of impulsive systems [8, 9]. The remainder of the paper is organized as follows: A brief introduction to switched linear systems is given in section 2. The stability-preserving method is presented in section 3 and demonstrated for a numerical example in section 4, followed by conclusions in section 5.

2 Preliminaries and state of the art

In the following a short overview of switched linear dynamical systems will be given. Additionally, two main approaches for model order reduction of switched systems are summarized.

2.1 Switched linear dynamical systems

Switched systems consist of a finite amount $k \in \mathbb{N}$ of continuous dynamical LTI subsystems which are activated and deactivated depending on a piecewise constant switching signal $\alpha \in \{1, 2, ..., k\}$. The state-space representation of a switched system Σ is:

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_\alpha \mathbf{x}(t) + \mathbf{B}_\alpha \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}_\alpha \mathbf{x}(t), \end{cases} \tag{1}$$

where the vectors $\mathbf{u}(t) \in \mathbb{R}^r$, $\mathbf{y}(t) \in \mathbb{R}^m$ and $\mathbf{x}(t) \in \mathbb{R}^n$ denote the input vector, output vector and state vector of the system, respectively. The matrix tuple $\{\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha\}$ defines the currently active subsystem. In this paper we apply arbitrary, time-dependent switching signals:

$$\alpha(t) : t \mapsto \{1, 2, ..., k\}. \tag{2}$$

Time-dependent switching between LTI subsystems is shown in Figure 1. Switching between the subsystems occurs at arbitrary instants of time t_e and is independent of the current system state $\mathbf{x}$. The time intervals where a subsystem remains active can be arbitrarily small, but the possibility of an infinite number of switches in finite time is excluded [5].

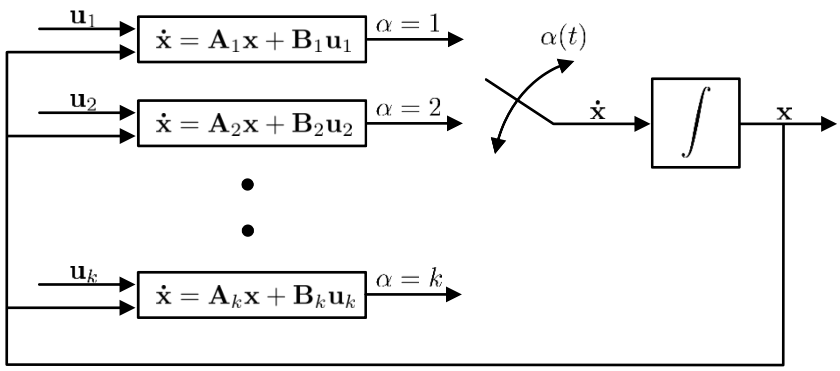

Figure 1: Time-dependent switching.

As far as the stability of switched systems is concerned, it has to be noted that arbitrary switching may lead to instability even if the subsystems are asymptotically stable [4, 5]. This is illustrated by the following simple example.

Example: Given are two LTI subsystems $\dot{\mathbf{x}}_1(t) = \mathbf{A}_1\mathbf{x}_1(t), \dot{\mathbf{x}}_2(t) = \mathbf{A}_2\mathbf{x}_2(t)$ with

$$\mathbf{A}_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}. \tag{3}$$

The eigenvalues of the subsystems are $\{-0.1 + \sqrt{2}\mathrm{i}, -0.1 - \sqrt{2}\mathrm{i}\}$. Hence, the subsystems are asymptotically stable. Nevertheless, arbitrary switching may lead to instability, see the time simulation depicted in the state-space shown in Figure 2.

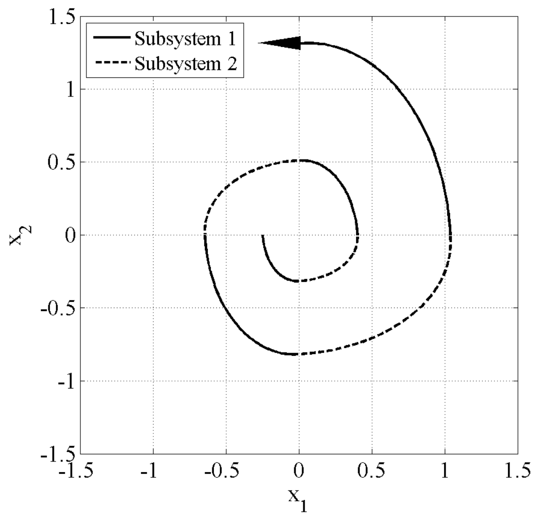

Figure 2: Unstable switched system.

The stability theory for switched systems is a well-examined. For arbitrary switching between subsystems the following theorem holds.

Theorem 1 ([4, 5])**.** *A switched system is asymptotically stable under arbitrary switching if and only if the subsystems share a common Lyapunov function.*

As far as a linear switched system (1) is concerned, asymptotic stability is guaranteed if the linear subsystems α share a common quadratic Lyapunov function:

$$\mathrm{V}_\alpha(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \quad \forall \quad \alpha \in \{1, 2, ..., k\}, \tag{4}$$

where $\mathbf{P}$ is a symmetric positive definite matrix.

Another possibility of switched systems are systems with impulse effects [8, 9]. Thereby, the state vector $\mathbf{x}(t)$ is reset at switching times t_e. The reset of the state vector for switching from subsystem $\alpha = j$ to $\alpha = i$ is using $\mathbf{R}_{j,i} \in \mathbb{R}^{n \times n}$:

$$\mathbf{x}_j(t_e^+) = \mathbf{R}_{j,i}\mathbf{x}_i(t_e^-). \tag{5}$$

Thereby, $\mathbf{x}_j(t_e^+)$ denotes the state vector of the subsystem $\alpha = j$ at the instant of time t_e and $\mathbf{x}_i(t_e^-)$ describes the state vector of the subsystem $\alpha = j$ at the same instant of time. As far as the asymptotic stability of switched systems with impulse effects is concerned, the following theorem holds:

Theorem 2 ([8, 9])**.** *The switched system with impulse effects is asymptotically stable if:*

1. *The reduced subsystems α have individual Lyapunov functions and, therefore, are asymptotically stable:*
$$\mathrm{V}_\alpha(\mathbf{x}_\alpha(t)) > 0 \quad \textit{and} \quad \dot{\mathrm{V}}_\alpha(\mathbf{x}_\alpha(t)) < 0.$$
2. *At switching times t_e from subsystem $\alpha = j$ to $\alpha = i$ the value of the Lyapunov function $\mathrm{V}_j(\mathbf{x}_{r,j}(t_e))$ does not exceed the value of the Lyapunov function $\mathrm{V}_i(\mathbf{x}_{r,i}(t_e))$:*
$$\mathrm{V}_j(\mathbf{x}_j(t_e^+)) \leq \mathrm{V}_i(\mathbf{x}_i(t_e^-)).$$

2.2 Methods for model order reduction of switched systems

Various methods for model order reduction of switched systems exist. In the following an overview of two approaches with the respective steps is given.

2.2.1 Reduction with common subspaces

One reduction method was proposed by Shaker et al. [12]. This method uses common subspaces $\mathcal{V}$ and $\mathcal{W}$ for all subsystems. Hence the subsystems have the same reduced state vector. At first a common generalized controllability and observability gramian is calculated by solving high-dimensional linear matrix inequalities (LMIs). These can be used to balance and truncate the subsystems with regard to generalized Hankel Singular Values. Therefore, this approach inherits the beneficial properties of truncated balanced realization (TBR). The advantages of this method are:

- Preservation of stability.
- Existence of H_∞ error bound.
- Error is independent of the switching frequency.

The disadvantages of this method are:

- Inaccurate approximation of the individual subsystems as common subspaces are used.
- Existence of a solution of the LMIs is not guaranteed.
- Application is limited to moderate high-order models because high-dimensional LMIs have to be solved.

2.2.2 Reduction with individual subspaces

In this section an approach of Diepold et al. [13] is presented. Contrary to the method of Shaker et al. [12] each subsystem is separately reduced. Hence individual subspaces $\mathcal{V}_\alpha$ and $\mathcal{W}_\alpha$ are calculated for each subsystem. Thereby, any projection-based method of model order reduction such as Krylov subspace methods, POD or TBR can be used. Therefore the reduced subsystems have different reduced state vectors $\mathbf{x}_{r,\alpha}(t)$. For that reason the state vector $\mathbf{x}_{r,\alpha}(t_e^-)$ of the previously active subsystem $\alpha(t_e^-)$ has to be projected using Petrov-Galerkin or Galerkin projection into the subspace of the currently active subsystem $\alpha(t_e^+)$ at each switching time t_e. The advantages of this method are:

- Accurate approximation of the subsystems.
- Reduction of very high-order switched systems with Krylov subspace methods or POD.

The disadvantages of this method are:

- Additional error at switching operations due to projection of the state vectors.
- Error depends on the switching frequency.
- No guarantee of stability for arbitrary switching signals.

According to Theorem 1, a reduced switched system is asymptotically stable if the subsystems share the same Lyapunov function. If individual subspaces are used for the subsystems, the reduced state vectors differ. Therefore the subsystem cannot have the same Lyapunov function even if the Lyapunov matrix is the same. The Lyapunov function of such a reduced switched system consists of the Lyapunov functions of the subsystems where only one subsystem with its corresponding Lyapunov function is active:

$$\mathrm{V}_{\Sigma_r}(\mathbf{x}_{r,\alpha}(t)) = \begin{cases} \mathrm{V}_1(\mathbf{x}_{r,1}(t)) & \text{if} \quad \alpha = 1, \\ & \vdots \\ \mathrm{V}_k(\mathbf{x}_{r,k}(t)) & \text{if} \quad \alpha = k. \end{cases} \tag{6}$$

2.2.3 Problem formulation

In this paper a method of model order reduction with individual subspaces, which inherits the advantages of the method of Diepold et al. and extends it to stability preservation, is proposed. The general idea is based on the theory of systems with impulse effects [8, 9]. For reduced order switched systems, the state vectors of the reduced subsystems lie in different subspaces. The reinitialization of the state vectors at switching times is done by projection which can be interpreted as a reset of the state vector. With analogy to Theorem 2 for switched systems with impulse effects the following theorem is proposed for reduced switched systems:

Theorem 3. *The reduced switched system with individual subspaces is asymptotically stable if the Lyapunov function* $\mathrm{V}_{\Sigma_r}(\mathbf{x}_{r,\alpha}(t))$ *decreases monotonically:*

1. *The reduced subsystems* α *have individual Lyapunov functions and, therefore, are asymptotically stable:*
$$\mathrm{V}_\alpha(\mathbf{x}_{r,\alpha}(t)) > 0 \quad \textit{and} \quad \dot{\mathrm{V}}_\alpha(\mathbf{x}_{r,\alpha}(t)) < 0.$$
2. *At switching times* t_e *from subsystem* $\alpha = j$ *to* $\alpha = i$ *the value of the Lyapunov functions, which may be discontinuous, will not increase:*
$$\mathrm{V}_j(\mathbf{x}_{r,j}(t_e)) \leq \mathrm{V}_i(\mathbf{x}_{r,i}(t_e)).$$

According to Theorem 3 the concept of this paper is that the Lyapunov function (6) must decrease monotonically. Therefore, the paper consists of two main steps. First, asymptotically stable reduced order subsystems are calculated in section 3.1. Next, it is ensured in section 3.2 and 3.3 that the value of the Lyapunov function does not increase at switching times t_e.

3 Main part

In this section we propose an approach for the stable model order reduction of switched linear systems with individual subspaces. The method is based on the approach in [13] and extends it for stability preservation. The proposed method is inspired by an approach for stability-preserving parametric model order reduction by matrix interpolation from [15]. In the following section, the three steps for the proposed method are described, where the calculations of step 1 and 2 are done offline and step 3 is performed online during the simulation.

3.1 Calculation of a set of stable reduced systems

Given is a switched system Σ comprising k high-dimensional LTI subsystems. The subsystems are reduced individually by projection-based model order reduction to the same order $q \ll n$ so that proper subspaces $\mathcal{V}_\alpha$ and $\mathcal{W}_\alpha$ are calculated for every subsystem α. The projection matrices $\mathbf{V}_\alpha \in \mathbb{R}^{n\times q}$ and $\mathbf{W}_\alpha \in \mathbb{R}^{n\times q}$ are suitably chosen and span the right subspaces $\mathcal{V}_\alpha$ and left subspaces $\mathcal{W}_\alpha$, respectively. This is called a Petrov-Galerkin projection and leads with the nonsingular matrix $\mathbf{W}_\alpha^T\mathbf{V}_\alpha$ to the following reduced switched system:

$$\Sigma_r : \begin{cases} \dot{\mathbf{x}}_{r,\alpha}(t) = \mathbf{A}_{r,\alpha}\mathbf{x}_{r,\alpha}(t) + \mathbf{B}_{r,\alpha}\mathbf{u}(t), \\ \mathbf{y}_{r,\alpha}(t) = \mathbf{C}_{r,\alpha}\mathbf{x}_{r,\alpha}(t), \end{cases} \tag{7}$$

where

$$\begin{aligned} \mathbf{A}_{r,\alpha} &= (\mathbf{W}_\alpha^T\mathbf{V}_\alpha)^{-1}\mathbf{W}_\alpha^T\mathbf{A}_\alpha\mathbf{V}_\alpha, \\ \mathbf{B}_{r,\alpha} &= (\mathbf{W}_\alpha^T\mathbf{V}_\alpha)^{-1}\mathbf{W}_\alpha^T\mathbf{B}_\alpha, \\ \mathbf{C}_{r,\alpha} &= \mathbf{C}_\alpha\mathbf{V}_\alpha. \end{aligned} \tag{8}$$

Well-known projection-based reduction methods are the Truncated Balanced Realization (TBR), Proper Orthogonal Decomposition (POD) or Krylov subspace methods. For detailed information we refer to [1] and references therein. We assume that the projection matrices $\mathbf{V}_\alpha$ and $\mathbf{W}_\alpha$ are orthogonal, which can be achieved with a Gram-Schmidt process.

In the following these low-dimensional subsystems are supposed to be Hurwitz stable, i.e. the eigenvalues of $\mathbf{A}_{r,\alpha}$ have negative real parts. This can be achieved by stable reduction methods like the TBR. The reduction of large-scale systems with POD or Krylov subspace methods in general does not lead to Hurwitz stable systems. However, several approaches have been proposed to stabilize the reduced systems via post-processing without significantly affecting the accuracy, for example by modifying the projection matrix $\mathbf{W}_\alpha$ in a suitable manner [16]. Consequently, the result of this step is a switched system with k stable reduced order LTI subsystems. As the LTI subsystems are asymptotically stable, there exists a quadratic Lyapunov function V_α with individual symmetric positive definite Lyapunov matrices $\mathbf{P}_\alpha > \mathbf{0}$, see [18]:[1]

$$\mathrm{V}_\alpha(\mathbf{x}_{r,\alpha}(t)) = \mathbf{x}_{r,\alpha}^T(t)\mathbf{P}_\alpha\mathbf{x}_{r,\alpha}(t) > 0 \quad \text{and} \quad \dot{\mathrm{V}}_{r,\alpha}(\mathbf{x}_{r,\alpha}(t)) < 0. \tag{9}$$

It has to be noted again that the state vectors of the subsystems $\mathbf{x}_{r,\alpha}(t)$ have a different interpretation as they lie in different subspaces $\mathcal{V}_\alpha$. Therefore, the subsystems do not share a common Lyapunov function even if the Lyapunov matrices $\mathbf{P}_\alpha$ are identical.

[1] We use the following notations. With $\mathbf{X} > \mathbf{0}$ a symmetric positive definite matrix $\mathbf{X}$ is denoted and with $\mathbf{X} < \mathbf{0}$ a symmetric negative definite matrix $\mathbf{X}$ is described.

3.2 Adaptation of the Lyapunov matrices

In section 2.1 we have seen that a switched system is not asymptotically stable in general, although the subsystems are asymptotically stable. If the subsystems share a common Lyapunov function, the switched system is asymptotically stable. However, for a switched system with individually reduced subsystems Σ_r it is not possible to find a common Lyapunov function since the state vectors differ. Consequently we propose the following approach.

We first suggest that the reduced subsystems are transformed in such a way that they share the Lyapunov matrix $\tilde{\mathbf{P}}_\alpha = \mathbf{I}$. This is done by state transformations $\mathbf{x}_{r,\alpha}(t) = \mathbf{L}_\alpha^{-1}\tilde{\mathbf{x}}_{r,\alpha}(t)$. The property $\tilde{\mathbf{P}}_\alpha = \mathbf{I}$ is – amongst others – known as contractivity [14]. It guarantees that the reduced state vector $\tilde{\mathbf{x}}_{r,\alpha}(t)$ shows no transient growth and its norm $|\tilde{\mathbf{x}}_{r,\alpha}(t)|$ decreases monotonically. Note that the state transformations do not change the transfer behavior of the reduced subsystems. A similar approach has been applied to linear parameter-varying systems [19].

For the introduction of the algorithm in this section we multiply the subsystems α of the reduced switched system Σ_r by regular matrices $\mathbf{L}_\alpha \in \mathrm{GL}(q, \mathbb{R})$ and introduce the changes of basis $\mathbf{x}_{r,\alpha}(t) = \mathbf{L}_\alpha^{-1}\tilde{\mathbf{x}}_{r,\alpha}(t)$. The resulting reduced switched system $\tilde{\Sigma}_r$ is:

$$\tilde{\Sigma}_r : \begin{cases} \dot{\tilde{x}}_{r,\alpha}(t) = \overbrace{\mathbf{L}_\alpha\mathbf{A}_{r,\alpha}\mathbf{L}_\alpha^{-1}}^{\tilde{\mathbf{A}}_{r,\alpha}}\tilde{\mathbf{x}}_{r,\alpha}(t) + \overbrace{\mathbf{L}_\alpha\mathbf{B}_{r,\alpha}}^{\tilde{\mathbf{B}}_{r,\alpha}}\mathbf{u}(t), \\ \mathbf{y}_{r,\alpha}(t) = \underbrace{\mathbf{C}_{r,\alpha}\mathbf{L}_\alpha^{-1}}_{\tilde{\mathbf{C}}_{r,\alpha}}\tilde{\mathbf{x}}_{r,\alpha}(t). \end{cases} \tag{10}$$

So far it is unknown how the matrices $\mathbf{L}_\alpha$ need to be chosen. We demand for the systems $\tilde{\Sigma}_r$ that they share the identity matrix as Lyapunov matrix $\tilde{\mathbf{P}}_\alpha = \mathbf{I}$:

$$\mathbf{I}\tilde{\mathbf{A}}_{r,\alpha} + \tilde{\mathbf{A}}_{r,\alpha}^T\mathbf{I} < \mathbf{0}. \tag{11}$$

Inserting the expression $\tilde{\mathbf{A}}_{r,\alpha} = \mathbf{L}_\alpha\mathbf{A}_{r,\alpha}\mathbf{L}_\alpha^{-1}$ leads to:

$$\mathbf{L}_\alpha\mathbf{A}_{r,\alpha}\mathbf{L}_\alpha^{-1} + \mathbf{L}_\alpha^{-T}\mathbf{A}_{r,\alpha}^T\mathbf{L}_\alpha^T < \mathbf{0}. \tag{12}$$

Introducing the decomposition $\mathbf{P}_\alpha = \mathbf{L}_\alpha^T\mathbf{L}_\alpha$ with $\mathbf{P}_\alpha > \mathbf{0}$, formula (12) can be written as:

$$\mathbf{P}_\alpha\mathbf{A}_{r,\alpha} + \mathbf{A}_{r,\alpha}^T\mathbf{P}_\alpha < \mathbf{0}. \tag{13}$$

These LMIs can easily be solved since the reduced models are low-dimensional [17]. They have a solution as the matrices $\mathbf{A}_{r,\alpha}$ are Hurwitz stable according to section 3.1.

To sum up, for each of the k reduced order subsystems of the reduced switched system Σ_r a low-dimensional Lyapunov equation needs to be solved. The matrices $\mathbf{L}_\alpha$ sought after can then be calculated, for example, via Cholesky decomposition of $\mathbf{P}_\alpha$. For this purpose MATLAB provides the command LYAPCHOL which directly returns the matrices $\mathbf{L}_\alpha$. The corresponding procedure is summarized in Algorithm 1. The result of this algorithm is a switched system with asymptotically stable reduced subsystems which share the same Lyapunov matrix $\tilde{\mathbf{P}}_\alpha = \mathbf{I}$. The Lyapunov function of the subsystem α is:

$$\begin{aligned} \mathrm{V}_\alpha(\tilde{\mathbf{x}}_{r,\alpha}(t)) &= \mathbf{x}_{r,\alpha}^T(t)\mathbf{P}_\alpha\mathbf{x}_{r,\alpha}(t) \\ &= \tilde{\mathbf{x}}_{r,\alpha}^T(t)\underbrace{\mathbf{L}_\alpha^{-T}\mathbf{P}_\alpha\mathbf{L}_\alpha^{-1}}_{\tilde{\mathbf{P}}_\alpha=\mathbf{I}}\tilde{\mathbf{x}}_{r,\alpha}(t) \\ &= \tilde{\mathbf{x}}_{r,\alpha}^T(t)\tilde{\mathbf{x}}_{r,\alpha}(t). \end{aligned} \tag{14}$$

Algorithm 1

Input: k matrices $\mathbf{A}_{r,\alpha}$

Output: k matrices $\mathbf{L}_\alpha$

1: **for** $\alpha = 1$ to k **do**

2: Choose $\mathbf{Q}_\alpha > \mathbf{0}$

3: Solve $\mathbf{P}_\alpha \mathbf{A}_{r,\alpha} + \mathbf{A}_{r,\alpha}^T \mathbf{P}_\alpha = -\mathbf{Q}_\alpha$ for $\mathbf{P}_\alpha > \mathbf{0}$

4: Compute $\mathbf{P}_\alpha = \mathbf{L}_\alpha^T \mathbf{L}_\alpha$

5: **end for**

The calculations in section 3.1 and 3.2 are performed offline, which means before simulating the reduced switched system $\tilde{\Sigma}_r$. Therefore, the system matrices of the reduced subsystems $\tilde{\mathbf{A}}_{r,\alpha}, \tilde{\mathbf{B}}_{r,\alpha}, \tilde{\mathbf{C}}_{r,\alpha}$ from (10) are calculated and stored after the second step.

3.3 Switching condition

This section deals with the time simulation of the reduced switched system $\tilde{\Sigma}_r$. During the time simulation only one reduced subsystem α is active. The switching operation from subsystem $\alpha(t_e^-)$ to subsystem $\alpha(t_e^+)$ takes place at time t_e. This is called the online phase. Since the state vectors of the reduced subsystems are described with different subspaces, the state vector of the subsystem $\alpha(t_e^+)$ has to be correctly initialized. This is done by projecting the state vector of the subsystem $\alpha(t_e^-)$ into the subspace $\mathcal{V}_{\alpha(t_e^+)}$ of the subsystem $\alpha(t_e^+)$ at the time point t_e. Consider a subsystem $\alpha(t_e^-) = i$ with the projection matrices $\mathbf{V}_i, \mathbf{W}_i$ and a subsystem $\alpha(t_e^-) = j$ with the projection matrices $\mathbf{V}_j, \mathbf{W}_j$. Then the initial state vector $\mathbf{x}_{r,j}(t_e^+)$ of the subsystem j at the switching point t_e is calculated by projecting the state vector $\mathbf{x}_{r,i}(t_e^-)$ into the subspace $\mathcal{V}_j$. In [13] Petrov-Galerkin projections $\mathcal{P}$ are used, which give the initial starting vector with $\mathbf{R}_{j,i} = (\mathbf{W}_j^T \mathbf{V}_j)^{-1} \mathbf{W}_j^T \mathbf{V}_i$:

$$\mathbf{x}_{r,j}(t_e^+) = (\mathbf{W}_j^T \mathbf{V}_j)^{-1} \mathbf{W}_j^T \mathbf{V}_i \mathbf{x}_{r,i}(t_e^-). \tag{15}$$

Although the reduced subsystems share a common Lyapunov matrix (14), the switched system can become unstable during simulation. The reason is that the Lyapunov function of the reduced switched system (6) can grow during the switching operation due to the projection of the state vector according to (15).

This shall be demonstrated by two examples. Considered are two subsystems i and j for which subspaces of reduced order $q = 1$ are calculated. The original state space has order $n = 2$. In Figure 3 the two-dimensional original state space and the subspaces of the two reduced subsystems are shown. Depicted are instant switching operations with infinitesimal time between the switching points t_e and the initial starting vectors of the new subspaces. They are calculated by Petrov-Galerkin projections. For example, during the switching from subsystem i to subsystem j, the state vector $\mathbf{x}_{r,i}(t_e^-)$ is projected orthogonal to the subspace $\mathcal{W}_j$ into the $\mathcal{V}_j$. This example demonstrates that the state vector can grow and the switched system can become unstable only because of switching operations using method from [13].

The next example is illustrated in Figure 4. It shows a sketch of the time simulation of two reduced subsystems of order $q = 2$. The figure is depicted in the reduced state space in such a way that the local coordinate systems of the reduced subsystems, which are generally non-orthogonal and lie in different subspaces, are mapped in the same two-dimensional space as orthogonal coordinate systems. Additionally shown are level sets of the Lyapunov functions

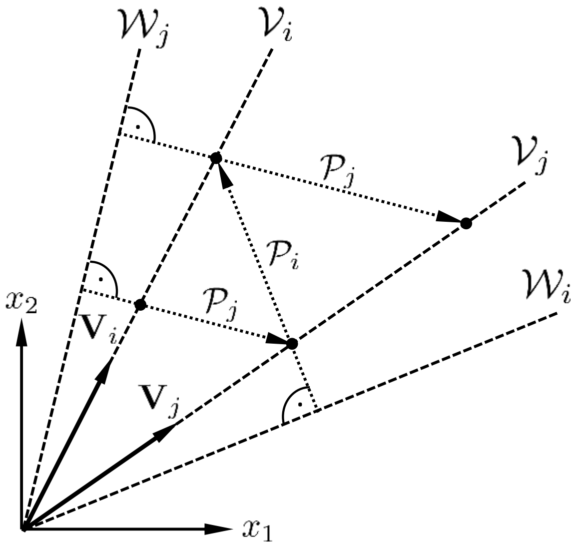

Figure 3: Calculation of new starting vectors with Petrov-Galerkin projections for an original 2-dimensional space and two subsystems of reduced order $q = 1$ [13].

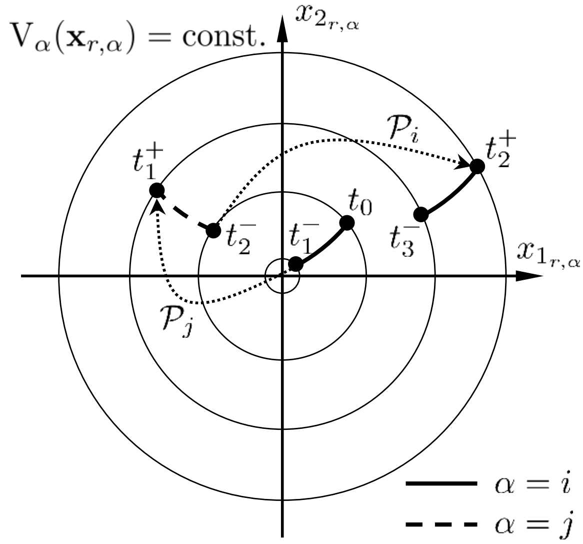

Figure 4: Sketch of a time simulation in the reduced state space with Petrov-Galerkin projection for two subsystems of reduced order $q = 2$ according to [13].

$\mathrm{V}_\alpha(\mathbf{x}_{r,\alpha}(t))$. According to (14) the level sets are circles. The subsystem i is simulated in the time interval $t \in [t_0, t_1]$. Therein, the Lyapunov function $\mathrm{V}_i(\mathbf{x}_{r,i}(t))$ decreases. At the time point t_1 it is switched to subsystem j. By projecting the state vector $x_{r,i}(t_1^-)$ to state vector $x_{r,j}(t_1^+)$ with the Petrov-Galerkin projection $\mathcal{P}_j$ the Lyapunov function jumps to a higher value. Afterwards the simulation of subsystem j is carried out until the time point t_2 and the Lyapunov function decays. The same procedure is done at time point t_2. Therefore the system becomes unstable just by switching operations.

The explanations above showed that the calculations in section 3.2 are not sufficient in order to guarantee asymptotic stability for the reduced switched system. In other words, the subsystems share the Lyapunov matrix and the switched systems would be asymptotically stable if they had the same state vectors. Nevertheless, switching might lead to instability as the state vector $\mathbf{x}_{r,\alpha}$ changes and hence the Lyapunov function might jump during the projection of the state vector to a higher value, where Theorem 3 is violated.

Owing to section 3.2 it is guaranteed that the Lyapunov function and the energy decreases for the individual subsystems. Additionally, the Lyapunov function of the reduced switched systems (6) is not allowed to increase during the switching operation at the time point t_e. This requirement is illustrated by the following inequality:

$$\mathrm{V}_j(\mathbf{x}_{r,j}(t_e^+)) \leq \mathrm{V}_i(\mathbf{x}_{r,i}(t_e^-)). \tag{16}$$

As a simple choice we demand that there is no jump of the Lyapunov functions during the switching operation:

$$\mathrm{V}_j(\mathbf{x}_{r,j}(t_e^+)) = \mathrm{V}_i(\mathbf{x}_{r,i}(t_e^-)). \tag{17}$$

Inserting the Lyapunov functions from (14) gives:

$$\mathbf{x}_{r,j}(t_e^+)^T \mathbf{x}_{r,j}(t_e^+) = \mathbf{x}_{r,i}(t_e^-)^T \mathbf{x}_{r,i}(t_e^-). \tag{18}$$

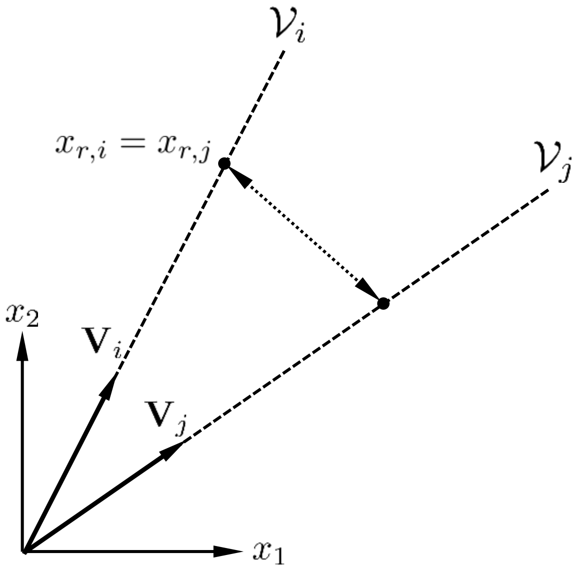

Figure 5: Calculation of new starting vectors with formula (19) for an original two-dimensional space and two subsystems of reduced order $q = 1$.

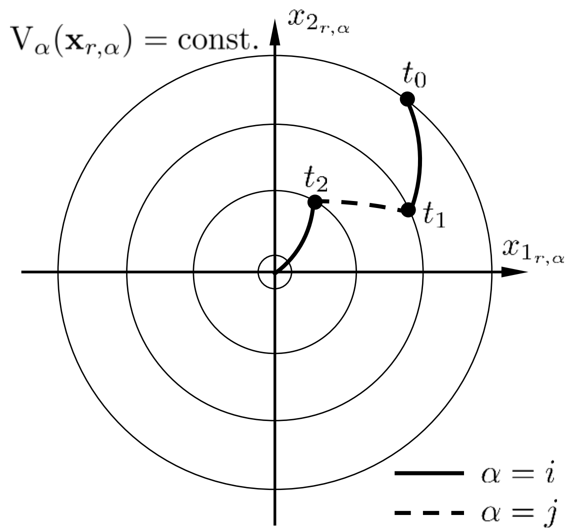

Figure 6: Sketch of a time simulation in reduced state space with formula (19) for two subsystems of reduced order $q = 2$.

Hence, owing to switching at the time point t_e the state vector of the new active subsystem is initialized by the state vector of the previously active subsystem with $\mathbf{R}_{j,i} = \mathbf{I} \in \mathbb{R}^{q \times q}$:

$$\mathbf{x}_{r,j}(t_e^+) = \mathbf{x}_{r,i}(t_e^-). \tag{19}$$

This kind of initialization guarantees stable switching but is not in agreement with projection-based model order reduction like Petrov-Galerkin (15) or Galerkin projection. Therefore the approximation of the original switched system might be less accurate at the time period after the switching operations. It is a conservative choice as it prevents the Lyapunov functions from jumping. The reduced switched system would be asymptotically stable as well if the Lyapunov functions jumped to smaller values during the switching operation.

With the result of this section, the two examples above shall be considered again. The instant switching between subsystems of reduced order $q = 1$ is plotted in Figure 5. As the reduced state vectors remain unchanged for infinitesimal time simulation between the switching points, the reduced switched system cannot become unstable. In Figure 6 the sketch of a time simulation of two reduced subsystems of order $q = 2$ is shown. The subsystem i is simulated in the time interval $t \in [t_0, t_1]$, where the Lyapunov function decreases. At the time point t_1 it is switched to subsystem j and the state vector of the subsystem j is initialized with formula (19). It can be seen that a jump of the Lyapunov function is impossible. Next the simulation of subsystem j is carried out until the time point t_2 and the Lyapunov function decays. Therefore the reduced switched system is asymptotically stable. It also becomes clear why it is important that the reduced subsystems share the same Lyapunov matrix. According to formula (14) the level sets of the Lyapunov functions $\mathrm{V}_\alpha(\mathbf{x}_{r,\alpha})$ are circles. Otherwise the level sets could be different ellipses and hence the reduced switched system could become unstable even if the initialization is done as proposed according to (19) as the Lyapunov function of subsystem i could decrease whereas the Lyapunov function of subsystem j grows.

4 Numerical Results

In this section a numerical example is given for the proposed method which shall be compared to the current methods from Diepold et al. [13] and Shaker et al. [12]. In the following a benchmark SISO model from [12] which consists of two subsystems of order 5 is used for evaluating the new method. The benchmark system is:

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_\alpha \mathbf{x}(t) + \mathbf{b}_\alpha \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{c}_\alpha^T \mathbf{x}(t), \end{cases} \tag{20}$$

with the first subsystem $\alpha = 1$:

$$\mathbf{A}_1 = \begin{pmatrix} -4.23 & 0.4654 & 1.305 & 0.313 & -1.461 \\ 0.4654 & -4.418 & 0.8745 & -0.9324 & -0.7062 \\ 1.305 & 0.8745 & -1.839 & -0.0083 & 0.6652 \\ 0.313 & -0.9324 & -0.0083 & -1.801 & -0.4979 \\ -1.461 & -0.7062 & 0.6652 & -0.4979 & -2.355 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} -0.1721 \\ -0.336 \\ 0.5415 \\ 0 \\ -0.5703 \end{pmatrix},$$
$$\mathbf{c}_1^T = \begin{pmatrix} -1.499 & -0.0503 & 0.553 & 0.0835 & 1.578 \end{pmatrix},$$

and the second subsystem $\alpha = 2$:

$$\mathbf{A}_2 = \begin{pmatrix} -5.055 & 0.4867 & 0.7761 & -3.765 & -2.702 \\ 0.4867 & -3.034 & 0.0537 & 0.6768 & 0.6030 \\ 0.7761 & 0.0537 & -1.392 & -0.0739 & 0.8858 \\ -3.765 & 0.6768 & -0.0739 & -5.26 & -1.886 \\ -2.702 & 0.603 & 0.8858 & -1.886 & -3.909 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -0.5081 \\ 0.8564 \\ 0.2685 \\ 0.625 \\ -1.047 \end{pmatrix},$$
$$\mathbf{c}_2^T = \begin{pmatrix} 1.536 & 0.4344 & -1.917 & 0 & 0 \end{pmatrix}.$$

The switched system which comprises the two subsystems is asymptotically stable as the subsystems share the same Lyapunov function $\mathrm{V}(\mathbf{x}) = \mathbf{x}^T\mathbf{P}_1\mathbf{x} = \mathbf{x}^T\mathbf{P}_2\mathbf{x} = \mathbf{x}^T\mathbf{I}\mathbf{x} = \mathbf{x}^T\mathbf{x}$.

In [12] the original switched system (20) is reduced to a switched system of order $q = 3$ applying the respective stability-preserving method. This results in a reduced switched system whose subsystems have the same reduced state vector $\mathbf{x}_r(t)$:

$$\Sigma_r : \begin{cases} \dot{\mathbf{x}}_r(t) = \mathbf{A}_{r,\alpha} \mathbf{x}_r(t) + \mathbf{b}_{r,\alpha} \mathbf{u}(t), \\ \mathbf{y}_r(t) = \mathbf{c}_{r,\alpha}^T \mathbf{x}(t). \end{cases} \tag{21}$$

The first reduced subsystem $\alpha = 1$ is:

$$\mathbf{A}_{r,1} = \begin{pmatrix} -1.031 & 0.0061 & -0.0811 \\ -0.1413 & -1.606 & 0.7891 \\ -0.1708 & 1.028 & -2.723 \end{pmatrix}, \quad \mathbf{b}_{r,1} = \begin{pmatrix} -0.4154 \\ 0.595 \\ 0.7314 \end{pmatrix},$$
$$\mathbf{c}_{r,1}^T = \begin{pmatrix} -0.2443 & -1.076 & 0.1176 \end{pmatrix},$$

and the second reduced subsystem $\alpha = 2$ is:

$$\mathbf{A}_{r,2} = \begin{pmatrix} -0.8714 & 0.0209 & 0.1824 \\ -0.153 & -1.652 & -0.864 \\ 0.0540 & -0.6046 & -2.7 \end{pmatrix}, \quad \mathbf{b}_{r,2} = \begin{pmatrix} 0.315 \\ 1.136 \\ 2.371 \end{pmatrix},$$
$$\mathbf{c}_{r,2}^T = \begin{pmatrix} 0.5949 & 0.5316 & -0.5847 \end{pmatrix}.$$

The reduced switched system of the proposed method and the method from Diepold et al. [13] consists of subsystems with different state vectors $\mathbf{x}_{r,\alpha}(t)$ and is of the form:

$$\Sigma_r : \begin{cases} \dot{\mathbf{x}}_{r,\alpha}(t) = \mathbf{A}_{r,\alpha}\mathbf{x}_{r,\alpha}(t) + \mathbf{b}_{r,\alpha}\mathbf{u}(t), \\ \mathbf{y}_{r,\alpha}(t) = \mathbf{c}_{r,\alpha}^T\mathbf{x}_{r,\alpha}(t). \end{cases} \tag{22}$$

We consider in the following various types of projection-based model order reduction methods and calculate reduced switched systems of order $q = 3$.

4.1 Comparison of stability

In order to compare the behavior of the different methods in terms of stability, a time-dependent switching signal with frequency 5Hz is used, which is depicted in Figure 7.

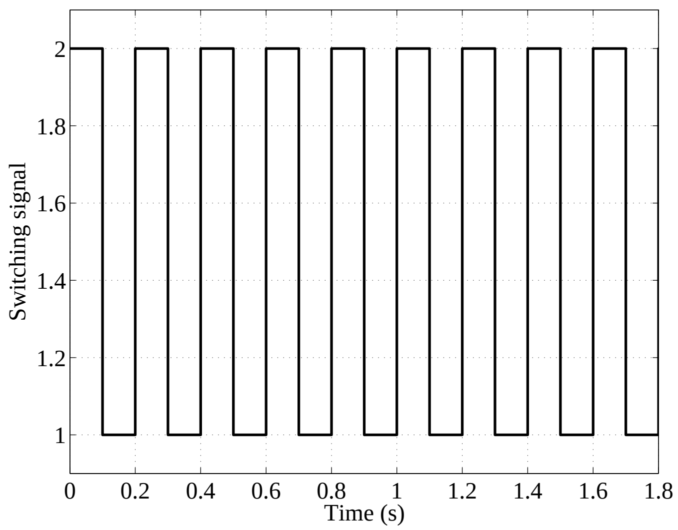

Figure 7: Time-dependent switching signal.

In the following only the methods which calculate different reduced state vectors – the proposed method and the method from [13] – are compared. In both cases the high-dimensional subsystems are reduced using TBR to subsystems of order $q = 3$. The Lyapunov function for the reduced switched system consists of the Lyapunov functions of the subsystems:

$$\mathrm{V}_\alpha(\mathbf{x}_{r,\alpha}(t)) = \begin{cases} \mathrm{V}_1(\mathbf{x}_{r,1}(t)) & \text{if} \quad \alpha = 1, \\ \mathrm{V}_2(\mathbf{x}_{r,2}(t)) & \text{if} \quad \alpha = 2. \end{cases} \tag{23}$$

In Figure 8 the value of the Lyapunov function of the reduced switched systems according to the method from [13] is shown for a time simulation with $\mathbf{u}(t) = 0$. The value of the Lyapunov function jumps to a higher level at the switching time points and decreases between the switching operations. Therefore, the reduced switched system is unstable for the switching signal from Figure 7. This is in accordance with Figure 4. In Figure 9 the value of the Lyapunov function is shown for the proposed method. The Lyapunov function decreases monotonically. Hence the reduced switched system is asymptotically stable, which agrees with Figure 6.

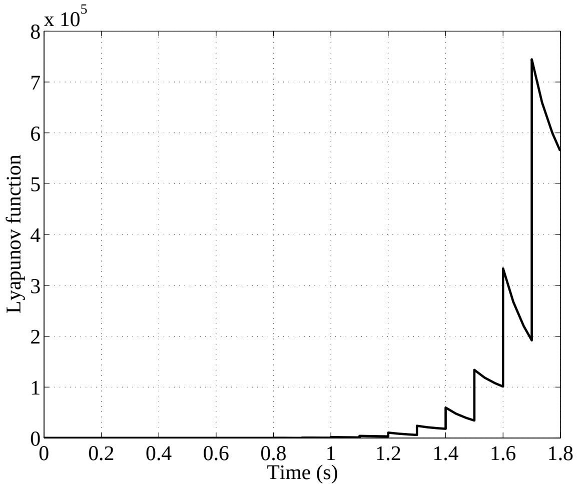

Figure 8: Development of the Lyapunov function of the low-dimensional switched system with the approach of [13].

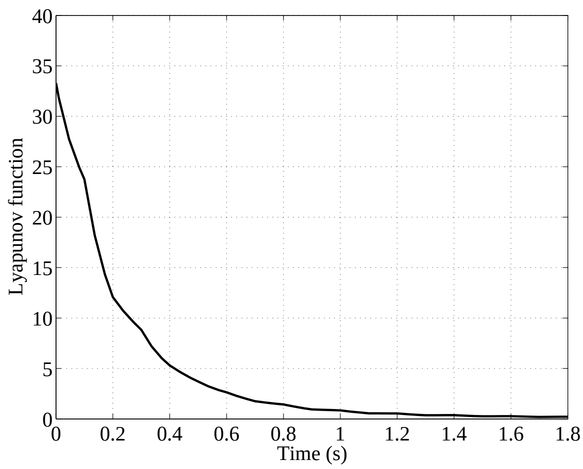

Figure 9: Development of the Lyapunov function of the low-dimensional switched system with the proposed approach.

4.2 Comparison of accuracy

For the comparison of the accuracy of the various methods a time-dependent switching signal, which is shown in Figure 10, is employed. The frequency of the signal decreases so that the methods can be evaluated for transient and steady-state behavior.

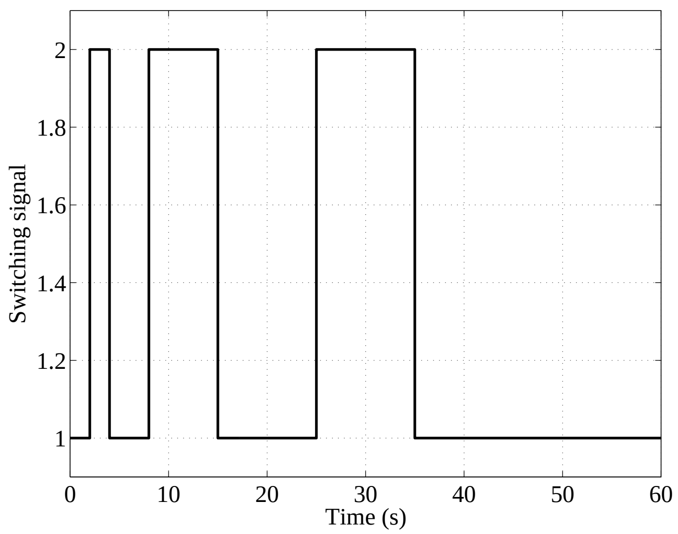

Figure 10: Time-dependent switching signal.

For the proposed method and the method from Diepold et al. [13] the subsystems were reduced using a one-sided Krylov subspace method with the reduced order $q = 3$ and an expansion point $s_0 = 0$. The step responses with $\mathbf{u}(t) = 1$ of the output of the original switched system $\mathbf{y}(t)$ and the different reduced switched systems $\mathbf{y}_r(t)$ (ROMs) are shown in Figure 11.
In terms of transient behavior, which means the accuracy directly after the switching operations, the proposed method delivers an inaccurate approximation of the original switched system. The reason is that condition (19) is rather conservative. The method from Diepold et al. [13] gives accurate results for switching from subsystem 1 to subsystem 2. There is a small error for

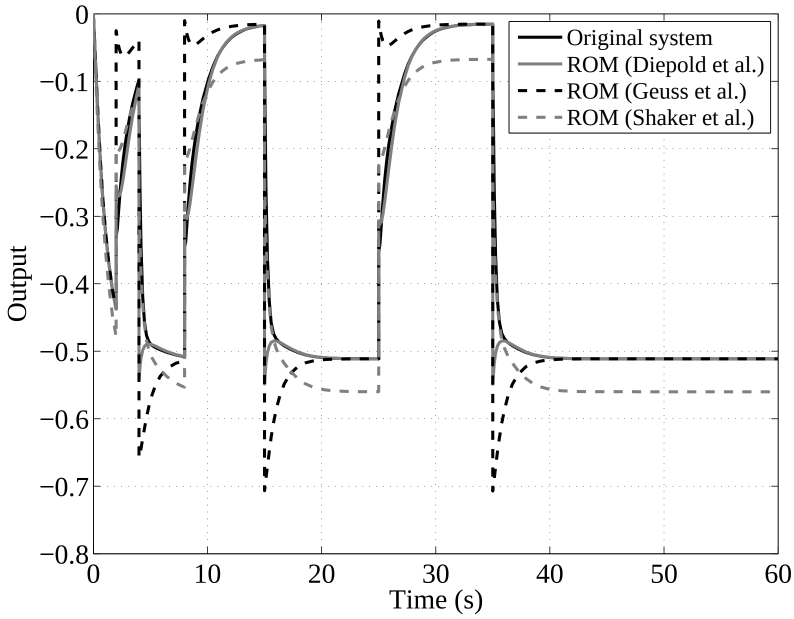

Figure 11: Step responses for the original switched system and various reduced switched systems (ROMs) for time-dependent switching.

switching from subsystem 2 to subsystem 1. The method from Shaker et al. [12] shows an accurate transient behavior except for a small error for switching from subsystem 1 to subsystem 2. In terms of steady-state behavior the method from Shaker et al. [12] is not stationary accurate. The method from Diepold et al. [13] and the proposed method deliver stationary accurate reduced switched systems as for both subsystems individual and, therefore, proper subspaces are chosen.

5 Conclusion and outlook

In this paper an approach has been proposed for stability-preserving model order reduction of switched systems. This approach is based on a method which uses different subspaces for the reduced subsystems and extends it to stability preservation for arbitrary switching. The proposed method needs the calculation of individual subspaces for the subsystems and the solution of low-dimensional Lyapunov equations. Additionally, the initialization of the reduced state vectors was proposed in such a way that the Lyapunov functions have the same value at the switching operation.

Although the proposed method delivers asymptotically stable reduced switched systems, the approximation becomes inaccurate for fast switching. In order to improve the accuracy at the switching time points less strict conditions concerning the Lyapunov function, which are at the moment quite conservative, might be found in step 3. Possible future work could also focus on step 2. In this paper the reduced subsystems share the identity matrix as Lyapunov matrix. Using optimization methods a common Lyapunov matrix could be found which delivers more accurate results.

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