Solution of the Peak Value Problem for Gabor's Theory of Communication

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***Abstract*—Since, for certain bounded signals, the common integral definition of the Hilbert transform may diverge, it was long thought that the Hilbert transform does not exist for general bounded signals. However, using a definition that is based on the $\mathcal{H}^1$-BMO$(\mathbb{R})$ duality, it is possible to define the Hilbert transform meaningfully for the space of bounded signals. Unfortunately, this abstract definition gives no constructive procedure for the calculation of the Hilbert transform. However, if the signals are additionally bandlimited, i.e., if we consider signals in $\mathcal{B}_\pi^\infty$, it was recently shown that an explicit formula for the calculation of the Hilbert transform does exist. Based on this result, we analyze the asymptotic growth behavior of the Hilbert transform of signals in $\mathcal{B}_\pi^\infty$ and solve the peak value problem of the Hilbert transform. It is shown that the order of growth of Hilbert transform of signals in $\mathcal{B}_\pi^\infty$ is at most logarithmic.**

I. Introduction

Classically, the Hilbert transform of a signal f is defined as the principal value integral

$$\begin{aligned}(Hf)(t) &= \frac{1}{\pi}\,\text{V.P.}\int_{-\infty}^{\infty}\frac{f(\tau)}{t-\tau}\mathrm{d}\tau\\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\int_{t-\frac{1}{\epsilon}}^{t-\epsilon}\frac{f(\tau)}{t-\tau}\mathrm{d}\tau+\int_{t+\epsilon}^{t+\frac{1}{\epsilon}}\frac{f(\tau)}{t-\tau}\mathrm{d}\tau\right). \quad (1)\end{aligned}$$

The above integral (1) can be used to define the Hilbert transform for more general spaces only if the integral converges for all signals from this space. However, the convergence of the integral is delicate and not guaranteed. In [1] the convergence of the integral (1) was analyzed for bounded bandlimited signals, and a subclass of signals for which it converges was identified. For bounded bandpass signals the Hilbert transform exists and is bounded. If f is a bounded bandpass signals, the distributional Fourier transform of which vanishes outside $[-\pi,-\epsilon\pi]\cup[\epsilon\pi,\pi]$, $0<\epsilon<1$, then f has a bounded Hilbert transform satisfying

$$\|Hf\|_\infty \le \left(A+\frac{2}{\pi}\log\left(\frac{1}{\epsilon}\right)\right)\|f\|_\infty, \quad (2)$$

where $A<4/\pi$ is a constant [2], [3]. This upper bound on the peak value of the Hilbert transform diverges as ϵ tends to zero. Probably, observations of this kind led to the conclusion "that an arbitrary bounded bandlimited function does not have a Hilbert transform..." [3]. Such a non-existence of the Hilbert transform for arbitrary bounded bandlimited signals would be problematic for all applications and theoretical concepts where the Hilbert transform of such signals is needed.

In several applications in communication theory and signal processing the Hilbert transform is an important operation. For example, the "analytical signal" [4], which was used in Dennis Gabor's "Theory of Communication" [5] and is nowadays a widely used concept in communications, is based on the Hilbert transform. Further, the Hilbert transform is a crucial component in the definition of the instantaneous amplitude and frequency of a signal [6], [7], the instantaneous phase of a signal [4], and the theory of modulation [3], [4].

According to the current state of the literature it is not clear whether all above concepts can be properly defined for general bounded bandlimited signals, because there is no constructive approach to define the Hilbert transform general bounded bandlimited signals. In this paper we use an extension of the Hilbert transform which is based on the $\mathcal{H}^1$-BMO$(\mathbb{R})$ duality [8], and develop a rigorous theory for the Hilbert transform of bounded bandlimited signals. As a consequence, the above concepts and ideas are well-defined for this space. We provide a constructive approach for the calculation of the Hilbert transform by giving an explicit formula. Based on this formula we further solve the peak-value problem of the Hilbert transform, and hence of the analytic signal, for the space of bounded bandlimited signals.

II. Notation

Let $\hat{f}$ denote the Fourier transform of a function f. $L^p(\mathbb{R})$, $1\le p<\infty$, is the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ is the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. For $0<\sigma<\infty$ let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon>0$ there exists a constant $C(\epsilon)$ with $|f(z)|\le C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z\in\mathbb{C}$. The Bernstein space $\mathcal{B}_\sigma^p$, $1\le p\le\infty$, consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., $\|\cdot\|_{\mathcal{B}_\sigma^p}=\|\cdot\|_p$. A signal in $\mathcal{B}_\sigma^p$ is called bandlimited to σ, and $\mathcal{B}_\sigma^\infty$ is the space of bandlimited signals that are bounded on the real axis. We call a signal in $\mathcal{B}_\pi^\infty$ bounded bandlimited signal.

III. The Operator Q

In this section we consider the LTI system $Q=DH$, which consists of the concatenation of the Hilbert transform H and the differential operator D, as an operator acting on $\mathcal{B}_\pi^2$. Since both operators $H:\mathcal{B}_\pi^2\to\mathcal{B}_\pi^2$ and $D:\mathcal{B}_\pi^2\to\mathcal{B}_\pi^2$ are stable

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LTI systems, $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$, as the concatenation of two stable LTI systems, is a stable LTI system. The system $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ has the frequency domain representation

$$(Qf)(t) = (DHf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_Q(\omega) \hat{f}(\omega) \, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega, \quad (3)$$

where

$$\hat{h}_Q(\omega) = \begin{cases} |\omega|, & |\omega| \le \pi \\ 0, & |\omega| > \pi. \end{cases}$$

Next, we show that the system $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ has also the mixed signal representation

$$(Qf)(t) = \sum_{k=-\infty}^{\infty} a_{-k} f(t-k), \quad (4)$$

where a_{-k}, $k \in \mathbb{Z}$, are certain coefficients, to be specified next. We call this representation mixed signal representation, because for a fixed $t \in \mathbb{R}$ we need the signal values on the discrete grid $\{t-k\}_{k \in \mathbb{Z}}$ in order to calculate $(Qf)(t)$. However, for different $t \in \mathbb{R}$ we need other signal values in general. As t ranges over $[0,1]$ we need all the signal values $f(\tau)$, $\tau \in \mathbb{R}$. The mixed signal representation (4) will be important for the results in Section IV, where we extend the Hilbert transform to $\mathcal{B}_\pi^\infty$.

In order to see the validity of the mixed signal representation (4), we consider the Fourier series of the 2π-periodic extension of $\hat{h}_Q$, i.e.,

$$\sum_{k=-\infty}^{\infty} a_k \, \mathrm{e}^{i\omega k}, \quad (5)$$

where the coefficients a_k, $k \in \mathbb{Z}$, are given by

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\omega| \, \mathrm{e}^{i\omega k} \, \mathrm{d}\omega = \begin{cases} \frac{\pi}{2}, & k = 0, \\ \frac{(-1)^k - 1}{\pi k^2}, & k \neq 0. \end{cases} \quad (6)$$

Since,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |a_k| &= \frac{\pi}{2} + 2 \sum_{k=1}^{\infty} \frac{|(-1)^k - 1|}{\pi k^2} \\ &= \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \\ &= \pi, \end{aligned} \quad (7)$$

we see that the Fourier series (5) is absolutely and uniformly convergent. Hence, starting with the frequency domain representation (3) of Qf, we obtain

$$\begin{aligned} (Qf)(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_Q(\omega) \, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \left(\sum_{k=-\infty}^{\infty} a_k \, \mathrm{e}^{i\omega k} \right) \mathrm{e}^{i\omega t} \, \mathrm{d}\omega \\ &= \sum_{k=-\infty}^{\infty} a_{-k} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \, \mathrm{e}^{i\omega(t-k)} \, \mathrm{d}\omega \\ &= \sum_{k=-\infty}^{\infty} a_{-k} f(t-k) \end{aligned} \quad (8)$$

for all $t \in \mathbb{R}$. The exchange of the integral and the sum in (8) was justified because the series (5) is uniformly convergent on $[-\pi, \pi]$.

A. Extension of Q to $\mathcal{B}_\pi^\infty$

So far we have considered the LTI system Q only acting on signals in $\mathcal{B}_\pi^2$. Next, we extend Q to a bounded operator Q^{E} acting on the larger space $\mathcal{B}_\pi^\infty$ of bandlimited signals that are bounded on the real axis. For the operator $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ we had the representations (3) and (4). However, the frequency domain representations which involves the Fourier transform of the signal makes no sense for signals in $\mathcal{B}_\pi^\infty$. The next theorem shows that the mixed signal representation (4) is still meaningful for signals in $\mathcal{B}_\pi^\infty$, because

$$Q^{\mathrm{E}} f = \sum_{k=-\infty}^{\infty} a_{-k} f(t-k), \quad (9)$$

is also a valid representation of the operator $Q^{\mathrm{E}} : \mathcal{B}_\pi^\infty \to \mathcal{B}_\pi^\infty$.

Theorem 1. *There exists a bounded linear operator $Q^E : \mathcal{B}_\pi^\infty \to \mathcal{B}_\pi^\infty$ with norm $\|Q^E\| = \pi$ that coincides with Q on $\mathcal{B}_\pi^2$, i.e., that satisfies $Q^E f = Qf$ for all $f \in \mathcal{B}_\pi^2$.*

Proof: We will show that the mixed signal representation (9), where the coefficients a_k are defined as in (6), defines a bounded linear operator that maps $\mathcal{B}_\pi^\infty$ into $\mathcal{B}_\pi^\infty$, and thus gives us the desired extension. The linearity of Q^{E} is obvious. It remains to show that the norm of Q^{E} satisfies $\|Q^{\mathrm{E}}\| = \pi$. Since $|(Q^{\mathrm{E}} f)(t)| \le \|f\|_\infty \sum_{k=-\infty}^{\infty} |a_k|$ for all $t \in \mathbb{R}$, it follows from (7) that $\|Q^{\mathrm{E}} f\|_\infty \le \pi \|f\|_\infty$. Moreover, for $t \in \mathbb{R}$, we have

$$\begin{aligned} (Q^{\mathrm{E}} \, \mathrm{e}^{i\pi \,\cdot\,})(t) &= \sum_{k=-\infty}^{\infty} a_{-k} \, \mathrm{e}^{i\pi(t-k)} \\ &= \sum_{k=-\infty}^{\infty} a_{-k} \, \mathrm{e}^{-i\pi k} \, \mathrm{e}^{i\pi t} \\ &= \hat{h}_Q(\pi) \, \mathrm{e}^{i\pi t} \\ &= \pi \, \mathrm{e}^{i\pi t}, \end{aligned}$$

which shows that $\|Q^{\mathrm{E}}\| \ge \|Q^{\mathrm{E}} \, \mathrm{e}^{i\pi \,\cdot\,}\|_\infty = \pi$. Thus, we have $\|Q^{\mathrm{E}}\| = \pi$. ■

IV. The Hilbert Transform for $\mathcal{B}_\pi^\infty$

Despite the convergence problems discussed in Section I, there is a way to define the Hilbert transform for signals in $\mathcal{B}_\pi^\infty$. This definition uses Fefferman's duality theorem, which states that the dual space of $\mathcal{H}^1$ is $\mathrm{BMO}(\mathbb{R})$ [9]. In addition to this rather abstract definition, we will also give a constructive procedure for the calculation of the Hilbert transform. We briefly review some definitions.

Definition 1. The space $\mathcal{H}^1$ denotes the Hardy space of all signals $f \in L^1(\mathbb{R})$ for which $Hf \in L^1(\mathbb{R})$. It is a Banach space endowed with the norm $\|f\|_{\mathcal{H}^1} := \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}$.

Definition 2. A function $f : \mathbb{R} \to \mathbb{C}$ is said to belong to $\text{BMO}(\mathbb{R})$, provided that it is locally in $L^1(\mathbb{R})$ and $\frac{1}{\mu(I)} \int_I |f(t) - m_I(f)| \mathrm{d}t \leq C_1$ for all bounded intervals I, where $m_I(f) := \frac{1}{\mu(I)} \int_I f(t) \mathrm{d}t$ and the constant C_1 is independent of I. μ denotes the Lebesgue measure.

For our further examinations, we need the important fact that the dual space of $\mathcal{H}^1$ is $\text{BMO}(\mathbb{R})$ [10, p. 245]. In order to state this duality, we use the space $\mathcal{H}^1_{\mathrm{D}} = \mathcal{H}^1 \cap \mathcal{S}$, which is dense in $\mathcal{H}^1$. By $\mathcal{S}$ we denote the usual Schwartz space of functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and fulfill $\sup_{t\in\mathbb{R}} |t^a \phi^{(b)}(t)| < \infty$ for all $a, b \in \mathbb{N} \cup \{0\}$.

Theorem 2 (Fefferman)**.** *Suppose $f \in BMO(\mathbb{R})$. Then the linear functional $\mathcal{H}^1_D \to \mathbb{C}$, $\phi \mapsto \int_{-\infty}^{\infty} f(t)\phi(t) \mathrm{d}t$ has a bounded extension to $\mathcal{H}^1$. Conversely, every continuous linear functional L on $\mathcal{H}^1$ is created in this way by a function $f \in BMO(\mathbb{R})$, which is unique up to an additive constant.*

The function $f \in \text{BMO}(\mathbb{R})$ in Theorem 2 is only unique up to an additive constant, because $\phi \in \mathcal{H}^1$ implies $\int_{-\infty}^{\infty} \phi(t) \mathrm{d}t = 0$. Therefore, it will be beneficial to identify two functions in $\text{BMO}(\mathbb{R})$ that differ only by a constant. We do this by introducing the equivalence relation $\sim$ on $\text{BMO}(\mathbb{R})$. We write $f \sim g$ if and only if $f(t) = g(t) + C_{\text{BMO}}$ for almost all $t \in \mathbb{R}$, where C_{BMO} is a constant. By $[f]$ we denote the equivalence class $[f] = \{g \in \text{BMO}(\mathbb{R}) : g \sim f\}$, and $\text{BMO}(\mathbb{R})/\mathbb{C}$ is the set of all equivalence classes in $\text{BMO}(\mathbb{R})$.

A possible extension of the Hilbert transform, which is based on the $\mathcal{H}^1$-$\text{BMO}(\mathbb{R})$ duality is given in the next definition [8].

Definition 3. We define the Hilbert transform $\mathfrak{H}f$ of $f \in L^\infty(\mathbb{R})$ to be the function in $\text{BMO}(\mathbb{R})/\mathbb{C}$ that generates the linear continuous functional

$$\langle \mathfrak{H}f, \phi \rangle = \int_{-\infty}^{\infty} f(t)(H\phi)(t) \mathrm{d}t, \quad \phi \in \mathcal{H}^1.$$

Note that this definition is very abstract, because it gives no information how to calculate the Hilbert transform $\mathfrak{H}f$. However, in [11] it was shown that for bandlimited signals in $f \in \mathcal{B}^\infty_\pi$ it is possible to explicitly calculate the Hilbert transform $\mathfrak{H}f$. Next, we will give this formula, which is based on the Q-operator from Section III.

We have seen that $Q^{\mathrm{E}} : \mathcal{B}^\infty_\pi \to \mathcal{B}^\infty_\pi$ is a bounded linear operator. Hence, for every $f \in \mathcal{B}^\infty_\pi$, the operator $\mathfrak{I}$ given by

$$(\mathfrak{I}f)(t) = \int_0^t (Q^{\mathrm{E}} f)(\tau) \mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{10}$$

is well defined. Since the operator $Q : \mathcal{B}^2_\pi \to \mathcal{B}^2_\pi$, as an operator on $\mathcal{B}^2_\pi$, was defined to be the concatenation of the Hilbert transform H and the differential operator D, it is clear that, for $g \in \mathcal{B}^2_\pi$, the integral of Qg as in (10) gives—up to a constant—the Hilbert transform Hg of g. For $g \in \mathcal{B}^2_\pi$ we have

$$\begin{aligned}(\mathfrak{I}g)(t) &= \int_0^t (Q^{\mathrm{E}} g)(\tau) \mathrm{d}\tau \\ &= \int_0^t (Qg)(\tau) \mathrm{d}\tau \\ &= \int_0^t (DHg)(\tau) \mathrm{d}\tau \\ &= (Hg)(t) - (Hg)(0),\end{aligned} \tag{11}$$

i.e., for every signal $g \in \mathcal{B}^2_\pi$, we have $(Hg)(t) = (\mathfrak{I}g)(t) + C_2(g)$, $t \in \mathbb{R}$, where $C_2(g)$ is a constant that depends on g.

Based on this observation one could conjecture that, for signals $f \in \mathcal{B}^\infty_\pi$, the integral $\mathfrak{I}f$ is somehow connected to the Hilbert transform $\mathfrak{H}f$ of f. In [11], it was shown that such a connection exists in the sense that $\mathfrak{I}f$ is a representative of the equivalence class $\mathfrak{H}f$.

Theorem 3. *Let $f \in \mathcal{B}^\infty_\pi$. Then we have $\mathfrak{H}f = [\mathfrak{I}f]$.*

Theorem 3 is very useful, because it enables us to compute the Hilbert transform of bounded bandlimited signals in $\mathcal{B}^\infty_\pi$ by using the constructive formula (10), instead of using the abstract Definition 3. This result is also the key for solving the peak vale problem of the Hilbert transform.

V. Peak Value Problem

The peak value of signals is important for many applications, e.g., for the hardware design in mobile communications. In the peak value problem we are interested in $\max_{|t|\leq T} |f(t)|$, i.e., in the peak value of a signal f on the interval $[-T, T]$. Next, we study the Hilbert transform of signals in $\mathcal{B}^\infty_\pi$, in particular its growth behavior on the real axis, and thereby solve the peak value problem for the Hilbert transform.

For all $f \in \mathcal{B}^\infty_\pi$, we have the upper bound

$$\begin{aligned}|(\mathfrak{I}f)(t)| &\leq \int_0^t |(Q^{\mathrm{E}} f)(\tau)| \mathrm{d}\tau \\ &\leq \|Q^{\mathrm{E}} f\|_\infty |t| \\ &\leq \pi \|f\|_\infty |t|,\end{aligned} \tag{12}$$

which shows that the asymptotic growth of the Hilbert transform $\mathfrak{H}f$ of signals $f \in \mathcal{B}^\infty_\pi$ is at most linear. More precisely, for all $f \in \mathcal{B}^\infty_\pi$ there exists a signal $g \in \text{BMO}(\mathbb{R})$ such that $\mathfrak{H}f = [g]$ and $g(t) = O(t)$.

On the other hand, it was shown in [12] that for the signal

$$f_1(t) = -\frac{2}{\pi} \int_0^\pi \frac{\sin(\omega t)}{\omega} \mathrm{d}\omega,$$

which is in $\mathcal{B}^\infty_\pi$, we have

$$|(\mathfrak{I}f_1)(t)| \geq \frac{2}{\pi} \left(\log(|t|) - \frac{\pi^2}{4} - 1 - \frac{1}{\pi} \right) \tag{13}$$

for all $t \in \mathbb{R}$ with $|t| \geq 1$. Thus, there are signals $f \in \mathcal{B}^\infty_\pi$, such that the growth of the Hilbert transform $\mathfrak{H}f$ is logarithmic, in the sense that there exists a signal $g \in \text{BMO}(\mathbb{R})$ such that $\mathfrak{H}f = [g]$ and $g(t) = \Omega(\log(t))$.

From this the question arises whether the asymptotically logarithmic growth is actually the maximum possible growth, i.e., whether the upper bound (12) can be improved. The next theorem gives a positive answer.

Theorem 4. *There exist two positive constants C_3 and C_4 such that for all $f \in \mathcal{B}_\pi^\infty$ and all $|t| \geq 2$ we have*

$$|(\mathfrak{I}f)(t)| \leq C_3 \log(1+|t|)\|f\|_\infty + C_4\|f\|_\infty.$$

Proof: Let $f \in \mathcal{B}_\pi^\infty$ be arbitrary but fixed. For $0 < \epsilon < 1$ consider the functions

$$f_\epsilon(t) = f((1-\epsilon)t)\frac{\sin(\epsilon\pi t)}{\epsilon\pi t}, \quad t \in \mathbb{R}.$$

We have $f_\epsilon \in \mathcal{B}_\pi^2$ and $\|f_\epsilon\|_\infty \leq \|f\|_\infty$ for all $0 < \epsilon < 1$, as well as $\lim_{\epsilon\to 0} f_\epsilon(t) = f(t)$ for all $t \in \mathbb{R}$, where the convergence is locally uniform.

First, we show that

$$(\mathfrak{I}f)(t) = \lim_{\epsilon\to 0}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau \tag{14}$$

for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ and $\delta > 0$ be arbitrary but fixed. Due to (7), there exists a $k_0 = k_0(\delta)$ such that

$$\sum_{|k|\geq k_0} |a_{-k}| < \frac{\delta}{3\|f\|_\infty}. \tag{15}$$

Further, there exists a $\epsilon_0 = \epsilon_0(\delta)$ such that

$$\max_{|k|<k_0} |f(t-k) - f_\epsilon(t-k)| \sum_{|k|<k_0} |a_{-k}| < \frac{\delta}{3} \tag{16}$$

for all $0 < \epsilon \leq \epsilon_0$. Using the pointwise convergence of the mixed signal representation for signals in $\mathcal{B}_\pi^\infty$, we have, for $0 < \epsilon \leq \epsilon_0$, that

$$\begin{aligned}
&|(Q^{\mathrm{E}}f)(t) - (Qf_\epsilon)(t)| \\
&\quad = \Bigg| \sum_{|k|<k_0} a_{-k} f(t-k) - \sum_{|k|<k_0} a_{-k} f_\epsilon(t-k) \\
&\qquad + \sum_{|k|\geq k_0} a_{-k} f(t-k) - \sum_{|k|\geq k_0} a_{-k} f_\epsilon(t-k) \Bigg| \\
&\quad \leq \Bigg| \sum_{|k|<k_0} a_{-k}(f(t-k) - f_\epsilon(t-k)) \Bigg| \\
&\qquad + \|f\|_\infty \sum_{|k|\geq k_0} |a_{-k}| + \|f_\epsilon\|_\infty \sum_{|k|\geq k_0} |a_{-k}| \\
&\quad < \max_{|k|<k_0} |f(t-k) - f_\epsilon(t-k)| \sum_{|k|<k_0} |a_{-k}| + \frac{2\delta}{3} \\
&\quad < \delta,
\end{aligned}$$

where we used (15) in the second to last inequality and (16) in the last inequality. It follows that

$$(Q^{\mathrm{E}}f)(t) = \lim_{\epsilon\to 0}(Qf_\epsilon)(t) \tag{17}$$

for all $t \in \mathbb{R}$, because $t \in \mathbb{R}$ was arbitrary. Further, since

$$\begin{aligned}
\|Qf_\epsilon\|_\infty &= \|Q^{\mathrm{E}} f_\epsilon\|_\infty \\
&\leq \|Q^{\mathrm{E}}\| \, \|f_\epsilon\|_\infty \\
&\leq \pi\|f\|_\infty,
\end{aligned}$$

we can apply Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned}
(\mathfrak{I}f)(t) &= \int_0^t (Q^{\mathrm{E}}f)(\tau)\mathrm{d}\tau \\
&= \int_0^t \lim_{\epsilon\to 0}(Qf_\epsilon)(\tau)\mathrm{d}\tau \\
&= \lim_{\epsilon\to 0}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau
\end{aligned}$$

for all $t \in \mathbb{R}$, which proves (14).

Equality (14) is a key observation. Due to the representation (10) and the properties of the operator Q, we can work with $\mathcal{B}_\pi^2$-functions in the following.

Next, we analyze the integral on the right-hand side of (14). We restrict ourselves to the case $t \geq 2$, the case $t \leq 2$ is treated analogously. Let $t \geq 2$ be arbitrary but fixed. Using (3), i.e., the frequency domain representation of Q, we obtain

$$\begin{aligned}
\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau &= \int_0^t \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega| \hat{f}_\epsilon(\omega)\, \mathrm{e}^{i\omega\tau}\, \mathrm{d}\omega\mathrm{d}\tau \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega|\hat{f}_\epsilon(\omega) \int_0^t \mathrm{e}^{i\omega\tau}\, \mathrm{d}\tau\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega|\hat{f}_\epsilon(\omega)\frac{\mathrm{e}^{i\omega t} - 1}{i\omega}\mathrm{d}\omega.
\end{aligned} \tag{18}$$

The order of integration was exchanged according to Fubini's theorem, which can be applied because

$$\begin{aligned}
\int_0^t \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega||\hat{f}_\epsilon(\omega)|\mathrm{d}\omega\mathrm{d}\tau &\leq |t|\pi\|f\|_{\mathcal{B}_\pi^2} \\
&< \infty.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega|\hat{f}_\epsilon(\omega)\frac{\mathrm{e}^{i\omega t} - 1}{i\omega}\mathrm{d}\omega \\
&\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{-i\,\mathrm{sgn}(\omega)\hat{\phi}(\omega)}_{=\hat{u}(\omega)} \hat{f}_\epsilon(\omega)\, \mathrm{e}^{i\omega t}\, \mathrm{d}\omega \\
&\qquad - \frac{1}{2\pi}\int_{-\pi}^{\pi} -i\,\mathrm{sgn}(\omega)\hat{\phi}(\omega)\hat{f}_\epsilon(\omega)\mathrm{d}\omega,
\end{aligned} \tag{19}$$

where the function

$$\hat{\phi}(\omega) = \begin{cases} 1, & |\omega| \leq \pi, \\ 2 + |\omega|/\pi, & \pi < |\omega| < 2\pi, \\ 0, & |\omega| \geq 2\pi, \end{cases}$$

was inserted without altering the integrals, because $\hat{\phi}(\omega) = 1$ for $\omega \in [-\pi, \pi]$. Using the abbreviation $\hat{u}(\omega) =$

$-i\,\mathrm{sgn}(\omega)\hat{\phi}(\omega)$ and applying the generalized Parseval equality, we obtain from (18) and (19) that

$$\begin{aligned}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau &= \int_{-\infty}^{\infty} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau - \int_{-\infty}^{\infty} f_\epsilon(\tau)u(-\tau)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau + \int_{-\infty}^{\infty} f_\epsilon(\tau)u(\tau)\mathrm{d}\tau, \end{aligned} \tag{20}$$

where u is given by

$$u(\tau) = \frac{1}{\pi\tau} + \frac{\sin(\pi\tau)-\sin(2\pi\tau)}{(\pi\tau)^2}, \quad \tau \in \mathbb{R}.$$

Dividing the integration range of the first and the second integral in (20) into three parts gives

$$\begin{aligned}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau &= \overbrace{\int_{|\tau|\le 1} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau}^{=(A_1)} + \overbrace{\int_{|\tau-t|\le 1} \ldots\,\mathrm{d}\tau}^{=(A_2)} + \overbrace{\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \ldots\,\mathrm{d}\tau}^{=(A_3)} \\ &+ \underbrace{\int_{|\tau|\le 1} f_\epsilon(\tau)u(\tau)\mathrm{d}\tau}_{=(B_1)} + \underbrace{\int_{|\tau-t|\le 1} \ldots\,\mathrm{d}\tau}_{=(B_2)} + \underbrace{\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \ldots\,\mathrm{d}\tau}_{=(B_3)}.\end{aligned}$$

For (A_1) we have

$$\begin{aligned}|(A_1)| &= \left|\int_{|\tau|\le 1} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau\right| \\ &\le \int_{|\tau|\le 1} |f_\epsilon(\tau)|\,|u(t-\tau)|\mathrm{d}\tau \\ &\le 2\|f_\epsilon\|_\infty \|u\|_\infty.\end{aligned}$$

The same calculation shows that

$$\begin{aligned}|(A_2)| &\le 2\|f_\epsilon\|_\infty \|u\|_\infty, \\ |(B_1)| &\le 2\|f_\epsilon\|_\infty \|u\|_\infty,\end{aligned}$$

and

$$|(B_2)| \le 2\|f_\epsilon\|_\infty \|u\|_\infty.$$

It remains to analyze $(A_3)+(B_3)$. We have

$$\begin{aligned}|(A_3)+(B_3)| &= \left|\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} f_\epsilon(\tau)(u(t-\tau)+u(\tau))\mathrm{d}\tau\right| \\ &\le \|f_\epsilon\|_\infty \left(\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \left|\frac{1}{\pi(t-\tau)} + \frac{1}{\pi\tau}\right|\mathrm{d}\tau \right. \\ &\qquad \left. + \int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \left(\frac{2}{\pi(t-\tau)^2} + \frac{2}{\pi\tau^2}\right)\mathrm{d}\tau\right) \\ &\le \|f_\epsilon\|_\infty \left(\frac{1}{\pi}\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \frac{|t|}{|t-\tau|\,|\tau|}\mathrm{d}\tau + \frac{8}{\pi}\right),\end{aligned}$$

because

$$\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \left(\frac{2}{\pi(t-\tau)^2} + \frac{2}{\pi\tau^2}\right)\mathrm{d}\tau \le \frac{8}{\pi}.$$

As for the remaining integral, we have

$$\begin{aligned}&\int_{\substack{|\tau-t|\ge 1\\|\tau|\ge 1}} \frac{|t|}{|t-\tau|\,|\tau|}\mathrm{d}\tau \\ &= \int_{-\infty}^{-1} \frac{t}{(\tau-t)\tau}\mathrm{d}\tau + \int_1^{t-1} \frac{t}{(t-\tau)\tau}\mathrm{d}\tau + \int_{t+1}^{\infty} \frac{t}{(\tau-t)\tau}\mathrm{d}\tau \\ &= \int_1^{t-1} \frac{t}{(t-\tau)\tau}\mathrm{d}\tau + 2\int_{t+1}^{\infty} \frac{t}{(\tau-t)\tau}\mathrm{d}\tau.\end{aligned}$$

Since

$$\begin{aligned}\int_{t+1}^{\infty} \frac{t}{(\tau-t)\tau}\mathrm{d}\tau &= \lim_{M\to\infty}\int_{t+1}^{M} \left(\frac{1}{\tau-t} - \frac{1}{\tau}\right)\mathrm{d}\tau \\ &= \lim_{M\to\infty}\left(\log\left(\frac{M-t}{M}\right) + \log(t+1)\right) \\ &= \log(t+1)\end{aligned}$$

and

$$\begin{aligned}\int_1^{t-1} \frac{t}{(t-\tau)\tau}\mathrm{d}\tau &= \int_1^{t-1} \frac{1}{t-\tau} + \frac{1}{\tau}\mathrm{d}\tau \\ &= 2\log(t-1),\end{aligned}$$

we obtain

$$|(A_3)+(B_3)| \le \|f_\epsilon\|_\infty \frac{4}{\pi}(\log(1+t)+2).$$

Combining the partial results gives

$$\begin{aligned}\left|\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau\right| &\le 8\|f_\epsilon\|_\infty\|u\|_\infty + \|f_\epsilon\|_\infty \frac{4}{\pi}(\log(1+t)+2) \\ &= C_3\log(1+t)\|f_\epsilon\|_\infty + C_4\|f_\epsilon\|_\infty \\ &\le C_3\log(1+t)\|f\|_\infty + C_4\|f\|_\infty.\end{aligned}$$

Finally, the assertion follows from (14). ■

Remark 1. The growth result in Theorem 4 implies that

$$\int_{-\infty}^{\infty} \frac{|(\mathfrak{I}f)(t)|^2}{(1+t^2)^\alpha}\mathrm{d}t < \infty$$

for all $\alpha > 1/2$. This shows that, for all $f \in \mathcal{B}_\pi^\infty$, the Hilbert transform $\mathfrak{H}f$ is in the Zakai class [13], in the sense that there exists a signal g in the Zakai class, satisfying $\mathfrak{H}f = [g]$.

A direct consequence of Theorem 4 is the following corollary, which solves the peak value problem.

Corollary 1. *For all $f \in \mathcal{B}_\pi^\infty$ there exists a constant $C_5 = C_5(f)$ such that for all $T > 2$ we have*

$$\max_{|t|\le T} |(\mathfrak{I}f)(t)| \le C_5\log(1+T). \tag{21}$$

Note that if we replace $\mathfrak{I}f$ in (21) with $\mathfrak{H}f$, the constant C_5 will also depend on the on the actual representative of $\mathfrak{H}f$.

Without proof, we present two further results about the peak value of the Hilbert transform for signals in the space $\mathcal{B}_{\pi,0}^\infty$, which consists of $\mathcal{B}_\pi^\infty$-signals f that vanish on the real axis at infinity, i.e., satisfy $\lim_{|t|\to\infty}|f(t)| = 0$.

Theorem 5. *For all $f \in \mathcal{B}_{\pi,0}^\infty$ we have*

$$\lim_{T\to\infty} \frac{1}{\log(1+T)} \max_{|t|\leq T}|(\mathfrak{H}f)(t)| = 0.$$

Theorem 6. *Let ϕ be an arbitrary function with $\lim_{t\to\infty}\phi(t) = 0$. Then there exists a signal $f_2 \in \mathcal{B}_{\pi,0}^\infty$ such that*

$$\limsup_{T\to\infty} \frac{1}{\phi(T)\log(1+T)} \max_{|t|\leq T}|(\mathfrak{H}f_2)(t)| = \infty.$$

Theorems 5 and 6 show that, for the space $\mathcal{B}_{\pi,0}^\infty$, the peak value of the Hilbert transform grows not as fast as $\log(1+T)$ but not "substantially" slower.

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