The information-flux method: a Petrov-Galerkin formulation based on maximum-entropy methods applied to convection-dominated problems

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Introduction

Convection-dominated convection-diffusion problems

Dominating convection can cause severe instabilities in flow computations. Such instabilities were already investigated for Finite Element Methods (FEMs) and various stabilizations, such as the Streamline Upwind Petrov-Galerkin (SUPG) method, were proposed, see e.g. [1]. However, for other methods like meshfree methods, no stabilization has been established so far.

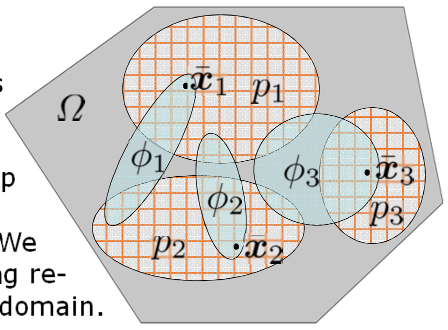
Maximum-entropy (max-ent) methods are least-biased schemes with some favourable properties, see e.g. [2]. For the convection-diffusion problem

$$\{-\kappa\Delta u(\boldsymbol{x}) + \boldsymbol{a}\cdot\nabla u(\boldsymbol{x}) = f(\boldsymbol{x}) | \boldsymbol{x}\in\Omega, u = 0 \text{ on } \partial\Omega\},$$

we developed a the stable *information-flux (inf-flux)* method and presented it, e.g. in [3,4]. Stability is achieved by incorporating information about analytical solutions of the homogeneous problem into the weighting function.

Information theory

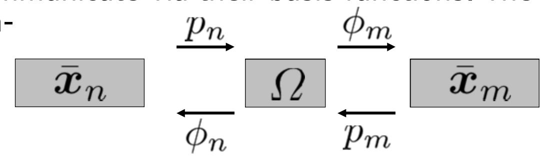
Communication channel



Let us consider a domain Ω with three nodes $\boldsymbol{x}_{1-3}$. The nodes communicate information to the domain via the corresponding solution basis functions p_{1-3}. Here, p_3 does not overlap with the weighting basis functions $\phi_{1,2}$. Thus, the nodal values $\bar{u}_{1,2}$ are independent of $\bar{u}_3$. We identify the weighting basis functions as being responsible for receiving information from the domain.

$$\begin{bmatrix} (\phi_1,\mathcal{L}p_1)_\Omega & (\phi_1,\mathcal{L}p_2)_\Omega & 0 \\ (\phi_2,\mathcal{L}p_1)_\Omega & (\phi_2,\mathcal{L}p_2)_\Omega & 0 \\ (\phi_3,\mathcal{L}p_1)_\Omega & (\phi_3,\mathcal{L}p_2)_\Omega & (\phi_3,\mathcal{L}p_3)_\Omega \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}$$

Altogether, we interpret the computational domain Ω as a communication channel through which the nodes communicate via their basis functions. The choice of the basis functions is fundamentally biasing the communication and thus also the results. Like e.g. in [2], the motivation to use max-ent schemes is to minimize this bias.



Maximum-entropy basis functions

Select least-biased local solution basis functions

Following [2], we want the solution basis functions to be non-negative and to fulfil the first-order consistency condition. This allows for an interpretation of $\{p_i\}$ as probability distributions. Other than that, we minimize the amount of information with the help of the information entropy. We determine a least-biased set of functions subject to aforementioned conditions:

$$\{p_i\} = \left\{ \arg\max -\sum_i p_i \ln\frac{p_i}{q_i} \middle| \sum_i p_i = 1, \sum_i p_i \bar{\boldsymbol{x}}_i = \boldsymbol{x} \right\}$$

In [2], this was shown to be an efficient to solve convex optimization problem. We obtain a convex approximation scheme with favourable properties like the weak Kronecker-δ property, for which we can, e.g. efficiently impose boundary conditions. q_i is the so-called prior function, a first estimate with which we can prescribe the locality of p_i. We choose a Gaussian prior of the form $q_i := \exp(-\gamma|\bar{\boldsymbol{x}}_i - \boldsymbol{x}|^2/h^2)$.

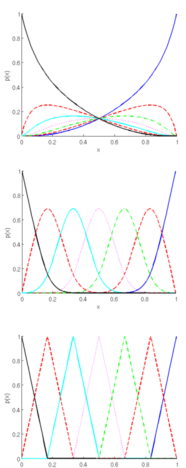


Fig. 1: Max-ent basis functions for Gaussian priors q_i with $\gamma = 0$ (i.e. a uniform prior; top), medium γ (middle), and $\gamma\to\infty$ (i.e. a Dirac prior; bottom).

Information-flux methods

Enhance weighting basis functions with selected information

From stabilized FEMs we know that an appropriate upwind weighting basis function can be the key to stability. Motivated by the analytical solution of the homogeneous 1-D convection-diffusion problem $u(x) = c_1\exp(\Upsilon x) + c_2$ with $\Upsilon = a/\kappa$, we here change the linear constraint for $\{p_i\}$ to $\sum_i \phi_i c_i = 0$ with

$$c_i = \frac{\exp((\bar{x}_i - x)\Upsilon^*) - 1}{\exp(\Upsilon^*) - 1}$$

and $\Upsilon^* = -\Upsilon$. Thus, the weighting basis functions are determined by:

$$\{\phi_i\} = \left\{ \arg\max -\sum_i \phi_i \ln\frac{\phi_i}{q_i} \middle| \sum_i \phi_i = 1, \sum_i \phi_i \boldsymbol{c}_i = 0 \right\}$$

Note that the constraint is designed such that for $\Upsilon\to 0$ (the diffusive limit), it becomes linear again, and $\phi_i = p_i$. Hence, an optimal (Bubnov-Galerkin) scheme is ensured for the diffusive limit, as well. See [3,4] for more details.

Results 1-D convection diffusion

Inf-flux methods: a seamless bridge to stabilized FEMs

Solution and weighting basis functions as well as the results of the inf-flux method for a 1-D problem are shown in Fig. 2 for medium locality $\gamma = 1.5$. The results are almost nodally exact. The small overshoot does not represent an instability. It is caused by the smooth approximation of $\{p_i\}$, and it vanishes for $\gamma\to\infty$, for which the inf-flux method approaches the SUPG method. This transition can also be seen in the convergence plots of Fig. 3. For lower γ, i.e. decreasing locality, the method becomes more and more accurate.

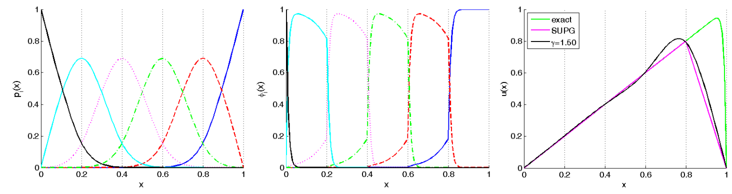


Fig 2: Solution (left) and upwind weighting (middle) basis functions and results (right) for $\gamma = 1.5$ on $\Omega = (0,1)$ with $a = 1$, $\kappa = 0.01$, and $f = 1$.

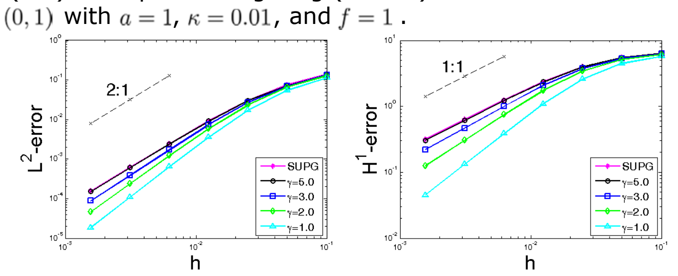


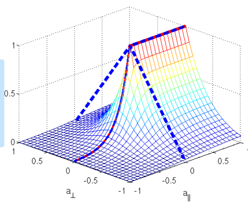
Fig 3: Convergence plots on $\Omega = (0,1)$ with $a = 1$, $\kappa = 0.01$, and $f = 1$ in L^2- and H^1-norm.

Multi-dimensional extensions and results

Streamline based approach:

Like for SUPG, we apply in streamline direction the constraint from the 1-D problem and linear constraints in all directions orthogonal to it. Thus, we do not account for any multi-dimensional effects of the convection.

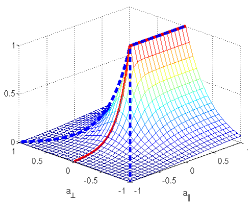
$$\boldsymbol{c}_i = \begin{bmatrix} \dfrac{\exp(\langle\bar{\boldsymbol{x}}_i - \boldsymbol{x}, \boldsymbol{\Upsilon}^*\rangle) - 1}{\exp||\boldsymbol{\Upsilon}^*|| - 1} \\ \langle \boldsymbol{e}^\perp, \bar{\boldsymbol{x}}_i - \boldsymbol{x}\rangle \end{bmatrix}$$



Freespace solution based approach:

We define the constraints by orthogonal freespace solutions symmetric to the streamline (see definition of eigenvector $\boldsymbol{\Upsilon}_\varphi$ in [4]). Thus, we account for multi-dimensional effects of the convection (see also [5]).

$$\boldsymbol{c}_i = \begin{bmatrix} \exp\left(\langle\bar{\boldsymbol{x}}_i - \boldsymbol{x}, \boldsymbol{\Upsilon}^*_{\varphi-}\rangle\right) - 1 \\ \exp\left(\langle\bar{\boldsymbol{x}}_i - \boldsymbol{x}, \boldsymbol{\Upsilon}^*_{\varphi+}\rangle\right) - 1 \end{bmatrix} \frac{1}{\exp||\boldsymbol{\Upsilon}^*||-1}$$



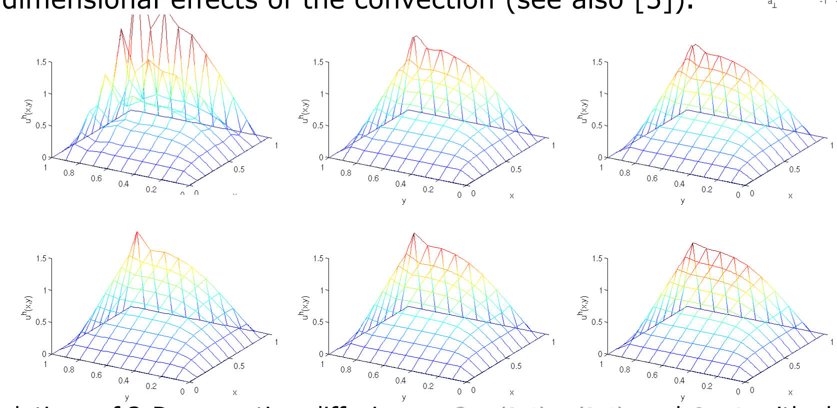


Fig 4: Solutions of 2-D convection-diffusion on $\Omega = (0,1)\times(0,1)$ and $f = 1$ with skew unit convection at 30° and $\kappa = 0.01$. Left: t: FEM, b: SUPG; middle: streamline based inf-flux, t: $\gamma = 1.5$, b: $\gamma\to\infty$; right: freespace solution based inf-flux, t: $\gamma = 1.5$, b: $\gamma\to\infty$.

Conclusions

We have presented the **information-flux method**, a new approach to stable computational methods for convection-diffusion problems. For a large variety of the locality parameter γ, **promising performance** has been observed, including a smooth **transition to stabilized FEMs** for $\gamma\to\infty$ and **superconvergence** for $\gamma < \infty$ for 1-D problems. Two **extensions to multi-dimensional problems** have been presented, one of which accounts for **multi-dimensional effects** of the convection. The method is currently extended to the **Navier-Stokes equations**; first results are obtained.

References

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