Semi-smooth Newton methods for finite strain plasticity

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Introduction

Computational plasticity at finite strains
• Highly nonlinear kinematics and material behavior
• Well-established local return-mappings schemes in the literature, e.g. [1,2]
• Difficult and only variationally inconsistent extension to general hyperelasticity
• Lack of stability at large time steps

Semi-smooth Newton methods for plasticity
• Global solution strategy in contrast to classical return-mapping schemes
• First approaches in [3] and further elaborated in [4,5] for small strain kinematics

Aim of this work
• Application of a semi-smooth Newton method to finite strain plasticity
• Incorporation of general hyperelasticity
• Improved stability compared to classical return-mapping methods
• Combination with other semi-smooth strategies, e.g. for (frictional) contact [6]

Mathematical description of finite strain plasticity
• Multiplicative split of the deformation gradient to define a plastic, locally stress-free intermediate configuration:

\[ \mathbf{F} = \mathbf{F}^p \mathbf{F}^s \]

• General hyperelasticity in terms of Mandel stresses:

\[ \Sigma = 2C^0 \frac{\partial C^0}{\partial \mathbf{F}} \]

• Von Mises yield function with linear isotropic and kinematic hardening:

\[ f^p = \sqrt{\frac{1}{2} (\sigma_1' + H' \mathbf{a}^k)} - \frac{2}{3} (\sigma_0 + H' \mathbf{a}^k) = \| \mathbf{\eta} \| - Y^p \]

• Evolution equations for internal variables:

\[ \dot{\mathbf{F}}^p \mathbf{F}^{-1} = \gamma \frac{\mathbf{\eta}}{\| \mathbf{\eta} \|} , \quad \dot{\mathbf{a}}^k = \frac{2}{3} \gamma , \quad \dot{\mathbf{a}}^k = -\gamma \frac{\mathbf{\eta}}{\| \mathbf{\eta} \|} \]

• Complementarity and consistency condition for the plastic flow:

\[ f^p \leq 0 , \quad \gamma \geq 0 , \quad \dot{f}^p \gamma = 0 , \quad \dot{f}^p \gamma = 0 \]

Reformulation as semi-smooth nonlinear system
• Introduction of the discrete plastic flow within one time step at each quadrature point as additional primary variables:

\[ \Delta \mathbf{L}^{P} = \int_{\mathbf{L}^{P}}^{t_{n+1}} \frac{\mathbf{\eta}}{\| \mathbf{\eta} \|} dt \]

• Evolution equations and inequality constraints are reformulated in terms of a nonlinear complementarity (NCP) function:

\[ C_{NCP} = \max \left( Y^p \eta + \sigma^p \Delta L^p \right)^+ \left( \eta - \min \left( \frac{\mathbf{\eta}}{\| \mathbf{\eta} \|} + \frac{\mathbf{\eta}}{\| \mathbf{\eta} \|} \frac{\partial C_{NCP}}{\partial \mathbf{\eta}} \Delta \mathbf{L}^{P} \right) \right) = 0 \]

• Piece-wise linear NCP-function, i.e. \( s = 0 \), is under further conditions (time integration, complementarity parameter) identical to return mapping methods
• Stabilization parameter \( 0 < s \leq \frac{1}{2} \) allows for significantly larger time steps (up to factor 25) without changing the numerical solution
• Except isotropy, no restrictions to the hyperelastic material law have to be made

Semi-smooth Newton method

Consistent linearization
• Application of a variant of Newton’s method based on a generalized definition of the derivatives in the semi-smooth NCP function [7]
• Linearization of the internal and external forces and the NCP function yield a block matrix system at each quadrature point:

\[ \left( \begin{array}{c} \frac{\partial f^p}{\partial \mathbf{u}} \frac{\partial f^p}{\partial f^p} \frac{\partial f^p}{\partial \mathbf{F}^0} \\ \frac{\partial f^p}{\partial \mathbf{\eta}} \frac{\partial f^p}{\partial \mathbf{\eta}} \frac{\partial f^p}{\partial \mathbf{\eta}} \end{array} \right) \left( \begin{array}{c} \Delta \mathbf{F}^{P} \Delta \mathbf{F}^{P} \Delta \mathbf{F}^{P} \\ \Delta \mathbf{\eta} \Delta \mathbf{\eta} \Delta \mathbf{\eta} \end{array} \right) = \left( \begin{array}{c} -\mathbf{f}_{int} - \mathbf{f}_{ext} \\ -\mathbf{C}^{p} \end{array} \right) \]

Static condensation
• Discontinuous interpolation of inner variables at each quadrature point enables static condensation of the plastic flow increment \( \Delta (\Delta L^p) \) at Gauss point level:

\[ \frac{\partial f^p}{\partial \mathbf{u}} - \frac{\partial f^p}{\partial \mathbf{\eta}} \left( \frac{\partial f^p}{\partial \mathbf{\eta}} \right)^{-1} \frac{\partial f^p}{\partial \mathbf{\eta}} \frac{\partial f^p}{\partial \mathbf{\eta}} \delta \mathbf{u} = -\mathbf{f}_{int} - \mathbf{f}_{ext} + \frac{\partial f^p}{\partial \mathbf{\eta}} \left( \frac{\partial f^p}{\partial \mathbf{\eta}} \right)^{-1} \frac{\partial f^p}{\partial \mathbf{\eta}} \delta \mathbf{\eta} \frac{\partial f^p}{\partial \mathbf{\eta}} \]

\[ \Delta \mathbf{F}^{P} = \left( \begin{array}{c} \frac{\partial f^p}{\partial \mathbf{u}} \frac{\partial f^p}{\partial \mathbf{\eta}} \end{array} \right) \left( \begin{array}{c} \Delta \mathbf{F}^{P} \Delta \mathbf{\eta} \end{array} \right) = \left( \begin{array}{c} -\mathbf{C}^{p} \end{array} \right) \]

• The remaining system only consists of the displacement degrees of freedom

Numerical results

Necking of a cylindrical bar
• Popular benchmark problem for finite strain plasticity in the literature, e.g. [1,2]
• Simulation with 960 F-bar elements to avoid volumetric locking

Cupping of a metal sheet
• A rigid hemisphere is pushed into a quadratic metal sheet
• Setting the stabilization parameter \( s = 1 \) permits to simulate the intrusion in only 25 equal time steps
• A semi-smooth Newton method is used for both plasticity and contact [6]

Conclusions
• Semi-smooth Newton method for finite strain plasticity results in a more robust formulation compared to classical return mapping schemes
• General isotropic hyperelasticity is naturally incorporated
• Unified treatment of plasticity and contact in a general framework of semi-smooth Newton methods

Outlook
• Further analysis of the interaction between plasticity and contact
• Application of the stabilization to frictional contact

References