The objective of this project is to derive a stabilized fixed-grid fluid discretization approach based on the extended Finite Element Method (XFEM) and to develop a robust fixed-grid fluid-structure interaction (FSI) scheme in 3D. Fixed-grid fluid-structure interaction methods are highly promising for a wide variety of industrial and biomedical applications since they allow for large motions, rotations and deformations of flexible structures or even topological changes without remeshing, contact of submersed structures (fluid-structure-contact interaction (FSCI)).

**Motivation**

- Systematic projection of solutions at time $t_{n+1}$ w.r.t. interface position $\Gamma(t)$
- Update using Aitken relaxation

**Governing equations**

- Fluid flow described in a fixed-grid Eulerian frame of reference using cut elements
  \[
  \rho^{(n+1)} \frac{\partial u}{\partial t} + \rho^{(n+1)} \nabla \cdot u + P^{(n+1)} \nabla \cdot v_{\text{conv}} + 2\mu \nabla \cdot \epsilon(u) = \rho^{(n+1)} b^{(n+1)} + \mu \nabla \cdot \epsilon(v_{\text{conv}}) \text{ in } \Omega^{(t)}
  \]
- Description of the structural motion in a Lagrangian framework
  \[
  \rho^{(n+1)} \frac{d d^{(n+1)}}{dt^2} = \mathbf{F}(\mathbf{S}) + \rho^{(n+1)} g^{(n+1)} \text{ in } \Omega^{(n+1)}
  \]
- FSI-coupling conditions
  \[
  [\sigma] = [u] - [u] = 0 \quad | \mathbf{n} | = (\mathbf{u}^{(n)} - \mathbf{u}^{(n+1)}) = 0 \text{ on } \Gamma_{PP}(t)
  \]

**Numerical Analysis and Results**

- Optimal spatial convergence in domain $L^2$-norms and interface $H^1$- and $H^1$-$\nabla$-norms for $u$ and $p$ in viscous and convection dominated cases.
- Good accuracy regarding the weak imposition of boundary conditions.
- Clear improvement of viscous and pressure fluxes at the interface compared to residual-based stabilized methods.
- Less sensitivity with respect to the interface position.
- Ghost penalty stabilization and face-oriented fluid stabilizations in the interface zone supersede any DOF-blocking strategy.
- Well-conditioned system matrices in viscous and convection dominated cases.

**Time integration involving moving domains**

A Semi-Lagrangian method for time integration

1. Tracking $\mathbf{X}$ back in time $t$ with $\mathbf{x} = \mathbf{X} - t \frac{\partial}{\partial t} \mathbf{u}(\mathbf{x}, t) \text{ d}t$
   \[
   \mathbf{z} = \mathbf{X} - \mathbf{b} = (1 - \delta_{ij}) \mathbf{u}(\mathbf{z}, t) \text{ d}t
   \]
2. Solution at time $t_{n+1}$ w.r.t. interface position $\Gamma^{n+1}$
   \[
   \sigma_{\Gamma_{PP}^{n+1}}(\mathbf{z}) = \sigma(\mathbf{z}) + \frac{\partial}{\partial t} \sigma_{\Gamma_{PP}^{n+1}}(\mathbf{z}) \frac{\mathbf{b}(\mathbf{z})}{\mathbf{b}(\mathbf{z})} + (1 - \delta_{ij}) \mathbf{u}(\mathbf{z}, t) \text{ d}t
   \]

**Spatial discretization using cut elements**

1. Volume-cell (3D polyhedral) representation of the physical fluid domain
   \[
   \Omega^{(t)} = \bigcup_{K=1}^{N_K} \bigcup_{l=1}^{N_l} \bigcup_{j=1}^{N_j} \bigcup_{i=1}^{N_i} \Omega_{Klji}^{(t)} \quad \Omega_{Klji}^{(t)} = \{ \mathbf{X} \in \Omega^{(t)} | \mathbf{X} \in K \land \mathbf{X} \notin \Omega_{\Gamma_{PP}}^{(t)} \land \mathbf{X} \notin \Omega_{\Gamma_{PB}}^{(t)} \land \mathbf{X} \notin \Omega_{\Gamma_{PB}}^{(t)} \land \mathbf{X} \notin \Omega_{\Gamma_{PB}}^{(t)} \}
   \]
2. Standard continuous approximations based on fluid volume-cells
   - Build volume-cell connections on $\equiv \Omega_{PP}^{(t)}$ between elements via common facets
   - Number of connections $C_{ij}$ per node $\mathbf{z}$ determines the number of required degrees of freedom
   - Assign the respective degree of freedom (DOF) to the volume-cells
3. Integration of weak formulation based on the divergence theorem
   - Hessian enrichment for cut elements
   - Integration only on $\partial \Omega^{(t)}$
   \[
   \mathbf{v} = \sum_{i=1}^{n+1} N_i \mathbf{v}^i \quad \mathbf{v}^i = \begin{cases} 1 \text{ in } \Omega^{(t)} & \text{ 1 else} 
   \end{cases}
   \]

**Face-oriented stabilized fixed-grid fluid formulation**

- Consistent Nitsche-type weak enforcement of interface conditions for viscous and convection dominated flows
- Find $(u^{n+1}, p^{n+1}) \in \mathcal{V}^{n+1} \times \mathcal{Q}^{n+1}$ such that $\mathbf{u}^{n+1}, p^{n+1} \in \mathcal{V}^{n+1} \times \mathcal{Q}^{n+1}$

\[
\left\{ \begin{align*}
\mathbf{u}^{n+1} - \mathbf{u}^{n} + \mathbf{P}^{n+1} \nabla \cdot \mathbf{v}_{\text{conv}} + 2\mu \nabla \cdot \epsilon(u^{n+1}) &= \mathbf{b}^{n+1} + \mu \nabla \cdot \epsilon(v_{\text{conv}}) & \text{ in } \Omega_{PP}^{n+1} \\
\mathbf{v}^{n+1} &= 0 & \Omega_{PB}^{n+1}
\end{align*} \right.
\]

- Ghost penalty stabilization and fluid stabilizations in the interface zone

\[
\gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) = \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) = \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) + \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1})
\]

\[
\gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) = \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) + \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) + \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1}) + \gamma_{\text{pen}}(\mathbf{u}^{n+1}, p^{n+1})
\]

**Partitioned Fluid-Structure Interaction**

Iterative Dirichlet-Neumann Coupling with Atkin relaxation

- Fluid-operator:
  \[
  \mathbf{F}_{t_{n+1}}^{n+1}(\mathbf{d}^{n+1}) = \mathbf{F}_{t_{n+1}}^{n+1}
  \]
  solve fluid flow for $(u_n, p_n) = \mathbf{F}_{t_{n+1}}^{n+1}$ with prescribed Dirichlet velocity $\mathbf{u}^{n+1} = \mathbf{d}^{n+1}$

- Structure-operator:
  \[
  \mathbf{d}_{t_{n+1}}^{n+1} = \mathbf{F}_{t_{n+1}}^{n+1}(\mathbf{f}^{n+1}) = \mathbf{d}_{t_{n+1}}^{n+1}
  \]
  solve structure for $\mathbf{d}_{t_{n+1}}^{n+1} = \mathbf{d}_{t_{n+1}}^{n+1}$ with prescribed Neumann interface forces $\mathbf{f}^{n+1}$

- Update using Atkin relaxation:
  \[
  \mathbf{d}_{t_{n+1}}^{n+1} = \mathbf{d}_{t_{n+1}}^{n+1} + \omega \left( \mathbf{d}_{t_{n+1}}^{n+1} - \mathbf{d}_{t_{n+1}}^{n+1} \right)
  \]

- Iterative coupling until convergence for each $t_{n+1}$.