A Three-Dimensional Nonlinear Finite Element Formulation for Geometrically Exact Kirchhoff Rods

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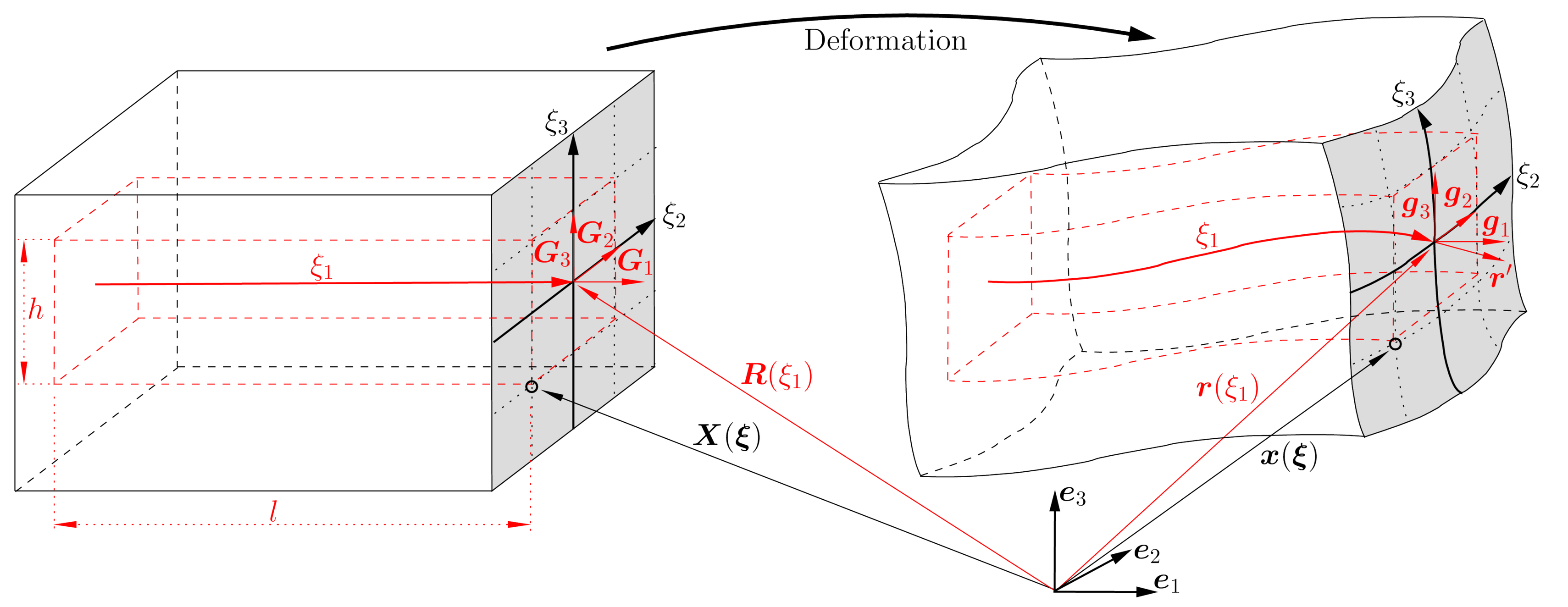
Introduction

Geometrically exact beam models

- Beams as structural models appropriate to describe the mechanics of bodies, whose length dimension is much larger than the transverse dimensions
- Geometrically exact (GE), if deformation and stress measures are work-paired
- Deduction of beam models from the 3D continuum:



- Constrained position vector due to Bernoulli assumption of rigid cross sections:

$$\mathbf{X}(\xi_1,\xi_2,\xi_3) = \mathbf{R}(\xi_1) + \xi_2\mathbf{G}_2(\xi_1) + \xi_3\mathbf{G}_3(\xi_1) \qquad \text{if} \quad h \ll l$$
$$\mathbf{x}(\xi_1,\xi_2,\xi_3) = \mathbf{r}(\xi_1) + \xi_2\mathbf{g}_2(\xi_1) + \xi_3\mathbf{g}_3(\xi_1)$$

- Model reduction from 3D Boltzmann continuum to 1D Cosserat continuum:

3 DoF: $\mathbf{x}(\boldsymbol{\xi}) \rightarrow$ 6 DoF: $\mathbf{r}(\xi_1),\ \mathbf{q}(\xi_1)$ with $\mathbf{g}_i = \boldsymbol{\Lambda}(\mathbf{q}(\xi_1))\mathbf{e}_i$ for $i = 1,2,3$

- Reduction from Reissner theory to Kirchhoff theory possible for slender beams:

6 DoF: $\mathbf{r}(\xi_1),\ \mathbf{q}(\xi_1) \rightarrow$ 4 DoF: $\mathbf{r}(\xi_1),\ \varphi(\xi_1)$ since $\mathbf{g}_1 \times \mathbf{r}' \doteq \mathbf{0}$

Aim

- Existing GE FEM beam formulations exclusively of Reissner type
- Development of a GE FEM beam formulation according to Kirchhoff theory

Governing Equations

Kinematics

- Description of the beam centerline and the material frame $\mathbf{g}_i$ via the 4 DoF $\mathbf{r}$, φ
- Introduction of a reference triad field, which is part of the solution:

$$\mathbf{g}_{ref,1} = \mathbf{r}'/||\mathbf{r}'||, \quad \mathbf{g}_{ref,2} = \mathbf{f}_2(\mathbf{r}), \quad \mathbf{g}_{ref,3} = \mathbf{f}_3(\mathbf{r})$$

- Determination of the material frame via a relative rotation of reference frame:

$$\mathbf{g}_1 = \mathbf{g}_{ref,1},\ \mathbf{g}_2 = \mathbf{g}_{ref,2}\cos\varphi + \mathbf{g}_{ref,3}\sin\varphi,\ \mathbf{g}_3 = \mathbf{g}_{ref,3}\cos\varphi - \mathbf{g}_{ref,2}\sin\varphi$$

- Derivation of geometrically exact spatial deformation measures:

$$\boldsymbol{\omega} = \begin{pmatrix} \tau + \varphi' - \tau_0 - \varphi_0' \\ (\mathbf{g}_2\cdot\boldsymbol{\kappa}) - (\mathbf{G}_2\cdot\boldsymbol{\kappa}_0) \\ (\mathbf{g}_3\cdot\boldsymbol{\kappa}) - (\mathbf{G}_3\cdot\boldsymbol{\kappa}_0) \end{pmatrix}_{\mathbf{g}_i} \quad \text{and} \quad \epsilon = ||\mathbf{r}'|| - 1$$

The torsion of the reference system and the centerline curvature are:

$$\tau = \mathbf{f}_2'(\mathbf{r})\cdot\mathbf{f}_3(\mathbf{r}) \quad \text{and} \quad \boldsymbol{\kappa} = \frac{\mathbf{r}'\times\mathbf{r}''}{||\mathbf{r}'||^2}$$

Constitutive law

- Derivation of spatial stress resultants from hyperelastic stored energy function:

$$\mathbf{n} = n\mathbf{g}_1 \quad \text{with} \quad n = \frac{\partial_{rel}\Pi_{int}(\epsilon,\boldsymbol{\omega})}{\partial\epsilon} \quad \text{and} \quad \mathbf{m} = \frac{\partial_{rel}\Pi_{int}(\epsilon,\boldsymbol{\omega})}{\partial\boldsymbol{\omega}}$$

$$\text{e.g.} \quad \Pi_{int}(\epsilon,\boldsymbol{\omega}) = \frac{1}{2}EA\epsilon^2 + \frac{1}{2}\boldsymbol{\omega}^T\mathbf{c}_m\boldsymbol{\omega} \quad \text{with} \quad \mathbf{c}_m = \text{diag}(GI_P, EI_2, EI_3)_{\mathbf{g}_i}$$

Weak form

$$\int_0^L \left[\left(\delta\alpha\mathbf{t} + \frac{\mathbf{r}'}{|\mathbf{r}'|^2}\times\delta\mathbf{r}'\right)'\cdot\mathbf{m} + \delta\mathbf{r}'\cdot\mathbf{n} - \delta\mathbf{r}\cdot\tilde{\mathbf{f}} - \left(\delta\alpha\mathbf{t} + \frac{\mathbf{r}'}{|\mathbf{r}'|^2}\times\delta\mathbf{r}'\right)\cdot\tilde{\mathbf{m}}\right] ds$$
$$- \Big[\delta\mathbf{r}\cdot\mathbf{f}\Big]_{\Gamma_\sigma} - \left[\left(\delta\alpha\mathbf{t} + \frac{\mathbf{r}'}{|\mathbf{r}'|^2}\times\delta\mathbf{r}'\right)\cdot\mathbf{m}\right]_{\Gamma_\sigma} = 0$$

with external line loads $\tilde{\mathbf{f}}$, $\tilde{\mathbf{m}}$ and primary variables $\mathbf{r}$, $\delta\mathbf{r}$, φ, $\delta\alpha$

Discretization

- Discretization of beam centerline with third order Hermite shape functions:

$$\mathbf{r}_h(\xi) = \sum_{i=1}^{2} H_d^i(\xi)\mathbf{d}^i + \frac{l_{ele}}{2}\sum_{i=1}^{2} H_t^i(\xi)\mathbf{t}^i$$

with nodal positions $\mathbf{d}^i$, nodal tangents $\mathbf{t}^i$ and the element length l_{ele}
→ Fulfilment of completeness, interpolation property, C^1-continuity at nodes

- Discretization of relative angle with third order Lagrange shape functions:

$$\varphi_h(\xi) = \sum_{i=1}^{4} L^i(\xi)\hat{\varphi}_i$$

Reference Triad Field

- Choice of reference triad as crucial step, that influences factors like continuity requirements, objectivity, convergence behaviour, robustness and complexity
- SR mapping defines triad $\mathbf{g}_{SR,j}$ out of given triad $\bar{\mathbf{g}}_i$ and given vector $\mathbf{g}_{SR,1}$:

$$\mathbf{g}_{SR,j} = \bar{\mathbf{g}}_j - \frac{\bar{\mathbf{g}}_j\cdot\mathbf{g}_{SR,1}}{1+\bar{\mathbf{g}}_1\cdot\mathbf{g}_{SR,1}}\left(\mathbf{g}_{SR,1} + \bar{\mathbf{g}}_1\right) \quad \text{for} \quad j = 2,3$$

- Direct application of the SR mapping leads to non-objectivity (SR element)
- An objective formulation follows with the procedure (NSRISR element):
  1) Define reference triads at nodal positions ξ_1^n and time t_k via SR mapping:

$$\mathbf{g}_{SR,j} = \mathbf{g}_{ref,j}(\xi_1^n, t_k),\ \bar{\mathbf{g}}_i = \mathbf{g}_{ref,i}(\xi_1^n, t_{k-1}),\ \mathbf{g}_{SR,1} = \mathbf{g}_1(\xi_1^n, t_k)$$

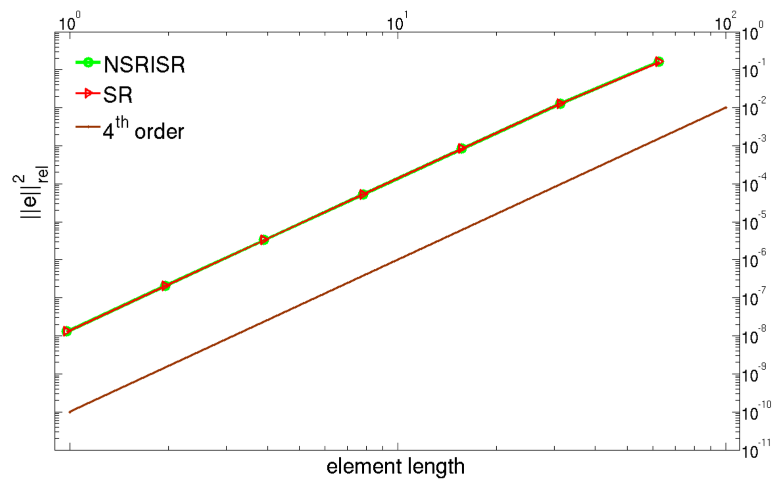
  2) Define interior reference triad field at position ξ_1 and time t_k via SR mapping:

$$\mathbf{g}_{SR,j} = \mathbf{g}_{ref,j}(\xi_1, t_k),\ \bar{\mathbf{g}}_i = \mathbf{g}_{ref,i}(\xi_1^n, t_k),\ \mathbf{g}_{SR,1} = \mathbf{g}_1(\xi_1, t_k)$$

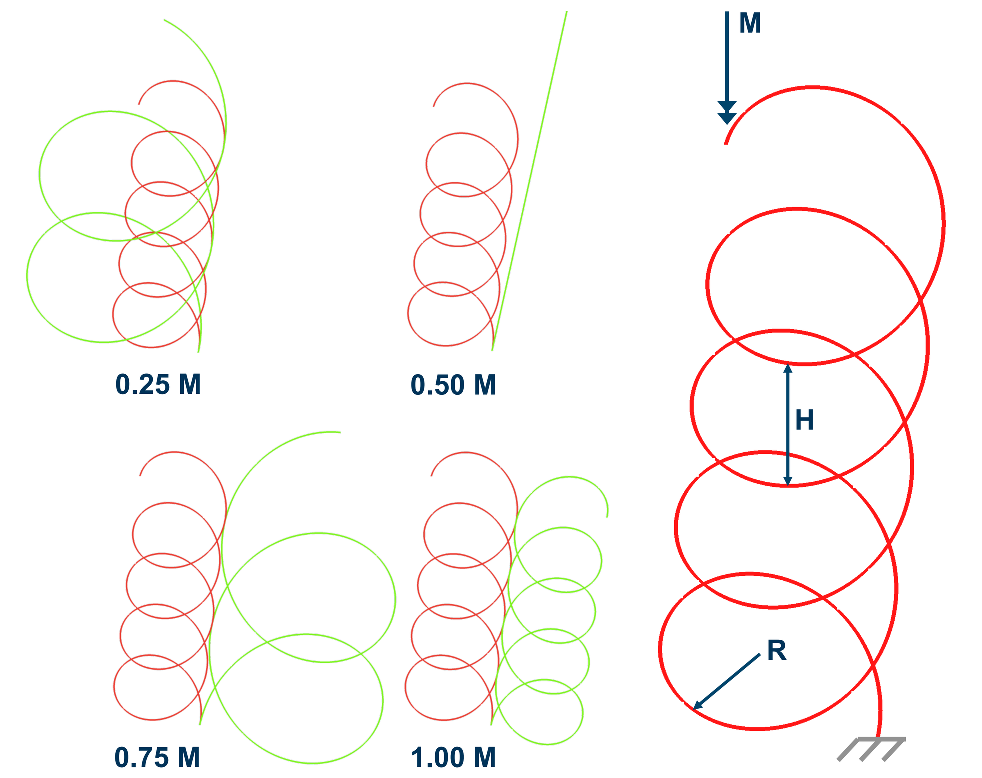
Numerical Examples

Inversion of an elastic helix

- Clamped helix: E=2G=1, $I=10^3$,
  with h=10, R=H=I/(9π√2)
- The external moment $M=18\pi EI_2/l$ exactly inverts the elastic helix



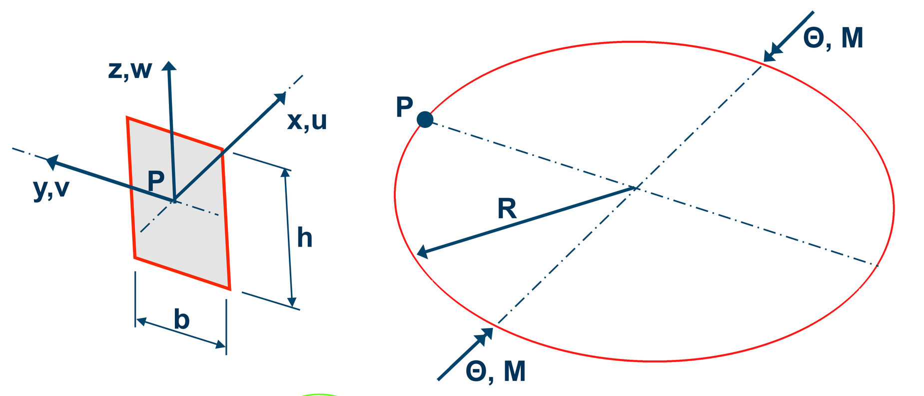
Convergence plot for SR and NSRISR element



Initial (red) and deformed (green) geometries

Instability of an elastic ring

- Circular ring: E=2.6G=2.1·10^7, R=20,
  with h=1.0, b=1/3, I_P=9.753·10^{-3}
- Displacement-controlled twist angle Θ / reaction moment M



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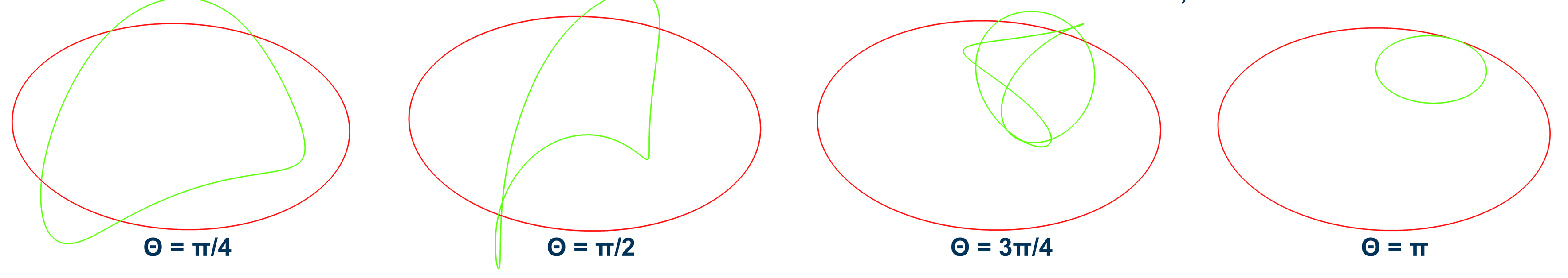

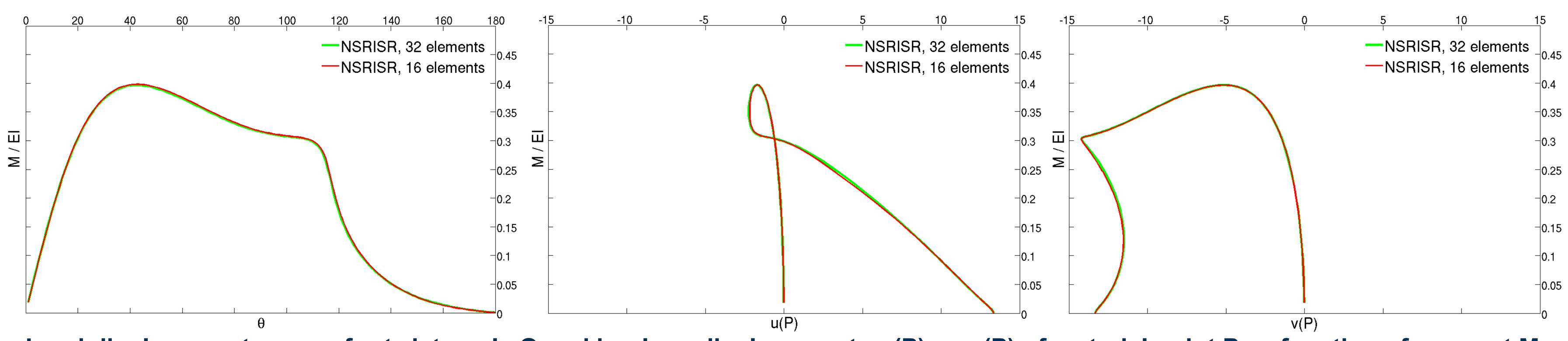

Load displacement curves for twist angle Θ and in-plane displacements u(P) an v(P) of material point P as function of moment M

Conclusions

Geometrically exact Kirchhoff beam element

- Strong enforcement of Kirchhoff constraint
- Kirchhoff kinematics based on objective reference triad interpolation
- C^1-continuity of beam geometry through Hermite interpolation
- Efficient model reduction possible for isotropic and torsion free applications
- Objectivity, path independence, consistent convergence order and locking avoidance theoretically predicted and numerically confirmed

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