Constitutive law

- Choice of reference triad as crucial step, that influences factors like continuity requirements, objectivity, convergence behaviour, robustness and complexity
- SR mapping defines triad \( \mathbf{g}_{SR,j} \) out of given triad \( \mathbf{g} \) and given vector \( \mathbf{g}_{SR,j} \):
  \[
  \mathbf{g}_{SR,j} = \mathbf{g} - \mathbf{e}_j + \frac{1}{2} \mathbf{g} \times \mathbf{e}_j
  \]
  for \( j = 1, 2, 3 \)
- Direct application of the SR mapping leads to non-objectivity (SR element)
- An objective formulation follows with the procedure (NSRIS element)
  1) Define reference triads at nodal positions \( \xi_i \) and time \( t_f \) via SR mapping:
     \[
     \mathbf{g}_{SR,j} = \mathbf{g}_{SR,j}^{\xi_i}(\xi_i, t_f), \quad \mathbf{g}_{BR,j}^{\xi_i}(\xi_i, t_f) - \mathbf{g}_{SR,j}^{\xi_i}(\xi_i, t_f) - \mathbf{g}_{SR,1}^{\xi_i}(\xi_i, t_f)
     \]
  2) Define interior reference triad field at position \( \xi_i \) and time \( t_f \) via SR mapping:
     \[
     \mathbf{g}_{SR,j}^{\xi_i}(\xi_i, t_f), \quad \mathbf{g}_{BR,j}^{\xi_i}(\xi_i, t_f) - \mathbf{g}_{SR,j}^{\xi_i}(\xi_i, t_f) - \mathbf{g}_{SR,1}^{\xi_i}(\xi_i, t_f)
     \]

Geometrically exact beam models

- Beams as structural models appropriate to describe the mechanics of bodies, whose length dimension is much larger than the transverse dimensions
- Geometrically exact (GE), if deformation and stress measures are work-paired
- Deduction of beam models from the 3D continuum:

Kinematics

- Description of the beam centerline and the material frame via the 4 DoF:
  \[
  \mathbf{g}_{BR,1} = f_1(\xi), \quad \mathbf{g}_{BR,2} = f_2(\xi), \quad \mathbf{g}_{BR,3} = f_3(\xi)
  \]
- Derivation of the material frame via a relative rotation of reference frame:
  \[
  \mathbf{g}_{BR,1} = \mathbf{g}_{BR,1}^{ref} f_1, \quad \mathbf{g}_{BR,2} = \mathbf{g}_{BR,2}^{ref} \cos \varphi + \mathbf{g}_{BR,3}^{ref} \sin \varphi, \quad \mathbf{g}_{BR,3} = \mathbf{g}_{BR,3}^{ref} \cos \varphi - \mathbf{g}_{BR,2}^{ref} \sin \varphi
  \]
- Derivation of geometrically exact spatial deformation measures:
  \[
  \omega = \frac{\tau_1}{\tau_2} x^2 y^2 \quad \text{and} \quad \epsilon = |\tau_1| + 1
  \]

Constitutive law

- Spacial of spatial stress resultants from hyperelastic stored energy function:
  \[
  \mathbf{n} = \mathbf{n}^{ref} f_1(\xi), \quad \mathbf{m} = \mathbf{m}^{ref} f_2(\xi)
  \]
- E.g. \( \mathbf{n}^{ref}(\xi, \omega) = \frac{1}{2} \mathbf{E} \mathbf{A} \mathbf{r}^2 + \mathbf{A} \mathbf{T} \mathbf{n} \mathbf{w} \mathbf{m}^{ref} \mathbf{m}^{ref} = \text{diag}(G, I_2, I_3)\mathbf{m}^{ref}
  \]
- Weak form
  \[
  \int_{\xi_1}^{\xi_2} \left[ \left( \mathbf{n} \mathbf{a} + \mathbf{m} \mathbf{r}^2 \right) - \mathbf{m} \mathbf{r} - \mathbf{a} \right] \mathbf{a} + \left( \mathbf{n} \mathbf{a} + \mathbf{m} \mathbf{r}^2 \right) \mathbf{a} \right] \mathbf{m} \right] \mathbf{r} = 0
  \]
with external line loads \( \mathbf{f}, \mathbf{m} \) and primary variables, \( \varphi, \psi, \delta \)

Numerical Examples

- Inversion of an elastic ring
  - Circular ring: \( E=2, \varphi=2 \cdot 10^4 \), \( R=20 \)
  - Clamped ring: \( h=1, b=1/3, \varphi=9, 75, 75, 3 \cdot 10^{-3} \)
- Instability of an elastic ring
  - Displacement-controlled twist angle \( \Theta \) / reaction moment \( M \)

Governing Equations

- Description of the beam centerline and the material frame \( \mathbf{g}_i \) via the 4 DoF \( r, \varphi \)
- Introduction of a reference triad field, which is part of the solution:
  \[
  \mathbf{g}_{BR,1} = f_1(\xi), \quad \mathbf{g}_{BR,2} = f_2(\xi), \quad \mathbf{g}_{BR,3} = f_3(\xi)
  \]
- Model reduction from 3D Boltzmann continuum to 1D Cosserat continuum:
  \[
  3 \text{ DoF: } \mathbf{x}(\xi) \rightarrow 6 \text{ DoF: } \mathbf{r}(\xi), \mathbf{q}(\xi) \quad \text{with } g_i = \lambda(q) \mathbf{g}_i, \quad \text{for } i = 1, 2, 3
  \]
- Reduction from Reissner theory to Kirchhoff theory possible for slender beams:
  \[
  6 \text{ DoF: } \mathbf{r}(\xi), \mathbf{q}(\xi) \rightarrow 4 \text{ DoF: } \mathbf{r}(\xi), \mathbf{q}(\xi) \quad \text{since } g_1 \times r^2 \equiv 0
  \]

Aim

- Existing GE FEM beam formulations exclusively of Reissner type
- Development of a GE FEM beam formulation according to Kirchhoff theory

\[ J. Simo, L. Vu Quoc, A three dimensional finite strain rod model. Part II: Computational aspects, Computer Methods in Applied Mechanics and Engineering 45 (1985) 55–70. \]