

Passage Time and Fluctuation Calculations for Subexponential Lévy Processes

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Abstract

We consider the passage time problem for Lévy processes, emphasising heavy tailed cases. Results are obtained under quite mild assumptions, namely, drift to $-\infty$ a.s. of the process, possibly at a linear rate (the finite mean case), but possibly much faster (the infinite mean case), together with subexponential growth. Local, and functional, versions of limit distributions are derived for the passage time itself, as well as for the position of the process just prior to passage, and the overshoot of a high level. Regular variation or maximum domain of attraction conditions, shown to be necessary for the kind of convergence behaviour we are interested in, are imposed on the positive tail of the canonical measure. Specialisation of the Lévy results to random walk situations is outlined.

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1 Introduction

In this paper we add to the literature on the passage time problem for Lévy processes, with special emphasis on heavy tailed cases. The overarching assumption is of a drift to $-\infty$ a.s. of the process, possibly at a linear rate, as is the case when the process has finite mean, but possibly at a much faster rate, when the mean is infinite but drift to $-\infty$ still obtains. To this will be added an assumption of subexponential growth together with regular variation or maximum domain of attraction conditions – heavy tails – “on the positive side”; on the negative side, we assume regular variation of the renewal measure of the descending ladder process, allowing

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both finite and infinite mean cases. We obtain very explicit and detailed descriptions of the asymptotic behaviours of the process, in these situations.

Our results are original in a number of respects. We give a very general treatment for Lévy processes, with results phrased in terms of the tail of the canonical measure of the process itself or its ladder processes. A point of comparison is with the paper of Asmussen and Klüppelberg (1996), who deal with ruin event calculations, mainly for random walks and the compound Poisson process, as used in insurance risk modelling. They also consider the case of subexponential tails, but with moment and other restrictions which we relax considerably. We treat *general Lévy processes*, and impose no overt moment conditions, though as a special case our results apply when the positive tail of the canonical measure is integrable (a finite mean for the positive jump process). We provide *local* as well as *functional* versions of the convergence results, so the results are new even in the finite mean case. (In the infinite mean case we know only of the paper by Klüppelberg and Kyprianou (2006), which deals with a special case.) The regular variation or maximum domain of attraction conditions we impose on the positive tail of the canonical measure are shown to be *necessary* as well as sufficient for the convergence. Subsidiary results in Proposition 3.1 (concerning the convergence of the overshoot for a general subordinator) and Proposition 3.2 (concerning connections between the regular variation or maximum domain of attraction behaviour of the upward ladder height measure as compared with the Lévy measure of the underlying process), are new, and should have interest outside this work.

An important area of application of results like these is in insurance risk, where positive jumps of the process under consideration represent claims on the insurance company's assets, while downward trending in the process represents premium income. In recent years there has been a recognition that operational risk claims in practice may be well modelled by a very heavy tailed distribution, perhaps even having an infinite mean (see Embrechts and Samorodnitsky (2003), Böcker and Klüppelberg (2010), and references in both papers). On the other hand, so that the company does not face ruin with probability 1, it is necessary to assume overall drift to $-\infty$ of the process, and since it's desirable to place minimum restrictions on income growth, we want to allow for the possibility of a heavy tailed distribution in the negative direction as well.

In the next section we introduce the setup and state the main results. Proofs are in Sections 3–5.

2 Setup and Main Results

Let $(X_t)_{t \geq 0}$, $X_0 = 0$, be a real-valued Lévy process on a probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with triplet $(\gamma, \sigma^2, \Pi_X)$, where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π_X is a Lévy measure on \mathbb{R} . Throughout, X is assumed to satisfy

$$\lim_{t \rightarrow \infty} X_t = -\infty \text{ a.s.} \quad (2.1)$$

Let $(H_t)_{t \geq 0}$ denote the ascending ladder height subordinator generated by X . In view of (2.1) the process $(H_t)_{t \geq 0}$ is defective, obtained from a nondefective subordinator \mathcal{H} by independent exponential killing with a rate $q > 0$. By this we mean there is a non-defective subordinator \mathcal{H} and an independent exponential variable e_q with expectation $1/q$ such that $(H_t)_{0 \leq t < L_\infty}$ has the distribution of $(\mathcal{H}_t)_{0 \leq t < e_q}$, where L_t , $t > 0$, is a local time of X ; cf. Bertoin (1995, Lemma VI.2, p.157). It follows that

$$P(H_t \leq x) = P(H_t \leq x, t < L_\infty) = e^{-qt} P(\mathcal{H}_t \leq x), \quad x > 0. \quad (2.2)$$

The descending ladder height subordinator, denoted by $(H_t^*)_{t \geq 0}$, is the ascending ladder height subordinator corresponding to the dual process $(X_t^*)_{t \geq 0} := (-X_t)_{t \geq 0}$. Under (2.1) the process $(H_t^*)_{t \geq 0}$ is proper, and the corresponding $q^* = 0$.

Let $\Pi_{\mathcal{H}}(\cdot)$ be the Lévy measure of \mathcal{H} , with tail $\bar{\Pi}_{\mathcal{H}}(x) = \Pi_{\mathcal{H}}\{(x, \infty)\}$, $x > 0$, assumed positive for all $x > 0$. Similarly, $\Pi_{H^*}(\cdot)$ is the Lévy measure of H^* , with tail $\bar{\Pi}_{H^*}$, and we write $d_{\mathcal{H}}$ and d_{H^*} for the drift coefficients of \mathcal{H} and H^* . We have $d_{\mathcal{H}} = d_H$ and $\Pi_{\mathcal{H}} = \Pi_H$. Let $\bar{\Pi}_X^+$ and $\bar{\Pi}_X^-$ be the positive and negative Lévy tails of X , equal to $\Pi_X\{(x, \infty)\}$ and $\Pi_X\{(-\infty, -x]\}$, $x > 0$. Write $\Pi_X^{(+)}$ and $\Pi_X^{(-)}$ for Π_X restricted to $(0, \infty)$ and $(-\infty, 0)$, respectively. Assume throughout that $\bar{\Pi}_X^+(x) > 0$ for all $x > 0$.

Our results will be phrased in terms of Π_X , $\Pi_{\mathcal{H}}$, and Π_{H^*} , or, more specifically, in terms of the behaviour of their tails for large values, and, after normalisation, we can regard these as being the tails of probability distributions. Then a condition applied to the tail of a probability measure can equally be applied to any of the probability measures defined by, e.g.,

$$\frac{\Pi_X(dx)\mathbf{1}_{\{x>1\}}}{\bar{\Pi}_X^+(1)}, \frac{\Pi_{\mathcal{H}}(dx)\mathbf{1}_{\{x>1\}}}{\bar{\Pi}_{\mathcal{H}}(1)}, \frac{\Pi_{H^*}(dx)\mathbf{1}_{\{x>1\}}}{\bar{\Pi}_{H^*}(1)} \quad (x \in \mathbb{R}). \quad (2.3)$$

We will need certain functionals of these tails, in particular

$$A_X^+(x) := \int_1^x \bar{\Pi}_X^+(y)dy \quad \text{and} \quad A_X^*(x) := \int_1^x \bar{\Pi}_X^-(y)dy, \quad x > 1; \quad (2.4)$$

and

$$A_{\mathcal{H}}(x) := \int_0^x \bar{\Pi}_{\mathcal{H}}(y)dy \quad \text{and} \quad A_{H^*}(x) := \int_0^x \bar{\Pi}_{H^*}(y)dy, \quad x > 0. \quad (2.5)$$

Particular classes of tail functions we are interested in are the regularly varying ones and the class of probability distributions in the maximum domain of attraction of the Gumbel distribution. Write $RV(\alpha)$ for the class of real valued functions regularly varying at ∞ with index $\alpha \in \mathbb{R}$, so that $RV(0)$ are the slowly varying functions. We refer to Bingham, Goldie and Teugels (1987) for definitions and properties of regularly varying functions.

Denote the tail of a distribution function F on $[0, \infty)$ by $\bar{F} = 1 - F$. Assume throughout that $\bar{F}(u) > 0$ for all $u > 0$. $\bar{F} \in RV(-\beta)$ for some $\beta \in (0, \infty)$ is equivalent to F being in the *maximum domain of attraction of a Fréchet distribution with parameter $\beta > 0$* , denoted $F \in \text{MDA}(\Phi_\beta)$. A positive random variable having distribution tail \bar{F} , assumed positive for all $u > 0$, is said to be in the *maximum domain of attraction of the Gumbel distribution*, which we denote as $\text{MDA}(\Lambda)$, with *auxiliary function* $a(u) > 0$, if

$$\frac{\bar{F}(u + a(u)x)}{\bar{F}(u)} \rightarrow e^{-x}, \quad x \geq 0. \quad (2.6)$$

(Here and throughout all limits are as $u \rightarrow \infty$ unless otherwise stated.) Useful properties of such distributions can be found in Bingham et al. (1987) p.410, Resnick (1987, Chapters 0 and 1), Embrechts, Klüppelberg and Mikosch (1997, Chapter 3), and de Haan and Ferreira (2006, Chapter 1). In particular, when (2.6) holds, F has finite moments of all orders, and the auxiliary function $a(u)$ satisfies $a(u) = o(u)$ and is *self-neglecting*, i.e. $a(u + Ka(u)) \sim a(u)$ for any fixed K . Typical distributions in $\text{MDA}(\Phi_\beta)$ are the Pareto distributions, while $\text{MDA}(\Lambda)$ includes the Weibull and lognormal. Further, it is well-known from extreme value theory (cf. Theorem 3.4.5

in Embrechts et al. (1997), or Theorem 1.1.6 in de Haan and Ferreira (2006)) that (2.6) can be extended to give that there is a function $0 < a(u) \rightarrow \infty$ and a positive random variable C such that

$$\frac{\overline{F}(u + a(u)x)}{\overline{F}(u)} \rightarrow P(C > x), \quad x > 0, \quad (2.7)$$

if and only if (for distributions with unbounded support to the right, as we have) $F \in \text{MDA}(\Phi_\beta)$ for some $\beta \in (0, \infty)$, or $\overline{F} \in \text{MDA}(\Lambda)$. Furthermore, $a(u)$ can be chosen as $a(u) = u$ in the first case, and as $a(u) = \int_u^\infty \overline{F}(y)dy/\overline{F}(u)$ (finite) in the second case, and C has a $Par(\beta)$ distribution (i.e. a Pareto distribution with parameter $\beta > 0$) having density $\beta(1+x)^{-\beta-1}$, $x > 0$, in the first case, and an $Exp(1)$ distribution in the second case.

We introduce also the class of *long-tailed distributions*, \mathcal{L} , and the subexponential class, \mathcal{S} . F (or its tail $\overline{F} = 1 - F$) is said to be in the class \mathcal{L} if

$$\frac{\overline{F}(u+x)}{\overline{F}(u)} \rightarrow 1, \quad \text{for } x \in (-\infty, \infty), \quad (2.8)$$

while F (or its tail \overline{F}) is said to be in the class \mathcal{S} of *subexponential* distributions if $F \in \mathcal{L}$ and

$$\frac{\overline{F^{2*}}(u)}{\overline{F}(u)} \rightarrow 2, \quad (2.9)$$

where $F^{2*} = F * F$. See, e.g., Klüppelberg, Kyprianou and Maller (2004), Sect. 1.3.2. We have $RV(\alpha) \subseteq \mathcal{L} \subseteq \mathcal{S}$ but $\text{MDA}(\Lambda)$ is not contained in \mathcal{S} (e.g., Goldie and Resnick (1988)).

Consistent with the convention noted in (2.3), abbreviate $\Pi_X^{(+)}(dx)\mathbf{1}_{\{x>1\}}/\overline{\Pi}_X^+(1) \in \text{MDA}(\Lambda)$ to $\Pi_X^{(+)} \in \text{MDA}(\Lambda)$ and $\Pi_{\mathcal{H}}(dx)\mathbf{1}_{\{x>1\}}/\overline{\Pi}_{\mathcal{H}}(1) \in \mathcal{S}$ to $\Pi_{\mathcal{H}} \in \mathcal{S}$, etc. With this notation, our second basic assumption is

$$\Pi_{\mathcal{H}} \in \mathcal{S}. \quad (2.10)$$

(2.10) is equivalent to $P(\mathcal{H}_1 \in \cdot) \in \mathcal{S}$ (e.g., Pakes (2004, 2007)), and it implies that

$$P(H_\infty > u) = P(\sup_{t \geq 0} X_t \leq u) \sim q^{-1} \overline{\Pi}_{\mathcal{H}}(u) \quad (2.11)$$

(from Lemma 3.5 of Klüppelberg et al. (2004)).

For $u > 0$ let

$$\tau_u := \inf\{t > 0 : X_t > u\}, \quad Z^{(u)} = -X_{\tau_u-}, \quad O^{(u)} = X_{\tau_u} - u \quad (2.12)$$

denote the passage time above level $u > 0$, the negative of the position reached just prior to passage, and the overshoot above the level. (The reason for taking $-X$ in the definition of Z will become apparent later.) Note that $P(\tau_u < \infty) = P(H_\infty > u) < 1$ for all $u > 0$ by (2.1), while $P(\tau_u < \infty) > 0$ for all $u > 0$ because of our assumption that $\overline{\Pi}_X^+(x) > 0$ for all $x > 0$. We use $P^{(u)}(\cdot) = P(\cdot | \tau_u < \infty)$, $u > 0$, defined in an elementary way, for the probability measure conditional on passage above u . We also use the notation $\overline{X}_t = \sup_{0 < s \leq t} X_s$, $t \geq 0$.

Recall the definition of $A_{H^*}(\cdot)$ in (2.5). Our third main assumption is of the form:

$$A_{H^*}(\cdot) \in RV(\gamma), \quad (2.13)$$

where the precise value of the index $\gamma \in [0, 1)$ will be specified later. By, e.g., Bingham et al. (1987), p.364, (2.13) is equivalent to $G^*(\cdot) \in RV(1 - \gamma)$, where G^* is the renewal measure for the strict decreasing ladder height process, and then we have, as $x \rightarrow \infty$,

$$A_{H^*}(x) \sim \frac{k_\gamma x}{G^*(x)} \in RV(\gamma), \text{ where } k_\gamma = \frac{1}{\Gamma(1 + \gamma)\Gamma(2 - \gamma)}. \quad (2.14)$$

We now state our two main results. Both assume (2.1) and (2.10), and the first assumes in addition that $A_{H^*} \in RV(0)$, that is, that A_{H^*} is slowly varying as $x \rightarrow \infty$. This implies that X_t^* is positively relatively stable as $t \rightarrow \infty$, so there is a continuous, increasing function $c(\cdot) \in RV(1)$ such that $X_t^*/c(t) \xrightarrow{P} 1$ as $t \rightarrow \infty$. This in turn implies that $(X^*(us)/c(u))_{0 \leq s < 1}$ converges weakly in $\mathbb{D}_0[0, 1]$ (i.e., in the sense of weak convergence of càdlàg functions on $[0, 1]$ with the Skorokhod topology) to the process $\mathbf{D}^{(0)}$, where $\mathbf{D}^{(0)}(s) \equiv s$. This situation includes the possibility of a finite mean for X_1^* . Write $b(\cdot)$ for the inverse function of $c(\cdot)$. We sometimes write $X^*(t)$ for X_t^* .

Theorem 2.1. *Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s., $\Pi_{\mathcal{H}} \in \mathcal{S}$, and $A_{H^*} \in RV(0)$.*

1. *Then the following are equivalent;*

(a) *$P^{(u)}(O^{(u)} \in a(u)dx)$ has a non-degenerate limit for some $a(u) > 0$, $a(u) \rightarrow \infty$;*

(b) *either $\bar{\Pi}_{\mathcal{H}} \in RV(1 - \beta)$ for some $\beta > 1$ and then (a) holds with $a(u) = u$ (Case (i)) or $\Pi_{\mathcal{H}} \in \text{MDA}(\Lambda)$, and then (a) holds with $a(u) = \int_u^\infty \bar{\Pi}_{\mathcal{H}}(y)dy/\bar{\Pi}_{\mathcal{H}}(u)$, (Case(ii));*

(c) *either $\bar{\Pi}_X^+ \in RV(-\beta)$ for some $\beta > 1$ (Case (i)) or $\Pi_X^{(+)} \in \text{MDA}(\Lambda)$ (Case (ii)), and $a(\cdot)$ may then be chosen as $a(u) = \int_u^\infty \bar{\Pi}_X^+(y)dy/\bar{\Pi}_X^+(u)$, $u > 0$.*

2. *When (a)–(c) hold, the $P^{(u)}$ -distribution of τ_u , restricted to the event $X(\tau_u-) < u$, has a density $g^{(u)}(\cdot)$ which satisfies*

$$\lim_{u \rightarrow \infty} b(a(u))g^{(u)}(tb(a(u))) = \begin{cases} \frac{\beta - 1}{(1 + t)^\beta} & \text{in Case (i),} \\ e^{-t} & \text{in Case (ii),} \end{cases} \quad (2.15)$$

uniformly on compacts. Moreover, conditioned on $\tau_u = tb(a(u))$, the $P^{(u)}$ -finite-dimensional distributions of the process

$$\left\{ \frac{X^*(s\tau_u)}{sc(\tau_u)}, 0 \leq s < 1 \right\}$$

converge to those of $\mathbf{D}^{(0)}$.

3. *Further: when (a)–(c) hold, under $P^{(u)}$ the process*

$$\mathbf{Y}^{(u)} := \left(\frac{Z^{(u)}}{a(u)}, \frac{O^{(u)}}{a(u)}, \frac{\tau_u}{b(a(u))}, \left(\frac{X^*(s\tau_u)}{a(u)} \right)_{0 \leq s < 1} \right) \quad (2.16)$$

converges weakly in $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$ to $(V, U, V, (V\mathbf{D}^{(0)}(s))_{0 < s < 1})$, where in Case (i)

$$P(V \in dz, U \in dy) = \frac{\beta(\beta - 1)dzdy}{(1 + z + y)^{\beta+1}}, \quad y, z > 0. \quad (2.17)$$

and in Case (ii)

$$P(V \in dz, U \in dy) = e^{-z-y}dzdy, \quad y, z > 0. \quad (2.18)$$

The assumption $A_{H^*} \in RV(0)$ is true in particular when $A_{H^*}(\infty) < \infty$, or, equivalently, when $EX_1^* < \infty$, so the case of a finite mean for EX_1^* is included in Theorem 2.1. This then constitutes a generalisation and extension of a result for the case of random walks and compound Poisson processes with finite mean in Asmussen and Klüppelberg (1996).

In our next result we replace the assumption $A_{H^*} \in RV(0)$ by the condition that $A_{H^*} \in RV(\gamma)$ for some $\gamma \in (0, 1)$. This can only happen when $E|X_1| = \infty$, and we will show that it is in fact equivalent, under our basic assumptions, to $\bar{\Pi}_X \in RV(\gamma - 1)$. It then follows that X^* is in the domain of attraction of \mathbf{D} , a standard stable subordinator of parameter $\bar{\gamma} := 1 - \gamma \in (0, 1)$. Let $c(\cdot)$ be such that $(X_{sc(u)}^*/c(u))_{s>0} \xrightarrow{D} \mathbf{D}$, and let $b(\cdot)$ denote the inverse function of $c(\cdot)$, so that $b(\cdot) \in RV(\bar{\gamma})$, and let $\widehat{\mathbf{D}}_{t,z}$ denote an associated “stable subordinator bridge”, which is a rescaled version of \mathbf{D} conditioned to be at $z > 0$ at time t ; viz,

$$P\left(\widehat{\mathbf{D}}_{t,z} \in \mathcal{B}\right) = P\left((D(ts))_{0 < s \leq 1} \in \mathcal{B} \mid D_t = z\right),$$

for Borel \mathcal{B} . Thus, with $h_t(x)dx = P(D_t \in dx)$ as the density of D , we have for $0 = s_0 < s_1 < s_2 \cdots < s_k < 1$, $y_0 = 0$, and $y_1 < y_2 < \cdots < y_k < z$,

$$P\left(\bigcap_{r=1}^k \{\widehat{D}_{t,z}(s_r) \in dy_r\}\right) = \frac{h_{t(1-s_k)}(z - y_k)}{h_t(z)} \prod_{r=1}^k h_{t(s_r - s_{r-1})}(y_r - y_{r-1}) dy_r. \quad (2.19)$$

We will use $\widehat{\mathbf{D}}_{W,V}$ in the obvious sense, where (W, V) are positive random variables independent of the family $\widehat{\mathbf{D}}_{t,z}$.

Theorem 2.2. *Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s., $\Pi_{\mathcal{H}} \in \mathcal{S}$, and $A_{H^*} \in RV(\gamma)$ with $\gamma \in (0, 1)$.*

1. *Then the following are equivalent;*

(a) *$P^{(u)}(O^{(u)} \in a(u)dx)$ has a non-degenerate limit for some $a(u) > 0$, $a(u) \rightarrow \infty$;*

(b) *either $\bar{\Pi}_{\mathcal{H}} \in RV(1 - \gamma - \beta)$ for some $\beta > 1 - \gamma$ and then (a) holds with $a(u) = u$ (Case (i)), or $\Pi_{\mathcal{H}} \in MDA(\Lambda)$ and then (a) holds with $a(u) = \int_u^\infty \bar{\Pi}_{\mathcal{H}}(y)dy / \bar{\Pi}_{\mathcal{H}}(u)$ (Case (ii));*

(c) *either $\bar{\Pi}_X^+ \in RV(-\beta)$ for some $\beta > 1 - \gamma$ (Case (i)) or $\Pi_X^{(+)} \in MDA(\Lambda)$ (Case (ii)), and $a(u)$ may then be chosen as $a(u) = \int_u^\infty \bar{\Pi}_X^+(y)dy / \bar{\Pi}_X^+(u)$, $u > 0$.*

2. *In addition to (a)–(c), further assume that for each $t > 0$, X_t has a non-lattice distribution. Then, uniformly for $0 < \Delta \leq \Delta_0$, $0 < z \leq \Delta_0$, and $0 < t \leq \Delta_0$, for any fixed Δ_0 ,*

$$\lim_{u \rightarrow \infty} a(u)b(a(u))P^{(u)}(Z^{(u)} \in (a(u)z, a(u)z + \Delta], \tau_u \in b(a(u))dt) = h_t(z)f(z)\Delta dt, \quad (2.20)$$

where, in Case (i),

$$f(z) = \frac{\Gamma(\beta)}{\Gamma(\gamma + \beta - 1)(1+z)^\beta}, \quad z > 0,$$

and in Case (ii)

$$f(z) = e^{-z}, \quad z > 0.$$

Moreover, with $0 = s_0 < s_1 < \cdots < s_{k-1} < 1$, and $I_i = (a(u)z_i, a(u)z_i + \Delta_i]$, $z_i > 0$, $1 \leq i \leq k$, write

$$A_k = \{X^*(s_i t b(a(u))) \in I_i, 1 \leq i < k\}.$$

Then, uniformly for $\Delta_i \in (0, \Delta_0]$, $t \in (0, \Delta_0]$, and $z_i \in (0, \Delta_0]$, $i = 1, 2, \dots, k$, we have

$$\begin{aligned} \lim_{u \rightarrow \infty} (a(u))^k b(a(u)) P^{(u)}(A_k, Z^{(u)} \in (a(u)z_k, a(u)z_k + \Delta_k], \tau_u \in b(a(u))dt) \\ = \theta(z_1, z_2, \dots, z_k, t) \prod_{i=1}^k \Delta_i dt. \end{aligned} \quad (2.21)$$

Here, with $z_0 = 0$ and $s_k = 1$,

$$\theta(z_1, z_2, \dots, z_k, t) = \prod_{i=1}^k h_{t(s_i - s_{i-1})}(z_i - z_{i-1}) f(z_k).$$

3. Further: assume (a)–(c), and that X_t has a non-lattice distribution for each $t > 0$. Then, under $P^{(u)}$, the process $\mathbf{Y}^{(u)}$ defined in (2.16) converges weakly in $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$ to $(V, U, W, (\widehat{\mathbf{D}}_{W,V}(s))_{0 < s < 1})$, where in Case (i)

$$P(V \in dz, U \in dy, W \in dt) = \frac{\Gamma(\beta + 1)}{\Gamma(\gamma + \beta - 1)(1 + z + y)^{\beta+1}} h_t(z) dz dy dt, \quad t, y, z > 0, \quad (2.22)$$

and in Case (ii)

$$P(V \in dz, U \in dy, W \in dt) = e^{-z-y} h_t(z) dz dy dt, \quad t, y, z > 0. \quad (2.23)$$

Remark 2.1. (i) The further assumption in Part 2 of Theorem 2.2, that for each $t > 0$, X_t has a non-lattice distribution, is equivalent to assuming that X is not a compound Poisson process whose step distribution takes values on a lattice. We can cover the lattice case also with only minor adjustments. Thus if the lattice has span 1, we need only restrict Δ to take integer values and replace $(a(u)z, a(u)z + \Delta]$ in (2.20) by $([a(u)z], [a(u)z] + \Delta]$, and similarly in (2.21) for a valid conclusion. The only difference in the proof is which version of a local limit theorem is used.

(ii) (2.17) and (2.18), and (2.22) and (2.23), show that, under the conditions of Theorem 2.1 and Theorem 2.2, the limiting distribution of $Z^{(u)}/a(u)$ (and of course those of $O^{(u)}/a(u)$ and $\tau_u/b(a(u))$) is concentrated on $[0, \infty)$. Thus, $\lim_{u \rightarrow \infty} P(Z^{(u)}/a(u) \leq -x) = 0$ for all $x > 0$. So it's convenient to define $Z^{(u)} = -X_{\tau_u}$ as we did in (2.12).

(iii) $\lim_{u \rightarrow \infty} P(Z^{(u)}/a(u) \leq 0) = 0$ implies that $\lim_{u \rightarrow \infty} P(X_{\tau_u} = u) = 0$, that is, X “creeps” over level u with probability tending to 0 as $u \rightarrow \infty$. This follows because, in order to creep with $Z^{(u)} > 0$, X would have to pass continuously over the interval $(0, u)$, or, equivalently \mathcal{H} would have to reach level u without any jumps. This probability is exponentially small, or zero if $(0, \infty)$ is regular for 0.

(iv) In general we cannot replace Condition (2.10) with simple conditions on Π_X directly; see the remark in Section 5 following the proof of Theorem 2.2.

(v) The marginal limiting distributions of the fluctuation quantities are easily computed from (2.17) and (2.18), and (2.22) and (2.23). The identities $t^\delta h_t(z) = h_1(z/t^\delta)$ and $\int_0^\infty h_t(z) dz = z^{-\gamma}/\Gamma(\bar{\gamma})$, where $\bar{\gamma} = 1 - \gamma$ (see Sato (1999), p. 261), are useful. Thus, for example, under the conditions of Case (i) of Theorem 2.2, the limiting values of $(Z^{(u)}, O^{(u)})$ and τ_u , suitably normalised, are

$$P(V \in dz, U \in dy) = \frac{\Gamma(\beta + 1)}{\Gamma(1 - \gamma)\Gamma(\gamma + \beta - 1)(1 + z + y)^{\beta+1}} dz dy, \quad y, z > 0, \quad (2.24)$$

and

$$P(W \in dt) = \frac{\Gamma(\beta)}{\Gamma(\gamma + \beta - 1)} \int_0^\infty \frac{h_1(z) dz}{(1 + t^{1/\bar{\gamma}} z)} dt, \quad t > 0. \quad (2.25)$$

It can be checked that no pair of (V, U, W) are independent, in Case (i). For Case (ii)

$$P(V \in dz, U \in dy) = \frac{z^{-\gamma} e^{-z-y}}{\Gamma(1-\gamma)} dz dy, \quad y, z > 0, \quad (2.26)$$

and

$$P(W \in dt) = \int_0^\infty e^{-t^{1/\bar{\gamma}} z} dz dt, \quad t > 0. \quad (2.27)$$

In this case, V is independent of U , U is independent of W , but V is not independent of W .

3 Preliminaries to the Proofs

Our first result applies to any defective subordinator, so we change notation slightly just for this result.

Proposition 3.1. *Let Y be any defective subordinator, obtained from a nondefective subordinator \mathcal{Y} with killing rate q , whose Lévy measure is Π_Y , with tail $\bar{\Pi}_Y$. Assume $\Pi_Y \in \mathcal{S}$. Write $P_Y^{(u)}$ for $P(\cdot | T_u^Y < \infty)$, where $T_u^Y = \inf\{t : Y_t > u\}$, $u > 0$, and put $O_Y^{(u)} = Y_{T_u^Y} - u$ on the event $\{T_u^Y < \infty\}$. Then $P_Y^{(u)}(O_Y^{(u)} \in a(u) dx)$ has a non-degenerate limit $P(O \in dx)$ for some $a(u) > 0$, $a(u) \rightarrow \infty$, iff either $\bar{\Pi}_Y \in RV(-\alpha)$ for some $\alpha > 0$, or $\Pi_Y \in \text{MDA}(\Lambda)$. Moreover, in the first case we can take $a(u) = u$ and O to have density $\alpha(1+x)^{-1-\alpha}$, and in the second case we can take $a(u) = \int_u^\infty \bar{\Pi}_Y(y) dy / \bar{\Pi}_Y(u) = o(u)$ and O to have density e^{-x} .*

Proof of Proposition 3.1: For the distribution of $O_Y^{(u)}$, decompose according to $\{T_u^Y = t\}$ and use the compensation formula for Poisson point processes to get

$$\begin{aligned} & P(O_Y^{(u)} > xa(u), T_u^Y < \infty) = P(Y_{T_u^Y} > u + xa(u), T_u^Y < \infty) \\ &= E \sum_{0 < t < L_\infty} \mathbf{1}_{\{Y_t > u + xa(u), T_u^Y = t\}} = E \int_0^\infty q e^{-qs} \sum_{0 < t < s} \mathbf{1}_{\{\mathcal{Y}_{t-} + \Delta \mathcal{Y}_t > u + xa(u), \mathcal{Y}_{t-} \leq u\}} ds \\ &= E \sum_{t > 0} e^{-qt} \mathbf{1}_{\{\mathcal{Y}_{t-} + \Delta \mathcal{Y}_t > u + xa(u), \mathcal{Y}_{t-} \leq u\}} \\ &= \int_0^\infty e^{-qt} \int_{(xa(u), \infty)} P(u \geq \mathcal{Y}_{t-} > (u + xa(u) - y) \vee 0) \Pi_Y(dy) dt \\ &= \int_0^\infty e^{-qt} \int_{(xa(u), \infty)} \Pi_Y(dy) \int_{(u + xa(u) - y) \vee 0 < z \leq u} P(\mathcal{Y}_{t-} \in dz) dt \\ &= \int_0^\infty e^{-qt} \int_{(0, u]} \int_{(u + xa(u) - z, \infty)} \Pi_Y(dy) P(\mathcal{Y}_t \in dz) dt. \end{aligned}$$

Using this, and writing $e(q)$ for an independent $Exp(q)$ random variable, we have for any

$C_0 > 0$

$$\begin{aligned}
P(O_Y^{(u)} > xa(u), T_u^Y < \infty) &= \int_0^\infty e^{-qt} \int_{(0,u]} P(\mathcal{Y}_t \in dz) \bar{\Pi}_Y(u + xa(u) - z) dt \\
&= q^{-1} \int_{(0,u]} P(\mathcal{Y}_{e(q)} \in dz) \bar{\Pi}_Y(u + xa(u) - z) \\
&= q^{-1} \left(\int_{(0,C_0]} + \int_{(C_0,u]} \right) P(\mathcal{Y}_{e(q)} \in dz) \bar{\Pi}_Y(u + xa(u) - z).
\end{aligned} \tag{3.1}$$

Assume at this stage that $\Pi_Y \in \mathcal{S}$. Then $\Pi_Y \in \mathcal{L}$, so we have

$$\bar{\Pi}_Y(u - z + xa(u)) \sim \bar{\Pi}_Y(u + xa(u)) \text{ uniformly for } z \in (0, C_0] \text{ and } x \geq 0. \tag{3.2}$$

Thus

$$\int_{(0,C_0]} P(\mathcal{Y}_{e(q)} \in dz) \bar{\Pi}_Y(u + xa(u) - z) \sim P(\mathcal{Y}_{e(q)} \leq C_0) \bar{\Pi}_Y(u + xa(u)). \tag{3.3}$$

Since $\Pi_Y \in \mathcal{S}$, we know from Lemma 3.5 of Klüppelberg et al. (2004) (with $\alpha = 0$) that $\bar{\Pi}_Y(u) \sim qP(T_u^Y < \infty)$. Given arbitrary $\varepsilon \in (0, 1)$, we can choose $C_0 > 0$ such that $P(\mathcal{Y}_{e(q)} > C_0) \leq \varepsilon$. Then for u large enough, again using (3.2),

$$\begin{aligned}
(1 + \varepsilon) \bar{\Pi}_Y(u) &\geq qP(T_u^Y < \infty) \\
&= \left(\int_{(0,C_0]} + \int_{(C_0,\infty)} \right) P(\mathcal{Y}_{e(q)} \in dz) \bar{\Pi}_Y(u - z) \\
&\geq (1 - \varepsilon) P(\mathcal{Y}_{e(q)} \leq C_0) \bar{\Pi}_Y(u) + \int_{(C_0,u]} P(\mathcal{Y}_{e(q)} \in dz) \bar{\Pi}_Y(u - z),
\end{aligned}$$

giving

$$\int_{(C_0,u]} P(\mathcal{Y}_{e(q)} \in dz) \bar{\Pi}_Y(u - z) \leq ((1 + \varepsilon) - (1 - \varepsilon)^2) \bar{\Pi}_Y(u) \leq 3\varepsilon \bar{\Pi}_Y(u).$$

From this, and (3.1) and (3.3), and since $\bar{\Pi}_Y(u) \sim qP(T_u^Y < \infty)$, we have

$$\begin{aligned}
P^{(u)} \left(O_Y^{(u)} > xa(u) \right) &= \frac{P \left(O_Y^{(u)} > xa(u), T_u^Y < \infty \right)}{P(T_u^Y < \infty)} \\
&= (1 + o(1)) P(\mathcal{Y}_{e(q)} \leq C_0) \frac{\bar{\Pi}_Y(u + xa(u) - z)}{\bar{\Pi}_Y(u)} + o(1).
\end{aligned} \tag{3.4}$$

As discussed in (2.7), the condition $\bar{\Pi}_Y \in RV(-\alpha)$ for some $\alpha > 0$, or $\Pi_Y \in \text{MDA}(\Lambda)$, is equivalent to the existence of $a(u) \rightarrow \infty$ such that

$$\frac{\bar{\Pi}_Y(u + xa(u))}{\bar{\Pi}_Y(u)} \rightarrow P(O > x), \tag{3.5}$$

and when it holds $a(u)$ and O have the stated properties. The conclusions of the proposition then follow from this and (3.4). \square

We will make use of Vigon's (2002) "équations amicales", which are

$$\bar{\Pi}_X^+(u) = \int_{(0,\infty)} \bar{\Pi}_{H^*}(y+u)\Pi_{\mathcal{H}}(dy) + d_{H^*}n(u), \quad u > 0, \quad (3.6)$$

and

$$\bar{\Pi}_X^-(u) = \int_{(0,\infty)} \bar{\Pi}_{\mathcal{H}}(y+u)\Pi_{H^*}(dy) + d_{\mathcal{H}}n^*(u) + q\bar{\Pi}_{H^*}(u), \quad u > 0, \quad (3.7)$$

where $n(\cdot)$, $n^*(\cdot)$ denote càdlàg versions of the densities of $\Pi_{\mathcal{H}}$, Π_{H^*} , defined if $d_{\mathcal{H}} > 0$, $d_{H^*} > 0$, respectively.

We are looking for limit theorems which will always include the convergence of the normed overshoot, so from now on we will add to our basic assumptions the following:

$$\bar{\Pi}_{\mathcal{H}} \in RV(-\alpha) \text{ for some } \alpha > 0 \text{ or } \Pi_{\mathcal{H}} \in \text{MDA}(\Lambda). \quad (3.8)$$

Results in Asmussen and Klüppelberg (1996) suggest that when $E|X_1| < \infty$, so that $EX_1 \in (-\infty, 0)$, and $EH_1^* < \infty$, we have (3.8) equivalent to

$$\bar{\Pi}_X^+ \in RV(-\beta) \text{ for some } \beta > 1 \text{ or } \Pi_X \in \text{MDA}(\Lambda). \quad (3.9)$$

We will prove this, and in fact a more general result, in the next proposition. At this stage we are not assuming $\Pi_{\mathcal{H}} \in \mathcal{S}$.

Proposition 3.2. *Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s. and $A_{H^*} \in RV(\gamma)$ with $\gamma \in [0, 1)$. Suppose (3.8) holds with $\alpha = \beta + \gamma - 1 > 0$. Then $\bar{\Pi}_X^+ \in RV(-\beta)$ or $\Pi_X \in \text{MDA}(\Lambda)$, equivalently,*

$$\frac{\bar{\Pi}_X(u + xa(u))}{\bar{\Pi}_X(u)} \rightarrow P(C > x), \quad x > 0, \quad (3.10)$$

where $a(u) = u$ and $P(C > x) = (1+x)^{-\beta}$ (Case (i)), or $a(u) = \int_u^\infty \bar{\Pi}_{\mathcal{H}}(y)dy/\bar{\Pi}_{\mathcal{H}}(u)$ and $P(C > x) = e^{-x}$ (Case (ii)). Further, in both cases we have, for some constants $c_{\gamma,\beta} \in (0, \infty)$ (whose values are made explicit in the proof),

$$\bar{\Pi}_X^+(u) \sim \frac{c_{\gamma,\beta}\bar{\Pi}_{\mathcal{H}}(u)A_{H^*}(a(u))}{a(u)}. \quad (3.11)$$

Moreover, in Case (ii) we can alternatively take $a(u) = \int_u^\infty \bar{\Pi}_X^+(y)dy/\bar{\Pi}_X^+(u)$, $u > 0$.

Proof of Proposition 3.2: Assume (2.1), and that (2.13) holds with $\gamma \in [0, 1)$.

The starting point is Vigon's équation amicale, (3.6), which we write as $\bar{\Pi}_X^+(u) = I(u) + d_{H^*}n(u)$, with

$$\begin{aligned} I(u) &= \int_{(0,\infty)} \Pi_{\mathcal{H}}(u+dy) \int_{(y,\infty)} \Pi_{H^*}(dz) = \int_{(0,\infty)} \Pi_{H^*}(dz) \int_{(0,z)} \Pi_{\mathcal{H}}(u+dy) \\ &= \int_{(0,\infty)} \Pi_{H^*}(a(u)dz) \Pi_{\mathcal{H}}((u, u+a(u)z]) \\ &= \left(\int_{(0,K]} + \int_{(K,\infty)} \right) \Pi_{H^*}(a(u)dz) \Pi_{\mathcal{H}}((u, u+a(u)z]) \\ &=: I_1(u) + I_2(u), \text{ say,} \end{aligned} \quad (3.12)$$

where $K > 0$. Recall the definition of A_{H^*} in (2.5), and note that

$$u\bar{\Pi}_{H^*}(u) \leq \int_0^u \bar{\Pi}_{H^*}(y)dy = A_{H^*}(u), \quad u > 0,$$

so we have by the regular variation of A_{H^*}

$$\frac{a(u)I_2(u)}{A_{H^*}(a(u))\bar{\Pi}_{\mathcal{H}}(u)} \leq \frac{a(u)\bar{\Pi}_{H^*}(Ka(u))}{A_{H^*}(a(u))} \leq \frac{a(u)A_{H^*}(Ka(u))}{KA_{H^*}(a(u))} \sim \frac{a(u)}{K^{1-\gamma}}.$$

Since $0 \leq \gamma < 1$ it follows that

$$\lim_{K \rightarrow \infty} \limsup_{u \rightarrow \infty} \frac{a(u)I_2(u)}{A_{H^*}(a(u))\bar{\Pi}_{\mathcal{H}}(u)} = 0. \quad (3.13)$$

Now assume (3.8) with $\alpha = \beta + \gamma - 1$. By (2.7) with \bar{F} replaced by $\bar{\Pi}_{\mathcal{H}}$, this implies

$$\frac{\Pi_{\mathcal{H}}\{(u, u + a(u)z]\}}{\bar{\Pi}_{\mathcal{H}}(u)} \rightarrow \int_0^z p(y)dy$$

uniformly for $z \in [0, K]$, where $p(\cdot)$ is the density of the limit random variable, C , specified in either case, $Par(\beta - 1 + \gamma)$ or $Exp(1)$. So the component $I_1(u)$ in (3.12) satisfies

$$\begin{aligned} I_1(u) &\sim \bar{\Pi}_{\mathcal{H}}(u) \int_0^K \Pi_{H^*}(a(u)dz) \int_0^z p(y)dy \\ &= \bar{\Pi}_{\mathcal{H}}(u) \int_0^K p(y)dy \int_y^K \Pi_{H^*}(a(u)dz) \\ &= \bar{\Pi}_{\mathcal{H}}(u) \int_0^K p(y)\bar{\Pi}_{H^*}(a(u)y)dy - \bar{\Pi}_{\mathcal{H}}(u)\bar{\Pi}_{H^*}(a(u)K) \int_0^K p(y)dy. \end{aligned} \quad (3.14)$$

(a) When $0 < \gamma < 1$, $A_{H^*} \in RV(\gamma)$ is equivalent, by the monotone density theorem (Bingham et al. (1987, Thm 1.7.2, p.39)), to $\bar{\Pi}_{H^*} \in RV(\gamma - 1)$, and then $\bar{\Pi}_{\mathcal{H}}(x) \sim \gamma x^{-1}A_{H^*}(x)$. So

$$\int_0^K p(y)\bar{\Pi}_{H^*}(a(u)y)dy \sim \frac{\gamma A_{H^*}(a(u))}{a(u)} \int_0^K p(y)y^{\gamma-1}dy, \quad (3.15)$$

and by taking $u \rightarrow \infty$ then $K \rightarrow \infty$ in (3.14) we conclude, for $0 < \gamma < 1$,

$$\lim_{K \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{a(u)I_1(u)}{A_{H^*}(a(u))\bar{\Pi}_{\mathcal{H}}(u)} = \gamma \int_0^\infty p(y)y^{\gamma-1}dy = \gamma E(C^{\gamma-1}). \quad (3.16)$$

(b) When $\gamma = 0$, so that A_{H^*} is slowly varying, we use the feature that $\lim_{x \downarrow 0} p(x) = p(0) > 0$ to argue, given arbitrary $\varepsilon > 0$, the existence of a $\delta_\varepsilon > 0$ such that for all large enough u ,

$$a(u) \int_0^{\delta_\varepsilon} p(y)\bar{\Pi}_{\mathcal{H}}(a(u)y)dy \leq p(0)(1 + \varepsilon)A_{H^*}(\delta_\varepsilon a(u)) \sim p(0)(1 + \varepsilon)A_{H^*}(a(u))$$

and

$$a(u) \int_0^{\delta_\varepsilon} p(y)\bar{\Pi}_{H^*}(a(u)y)dy \geq p(0)(1 - \varepsilon)A_{H^*}(\delta_\varepsilon a(u)) \sim p(0)(1 - \varepsilon)A_{H^*}(a(u)).$$

A_{H^*} slowly varying implies $x\bar{\Pi}_{H^*}(x) = o(A_{H^*}(x))$ as $x \rightarrow \infty$, so with δ_ε fixed we can argue

$$\begin{aligned} \int_{\delta_\varepsilon}^K p(y)\bar{\Pi}_{H^*}(a(u)y)dy &= o\left(\frac{1}{a(u)} \int_{\delta_\varepsilon}^K p(y)A_{H^*}(a(u)y)\frac{dy}{y}\right) \\ &= o\left(\frac{A_{H^*}(a(u))}{a(u)}\right), \end{aligned}$$

and we deduce for $\gamma = 0$ that

$$\lim_{K \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{a(u)I_1(u)}{A_{H^*}(a(u))\bar{\Pi}_{\mathcal{H}}(u)} = p(0). \quad (3.17)$$

Thus in all cases we have

$$I(u) \sim \frac{c(\gamma, \beta)A_{H^*}(a(u))\bar{\Pi}_{\mathcal{H}}(u)}{a(u)} \quad (3.18)$$

for a constant $c(\gamma, \beta) \in (0, \infty)$ which we can evaluate as follows.

(a) When $\gamma \in (0, 1)$, in Case (i)

$$c(\gamma, \beta) = \gamma E(C^{\gamma-1}) = \gamma\beta \int_0^\infty \frac{x^{\gamma-1}dx}{(1+x)^{\beta+1}} = \frac{\Gamma(\gamma+1)\Gamma(\beta-\gamma+1)}{\Gamma(\beta)}, \quad (3.19)$$

while in Case (ii)

$$c(\gamma, \beta) = \gamma E(C^{\gamma-1}) = \Gamma(\gamma+1). \quad (3.20)$$

(b) When $\gamma = 0$, $p(0) = \beta$ in Case (i), and in Case (ii), $p(0) = 1$, so we set $c(0, \beta) = \beta$ in Case (i), and $c(0, \beta) = 1$ in Case (ii).

Now integrate (3.6) and use the estimate (3.18) to get

$$\begin{aligned} \int_u^\infty \bar{\Pi}_X^+(y)dy &= \int_u^\infty I(v)dv + d_{H^*}\bar{\Pi}_{\mathcal{H}}(u) \\ &\sim c(\gamma, \beta) \int_u^\infty \frac{A_{H^*}(a(v))\bar{\Pi}_{\mathcal{H}}(v)}{a(v)}dv + d_{H^*}\bar{\Pi}_{\mathcal{H}}(u). \end{aligned} \quad (3.21)$$

Assume in addition that $\bar{\Pi}_{\mathcal{H}} \in RV(1-\gamma-\beta)$. This together with $A_{H^*} \in RV(\gamma)$ means that the product $\bar{\Pi}_{\mathcal{H}}A_{H^*} \in RV(1-\beta)$. Then, taking $a(u) = u$ in this case, (3.21) gives

$$\frac{1}{\bar{\Pi}_{\mathcal{H}}(u)A_{H^*}(u)} \int_u^\infty \bar{\Pi}_X^+(y)dy \sim c(\gamma, \beta) \int_1^\infty v^{-\beta}dv + \frac{d_{H^*}}{A_{H^*}(u)}. \quad (3.22)$$

In either case, $A_{H^*}(\infty) = \infty$ or $A_{H^*}(\infty) < \infty$, we can use the monotone density theorem again to deduce from this that $\bar{\Pi}_X^+ \in RV(-\beta)$, and hence that (3.10) holds with $a(u) = u$.

Alternatively, suppose $\bar{\Pi}_{\mathcal{H}} \in \text{MDA}(\Lambda)$. In this case, (3.21) gives

$$\int_{u+xa(u)}^\infty \bar{\Pi}_X^+(y)dy \sim c(\gamma, \beta) \int_{u+xa(u)}^\infty \frac{A_{H^*}(a(v))\bar{\Pi}_{\mathcal{H}}(v)}{a(v)}dv + d_{H^*}\bar{\Pi}_{\mathcal{H}}(u+xa(u)), \quad x \geq 0.$$

Change variable by $v = u + v'a(u)$ on the RHS. Since $a(\cdot)$ is self-neglecting, we have $a(v) = a(u + v'a(u)) \sim a(u)$, so by the regular variation of A_{H^*} ,

$$\frac{A_{H^*}(a(v))}{a(v)} \sim \frac{A_{H^*}(a(u))}{a(u)},$$

and since $\bar{\Pi}_{\mathcal{H}} \in \text{MDA}(\Lambda)$,

$$\bar{\Pi}_{\mathcal{H}}(v) = \bar{\Pi}_{\mathcal{H}}(u + v'a(u)) \sim e^{-v'} \bar{\Pi}_{\mathcal{H}}(u).$$

Thus for $x \geq 0$

$$\begin{aligned} \frac{1}{\bar{\Pi}_{\mathcal{H}}(u)} \int_{u+xa(u)}^{\infty} \bar{\Pi}_X^+(y) dy &\sim c(\gamma, \beta) a(u) \int_x^{\infty} \frac{A_{H^*}(a(v)) \bar{\Pi}_{\mathcal{H}}(v)}{a(v) \bar{\Pi}_{\mathcal{H}}(u)} dv' + d_{H^*} \frac{\bar{\Pi}_{\mathcal{H}}(u + xa(u))}{\bar{\Pi}_{\mathcal{H}}(u)} \\ &\sim c(\gamma, \beta) A_{H^*}(a(u)) \int_x^{\infty} e^{-v'} dv' + e^{-x} d_{H^*}, \end{aligned} \quad (3.23)$$

which, applied with $x = 0$, also gives

$$\frac{\int_{u+xa(u)}^{\infty} \bar{\Pi}_X^+(y) dy}{\int_u^{\infty} \bar{\Pi}_X^+(y) dy} \rightarrow e^{-x}, \quad x \geq 0.$$

Applying Thm 2.7.3(b) p.110 of de Haan (1970) we get

$$\frac{\bar{\Pi}_X^+(u + xa(u))}{\bar{\Pi}_X^+(u)} \rightarrow e^{-x}, \quad x \geq 0,$$

which is (3.10) in this case, and this implies

$$\frac{\int_{u+xa(u)}^{\infty} \bar{\Pi}_X^+(y) dy}{a(u) \bar{\Pi}_X^+(u)} \rightarrow e^{-x}, \quad x \geq 0, \quad (3.24)$$

hence

$$a(u) \sim \frac{\int_u^{\infty} \bar{\Pi}_X^+(y) dy}{\bar{\Pi}_X^+(u)}, \quad (3.25)$$

as claimed for this case.

It remains to prove (3.11). In Case (i), when $\bar{\Pi}_{\mathcal{H}} \in RV(1 - \gamma - \beta)$ and $\bar{\Pi}_X \in RV(-\beta)$, the relation (3.22) gives

$$\bar{\Pi}_X^+(u) \sim \frac{\beta - 1}{u} \int_u^{\infty} \bar{\Pi}_X^+(y) dy \sim \left(c(\gamma, \beta) + \frac{(\beta - 1) d_{H^*}}{A_{H^*}(u)} \right) \frac{\bar{\Pi}_{\mathcal{H}}(u) A_{H^*}(u)}{u}. \quad (3.26)$$

- (a) When $\gamma \in (0, 1)$, this implies (3.11) with $c_{\gamma, \beta} = c(\gamma, \beta) + (\beta - 1) d_{H^*} / EH_1^*$, for $EH_1^* \leq \infty$.
(b) When $\gamma = 0$, $c_{0, \beta} = c(0, \beta)$ for $EH_1^* = \infty$ and

$$\begin{aligned} c_{0, \beta} &= c(0, \beta) + \frac{(\beta - 1) d_{H^*}}{EH_1^* - d_{H^*}} = \beta + \frac{(\beta - 1) d_{H^*}}{EH_1^* - d_{H^*}} \\ &= \frac{\beta EH_1^* - d_{H^*}}{EH_1^* - d_{H^*}} = \frac{\beta EH_1^* - d_{H^*}}{A_{H^*}(\infty)}, \end{aligned}$$

for $EH_1^* < \infty$. In Case (ii), when $\Pi_X \in \text{MDA}(\Lambda)$, (3.23) and (3.24) give

$$\bar{\Pi}_X^+(u) \sim \frac{1}{a(u)} \int_u^{\infty} \bar{\Pi}_X^+(y) dy \sim \left(c(\gamma, \beta) + \frac{d_{H^*}}{A_{H^*}(a(u))} \right) \frac{\bar{\Pi}_{\mathcal{H}}(u) A_{H^*}(a(u))}{a(u)}. \quad (3.27)$$

(a) When $\gamma \in (0, 1)$ this implies (3.11) with $c_{\gamma, \beta} = c(\gamma, \beta) + d_{H^*}/EH_1^*$, for $EH_1^* \leq \infty$. (b) When $\gamma = 0$, $c_{0, \beta} = 1$ for $EH_1^* = \infty$ and

$$\begin{aligned} c_{0, \beta} &= c(0, \beta) + \frac{d_{H^*}}{EH_1^* - d_{H^*}} = 1 + \frac{d_{H^*}}{EH_1^* - d_{H^*}} \\ &= \frac{EH_1^*}{EH_1^* - d_{H^*}} = \frac{EH_1^*}{A_{H^*}(\infty)}. \end{aligned}$$

for $EH_1^* < \infty$. This completes Proposition 3.2. \square

It is important for our analysis that the condition $A_{H^*} \in RV(\gamma)$ can be expressed in terms of the left-hand tail $\bar{\Pi}_X^-$. Doney (2007, Cor. 4, p.31) (interchange $+/-$ in his result) shows that, when $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s., $E|X_1| < \infty$ iff $EH_1^* < \infty$, and then $E|X_1| = qEH_1^*$. The following proposition generalises this, allowing for $EH_1^* = \infty$.

Proposition 3.3. *Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s and $A_X^*(\infty) = \infty$, or, equivalently, $EH_1^* = \infty$. Then*

$$\lim_{x \rightarrow \infty} \frac{A_X^*(x)}{A_{H^*}(x)} = q. \quad (3.28)$$

Proof of Proposition 3.3: Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s and $A_{H^*}(\infty) = \infty$. The integral term in (3.7) can be written as

$$\int_{(0, \infty)} (\bar{\Pi}_{H^*}(u) - \bar{\Pi}_{H^*}(y+u)) \Pi_{\mathcal{H}}(dy) = \int_{(0, \infty)} \bar{\Pi}_{\mathcal{H}}(y) dy (\bar{\Pi}_{H^*}(u) - \bar{\Pi}_{H^*}(y+u))$$

after integrating by parts. So, by integrating (3.7), we have

$$A_X^*(x) - q \int_1^x \bar{\Pi}_{H^*}(u) du = d_{\mathcal{H}}(\bar{\Pi}_{H^*}(1) - \bar{\Pi}_{H^*}(x)) + I(x), \quad (3.29)$$

where

$$I(x) = \int_{(0, \infty)} \Pi_{\mathcal{H}}(dy) \int_1^x (\bar{\Pi}_{H^*}(u) - \bar{\Pi}_{H^*}(y+u)) du.$$

The inner integral can be written as

$$\left(\int_1^x - \int_{1+y}^{x+y} \right) \bar{\Pi}_{H^*}(u) du = \left(\int_1^{1+y} - \int_x^{x+y} \right) \bar{\Pi}_{H^*}(u) du,$$

so

$$\begin{aligned} I(x) &= \int_0^\infty \Pi_{\mathcal{H}}(dy) \int_0^y (\bar{\Pi}_{H^*}(1+w) - \bar{\Pi}_{H^*}(x+w)) dw \\ &= \int_0^\infty \bar{\Pi}_{\mathcal{H}}(w) (\bar{\Pi}_{H^*}(1+w) - \bar{\Pi}_{H^*}(x+w)) dw \\ &\leq \bar{\Pi}_{H^*}(1) \int_0^K \bar{\Pi}_{\mathcal{H}}(w) dw + \bar{\Pi}_{\mathcal{H}}(K) \int_K^\infty \int_{1+w < z \leq x+w} \Pi_{H^*}(dz) dw, \end{aligned}$$

where $K > 0$. The last integral here can be written as

$$\begin{aligned} & \left(\int_{1+K < z \leq x+K} \int_K^{z-1} + \int_{z > x+K} \int_{z-x}^{z-1} \right) dw \Pi_{H^*}(dz) \\ &= \left(\int_{1+K < z \leq x+K} (z-1-K) + (x-1) \int_{z > x+K} \right) \Pi_{H^*}(dz) \\ &= \int_{1+K}^{x+K} \bar{\Pi}_{H^*}(z) dz. \end{aligned}$$

It follows that

$$I(x) \leq \bar{\Pi}_{H^*}(1) \int_0^K \bar{\Pi}_{\mathcal{H}}(w) dw + \bar{\Pi}_{\mathcal{H}}(K) \int_{1+K}^{x+K} \bar{\Pi}_{H^*}(w) dw. \quad (3.30)$$

Now note that the second term in (3.30) is bounded above by

$$\bar{\Pi}_{\mathcal{H}}(K) \left(\int_0^x + \int_x^{x+K} \right) \bar{\Pi}_{H^*}(w) dw \leq \bar{\Pi}_{\mathcal{H}}(K) (A_{H^*}(x) + K \bar{\Pi}_{\mathcal{H}}(K)).$$

Since $A_{H^*}(\infty) = \infty$, when we divide by $A_{H^*}(x)$ and let $x \rightarrow \infty$ and then $K \rightarrow \infty$ in (3.30) we get $\lim_{x \rightarrow \infty} I(x)/A_{H^*}(x) = 0$. Then (3.28) follows from (3.29). \square

Remark 3.1. We mention that a random walk version of Proposition 3.3 is (in a different notation) in Lemma 1 of Denisov, Foss and Korshunov (2004).

4 The Case $\gamma = 0$ (including Finite Mean).

Assume (2.1) and (2.13) with $\gamma = 0$, so $A_{H^*} \in RV(0)$, or, equivalently, $x \bar{\Pi}_{H^*}(x) = o(A_{H^*}(x))$ as $x \rightarrow \infty$. Now (e.g., use Theorem 4.4 of Doney and Maller (2002) with $+/-$ interchanged) (2.1) implies

$$\frac{x \bar{\Pi}^+(x)}{A_X^*(x)} \leq \frac{A_X^+(x)}{A_X^*(x)} \rightarrow 0 \text{ as } x \rightarrow \infty \quad (4.1)$$

if $A_X^*(\infty) = \infty$, otherwise $A_X^*(\infty) < \infty$ and then $A_X^+(\infty) < \infty$ and $\lim_{x \rightarrow \infty} x \bar{\Pi}^+(x) = 0$. Thus, with

$$A(x) := \gamma + \bar{\Pi}^+(1) - \bar{\Pi}^-(1) + A_X^+(x) - A_X^*(x), \quad x > 0,$$

we see that $x \bar{\Pi}(x) = o(-A_X(x))$ as $x \rightarrow \infty$, and this means that X_t^* is positively relatively stable as $t \rightarrow \infty$. Consequently, there is a continuous, increasing function $c(\cdot) \in RV(1)$ such that $X_t^*/c(t) \xrightarrow{P} 1$ as $t \rightarrow \infty$. The function $c(\cdot)$ can be chosen to satisfy

$$c(x) = x A_X^*(c(x)),$$

and its inverse function $b(\cdot) := c^{-1}(\cdot)$ is given by

$$b(y) = \frac{y}{A_X^*(y)}, \quad y > 0.$$

Employing Proposition 3.3, we see that

$$b(y) = \frac{y}{A_X^*(y)} \sim \frac{y}{q A_{H^*}(y)} \text{ as } y \rightarrow \infty, \quad (4.2)$$

when $A_{H^*}(\infty) \leq \infty$. When $A_{H^*}(\infty) < \infty$, and so $EX_1 \in (-\infty, 0)$, we simply take $c(x) = |EX_1|x$ and $b(x) = x/|EX_1|$, $x > 0$.

We define another norming function by $r(u) = b(a(u))$, and note that $c(r(u)) = a(u)$ and

$$r(u) \sim \frac{a(u)}{qA_{H^*}(a(u))} \quad (4.3)$$

when $A_{H^*}(\infty) = \infty$, and

$$r(u) \sim \frac{a(u)}{|EX_1|} = \frac{a(u)}{qEH_1^*} \quad (4.4)$$

when $A_{H^*}(\infty) < \infty$. The function $r(u)$ turns out to be the right norming for τ_u in the present situation.

Proof of Theorem 2.1: Assume (2.1) and (2.10), and that (2.13) holds with $\gamma = 0$. Then Parts 1(a) and 1(b) of the theorem are equivalent by Proposition 3.1 applied to the subordinator $\mathcal{Y} := \mathcal{H}$, and Part 1(c) follows from Part 1(b) by Proposition 3.2. We now show that Part 1(c) implies Part 2.

Proposition 4.1. *Assume (2.1) and (2.10), and additionally that $A_{H^*} \in RV(0)$, and either (i) $\bar{\Pi}_X^+ \in RV(-\beta)$, where $\beta > 1$, or (ii) $\Pi_X \in \text{MDA}(\Lambda)$. Then the conclusions of Part 2 of Theorem 2.1 hold.*

Proof of Proposition 4.1: A slight extension of a result proved in Doney and Rivero (2012) states that, on the event $X_{\tau_u-} < u$, the joint distribution of (τ_u, X_{τ_u-}) is given by

$$P(\tau_u \in dt, X_{\tau_u-} \in dy) = P(X_t \in dy, \bar{X}_t \leq u) \bar{\Pi}_X^+(u - y) dt. \quad (4.5)$$

(Recall that $\bar{X}_t = \sup_{0 < s \leq t} X_s$, $t \geq 0$, and $Z^{(u)} = -X_{\tau_u-} = X_{\tau_u-}^*$). So we have, for $t > 0$, $u > 0$, $\varepsilon > 0$,

$$\begin{aligned} & P(\tau_u \in r(u)dt, Z^{(u)} \in [(1 - \varepsilon)c(\tau(u)), (1 + \varepsilon)c(\tau(u))]) \\ &= \int_{[(1-\varepsilon)c(tr(u)), (1+\varepsilon)c(tr(u))]} \bar{\Pi}_X^+(u + y) P(X_{tr(u)}^* \in dy, \bar{X}_{tr(u)} \leq u) dt. \end{aligned}$$

Under the assumptions of the proposition (3.10) holds, and also $c(\cdot) \in RV(1)$ implies $c(tr(u)) \sim t(c(r(u)) = ta(u))$. So the last integral is asymptotically equivalent to

$$\begin{aligned} & \int_{[(1-\varepsilon)t, (1+\varepsilon)t]} \bar{\Pi}_X^+(u + ya(u)) P(X_{tr(u)}^* \in a(u)dy, \bar{X}_{tr(u)} \leq u) \\ & \sim r(u) \bar{\Pi}_X^+(u) \int_{[(1-\varepsilon)t, (1+\varepsilon)t]} P(C > y) P(X_{tr(u)}^* \in a(u)dy, \bar{X}_{tr(u)} \leq u) \\ & = r(u) \bar{\Pi}_X^+(u) \int_{[(1-\varepsilon)t, (1+\varepsilon)t]} P(C > y) P\left(\frac{X_{tr(u)}^*}{c(r(u))} \in dy, \frac{\bar{X}_{tr(u)}}{c(r(u))} \leq \frac{u}{a(u)}\right) \\ & = r(u) \bar{\Pi}_X^+(u) \left\{ \int_{[1-\varepsilon, 1+\varepsilon]} P(C > ty) P\left(\frac{X_{tr(u)}^*}{tc(r(u))} \in dy\right) + o(1) \right\}, \end{aligned}$$

where we use the fact that $a(u) \leq u$ for large u to see that $P(\bar{X}_{tr(u)}/c(r(u)) > u/a(u)) \rightarrow 0$. Next, since

$$P\left(\left|\frac{X_{tr(u)}^*}{tc(r(u))} - 1\right| \leq \eta\right) \rightarrow 1,$$

for arbitrarily small $\eta > 0$ and all $t > 0$, we deduce that

$$\int_{[1-\epsilon, 1+\epsilon]} P(C > ty) P\left(X_{tr(u)}^* \in tc(r(u))dy\right) = P(C > t) + o(1),$$

so that

$$\begin{aligned} & P^{(u)}(\tau_u \in r(u)dt, Z^{(u)} \in [(1-\epsilon)c(\tau_u), (1+\epsilon)c(\tau_u)]) \\ & \sim \frac{r(u)\bar{\Pi}_X^+(u)P(C > t)dt}{P(\tau_u < \infty)} \\ & \sim \frac{a(u)\bar{\Pi}_X^+(u)P(C > t)dt}{\bar{\Pi}_{\mathcal{H}}(u)A_{H^*}(u)} \quad (\text{by (2.11)}) \\ & \rightarrow c_{0,\beta}P(C > t)dt \quad (\text{by (3.11)}). \end{aligned}$$

The evaluation of $c_{0,\beta}$ shows that the limit here is a probability density function, and since it does not depend on ϵ , we deduce that (2.15) holds, and also that, conditioned on $\tau_u = tr(u)$, the $P^{(u)}$ -distribution of $X^*(\tau_u-)/c(\tau_u)$ converges to the distribution concentrated on 1.

To extend this to the k -dimensional distributions, we take $0 < s_1 < s_2 < \dots < s_{k-1} < 1$, set

$$A_k := \left\{1 - \epsilon \leq \frac{X^*(s_i\tau_u)}{s_i c(\tau_u)} \leq 1 + \epsilon \text{ for } 1 \leq i \leq k-1\right\},$$

and apply the previous argument to

$$P(A_k, \tau_u \in r(u)dt, Z^{(u)} \in [(1-\epsilon)c(\tau_u), (1+\epsilon)c(\tau_u)]).$$

We find that

$$P^{(u)}(A_k, \tau_u \in r(u)dt, Z^{(u)} \in [(1-\epsilon)c(\tau_u), (1+\epsilon)c(\tau_u)]) \rightarrow c_{0,\beta}P(C > t)dt,$$

and the convergence of the k -dimensional distributions follows.

To include the behaviour of the overshoot, we need the following result.

Lemma 4.1. *For $u > 0$, $z \geq 0$, and $y \geq 0$ we have*

$$P^{(u)}(Z^{(u)} \in dy, O_u > z) = P^{(u)}(Z^{(u)} \in dy) \frac{\bar{\Pi}_X^+(u+y+z)}{\bar{\Pi}_X^+(u+y)}.$$

Proof of Lemma 4.1: Using the quintuple law in Doney and Kyprianou (2006) twice (see also Griffin and Maller (2011)), we see that for $y \geq 0$,

$$\begin{aligned} P^{(u)}(Z^{(u)} \in dy, O_u > z) &= \int_{w=0}^u G(dw)G^*(u-w-dy)\bar{\Pi}_X^+(u+y+z) \\ &= \int_{w=0}^u G(dw)G^*(u-w-dy)\bar{\Pi}_X^+(u+y) \frac{\bar{\Pi}(u+y+z)}{\bar{\Pi}_X^+(u+y)} \\ &= P^{(u)}(Z^{(u)} \in dy) \frac{\bar{\Pi}_X^+(u+y+z)}{\bar{\Pi}(u+y)}. \end{aligned}$$

(Note that there is no issue of creeping to take into account since we keep $X_{\tau_u} > u$.)

Corollary 4.1. *Under the assumptions of Proposition 4.1, the $P^{(u)}$ -finite-dimensional distributions $\mathbf{Y}^{(u)}$, defined in (2.16), converge to those of $(V, U, V, (\mathbf{VD}^{(0)}(s))_{0 < s < 1})$.*

Proof of Corollary 4.1: The result for

$$\left(\frac{Z^{(u)}}{a(u)}, \frac{\tau_u}{b(a(u))}, \left(\frac{X^*(s\tau_u)}{a(u)} \right)_{0 < s < 1} \right)$$

is immediate from Proposition 4.1, and since, given $Z^{(u)}$, $O^{(u)}$ is dependent of the pre- τ_u σ -field, we need only check that

$$P(O^{(u)} > xa(u) | Z^{(u)} = a(u)z) \rightarrow \begin{cases} \left(\frac{1+z}{1+z+x} \right)^\beta & \text{in Case (i),} \\ e^{-x} & \text{in Case (ii).} \end{cases}$$

But this is immediate from Lemma 4.1.

In particular, we have that the $P^{(u)}$ -distribution of $O^{(u)}$ converges to that of U , so 1(a), and hence the assumption of Proposition 3.1 holds, so 1(b) also holds. Thus Parts 1(a)–1(c) are proved equivalent.

Finally, we show that the convergence in this result can be replaced by weak convergence on the Skorokhod space.

Proposition 4.2. *Under the assumptions of Proposition 4.1, the $P^{(u)}$ -distribution of $\mathbf{Y}^{(u)}$ converges weakly on $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$ as $u \rightarrow \infty$.*

Proof of Proposition 4.2: Put $\mathbf{Y}^{(u)} = (W^{(u)}, \mathbf{X}^{(u)})$, where

$$\mathbf{X}^{(u)} = \left(\frac{X^*(s\tau_u)}{a(u)} \right)_{0 < s < 1}.$$

We need only prove tightness. This will follow if we can show that for any $\varepsilon > 0$ there is a compact subset of K of $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$ such that $\limsup_{u \rightarrow \infty} P^{(u)}(Y^{(u)} \in K^c) \leq \varepsilon$. We will do this with $K = K_1 \times K_2$, where $K_1 \subset \mathbb{R}^3$ is of the form $\{1/D < x_r < D, r = 1, 2, 3\}$, $K_2 \subset \mathbb{D}_0[0, 1]$ will be specified later, and D is fixed with $P^{(u)}(W^{(u)} \in K_1^c) \leq \varepsilon/2$ for large u . So it suffices to show that $\limsup_{u \rightarrow \infty} P^{(u)}(\mathbf{Y}^{(u)} \in K_1 \times K_2^c) \leq \varepsilon/2$. This probability is dominated by

$$P^{(u)} \left(B \cap (\mathbf{X}^{(u)} \in K_2^c) \right)$$

where

$$B = \left\{ \frac{\tau_u}{r(u)} \in (D^{-1}, D), \frac{Z^{(u)}}{a(u)} \in (D^{-1}, D) \right\}.$$

But (recall $c(r(u)) = a(u)$)

$$\begin{aligned} & P^{(u)} \left(\mathbf{X}^{(u)} \in K_2^c, B \right) \\ & \leq \frac{1}{P(\tau_u < \infty)} \int_{r(u)/D}^{r(u)D} \int_{z \in (D^{-1}, D)} dt P \left(X_t^* \in a(u)dz, \mathbf{X}^{(u)} \in K_2^c \right) \bar{\Pi}_X^+(u + a(u)z) \\ & \leq \frac{\bar{\Pi}_X^+(u)}{P(\tau_u < \infty)} \int_{r(u)/D}^{r(u)D} dt P \left(\left(\frac{X_{st}^*}{a(u)}, 0 \leq s < 1 \right) \in K_2^c \right) \\ & = \frac{\bar{\Pi}_X^+(u)r(u)}{P(\tau_u < \infty)} \int_{1/D}^D dt P \left(\left(\frac{X_{rst}^*}{c(r(u))}, 0 \leq s < 1 \right) \in K_2^c \right). \end{aligned}$$

We know from (4.3) that $r(u) \sim q^{-1}a(u)/A_{H^*}(u)$ so

$$\limsup_{u \rightarrow \infty} \frac{r(u)\bar{\Pi}_X^+(u)}{P(\tau_u < \infty)} < \infty.$$

Also, since $(X_{ys}^*/c(y))_{0 \leq s < 1}$ is tight as $y \rightarrow \infty$, we can choose K_2 such that when $D^{-1}a(u)$ is sufficiently large,

$$P\left(\sup_{t \in (D^{-1}, D)} \left(\frac{X_{r(u)st}^*}{c(r(u))}, 0 \leq s < 1\right) \in K_2^c\right) \leq \varepsilon,$$

and the result follows. \square

5 The case $0 < \gamma < 1$ (Infinite Mean)

Throughout our standing assumptions (and notations) will be those of Theorem 2.2, namely, (2.1) and (2.10) hold, and (2.13) holds with $\gamma \in (0, 1)$. By the monotone density theorem, the latter is equivalent to

$$\bar{\Pi}_X^-(x) \sim \gamma x^{-1}A_X^*(x) \in RV(\gamma - 1) \text{ as } x \rightarrow \infty. \quad (5.1)$$

From (4.1) we deduce $\lim_{x \rightarrow \infty} \bar{\Pi}_X^+(x)/\bar{\Pi}_X^-(x) = 0$. This together with (5.1) means that X^* is in the domain of attraction of a standard stable subordinator, \mathbf{D} , of parameter $\bar{\gamma} := 1 - \gamma \in (0, 1)$. Thus we can find a continuous, increasing function $c(\cdot)$ such that $(X_{sc(u)}^*/c(u))_{s>0} \xrightarrow{D} \mathbf{D}$, and one can check that

$$u\bar{\Pi}_X^-(c(u)) \rightarrow 1/\Gamma(\gamma).$$

Write $b(\cdot)$ for the inverse of $c(\cdot)$, so that $b(\cdot) \in \mathcal{R}_{\bar{\gamma}}$, and in fact

$$b(u) \sim \frac{1}{\Gamma(\gamma)\bar{\Pi}_X^-(u)}. \quad (5.2)$$

Put $r(u) = b(a(u))$, so that $c(r(u)) = a(u)$, and

$$r(u) \sim \frac{1}{\Gamma(\gamma)\bar{\Pi}_X^-(a(u))} \sim \frac{a(u)}{\Gamma(1 + \gamma)A_X^*(a(u))}. \quad (5.3)$$

Note also that a version of Stone's stable local limit theorem (see Prop. 13 of Doney and Rivero (2012)) implies that

$$P(X_{tr(u)}^* \in (a(u)z, a(u)z + \Delta]) = \frac{\Delta}{a(u)} (h_t(z) + o(1)) \quad (5.4)$$

as $u \rightarrow \infty$, uniformly for $\Delta \in (0, \Delta_0]$, $t \in (0, \Delta_0]$ and $z \in \mathbb{R}$, for any fixed $\Delta_0 > 0$.

We have already proved Part 1 of Theorem 2.2, except for the implication from Part 1(c) to Part 1(a), and we now show that Part 1(c) implies Part 2.

Proposition 5.1. *Assume (2.1) and (2.10), and that $A_{H^*} \in RV(\gamma)$ with $\gamma \in (0, 1)$. Suppose either (i) $\bar{\Pi}_X^+(x) \in RV(-\beta)$, where $\beta > 1 - \gamma$, or (ii) $\bar{\Pi}_X^+(x) \in \text{MDA}(\Lambda)$ and $\bar{\Pi}_{\mathcal{H}} \in \mathcal{S}$. Then Part 2 of Theorem 2.2 holds.*

Proof of Proposition 5.1: From (4.5) we have

$$\begin{aligned}
& P(\tau_u \in r(u)dt, Z^{(u)} \in [za(u), za(u) + \Delta]) \\
&= \int_{y \in [0, \Delta]} \bar{\Pi}_X^+(u + za(u) + y) P(X_{tr(u)}^* \in za(u) + dy, \bar{X}_{tr(u)} \leq u) dt \\
&\sim \bar{\Pi}_X^+(u + za(u)) \int_{y \in [0, \Delta]} P(X_{tr(u)}^* \in za(u) + dy, \bar{X}_{tr(u)} \leq u) dt \\
&\sim \bar{\Pi}_X^+(u) P(C > z) P(X_{tr(u)}^* \in [za(u), za(u) + \Delta], \bar{X}_{tr(u)} \leq u) dt.
\end{aligned}$$

Write

$$P(X_{tr(u)}^* \in [za(u), za(u) + \Delta], \bar{X}_{tr(u)} \leq u) = P_1(u) - P_2(u),$$

where, by (5.4),

$$P_1 = P(X_{tr(u)}^* \in [za(u), za(u) + \Delta]) = \frac{\Delta}{a(u)} (h_t(z) + o(1)), \quad (5.5)$$

and we will show that

$$P_2 = P(X_{tr(u)}^* \in [za(u), za(u) + \Delta], \bar{X}_{tr(u)} > u) = o\left(\frac{\Delta}{a(u)}\right). \quad (5.6)$$

To do this write $P_2(u) = P_2^{(1)}(u) + P_2^{(2)}(u)$, and argue as follows:

$$\begin{aligned}
& P_2^{(1)}(u) = P(\tau_u \leq tr(u)/2, X_{tr(u)}^* \in (za(u), za(u) + \Delta]) \\
&= \int_{0 \leq s \leq tr(u)/2} \int_{y > 0} P(\tau_u \in ds, O^{(u)} \in dy) P(X_{tr(u)-s}^* \in (u + y + za(u), u + y + za(u) + \Delta]) \\
&\leq \int_{0 \leq s \leq tr(u)/2} \int_{y > 0} P(\tau_u \in ds, O^{(u)} \in dy) \frac{C\Delta}{c(tr(u) - s)} \\
&\leq \frac{C'\Delta}{c(tr(u))} P(\tau_u < \infty) \\
&= o\left(\frac{\Delta}{c(tr(u))}\right).
\end{aligned}$$

Introduce $\tau^*(u) = \min\{s : X_s^* > u\}$ and $\sigma_{tr(u)}(u) = \max\{s \leq tr(u) : X_s > u\}$, and use duality

to write

$$\begin{aligned}
P_2^{(2)}(u) &= \int_{[0,\Delta]} P\left(tr(u)/2 < \tau_u \leq tr(u), X_{tr(u)}^* \in a(u)z + dy\right) \\
&\leq \int_{[0,\Delta]} P\left(tr(u)/2 < \sigma_{tr(u)}(u) \leq tr(u), X_{tr(u)}^* \in a(u)z + dy\right) \\
&= \int_{[0,\Delta]} P\left(0 < \tau^*(u + a(u)z + y) < tr(u)/2, X_{tr(u)}^* \in a(u)z + dy\right) \\
&\leq P\left(0 < \tau^*(u + a(u)z) < tr(u)/2, X_{tr(u)}^* \in (a(u)z, a(u)z + \Delta]\right) \\
&= \int_{0 \leq v \leq tr(u)/2} \int_{y > 0} P(\tau^*(u + a(u)z) \in dv), X_v^* \in u + a(u)z + dy \\
&\quad \times P(X_{tr(u)-v} \in (u + y - \Delta, u + y]) \\
&= o(1) \int_{0 \leq v \leq tr(u)/2} P(\tau^*(u + a(u)z) \in dv) \frac{\Delta}{c(tr(u) - v)} \\
&= o\left(\frac{\Delta}{c(tr(u))}\right).
\end{aligned}$$

Here we used the strong Markov property at $\tau^*(u + a(u)z)$ and equated $P(X_{tr(u)-v} \in (u + y - \Delta, u + y])$ with $P(X_{tr(u)-v}^* \in (-u - y + \Delta, -u - y])$. Since $c(tr(u)) = O(a(u))$, this gives (5.6).

Also from (3.11) we deduce, in Case (i),

$$c_{\gamma,\beta} \bar{\Pi}_{\mathcal{H}}(u) \sim qr(u) \bar{\Pi}_X^+(u), \quad (5.7)$$

where, in this case, by (3.26), $c_{\gamma,\beta} = \Gamma(\gamma + \beta - 1)/\Gamma(\beta)$. From (2.11), (5.5), (5.6) and (5.7) we see that

$$\begin{aligned}
P^{(u)}(Z^{(u)} \in (a(u)z, a(u)z + \Delta], \tau_u \in r(u)dt) &\sim (1+z)^{-\beta} \bar{\Pi}_X^+(u) \frac{qh_t(z)\Delta}{\bar{\Pi}_{\mathcal{H}}(u)a(u)} dt \\
&= (1+z)^{-\beta} \frac{h_t(z)\Gamma(\beta)\Delta}{\Gamma(\gamma + \beta - 1)r(u)a(u)} = \frac{h_t(z)f(z)\Delta}{r(u)a(u)} dt.
\end{aligned}$$

In Case (ii) we get from (3.11)

$$P^{(u)}(Z^{(u)} \in (a(u)z, a(u)z + \Delta], \tau_u \in r(u)dt) \sim e^{-z} \frac{h_t(z)\Delta}{r(u)a(u)} dt = \frac{h_t(z)f(z)\Delta}{r(u)a(u)} dt,$$

and (2.20) is established. Notice also that since $h_t(\cdot)$ vanishes on the negative half-line, the previous estimates show that $P^{(u)}(-Z^{(u)} \in (a(u)z, a(u)z + \Delta], \tau_u \in r(u)dt)/dt$ is uniformly $o(r(u)^{-1}a(u)^{-1})$ for $z \in (0, \Delta_0]$ and $t \in (0, \Delta_0]$.

For $k \geq 1$ we assume first that $z_1 < z_2 < \dots < z_k$ and write (2.21) as

$$r(u)(a(u))^k P^{(u)}\left(\bigcap_1^k A_i \cap B\right) = \theta_k(z_1, z_2, \dots, z_k, t) \left(\prod_{i=1}^k \Delta_i + o(1)\right) dt,$$

where

$$A_i := \{X^*(s_i tr(u) -) \in (a(u)z_i, a(u)z_i + \Delta_i]\}, \quad 1 \leq i \leq k, \quad \text{and} \quad B := \{\tau(u) \in r(u)dt\}.$$

As before, we have

$$P\left(\bigcap_{i=1}^k A_i \cap B\right) \sim P\left(\bigcap_{i=1}^k A_i^{(n)}\right) \bar{\Pi}_X^+(u + a(u)z_k) dt, \quad (5.8)$$

where $A_i^{(n)} := \{X^*(s_i tr(u)) \in (a(u)z_i, a(u)z_i + \Delta_i], \tau_u > tr(u)\}$. But we can also write

$$\{\tau_u > r\} = \bigcap_{i=1}^k \{\tau_u \notin (rs_{i-1}, rs_i]\},$$

where we recall $s_0 = 0$. Note that each $r(u)(s_i - s_{i-1}) \rightarrow \infty$ uniformly as $u \rightarrow \infty$. So by the Markov property and stationarity we have

$$\begin{aligned} P\left(\bigcap_{i=1}^k A_i^{(n)}\right) &= \int_{a(u)z_{k-1}}^{a(u)z_{k-1} + \Delta_{k-1}} P(X^*(tr(u)s_k) \in (a(u)z_k, a(u)z_k + \Delta_k] | X^*(tr(u)s_{k-1}) = y) \\ &\quad \times P\left(\bigcap_{i=1}^{k-1} A_i^{(n)}, X^*(tr(u)s_{k-1}) \in dy\right) \\ &= \int_{a(u)z_{k-1}}^{a(u)z_{k-1} + \Delta_{k-1}} P(X^*(r(s_k - s_{k-1})) \in (a(u)z_k - y, a(u)z_k - y + \Delta_k]) \\ &\quad \times P\left(\bigcap_{i=1}^{k-1} A_i^{(n)}, X^*(rs_{k-1}) \in dy\right) \\ &= \frac{\Delta_k}{a(u)} \left(h_{t(s_k - s_{k-1})}((z_k - z_{k-1})) + o(1) \right) \times P^{(u)}\left(\bigcap_{i=1}^{k-1} A_i^{(n)}\right), \end{aligned}$$

where the last line uses the $k = 1$ result. Repeating this argument a further $k - 1$ times gives

$$P\left(\bigcap_{i=1}^k A_i^{(n)}\right) = (a(u))^{-k} \prod_{i=1}^k \Delta_i \left(\prod_{i=1}^k h_{t(s_i - s_{i-1})}(z_i - z_{i-1}) + o(1) \right),$$

and the result then follows from (5.8) and the previous calculation. Clearly if any $z_i \leq z_{i-1}$ the calculation is still valid, but the above product vanishes.

Using this local result and Lemma 4.1 we easily obtain the convergence of the finite-dimensional distributions, as claimed in Part 3.

Now argue as follows. (2.20) implies that $Z^{(u)}/a(u)$ has a proper limiting distribution. By Lemma 4.1 this means that $(Z^{(u)}/a(u), O^{(u)}/a(u))$ has a proper limiting distribution, thus, in particular, $O^{(u)}/a(u)$ has a proper limiting distribution. From Proposition 3.1 we then deduce properties 1(a) and 1(b), and the proof of Theorem 2.2 is completed by repeating the tightness argument of the previous section, almost word for word. \square

Remark 5.1. Assumption (2.10), that $\mathcal{H} \in \mathcal{S}$, is only needed for application of Proposition 3.1, where it is used in effect to deduce that $\bar{\Pi}_{\mathcal{H}}(u) \sim qP(\tau_u < \infty)$ via (2.11). We could replace assumption (2.10) with $\bar{\Pi}_{\mathcal{H}}(u) \sim qP(\tau_u < \infty)$ throughout. But general necessary and sufficient conditions for the latter in terms of more basic quantities are currently not known.

Further note that $\bar{\Pi}_{\mathcal{H}}(u)$ is not asymptotically equivalent to the more basic quantity $\bar{\Pi}_X^+(u)$ in our situation. Vigon's "équation amicale inversée" is

$$\bar{\Pi}_{\mathcal{H}}(u) = \int_{(0,\infty)} \bar{\Pi}_X^+(y+u)G^*(dy) \quad (5.9)$$

(recall that G^* is the renewal measure in the downgoing ladder height process H^*). Under the assumption $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s., $G^*(\infty) = \infty$, and it's not hard to show from (5.9) that either $\bar{\Pi}_{\mathcal{H}} \in \mathcal{L}$ (see (2.8), or $\bar{\Pi}_X^+ \in \mathcal{L}$ implies $\bar{\Pi}_{\mathcal{H}}(u)/\bar{\Pi}_X^+(u) \rightarrow \infty$.

In general, a sufficient condition for $\bar{\Pi}_{\mathcal{H}} \in \mathcal{S}$ is $\bar{\Pi}_X^+ \in \mathcal{D} \cap \mathcal{L}$, where \mathcal{D} is the class of dominatedly varying functions, i.e, those for which $\limsup_{x \rightarrow \infty} \bar{\Pi}_X^+(x/2)/\bar{\Pi}_X^+(x) < \infty$; see [15], p.11. So we can replace Assumption (2.10) by $\bar{\Pi}_X^+ \in \mathcal{D} \cap \mathcal{L}$ throughout. In particular, $\bar{\Pi}_X^+ \in \mathcal{D}$ if $\bar{\Pi}_X^+$ is regularly varying with index $-\alpha$, where $\alpha \geq 0$, as $x \rightarrow \infty$.

Further connections between $\bar{\Pi}_{\mathcal{H}}$ and $\bar{\Pi}_X$ are in Prop. 5.4 of Klüppelberg et al. (2004) and the related discussion.

6 Random walks

We can specialize our results to the case that X is a compound Poisson process of the form $X_t = S_{N_t}$, where $(S_n, n \geq 0)$ is a random walk and $(N_t, t \geq 0)$ is an independent Poisson counting process of unit rate. Then, writing Z_n and Z_n^* for the n th strict increasing and weak decreasing ladder heights in S , we have also that $H_t = Z_{N_t}$ and $H_t^* = Z_{N_t}^*$ for all $t \geq 0$. Then our basic assumptions, (2.1) and (2.10) are equivalent to

$$S_n \xrightarrow{a.s.} -\infty \text{ and } J \in \mathcal{S},$$

where $J(dx) = P(Z_1 \in dx | Z_1 \in (0, \infty))$. It is also clear that, with $\tau^S(u) := \inf\{n : S_n > u\}$, we have the identity

$$\tau_u = \sum_1^{\tau^S(u)} e_i,$$

where the e_i are i.i.d. $Exp(1)$ random variables. Clearly the event $\{\tau_u < \infty\}$ coincides a.s. with the event $\{\tau^S(u) < \infty\}$, so $P^{(u)}(\cdot)$ has an unambiguous meaning, and furthermore it is straightforward to show that for any $r(u) \rightarrow \infty$, $u \rightarrow \infty$, the statements

$$r(u)P^{(u)}(\tau^S(u) = [tr(u)]) \rightarrow g(t)$$

and

$$r(u)P^{(u)}(\tau_u \in r(u)dt) \rightarrow g(t)dt$$

are equivalent. Also the spatial quantities $Z_S^{(u)} := S^*(\tau^S(u))$ and $O_S^{(u)} := S(\tau^S(u)) - u$ coincide with $Z^{(u)}$ and $O^{(u)}$.

We claim that this allows us to deduce versions of Theorems 2.2 and 2.1 for random walks, with very minor changes. Specifically, if F is the distribution of S_1 and we replace Π and $\Pi_{\mathcal{H}}$ in those results by F and J , then Theorem 2.1 requires only replacing $g^{(u)}(tr(u))$ by $P^{(u)}(\tau^S(u) = [tr(u)])$, and Theorem 2.2 requires only an analogous change to (2.20).

Alternatively, we can prove the random walk results by repeating the Lévy process proof, with appropriate changes.

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