Fast Tracking Control for the Thrust Direction of a Quadrotor Helicopter

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In this paper an energy based tracking control for the thrust direction of a quadrotor helicopter is presented. The concept can be considered as a direct extension of the setpoint control presented in [6], since by the application of appropriate feedforward control the open loop structure of the setpoint control case is preserved. In particular the tracking dynamics are still autonomous. Moreover, some modifications compared to [6] are introduced, which comprise the addition of integral action to the controller part that regulates the angular velocity around the thrust axis. Additionally, the whole control problem is reformulated using the reduced attitude parametrization presented in [4], which is better suited for this particular control problem than the quaternion representation used before. Global and local analysis of all equilibrium points is provided and shows that the desired equilibrium, corresponding to a zero tracking error, is almost globally asymptotically and locally exponentially stable.

1 Introduction

A quadrotor helicopter is a highly maneuverable vertical take-off and landing aircraft, which offers the ability of hovering. As shown in Fig. 1, it is basically a rigid body with four rotors arranged in a common plane which generate thrust forces and drag moments. The effects of the four single rotors can be summarized in the center of gravity as a total thrust F perpendicular to the plane and a torque vector $\boldsymbol{\tau} = \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^T$. Since the direction of the thrust is body-fixed, the execution of almost all translational motions requires tilting the whole quadrotor helicopter systematically. Consequently, a desired thrust direction is usually the remote control command of a human operator or the output of a higher level position controller and has to be tracked by a lower level thrust direction controller. In the latter case, especially if a high bandwidth position controller is considered, exact tracking of the thrust command is required. In this paper we will combine a suitable feedforward control with the approved error feedback from the setpoint control [6] to achieve asymptotic convergence of the actual thrust direction to its moving reference direction. Moreover, integral action for disturbance rejection is added to the controller part regulating the angular velocity around the thrust axis. The controller design gives rise to a continuous state feedback law and is based on an energy shaping approach. The potential energy as well as the damping functions proposed in this paper are slightly modified compared to the ones assigned in [6]. The modifications significantly facilitate the local analysis of the closed loop equilibrium points while their influence on the controller perfomance is negligible. Finally, the problem is reformulated using the reduced attitude parametrization presented in [4] instead of the quaternion representation. The new parametrization contributes to a simplified stability analysis since it allows for examining equilibrium points instead of equilibrium sets.
The control task belongs to the field of reduced attitude control, which has been studied for example in [4, 3, 2, 13, 7]. While applications encompass for instance the spin axis stabilization of spacecrafts [2, 13] or the stabilization of the inverted 3D pendulum [3], reduced attitude control has been rarely used in the field of unmanned aerial vehicles, an exception is [7]. The

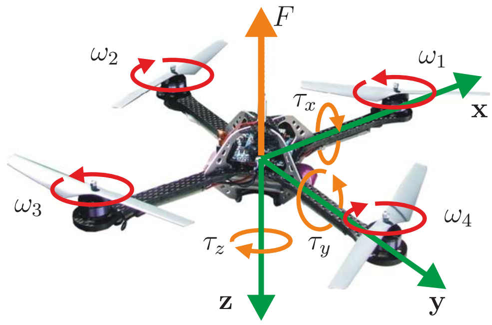

Figure 1: Quadrotor with body-fixed frame $B = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and control inputs $F, \tau_x, \tau_y, \tau_z$.

energy based approach presented here can be roughly assigned to the very general concept presented in [3] but focuses on indicating explicit energy and damping functions. Compared to other control concepts, energy based control admits a transparent physical interpretation of the closed loop. The static forces or torques of the control law can be viewed as the effects of virtual (possibly nonlinear) spring elements, which correspond to the potential energy that is shaped by the control engineer. This process determines the equilibria of the closed loop and also admits the consideration of control input constraints. By injecting an appropriate damping in the closed loop system, the dynamic forces or torques are designed. This way the energy dissipation and thus the transient behavior is controlled. The control concept presented here, places emphasis on a sophisticated damping strategy that aims on a fast transient behavior.
In Section 2 we briefly introduce the notation and the definitions used in the following. A detailed problem statement together with the derivation of the tracking dynamics is given in Section 3. Based on an energy shaping approach, the thrust direction controller is developed in Section 4, before almost global asymptotic stability of the equilibrium, corresponding to a zero tracking error, is proven in Section 5. Finally, conclusions are drawn in Section 6.

2 Nomenclature and Definitions

Scalars are indicated as italic letters, whereas vectors, matrices and composite quantities are indicated by upright bold letters. Any physical vector $\mathbf{a} \in \mathbb{R}^3$ has meaning even without concrete numerical values and is thus referred to as an abstract vector. To assign numerical values to an abstract vector a suitable coordinate frame has to be chosen. All coordinate frames used are right-handed Cartesian coordinate systems and identified by uppercase italic letters. The representation of an abstract vector $\mathbf{a} \in \mathbb{R}^3$ with respect to a certain frame $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is denoted by $\mathbf{a}_E$. The elements of a vector $\mathbf{a}_E$ are identified by $\mathbf{a}_E = \begin{bmatrix} a_{Ex} & a_{Ey} & a_{Ez} \end{bmatrix}^T$ and by $\mathbf{a}_{Exy}$ we mean $\mathbf{a}_{Exy} = \begin{bmatrix} a_{Ex} & a_{Ey} \end{bmatrix}^T$. For some vectors, which are exclusively represented in one coordinate frame, the basis designation will be dropped. Additionally, we define the basis independent unit vectors $\mathbf{e}_x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{e}_y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\mathbf{e}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The transformation from a frame E to another frame E' is given by a rotation matrix $\mathbf{R}_{E'E} \in SO(3)$, where $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} : \mathbf{R}^T\mathbf{R} = \mathbf{I}_3, \det(\mathbf{R}) = 1\}$ is the special orthogonal group and $\mathbf{I}_i$, $i \in \mathbb{N}$ denotes the $i \times i$ identity matrix. The angular velocity of a frame E' with respect to a frame E given in a frame E'' is denoted by $\boldsymbol{\omega}_{E''}^{EE'} \in \mathbb{R}^3$. The skew symmetric operator $\langle\!\langle \cdot \rangle\!\rangle : \mathbb{R}^3 \to \mathfrak{so}(3)$, where $\mathfrak{so}(3) = \{\mathbf{K} \in \mathbb{R}^{3\times3} : \mathbf{K}^T = -\mathbf{K}\}$ is defined such that $\langle\!\langle \mathbf{a} \rangle\!\rangle \mathbf{b} = \mathbf{a} \times \mathbf{b}$ reflects the cross product for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Moreover, we define the function $\Lambda_{\zeta_l}^{\zeta_u} : [0,\pi] \to [0, \sin(\zeta_l)]$ as

$$\Lambda_{\zeta_l}^{\zeta_u}(\zeta) = \begin{cases} \sin(\zeta) & \text{if} \quad 0 \le \zeta \le \zeta_l\,, \\ \sin(\zeta_l) & \text{if} \quad \zeta_l < \zeta \le \zeta_u\,, \\ \frac{\sin(\zeta_l)}{\sin(\zeta_u)}\sin(\zeta) & \text{if} \quad \zeta_u < \zeta \le \pi\,, \end{cases} \tag{1}$$

where $\zeta_l, \zeta_u \in \mathbb{R}_+$ are constants. We also use the function $\chi_{\zeta_1}^{\zeta_2} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$\chi_{\zeta_1}^{\zeta_2}(\zeta, \psi_1(\zeta, \mathbf{a}), \psi_2(\zeta, \mathbf{a})) = \begin{cases} \psi_1(\zeta, \mathbf{a}) & \text{if } \zeta \le \zeta_1\,, \\ \frac{(\zeta_2-\zeta)\psi_1(\zeta_1,\mathbf{a})+(\zeta-\zeta_1)\psi_2(\zeta_2,\mathbf{a})}{\zeta_2-\zeta_1} & \text{if } \zeta_1 < \zeta \le \zeta_2\,, \\ \psi_2(\zeta, \mathbf{a}) & \text{if } \zeta_2 < \zeta\,, \end{cases} \tag{2}$$

which provides a linear interpolation between the scalar functions $\psi_1(\zeta, \mathbf{a})$ and $\psi_2(\zeta, \mathbf{a})$ with respect to ζ in the interpolation region defined by ζ_1 and ζ_2. For some $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$ we moreover define

$$\chi_{\zeta_1,\zeta_2}^{\zeta_3,\zeta_4}(\zeta, \psi_1(\zeta,\mathbf{a}), \psi_2(\zeta,\mathbf{a})) := \chi_{\zeta_1}^{\zeta_2}\left(\zeta, \psi_1(\zeta,\mathbf{a}), \chi_{\zeta_3}^{\zeta_4}(\zeta, \psi_2(\zeta,\mathbf{a}), \psi_1(\zeta,\mathbf{a}))\right)\,, \tag{3}$$

which provides a linear interpolation from $\psi_1(\zeta, \mathbf{a})$ to $\psi_2(\zeta, \mathbf{a})$ and back to $\psi_1(\zeta, \mathbf{a})$.
We will frequently encounter the case that a (scalar, vector or matrix) quantity a can be given as a function $f(\cdot)$ of coordinates $\mathbf{b}$, i.e. $a = f(\mathbf{b})$, and also as a function $\tilde{f}(\cdot)$ of coordinates $\mathbf{c}$, i.e. $a = \tilde{f}(\mathbf{c})$. With a slight abuse of notation we will write $a(\mathbf{b})$ to refer to $f(\mathbf{b})$ and $a(\mathbf{c})$ to refer to $\tilde{f}(\mathbf{c})$. Sometimes, we will also drop the argument and in that case writing a may refer to $f(\mathbf{b})$ or $\tilde{f}(\mathbf{c})$ depending on the context.
Finally, some properties of the skew symmetric operator $\langle\!\langle \cdot \rangle\!\rangle$ that will be needed in the following are stated. They can be found for example in [11]. By the skew symmetry it holds that $\langle\!\langle \mathbf{a} \rangle\!\rangle = -\langle\!\langle \mathbf{a} \rangle\!\rangle^T$. Since $\langle\!\langle \mathbf{a} \rangle\!\rangle\mathbf{b}$ reflects the cross product $\mathbf{a}\times\mathbf{b}$ we also have that $\langle\!\langle \mathbf{a} \rangle\!\rangle\mathbf{b} = -\langle\!\langle \mathbf{b} \rangle\!\rangle\mathbf{a}$. Moreover, for any rotation matrix $\mathbf{R}_{E'E}$ and any vector $\mathbf{a} \in \mathbb{R}^3$ it holds that

$$\mathbf{R}_{E'E}\langle\!\langle \mathbf{a} \rangle\!\rangle\mathbf{R}_{E'E}^T = \langle\!\langle \mathbf{R}_{E'E}\mathbf{a} \rangle\!\rangle\,. \tag{4}$$

Eventually, let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, then the Grassman identy is

$$\langle\!\langle \mathbf{a} \rangle\!\rangle(\langle\!\langle \mathbf{b} \rangle\!\rangle\mathbf{c}) = (\mathbf{a}^T\,\mathbf{c})\mathbf{b} - (\mathbf{a}^T\mathbf{b})\mathbf{c}\,. \tag{5}$$

3 Problem Statement and Reduced Attitude Tracking Dynamics

Regarding the attitude, we model the quadrotor helicopter as a rigid body actuated in torque. This commonly used model exploits the generally accepted assumption that there exists a known one to one relation $\begin{bmatrix} F & \boldsymbol{\tau}^T \end{bmatrix}^T = \mathbf{f}(\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2)$ (see e.g. [12]) between the magnitude F of the thrust force, the torque vector $\boldsymbol{\tau}$ and the squares of the rotor angular rates ω_i, $i \in \{1,2,3,4\}$, which are the real control variables. Furthermore, the model neglects some minor effects like the gyroscopic torques of the rotors or the flapping dynamics. We distinguish an inertial north east down coordinate frame $I = \{\mathbf{e}_{north}, \mathbf{e}_{east}, \mathbf{e}_{down}\}$ and a body-fixed frame $B = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ attached to the center of gravity of the quadrotor and oriented as shown in Fig. 1. Then, the rigid body attitude dynamics read

$$\dot{\mathbf{R}}_{BI} = -\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle\mathbf{R}_{BI}\,, \tag{6}$$

$$\mathbf{J}\dot{\boldsymbol{\omega}}_B^{IB} = -\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle\mathbf{J}\boldsymbol{\omega}_B^{IB} + \boldsymbol{\tau} + \hat{\tau}_z\mathbf{e}_z\,, \tag{7}$$

where $\mathbf{J} = \mathbf{J}^T > \mathbf{0}$ is the moment of inertia matrix given in B, $\boldsymbol{\tau} = \boldsymbol{\tau}_B$ is the control torque and $\hat{\tau}_z$ is a constant and bounded disturbance torque acting on the body-fixed z-axis. Such a constant torque $\hat{\tau}_z$ can occur during hovering, for instance when the drag moments caused by the rotors are unbalanced due to slight inhomogeneities. In the following we assume that the states $\mathbf{R}_{BI}$ and $\boldsymbol{\omega}_B^{IB}$ are accessible either by direct measurements or as the output of an appropriate data fusion.

The unit vector pointing along the body-fixed z-axis is denoted by $\mathbf{z}$. As the thrust vector always points in the direction $-\mathbf{z}$, aligning the thrust vector is equivalent with aligning $\mathbf{z}$. Accordingly, the control objective is to make the vector $\mathbf{z}$ track a desired time-varying direction $\mathbf{z}_d$, or more technically $\mathbf{z} \to \mathbf{z}_d$ as $t \to \infty$. Even if $\mathbf{z} \equiv \mathbf{z}_d$ holds the quadrotor can still rotate around its z-axis and we want that motion to decay. Hence we additionally claim $\boldsymbol{\omega}_B^{IB} \to \begin{bmatrix} \omega_{Bx}^{IB} & \omega_{By}^{IB} & 0 \end{bmatrix}^T$ as $t \to \infty$. Since the desired direction $\mathbf{z}_d$ is usually the output of a higher level position controller, its representation with respect to the inertial frame $\mathbf{z}_{dI}$ as well as its time derivatives $\dot{\mathbf{z}}_{dI}, \ddot{\mathbf{z}}_{dI}$ are considered to be known and bounded.

The desired z-axis direction given in the body-fixed frame $\mathbf{z}_{dB} = \mathbf{R}_{BI}\mathbf{z}_{dI}$ can be interpreted as a reduced attitude error suitable for our purposes, [4]. In terms of $\mathbf{z}_{dB} \in \mathcal{S}^2$, where $\mathcal{S}^2 = \{\mathbf{a} \in \mathbb{R}^3 : \mathbf{a}^T\mathbf{a} = 1\}$, the first control objective reads $\mathbf{z}_{dB} \to \mathbf{z}_B = \mathbf{e}_z$ as $t \to \infty$. To derive the reduced attitude error dynamics, we compute the time derivative of $\mathbf{z}_{dB}$ using (6), (5) together with the fact that $\dot{\mathbf{z}}_{dI} \perp \mathbf{z}_{dI}$ and (4),

$$
\begin{aligned}
\dot{\mathbf{z}}_{dB} &= \dot{\mathbf{R}}_{BI}\mathbf{z}_{dI} + \mathbf{R}_{BI}\dot{\mathbf{z}}_{dI} = -\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle \mathbf{R}_{BI}\mathbf{z}_{dI} + \mathbf{R}_{BI} \, \langle\!\langle \langle\!\langle \mathbf{z}_{dI} \rangle\!\rangle \dot{\mathbf{z}}_{dI} \rangle\!\rangle \mathbf{z}_{dI} = \\
&= -\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle \mathbf{R}_{BI}\mathbf{z}_{dI} + \langle\!\langle \mathbf{R}_{BI} \langle\!\langle \mathbf{z}_{dI} \rangle\!\rangle \dot{\mathbf{z}}_{dI} \rangle\!\rangle \, \mathbf{R}_{BI}\mathbf{z}_{dI} = \\
&= \langle\!\langle -\boldsymbol{\omega}_B^{IB} + \mathbf{R}_{BI} \langle\!\langle \mathbf{z}_{dI} \rangle\!\rangle \dot{\mathbf{z}}_{dI} \rangle\!\rangle \mathbf{z}_{dB} = -\langle\!\langle \boldsymbol{\omega}_B^{IB} - \boldsymbol{\omega}_{dB} \rangle\!\rangle \mathbf{z}_{dB} \\
&= -\langle\!\langle \tilde{\boldsymbol{\omega}}_B \rangle\!\rangle \mathbf{z}_{dB} = \langle\!\langle \mathbf{z}_{dB} \rangle\!\rangle \tilde{\boldsymbol{\omega}}_B \ .
\end{aligned}
\tag{8}
$$

Note that the angular velocity $\boldsymbol{\omega}_{dB} = \mathbf{R}_{BI}\langle\!\langle \mathbf{z}_{dI} \rangle\!\rangle \dot{\mathbf{z}}_{dI}$ does not indicate the rotational motion between two particular coordinate frames, though it could be interpreted as the angular rate of a desired frame D with respect to the inertial frame I, $\boldsymbol{\omega}_{dB} = \boldsymbol{\omega}_B^{ID}$. However this virtual frame D is not fully determined, since we only know about its z-axis $\mathbf{z}_d$. Thus, any rotation matrix $\mathbf{R}_{ID}$ solving $\mathbf{z}_{dI} = \mathbf{R}_{ID}\mathbf{e}_z$ would characterize an admissible frame D. Moreover, not even the angular velocity of such a frame D is unique, since all angular rates satisfying $\boldsymbol{\omega}_B^{ID} = \boldsymbol{\omega}_{dB} + a \cdot \mathbf{z}_{dB}$, where a is arbitrary would lead to the same $\dot{\mathbf{z}}_{dB}$. Consequently, the error angular velocity $\tilde{\boldsymbol{\omega}}_B = \boldsymbol{\omega}_B^{IB} - \boldsymbol{\omega}_{dB}$ does not characterize the motion between two properly defined coordinate frames but nevertheless represents an appropriate angular velocity error, regarding our control problem. Note that $\tilde{\boldsymbol{\omega}}_B = \mathbf{0}$ implies $\omega_{Bz}^{IB} = 0$ if $\mathbf{z}_{dB} = \mathbf{e}_z$. Using (7), the derivative of $\tilde{\boldsymbol{\omega}}_B$ with respect to time is identified as

$$
\dot{\tilde{\boldsymbol{\omega}}}_B = \dot{\boldsymbol{\omega}}_B^{IB} - \dot{\boldsymbol{\omega}}_{dB} = \mathbf{J}^{-1}\left(-\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle \mathbf{J}\boldsymbol{\omega}_B^{IB} + \boldsymbol{\tau} + \hat{\tau}_z\mathbf{e}_z\right) - \dot{\boldsymbol{\omega}}_{dB} \ , \tag{9}
$$

where $\dot{\boldsymbol{\omega}}_{dB} = -\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle \mathbf{R}_{BI}\langle\!\langle \mathbf{z}_{dI} \rangle\!\rangle \dot{\mathbf{z}}_{dI} + \mathbf{R}_{BI}\langle\!\langle \mathbf{z}_{dI} \rangle\!\rangle \ddot{\mathbf{z}}_{dI}$.

4 Controller Design

In this section, we will first execute an input transformation, such that the controller presented in [6] could be applied without any changes in the undisturbed case. We will then restate this controller with some slight modifications concerning additional integral action for disturbance rejection and minor changes to the potential energy and the damping.

By inserting the input transformation

$$\boldsymbol{\tau} = k(\mathbf{w})\langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle \mathbf{J}\boldsymbol{\omega}_B^{IB} - (k(\mathbf{w})-1)\left(\langle\!\langle \boldsymbol{\omega}_{dB} \rangle\!\rangle \mathbf{J}(\boldsymbol{\omega}_B^{IB} - \boldsymbol{\omega}_{dB}) + \langle\!\langle \boldsymbol{\omega}_B^{IB} \rangle\!\rangle \mathbf{J}\boldsymbol{\omega}_{dB}\right) + \mathbf{J}\dot{\boldsymbol{\omega}}_{dB} + \tilde{\boldsymbol{\tau}} \ , \quad (10)$$

into (9) one obtains

$$\dot{\tilde{\boldsymbol{\omega}}}_B = \mathbf{J}^{-1}\left((k(\mathbf{w})-1)\langle\!\langle \tilde{\boldsymbol{\omega}}_B \rangle\!\rangle \mathbf{J}\tilde{\boldsymbol{\omega}}_B + \tilde{\boldsymbol{\tau}} + \hat{\tau}_z \mathbf{e}_z\right) \ , \quad (11)$$

where $\tilde{\boldsymbol{\tau}}$ is the new control input and $k(\mathbf{w})$ is any locally Lipschitz continuous function of the state $\mathbf{w}$ defined below. For the sake of convenience, the function $k(\mathbf{w})$ is often chosen constant. For example $k = 1$ will cancel the coriolis term, whereas $k = 0$ yields $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}}$ in the case of a setpoint control (when $\mathbf{z}_{dI}$ is constant or considered to be constant). It will turn out that the stability properties of the closed loop do not depend on the particular choice of k.
If $\hat{\tau}_z = 0$, the control law presented in [6] could now be directly applied to $\tilde{\boldsymbol{\tau}}$ and would achieve asymptotic tracking of $\mathbf{z}_d$. In the following some modifications of that control law will be introduced, but we adopt the input constraint

$$\|\tilde{\boldsymbol{\tau}}_{xy}\| \leq \bar{\tau}_{xy} \ , \quad |\tilde{\tau}_z| \leq \bar{\tau}_z \ , \quad (12)$$

where $\bar{\tau}_{xy}$ and $\bar{\tau}_z$ are positive constants and $\bar{\tau}_{xy} \gg \bar{\tau}_z$ holds. Although $\tilde{\boldsymbol{\tau}}$ is not the actual control torque, we nevertheless consider the constraint, since, as mentioned above, $\tilde{\boldsymbol{\tau}}$ equals $\boldsymbol{\tau}$ if k is chosen to be zero and a setpoint control is considered. Moreover, the constraint can be easily dropped if desired. Due to the similarity of (11) and (7), we will sloppily refer to $\tilde{\boldsymbol{\tau}}$ as the control torque from time to time, while it is always clear that the actual torque is $\boldsymbol{\tau}$. Eventually, we assume

$$\hat{\tau}_z \in [-b, b] \ , \quad 0 < b < \bar{\tau}_z \quad (13)$$

holds for the constant disturbance $\hat{\tau}_z$.
The first modification compared to [6] concerns the optional augmentation of the controller by an integrator state Ω for disturbance rejection, which satisfies

$$\dot{\Omega} = \tilde{\omega}_{Bz} \ . \quad (14)$$

Together with (8), (11) the preceding equation represents the open loop dynamics and the corresponding state vector is $\mathbf{w} = \begin{bmatrix}\mathbf{z}_{dB}^T & \Omega & \tilde{\boldsymbol{\omega}}_B^T\end{bmatrix}^T \in \mathbb{W} = \mathcal{S}^2 \times \mathbb{R} \times \mathbb{R}^3$. Since we want to consider the integrator dynamics (14) as an optional feature, we will discuss the case without integral action at the end of each section. Without stating it explicitly, we assume that the system is undisturbed ($\hat{\tau}_z = 0$) in this case.

4.1 Shaping of the Potential Energy

We will now briefly state the control law, which is largely identical to the control law given in [6] and subsequently discuss the modifications made. For a detailed motivation and discussion of the controller properties, we refer to the aforementioned article. Since the controller design is based on an energy shaping approach (see e.g. [10]), it is constructed such that the closed loop system is described by means of an assigned continuously differentiable energy function $V(\mathbf{w})$, which has a strict minimum at the desired equilibrium point $\mathbf{w}_d = \begin{bmatrix}\mathbf{e}_z^T & \Omega_d & \mathbf{0}^T\end{bmatrix}^T$. Since the state Ω was introduced for disturbance rejection, the constant value Ω_d is not known a priori but depends on the disturbance $\hat{\tau}_z$. In the following we will assign an energy function

$$V(\mathbf{w}) = E_{rot}(\tilde{\boldsymbol{\omega}}_B) + E_{pot}(\mathbf{z}_{dB}, \Omega) = \frac{1}{2}\tilde{\boldsymbol{\omega}}_B^T \mathbf{J}\tilde{\boldsymbol{\omega}}_B + E_{pot}(\mathbf{z}_{dB}, \Omega) \ , \quad (15)$$

which is composed of a kinetic and a potential energy part and fulfills $V(\mathbf{w}_d) = 0$ and $V(\mathbf{w}) > 0$ if $\mathbf{w} \neq \mathbf{w}_d$. Moreover, V will be constructed such that all its sublevel sets are compact.
The potential energy E_{pot} necessarily needs a component depending on the alignment error of the thrust axis. A natural error function for the alignment error is the angle $\varphi = \arccos(\mathbf{z}_B^T \mathbf{z}_{dB}) = \arccos(\mathbf{e}_z^T \mathbf{z}_{dB}) = \arccos(z_{dBz}) \in [0, \pi]$ between $\mathbf{z}$ and $\mathbf{z}_d$. We propose a potential energy of the form

$$E_{pot}(\varphi, \Omega) = E_\varphi(\varphi) + E_\Omega(\Omega) + E_{\hat{\tau}_z}(\Omega) \ , \tag{16}$$

where

$$E_\varphi(\varphi) = c_\varphi \int_0^\varphi \Lambda_{\varphi_l}^{\varphi_u}(\zeta)\, d\zeta \ , \quad E_\Omega(\Omega) = \int_{\Omega_d}^{\Omega} \varpi(\zeta)\, d\zeta \ , \quad E_{\hat{\tau}_z}(\Omega) = \int_{\Omega_d}^{\Omega} -\hat{\tau}_z \, d\zeta \ . \tag{17}$$

Therein, $c_\varphi < \frac{\bar{\tau}_{xy}}{\sin(\varphi_l)}$ is a positive constant and $\Lambda_{\varphi_l}^{\varphi_u}(\zeta)$ is defined in (1). The energy E_φ can be considered as the potential energy caused by a torsion spring with a nonlinear spring characteristic $-c_\varphi \cdot \Lambda_{\varphi_l}^{\varphi_u}(\varphi)$, which is arranged between the actual and the desired thrust direction. Regarding Ω as a deflection around the body-fixed z-axis, $E_{\hat{\tau}_z}$ is the potential energy component caused by the disturbance $\hat{\tau}_z$. Finally, E_Ω can be interpreted as the potential of a spring with spring characteristic $-\varpi(\Omega)$. We claim that $\varpi : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous odd function additionally satisfying

$$\frac{d\varpi(\Omega)}{d\Omega} \geq 0, \qquad \max(|\varpi(\Omega)|) < \bar{\tau}_z, \qquad \forall \Omega, \tag{18}$$

$$\frac{d\varpi(\Omega)}{d\Omega} > 0, \qquad \max(|\varpi(\Omega)|) > b, \qquad \text{if } \Omega \in [-\bar{\Omega}, \bar{\Omega}], \tag{19}$$

where $\bar{\Omega}$ is a positive constant. A simple function satisfying the requirements would be the saturated linear spring characteristic

$$c_\Omega \cdot sat_{\bar{\Omega}}(\Omega) = \begin{cases} c_\Omega \cdot \Omega & \text{if } \Omega \in [-\bar{\Omega}, \bar{\Omega}], \\ c_\Omega \cdot \operatorname{sgn}(\Omega) \cdot \bar{\Omega} & \text{otherwise,} \end{cases} \tag{20}$$

where the constant c_Ω fulfills $\frac{\bar{\tau}_z}{\bar{\Omega}} > c_\Omega > \frac{b}{\bar{\Omega}} > 0$. The constraints (18) and (19) imply that for any admissible $\hat{\tau}_z$ there exists a $\Omega_d \in [-\bar{\Omega}, \bar{\Omega}]$ such that $\varpi(\Omega_d) - \hat{\tau}_z = 0$. In other words, the virtual spring is able to compansate the disturbance at a certain deflection Ω_d.
In order to take the derivative of V with respect to time, the time derivative of the error angle φ is needed. Using (8) and the unit length of $\mathbf{z}_B$, one obtains

$$\dot{\varphi} = \frac{-\dot{z}_{dBz}}{\sqrt{1 - z_{dBz}^2}} = -\underbrace{\frac{1}{\sqrt{z_{dBx}^2 + z_{dBy}^2}} \begin{bmatrix} -z_{dBy} \\ z_{dBx} \\ 0 \end{bmatrix}^T}_{\mathbf{e}_\varphi^T} \tilde{\boldsymbol{\omega}}_B = -\mathbf{e}_\varphi^T \tilde{\boldsymbol{\omega}}_B \ , \tag{21}$$

where $\mathbf{e}_\varphi$ obviously is a unit vector. Note that $\dot{\varphi}$ only depends on the first two elements of $\tilde{\boldsymbol{\omega}}_B$ but not on $\tilde{\omega}_{Bz}$. Accordingly, computing the time derivative of the energy function $V(\mathbf{w})$ yields

$$\begin{aligned} \dot{V} &= \tilde{\boldsymbol{\omega}}_B^T \mathbf{J} \dot{\tilde{\boldsymbol{\omega}}}_B + \frac{\partial E_\varphi(\varphi)}{\partial \varphi} \dot{\varphi} + \left(\frac{\partial E_\Omega(\Omega)}{\partial \Omega} + \frac{\partial E_{\hat{\tau}_z}(\Omega)}{\partial \Omega} \right) \dot{\Omega} \\ &= \tilde{\boldsymbol{\omega}}_B^T (\tilde{\boldsymbol{\tau}} + \hat{\tau}_z \mathbf{e}_z) - \underbrace{\left(\overbrace{\frac{\partial E_\varphi(\varphi)}{\partial \varphi} \mathbf{e}_\varphi^T}^{(\mathbf{T}_\varphi^\varphi)^T} - \overbrace{\frac{\partial E_\Omega(\Omega)}{\partial \Omega} \mathbf{e}_z^T}^{(\mathbf{T}_z^\Omega)^T} \right)}_{\mathbf{T}^T} \tilde{\boldsymbol{\omega}}_B - \underbrace{\hat{\tau}_z \mathbf{e}_z^T}_{(\mathbf{T}_z^{\hat{\tau}_z})^T} \tilde{\boldsymbol{\omega}}_B \ , \end{aligned} \tag{22}$$

where (11), (14), and (21) have been used. The vector $\mathbf{T} = \mathbf{T}_\varphi^\varphi + \mathbf{T}_z^\Omega$ denotes the torque field caused by $E_\varphi + E_\Omega$ whereas $\mathbf{T}_z^{\hat{\tau}_z}$ is the torque resulting from $E_{\hat{\tau}_z}$. Now, by choosing the control law

$$\tilde{\boldsymbol{\tau}} = \mathbf{T} - \mathbf{D}(\mathbf{w})\tilde{\boldsymbol{\omega}}_B \, , \tag{23}$$

where $\mathbf{D}(\mathbf{w}) \geq \mathbf{0}$ is a state dependent damping matrix and inserting it into (22), one obtains

$$\dot{V} = -\tilde{\boldsymbol{\omega}}_B^T \mathbf{D}(\mathbf{w})\tilde{\boldsymbol{\omega}}_B \leq \mathbf{0} \, . \tag{24}$$

It follows from (24) that the sublevel sets of V are not only compact[1] but also positively invariant, which proves global stability of the desired equilibrium.
In contrast to [6], a slightly modified potential energy component E_φ was chosen in (17). Instead of a linear increasing and decreasing integrand, here an integrand increasing and decreasing with the sine function is chosen. While the deviation from the linear function is negligible for typical values of φ_l and φ_u, this choice significantly facilitates linearizing the closed loop system around its equilibrium points for the analysis of the local properties.
If one forgoes the optional integral dynamics (14), one simply omits $E_\Omega(\Omega)$ and $E_{\hat{\tau}_z}(\Omega)$ in the potential energy.

4.2 Damping Injection

The damping matrix $\mathbf{D}(\mathbf{w})$ needed to complete the control law (23) is adopted from [6] with some minor changes for $\varphi > \varphi_u - \Delta\varphi$. The basic idea is to decompose the angular velocity $\tilde{\boldsymbol{\omega}}_B$ into components having a different meaning in view of the control task and to damp them individually. Introducing the orthonormal basis vectors

$$\mathbf{e}_\perp = \frac{1}{\sqrt{z_{dBx}^2 + z_{dBy}^2}} \begin{bmatrix} z_{dBx} \\ z_{dBy} \\ 0 \end{bmatrix}, \quad \mathbf{e}_\varphi = \frac{1}{\sqrt{z_{dBx}^2 + z_{dBy}^2}} \begin{bmatrix} -z_{dBy} \\ z_{dBx} \\ 0 \end{bmatrix}, \quad \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{25}$$

the decomposition of $\tilde{\boldsymbol{\omega}}_B$ reads

$$\tilde{\boldsymbol{\omega}}_B = \mathbf{e}_\perp^T \tilde{\boldsymbol{\omega}}_B \mathbf{e}_\perp + \mathbf{e}_\varphi^T \tilde{\boldsymbol{\omega}}_B \mathbf{e}_\varphi + \mathbf{e}_z^T \tilde{\boldsymbol{\omega}}_B \mathbf{e}_z = \omega_\perp \mathbf{e}_\perp + \omega_\varphi \mathbf{e}_\varphi + \omega_z \mathbf{e}_z \, . \tag{26}$$

Note that according to (21), it holds that $\omega_\varphi = -\dot{\varphi}$ and that the other components $\omega_\perp$ and $\omega_z = \tilde{\omega}_{Bz}$ do not influence $\dot{\varphi}$. By choosing a damping matrix of the form

$$\mathbf{D}(\mathbf{w}) = \kappa_{xy}(\mathbf{w})\left(d_\varphi(\mathbf{w})\mathbf{e}_\varphi \mathbf{e}_\varphi^T + d_\perp \mathbf{e}_\perp \mathbf{e}_\perp^T\right) + \kappa_z(\mathbf{w}) d_z \mathbf{e}_z \mathbf{e}_z^T = \begin{bmatrix} \kappa_{xy}(\mathbf{w})\mathbf{D}_{xy}(\mathbf{w}) & \mathbf{0} \\ \mathbf{0} & \kappa_z(\mathbf{w}) d_z \end{bmatrix} \geq 0 \, , \tag{27}$$

the damping coefficients d_φ, $d_\perp$ and d_z allow an individual damping of ω_φ, $\omega_\perp$ and ω_z. The submatrix $\mathbf{D}_{xy}(\mathbf{w})$ in (27) reads

$$\mathbf{D}_{xy}(\mathbf{w}) = \frac{d_\varphi(\mathbf{w})}{z_{dBx}^2 + z_{dBy}^2} \begin{bmatrix} z_{dBy}^2 & -z_{dBx} z_{dBy} \\ -z_{dBx} z_{dBy} & z_{dBx}^2 \end{bmatrix} + \frac{d_\perp}{z_{dBx}^2 + z_{dBy}^2} \begin{bmatrix} z_{dBx}^2 & z_{dBx} z_{dBy} \\ z_{dBx} z_{dBy} & z_{dBy}^2 \end{bmatrix} \tag{28}$$

[1] One may verify the compactness of the sublevel sets by taking into account that the reduced attitude space $\mathcal{S}^2$ is compact itself, E_{rot} is a radially unbounded function of $\tilde{\boldsymbol{\omega}}_B$ and $E_\Omega(\Omega) + E_{\hat{\tau}_z}(\Omega)$ is an unbounded function of $|\Omega - \Omega_d|$.

and the gains κ_{xy} and κ_z serve to saturate the damping torques if necessary. They are defined as

$$\kappa_{xy} = \min_{\kappa>0, \|\mathbf{T}_{xy} - \kappa \cdot \mathbf{D}_{xy} \tilde{\boldsymbol{\omega}}_{Bxy}\| = \bar{\tau}_{xy}} (1, \kappa) , \qquad \kappa_z = \min_{\kappa>0, |T_z - \kappa \cdot d_z \tilde{\omega}_{Bz}| = \bar{\tau}_z} (1, \kappa) \tag{29}$$

and guarantee compliance with (12). Since $\omega_\perp$ and ω_z do not contribute to a decrease of φ they should always be damped. Accordingly $d_\perp$ and d_z are chosen to be positive constants. The choice of the damping coefficient d_φ given below, is guided by some simplifying considerations on the motion of the controlled system. These simplifications merely serve as tool for the design of d_φ and do not affect the stability analysis in Section 5. First, we assume that the system is undisturbed, i.e. $\hat{\tau}_z = 0$. Second, regarding the symmetry properties of a quadrotor, one can assume that $\mathbf{J} \approx \hat{\mathbf{J}} = \mathrm{diag}(\hat{J}_1, \hat{J}_1, \hat{J}_2) > \mathbf{0}$ approximately holds. Since moreover $\bar{\tau}_{xy} \gg \bar{\tau}_z$, one will usually construct the energies E_φ and E_Ω such that $\|\mathbf{T}_\varphi^\varphi\| \gg \|\mathbf{T}_z^\Omega\|$ holds in the better part of the state space. It is therefore plausible to assume that the influence of $\mathbf{T}_z^\Omega$ on the motion of the quadrotor is rather negligible compared to $\mathbf{T}_\varphi^\varphi$. Assuming $\mathbf{T}_z^\Omega \approx 0$ and an initial angular velocity around the axis $\mathbf{e}_\varphi$, i.e. $\tilde{\boldsymbol{\omega}}_B \parallel \mathbf{e}_\varphi$, one concludes from (11), (23), (22) and (27) that $\dot{\tilde{\boldsymbol{\omega}}}_B \parallel \mathbf{e}_\varphi$ also holds. Thus, the rotational motion occurs only around one constant axis given by $\mathbf{e}_\varphi$ and accordingly it can be completely characterized using the angle φ. Assuming furthermore that no saturation occurs, i.e. $\kappa_{xy} = 1$, one obtains

$$\hat{J}_1 \ddot{\varphi} = -T_\varphi - d_\varphi(\mathbf{w}) \cdot \dot{\varphi} \quad \text{if } \varphi \neq 0 \,,^2 \tag{30}$$

where $T_\varphi = \|\mathbf{T}_\varphi^\varphi\|$. According to (22), (17) and (1) we have $T_\varphi = c_\varphi \sin(\varphi)$ for $0 \le \varphi \le \varphi_l$. For typical values of φ_l we can furthermore use the small-angle approximation to obtain

$$\hat{J}_1 \ddot{\varphi} = -c_\varphi \varphi - d_\varphi(\mathbf{w}) \cdot \dot{\varphi} \quad \text{if } 0 < \varphi \le \varphi_l \,. \tag{31}$$

By choosing

$$d_\varphi(\varphi, \dot{\varphi}, T_\varphi) = \chi_{\varphi_l, \varphi_l + \Delta\varphi}^{\varphi_u - \Delta\varphi, \varphi_u} \left(\varphi, \delta_\varphi, d_\varphi^*(\varphi, \dot{\varphi}, T_\varphi)\right) , \tag{32}$$

where δ_φ and $\Delta\varphi < \frac{\varphi_u - \varphi_l}{2}$ are positive constants, the dynamics (31) are rendered linear. This certainly is a desirable behavior for small alignment errors of the thrust vector. In contrast to d_φ given in [6], here in (32) the damping is rendered constant not only for $\varphi \le \varphi_l$ but also for $\varphi \ge \varphi_u$. This together with choosing

$$d_\perp = \delta_\varphi \,, \tag{33}$$

ensures that, according to (28), the matrix $\mathbf{D}_{xy}$ becomes constant,

$$\mathbf{D}_{xy} = \delta_\varphi \mathbf{I}_2 > 0 \quad \text{if } \varphi \le \varphi_l \text{ or } \varphi \ge \varphi_u. \tag{34}$$

This way, difficulties with determining $\mathbf{e}_\varphi$ and $\mathbf{e}_\perp$ near $\varphi = 0$ and $\varphi = \pi$, where $z_{dBx}^2 + z_{dBy}^2 = 0$, are effectively omitted.

A sophisticated damping strategy d_φ^* is applied in the region $\varphi_l + \Delta\varphi < \varphi < \varphi_u - \Delta\varphi$. There, a strategy similar to the bang-bang solution of a time optimal control is applied. This requires to indicate a switching curve $s_\varphi(\varphi) < 0$, where the transition from acceleration ($\ddot{\varphi} < 0$) to deceleration ($\ddot{\varphi} > 0$) occurs. If $\dot{\varphi} > s(\varphi)$, the damping d_φ^* is chosen to enable maximum acceleration based on (30). In case of $\dot{\varphi} > 0$, this means supporting the torque $-T_\varphi$ by a positive damping $d_\varphi^* > 0$, such that the maximum torque $\bar{\tau}_{xy}$ is exploited. In case of $\dot{\varphi} < 0$, the damping is set to

[2]Note that $\varphi = 0$ has to be excluded, since we have defined φ to be positive or equal to zero. Accordingly $\dot{\varphi} \ge 0$ must hold for $\varphi = 0$. The solutions of the given differential equation do not necessarily satisfy this requirement.

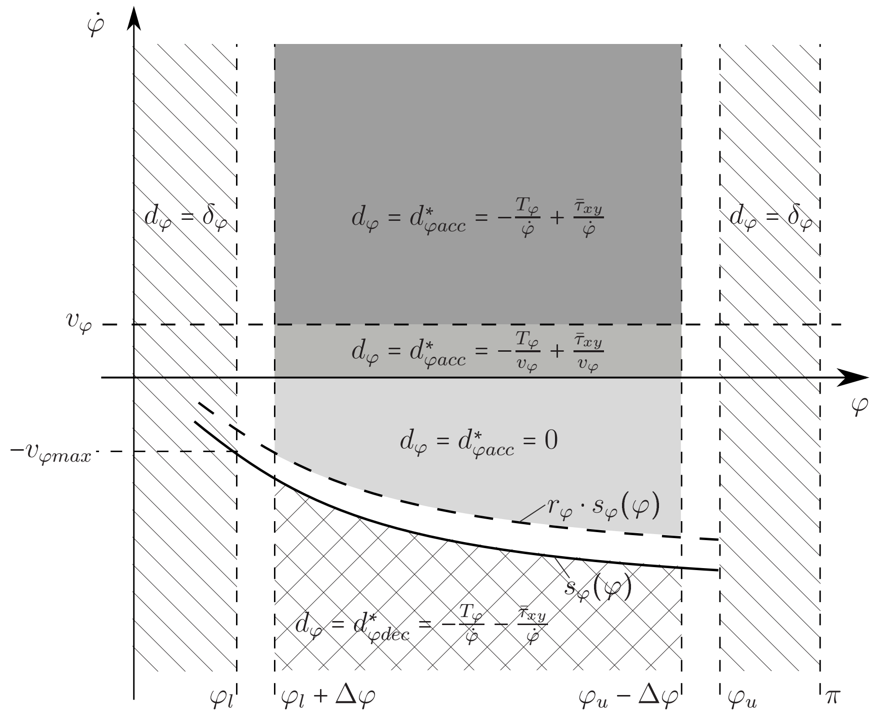

Figure 2: Visualization of the damping $d_\varphi(\varphi, \dot{\varphi}, T_\varphi)$ in the phase plane. Hatched areas: Constant damping. Shaded areas: Acceleration supporting damping. Crosshatched areas: Deceleration supporting damping. White areas indicate regions, where the damping is interpolated.

zero to avoid counteracting $-T_\varphi$. If $\dot{\varphi} < s(\varphi) < 0$ maximum deceleration is desired, which can be achieved by choosing $d^*_\varphi > 0$ so high that $-T_\varphi$ is overcompensated and $\bar{\tau}_{xy}$ is used to slow down. Summing up, we choose

$$d^*_\varphi(\varphi, \dot{\varphi}, T_\varphi) = \chi^{r_\varphi \cdot s_\varphi(\varphi)}_{s_\varphi(\varphi)}\left(\dot{\varphi}, d^*_{\varphi dec}(\dot{\varphi}, T_\varphi), d^*_{\varphi acc}(\dot{\varphi}, T_\varphi)\right), \tag{35}$$

where

$$d^*_{\varphi acc}(\dot{\varphi}, T_\varphi) = \begin{cases} -\frac{T_\varphi}{\dot{\varphi}} + \frac{\bar{\tau}_{xy}}{\dot{\varphi}} & \text{if } \dot{\varphi} > v_\varphi \\ -\frac{T_\varphi}{v_\varphi} + \frac{\bar{\tau}_{xy}}{v_\varphi} & \text{if } v_\varphi \geq \dot{\varphi} > 0 \\ 0 & \text{if } 0 \geq \dot{\varphi}, \end{cases} \quad d^*_{\varphi dec}(\dot{\varphi}, T_\varphi) = -\frac{T_\varphi}{\dot{\varphi}} - \frac{\bar{\tau}_{xy}}{\dot{\varphi}}, \tag{36}$$

and r_φ as well as v_φ are positive constants. It is clear from (35) that $0 < r_\varphi < 1$ defines a region of interpolation between $d^*_{\varphi acc}$ and $d^*_{\varphi dec}$ in order to render the resulting torque continuous. An examination of $d^*_{\varphi acc}$ reveals that the small constant $v_\varphi > 0$ prevents the damping from growing unbounded when $\dot{\varphi}$ approaches zero. The switching curve which is used in (35) reads

$$s_\varphi(\varphi) = -\sqrt{v^2_{\varphi max} - 2\hat{J}_1^{-1}\bar{\tau}_{xy}(\varphi_l - \varphi)} < 0\,. \tag{37}$$

This curve is simply the phase-plane trajectory $\dot{\varphi}(\varphi)$ solving the differential equation $\hat{J}_1\ddot{\varphi} = \bar{\tau}_{xy}$ and passing through the point $\dot{\varphi}(\varphi_l) = -v_{\varphi max} < 0$. Figure 2 visualizes the applied damping strategy for d_φ in the phase plane.

5 Stability Properties

In this section we prove local exponential and almost global asymptotic stability of the desired equilibrium. This is done by showing first that apart from the desired equilibrium another undesired equilibrium exists. The analysis of their local properties shows that the desired equilibrium is locally exponentially stable and the undesired equilibrium is a hyperbolic fixed point. Using LaSalle's invariance principle we establish that the undesired equilibrium is unstable and the desired equilibrium exhibits a region of attraction that covers the whole state space except for a manifold with Lebesgue measure zero. As it is analyzed in [1], this so called almost global asymptotic stability is the best we can achieve with a control law that is continuous over the whole state space.

By inserting the control law (23) derived in the previous section into the open loop dynamics (8), (11), (14) and setting the left hand side to zero, the equilibrium points are identified as

$$\mathbf{w}_d = \begin{bmatrix} \mathbf{e}_z^T & \Omega_d & \mathbf{0}^T \end{bmatrix}^T , \qquad \mathbf{w}_u = \begin{bmatrix} -\mathbf{e}_z^T & \Omega_d & \mathbf{0}^T \end{bmatrix}^T , \tag{38}$$

where $\mathbf{w}_d$ is the desired equilibrium and $\mathbf{w}_u$ is an undesired equilibrium since $\mathbf{z}_{dB} = -\mathbf{e}_z$ implies $\varphi = \pi$.

5.1 Local Properties

To derive the local properties, the closed loop system is linearized around $\mathbf{w}_d$ and $\mathbf{w}_u$. Observe that $\mathbf{z}_{dB}$ has only two degrees of freedom since it evolves on $\mathcal{S}^2$. Consequently, it can be locally represented by $\mathbf{z}_{dBxy}$ in a neighborhood of the equilibria. Near $\mathbf{w}_d$, where $\mathbf{z}_{dB} \approx \mathbf{e}_z$, it holds that $z_{dBz} = \sqrt{1-(z_{dBx}^2+z_{dBy}^2)}$, whereas near $\mathbf{w}_u$, where $\mathbf{z}_{dB} \approx -\mathbf{e}_z$, it holds that $z_{dBz} = -\sqrt{1-(z_{dBx}^2+z_{dBy}^2)}$. Accordingly, from the first two lines of (8) one obtains

$$\begin{bmatrix} \dot{z}_{dBx} \\ \dot{z}_{dBy} \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_{Bz} z_{dBy} - \tilde{\omega}_{By}\sqrt{1-(z_{dBx}^2+z_{dBy}^2)} \\ -\tilde{\omega}_{Bz} z_{dBx} + \tilde{\omega}_{Bx}\sqrt{1-(z_{dBx}^2+z_{dBy}^2)} \end{bmatrix} \quad \text{in a neighborhood of } \mathbf{w}_d \text{ and} \tag{39}$$

$$\begin{bmatrix} \dot{z}_{dBx} \\ \dot{z}_{dBy} \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_{Bz} z_{dBy} + \tilde{\omega}_{By}\sqrt{1-(z_{dBx}^2+z_{dBy}^2)} \\ -\tilde{\omega}_{Bz} z_{dBx} - \tilde{\omega}_{Bx}\sqrt{1-(z_{dBx}^2+z_{dBy}^2)} \end{bmatrix} \quad \text{in a neighborhood of } \mathbf{w}_u. \tag{40}$$

Recalling that $\varphi \in [0,\pi]$ and hence $\sqrt{z_{dBx}^2+z_{dBy}^2} = \sin(\varphi)$, the torque $\mathbf{T}_\varphi^\varphi$ around $\mathbf{w}_d$ can be indicated as

$$\mathbf{T}_\varphi^\varphi = c_\varphi \sin(\varphi)\cdot \mathbf{e}_\varphi = c_\varphi \sin(\varphi)\cdot \frac{1}{\sin(\varphi)} \begin{bmatrix} -z_{dBy} \\ z_{dBx} \\ 0 \end{bmatrix} = c_\varphi \begin{bmatrix} -z_{dBy} \\ z_{dBx} \\ 0 \end{bmatrix} \tag{41}$$

and analogous computations in a neighborhood of $\mathbf{w}_u$ yield

$$\mathbf{T}_\varphi^\varphi = c_\varphi \frac{\sin(\varphi_l)}{\sin(\varphi_u)} \sin(\varphi)\cdot \mathbf{e}_\varphi = c_\varphi \frac{\sin(\varphi_l)}{\sin(\varphi_u)} \begin{bmatrix} -z_{dBy} \\ z_{dBx} \\ 0 \end{bmatrix} . \tag{42}$$

5.1.1 Local Stability properties of the desired equilibrium $\mathbf{w}_d$

For the linearization around $\mathbf{w}_d$, it proves advantageous to define the reduced state vector $\mathbf{w}_r^d = \begin{bmatrix} z_{dBy} & -z_{dBx} & \Omega & \tilde{\boldsymbol{\omega}}_B^T \end{bmatrix}^T = \begin{bmatrix} \mathfrak{z}_d^T & \tilde{\boldsymbol{\omega}}_B^T \end{bmatrix}^T$. Note in particular the order and the sign of the first two components. In terms of $\mathbf{w}_r^d$ the desired equilibrium reads $\bar{\mathbf{x}}_r^d = \begin{bmatrix} \mathbf{0}^T & \Omega_d & \mathbf{0}^T \end{bmatrix}^T$. Linearizing (39) and (14) around $\mathbf{w}_r^d = \bar{\mathbf{x}}_r^d$ yields

$$\Delta\dot{\mathfrak{z}}_d = \Delta\tilde{\boldsymbol{\omega}}_B \tag{43}$$

and the linearization of (11), considering (23), (41), (27) and (34) results in

$$\Delta\dot{\tilde{\boldsymbol{\omega}}}_B = -\mathbf{J}^{-1}\mathbf{C}_d\Delta\mathfrak{z}_d - \mathbf{J}^{-1}\mathbf{B}\Delta\tilde{\boldsymbol{\omega}}_B\ , \tag{44}$$

where $\mathbf{C}_d = \mathrm{diag}(c_\varphi, c_\varphi, \frac{d\varpi(\Omega)}{d\Omega}|_{\Omega=\Omega_d}) > \mathbf{0}$ and $\mathbf{B} = \mathrm{diag}(\delta_\varphi, \delta_\varphi, d_z) > \mathbf{0}$. Combining (43) and (44) yields

$$\Delta\dot{\mathbf{x}}_r^d = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{J}^{-1}\mathbf{C}_d & -\mathbf{J}^{-1}\mathbf{B} \end{bmatrix}}_{\mathbf{A}_d} \Delta\mathbf{x}_r^d\ . \tag{45}$$

Since $\mathbf{J}^{-1}$ is positive definite, there exists a nonsingular real matrix M, such that $\mathbf{J}^{-1} = \mathrm{M}\mathrm{M}^T$. By defining $\tilde{\mathbf{C}}_d = \mathrm{M}^T\mathbf{C}_d\mathrm{M} > \mathbf{0}$ and $\tilde{\mathbf{B}} = \mathrm{M}^T\mathbf{B}\mathrm{M} > \mathbf{0}$ one can rewrite $\mathbf{A}_d$ as

$$\mathbf{A}_d = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathrm{M}\tilde{\mathbf{C}}_d\mathrm{M}^{-1} & -\mathrm{M}\tilde{\mathbf{B}}\mathrm{M}^{-1} \end{bmatrix} = \begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{M} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\tilde{\mathbf{C}}_d & -\tilde{\mathbf{B}} \end{bmatrix}}_{\tilde{\mathbf{A}}_d} \begin{bmatrix} \mathrm{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathrm{M}^{-1} \end{bmatrix}\ . \tag{46}$$

Since the eigenvalues of $\mathbf{A}_d$ and $\tilde{\mathbf{A}}_d$ are the same, we can restrict the analysis to $\tilde{\mathbf{A}}_d$. Let $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1^T & \mathbf{v}_2^T \end{bmatrix}^T$, where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^3$, be any eigenvector of $\tilde{\mathbf{A}}_d$ corresponding to the eigenvalue λ. Then, $\tilde{\mathbf{A}}_d\mathbf{v} = \lambda\mathbf{v}$ implies $\mathbf{v}_2 = \lambda\mathbf{v}_1$ and $\tilde{\mathbf{B}}\mathbf{v}_2 + \tilde{\mathbf{C}}_d\mathbf{v}_1 = -\lambda\mathbf{v}_2$. Inserting the first equation into the second yields $\lambda^2\mathbf{v}_1 + \lambda\tilde{\mathbf{B}}\mathbf{v}_1 + \tilde{\mathbf{C}}_d\mathbf{v}_1 = \mathbf{0}$. Multiplying by the complex conjugate $\bar{\mathbf{v}}_1^T$ from the left results in

$$\bar{\mathbf{v}}_1^T\mathbf{v}_1\lambda^2 + \bar{\mathbf{v}}_1^T\tilde{\mathbf{B}}\mathbf{v}_1\lambda + \bar{\mathbf{v}}_1^T\tilde{\mathbf{C}}_d\mathbf{v}_1 = a\lambda^2 + b\lambda + c = 0\ . \tag{47}$$

From the positive definiteness of $\tilde{\mathbf{C}}_d$ and $\tilde{\mathbf{B}}$, it follows that $a, b, c > 0$ and from the Routh-Hurwitz criterion for polynomials of order two, all solutions of (47) lie in the left complex half plane. This proves asymptotic stability of $\mathbf{w}_d$ with local exponential convergence, [8, Theorem 4.15].

In the case that no integral state Ω is desired, the linearization around the desired equilibrium yields a dynamic matrix $\bar{\mathbf{A}}_d$, which is obtained from $\mathbf{A}_d$ by deleting the third row and the third column. Let $\mathbf{A}_0$ be the matrix, where the third column of $\mathbf{A}_d$ has been replaced by zeros, then by a Laplace expansion along the third column of the matrix $(\lambda\mathbf{I}_6 - \mathbf{A}_0)$, it can be seen that $\det(\lambda\mathbf{I}_6 - \mathbf{A}_0) = \lambda\det(\lambda\mathbf{I}_5 - \bar{\mathbf{A}}_d)$. Thus, all eigenvalues of $\bar{\mathbf{A}}_d$ are eigenvalues of $\mathbf{A}_0$, but $\mathbf{A}_0$ has an additional zero eigenvalue. It holds that

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{J}^{-1}\mathbf{C}_0 & -\mathbf{J}^{-1}\mathbf{B} \end{bmatrix}\ , \tag{48}$$

where $\mathbf{C}_0 = \mathrm{diag}(c_\varphi, c_\varphi, 0) \geq \mathbf{0}$ has rank two. Analogously as before one derives the conditional equation (47), where $\tilde{\mathbf{C}}_d$ is now replaced by the rank two matrix $\tilde{\mathbf{C}}_0 = \mathrm{M}^T\mathbf{C}_0\mathrm{M} \geq \mathbf{0}$. As one can easily verify that $\mathbf{A}_0$ has exactly one zero eigenvalue, the vector $\mathbf{v}_{1,1}$ yielding $c = \bar{\mathbf{v}}_{1,1}^T\tilde{\mathbf{C}}_0\mathbf{v}_{1,1} = 0$

must be one of the three vectors $\mathbf{v}_{1,i}$ solving $\lambda^2\mathbf{v}_{1,i} + \lambda\tilde{\mathbf{B}}\mathbf{v}_{1,i} + \tilde{\mathbf{C}}_0\mathbf{v}_{1,i} = \mathbf{0}$, $i \in \{1,2,3\}$. This can be understood by noting that $c = 0$ in (47) results in $\lambda(\lambda a + b) = 0$ which gives rise to the zero eigenvalue and additionally to a negative eigenvalue $\lambda = -\frac{a}{b}$. Since $\mathbf{A}_0$ has no further zero eigenvalues, $\mathbf{v}_{1,2}$ and $\mathbf{v}_{1,3}$ must be linearly independent of $\mathbf{v}_{1,1}$ and thus yield $c > 0$. Using again the Routh-Hurwitz criterion we conclude that all remaining eigenvalues are in the left half plane. It follows that all eigenvalues of $\bar{\mathbf{A}}_d$ have a negative real part and hence the desired closed loop equilibrium is locally exponentially stable.

5.1.2 Local properties of the undesired equilibrium $\mathbf{w}_u$

For the linearization around $\mathbf{w}_u$ another reduced state vector $\mathbf{w}_r^u = \begin{bmatrix}-z_{dBy} & z_{dBx} & \Omega & \tilde{\boldsymbol{\omega}}_B^T\end{bmatrix}^T$ is defined, where the sign of the first two components has been changed. Linearizing around the undesired equilibrium, which corresponds to $\bar{\mathbf{x}}_r^u = \begin{bmatrix}\mathbf{0}^T & \Omega_d & \mathbf{0}^T\end{bmatrix}^T$ yields

$$\Delta\dot{\mathbf{x}}_r^u = \underbrace{\begin{bmatrix}\mathbf{0} & \mathbf{I}_3 \\ -\mathbf{J}^{-1}\mathbf{C}_u & -\mathbf{J}^{-1}\mathbf{B}\end{bmatrix}}_{\mathbf{A}_u}\Delta\mathbf{x}_r^u \; , \tag{49}$$

where now the equations (40) and (42) have been used and accordingly the stiffness matrix $\mathbf{C}_u = \mathrm{diag}\left(-c_\varphi\frac{\sin(\varphi_l)}{\sin(\varphi_u)}, -c_\varphi\frac{\sin(\varphi_l)}{\sin(\varphi_u)}, \frac{d\varpi(\Omega)}{d\Omega}\big|_{\Omega=\Omega_d}\right)$. As it was shown before, the eigenvalues of $\mathbf{A}_u$ coincide with the eigenvalues of

$$\tilde{\mathbf{A}}_u = \begin{bmatrix}\mathbf{0} & \mathbf{I}_3 \\ -\tilde{\mathbf{C}}_u & -\tilde{\mathbf{B}}\end{bmatrix} \; , \tag{50}$$

where $\tilde{\mathbf{C}}_u = \mathbf{M}^T\mathbf{C}_u\mathbf{M}$. The eigenvalues of $\tilde{\mathbf{A}}_u$ in turn satisfy

$$\bar{\mathbf{v}}_1^T\mathbf{v}_1\lambda^2 + \bar{\mathbf{v}}_1^T\tilde{\mathbf{B}}\mathbf{v}_1\lambda + \bar{\mathbf{v}}_1^T\tilde{\mathbf{C}}_u\mathbf{v}_1 = a\lambda^2 + b\lambda + c = 0 \; . \tag{51}$$

Note that $\mathbf{A}_u$ obviously has full rank and hence no zero eigenvalues can exist, which implies $c \neq 0$. Moreover, from $\tilde{\mathbf{B}} > 0$ it follows that $b > 0$ and accordingly no eigenvalues can lie on the imaginary axis. One concludes that $\mathbf{w}_u$ is a hyperbolic fixed point and hence no invariant center manifold $W^c(\mathbf{w}_u)$ exists, [9, Appendix B]. It will be proven in the next section that $\mathbf{w}_u$ is unstable and hence an unstable invariant manifold $W^u(\mathbf{w}_u)$ of at least dimension one must exist. This in turn limits the stable invariant manifold $W^s(\mathbf{w}_u)$ to be of a smaller dimension than the state space.
The same properties can be established for the case without the integral state Ω by proceeding analogously as in the previous section. Since $\tilde{\mathbf{C}}_{u0}$ now has two negative and one zero eigenvalue, it is easily proven that $\bar{\mathbf{A}}_u$ must posses exactly three eigenvalues in the left and two eigenvalues in the right complex half plane. Thus, there exists a stable invariant manifold of dimension three and an unstable invariant manifold of dimension two.

5.2 Global Stability Properties

In this section we prove almost global asymptotic stability of the desired equilibrium following the lines of [5]. Using LaSalle's invariance principle (see e.g.[8, Theorem 4.4]) this is done by showing first that all solutions converge to $\mathbf{w}_d$ or $\mathbf{w}_u$ and second that $\mathbf{w}_u$ is unstable and hence can only posses a stable invariant manifold of Lebesgue measure zero.
Recalling that all level sets of V are positively invariant and compact, we apply LaSalle's invariance principle by showing that the set $\mathcal{E} := \{\mathbf{w} \in \mathbb{W} : \dot{V}(\mathbf{w}) = 0\}$ contains no invariant sets apart from the equilibrium points $\mathbf{w}_d$ and $\mathbf{w}_u$. Inserting (27) in (24) and using (26) yields

$$\dot{V}(\mathbf{w}) = -\kappa_{xy}(\mathbf{w})\left(d_\varphi(\mathbf{w})\omega_\varphi^2 + d_\perp\omega_\perp^2\right) - \kappa_z(\mathbf{w})d_z\omega_z^2\,. \tag{52}$$

Taking into account that $\kappa_{xy}, \kappa_z > 0$ the function $\dot{V}$ only vanishes if $d_\varphi(\mathbf{w})\omega_\varphi^2 = d_\perp\omega_\perp^2 = d_z\omega_z^2 = 0$ holds. Outside the equilibrium points, in the set $\widetilde{\mathbb{W}} = \mathbb{W}\backslash\{\mathbf{w}_d, \mathbf{w}_u\}$, this condition is only fulfilled in the subset $\mathcal{E}_1$, which is the set of all states with zero angular velocity, and in the set $\mathcal{E}_2$ in which $\tilde{\boldsymbol{\omega}}_B \neq \mathbf{0}$ while $\omega_\perp = \omega_z = d_\varphi(\mathbf{w})\omega_\varphi^2 = 0$. Thus $\mathcal{E}_2$ is the set where $d_\varphi(\mathbf{w}) = 0$. Explicitly stated, the sets $\mathcal{E}_1$, $\mathcal{E}_2$ are

$$\begin{aligned}
&\mathcal{E}_1 = \left\{\mathbf{w} \in \widetilde{\mathbb{W}} : \tilde{\boldsymbol{\omega}}_B = \mathbf{0}\right\}, \\
&\mathcal{E}_2 = \left\{\mathbf{w} \in \widetilde{\mathbb{W}} : \varphi_l + \Delta\varphi \leq \varphi \leq \varphi_u - \Delta\varphi,\, r_\varphi s_\varphi(\varphi) \leq \dot{\varphi} < 0,\, \omega_\perp = 0,\, \omega_z = 0\right\}\,.
\end{aligned} \tag{53}$$

Next, we show that the equilibrium points $\mathbf{w}_d$ and $\mathbf{w}_u$ are the only invariant sets in $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\mathbf{w}_d, \mathbf{w}_u\}$.

Imagine that $\mathbf{w} \in \mathcal{E}_1$, then $\tilde{\boldsymbol{\tau}} \neq \mathbf{0}$ and since $\tilde{\boldsymbol{\omega}}_B = \mathbf{0}$, we have $\dot{\tilde{\boldsymbol{\omega}}} \neq \mathbf{0}$ from (11). Thus, the state will exit the subset $\mathcal{E}_1$ instantaneously, which shows that no invariant sets are contained in $\mathcal{E}_1$.

Now, let us assume $\mathbf{w}$ can stay inside the subset $\mathcal{E}_2$ for all times. Recalling that $\omega_\varphi = -\dot{\varphi}$ it follows from (26) that

$$\tilde{\boldsymbol{\omega}}_B = -\dot{\varphi}\,\mathbf{e}_\varphi \neq \mathbf{0}\,. \tag{54}$$

A lower bound $\underline{\omega}$ for the angular velocity can be derived solving

$$E_{rot0} = \frac{1}{2}\tilde{\boldsymbol{\omega}}_{B0}^T\mathbf{J}\tilde{\boldsymbol{\omega}}_{B0} = \frac{1}{2}\bar{\lambda}(\mathbf{J})\underline{\omega}^2 > 0\,, \tag{55}$$

where $\bar{\lambda}(\mathbf{J})$ denotes the largest eigenvalue of $\mathbf{J}$, E_{rot0} is the rotational energy and $\tilde{\boldsymbol{\omega}}_{B0}$ the angular velocity at the time $t = 0$. Since in the set $\mathcal{E}_2$ no damping occurs, using (23) and (22) the torque can be identified as $\tilde{\boldsymbol{\tau}} = \mathbf{T} = \mathbf{T}_\varphi^\varphi + \mathbf{T}_z^\Omega = k_\varphi \cdot \mathbf{e}_\varphi + k_z \cdot \mathbf{e}_z$, where $k_\varphi > 0$. Now as long as $\mathbf{w} \in \mathcal{E}_2$, it holds that $\dot{E}_{rot} = \tilde{\boldsymbol{\omega}}_B^T\tilde{\boldsymbol{\tau}} = -\dot{\varphi}k_\varphi > 0$ and consequently

$$\underline{\omega}^2 \leq \tilde{\boldsymbol{\omega}}_B^T\tilde{\boldsymbol{\omega}}_B = \dot{\varphi}^2\,. \tag{56}$$

Based on (37), a lower bound for $\dot{\varphi}$ is given by

$$-L = s_\varphi(\pi) < \dot{\varphi} < 0\,. \tag{57}$$

Making use of (57), we can hence extend (56) to obtain

$$\underline{\omega}^2 \leq \dot{\varphi}^2 < -L\dot{\varphi}. \tag{58}$$

Now, let $\varphi_0 \in [\varphi_l + \Delta\varphi, \varphi_u - \Delta\varphi]$ be the error angle at $t = 0$. The time $\tilde{t}$ solving the equation

$$\varphi_l + \Delta\varphi = \varphi_0 + \int_0^{\tilde{t}} \dot{\varphi}\,dt \tag{59}$$

certainly is an upper bound of the time at which the state must leave $\mathcal{E}_2$. By inserting (58) in (59) and evaluating the integral, we obtain the inequality

$$\varphi_l + \Delta\varphi \leq \varphi_0 - \frac{\underline{\omega}^2}{L}\tilde{t}\,, \tag{60}$$

which reveals that $\tilde{t}$ itself is upper bounded by

$$\tilde{t} \leq L \frac{\varphi_0 - (\varphi_l + \Delta\varphi)}{\underline{\omega}^2} \; . \tag{61}$$

This proves our assumption wrong, since $\mathcal{E}_2$ is left in any case and accordingly cannot contain any invariant sets. Moreover, also the union of $\mathcal{E}_1$ and $\mathcal{E}_2$ cannot contain any invariant sets. This is due to the fact that, although the state may cross over from $\mathcal{E}_1$ into $\mathcal{E}_2$, the reverse transition is not possible. Hence, $\{\mathbf{w}_d, \mathbf{w}_u\}$ represents the largest invariant set contained in $\mathcal{E}$ and according to LaSalle's invariance principle every trajectory converges to either to $\mathbf{w}_d$ or to $\mathbf{w}_u$.
In order to prove almost global asymptotic stability of the desired equilibrium $\mathbf{w}_d$, we first notice that $V(\mathbf{w}_d) = 0$ and $V(\mathbf{w}_u) = E_\varphi(\pi) > 0$. If we choose any initial state $\mathbf{w}_0$, such that $V(\mathbf{w}_0) < E_\varphi(\pi)$, we exclude $\mathbf{w}_u$ from the initial sublevel set of $V(\mathbf{w})$ and the solution can only approach $\mathbf{w}_d$. Hence, $\mathbf{w}_d$ is an asymptotically stable equilibrium point. Since the set $\{\mathbf{w} \in \mathbb{W} : V(\mathbf{w}) < E_\varphi(\pi)\}$ is adjacent to $\mathbf{w}_u$, the preceding argumentation also proves that $\mathbf{w}_u$ is unstable and according to the analysis in Section 5.1.2 its stable invariant manifold $W^s(\mathbf{w}_u)$ must be of smaller dimension than the state space, i.e. $\dim(W^s(\mathbf{w}_u)) < 6$. It is known that an m-dimensional invariant manifold of an n-dimensional system has Lebesgue measure zero if $m < n$, see e.g. [9, Appendix B]. This proves almost global asymptotic stability of $\mathbf{w}_d$.
If the integral state Ω is omitted, the stability proof remains completely unchanged.

6 Conclusion

In this paper we have presented a fast tracking controller for the thrust direction of a quadrotor helicopter. The nonlinear controller is an extension of the setpoint controller presented in [6] that involves the augmentation of the controller by integral action concerning the angular rate around the thrust axis. Furthermore, minor modifications of the potential energy and the damping have been introduced, which are advantageous for the local stability analysis, while their impact on the controller performance is negligible. If the integral state is omitted, the closed loop dynamics of the tracking control problem and the setpoint control problem hardly differ. Therefore, we refer the interested reader to [6] for a performance analysis based on simulation results. Compared to [6] the control problem has been additionally reformulated using the reduced attitude parametrization given in [4] enabling a simplified stability analysis. Local exponential and almost global asymptotic stability of the desired equilibrium, corresponding to a zero tracking error, has been proven.

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