Finite-Length Scaling of Convolutional LDPC Codes

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Motivation and Example

- Capacity-reaching LDPC codes exist
- The optimal parameters are known for long block lengths

Question:
What is the best performance for finite block lengths?

LDPC Codes [1]

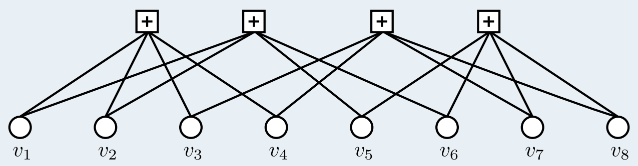
Definition: Low-density Parity-Check Code (Gallager,1962)
Low-density parity-check codes are codes specified by a matrix containing mostly 0's and only a small number of 1's.

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Example: $(2, 4)$ regular code

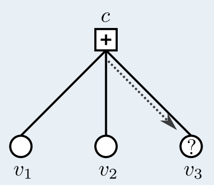
- **Regular** (l, r) codes:
  l ones in every column, respectively r ones in every row
- **Irregular** codes:
  Edge degree distributions described by polynomials
  $\lambda(x) = \sum_{i=1}^{d_v} \lambda_i x^{i-1}$, $\rho(x) = \sum_{i=1}^{d_c} \rho_i x^{i-1}$

Graphical Representation as Tanner Graph (Tanner,1981):

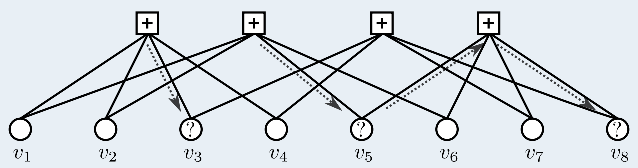


Properties
- **LDPC codes can reach capacity**
- **The decoding complexity stays linear**

Decoding

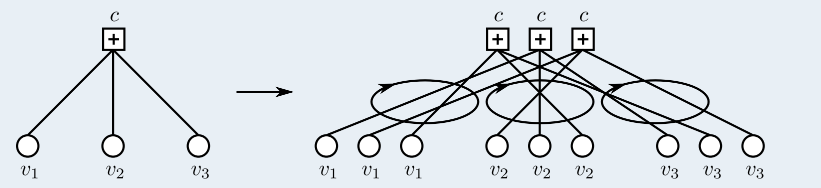


Check equation for check node c:
$\sum_{k \in N(c)} v_k = 0 \mod 2$



If variable nodes are erased due to the transmission over a binary erasure channels (BEC) they can be iteratively restored with the help of the knowledge of the rest of the graph of the code.

Protograph Based Construction [2]

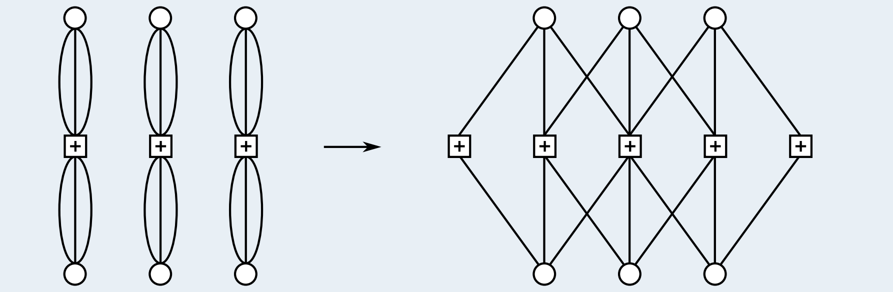


- Small Tanner graph are used as a "blue print" of the structure
- This structure gets copied several times
- Similar connections are randomly permuted to obtain larger girths which avoids dependencies during the iterative decoding

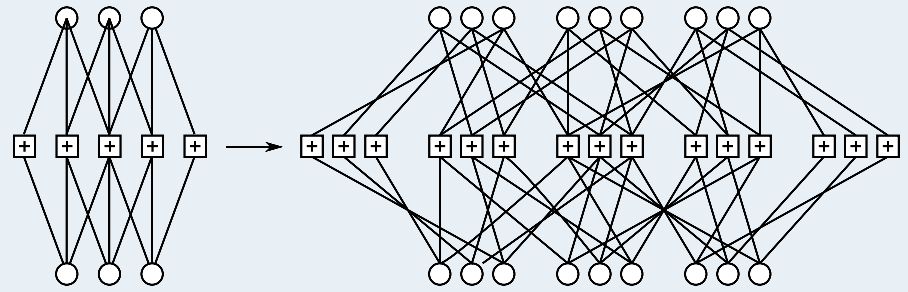
$\Rightarrow$ "copy-and-permute"

Advantages
- The protograph representation can be used for analysis

Codes Based on Coupled Protographs [3]



1. Choose a simple protograph
2. Couple multiple protographs to a spatially coupled protograph



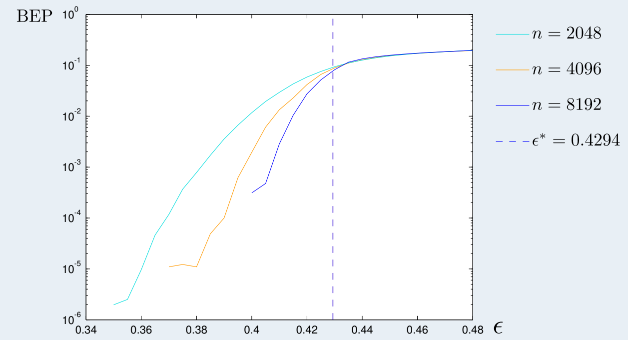
3. Lift the coupled protograph with the "copy-and-permute" operation

The convolutional-like band matrix $\mathbf{H}$ consists of submatrices $\mathbf{H}_{i,j}$ which are permutation matrices for edge permutations:

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & & & & & \\ \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & & & \\ \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & \\ & & \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \\ & & & & \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \ddots \end{pmatrix}$$

Advantages
- Systematic encoding is possible
- **The MAP threshold can be reached with iterative belief propagation (BP) decoding [4]**

Finite-Length Performance of LDPC Codes



Transmission over a binary erasure channel BEC(ϵ), $(r, l) = (3, 6)$, $n = 2048$.

- In the error floor region "small" local weaknesses (small loops in the Tanner graph) have the highest influence
- The decoding behavior in the waterfall region is dominated from "large" (non resolvable clusters in the graph) failures of the code.
- The best possible performance for very large blocks is the threshold of iterative BP decoding: $\epsilon^* = 0.4294$
- We have a smooth transition instead of a single sharp step

Question:
Can we predict the finite block length performance?

▶ Option 1: Plain Simulation

Transmissions over a channel are simulated to obtain the error correction ability of a code $\mathcal{C}$.

\+ The actual performance is obtained
\- High computational Complexity
\- Little insight for Code Optimization is gained

▶ Option 2: Density evolution

The erasure probability densities of the BP decoding are tracked for the degree distribution of a certain class of irregular codes. If the erasure probability converges to 0, the decoding succeeds.

\+ The Decoding process can be accurately tracked
\+ A lot of insight for code design is gained
\- High computational Complexity
\- The results hold for large (infinite) block lengths only

▶ Option 3: EXIT analysis

The exchange and gain of mutual information during the iterations between check nodes and variable nodes is tracked for the degree distribution of a certain class of irregular codes. If the mutual information is increased to 1 during the BP decoding, the decoding succeeds.

\+ The decoding can be accurately tracked with less complexity
\+ The resulting curves can be directly used as a design tool
\- Large (infinite) block lengths are assumed

Option 4: Conjecture of the Scaling Law [5]

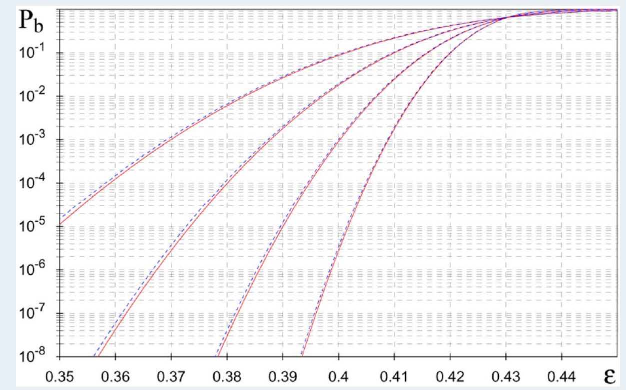
Scaling laws stem from statistical physics where a system follows a control parameter in a very specific way around a phase transition. Around the threshold there holds a scaling law for iteratively decoded LDPC codes:

$$P_B \cong Q\left(\frac{\sqrt{n}(\epsilon^* - \epsilon)}{\alpha}\right)$$

A refined scaling law is also conjectured:

$$P_B \cong Q\left(\frac{\sqrt{n}(\epsilon^* - \beta n^{-\frac{2}{3}} - \epsilon)}{\alpha}\right)$$

The parameters α and β depend on the code ensemble.



$(3, 6)$ regular ensemble, $n = 1024$, $n = 2048$, $n = 4096$ and $n = 8192$ [5].

How do we obtain the Parameters?

There exist closed form expressions for regular ensembles:

$$\alpha = \epsilon^* \sqrt{\frac{l-1}{l}\left(\frac{1}{x^*} - \frac{1}{y^*}\right)},$$

where x^*, y^* are erasure probabilities in the graph at the decoding threshold obtained from density evolution: $y^* = 1 - (1 - x^*)^{r-1}$

Unfortunately, no closed forms exist for our code constructions.

Task
How do we take into account the special protograph based S-LDPC structure?

References

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[3] M. Lentmaier, G. P. Fettweis, K. S. Zigangirov, and D. J. Costello, "Approaching Capacity with Asymptotically Regular LDPC Codes," *Proc. Inf. Theory and Applicat. Workshop (ITA)*, pp. 173—177, 2009. [Online].

[4] S. Kudekar and T. J. Richardson, "Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well over the BEC," *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, pp. 1–29, 2010. [Online].

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Alexander von Humboldt Stiftung / Foundation

Technische Universität München