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Titel: Weak stationarity of Ornstein-Uhlenbeck processes with stochastic speed of mean reversion

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Abstract. When modeling energy prices with the Ornstein-Uhlenbeck process, it was shown in Barlow, Gusev, and Lai [2] and Zapranis and Alexandris [16] that there is a large uncertainty attached to the estimation of the speed of mean-reversion and that it is not constant but may vary considerably over time. In this paper we generalise the Ornstein-Uhlenbeck process to allow for the speed of mean reversion to be stochastic. We suppose that the mean-reversion is a Brwonian stationary process. We apply Malliavin calculus in our computations and we show that this generalised Ornstein-Uhlenbeck process is stationary in the weak sense. Moreover we compute the instantaneous rate of change in the mean and in the squared fluctuations of the genaralised Ornstein-Uhlenbeck process given its initial position. Finally, we derive the chaos expansion of this generalised Ornstein-Uhlenbeck process.

1. Introduction

An Ornstein-Uhlenbeck (OU) process X is defined as the solution of the stochastic differential equation

$$dX(t) = -\alpha X(t)\, dt + \sigma\, dW(t), \tag{1.1}$$

where W is a Brownian motion and σ and α are two positive constants. Such processes have applications, for example in the areas of physics and finance. The process X has a drift which will push it towards its long-term mean level at the origin, while the Brownian component introduce random fluctuations. It is well-known that X has a Gaussian stationary distribution.

The parameter α is sometimes referred to as the speed of mean-reversion. In practical applications, it may be hard to identify precisely the speed of mean-reversion α. For instance, in modelling energy prices based on OU processes, Barlow, Gusev, and Lai [2] studied the problem of estimating parameters based on historical data. They found out that there is a large uncertainty attached to the estimation of the speed of mean-reversion. In their study of Paris daily temperature data, Zapranis and Alexandridis [16] showed by means of wavelet analysis that the mean reversion rate is not constant but may vary considerably over time.

In this paper we generalize the OU dynamics in (1.1) to allow for a speed of mean-reversion α being a stochastic process. We study the weak stationarity of this *generalized OU process*. In other words, we analyze the stationarity properties of the mean, the variance, and the covariance of the generalized OU process for simple specifications of α. Specifically, in our analysis we suppose α is a Brownian stationary process. Brownian

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stationary processes are themselves an extension of OU-processes, and have been intensively studied in the area of turbulence and finance (see Barndorff-Nielsen and Schmiegel [5], or Barndorff-Nielsen et al. [4] for the case of energy markets). Despite the fact that α may attain negative values, we are able to show the stationarity of the mean, the variance, and the covariance of X when the average speed of mean-reversion is sufficiently larger than its variance. Explicit conditions for these results to hold are derived. In our analysis, some aspects of Malliavin Calculus are applied.

In order to describe the behavior of the first and second moment of increments of the generalised Ornstein-Uhlenbeck process X, we derive the instantaneous rate of change in the mean of X given the initial position of the process. We show that the latter is given in terms of the mean of α. Moreover we compute the instantaneous rate of change in the squared fluctuations of X given its initial position and we show that this is given in terms of the volatility σ. Hence, locally our process behaves like a classical Ornstein-Uhlenbeck process.

We further compute the chaos expansion of the generalized OU process. We show that for a specific choice of the process α, the chaos of order 1 converges pointwise to a function in $L^2(\mathbb{R})$. However, it does not converge in $L^2(\mathbb{R})$.

A generalized Ornstein-Uhlenbeck (OU) process is sometimes in the literature defined by

$$V(t) = \int_{-\infty}^{t} \mathrm{e}^{-L(s-)}\, dU(s), \tag{1.2}$$

where $\{L, U\}$ is a bivariate Lévy process. See Carmona, Petit, and Yor [9] and Lindner and Sato [13] for basic properties of such processes. These processes have been applied in many areas, in particular in option pricing (see e.g. Yor [15]) or in insurance (see Dufresne [10]). The explicit dynamics V in (1.2) does not solve an OU-type stochastic differential equation and L is not immediately interpretable as a speed of mean-reversion. Another path of study is the so-called *quasi* Ornstein-Uhlenbeck processes, which are defined as processes X solving a stochastic differential equation of the type (1.1), however, with W being a general noise process with stationary increments (see Barndorff-Nielsen and Basse O'Connor [3]).

The paper is organized as follows. In Section 2, we describe the generalized Ornstein-Uhlenbeck process we consider in our analysis. In Section 3, we show that the mean and variance of this model class is stationary under some mild conditions on the model parameters. In Section 3 we compute the instantaneous rate of change of the first and the second moment of increments of the generalised Ornstein-Uhlenbeck process X. In Section 4 we derive the chaos expansion of the model.

2. Generalized Ornstein-Uhlenbeck processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\mathcal{F}_t$, for $t \geq 0$. We denote by $W = \{W(t)\}_{t\geq 0}$ a standard Brownian motion. We write D_t for the Malliavin derivative at time $t \geq 0$, defined on the subspace $\mathbb{D}^{1,2}$ of $L^2(P) := L^2(\Omega, \mathcal{F}, P)$. A stochastic process Y is said to be Skorohod integrable on $[0,T]$, for some $T<\infty$, if

$$\left|\mathbb{E}\left[\int_0^T (D_t Z)Y(t)\,dt\right]\right| \le c\|Z\|_2\,,$$

for all $Z\in\mathbb{D}^{1,2}$, where c is a constant depending on Y and $\|\cdot\|_2$ is the $L^2(P)$-norm. We denote the Skorohod integral of Y over $[0,T]$ by

$$\int_0^T Y(t)\,\delta W(t)\,.$$

If Y is $\mathcal{F}_t$-adapted, the Skorohod integral of Y coincides with the Itô integral, that is

$$\int_0^T Y(t)\,\delta W(t) = \int_0^T Y(t)\,dW(t)\,.$$

For a thorough introduction to Malliavin Calculus, we refer the reader to Nualart [14].

Introduce the *generalized Ornstein-Uhlenbeck process* (GOU) X as the solution to the stochastic differential equation

$$dX(t) = -\alpha(t)X(t)\,dt + \sigma\,dW(t)\,, \tag{2.1}$$

where σ is a positive constant and α is an $\mathcal{F}_t$-adapted stochastic process. Notice that α is *not* restricted to be positive, but may attain negative values as well. The intial condition $X(0)=X_0$ is assumed to be a constant. In the next Proposition we derive the explicit solution to (2.1).

Proposition 2.1. *Assume that α is integrable on $[0,T]$ for a given $T<\infty$ and*

$$\mathbb{E}\left[\exp\left(2\int_0^T \alpha(u)\,du\right)\right] < \infty\,.$$

Then, for $t\le T$, the $\mathcal{F}_t$-adapted process

$$X(t) = X_0\exp\left(-\int_0^t \alpha(u)\,du\right) + \exp\left(-\int_0^t \alpha(u)\,du\right)\int_0^t \sigma\exp\left(\int_0^s \alpha(u)\,du\right)dW(s)\,,$$

is a solution to (2.1).

Proof. We apply the Itô Formula to obtain

$$\begin{aligned} d\left(X(t)\exp\left(\int_0^t \alpha(u)\,du\right)\right) &= \alpha(t)X(t)\exp\left(\int_0^t \alpha(u)\,du\right)dt + \exp\left(\int_0^t \alpha(u)\,du\right)dX(t) \\ &= \sigma\exp\left(\int_0^t \alpha(u)\,du\right)dW(t)\,. \end{aligned}$$

Integrating yields the representation of X. Note that the stochastic integral is well-defined in Itô sense due to the $\mathcal{F}_t$-adaptedness of α and the integrability condition. ∎

Let the explicit dynamics X in Prop. 2.1 be our GOU process. Notice that in the case α is deterministic, it is usual to write

$$X(t) = X_0 \exp\left(-\int_0^t \alpha(u)\,du\right) + \int_0^t \sigma \exp\left(-\int_s^t \alpha(u)\,du\right)\,dW(s)\,.$$

That is, moving the exponential inside the Itô integral. In the general case, where α is stochastic, this is no longer a valid representation as $\exp(\int_0^t \alpha(u)\,du)$ is anticipating. However, the next Proposition shows that we can move the exponential inside the integral when we interpret the stochastic integral in the sense of Skorohod. Moreover, we get an additional drift term involving the Malliavin derivative of the exponential.

Proposition 2.2. *Assume for every* $0 \leq t \leq T$*, that* $\exp(-\int_0^t \alpha(u)\,du) \in \mathbb{D}^{1,2}$, $\alpha(t) \in \mathbb{D}^{1,2}$, $\exp\left(-\int_s^t \alpha(u)du\right)$ *is Skorohod integrable, and*

$$\mathbb{E}\left[\int_0^t \exp\left(-2\int_s^t \alpha(u)\,du\right)\,ds\right] < \infty\,. \tag{2.2}$$

Then, for every $0 \leq t \leq T$ *we have a.s.*

$$X(t) = X_0 \exp\left(-\int_0^t \alpha(u)\,du\right) + \int_0^t \sigma \exp\left(-\int_s^t \alpha(u)\right)\,\delta W(s) - \int_0^t \int_s^t \sigma(D_s\alpha(v)) \exp\left(-\int_s^t \alpha(u)\,du\right)\,dv\,ds\,.$$

Proof. Applying the integration by parts formula (1.49) in Nualart [14], we find from the assumptions in the Proposition that

$$\begin{aligned}
&\exp\left(-\int_0^t \alpha(u)\,du\right)\int_0^t \exp\left(\int_0^s \alpha(u)\,du\right)\,dW(s) \\
&\quad = \int_0^t \exp\left(-\int_s^t \alpha(u)\,du\right)\,\delta W(s) + \int_0^t \left(D_s \exp\left(-\int_0^t \alpha(u)\,du\right)\right)\exp\left(\int_0^s \alpha(u)\,du\right)\,ds\,.
\end{aligned}$$

By the chain rule of the Malliavin derivative, we find

$$\begin{aligned}
D_s \exp\left(-\int_0^t \alpha(u)\,du\right) &= -\left(D_s \int_0^t \alpha(v)\,dv\right)\exp\left(-\int_0^t \alpha(u)\,du\right) \\
&= -\int_0^t D_s\alpha(v)\,dv \exp\left(-\int_0^t \alpha(u)\,du\right)\,.
\end{aligned}$$

Since $\alpha(v)$ is $\mathcal{F}_v$-measurable, we find that $D_s\alpha(v) = 0$ *a.s.* for $s > v$. Thus, *a.s.*,

$$\int_0^t D_s\alpha(v)\,dv = \int_s^t D_s\alpha(v)\,dv\,.$$

This proves the Proposition. $\square$

As the Skorohod integral has mean zero, we find from the Fubini-Tonelli theorem that (2.3)

$$\mathbb{E}\left[X(t)\right] = X_0\mathbb{E}\left[\exp\left(-\int_0^t \alpha(u)\,du\right)\right] - \sigma\int_0^t\int_s^t \mathbb{E}\left[(D_s\alpha(v))\exp\left(-\int_s^t \alpha(u)\,du\right)\right]\,dv\,ds\,.$$

The second moment can be expressed by using the "isometry" for Skorohod integrals and integration by parts. This is the purpose of the next Proposition.

Proposition 2.3. *Assume that* $\alpha \in \mathbb{D}^{1,2}$ *and that for every* $0 \le t \le T$, $D_w\alpha(v)\mathrm{e}^{-\int_w^t \alpha(u)\,du} \in \mathbb{D}^{1,2}$, $\mathrm{e}^{-\int_0^t \alpha(u)du} \in \mathbb{D}^{1,2}$, $\mathrm{e}^{-\int_s^t \alpha(u)du}$ *is Skorohod integrable, and* $\mathrm{e}^{-\int_0^t \alpha(u)du}\mathrm{e}^{-\int_s^t \alpha(u)du} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. *Then,*

$$\begin{aligned}
\mathbb{E}\left[X^2(t)\right] &= X_0^2\mathbb{E}\left[\mathrm{e}^{-2\int_0^t \alpha(u)\,du}\right] + \sigma^2\int_0^t \mathbb{E}\left[\mathrm{e}^{-2\int_s^t \alpha(u)\,du}\right]\,ds \\
&\quad + \sigma^2\int_0^t\int_0^t \mathbb{E}\left[\int_s^t D_v\alpha(w)\,dw\mathrm{e}^{-\int_s^t \alpha(u)\,du}\int_v^t D_s\alpha(w)\,dw\mathrm{e}^{-\int_v^t \alpha(u)\,du}\right]\,dv\,ds \\
&\quad + \sigma^2\mathbb{E}\left[\left(\int_0^t\int_s^t (D_s\alpha(v))\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,dv\,ds\right)^2\right] \\
&\quad - 4X_0\sigma\int_0^t\int_s^t \mathbb{E}\left[(D_s\alpha(v))\mathrm{e}^{-\int_0^t \alpha(u)\,du - \int_s^t \alpha(u)\,du}\right]\,dv\,ds \\
&\quad - 2\sigma^2\int_0^t\int_0^t\int_w^t \mathbb{E}\left[D_s\left((D_w\alpha(v))\mathrm{e}^{-\int_w^t \alpha(u)\,du}\right)\,dv\,dw\mathrm{e}^{-\int_s^t \alpha(u)\,du}\right]\,ds\,.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
X^2(t) &= X_0^2\mathrm{e}^{-2\int_0^t \alpha(u)\,du} + \left(\int_0^t \sigma\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s)\right)^2 + \left(\int_0^t\int_s^t \sigma(D_s\alpha(v))\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,dv\,ds\right)^2 \\
&\quad + 2X_0\mathrm{e}^{-\int_0^t \alpha(u)\,du}\int_0^t \sigma\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s) \\
&\quad - 2X_0\mathrm{e}^{-\int_0^t \alpha(u)\,du}\int_0^t\int_s^t \sigma(D_s\alpha(v))\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,dv\,ds \\
&\quad - 2\int_0^t\int_s^t \sigma(D_s\alpha(v))\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,dv\,ds\int_0^t \sigma\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s)\,.
\end{aligned}$$

Now, the integration by parts in Eq. (1.48) of Nualart [14] and under the assumptions mentioned in the Proposition, we get

$$\begin{aligned}
\mathrm{e}^{-\int_0^t \alpha(u)\,du}\int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s) &= \int_0^t \mathrm{e}^{-\int_0^t \alpha(u)\,du - \int_s^t \alpha(u)\,du}\,\delta W(s) \\
&\quad + \int_0^t (D_s\mathrm{e}^{-\int_0^t \alpha(u)\,du})\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds \\
&= \int_0^t \mathrm{e}^{-\int_0^t \alpha(u)\,du - \int_s^t \alpha(u)\,du}\,\delta W(s)
\end{aligned}$$

$$- \int_0^t \int_s^t (D_s\alpha(v))\mathrm{e}^{-\int_0^t \alpha(u)\,du - \int_s^t \alpha(u)\,du}\, dv\, ds\,.$$

Taking expectations, gives

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^t \alpha(u)\,du} \int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\, \delta W(s)\right] = - \int_0^t \int_s^t \mathbb{E}\left[(D_s\alpha(v))\mathrm{e}^{-\int_0^t \alpha(u)\,du - \int_s^t \alpha(u)\,du}\right] dv\, ds\,.$$

A similar argument shows that

$$\begin{aligned}\mathbb{E}&\left[\int_0^t \int_w^t (D_w\alpha(v))\mathrm{e}^{-\int_w^t \alpha(u)\,du}\, dv\, dw \int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\, \delta W(s)\right] \\ &= \int_0^t \int_0^t \int_w^t \mathbb{E}\left[D_s\left((D_w\alpha(v))\mathrm{e}^{-\int_w^t \alpha(u)\,du}\right) dv\, dw \mathrm{e}^{-\int_s^t \alpha(u)\,du}\right] ds\,.\end{aligned}$$

Applying the covariance formula for Skorohod integrals in Eq. (1.48) of Nualart [14], it holds that

$$\begin{aligned}\mathbb{E}&\left[\left(\int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\, \delta W(s)\right)^2\right] \\ &= \int_0^t \mathbb{E}\left[\mathrm{e}^{-2\int_s^t \alpha(u)\,du}\right] ds \\ &\quad + \int_0^t \int_0^t \mathbb{E}\left(D_v\mathrm{e}^{-\int_s^t \alpha(u)\,du})(D_s\mathrm{e}^{-\int_v^t \alpha(u)\,du})\right] dv\, ds \\ &= \int_0^t \mathbb{E}\left[\mathrm{e}^{-2\int_s^t \alpha(u)\,du}\right] ds \\ &\quad + \int_0^t \int_0^t \mathbb{E}\left[\int_s^t D_v\alpha(w)\, dw \mathrm{e}^{-\int_s^t \alpha(u)\,du} \int_v^t D_s\alpha(w)\, dw \mathrm{e}^{-\int_v^t \alpha(u)\,du}\right] dv\, ds\,.\end{aligned}$$

After collecting terms the statement is proved. □

In the next section we specify α to be a Brownian stationary process and analyse the weak stationarity of X.

3. Weak stationarity

First, we recall the definition of weak stationarity (see for example Kloeden and Platen [12]).

Definition 3.1. *Let X be a stochastic process, μ be a constant, and $c : \mathbb{R} \to \mathbb{R}$ be a function. Then X is called weakly stationary if its mean, its variance, and its covariance satisfy*

$$\mathbb{E}[X(t)] = \mu, \quad Var[X(t)] = c(0), \quad Cov[X(t), X(t-\delta)] = c(\delta),$$

for all $t \in [0,T]$ and $\delta > 0$.

As we see from the definition, weak stationary processes are defined by the property that their first and second moments are not affected by a shift of the time variable. The purpose of this section is to prove the weak stationarity of the GOU process, when t goes to ∞. Let $g : \mathbb{R}_+ \mapsto \mathbb{R}$ be a measurable function such that

$$\int_0^\infty g^2(u)\, du < \infty\,. \tag{3.1}$$

For a constant $\mu \in \mathbb{R}$, define α to be the Brownian stationary (BS) process

$$\alpha(t) = \mu + \int_{-\infty}^t g(t-s)\, dW(s)\,. \tag{3.2}$$

By the condition of g, we immediately see by the Itô isometry and properties of Wiener integration that $\alpha(t)$ is a Gaussian $\mathcal{F}_t$-adapted process with mean μ and variance $\int_0^\infty g^2(u)\, du$ independent of time t, thus being stationary. A typical example is to consider $g(u) = \eta \exp(-\beta u)$, for positive constants η and β. Thus α is the stationary solution of the OU process solving the stochastic differential equation

$$d\alpha(t) = \beta(\mu - \alpha(t))\, dt + \eta\, dW(t)\,. \tag{3.3}$$

In this specific case, the speed of mean reversion α is itself an OU process. If W has a positive increment, then so does α, and thus increasing the speed of mean reversion of X. The process α will decrease exponentially fast back at a rate β towards its long-term mean level μ, being the average speed of mean reversion for X. A negative increment of W will push α downwards, yielding a slower mean reversion of X. Thus, positive increments of W implies faster mean-reversion, whereas negative increments means slower mean reversion. We can turn this relationship around by supposing η to be negative, meaning that α depends on W opposite to X. Note that α can become negative, as the Wiener integral in (3.2) is normally distributed and therefore takes values on the whole real line. However, for relatively large and positive values of μ, the probability of negative speeds of mean reversion will become small. As we shall see below, the mean, the variance, and the covariance of X are stationary when a specific relationship in the size between μ and g holds.

Observe that for any $0 \le s < t < \infty$, Tonelli's Theorem and Cauchy-Schwarz's inequality along with the Itô isometry yield

$$\begin{aligned}\mathbb{E}\left[\int_s^t \left|\int_{-\infty}^u g(u-v)\, dW(v)\right| du\right] &\le \int_s^t \mathbb{E}\left[\left(\int_{-\infty}^u g(u-v)\, dW(v)\right)^2\right]^{1/2} du \\ &= \sqrt{\int_0^\infty g^2(u) du} \times (t-s) < \infty\,.\end{aligned}$$

Thus, $\alpha(u)$ is integrable on any interval $[s,t]$, $0 \le s < t < \infty$. Remark that the process α is in general not a semimartingale. In fact, by a suitable choice of g (see *e.g.* Alos et al. [1]) we can allow for α to be a fractional Brownian motion. It is possible to show that α is a semimartingale when $g(0)$ is well-defined and g is absolutely continuous and has a square integrable derivative defined almost everywhere (see Basse and Pedersen [6] and Benth and Ejyolfsson [8]). For example, the interesting case of continuous-time autoregressive moving average processes satisfy these properties (see Benth and Šaltytė Benth [7] for an application of these processes to weather modelling).

Introduce the function $h(x,y)$ for $0 \leq x \leq y$ by

$$h(x,y) = \int_x^y g(u)\,du. \tag{3.4}$$

We have the following useful Lemma.

Lemma 3.2. *Assume that $\int_0^\infty h^2(u,x+u)\,du < \infty$ for any $x > 0$. Then for every $0 \leq s \leq t$*

$$\int_s^t \alpha(u)\,du = \mu(t-s) + \int_{-\infty}^s h(s-u,t-u)\,dW(u) + \int_s^t h(0,t-u)\,dW(u)\,. \tag{3.5}$$

Proof. First, observe that the condition on the function h ensures that the first Wiener integral on the right-hand side of (3.5) is well-defined since,

$$\int_{-\infty}^s h^2(s-u,t-u)\,du = \int_0^\infty h^2(u,u+t-s)\,du\,.$$

After appealing to the Cauchy-Schwartz inequality we find,

$$\int_s^t h^2(0,t-u)\,du = \int_0^{t-s} h^2(0,u)\,du \leq \int_0^\infty g^2(u)\,du \int_0^{t-s} u\,du < \infty\,,$$

and therefore the second Wiener integral is also well-defined.

By definition of α, we find

$$\int_s^t \alpha(u)\,du = \mu(t-s) + \int_s^t \int_{-\infty}^u g(u-v)\,dW(v)\,du\,.$$

But

$$\int_{-\infty}^u g(u-v)\,dW(v) = \int_{-\infty}^s g(u-v)\,dW(v) + \int_s^u g(u-v)\,dW(v)\,.$$

By the stochastic Fubini Theorem, we find

$$\int_s^t \int_s^u g(u-v)\,dW(v)\,du = \int_s^t \int_v^t g(u-v)\,du\,dW(v) = \int_s^t \int_0^{t-v} g(u)\,du\,dW(v)\,,$$

and hence,

$$\int_s^t \int_s^u g(u-v)\,dW(v)\,du = \int_s^t h(0,t-v)\,dW(v)\,.$$

By the assumption on h, we again apply the stochastic Fubini theorem to find

$$\int_s^t \int_{-\infty}^s g(u-v)\,dW(v)\,du = \int_{-\infty}^s \int_s^t g(u-v)\,du\,dW(v) = \int_{-\infty}^s \int_{s-v}^{t-v} g(u)\,du\,dW(v)\,.$$

Hence, the Lemma follows after using the definition of h in the last integral. $\square$

We observe that $\int_s^t \alpha(u)\,du$ is represented as a sum of two independent Wiener integrals. This will enable us to compute exponential moments of $\int_s^t \alpha(u)\,du$ easily. Remark that from the Cauchy-Schwartz inequality

$$\left(\int_u^{u+t-s} g(v)\,dv\right)^2 = \left(\int_u^{u+t-s} 1\cdot g(v)\,dv\right)^2 \leq x\int_u^{u+x} g^2(v)\,dv\,,$$

and therefore a sufficient condition for $\int_0^\infty h^2(u,u+x)\,du$ to be finite is

$$\int_0^\infty \int_u^{u+x} g^2(v)\,dv\,du < \infty\,. \tag{3.6}$$

As an example, consider α to be an OU process, with $g(v) = \exp(-\beta v)$, for a constant $\beta > 0$. Then

$$\int_u^{u+x} g^2(v)\,dv = \frac{1}{2\beta}\left(1 - \mathrm{e}^{-2\beta x}\right)\mathrm{e}^{-2\beta u}\,,$$

and hence (3.6) holds.

The Malliavin derivative of $\alpha(u)$ is simple to compute. It holds that

$$D_s\alpha(v) = g(v-s)\,, \tag{3.7}$$

for $s < v$. In the case $s > v$, $D_s\alpha(v) = 0$. There is a potential problem at $v = s$ since g might not be defined there. However, as we are going to integrate expressions like $D_s\alpha(v)$ with respect to the Lebesgue measure, we leave the Malliavin derivative undefined for this singular point. If $g(0)$ is well-defined, there is no problem.

Using (3.7), we find

$$\begin{aligned}\int_0^t\int_s^t D_s\alpha(v)\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,dv\,ds &= \int_0^t\int_s^t g(v-s)\,dv\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds\\ &= \int_0^t\int_0^{t-s} g(v)\,dv\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds\\ &= \int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds\,.\end{aligned}$$

We conclude from Prop. 2.2 that

$$X(t) = X_0\mathrm{e}^{-\int_0^t \alpha(u)\,du} + \int_0^t \sigma\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s) - \int_0^t \sigma h(0,t-s)\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds\,, \tag{3.8}$$

where $\int_s^t \alpha(u)\,du$ is expressed in Lemma 3.2. Note that the exponential integrability (2.2) in Prop. 2.2 which is a condition on $\int_s^t \alpha(u)\,du$ is satisfied as this is a normal random variable which has finite exponential moments of all orders. The conditions of Malliavin differentiability and Skorohod integrability are also readily verified in this explicit case. Recalling (3.5), it follows from Theorem 2.2.1 in Nualart [14] that $\int_s^t \alpha(u)\,du$ is Malliavin differentiable. By the chain rule, so is $\exp(-\int_s^t \alpha(u)\,du)$. We know from Proposition 1.3.1 of Nualart [14] that the space of Malliavin differentiable random variables is included in the domain of the Skorohod integral. This ensures the Skorohod integrability of $\exp(-\int_s^t \alpha(u)\,du)$.

In the next subsection we study the stationarity of the mean of the GOU process.

3.1. **Stationarity of the mean.** We compute the expectation of X and show that it has a limit when t goes to ∞.

Proposition 3.3. *The expected value of X is*

$$\mathbb{E}[X(t)] = X_0 \exp\left(-\mu t + \frac{1}{2}\int_0^\infty h^2(u, t+u)\,du + \frac{1}{2}\int_0^t h^2(0,u)\,du\right) - \int_0^t \sigma h(0,v) \exp\left(-\mu v + \frac{1}{2}\int_0^\infty h^2(u, v+u)\,du + \frac{1}{2}\int_0^v h^2(0,u)\,du\right) dv\,.$$

If,

$$\lim_{t\to\infty} \exp\left(-\mu t + \frac{1}{2}\int_0^\infty h^2(u, t+u)\,du + \frac{1}{2}\int_0^t h^2(0,u)\,du\right) = 0\,, \tag{3.9}$$

and

$$\int_0^\infty \sigma|h(0,v)| \exp\left(-\mu v + \frac{1}{2}\int_0^\infty h^2(u, v+u)\,du + \frac{1}{2}\int_0^v h^2(0,u)\,du\right) dv < \infty\,, \tag{3.10}$$

then $\mathbb{E}[X(t)]$ has a limit when $t \to \infty$ given by

$$\lim_{t\to\infty} \mathbb{E}[X(t)] = -\int_0^\infty \sigma h(0,v) \exp\left(-\mu v + \frac{1}{2}\int_0^\infty h^2(u, v+u)\,du + \frac{1}{2}\int_0^v h^2(0,u)\,du\right) dv\,.$$

Proof. As the expectation of the Skorohod integral is zero, we have by Fubini-Tonelli's theorem

$$\mathbb{E}[X(t)] = X_0 \mathbb{E}\left[e^{-\int_0^t \alpha(u)\,du}\right] - \int_0^t \sigma h(0, t-s) \mathbb{E}\left[e^{-\int_s^t \alpha(u)\,du}\right] ds\,.$$

From Lemma 3.2, we find by independence of the Wiener integrals, that

$$\begin{aligned}\mathbb{E}\left[e^{-\int_0^t \alpha(u)\,du}\right] &= e^{-\mu t}\mathbb{E}\left[e^{-\int_{-\infty}^0 h(-u,t-u)\,dW(u)}\right]\mathbb{E}\left[e^{-\int_0^t h(0,t-u)\,dW(u)}\right] \\ &= e^{-\mu t}\exp\left(\frac{1}{2}\int_{-\infty}^0 h^2(-u, t-u)\,du + \frac{1}{2}\int_0^t h^2(0, t-u)\,du\right) \\ &= e^{-\mu t}\exp\left(\frac{1}{2}\int_0^\infty h^2(u, t+u)\,du + \frac{1}{2}\int_0^t h^2(0,u)\,du\right)\,.\end{aligned}$$

Similarly, we find

$$\mathbb{E}\left[e^{-\int_s^t \alpha(u)\,du}\right] = e^{-\mu(t-s)}\exp\left(\frac{1}{2}\int_0^\infty h^2(u, t-s+u)\,du + \frac{1}{2}\int_0^{t-s} h^2(0,u)\,du\right)\,. \tag{3.11}$$

Hence, the expression for $\mathbb{E}[X(t)]$ follows. It is simple to see that the integrability conditions imply the limit as claimed. □

It is easily seen from the conditions in Prop. 3.3 that $\mu > 0$ is a necessary condition in order to have a limiting expectation. Notice that the limiting expectation of X may become negative. For example, if g in the definition of the process α is positive, then trivially $h(0,v)$ is positive and we have a negative limiting expectation of X. This is in sharp contrast to the classical OU process with a constant (and positive) speed of mean reversion α, as this has zero expectation in stationarity. We attribute the negative expected limiting value of X to the probability (however small) that α itself can be negative. However, as μ is positive, the mean of α will be positive as well. A negative α gives a non-stationary behavior, which will locally occur for X during times when α crosses zero from above

Let us consider the case where α is an OU process. In the following lemma we compute the stationary mean of the GOU process for this specific choice of α.

Lemma 3.4. *Let α be an OU process as in* (3.3)*. If*

$$\mu\beta > \frac{\eta^2}{2\beta}\,, \tag{3.12}$$

then

$$\lim_{t\to\infty}\mathbb{E}[X(t)] = -\int_0^\infty \frac{\sigma\eta}{\beta}(1-\mathrm{e}^{-\beta v})\mathrm{e}^{-\left(\mu-\frac{\eta^2}{2\beta^2}\right)v-\frac{\eta^2}{\beta^3}\left(1-\mathrm{e}^{-\beta v}\right)+\frac{\eta^2}{4\beta^3}\left(1-\mathrm{e}^{-2\beta v}\right)+\frac{\eta^2}{4\beta^3}\left(1-\mathrm{e}^{-\beta v}\right)^2}\,dv\,.$$

Proof. Let α be an OU process, with $g(u) = \eta\exp(-\beta u)$, for constants η and β with $\beta > 0$. Then

$$h(x,y) = \int_x^y \eta\mathrm{e}^{-\beta u}\,du = \frac{\eta}{\beta}\left(\mathrm{e}^{-\beta x}-\mathrm{e}^{-\beta y}\right)\,.$$

Hence,

$$\int_0^\infty h^2(u,t+u)\,du = \frac{\eta^2}{\beta^2}\int_0^\infty\left(\mathrm{e}^{-\beta u}-\mathrm{e}^{-\beta(u+t)}\right)^2\,du = \frac{\eta^2}{2\beta^3}\left(1-\mathrm{e}^{-\beta t}\right)^2\,.$$

Next,

$$\int_0^t h^2(0,u)\,du = \frac{\eta^2}{\beta^2}\int_0^t\left(1-\mathrm{e}^{-\beta u}\right)^2\,du = \frac{\eta^2}{\beta^2}t-\frac{2\eta^2}{\beta^3}\left(1-\mathrm{e}^{-\beta t}\right)+\frac{\eta^2}{2\beta^3}\left(1-\mathrm{e}^{-2\beta t}\right)\,.$$

Therefore, it holds that

$$\lim_{t\to\infty}\exp\left(-\mu t+\frac{1}{2}\int_0^\infty h^2(u,t+u)\,du+\frac{1}{2}\int_0^t h^2(0,u)\,du\right) = 0\,,$$

if and only if (3.12) holds true. Moreover,

$$\begin{aligned}-\mu v+\frac{1}{2}\int_0^\infty h^2(u,v+u)\,du+\frac{1}{2}\int_0^v h^2(0,u)\,du \\ = -\left(\mu-\frac{\eta^2}{2\beta^2}\right)v-\frac{\eta^2}{\beta^3}\left(1-\mathrm{e}^{-\beta v}\right)+\frac{\eta^2}{4\beta^3}\left(1-\mathrm{e}^{-2\beta v}\right)+\frac{\eta^2}{4\beta^3}\left(1-\mathrm{e}^{-\beta v}\right)^2\,.\end{aligned}$$

This implies,

$$\int_0^\infty h(0,v)\exp\left(-\mu v+\frac{1}{2}\int_0^\infty h^2(u,v+u)\,du+\frac{1}{2}\int_0^v h^2(0,u)\,du\right)\,dv$$
$$\le c\int_0^\infty \left(1-\mathrm{e}^{-\beta v}\right)\exp\left(-\left(\mu-\frac{\eta^2}{2\beta^2}\right)v\right)\,dv\,,$$

for some constant $c>0$. But under condition (3.12) the integral is finite and we prove the statement. ☐

Note that by the definition of g, the stationary variance of α is $\eta^2/2\beta$. Hence for X to have a stationary mean value, the stationary mean of α times its speed of mean reversion must be larger than the stationary variance of α. In the case $\eta>0$, this stationary mean value becomes negative, whereas $\eta<0$ gives a positive stationary mean since g and therefore $h(0,v)$ are negative. If $\eta<0$ an increase in X due to a positive increment of W occurs in parallel to a decrease in α. This would mean that X is pushed away from its mean and reverts slower giving the rational for a positive stationary mean of the process X.

3.2. **Stationarity of the variance.** In this subsection we analyze the second moment of X and its limiting behavior. To reduce the number of terms, we suppose that $X_0=0$, which gives

$$X(t)=\int_0^t \sigma\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)-\int_0^t\sigma h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\,.$$

We find $\mathbb{E}[X^2(t)]=I_1-2I_2+I_3$ where

$$I_1=\mathbb{E}\left[\left(\int_0^t \sigma\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)\right)^2\right]\,, \tag{3.13}$$

$$I_2=\mathbb{E}\left[\int_0^t \sigma^2\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\right]\,, \tag{3.14}$$

$$I_3=\mathbb{E}\left[\left(\int_0^t \sigma h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\right)^2\right]\,. \tag{3.15}$$

We want to compute the three expectations (3.13)-(3.15). The approach is based on the same ideas as when we calculated the expectation of X.

A key element in our derivations of the variance is the expectation of terms like $\exp(-\int_s^t\alpha(u)\,du-\int_v^t\alpha(u)\,du)$ for $s,v\in[0,t]$. This is the content of the next Lemma.

Lemma 3.5. *It holds for $v\le s\le t$,*

$$\ln\mathbb{E}\left[\exp\left(-\int_s^t\alpha(u)\,du-\int_v^t\alpha(u)\,du\right)\right]=-2\mu(t-s)-\mu(s-v)+H(s-v,t-s)\,, \tag{3.16}$$

where

$$H(x,y) = \frac{1}{2}\int_0^\infty \left(h(x+u, x+y+u) + h(u, x+y+u)\right)^2 \, du$$
$$+ \frac{1}{2}\int_0^x \left(h(u, y+u) + h(0, y+u)\right)^2 \, du + 2\int_0^y h^2(0,u)\, du\,,$$

for $x, y \geq 0$.

Proof. As

$$-\int_s^t \alpha(u)\, du = -\mu(t-s) - \int_{-\infty}^s h(s-u, t-u)\, dW(u) - \int_s^t h(0, t-u)\, dW(u)\,,$$

we have for $v \leq s$

$$\begin{aligned}
-\int_s^t \alpha(u)\, du - \int_v^t \alpha(u)\, du \\
&= -\mu(t-s) - \mu(t-v) \\
&\quad - \left(\int_{-\infty}^s h(s-u, t-u)\, dW(u) + \int_{-\infty}^v h(v-u, t-u)\, dW(u)\right) \\
&\quad - \left(\int_s^t h(0, t-u)\, dW(u) + \int_v^t h(0, t-u)\, dW(u)\right) \\
&= -2\mu(t-s) - \mu(s-v) \\
&\quad - \left(\int_{-\infty}^v h(s-u, t-u) + h(v-u, t-u)\, dW(u) + \int_v^s h(s-u, t-u)\, dW(u)\right) \\
&\quad - \left(2\int_s^t h(0, t-u)\, dW(u) + \int_v^s h(0, t-u)\, dW(u)\right) \\
&= -2\mu(t-s) - \mu(s-v) \\
&\quad - \int_{-\infty}^v h(s-u, t-u) + h(v-u, t-u)\, dW(u) \\
&\quad - \int_v^s h(s-u, t-u) + h(0, t-u)\, dW(u) - 2\int_s^t h(0, t-u)\, dW(u)\,.
\end{aligned}$$

The three Wiener integrals will be independent by the properties of Brownian motion. Hence, the results follows after using the fact that the exponential of a Wiener integral is lognormally distributed. □

We remark that the case $v > s$ is covered by the above result by simply interchanging the roles of v and s.

Let us now compute the three expectations (3.13)-(3.15). We start with the expectation in (3.13), which is computed in the following Lemma.

Lemma 3.6. *It holds that*

$$\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s)\right)^2\right] = \int_0^t \mathrm{e}^{-2\mu u + H(0,u)}\,du + \int_0^t \int_0^{t-v} h(0,v)h(u,u+v)\mathrm{e}^{-2\mu v - \mu u + H(u,v)}\,du\,dv + \int_0^t \int_0^v h(u,v)h(0,v-u)\mathrm{e}^{-2\mu(v-u)-\mu u + H(u,v-u)}\,du\,dv\,.$$

Proof. By the formula for the variance of a Skorohod integral (Proposition 1.3.1 in Nualart [14]), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s)\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\int_s^t \alpha(u)\,du}\,ds\right] + \mathbb{E}\left[\int_0^t \int_0^t \left(D_v \mathrm{e}^{-\int_s^t \alpha(u)\,du}\right)\left(D_w \mathrm{e}^{-\int_v^t \alpha(u)\,du}\right)dv\,dw\right] \\
&= \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\int_s^t \alpha(u)\,du}\,ds\right] \\
&\quad + \int_0^t \int_0^t \mathbb{E}\left[\mathrm{e}^{-\int_s^t \alpha(u)\,du - \int_v^t \alpha(u)\,du}\right]\left(\int_s^t D_v\alpha(u)\,du\right)\left(\int_v^t D_w\alpha(u)\,du\right)dv\,dw\,.
\end{aligned}$$

Here we have applied the chain rule for Malliavin differentiation together with the Fubini Theorem. Since $D_v\alpha(u) = g(u-v)\mathbf{1}(u>v)$, it holds

$$\begin{aligned}
\int_s^t D_v\alpha(u)\,du &= \int_s^t g(u-v)\mathbf{1}(u>v)\,du = \int_{\max(v,s)}^t g(u-v)\,du \\
&= \int_{\max(v,s)-v}^{t-v} g(w)\,dw = h(\max(v,s)-v, t-v)\,. \qquad (3.17)
\end{aligned}$$

Similarly,

$$\int_v^t D_w\alpha(u)\,du = h(\max(v,w)-w, t-w)\,.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{-\int_s^t \alpha(u)\,du}\,\delta W(s)\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\int_s^t \alpha(u)\,du}\,ds\right] \\
&\quad + \int_0^t \int_0^t \mathbb{E}\left[\mathrm{e}^{-\int_s^t \alpha(u)\,du - \int_v^t \alpha(u)\,du}\right] h(\max(s,v)-s, t-s)h(\max(s,v)-v, t-v)\,dv\,ds \\
&= \int_0^t \mathrm{e}^{-2\mu(t-s)+H(0,t-s)}\,ds
\end{aligned}$$

$$+\int_0^t\int_0^s h(0,t-s)h(s-v,t-v)\mathrm{e}^{-2\mu(t-s)-\mu(s-v)+H(s-v,t-s)}\,dv\,ds$$
$$+\int_0^t\int_s^t h(v-s,t-s)h(0,t-v)\mathrm{e}^{-2\mu(t-v)-\mu(v-s)+H(v-s,t-v)}\,dv\,ds\,,$$

where in the latter we applied Lemma 3.5. Hence, the statement of the Lemma follows after a change of variables in the integrals. □

We derive the second expectation (3.14) in the next Lemma.

Lemma 3.7. *It holds that*

$$\mathbb{E}\left[\int_0^t \mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\right]=$$
$$-\int_0^t\int_0^{t-v} h(0,u+v)h(0,v)\mathrm{e}^{-2\mu v-\mu u+H(u,v)}\,du\,dv$$
$$-\int_0^t\int_0^{v} h(0,v-u)h(u,v)\mathrm{e}^{-2\mu(v-u)-\mu u+H(u,v-u)}\,du\,dv\,.$$

Proof. Using integration by parts for Skorohod integrals together with the Fubini Theorem, we find

$$\mathbb{E}\left[\int_0^t \mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\right]$$
$$=\int_0^t h(0,t-v)\mathbb{E}\left[\mathrm{e}^{-\int_v^t\alpha(u)\,du}\int_0^t \mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)\right]dv$$
$$=\int_0^t h(0,t-v)\mathbb{E}\left[\int_0^t \mathrm{e}^{-\int_s^t\alpha(u)\,du-\int_v^t\alpha(u)\,du}\,\delta W(s)\right]dv$$
$$+\int_0^t h(0,t-v)\mathbb{E}\left[\int_0^t\left(D_s\mathrm{e}^{-\int_v^t\alpha(u)\,du}\right)\mathrm{e}^{-\int_s^t\alpha(u)\,du}ds\right]dv\,.$$

The Skorohod integral has zero expectation. Hence,

$$\mathbb{E}\left[\int_0^t \mathrm{e}^{-\int_s^t\alpha(u)\,du}\,\delta W(s)\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\right]$$
$$=-\int_0^t\int_0^t h(0,t-v)h(\max(v,s)-s,t-s)\mathbb{E}\left[\mathrm{e}^{-\int_v^t\alpha(u)\,du-\int_s^t\alpha(u)\,du}\right]ds\,dv\,,$$

after using the chain rule for the Malliavin derivative and the fact that $D_s\alpha(u)=g(u-s)\mathbf{1}(u>s)$. Invoking Lemma 3.5 proves the result. □

Finally, we derive the expectation in (3.15).

Lemma 3.8. *It holds*

$$\mathbb{E}\left[\left(\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t\alpha(u)\,du}\,ds\right)^2\right]$$

$$= \int_0^t \int_0^{t-v} h(0,v)h(0,u+v)\mathrm{e}^{-2\mu v-\mu u+H(u,v)}\,du\,dv$$
$$+ \int_0^t \int_0^{v} h(0,v)h(0,v-u)\mathrm{e}^{-2\mu(v-u)-\mu u+H(u,v-u)}\,du\,dv\,.$$

Proof. We find,

$$\mathbb{E}\left[\left(\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds\right)^2\right]$$
$$= \mathbb{E}\left[\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t \alpha(u)\,du}ds \int_0^t h(0,t-v)\mathrm{e}^{-\int_v^t \alpha(u)\,du}\,dv\right]$$
$$= \int_0^t \int_0^t h(0,t-s)h(0,t-v)\mathbb{E}\left[\mathrm{e}^{-\int_s^t \alpha(u)\,du-\int_v^t \alpha(u)\,du}\right]\,dv\,ds\,.$$

Appealing to Lemma 3.5 and doing a change of variables prove the statement of the Lemma. □

Collecting together the results for the expectations in (3.13)-(3.15) calculated in Lemmas 3.6-3.8, we get

Proposition 3.9. *The second moment of X is*

$$\mathbb{E}\left[X^2(t)\right] = \sigma^2 \int_0^t \mathrm{e}^{-2\mu u+H(0,u)}\,du$$
$$+ \sigma^2 \int_0^t \int_0^v \mathrm{e}^{-2\mu u-\mu(v-u)+H(v-u,u)}h(0,u)(h(v-u,v)+3h(0,v))\,du\,dv$$
$$+ \sigma^2 \int_0^t \int_0^v \mathrm{e}^{-2\mu(v-u)-\mu u+H(u,v-u)}h(0,v-u)(3h(u,v)+h(0,v))\,du\,dv\,.$$

If

$$\int_0^\infty \mathrm{e}^{-2\mu u+H(0,u)}\,du < \infty$$
$$\int_0^\infty \int_0^v |h(0,u)||h(v-u,v)+3h(0,v)|\mathrm{e}^{-2\mu u-\mu(u+v)+H(v-u,u)}\,du\,dv < \infty$$
$$\int_0^\infty \int_0^v |h(0,v-u)||3h(u,v)+h(0,v)|\mathrm{e}^{-2\mu(v-u)-\mu u+H(u,v-u)}\,du\,dv < \infty\,,$$

then the second moment of X has a limit given by

$$\lim_{t\to\infty} \mathbb{E}\left[X^2(t)\right] = \sigma^2 \int_0^\infty \mathrm{e}^{-2\mu u+H(0,u)}\,du$$
$$+ \sigma^2 \int_0^\infty \int_0^v h(0,u)(h(v-u,v)+3h(0,v))\mathrm{e}^{-2\mu u-\mu(u+v)+H(v-u,u)}\,du\,dv$$
$$+ \sigma^2 \int_0^\infty \int_0^v h(0,v-u)(3h(u,v)+h(0,v))\mathrm{e}^{-2\mu(v-u)-\mu u+H(u,v-u)}\,du\,dv\,.$$

Proof. The results of the Lemmas 3.6-3.8 give

$$\begin{aligned}\mathbb{E}[X^2(t)] = \sigma^2 \int_0^t \mathrm{e}^{-2\mu u + H(0,u)}\, du \\ + \sigma^2 \int_0^t \int_0^{t-v} h(0,v)\,(h(u,u+v) + 3h(0,u+v))\, \mathrm{e}^{-2\mu v - \mu u + H(u,v)}\, du\, dv \\ + \sigma^2 \int_0^t \int_0^{v} h(0,v-u)\,(h(0,v) + 3h(u,v))\, \mathrm{e}^{-2\mu(v-u) - \mu u + H(u,v-u)}\, du\, dv\,.\end{aligned}$$

Note from calculus that for sufficiently "nice" functions $q(t,v)$, it holds

$$\int_0^t q(t,v)\, dv = \int_0^t q(v,v)\, dv + \int_0^t \int_0^v \frac{\partial q}{\partial v}(v,u)\, du\, dv\,. \tag{3.18}$$

Hence, letting

$$q(t,v) = \int_0^{t-v} h(0,v)\,(h(u,u+v) + 3h(0,u+v))\, \mathrm{e}^{-2\mu v - \mu u + H(u,v)}\, du\,,$$

we have $q(v,v) = 0$ and

$$\frac{\partial q}{\partial t}(t,v) = h(0,v)\,(h(t-v,t) + 3h(0,t-v))\, \mathrm{e}^{-2\mu v - \mu(t-v) + H(t-v,v)}\,.$$

Using (3.18), the Proposition follows. □

In the latter proposition we showed the stationarity of the second moment of the GOU process. The stationarity of the variance follows immediately from Prop. 3.3, where besides of the integrability conditions of Prop. 3.9, we have to assume that (3.9) and (3.10) hold.

In the next lemma we investigate the conditions of stationarity of the second moment of X, where we restrict our attention to the OU specification.

Lemma 3.10. *Let α be an OU process as in* (3.3). *If*

$$\mu > \frac{\eta^2}{\beta^2}, \tag{3.19}$$

then

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}[X^2(t)] = \sigma^2 \int_0^\infty \mathrm{e}^{-2\mu u + H(0,u)}\, du \\ + \int_0^\infty \int_0^v \frac{\sigma^2\eta^2}{\beta^2}(\mathrm{e}^{-\beta(v-u)} - 4\mathrm{e}^{-\beta v} + 3)(1 - \mathrm{e}^{-\beta u})\mathrm{e}^{-2\mu u - \mu(v-u) + H(v-u,u)}\, du\, dv \\ + \int_0^\infty \int_0^v \frac{\sigma^2\eta^2}{\beta^2}(1 - \mathrm{e}^{-\beta(v-u)})(3\mathrm{e}^{-\beta u} - 4\mathrm{e}^{-\beta v} + 1)\mathrm{e}^{-2\mu(v-u) - \mu u + H(u,v-u)}\, du\, dv\,,\end{aligned}$$

where

$$H(x,y) = \frac{\eta^2}{4\beta^3}\left(1 - 2\mathrm{e}^{-\beta(x+y)} + \mathrm{e}^{-\beta x}\right)^2 + \frac{\eta^2}{2\beta^2}x + \frac{\eta^2}{4\beta^3}(1 - 4\mathrm{e}^{-\beta y} + 4\mathrm{e}^{-2\beta y})(1 - \mathrm{e}^{-2\beta x})$$

$$+\frac{\eta^2}{\beta^3}(1-2\mathrm{e}^{-\beta y})(1-\mathrm{e}^{-\beta x})+\frac{2\eta^2}{\beta^2}y-\frac{4\eta^2}{\beta^3}(1-\mathrm{e}^{-\beta y})+\frac{\eta^2}{\beta^3}(1-\mathrm{e}^{-2\beta y})\,. \tag{3.20}$$

Proof. We consider we have an OU process. Thus $g(u)=\eta\exp(-\beta u)$, for η and β constants and $\beta>0$. As we have seen earlier,

$$h(x,y)=\frac{\eta}{\beta}\left(\mathrm{e}^{-\beta x}-\mathrm{e}^{-\beta y}\right)\,.$$

We use this to compute $H(x,y)$. First, we find

$$\int_0^y h^2(0,u)\,du=\frac{\eta^2}{\beta^2}y-\frac{2\eta^2}{\beta^3}(1-\mathrm{e}^{-\beta y})+\frac{\eta^2}{2\beta^3}(1-\mathrm{e}^{-2\beta y})\,.$$

Next,

$$\begin{aligned}\int_0^x\left(h(u,y+u)+h(0,y+u)\right)^2\,du&=\frac{\eta^2}{\beta^2}x+\frac{\eta^2}{2\beta^3}(1-4\mathrm{e}^{-\beta y}+4\mathrm{e}^{-2\beta y})(1-\mathrm{e}^{-2\beta x})\\&\quad+\frac{2\eta^2}{\beta^3}(1-2\mathrm{e}^{-\beta y})(1-e^{-\beta x})\,.\end{aligned}$$

Finally,

$$\int_0^\infty\left(h(x+u,x+y+u)+h(u,x+y+u)\right)^2\,du=\frac{\eta^2}{2\beta^3}\left(1-2\mathrm{e}^{-\beta(x+y)}+\mathrm{e}^{-\beta x}\right)^2\,.$$

Therefore, we find the expression (3.20) for $H(x,y)$. Consider now the first term in the second moment of X. We find

$$\int_0^\infty \mathrm{e}^{-2\mu u+H(0,u)}\,du\le c\int_0^t\exp\left(-2\left(\mu-\frac{\eta^2}{\beta^2}\right)u\right)\,du<\infty\,,$$

as long as $\mu>\eta^2/\beta^2$. Hence, the limit of the first term exists when $t\to\infty$ under this restriction on the parameters. Next we consider the second term in the second moment of X. It holds,

$$\begin{aligned}&\int_0^\infty\int_0^v\frac{\sigma^2\eta^2}{\beta^2}|\mathrm{e}^{-\beta(v-u)}-4\mathrm{e}^{-\beta v}+3||1-\mathrm{e}^{-\beta u}|\mathrm{e}^{-2\mu u-\mu(v-u)+H(v-u,u)}\,du\,dv\\&\quad\le c\int_0^\infty\int_0^v|\mathrm{e}^{-u(\mu-3\frac{\eta^2}{2\beta^2})}\mathrm{e}^{-v(\mu-\frac{\eta^2}{2\beta^2})}|\,du\,dv\\&\quad\le c\int_0^\infty\Big|\frac{\mathrm{e}^{-2v(\mu-\frac{\eta^2}{\beta^2})}-\mathrm{e}^{-v(\mu-\frac{\eta^2}{2\beta^2})}}{-\mu+\frac{3\eta^2}{2\beta^2}}\Big|\,dv<\infty,\end{aligned}$$

as long as $\mu>\eta^2/\beta^2$. Finally we consider the last term in the second moment of X. We find

$$\begin{aligned}&\int_0^\infty\int_0^v\frac{\sigma^2\eta^2}{\beta^2}|1-\mathrm{e}^{-\beta(v-u)}||3\mathrm{e}^{-\beta u}-4\mathrm{e}^{-\beta v}+1|\mathrm{e}^{-2\mu(v-u)-\mu u+H(u,v-u)}\,du\,dv\\&\quad\le c\int_0^\infty\int_0^v\mathrm{e}^{-2v(\mu-\frac{\eta^2}{\beta^2})}\mathrm{e}^{-u(-\mu+\frac{3\eta^2}{2\beta^2})}\,du\,dv\end{aligned}$$

$$\leq c \int_0^\infty \mathrm{e}^{-v(\mu - \frac{\eta^2}{2\beta^2})} - \mathrm{e}^{-2v(\mu + \frac{\eta^2}{\beta^2})}\, dv < \infty,$$

as long as $\mu > \eta^2/\beta^2$. With this condition on the parameters, the third term in the expression of the second moment of X has a limit as time goes to infinity. □

Therefore, the second moment of X has a stationary limit as long as $\mu\beta$ is greater than twice the stationary variance of α, which we recall as $\eta^2/2\beta$. This condition is more restrictive than the one ensuring the stationarity of the mean. Thus, it obviously implies the stationarity of the expectation of X. In conclusion, when we specify α to be the OU process given by (3.3) then X has a stationary limit for the expectation and variance if

$$\mu\beta > \frac{\eta^2}{\beta}\,.$$

Observe that for fixed η, the stationary variance of α is decreasing with increasing speed of mean reversion β. On the other hand, for fixed μ, the expression $\mu\beta$ is obviously increasing with β. Hence, the condition for stationarity is less restrictive for models with high speed of mean reversion in the α than those of slow. Note that the slower the mean reversion in α becomes, the closer α gets to a non-stationary drifted Brownian motion. Hence, to obtain stationarity for small β's, one must have sufficiently high mean levels μ, and/or, sufficiently small noise level η in the dynamics of α.

3.3. **Stationarity of the covariance.** In this subsection we compute the covariance of X and prove its stationarity. Notice that we are going to present the proofs of all the results of this subsection in an Appendix since they follow the same lines as the proofs of the results presented in Subsection 3.2.

Suppose that $X_0 = 0$. Then we have for $\delta > 0$, $\mathbb{E}[X(t)X(t-\delta))] = I_1 - I_2 - I_3 + I_4$, where

$$I_1 = \mathbb{E}\Big[\int_0^t \sigma \mathrm{e}^{-\int_s^t \alpha(u)du}\delta W(s) \int_0^{t-\delta} \sigma \mathrm{e}^{-\int_s^{t-\delta}\alpha(u)du}\delta W(s)\Big]\,, \tag{3.21}$$

$$I_2 = \mathbb{E}\Big[\int_0^t \sigma \mathrm{e}^{-\int_s^t \alpha(u)du}\delta W(s) \int_0^{t-\delta} \sigma h(0,t-\delta-s) \mathrm{e}^{-\int_s^{t-\delta}\alpha(u)du}ds)\Big]\,, \tag{3.22}$$

$$I_3 = \mathbb{E}\Big[\int_0^t \sigma h(0,t-s) \mathrm{e}^{-\int_s^t \alpha(u)du}ds \int_0^{t-\delta} \sigma \mathrm{e}^{-\int_s^{t-\delta}\alpha(u)du}\delta W(s)\Big]\,, \tag{3.23}$$

$$I_4 = \mathbb{E}\Big[\int_0^t \sigma h(0,t-s) \mathrm{e}^{-\int_s^t \alpha(u)du}ds \int_0^{t-\delta} \sigma h(0,t-\delta-s) \mathrm{e}^{-\int_s^{t-\delta}\alpha(u)du}ds)\Big]\,. \tag{3.24}$$

We want to compute the expectations (3.21)-(3.24). The approach is based on the same ideas as when we calculated the second moment of X. We first compute the expectation of the term $\exp(-\int_\eta^t \alpha(u)du - \int_\omega^v \alpha(u)du)$, $0 \leq v < t \leq T$. This is the content of the next lemma.

Lemma 3.11. *It holds for $v \le t$, $\omega \le \eta$, $\eta \le v$,*

$$\ln \mathbb{E}\Big[\exp\Big(-\int_\eta^t \alpha(u)du - \int_\omega^v \alpha(u)du\Big)\Big] = -\mu(t-\eta) - \mu(v-\omega) + \widetilde{H}(\eta-\omega, t-\eta, v-\eta), \tag{3.25}$$

where

$$\begin{aligned}\widetilde{H}(x,y,z) = &\frac{1}{2}\int_0^\infty \{h(x+u, x+y+u) + h(u, z+x+u)\}^2 du \\ &+ \frac{1}{2}\int_0^x \{h(u, u+y) + h(0, u+z)\}^2 du \\ &+ \frac{1}{2}\int_0^z \{h(0, y-u) + h(0, z-u)\}^2 du + \frac{1}{2}\int_0^{y-z} h^2(0, y-z-u) du,\end{aligned}$$

for $x, y, z \ge 0$. Moreover, it holds for $\eta \le \omega \le v \le t$,

$$\ln \mathbb{E}\Big[\exp\Big(-\int_\eta^t \alpha(u)du - \int_\omega^v \alpha(u)du\Big)\Big] = -\mu(t-\eta) - \mu(v-\omega) + \widetilde{H}(\omega-\eta, t-\omega, v-\omega)\,. \tag{3.26}$$

Notice that $\widetilde{H}(x,y,y) = H(x,y)$. Thus in case $t = v$ in (3.25) and (3.26), these two equations are the same as Eq. (3.16) in Lemma 3.5.

Let us now compute the expectations (3.21)-(3.24). We start with the expectation in (3.21) which we compute in the following lemma.

Lemma 3.12. *It holds for $\delta > 0$,*

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t \sigma e^{-\int_\omega^t \alpha(u)du}\delta W(\omega)\int_0^{t-\delta}\sigma e^{-\int_\omega^{t-\delta}\alpha(u)du}\delta W(\omega)\Big] \\ &= \sigma^2\int_\delta^t \exp(-\mu v - \mu(v-\delta) + \widetilde{H}(0, v, v-\delta))\,dv \\ &\quad + \sigma^2\int_\delta^t\int_0^{t-s}\exp(-\mu s - \mu(s+v-\delta) + \widetilde{H}(v, s, s-\delta))h(v, s+v)h(0, s-\delta)\,dv\,ds \\ &\quad + \sigma^2\int_\delta^t\int_0^{s-\delta}\exp(-\mu s - \mu(s-v-\delta) + \widetilde{H}(v, s-v, s-v-\delta))h(0, s-v)h(v, s-\delta)\,dv\,ds\,.\end{aligned}$$

We derive the second expectation (3.22) in the next Lemma.

Lemma 3.13. *It holds that for $\delta > 0$,*

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t \sigma e^{-\int_\omega^t \alpha(u)du}\delta W(\omega)\int_0^{t-\delta}\sigma h(0, t-\delta-\omega)e^{-\int_\omega^{t-\delta}\alpha(u)du}d\omega\Big] \\ &= -\sigma^2\int_\delta^t\int_0^{t-s} h(0, s+v-\delta)h(0, s-\delta)\exp(-\mu(s+v-\delta) - \mu s + \widetilde{H}(v, s, s-\delta))\,dv\,ds \\ &\quad - \sigma^2\int_\delta^t\int_0^{s-\delta} h(0, s-v-\delta)h(v, s-\delta)\exp(-\mu(s-v-\delta) - \mu s\end{aligned}$$

$$+ \widetilde{H}(v, s-v, s-v-\delta))\, dv\, ds\,.$$

We derive the expectation (3.23) in the next Lemma.

Lemma 3.14. *It holds for $\delta > 0$,*

$$\mathbb{E}\Big[\int_0^t \sigma h(0,t-\omega)\mathrm{e}^{-\int_\omega^t \alpha(u)du} d\omega \int_0^{t-\delta} \sigma \mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\delta W(\omega)\Big]$$
$$= -\sigma^2 \int_\delta^t \int_0^{t-s} h(0,s+v)h(0,s)\exp(-\mu(s+v)-\mu(v-\delta)+\widetilde{H}(v,s,s-\delta))\, dv\, ds$$
$$-\sigma^2 \int_\delta^t \int_0^{s-\delta} h(0,s-v)h(v,s)\exp(-\mu(s-v)-\mu(s-\delta)+\widetilde{H}(v,s-v,s-v-\delta))\, dv\, ds\,.$$

We derive the expectation (3.24) in the next Lemma.

Lemma 3.15. *It holds for $\delta > 0$,*

$$\mathbb{E}\Big[\int_0^t \sigma h(0,t-\eta)\mathrm{e}^{-\int_\eta^t \alpha(u)du} d\eta \int_0^{t-\delta} \sigma h(0,t-\delta-\omega)\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du} d\omega)\Big]$$
$$= \sigma^2 \int_\delta^t \int_0^{t-s} h(0,s)h(0,s+v-\delta)\exp(-\mu s-\mu(s+v-\delta)+\widetilde{H}(v,s,s-\delta))\, dv\, ds$$
$$+ \sigma^2 \int_\delta^t \int_0^{s-\delta} h(0,s)h(0,s-v-\delta)\exp(-\mu s-\mu(s-v-\delta)$$
$$+ \widetilde{H}(v,s-v,s-v-\delta))\, dv\, ds$$
$$+ \sigma^2 \Big(\int_0^\delta h(0,\delta-v)G(\delta-v)dv\Big)\Big(\int_0^{t-\delta} h(0,v)G(v)\, dv\Big)\,,$$

where $G(x) = \exp\Big\{-\mu x + \frac{1}{2}\int_0^\infty h^2(u,x+u)du + \frac{1}{2}\int_0^x h^2(0,u)du\Big\}$, *for* $x \geq 0$.

Collecting together the expectations studied in Lemmas 3.11-3.15, we compute $\mathbb{E}[X(t)X(t-\delta)]$, for $\delta > 0$ in the following proposition.

Proposition 3.16. *It holds for $\delta > 0$,*

$$\mathbb{E}[X(t)X(t-\delta)] = \sigma^2 \int_\delta^t \exp(-\mu v - \mu(v-\delta) + \widetilde{H}(0,v,v-\delta))\, dv$$
$$+ \sigma^2 \int_\delta^t \int_0^s \exp(-\mu v - \mu(s-\delta) + \widetilde{H}(s-v,v,v-\delta))\Big\{h(0,v-\delta)$$
$$[h(s-v,s)+h(0,s-\delta)] + h(0,v)[h(0,s)+h(0,s-\delta)]\Big\}\, dv\, ds$$
$$+ \sigma^2 \int_\delta^t \int_0^{s-\delta} \exp(-\mu s - \mu(s-v-\delta) + \widetilde{H}(v,s-v,s-v-\delta))\Big\{h(0,s-v)$$

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$$[h(v,s-\delta)+h(v,s)]+h(0,s-v-\delta)[h(v,s-\delta)+h(0,s)]\Big\}\,dv\,ds$$
$$+\sigma^2\Big(\int_0^\delta h(0,\delta-v)G(\delta-v)dv\Big)\Big(\int_0^{t-\delta} h(0,v)G(v)\,dv\Big)\,.$$

If

$$\int_\delta^\infty \exp(-\mu v-\mu(v-\delta)+\widetilde{H}(0,v,v-\delta))\,dv<\infty\,,$$
$$\int_\delta^\infty\int_0^s \exp(-\mu v-\mu(s-\delta)+\widetilde{H}(s-v,v,v-\delta))\Big|h(0,v-\delta)$$
$$[h(s-v,s)+h(0,s-\delta)]+h(0,v)[h(0,s)+h(0,s-\delta)]\Big|\,dv\,ds<\infty\,,$$
$$\int_\delta^\infty\int_0^{s-\delta} \exp(-\mu s-\mu(s-v-\delta)+\widetilde{H}(v,s-v,s-v-\delta))$$
$$\Big|h(0,s-v)[h(v,s-\delta)+h(v,s)]+h(0,s-v-\delta)[h(v,s-\delta)+h(0,s)]\Big|\,dv\,ds\;<\infty,$$
$$\int_0^\infty |h(0,v)G(v)|\,dv\;<\infty,$$

then

$$\lim_{t\to\infty}\mathbb{E}[X(t)X(t-\delta)]$$
$$=\sigma^2\int_\delta^\infty \exp(-\mu v-\mu(v-\delta)+\widetilde{H}(0,v,v-\delta))\,dv$$
$$+\sigma^2\int_\delta^\infty\int_0^s \exp(-\mu v-\mu(s-\delta)+\widetilde{H}(s-v,v,v-\delta))\Big\{h(0,v-\delta)$$
$$[h(s-v,s)+h(0,s-\delta)]+h(0,v)[h(0,s)+h(0,s-\delta)]\Big\}\,dv\,ds$$
$$+\sigma^2\int_\delta^\infty\int_0^{s-\delta} \exp(-\mu s-\mu(s-v-\delta)+\widetilde{H}(v,s-v,s-v-\delta))\Big\{h(0,s-v)$$
$$[h(v,s-\delta)+h(v,s)]+h(0,s-v-\delta)[h(v,s-\delta)+h(0,s)]\Big\}\,dv\,ds$$
$$+\sigma^2\Big(\int_0^\delta h(0,\delta-v)G(\delta-v)dv\Big)\Big(\int_0^\infty h(0,v)G(v)\,dv\Big)\,.$$

Considering the integrability conditions of the latter proposition together with the integrability conditions of Prop. 3.9, we prove the stationarity of the covariance and thus the weak stationarity of the GOU process.

Notice that when we consider $\delta=0$ in Prop. 3.16, we recover the expression for the second moment studied in Prop. 3.9. We chose to present the results for the second moment in Subsection 3.2 to be more didactic.

In the next lemma we investigate the conditions of stationarity of the covariance where we restrict our attention to the case where α is an OU process.

Lemma 3.17. *Let α be an OU process as in* (3.3)*. If*

$$\mu > \frac{\eta^2}{\beta^2},$$

then

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}[X(t)X(t-\delta)] \\
&\quad = \sigma^2 \int_\delta^\infty \exp(-\mu v - \mu(v-\delta) + \widetilde{H}(0,v,v-\delta))\,dv \\
&\qquad + \sigma^2 \int_\delta^\infty \int_0^s \exp(-\mu v - \mu(s-\delta) + \widetilde{H}(s-v,v,v-\delta))\Big\{h(0,v-\delta) \\
&\qquad\quad [h(s-v,s) + h(0,s-\delta)] + h(0,v)[h(0,s) + h(0,s-\delta)]\Big\}\,dv\,ds \\
&\qquad + \sigma^2 \int_\delta^\infty \int_0^{s-\delta} \exp(-\mu s - \mu(s-v-\delta) + \widetilde{H}(v,s-v,s-v-\delta))\Big\{h(0,s-v) \\
&\qquad\quad [h(v,s-\delta) + h(v,s)] + h(0,s-v-\delta)[h(v,s-\delta) + h(0,s)]\Big\}\,dv\,ds \\
&\qquad + \sigma^2\Big(\int_0^\delta h(0,\delta-v)G(\delta-v)dv\Big)\Big(\int_0^\infty h(0,v)G(v)\,dv\Big).
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{H}(x,y,z) = {}& \frac{\eta^2}{4\beta^3}\Big(1 - e^{\beta(x+y)} - e^{-\beta(x+z)} + e^{-\beta x}\Big)^2 + \frac{\eta^2}{\beta^3}(1-e^{-\beta x})(1 - e^{-\beta z} - e^{-\beta y}) + \frac{\eta^2}{2\beta^2}x \\
&+ \frac{\eta^2}{4\beta^3}(1-e^{-2\beta x})\Big(1 + e^{-2\beta y} - 2e^{-\beta y} + e^{-2\beta z} - 2e^{-\beta z} + 2e^{-\beta(y+z)}\Big) \\
&+ \frac{2\eta^2}{\beta^2}z + \frac{\eta^2}{4\beta^3}(1-e^{-2\beta z})(1 + e^{-2\beta(y-z)} + 2e^{-\beta(y-z)}) \\
&+ \frac{2\eta^2}{\beta^3}e^{-\beta y}(1-e^{\beta z}) + \frac{2\eta^2}{\beta^3}(e^{-\beta z} - 1) + \frac{\eta^2}{2\beta^2}(y-z) - \frac{3\eta^2}{4\beta^3}(1 - e^{-2\beta(y-z)}).
\end{aligned}$$

and

$$G(x) = \exp\Big(-\mu x + \frac{\eta^2}{4\beta^3}(1-e^{-\beta x})^2 + \frac{\eta^2}{2\beta^2}x - \frac{\eta^2}{\beta^3}(1-e^{-\beta x}) + \frac{\eta^2}{4\beta^3}(1-e^{-2\beta x})\Big).$$

As a conclusion, when we consider α to be an OU process, then the GOU process is weakly stationary if $\mu\beta$ is greater than twice the stationary variance of α. Thus, it is weakly stationary if we impose the condition

$$\mu\beta > \frac{\eta^2}{\beta},$$

on the parameters of the model.

4. Local behavior

In this section we study the instantaneous rate of change in the mean and in the squared fluctuations of the process X given that $X(s) = x$, where $x \in \mathbb{R}$. We present the proofs of the results studied in this section in the Appendix as they are similar to the arguments in the previous section.

We compute in the next lemma the mean of the increment $X(t) - X(s)$, $t \geq s$, given that $X(s) = x$.

Lemma 4.1. *We have for $s \leq t$,*

$$\mathbb{E}[X(t) - X(s) | X(s) = x] = x\big(f(t-s) - 1\big) + \widetilde{f}(t-s),$$

where

$$f(v) = \exp\Big\{ -\mu v + \frac{1}{2}\int_0^\infty h^2(u, v+u)du + \frac{1}{2}\int_0^v h^2(0,u)du \Big\} \tag{4.1}$$

and

$$\widetilde{f}(v) = -\int_0^v \sigma h(0,u) f(u) du\,.$$

From this Lemma we can compute the instantaneous rate of change in the mean of the process X, given that $X(s) = x$. This is the content of the next Proposition.

Proposition 4.2. *Assume that for all $t \in [0, T]$,*

$$h^2(u, t-s+u) \leq \zeta(s,u) \tag{4.2}$$

and

$$|g(t-s+u)| \leq \psi(s,u), \tag{4.3}$$

where $\int_0^\infty \zeta(s,u)du < \infty$ and $\int_0^\infty \psi(s,u)du < \infty$. Moreover assume

$$(\mathcal{A}) \qquad \int_0^\infty |h(0,u)f(u)| < \infty\,, \qquad \int_0^\infty h^2(0,u) < \infty\,.$$

Then we have for $s \leq t$,

$$a(x) = \lim_{t \to s} \frac{\mathbb{E}[X(t) - X(s) | X(s) = x]}{t-s} = -\mu x\,. \tag{4.4}$$

As we can see, locally the drift of X is behaving like the drift of an Ornstein-Uhlenbeck process. The speed of mean reversion becomes equal to the mean of α, namely μ.

In the next lemma we compute the squared fluctuations of $X(t) - X(s)$, $t \geq s$ given that $X(s) = x$.

Lemma 4.3. *We have for $s \leq t$,*

$$\mathbb{E}\big[(X(t)-X(s))^2]|X(s)=x\big] = \mathcal{I}_1 - 2\mathcal{I}_2 + \mathcal{I}_3 - 2x\mathcal{I}_4 + 2x\mathcal{I}_5 + x^2\mathcal{I}_6\,,$$

where

$$\mathcal{I}_1 = \mathbb{E}\Big[\Big\{\int_s^t \sigma \exp\big(-\int_u^t \alpha(v)dv\big)\delta W(u)\Big\}^2\Big]\,,$$

$$\mathcal{I}_2 = \quad \mathbb{E}\Big[\Big\{\int_s^t \sigma^2 \exp(-\int_u^t \alpha(v)dv)\delta W(u)\Big\}\Big\{\int_s^t h(0,t-u)\exp(-\int_u^t \alpha(v)dv)du)\Big\}\Big]\,,$$

$$\mathcal{I}_3 = \mathbb{E}\Big[\Big(\int_s^t \sigma h(0,t-u)\exp\big(-\int_u^t \alpha(v)dv\big)du\Big)^2\Big]\,,$$

$$\mathcal{I}_4 = \mathbb{E}\Big[\int_s^t \sigma h(0,t-u)\exp(-\int_u^t \alpha(v)dv)du\big\{\exp(-\int_s^t \alpha(v)dv)-1\big\}\Big]\,,$$

$$\mathcal{I}_5 = \mathbb{E}\Big[\exp(-\int_s^t \alpha(v)dv)\int_s^t \sigma\exp(-\int_u^t \alpha(v)dv)\delta W(v)\Big]\,,$$

$$\mathcal{I}_6 = \mathbb{E}\Big[\Big(\exp\big(-\int_s^t \alpha(v)dv\big)-1\Big)^2\Big]\,.$$

With this Lemma at hand, we can consider the instantaneous rate of change of the squared fluctuations of the process X given that $X(s)=x$.

Proposition 4.4. *Assume that* (4.2), (4.3), *and* $(\mathcal{A})$ *holds true. If moreover,*

$$(\mathcal{B}) \qquad \int_0^\infty \exp(-2\mu v + H(0,v))dv < \infty,$$

$$\int_0^\infty \int_0^v |h(0,u)|\mathrm{e}^{-2\mu(u)-\mu(v-u)+H(v-u,u)}\big|3h(v-u,v)+h(0,v)\big|du dv < \infty,$$

$$\int_0^\infty \int_0^v |h(0,v-u)|\mathrm{e}^{-2\mu(v-u)-\mu(u)+H(u,v-u)}\big|3h(0,v)+h(u,v)\big|du dv < \infty\,.$$

Moreover, if

$$(\mathcal{C}) \qquad \int_0^\infty |h(0,v)|\mathrm{e}^{-2\mu v+H(0,v)}dv < \infty\,,$$

$$\int_0^\infty \int_0^v \Big|h(0,u)\frac{\partial}{\partial v}H(v-u,u)\Big|\mathrm{e}^{-2\mu u-\mu(v-u)+H(v-u,u)}du\,dv\ < \infty\,,$$

$$\int_0^\infty \Big|h(0,v)\frac{\partial}{\partial v}H(0,v))\Big|\mathrm{e}^{-2\mu v+H(0,v)}dv < \infty\,,$$

$$\int_0^\infty \int_0^v h(0,u)\Big\{\mathrm{e}^{-2\mu u-\mu(v-u)+H(v-u,u)}\big(\frac{\partial}{\partial v}H(v-u,u)\big)^2 + \frac{\partial^2}{\partial v^2}H(v-u,u)\Big\}du\,dv\ < \infty\,.$$

Then we have for $s \le t$

$$b^2(x) = \lim_{t\to s} \frac{\mathbb{E}[(X(t)-X(s))^2 | X(s)=x]}{t-s} = \sigma^2 .$$

The variance of the process X will locally behave as a constant, and therefore we may view the process X as an Ornstein-Uhlenbeck process for small increments. Notice that for the so-called diffusion processes (see for example Kloeden and Platen [12] for more about such processes), the quantity $a(x)$ is called the drift of the diffusion and $b(x)$ its diffusion coefficient at position x. These two terms describe the behavior of the first and second moment of increments of the process X over an infinitesimal time interval $[s, s+ds]$.

Now we consider the case where α is the OU process given in (3.3). Under the assumption

$$\mu\beta > \frac{\eta^2}{\beta^2} ,$$

the results of Propositions 4.2 and 4.4 hold true. In fact when α is an OU process then assumptions (4.2), (4.3), $(\mathcal{A})$, and $(\mathcal{B})$ follow immediately from Lemmas 3.4 and 3.10. To prove $(\mathcal{C})$ we observe that $\frac{\partial}{\partial v} H(v-u,u)$ and $\frac{\partial^2}{\partial v^2} H(v-u,u)$ are bounded uniformly in u and v and thus $(\mathcal{C})$ becomes equivalent to $(\mathcal{B})$.

5. Chaos expansion

In this Section we develop the chaos expansion of X. We denote by $I_n(\phi_n)$, for a natural number n, the nth chaos with kernel function $\phi_n \in L^2(\mathbb{R}^n)$ being symmetric. That is,

$$\begin{aligned} I_n(\phi_n) &= \int_{\mathbb{R}^n} \phi_n(u_1,\dots,u_n) dW^{\otimes n}(u_1,\dots,u_n) \\ &= n! \int_{-\infty}^{\infty} \int_{-\infty}^{u_{n-1}} \cdots \int_{-\infty}^{u_2} \phi_n(u_1,\dots,u_n)\, dW(u_1)\cdots dW(u_n) . \end{aligned}$$

Note that in particular, $I_0(\phi_0)) = \phi_0$ is simply a constant and

$$I_1(\phi_1) = \int_{\mathbb{R}} \phi_1(u)\, dW(u) ,$$

a Gaussian random variable with zero mean and variance being the $L^2(\mathbb{R})$-norm of ϕ_1. Throughout this Section we assume that the integrability condition on h in Lemma 3.2 holds.

Our first result concerns the chaos expansion of $-\int_s^t \alpha(u)\, du$.

Lemma 5.1. *It holds that*

$$-\int_s^t \alpha(u)\, du = -\mu(t-s) + I_1(\phi_{s,t}) ,$$

for

$$\phi_{s,t}(u) = -h(s-u, t-u)\mathbf{1}(u \le s) - h(0, t-u)\mathbf{1}(s \le u \le t) . \tag{5.1}$$

Proof. From Lemma 3.2, we find

$$-\int_s^t \alpha(u)\,du = -\mu(t-s) - \int_{-\infty}^s h(s-u,t-u)\,dW(u) - \int_s^t h(0,t-u)\,dW(u)$$
$$= -\mu(t-s) + \int_{\mathbb{R}} \phi_{s,t}(u)\,dW(u)\,.$$

Hence, the result follows. □

Next, we state explicitly the chaos expansion of $\exp\left(-\int_s^t \alpha(u)\,du\right)$.

Lemma 5.2. *It holds*

$$\exp\left(-\int_s^t \alpha(u)\,du\right) = \sum_{n=0}^{\infty} I_n\left(\frac{1}{n!}f(t-s)\phi_{s,t}^{\otimes n}\right)\,,$$

where f is given by (4.1)

Proof. From the theory of chaos expansions, it holds (see Nualart [14])

$$\exp\left(I_1(\phi_{s,t}) - \frac{1}{2}|\phi_{s,t}|_2^2\right) = \sum_{n=0}^{\infty}\frac{1}{n!}I_n\left(\phi_{s,t}^{\otimes n}\right)\,,$$

where $|\cdot|_2$ denotes the $L^2(\mathbb{R})$-norm. But,

$$|\phi_{s,t}|_2^2 = \int_{\mathbb{R}} \left(h(s-u,t-u)\mathbf{1}(u\le s) + h(0,t-u)\mathbf{1}(s\le u\le t)\right)^2\,du$$
$$= \int_0^{\infty} h^2(u,t-s+u)\,du + \int_0^{t-s} h^2(0,u)\,du\,.$$

Hence, the result follows from Lemma 5.1. □

Using this, we can compute the chaos of the "drift-term" of X.

Lemma 5.3. *Assume that the function*

$$v \mapsto h(0,v)\exp\left(-\mu v + \int_0^{\infty} h^2(u,v+u)\,du + \int_0^v h^2(0,u)\,du\right)$$

is integrable on $[0,t]$. *Then the integral* $\int_0^t h(0,t-s)\exp(-\int_s^t \alpha(u)\,du)\,ds$ *is well-defined as a Bochner integral in* $L^2(\Omega,\mathcal{F},P)$. *Moreover, it holds*

$$\int_0^t h(0,t-s)\mathrm{e}^{-\int_s^t \alpha(u)\,du}\,ds = \sum_{n=0}^{\infty} I_n\left(\frac{1}{n!}\int_0^t h(0,t-s)f(t-s)\phi_{s,t}^{\otimes n}\,ds\right)\,,$$

where f is defined in Lemma 5.2 and $\phi_{s,t}$ in (5.1).

Proof. Recall that

$$\exp\left(-\int_s^t \alpha(u)\,du\right) = \exp\left(-\mu(t-s) + I_1(\phi_{s,t})\right)\,,$$

and hence

$$\mathbb{E}\left[\exp\left(-2\int_s^t \alpha(u)\,du\right)\right]^{1/2} = \exp\left(-\mu(t-s) + |\phi_{s,t}|_2^2\right)$$
$$= \exp\left(-\mu(t-s) + \int_0^\infty h^2(u, t-s+u)\,du + \int_0^{t-s} h^2(0,u)\,du\right),$$

from the proof of Lemma 5.2. But then the condition implies that $s \mapsto h(0, t-s)\exp\left(-\int_s^t \alpha(u)\,du\right)$ is Bochner integrable on $[0,t]$. Invoking Prop. 8.1 and the following discussion on page 281 in Hida *et al.* [11], the chaos expansion of a Bochner integral is given by integrating the chaos functions. Hence, the result follows. □

Finally, we compute the chaos expansion of the Skorohod integral in the next Lemma.

Lemma 5.4. *Assume that* $e^{-\int_s^t \alpha(u)\,du}$ *is Skorohod integrable. It holds that*

$$\int_0^t e^{-\int_s^t \alpha(u)\,du}\,\delta W(s) = \sum_{n=0}^\infty I_{n+1}\left(\frac{1}{n!}\psi_{n,t}\right),$$

where $\psi_{n,t} \in L^2(\mathbb{R}^{n+1})$ *is the symmetrization of the function*

$$\widetilde{\psi}_{n,t}(u_1,\ldots,u_n,u_{n+1}) = \mathbf{1}_{[0,t]}(u_{n+1}) f(t-u_{n+1})\phi^{\otimes n}_{u_{n+1},t}(u_1,\ldots,u_n).$$

Proof. This follows from Prop. 1.3.3 in Nualart [14]. □

Collecting together the results in the Lemmas above, we find the chaos expansion of $X(t)$ to be

$$X(t) = X_0 f(t) + \int_0^t \sigma h(0,u) f(u)\,du$$
$$+ \sum_{n=1}^\infty I_n\left(\frac{1}{n!}\left(X_0 f(t)\phi^{\otimes n}_{0,t} + \int_0^t \sigma h(0,t-s) f(t-s)\phi^{\otimes n}_{s,t}\,ds + \sigma n \psi_{n-1,t}\right)\right). \tag{5.2}$$

Note that the integral in the chaos functions can be alternatively written as

$$\int_0^t h(0,t-s) f(t-s)\phi^{\otimes n}_{s,t}\,ds = \int_0^t h(0,u) f(u)\phi^{\otimes n}_{t-u,t}\,du$$

after a change of variables.

We consider the case where α is the OU process given by (3.3). In this case, $g(u) = \eta e^{-\beta u}$ and $\exp(-\int_s^t \alpha(u)du)$ is skorohod integrable as it was already discussed in Section 3. Since we are working in the setting of $L^2(\Omega, \mathcal{F}, P)$-random variables, we suppose that $\mu\beta > \eta^2/\beta$, which we recall from the previous Section that it is a sufficient condition for the second moment of $X(t)$ to exist for all $t \geq 0$. We compute a chaos expansion for a random variable X in this particular case of specification of g and investigate the kernel functions in the chaos expansion.

Let us start with considering the zeroth order chaos, which is the mean of X. We recall the expression of $h(x,y)$ and $f(x)$ in (3.4) and (4.1), resp. We know the limit of the zeroth order chaos (the mean) from Prop. 3.3, giving

$$\lim_{t\to\infty} X_0 f(t) + \int_0^t h(0,u)f(u)\,du = \int_0^\infty h(0,u)f(u)\,du\,.$$

Next, we focus on chaos 1, that is the case $n=1$ for simplicity. In the following lemma we compute the point wise limit of the kernel of chaos 1.

Lemma 5.5. *Assume $\mu\beta > \eta^2/2\beta$. The limit of the zero order chaos is given by*

$$\lim_{t\to\infty} X_0 f(t)\phi_{0,t}(u) + \int_0^t \sigma h(0,t-s)\phi_{s,t}(u)\,ds = k\mathrm{e}^{\beta u}\,\mathbf{1}_{(u\le 0)}\,, \tag{5.3}$$

where $k = -\int_0^\infty \widetilde{h}(u)\,du + \beta\int_0^\infty\int_0^v \widetilde{h}(s)\mathrm{e}^{-\beta(v-s)}\,ds\,dv$.

Proof. As we have from (5.1)

$$\begin{aligned}
\phi_{s,t}(u) &= -h(s-u,t-u)\mathbf{1}(u\le s) - h(0,t-u)\mathbf{1}(s\le u\le t)\\
&= -\int_{s-u}^{t-u} g(v)\,dv\mathbf{1}(u\le s) - \int_0^{t-u} g(v)\,dv\mathbf{1}(s\le u\le t)\\
&= \frac{\eta}{\beta}\mathrm{e}^{\beta u}\left(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s}\right)\mathbf{1}(u\le s) + \frac{\eta}{\beta}\left(\mathrm{e}^{-\beta(t-u)}-1\right)\mathbf{1}(s\le u\le t)\,.
\end{aligned}$$

for $u\in\mathbb{R}$. Let $\widetilde{h}(v) = \frac{\eta}{\beta}h(0,v)f(v)$. Note that we have

$$\begin{aligned}
h(0,u)f(u) &= \frac{\eta}{\beta}(1-\mathrm{e}^{-\beta u})\exp\Big(-\mu u + \frac{1}{2}\int_0^\infty h^2(v,u+v)dv + \frac{1}{2}\int_0^u h^2(0,v)dv\Big)\\
&= \frac{\eta}{\beta}(1-\mathrm{e}^{-\beta u})\exp\big(-\mu u - \frac{\eta^2}{2\beta^3} + \frac{\eta^2}{2\beta^3}\mathrm{e}^{-\beta u} + \frac{\eta^2}{2\beta^2}u\big)\,.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^t h(0,t-s)f(t-s)\phi_{s,t}(u)\,ds\\
&\quad= \int_0^t \widetilde{h}(t-s)\mathrm{e}^{\beta u}(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\mathbf{1}(u\le s)\,ds + \int_0^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta(t-u)}-1)\mathbf{1}(s\le u\le t)\,ds\\
&\quad= \begin{cases}
\int_0^t \widetilde{h}(t-s)\mathrm{e}^{\beta u}(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\,, & u\le 0\,,\\
\int_u^t \widetilde{h}(t-s)\mathrm{e}^{\beta u}(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds + \int_0^u \widetilde{h}(t-s)(\mathrm{e}^{-\beta(t-u)}-1)\,ds\,, & u\in[0,t]\,,\\
0\,, & u>t\,.
\end{cases}
\end{aligned}$$

Consider first $u<0$. After a simple change of variables, it holds,

$$\int_0^t \widetilde{h}(t-s)\mathrm{e}^{\beta u}(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds = \mathrm{e}^{\beta u}\left(-\int_0^t \widetilde{h}(s)\mathrm{e}^{-\beta(t-s)}\,ds + \mathrm{e}^{-\beta t}\int_0^t \widetilde{h}(s)\,ds\right)\,.$$

Since under the assumed condition, $\lim_{t\to\infty}\int_0^t \widetilde{h}(s)\,ds = \frac{\eta}{\beta}\int_0^\infty h(0,s)f(s)\,ds$, then

$$\lim_{t\to\infty} \mathrm{e}^{-\beta t}\int_0^t \widetilde{h}(s)\,ds = 0\,.$$

From the "key formula" in (3.18), we find

$$\begin{aligned}\int_0^t \widetilde{h}(s)\mathrm{e}^{-\beta(t-s)}\,ds &= \int_0^t \widetilde{h}(s)\,ds + \int_0^t\int_0^v \widetilde{h}(u)(-\beta)\mathrm{e}^{-\beta(v-u)}\,du\,dv \\ &= \int_0^t \widetilde{h}(s)\,ds - \beta\int_0^t \mathrm{e}^{-\beta v}\int_0^v \widetilde{h}(u)\mathrm{e}^{\beta u}\,du\,dv\,.\end{aligned}$$

Notice that the first term on the right-hand side has a limit as time goes to infinity. To have a limit for the second term, we need that the function $v \mapsto \exp(-\beta v)\int_0^v \widetilde{h}(u)\exp(\beta u)\,du$ is integrable on $\mathbb{R}_+$. But this is indeed the case, since

$$\begin{aligned}\mathrm{e}^{-\beta v}\int_0^v \widetilde{h}(u)\mathrm{e}^{\beta u}\,du &\le c\mathrm{e}^{-\beta v}\int_0^v \mathrm{e}^{\beta u-\mu u+\eta^2 u/(2\beta^2)}\,du \\ &\le c\frac{\mathrm{e}^{-\beta v}-\mathrm{e}^{-(\mu-\frac{\eta^2}{2\beta^2})v}}{\mu-\frac{\eta^2}{2\beta^2}-\beta}\,,\end{aligned}$$

which is integrable on $\mathbb{R}_+$ by assumptions on the parameters. We can therefore conclude that for every $u<0$, there exists a limit as $t\to\infty$ given by

$$\lim_{t\to\infty}\int_0^t \widetilde{h}(t-s)\phi_{s,t}(u)\,ds = \left(-\int_0^\infty \widetilde{h}(u)\,du + \beta\int_0^\infty \mathrm{e}^{-\beta v}\int_0^v \widetilde{h}(s)\mathrm{e}^{\beta s}\,ds\,dv\right)\mathrm{e}^{\beta u}\,. \tag{5.4}$$

Next, let us consider a fixed $u\in\mathbb{R}_+$. Note that for $t>u$,

$$\widetilde{h}(t-s) \le c\mathrm{e}^{-(\mu-\eta^2/(2\beta^2))(t-s)}\,,$$

and hence for fixed u we will find that $\int_0^u \widetilde{h}(t-s)\,ds \to 0$, when $t\to\infty$. But this yields that pointwise in $u\in\mathbb{R}_+$, $\int_0^u \widetilde{h}(t-s)(\mathrm{e}^{-\beta(t-u)}-1)\,ds \to 0$, when $t\to\infty$. Next, after a change of variables and some straightforward manipulations,

$$\int_u^t \widetilde{h}(t-s)\mathrm{e}^{\beta u}(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds = -\int_0^{t-u} \mathrm{e}^{\beta u}\mathrm{e}^{-\beta t}\widetilde{h}(v)(\mathrm{e}^{\beta v}-1)\,dv\,. \tag{5.5}$$

But for $v\le t-u$, we have

$$\begin{aligned}|\mathrm{e}^{\beta u}\mathrm{e}^{-\beta t}\widetilde{h}(v)(\mathrm{e}^{\beta v}-1)| &\le c\mathrm{e}^{\beta u}\mathrm{e}^{-\beta t}\mathrm{e}^{\beta v}\mathrm{e}^{-(\mu-\eta^2(2\beta^2))v} \\ &\le c\mathrm{e}^{\beta u}\mathrm{e}^{-\beta t}\ \mathrm{e}^{\beta(t-u)}\mathrm{e}^{-(\mu-\eta^2/(2\beta^2))v}\,,\end{aligned}$$

which is integrable in $\mathbb{R}_+$. Thus we can take the limit inside the integral in (5.5) and we prove that the limit of $\int_u^t \widetilde{h}(t-s)\mathrm{e}^{\beta u}(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds$ goes to 0 when t goes to infinity and the statement is proved. □

We note further that the pointwise limit in (5.3) is indeed a function in $L^2(\mathbb{R})$. One could hope that the kernel of chaos 1 converges not only pointwise to the $L^2(\mathbb{R})$-function $\exp(\beta u)\mathbf{1}(u<0)$, but also in $L^2(\mathbb{R})$. We demonstrate that this is *not* the case. Indeed, we have

$$\begin{aligned}
&\Big|\int_0^t \widetilde{h}(t-s)\phi_{s,t}(\cdot)\,ds - k\mathrm{e}^{\beta\cdot}\mathbf{1}(\cdot<0)\Big|_{L^2(\mathbb{R})} \\
&\quad = \Big|\int_0^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta\cdot}\mathbf{1}(\cdot<0) - k\mathrm{e}^{\beta\cdot}\mathbf{1}(\cdot<0) \\
&\qquad + \int_\cdot^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta\cdot}\mathbf{1}(0\le\cdot\le t) \\
&\qquad + \int_0^\cdot \widetilde{h}(t-s)(\mathrm{e}^{-\beta(t-\cdot)}-1)\,ds\mathbf{1}(0\le\cdot\le t)\Big|^2_{L^2(\mathbb{R})} \\
&\quad = \frac{1}{2\beta}\Big|\int_0^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds - k\Big|^2 \\
&\qquad + \int_0^t \Big(\int_u^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta u} + \int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)^2\,du
\end{aligned}$$

The first term goes to zero by definition of k (using the arguments as above). We consider the second term. We have

$$\begin{aligned}
&\int_0^t \Big(\int_u^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta u} + \int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)^2\,du \\
&\quad = \int_0^t \Big(\int_u^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta u}\Big)^2 du + \int_0^t \Big(\int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)^2\,du \\
&\qquad + 2\int_0^t \Big(\int_u^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta u}\Big)\Big(\int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)\,du.
\end{aligned} \tag{5.6}$$

However, for $s\le t$, we have

$$\widetilde{h}(t-s) \ge c(1-e^{-\beta(t-s)})\mathrm{e}^{-t(\mu-\frac{\eta^2}{2\beta})}\mathrm{e}^{s(\mu-\frac{\eta^2}{2\beta})},$$

where c is a strictly positive constant. Thus

$$\Big(\int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)^2 \ge c(1-e^{-\beta t})^4\mathrm{e}^{-2t(\mu-\frac{\eta^2}{2\beta})}(\mathrm{e}^{2u(\mu-\frac{\eta^2}{2\beta})}-1).$$

Integrating and taking the limit when t goes to infinity we get

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t \Big(\int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)^2\,du &\ge \lim_{t\to\infty}\int_0^t c(1-e^{-\beta t})^4\mathrm{e}^{-2t(\mu-\frac{\eta^2}{2\beta})}(\mathrm{e}^{2u(\mu-\frac{\eta^2}{2\beta})}-1)\,du \\
&\ge \lim_{t\to\infty} c(1-e^{-\beta t})^4\mathrm{e}^{-2t(\mu-\frac{\eta^2}{2\beta})}(\mathrm{e}^{2t(\mu-\frac{\eta^2}{2\beta})}-1-t) \\
&= c,
\end{aligned}$$

where c is a strictly positive constant. Since the first and third terms in (5.6) are positive then we have

$$\lim_{t\to\infty}\int_0^t \Big(\int_u^t \widetilde{h}(t-s)(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\beta s})\,ds\mathrm{e}^{\beta u}+\int_0^u \widetilde{h}(t-s)\,ds(\mathrm{e}^{-\beta(t-u)}-1)\Big)^2 du > 0$$

Therefore the kernel function of chaos 1 converges pointwise to a square-integrable function, but not in L^2.

6. Appendix

Proof of Lemma 3.11. We have for $v \le t$, $\omega \le \eta$,

$$\begin{aligned}
-\int_\eta^t \alpha(u)du-\int_\omega^v \alpha(u)du &= -\mu(t-\eta)-\mu(v-\omega)-\int_{-\infty}^{\omega} h(\eta-u,t-u)dW(u)\\
&\quad-\int_{-\infty}^{\omega} h(\omega-u,v-u)dW(u)-\int_\omega^\eta h(\eta-u,t-u)dW(u)\\
&\quad-\int_\eta^v h(0,t-u)dW(u)-\int_\eta^v h(0,v-u)dW(u)\\
&\quad-\int_v^t h(0,t-u)dW(u)-\int_\omega^\eta h(0,v-u)dW(u)\\
&= -\mu(t-\eta)-\mu(v-\omega)\\
&\quad-\int_{-\infty}^{\omega}\{h(\eta-u,t-u)+h(\omega-u,v-u)\}dW(u)\\
&\quad-\int_\eta^v\{h(0,t-u)+h(0,v-u)\}dW(u)\\
&\quad-\int_\omega^\eta\{h(\eta-u,t-u)+h(0,v-u)\}dW(u)-\int_v^t h(0,t-u)dW(u)\,.
\end{aligned}$$

The result follows after using that the exponential of a Wiener integral is lognormally distributed. In the same way we can prove the formula for $\eta \le \omega \le v \le t$ and the result follows.

Proof of Lemma 3.12. By Prop. 1.3.1 in Nualart [14] we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \sigma\mathrm{e}^{-\int_\omega^t \alpha(u)du}\delta W(\omega)\int_0^{t-\delta}\sigma\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\delta W(\omega)\Big]\\
&\quad=\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}\mathrm{e}^{-\int_\omega^t \alpha(u)du}\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}d\omega\Big]\\
&\qquad+\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}\int_0^{t-\delta}(D_\omega\mathrm{e}^{-\int_\eta^t \alpha(u)du})(D_\eta\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du})d\omega d\eta\Big]\\
&\quad=\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}\mathrm{e}^{-\int_\omega^t \alpha(u)du}\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}d\omega\Big]
\end{aligned}$$

$$+\sigma^2\int_0^{t-\delta}\int_0^{t-\delta}\mathbb{E}\Big[\mathrm{e}^{-\int_\eta^t\alpha(u)du}\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\Big]\Big(\int_\eta^t D_\omega\alpha(u)du\Big)\Big(\int_\omega^{t-\delta}D_\eta\alpha(u)du\Big)\,d\omega\,d\eta\,.$$

Here above we have applied the chain rule for Malliavin differentiation and the Fubini theorem. We get the result from (3.17), Lemma 3.11, and considering a measure change.

Proof of Lemma 3.13. Using the duality formula (Eq. (1.42) in Nulalart [14]) and the Fubini Theorem, we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \sigma\mathrm{e}^{-\int_\omega^t\alpha(u)du}\delta W(\omega)\int_0^{t-\delta}\sigma h(0,t-\delta-\omega)\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}d\omega\Big]\\
&\quad=\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}D_\eta\Big(\int_0^{t-\delta}h(0,t-\delta-\omega)\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}d\omega\Big)\mathrm{e}^{-\int_\eta^t\alpha(u)du}\,d\eta\Big]\\
&\quad=-\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}\int_0^{t-\delta}h(0,t-\delta-\omega)\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}D_\eta\Big(\int_\omega^{t-\delta}\alpha(u)du\Big)\mathrm{e}^{-\int_\eta^t\alpha(u)du}\,d\omega\,d\eta\Big]\\
&\quad=-\sigma^2\int_0^t\int_0^{t-\delta}h(0,t-\delta-\omega)h(\max(\eta,\omega)-\eta,t-\delta-\eta)\mathbb{E}\Big[\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\mathrm{e}^{-\int_\eta^t\alpha(u)du}\Big]\,d\omega\,d\eta
\end{aligned}$$

and we get the result by applying Lemma 3.11 and considering a measure change.

Proof of Lemma 3.14. Using the duality formula (Eq. (1.42) in Nulalart [14]) and the Fubini Theorem, we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \sigma h(0,t-\omega\mathrm{e}^{-\int_\omega^t\alpha(u)du}d\omega\int_0^{t-\delta}\sigma\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\delta W(\omega)\Big]\\
&\quad=\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}D_\eta\Big(\int_0^t h(0,t-\omega)\mathrm{e}^{-\int_\omega^t\alpha(u)du}d\omega\Big)\mathrm{e}^{-\int_\eta^{t-\delta}\alpha(u)du}\,d\eta\Big]\\
&\quad=-\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}\int_0^{t-\delta}h(0,t-\omega)\mathrm{e}^{-\int_\omega^t\alpha(u)du}D_\eta\Big(\int_\omega^t\alpha(u)du\Big)\mathrm{e}^{-\int_\eta^{t-\delta}\alpha(u)du}\,d\omega\,d\eta\Big]\\
&\quad=-\sigma^2\int_0^{t-\delta}\int_0^{t-\delta}h(0,t-\omega)h(\max(\eta,\omega)-\eta,t-\eta)\mathbb{E}\Big[\mathrm{e}^{-\int_\omega^t\alpha(u)du}\mathrm{e}^{-\int_\eta^{t-\delta}\alpha(u)du}\Big]\,d\omega\,d\eta
\end{aligned}$$

and we get the result by applying Lemma 3.11 and considering a measure change.

Proof of Lemma 3.15. From (3.11), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t\sigma h(0,t-\eta)\mathrm{e}^{-\int_\eta^t\alpha(u)du}d\eta\int_0^{t-\delta}\sigma h(0,t-\delta-\omega)\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\,d\omega\Big]\\
&\quad=\sigma^2\mathbb{E}\Big[\int_0^{t-\delta}\int_0^{t-\delta}h(0,t-\eta)h(0,t-\delta-\omega)\mathrm{e}^{-\int_\eta^t\alpha(u)du}\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\,d\omega\,d\eta\Big]\\
&\qquad+\sigma^2\mathbb{E}\Big[\int_{t-\delta}^t h(0,t-\eta)\mathrm{e}^{-\int_\eta^t\alpha(u)du}\,d\eta\int_0^{t-\delta}h(0,t-\delta-\omega)^{-\int_\omega^{t-\delta}\alpha(u)du}\,d\omega\Big]
\end{aligned}$$

$$= \sigma^2 \int_0^{t-\delta} \int_0^{t-\delta} h(0,t-\eta)h(0,t-\delta-\omega)\mathbb{E}\Big[\mathrm{e}^{-\int_\eta^t \alpha(u)du}\mathrm{e}^{-\int_\omega^{t-\delta}\alpha(u)du}\Big]\, d\omega\, d\eta$$
$$+ \sigma^2 \int_{t-\delta}^{t} h(0,t-\eta)\mathrm{e}^{-\mu(t-\eta)+\frac{1}{2}\int_0^\infty h^2(u,t-\eta+u)du+\frac{1}{2}\int_0^{t-\eta} h^2(0,u)du}\, d\eta$$
$$\times \int_0^{t-\delta} h(0,t-\delta-\omega)\mathrm{e}^{-\mu(t-\delta-\omega)+\frac{1}{2}\int_0^\infty h^2(u,t-\delta-\omega+u)du+\frac{1}{2}\int_0^{t-\delta-\omega} h^2(0,u)du}\, d\omega$$

and the result follows by a change of variable and by appealing to Lemma 3.11.

Proof of Proposition 3.16. Collecting together the results for the expectations in (3.21)-(3.24) in Lemmas 3.12-3.15 and using (3.18) we get the result for $\mathbb{E}[X(t)X(t-\delta)]$, $\delta > 0$. Then applying the dominated convergence theorem we get the limit when t goes to ∞.

Proof of Lemma 3.17. We have

$$\int_0^\infty (h(x+u,x+y+u)+h(x,z+x+u))^2 du = \frac{\eta^2}{2\beta^2}\{1-\mathrm{e}^{-\beta(x+y)}-\mathrm{e}^{-\beta(x+z)}+\mathrm{e}^{-\beta x}\}^2,$$

$$\int_0^x \{h(u,u+y)+h(0,u+z)\}^2 du$$
$$= \frac{\eta^2}{2\beta^3}\Big(1-\mathrm{e}^{-2\beta x}\Big)\Big(1+\mathrm{e}^{-2\beta y}-2\mathrm{e}^{-\beta y}+\mathrm{e}^{-2\beta z}-2\mathrm{e}^{-\beta z}+2\mathrm{e}^{-\beta(y+z)}\Big)$$
$$+\frac{2\eta^2}{\beta^3}(1-\mathrm{e}^{-\beta x})(1-\mathrm{e}^{-\beta z}-\mathrm{e}^{-\beta y})+\frac{\eta^2}{\beta^2}x,$$

$$\int_0^z \{h(0,y-u)+h(0,z-u)\}^2 du = \frac{4\eta^2}{\beta^2}z+\frac{\eta^2}{2\beta^3}(1-\mathrm{e}^{-2\beta z})(1+\mathrm{e}^{-2\beta(y-z)}+2\mathrm{e}^{-\beta(y-z)})$$
$$+\frac{4\eta^2}{\beta^3}\mathrm{e}^{-\beta y}(1-\mathrm{e}^{\beta z})+\frac{4\eta^2}{\beta^3}(\mathrm{e}^{-\beta z}-1),$$

$$\int_0^{y-z} h^2(0,y-z-u)du = \frac{\eta^2}{\beta^2}(y-z)-\frac{3\eta^2}{2\beta^3}(1-\mathrm{e}^{-2\beta(y-z)}).$$

The expressions for $\widetilde{H}(x,y,z)$ and $G(x)$ follow immediately. Thus the statement is proved.

Proof of Proposition 4.1. Recall the expression of X in (3.8). We have

$$X(t)-X(s) = X(s)(\mathrm{e}^{-\int_s^t \alpha(v)dv}-1)+\int_s^t \sigma\mathrm{e}^{-\int_u^t \alpha(v)dv}\delta W(u)$$
$$-\int_s^t \sigma h(0,t-u)\mathrm{e}^{-\int_u^t \alpha(v)dv}du\,. \tag{6.1}$$

Taking the conditional expectation, we get using (3.11)

$$
\begin{aligned}
\mathbb{E}[X(t)-X(s)|X(s)=x] &= x\mathbb{E}[\mathrm{e}^{-\int_s^t \alpha(v)dv}-1] - \int_s^t \sigma h(0,t-u)\mathbb{E}[\mathrm{e}^{-\int_u^t \alpha(v)dv}]du \\
&= x(\mathrm{e}^{-\mu(t-s)+\frac{1}{2}\int_0^\infty h^2(u,t-s+u)du+\frac{1}{2}\int_0^{t-s} h^2(0,u)du}-1) \\
&\quad - \int_s^t \sigma h(0,t-u)\mathrm{e}^{-\mu(t-u)}\mathrm{e}^{\frac{1}{2}\int_0^\infty h^2(\eta,t-u+\eta)d\eta+\frac{1}{2}\int_0^{t-u} h^2(0,\eta)d\eta}du \\
&= x(\mathrm{e}^{-\mu(t-s)+\frac{1}{2}\int_0^\infty h^2(u,t-s+u)du+\frac{1}{2}\int_0^{t-s} h^2(0,u)du}-1) \\
&\quad - \int_0^{t-s} \sigma h(0,v)\mathrm{e}^{-\mu v+\frac{1}{2}\int_0^\infty h^2(u,v+u)du+\frac{1}{2}\int_0^{v} h^2(0,u)du}dv
\end{aligned}
$$

and the result follows.

Proof of Proposition 4.2. We have from the definition of the function h that $h(u,t-s+u)=\int_u^{t-s+u} g(v)dv$. If $\int_0^\infty |g(v)|dv<\infty$ then by dominated convergence theorem, we deduce that $h^2(u,t-s+u)$ and $\int_0^{t-s} h^2(0,u)du$ converge to 0 when t goes to s. Hence by (4.2) and applying again the dominated convergence theorem we deduce that $f(t-s)$ vanishes when t goes to s. Thus to compute the limit in (4.4), we use the l'Hôpital's rule. We compute

$$
\frac{\partial f(t-s)}{\partial t} = f(t-s)\Big(-\mu+\int_0^\infty h(u,t-s+u)g(t-s+u)du+\frac{1}{2}h^2(0,t-s)\Big)\,.
$$

$$
\frac{\partial \widetilde{f}(t-s)}{\partial t} = -\sigma h(0,t-s)f(t-s)\,.
$$

Then from (4.3) and following the same arguments as before the result follows.

Proof of Lemma 4.3. The result follows immediately using the expression of $X(t)-X(s)$ (Eq. (6.1)).

Proof of Proposition 4.4. Following the same computations as in the proofs of Lemma 3.6, Lemma 3.7, and Lemma 3.8, it holds that

$$
\begin{aligned}
\mathcal{I}_1 &= \int_s^t \sigma^2\mathrm{e}^{-2\mu(t-u)+H(0,t-u)}du \\
&\quad + \int_s^t\int_s^u \sigma^2 h(0,t-u)h(u-v,t-v)\mathrm{e}^{-2\mu(t-u)-\mu(u-v)+H(u-v,t-v)}dvdu \\
&\quad + \int_s^t\int_u^t \sigma^2 h(v-u,t-u)h(0,t-v)\mathrm{e}^{-2\mu(t-v)-\mu(v-u)+H(v-u,t-v)}dvdu\,,
\end{aligned}
$$

$$
\mathcal{I}_2 = -\int_s^t\int_s^u \sigma^2 h(0,t-u)h(u-v,t-v)\mathrm{e}^{-2\mu(t-u)-\mu(u-v)+H(u-v,t-u)}dvdu
$$

$$-\int_s^t\int_u^t \sigma^2 h(0,t-v)h(0,t-u)\mathrm{e}^{-2\mu(t-v)-\mu(v-u)+H(v-u,t-v)}dvdu\,,$$

and

$$\mathcal{I}_3=\int_s^t\int_s^u \sigma^2 h(0,t-u)h(0,t-v)\mathrm{e}^{-2\mu(t-u)-\mu(u-v)+H(u-v,t-u)}dvdu$$
$$-\int_s^t\int_u^t \sigma^2 h(0,t-v)h(0,t-u)\mathrm{e}^{-2\mu(t-v)-\mu(v-u)+H(v-u,t-v)}dvdu\,.$$

Thus collecting $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ together, we get

$$\begin{aligned}\mathcal{I}_1-2\mathcal{I}_2+\mathcal{I}_3=&\int_0^{t-s}\sigma^2\exp(-2\mu v+H(0,v))dv\\&+\int_0^{t-s}\int_0^v \sigma^2 h(0,u)\mathrm{e}^{-2\mu(u)-\mu(v-u)+H(v-u,u)}\big(3h(v-u,v)+h(0,v)\big)dudv\\&+\int_0^{t-s}\int_0^v \sigma^2 h(0,v-u)\mathrm{e}^{-2\mu(v-u)-\mu(u)+H(u,v-u)}\big(3h(0,v)+h(u,v)\big)dudv\,.\end{aligned}$$

By $(\mathcal{B})$ and applying dominated convergence theorem, the latter vanishes when t goes to s. Moreover we have

$$\begin{aligned}\frac{\partial(\mathcal{I}_1-2\mathcal{I}_2+\mathcal{I}_3)}{\partial t}=&\,\sigma^2\exp(-2\mu(t-s)+H(0,t-s))\\&+\int_0^{t-s}\sigma^2 h(0,u)\mathrm{e}^{-2\mu(u)-\mu(v-u)+H(v-u,u)}(3h(v-u,v)+h(0,v))dudv\\&+\int_0^{t-s}\sigma^2 h(0,v-u)\mathrm{e}^{-2\mu(v-u)-\mu(u)+H(u,v-u)}(3h(0,v)+h(u,v))dudv\,.\end{aligned}$$

Thus using $(\mathcal{C})$ and l'Hôpital's rule, we get $\lim_{t\to s}\frac{\mathcal{I}_1-2\mathcal{I}_2+\mathcal{I}_3}{t-s}=\sigma^2$. In other hand, using Lemma 3.5, we get

$$\begin{aligned}\mathcal{I}_4=\sigma\int_0^t h(0,t-u)(&\mathrm{e}^{-2\mu(t-u)-\mu(u-s)+H(u-s,t-u)}\\&-\mathrm{e}^{\mu(t-u)+\frac{1}{2}\int_0^\infty h^2(v,t-u+v)dv+\frac{1}{2}\int_0^{t-u}h^2(0,v)dv})du\end{aligned}$$

Using integration by parts for Skorohod integral and Lemma 3.5, we get

$$\mathcal{I}_5=-\sigma\int_s^t h(0,t-u)\mathrm{e}^{-2\mu(t-u)-\mu(u-s)+H(u-s,t-u)}du\,.$$

Hence collecting $\mathcal{I}_4$ and $\mathcal{I}_6$ together, we find

$$-2x\mathcal{I}_4+2x\mathcal{I}_5=-4x\sigma\int_0^{t-s}h(0,v)\mathrm{e}^{-2\mu v-\mu(t-v)+H(t-v,v)}dv-2x\widetilde{f}(t-s)\,,$$

where $\widetilde{f}(v)$ is as defined in Prop. 4.2. Using the "Key formula" in (3.18), we have

$$-2x\mathcal{I}_4 + 2x\mathcal{I}_5 = -4x\sigma \int_0^{t-s} h(0,v)\mathrm{e}^{-2\mu v + H(0,v)} dv - 2x\widetilde{f}(t-s)$$
$$- 4x\sigma \int_0^{t-s} \int_0^{v} h(0,u)\mathrm{e}^{-2\mu u - \mu(v-u) + H(v-u,u)} \Big(-\mu + \frac{\partial}{\partial v} H(v-u,u) \Big) du\, dv\,.$$

Using $(\mathcal{C})$ and dominated convergence theorem the latter vanishes when t goes to 0. We compute using (3.18),

$$\frac{\partial(-2x\mathcal{I}_4 + 2x\mathcal{I}_5)}{\partial t} = -4x\sigma h(0,t-s)\mathrm{e}^{-2\mu(t-s) + H(0,t-s)} - 2x\sigma h(0,t-s) f(t-s)$$
$$- 4x\sigma \int_0^{t-s} h(0,u)\mathrm{e}^{-2\mu u - \mu(t-s-u) + H(t-s-u,u)}$$
$$\Big(-\mu + \frac{\partial}{\partial(t-s)} H(t-s-u,u) \Big) du$$
$$= -4x\sigma h(0,t-s)\mathrm{e}^{-2\mu(t-s) + H(0,t-s)} - 2x\sigma h(0,t-s) f(t-s)$$
$$- 4x\sigma \int_0^{t-s} h(0,v)\mathrm{e}^{-2\mu v + H(0,v)} \Big(-\mu + \frac{\partial}{\partial v} H(0,v) \Big)\, dv$$
$$- 4x\sigma \int_0^{t-s} \int_0^{v} h(0,u) \Big\{ \mathrm{e}^{-2\mu u - \mu(v-u) + H(v-u,u)}$$
$$\Big(-\mu + \frac{\partial}{\partial v} H(v-u,u) \Big)^2 + \frac{\partial^2}{\partial v^2} H(v-u,u) \Big\}\, du\, dv\,.$$

Applying the l'Hôpital's rule, $(\mathcal{C})$, and the dominated convergence theorem, we get

$$\lim_{t\to s} \frac{-2x\mathcal{I}_4 + 2x\mathcal{I}_5}{t-s} = 0.$$

Using the same computations as before, it is easy to show that if (4.2), (4.3), and $(\mathcal{A})$ hold true then $\lim_{t\to s} x^2 \frac{\mathcal{I}_6}{t-s} = 0$ and the statement is proved.

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