Integrability conditions for space-time stochastic integrals: theory and applications

Carsten Chong* and Claudia Klüppelberg†

May 14, 2014

Abstract

We derive explicit integrability conditions for stochastic integrals taken over time and space driven by a random measure. Our main tool is a canonical decomposition of a random measure which extends the results from the purely temporal case. We show that the characteristics of this decomposition can be chosen as predictable strict random measures, and we compute the characteristics of the stochastic integral process. We apply our conditions to a variety of examples, in particular to ambit processes, which represent a rich model class.

AMS 2010 Subject Classifications: primary: 60G48, 60G57, 60H05
secondary: 60G10, 60G60, 60H15

Keywords: ambit process, continuous-time moving average, integrability conditions, Lévy basis, martingale measure, predictable characteristics, random measure, stochastic integration, stochastic partial differential equation, supCARMA, supCOGARCH, supOU, Volterra process

*Center for Mathematical Sciences, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany, e-mail: carsten.chong@tum.de, url: www-m4.ma.tum.de/personen/doktoranden/carsten-chong/

†Center for Mathematical Sciences, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany, e-mail: cklu@ma.tum.de, url: www-m4.ma.tum.de/

1 Introduction

Following Itô's seminal paper [26], stochastic integration theory w.r.t. semimartingales was brought to maturity during the 1970s and 1980s. One of the fundamental results in this area is the Bichteler-Dellacherie theorem, which shows the equivalence between the class of semimartingales and the class of finite L^0-random measures. As a consequence, semimartingales constitute the largest class of integrators that allow for stochastic integrals of predictable integrands satisfying the dominated convergence theorem. The natural analogue to semimartingale integrals in a space-time setting are integrals of the form

$$\int_{\mathbb{R}\times E} H(t,x)\, M(\mathrm{d}t,\mathrm{d}x), \tag{1.1}$$

where E is some space and M is an L^0-random measure on $\mathbb{R}\times E$. The construction of such integrals is discussed in [15] in its full generality, so the theory is complete from this point of view.

However, whether H is integrable w.r.t. M or not, depends on whether

$$\lim_{r\to 0}\sup\left\{\mathbb{E}\left[\left|\int S\,\mathrm{d}M\right|\wedge 1\right] : |S|\leq |rH|, S \text{ is a simple integrand}\right\}=0 \tag{1.2}$$

or not, a property which is hard to check. Thus, the aim of this paper is to characterize (1.2) in terms of equivalent conditions, which can be verified in concrete situations. In the purely temporal case, this subject is addressed in [12]. The result there is obtained by using the local semimartingale characteristics corresponding to a random measure. Our approach parallels this method, but it turns out that the notion of characteristics in the space-time setting is much more complex. We will show that, if M has different times of discontinuity (cf. Definition 3.1 below), we can associate a characteristic triplet to it consisting of strict random measures (cf. Definition 2.1(3)) that are jointly σ-additive in space and time. Moreover, we will determine the characteristics of stochastic integral processes, which is more involved than in the temporal case, since a concept is needed to merge space and time appropriately. Having achieved this step, integrability conditions in the same fashion as in [12, 46] can be given for space-time integrals. We will also compare our results to those of [46], [51] and [27].

Applications of our theoretical results will be chosen from the class of ambit processes

$$Y(t,x) := \int_{\mathbb{R}\times\mathbb{R}^d} h(t,s;x,y)\sigma(s,y)\, M(\mathrm{d}s,\mathrm{d}y), \quad t\in\mathbb{R}, x\in\mathbb{R}^d, \tag{1.3}$$

which have been suggested for modelling physical space-time phenomena like turbulence, see e.g. [9]. In the case, where $\sigma = 1$ and M is a Lévy basis (see Remark 4.4), such multiparameter integrals have already been investigated by many authors: for instance, [17, 36, 47] discuss path properties of the resulting process Y, while [23, 40] address the extremal behaviour of Y; mixing conditions are examined in [25].

As a broad model class, the applications of ambit processes go far beyond turbulence modelling. For example, [43] describes the movement of relativistic quantum particles by equations of the form (1.3). Moreover, solutions to stochastic partial differential equations driven by random noise are often of the form (1.3), cf. [9, 51] and Section 5.2. Furthermore, stochastic processes like forward contracts in bond and electricity markets based on a Heath-Jarrow-Morten approach also rely on a spatial structure, cf. [10, 2]. Other applications include brain imaging [30] and tumor growth [5, 29].

The concept of an ambit process has also been successfully invoked to define superpositions of stochastic processes like Ornstein-Uhlenbeck processes or, more generally, continuous-time ARMA (CARMA) processes. In these models, only integrals of deterministic integrands w.r.t. Lévy bases are involved, so the integration theory of [46] is sufficient. Our integrability conditions, however, allow for a volatility modulation of the noise, which generates a greater model flexibility. Moreover, in [13] ambit processes have been used to define superpositions of continuous-time GARCH (COGARCH) processes. In its simplest case superposition leads to multi-factor models, economically and statistically necessary extensions of the one-factor models; cf. [28]. As we shall see, the supCOGARCH model again needs the integrability criteria we have developed since for this model the volatility σ and the random measure M are not independent.

Our paper is organized as follows. Section 2 introduces the notation and gives a summary on the concept of a random measure and its stochastic integration theory. Section 3 derives a canonical decomposition for random measures as known for semimartingales and calculates the characteristic triplet of stochastic integral processes. Section 4 presents integrability conditions in terms of the characteristics from Section 3. Section 5 is dedicated to examples to highlight our results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, P)$ be a stochastic basis satisfying the usual assumptions of completeness and right-continuity. Denote the base space by $\bar{\Omega} := \Omega \times \mathbb{R}$ and the optional (resp. predictable) σ-field on $\bar{\Omega}$ by $\mathcal{O}$ (resp. $\mathcal{P}$). Furthermore, fix some Lusin space E, equipped with its Borel σ-field $\mathcal{E}$. Using the abbreviations $\tilde{\Omega} := \Omega \times \mathbb{R} \times E$ and $\tilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{E}$ (resp. $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$), we call a function $H\colon \tilde{\Omega} \to \mathbb{R}$ **optional** (resp. **predictable**) if it is $\tilde{\mathcal{O}}$-measurable (resp. $\tilde{\mathcal{P}}$-measurable). We will often use the symbols $\mathcal{O}$ and $\mathcal{P}$ (resp. $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{P}}$) also for the collection of optional and predictable functions from $\bar{\Omega}$ (resp. $\tilde{\Omega}$) to $\mathbb{R}$. We refer to [27, Chap. I and II] for all notions not explicitly explained.

Some further notational conventions: we write $A_t := A \cap (\Omega \times (-\infty, t])$ for $A \in \mathcal{P}$, and $\tilde{A}_t := A \cap (\Omega \times (-\infty, t] \times E)$ for $\tilde{A} \in \tilde{\mathcal{P}}$. $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ denotes the collection of bounded Borel sets in $\mathbb{R}^d$. Next, if μ is a signed measure and X a finite variation process, we write $|\mu|$ and $|X|$ for the variation of μ and the variation process of X, respectively. Finally, we equip $L^p = L^p(\Omega, \mathcal{F}, P)$, $p \in [0, \infty)$, with the topology induced by

$$\|X\|_p := \mathbb{E}[|X|^p]^{1/p}, \quad p \geq 1, \quad \|X\|_p := \mathbb{E}[|X|^p], \quad 0 < p < 1, \quad \|X\|_0 := \mathbb{E}[|X| \wedge 1]$$

for $X \in L^p$. Among several definitions of a random measure in the literature, the following two are the most frequent ones: in essence, a random measure is either a random variable whose realizations are measures on some measurable space (e.g. [27, 31]) or it is a σ-additive set function with values in the space L^p (e.g. [15, 33, 39, 46, 51]). Our terminology is as follows:

Definition 2.1 Let $(\tilde{O}_k)_{k\in\mathbb{N}}$ be a sequence of sets in $\tilde{\mathcal{P}}$ with $\tilde{O}_k \uparrow \tilde{\Omega}$. Set $\tilde{\mathcal{P}}_M := \bigcup_{k=1}^\infty \tilde{\mathcal{P}}|_{\tilde{O}_k}$, which is the collection of all sets $A \in \tilde{\mathcal{P}}$ such that $A \subseteq \tilde{O}_k$ for some $k \in \mathbb{N}$.

(1) An L^p**-random measure** on $\mathbb{R} \times E$ is a mapping $M\colon \tilde{\mathcal{P}}_M \to L^p$ satisfying:

(a) $M(\emptyset) = 0$ a.s.,

(b) For every sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\tilde{\mathcal{P}}_M$ with $\bigcup_{i=1}^\infty A_i \in \tilde{\mathcal{P}}_M$ we have

$$M\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty M(A_i) \quad \text{in } L^p.$$

(c) For all $A \in \tilde{\mathcal{P}}_M$ with $A \subseteq \tilde{\Omega}_t$ for some $t \in \mathbb{R}$, the random variable $M(A)$ is $\mathcal{F}_t$-measurable.

(d) For all $A \in \tilde{\mathcal{P}}_M$, $t \in \mathbb{R}$ and $F \in \mathcal{F}_t$, we have

$$M\big(A \cap (F \times (t, \infty) \times E)\big) = 1_F M\big(A \cap (\Omega \times (t, \infty) \times E)\big) \quad \text{a.s.}$$

(2) If $p = 0$, we only say **random measure**; if $\tilde{O}_k$ can be chosen as $\tilde{\Omega}$ for all $k \in \mathbb{N}$, M is called a **finite** random measure; and finally, if E consists of only one point, M is called a **null-spatial** random measure.

(3) A **strict random measure** is a signed transition kernel $\mu(\omega, \mathrm{d}t, \mathrm{d}x)$ from $(\Omega, \mathcal{F})$ to $(\mathbb{R} \times E, \mathcal{B}(\mathbb{R}) \otimes \mathcal{E})$ with the following properties:

(a) There is a strictly positive function $V \in \tilde{\mathcal{P}}$ such that $\int_{\mathbb{R}\times E} V(t, x)\,|\mu|(\mathrm{d}t, \mathrm{d}x) \in L^1$.

(b) For $\tilde{\mathcal{O}}$-measurable functions W such that W/V is bounded, the process

$$W * \mu_t := \int_{(-\infty,t]\times E} W(s, x)\,\mu(\mathrm{d}s, \mathrm{d}x), \quad t \in \mathbb{R},$$

is optional.

Remark 2.2

(1) If we can choose $O_k = \Omega \times O'_k$ with $O'_k \uparrow \mathbb{R} \times E$, one popular choice for $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is the **natural filtration** $(\mathcal{F}^M_t)_{t\in\mathbb{R}}$ of M which is the smallest filtration satisfying the usual assumptions such that for all $t \in \mathbb{R}$ we have $M(\Omega \times B) \in \mathcal{F}^M_t$ if $B \subseteq ((-\infty, t] \times E) \cap O'_k$ with some $k \in \mathbb{N}$.

(2) If μ is a positive transition kernel in Definition 2.1(3), μ is an optional $\tilde{\mathcal{P}}$-σ-finite random measure in the sense of [27, Chap. II], where also the predictable compensator of a strict random measure is defined. Obviously, a strict random measure *is* a random measure. For more details on that, see also [15, Ex. 5 and 6]. □

Stochastic integration theory in space-time w.r.t. L^p-random measures is discussed in [15], see also [14]. The special case of L^2-integration theory is also discussed in [21, 51]. Let us recall the details involved: a **simple integrand** is a function $\tilde{\Omega} \to \mathbb{R}$ of the form

$$S := \sum_{i=1}^{r} a_i 1_{A_i}, \quad r \in \mathbb{N}, a_i \in \mathbb{R}, A_i \in \tilde{\mathcal{P}}_M, \tag{2.1}$$

for which the stochastic integral w.r.t. M is canonically defined as

$$\int S\,\mathrm{d}M := \sum_{i=1}^{r} a_i M(A_i). \tag{2.2}$$

Now consider the collection $\mathcal{S}^{\uparrow}_M$ of positive functions $\tilde{\Omega} \to \mathbb{R}$ which are the pointwise supremum of simple integrands and define the **Daniell mean** $\|\cdot\|^{\mathrm{D}}_{M,p} \colon \mathbb{R}^{\tilde{\Omega}} \to [0, \infty]$ by

- $\|K\|^{\mathrm{D}}_{M,p} := \sup_{S\in\mathcal{S}_M, |S|\leq K} \left\| \int S\,\mathrm{d}M \right\|_p$, if $K \in \mathcal{S}^{\uparrow}_M$, and
- $\|H\|^{\mathrm{D}}_{M,p} := \inf_{K\in\mathcal{S}^{\uparrow}_M, |H|\leq K} \|K\|^{\mathrm{D}}_{M,p}$ for arbitrary functions $H \colon \tilde{\Omega} \to \mathbb{R}$.

An arbitrary function $H\colon \tilde{\Omega} \to \mathbb{R}$ is called **integrable** w.r.t. M if there is a sequence of simple integrands $(S_n)_{n\in\mathbb{N}}$ such that $\|H - S_n\|^{\mathrm{D}}_{M,p} \to 0$ as $n \to \infty$. Then the **stochastic integral** of H w.r.t. M defined by

$$\int H \,\mathrm{d}M := \int_{\mathbb{R}\times E} H(t,x)\, M(\mathrm{d}t, \mathrm{d}x) := \lim_{n\to\infty} \int S_n \,\mathrm{d}M \tag{2.3}$$

exists in L^p and does not depend on the choice of $(S_n)_{n\in\mathbb{N}}$. The collection of integrable functions is denoted by $L^{1,p}(M)$ and can be characterized as follows [14, Thm. 3.4.10 and 3.2.24]:

Theorem 2.3. *Let $F^{1,p}(M)$ be the collection of functions H with $\|rH\|^{\mathrm{D}}_{M,p} \to 0$ as $r \to 0$. If we identify two functions coinciding up to a set whose indicator function has Daniell mean 0, then*

$$L^{1,p}(M) = \tilde{\mathcal{P}} \cap F^{1,p}(M). \tag{2.4}$$

Moreover, the following dominated convergence theorem holds: Let $(H_n)_{n\in\mathbb{N}}$ be a sequence in $L^{1,p}(M)$ converging pointwise to some limit H. If there exists some function $F \in F^{1,p}(M)$ with $|H_n| \le F$ for each $n \in \mathbb{N}$, both H and H_n are integrable with $\|H - H_n\|^{\mathrm{D}}_{M,p} \to 0$ as $n \to \infty$ and

$$\int H \,\mathrm{d}M = \lim_{n\to\infty} \int H_n \,\mathrm{d}M \quad \text{in } L^p. \tag{DCT}$$

Given a predictable function $H \in \tilde{\mathcal{P}}$, we can obviously define a new random measure $H.M$ in the following way:

$$K \in L^{1,0}(H.M) :\Leftrightarrow KH \in L^{1,0}(M), \qquad \int K \,\mathrm{d}(H.M) := \int KH \,\mathrm{d}M. \tag{2.5}$$

This indeed defines a random measure provided there exists a sequence $(\tilde{O}_k)_{k\in\mathbb{N}} \subseteq \tilde{\mathcal{P}}$ with $\tilde{O}_k \uparrow \tilde{\Omega}$ and $1_{\tilde{O}_k} \in L^{1,0}(H.M)$ for all $k \in \mathbb{N}$. But this construction does not extend the class $L^{1,0}(M)$ of integrable functions w.r.t. M. However, as shown in [15, §3], $L^{1,p}(M)$ can indeed be extended further in the following way. Given an L^p-random measure M, fix some $\tilde{\mathcal{P}}$-measurable function H such that:

$$\text{There exists a predictable process } K\colon \bar{\Omega} \to \mathbb{R}, K > 0, \text{ such that } KH \in L^{1,p}(M). \tag{2.6}$$

Now set $\bar{O}_k := \{K \ge k^{-1}\}$ for $k \in \mathbb{N}$, which obviously defines predictable sets increasing to $\bar{\Omega}$, and then $\mathcal{P}_{H\cdot M} := \{A \in \mathcal{P} \colon A \subseteq \bar{O}_k \text{ for some } k \in \mathbb{N}\}$. Then we define a new null-spatial L^p-random measure by

$$H \cdot M \colon \mathcal{P}_{H\cdot M} \to L^p, (H \cdot M)(A) := \int 1_A H \,\mathrm{d}M.$$

The following is known from [15], see also [12, Thm. 2.4]:

(1) If $H \in L^{1,p}(M)$, $H \cdot M$ is a finite L^p-random measure and $\int 1 \,\mathrm{d}(H \cdot M) = \int H \,\mathrm{d}M$.

(2) If $G\colon \bar{\Omega} \to \mathbb{R}$ is a predictable process, we have $G \in L^{1,p}(H \cdot M)$ if and only if $\|rGH\|_{M,p} \to 0$ as $r \to 0$, where for every $\tilde{\mathcal{P}}$-measurable function we set

$$\|H\|_{M,p} := \sup_{\substack{F\colon \bar{\Omega}\to\mathbb{R} \text{ predictable,} \\ |F|\le 1, FH \in L^{1,p}(M)}} \left\| \int FH \,\mathrm{d}M \right\|_p. \tag{2.7}$$

In this case we have $\int G \,\mathrm{d}(H \cdot M) = \int GH \,\mathrm{d}M$.

Therefore, it is reasonable to extend the set of **integrable** functions w.r.t. M from $L^{1,p}(M)$ to

$$L^p(M) = \{H \in \tilde{\mathcal{P}} \colon H \text{ satisfies (2.6) and } \|rH\|_{M,p} \stackrel{r\to 0}{\longrightarrow} 0\} \tag{2.8}$$

by setting

$$\int H \,\mathrm{d}M := (H \cdot M)(\bar{\Omega}), \quad H \in L^p(M).$$

We remark that in the null-spatial case $L^{1,0}(M) = L^0(M)$. But in general, the inclusion $L^{1,p}(M) \subseteq L^p(M)$ is strict, see [15, §3b] and Example 4.7 below.

Let us also remark that [20] introduces a stochastic integral for a Gaussian random measure where the integrands are allowed to be distribution-valued. It is still an open question whether it is possible to extend this to the general setting of L^p-random measures, in particular if $p < 2$; we do not pursue this direction in the present paper.

In the sequel we will frequently use the following fact [12, Ex. 4.1]: If M is a finite random measure, the process $(M(\tilde{\Omega}_t))_{t\in\mathbb{R}}$ has a càdlàg modification, which is then a semimartingale up to infinity w.r.t. to the underlying filtration (see [12, Section 3] for a definition). This semimartingale will be also be denoted by $M = (M_t)_{t\in\mathbb{R}}$.

3 Predictable characteristics of random measures

Let us introduce three important subclasses of random measures:

Definition 3.1 Let M be a random measure where $\tilde{O}_k = O_k \times E_k$ with $O_k \uparrow \bar{\Omega}$ and $E_k \uparrow E$. Set $\mathcal{E}_M := \bigcup_{k=1}^{\infty} \mathcal{E}|_{E_k}$.

(1) M has **different times of discontinuity** if for all $k \in \mathbb{N}$ and disjoint sets $U_1, U_2 \in \mathcal{E}_M$ the semimartingales $1_{O_k \times U_i} \cdot M$, $i = 1, 2$, a.s. never jump at the same time.

(2) M is called **orthogonal** if for all pairs of disjoint sets $U_1, U_2 \in \mathcal{E}_M$ and $k \in \mathbb{N}$ we have $[(1_{O_k\times U_1} \cdot M)^{\mathrm{c}}, (1_{O_k\times U_2} \cdot M)^{\mathrm{c}}] = 0$.

(3) M has **no fixed time of discontinuity** if for all $U \in \mathcal{E}_M$, $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we have $\Delta(1_{O_k\times U} \cdot M)_t = 0$ a.s.

In the next theorem we prove a canonical decomposition for random measures with different times of discontinuity generalizing the results of [27] and [12]. Without this extra assumption on the random measure, only non-explicit results such as [15, Thm. 4.21] or results for $p \geq 2$ as in [35, Thm. 1] are known. We also remark that the integrability conditions in Theorem 4.1 will be stated in terms of this decomposition. Some notation beforehand: we write $\mathcal{B}_0(\mathbb{R})$ for the collection of Borel sets on $\mathbb{R}$ which are bounded away from 0. Furthermore, if X is a semimartingale up to infinity, we write $\mathfrak{B}(X)$ for its first characteristic, $[X]$ for its quadratic variation, X^{c} for its continuous part (all of them starting at $-\infty$ with 0), μ^X for its jump measure and ν^X for its predictable compensator. Finally, if $U \in \mathcal{E}$, $M|_U$ denotes the random measure given by $M|_U(A) = M(A \cap (\bar{\Omega} \times U))$ for $A \in \tilde{\mathcal{P}}_M$.

Theorem 3.2. *Let M have different times of discontinuity.*

(1) The mappings

$$B(A) := \mathfrak{B}(1_A \cdot M)_\infty, \quad M^{\mathrm{c}}(A) := (1_A \cdot M)^{\mathrm{c}}_\infty, \quad A \in \tilde{\mathcal{P}}_M,$$

are random measures, the mapping

$$C(A;B) := [(1_A \cdot M)^{\mathrm{c}}, (1_B \cdot M)^{\mathrm{c}}]_\infty, \quad A \in \tilde{\mathcal{P}}_M,$$

is a random bimeasure (i.e. a random measure in both arguments when the other one is fixed) and

$$\mu(A,V) := \mu^{1_A \cdot M}(\mathbb{R} \times V), \quad \nu(A,V) := \nu^{1_A \cdot M}(\mathbb{R} \times V), \quad A \in \tilde{\mathcal{P}}_M, V \in \mathcal{B}_0(\mathbb{R}), \tag{3.1}$$

*can be extended to random measures on $\tilde{\mathcal{P}}_M \otimes \mathcal{B}_0(\mathbb{R})$. Moreover, (B,C,ν) can be chosen as predictable strict random (bi-)measures and form the **characteristic triplet** of M.*

(2) *Let $A \in \tilde{\mathcal{P}}_M$ and τ be a truncation function (i.e. a bounded function with $\tau(y) = y$ in a neighbourhood of 0). Then $1_A(t,x)(y-\tau(y))$ (resp. $1_A(t,x)\tau(y)$) is integrable w.r.t. μ (resp. $\mu - \nu$), and we have*

$$\begin{aligned} M(A) = B(A) + M^{\mathrm{c}}(A) + \int_{\mathbb{R}\times E\times\mathbb{R}} 1_A(t,x)(y-\tau(y))\,\mu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y) + \\ + \int_{\mathbb{R}\times E\times\mathbb{R}} 1_A(t,x)\tau(y)\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y), \end{aligned} \tag{3.2}$$

(3) *There are a positive predictable strict random measure $A(\omega,\mathrm{d}t,\mathrm{d}x)$, a $\tilde{\mathcal{P}}$-measurable function $b(\omega,t,x)$ and a transition kernel $K(\omega,t,x,\mathrm{d}y)$ from $(\tilde{\Omega},\tilde{\mathcal{P}})$ to $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that for a.e. $\omega \in \Omega$*

$$B(\omega,\mathrm{d}t,\mathrm{d}x) = b(\omega,t,x)\,A(\omega,\mathrm{d}t,\mathrm{d}x), \quad \nu(\omega,\mathrm{d}t,\mathrm{d}x,\mathrm{d}y) = K(\omega,t,x,\mathrm{d}y)\,A(\omega,\mathrm{d}t,\mathrm{d}x).$$

For the proof of Theorem 3.2 let us recall the semimartingale topology of [22] on the space $\mathcal{SM}$ of semimartingales up to infinity, which is induced by

$$\|X\|_{\mathcal{SM}} := \sup_{|H|\le 1, H\in\mathcal{P}} \left\| \int_{-\infty}^{\infty} H_t\,\mathrm{d}X_t \right\|_0, \quad X \in \mathcal{SM}.$$

The following results are known:

Lemma 3.3.

(1) *Let $(X^n)_{n\in\mathbb{N}} \subseteq \mathcal{SM}$ and (B^n, C^n, ν^n) denote the semimartingale characteristics of X^n. If $X^n \to 0$ in $\mathcal{SM}$, then each of the following semimartingale sequences converges to 0 in $\mathcal{SM}$ as well: B^n, $X^{\mathrm{c},n}$, C^n, $[X^n]$, $(y-\tau(y)) * \mu^n$ and $\tau(y) * (\mu^n - \nu^n)$.*

(2) *If $W(\omega,t,y)$ is a positive bounded predictable function, then $W * \mu^n \to 0$ in probability if and only if $W * \nu^n \to 0$ in probability. Similarly, $W * \mu^n < \infty$ a.s. if and only if $W * \nu^n < \infty$ a.s.*

(3) *The collection of predictable finite variation processes is closed under the semimartingale topology.*

For the first part of this lemma, see [12, Thm. 4.10] and [22, p. 276]. The second part is taken from [12, Lemmas 4.8 and 4.12]. The third assertion is proved in [38, Thm. IV.7].

Proof of Theorem 3.2. Let $k \in \mathbb{N}$ and consider the set function $(S,U) \mapsto B(S \times U)$ from the semiring $\mathcal{H} := \mathcal{P}|_{O_k} \times \mathcal{E}|_{E_k}$ to L^0. Obviously, it is finitely additive in each component: for fixed

U, additivity in time holds by the definition of B, while for fixed S, additivity in space is due to the assumption of different times of discontinuity. By a straightforward induction argument this implies that B is also finitely additive jointly in space and time. Next, let

$$\mathcal{R}(\mathcal{H}) = \left\{ \bigcup_{n=1}^{N} C_n \colon N \in \mathbb{N}, C_n \in \mathcal{H} \text{ pairwise disjoint} \right\}$$

denote the ring generated by $\mathcal{H}$. Setting $B(\bigcup_{n=1}^{N} C_n) := \sum_{n=1}^{N} B(C_n)$ one obtains a well-defined extension of B to $\mathcal{R}(\mathcal{H})$, which is consistent with the original definition of B and still finitely additive. Furthermore, since $\mathcal{R}(\mathcal{H})$ contains $O_k \times E_k$, we can further extend B to a measure on $\sigma(\mathcal{H}) = \tilde{\mathcal{P}}|_{\tilde{O}_k}$ using [34, Thm. B.1.1]. We only have to show the implication

$$(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}(\mathcal{H}) \text{ with } \limsup_{n\to\infty} A_n = \emptyset \implies \lim_{n\to\infty} B(A_n) = 0 \quad \text{in } L^0. \tag{3.3}$$

In fact, under the assumption on the left-hand side of (3.3), $1_{A_n} \cdot M \to 0$ in $\mathcal{SM}$:

$$\begin{aligned} \|1_{A_n} \cdot M\|_{\mathcal{SM}} &= \sup_{|H|\leq 1, H\in\mathcal{P}} \left\| \int H \,\mathrm{d}(1_{A_n} \cdot M) \right\|_0 = \sup_{|H|\leq 1, H\in\mathcal{P}} \left\| \int H 1_{A_n} \,\mathrm{d}M \right\|_0 \\ &\leq \sup_{S\in\mathcal{S}_M, |S|\leq 1_{A_n}} \left\| \int S \,\mathrm{d}M \right\|_0 = \|1_{A_n}\|_{M,0}^{\mathrm{D}} \stackrel{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

by (DCT) with $1_{O_k\times E_k}$ as dominating function. Using Lemma 3.3(1), Equation (3.3) follows.

This extension still coincides with the definition of B in Theorem 3.2: From the construction given in the proof of [34, Thm. B.1.1], we know that given $A \in \tilde{\mathcal{P}}|_{\tilde{O}_k}$, there is a sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{R}(\mathcal{H})$ with $\limsup((A \setminus A_n) \cup (A_n \setminus A)) = \emptyset$ and $B(A_n) \to B(A)$ in L^0 as $n \to \infty$. As above we obtain $1_{A_n} \cdot M \to 1_A \cdot M$ in $\mathcal{SM}$, which implies the assertion. And of course, B is unique and $B(A)$ does not depend on the choice of $k \in \mathbb{N}$ with $A \subseteq O_k$.

Finally, we prove that B corresponds to a predictable strict random measure. By [15, Thm. 4.10] it suffices to show that for $H \in L^{1,0}(B)$ the semimartingale $H \cdot B$ is predictable and has finite variation on bounded intervals. If $H \in \mathcal{S}_M$, this follows from linearity and the fact that the first characteristic of a semimartingale up to infinity is a predictable finite variation process. In the general case choose a sequence $(S_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_M$ with $S_n \to H$ pointwise and $|S_n| \leq H$ for all $n \in \mathbb{N}$. As $n \to \infty$, we have $S_n \cdot B \to H \cdot B$ in $\mathcal{SM}$ by (DCT). By Lemma 3.3(3) we conclude that also $H \cdot B$ is a predictable finite variation process.

For C we fix one argument and apply the same procedure to the other argument; for M^{c} we refer to [15, Thm. 4.13]. Let us proceed to μ and ν, where in both cases we first fix some $V \in \mathcal{B}_0(\mathbb{R})$ with $\inf\{|x| \colon x \in V\} \geq \epsilon > 0$ and $\epsilon < 1$. In order to apply the same construction scheme as for B, only the proof of (3.3) is different for μ and ν. To this end, let $(A_n)_{n\in\mathbb{N}}$ be as on the left-hand side of (3.3), that is, $1_{A_n} \cdot M \to 0$ in $\mathcal{SM}$. Now define $\tilde{\tau}(y) = (y \wedge \epsilon) \vee (-\epsilon)$ and choose $K > 1$ such that $|\tilde{\tau}(y)| \leq K(y^2 \wedge 1)$ for $|y| \geq \epsilon$. Then

$$\begin{aligned} \|\mu(A_n, V)\|_0 &= \left\| \frac{1_V(y)}{|\tilde{\tau}(y)|} |\tilde{\tau}(y)| * \mu_\infty^{1_{A_n}\cdot M} \right\|_0 \leq \epsilon^{-1} \left\| 1_V(y) |\tilde{\tau}(y)| * \mu_\infty^{1_{A_n}\cdot M} \right\|_0 \\ &\leq K\epsilon^{-1} \left\| (y^2 \wedge 1) * \mu_\infty^{1_{A_n}\cdot M} \right\|_0 \leq K\epsilon^{-1} \left\| [1_{A_n} \cdot M]_\infty \right\|_0 \to 0, \end{aligned}$$

where the last step follows from Lemma 3.3(1). Part (2) of the same lemma yields that also $\nu(A_n, V) \to 0$ in L^0 as $n \to \infty$. Consequently, [15, Thm. 4.12] shows that $\mu(\cdot, V)$ and $\nu(\cdot, V)$ can be chosen as positive strict random measures. Observing that $\mu(A, \cdot)$ (resp. $\nu(A, \cdot)$) is clearly also a positive (and predictable) strict random measure for given $A \in \tilde{\mathcal{P}}_M$, μ (resp. ν) can be extended to a positive (and predictable) strict random measure on the product $\tilde{\mathcal{P}}_M \otimes \mathcal{B}_0(\mathbb{R})$ (see [46, Prop. 2.4]). Of course, ν is the predictable compensator of μ.

The integrability of $1_A(t,x)(y-\tau(y))$ (resp. $1_A(t,x)\tau(y)$) w.r.t. μ (resp. $\mu-\nu$) is an obvious consequence of (3.1) and the corresponding statements in the null-spatial case. The canonical decomposition of M follows since both sides of (3.2) are random measures coinciding on $\mathcal{H}$.

Finally, part (3) of Theorem 3.2 can be proved analogously to [27, Prop. II.2.9]. □

Remark 3.4 If M is additionally orthogonal, we have $C(A;B) = C(A \cap B; A \cap B)$ for all $A, B \in \tilde{\mathcal{P}}_M$. Consequently, we may identify C with $C(A) := [(1_A \cdot M)^{\mathrm{c}}]_\infty$ for $A \in \tilde{\mathcal{P}}_M$. Of course, C can then be chosen as a predictable strict random measure. □

Next we calculate the characteristics introduced in Theorem 3.2 in two concrete situations: first, for the random measure of a stochastic integral process, and second, for a random measure under an absolutely continuous change of measure. Although the results in both cases are comparable with the purely temporal setting, the first task turns out to be the more difficult one. Moreover, the characteristics for stochastic integral processes are of particular importance for our integrability conditions in Section 4.

Beforehand, we need some bimeasure theory: it is well-known that bimeasures cannot be extended to measures on the product σ-field in general and that integration theory w.r.t. bimeasures differs from integration theory w.r.t. measures. Following [18], let two measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1,2$, and a bimeasure $\beta\colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathbb{R}$ be given. We call a pair (f_1, f_2) of $\mathcal{F}_i$-measurable functions f_i, $i = 1,2$, **strictly β-integrable** if

(1) f_1 (resp. f_2) is integrable w.r.t. $\beta(\cdot;B)$ for all $B \in \mathcal{F}_2$ (resp. $\beta(A;\cdot)$ for all $A \in \mathcal{F}_1$),

(2) f_2 is integrable w.r.t. the measure $B \mapsto \int_{\Omega_1} f_1(\omega_1)\,\beta(\mathrm{d}\omega_1;B)$ and f_1 is integrable w.r.t. the measure $A \mapsto \int_{\Omega_2} f_2(\omega_2)\,\beta(A;\mathrm{d}\omega_2)$,

(3) for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, the following integrals are equal:

$$\int_A f_1(\omega_1)\left(\int_B f_2(\omega_2)\,\beta(\mathrm{d}\omega_1;\mathrm{d}\omega_2)\right) = \int_B f_2(\omega_2)\left(\int_A f_1(\omega_1)\,\beta(\mathrm{d}\omega_1;\mathrm{d}\omega_2)\right). \tag{3.4}$$

The **strict β-integral** of $(f_1;f_2)$ on $(A;B)$, denoted by $\int_{(A;B)}(f_1;f_2)\,\mathrm{d}\beta$, is then defined as the common value (3.4).

The next theorem determines the characteristics of stochastic integral processes, which is [27, Prop. IX.5.3] in the null-spatial case.

Theorem 3.5. *Let M be a random measure with different times of discontinuity and $H \in \tilde{\mathcal{P}}$ satisfy* (2.6) *with some $K > 0$. Then the null-spatial random measure $H \cdot M$ has characteristics $(B^{H\cdot M}, C^{H\cdot M}, \nu^{H\cdot M})$ given by*

$$B^{H\cdot M}(A) = (H\cdot B)(A) + \int_{\mathbb{R}\times E\times\mathbb{R}} 1_A(t)[\tau(H(t,x)y) - H(t,x)\tau(y)]\,\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y), \tag{3.5}$$

$$C^{H\cdot M}(A) = \int_{\mathbb{R}} K_t^{-2}\,\mathrm{d}\left(\int_{(A_t\times E;A_t\times E)} (HK;HK)\,\mathrm{d}C\right), \tag{3.6}$$

$$W(t,y) * \nu^{H\cdot M} = W(t,H(t,x)y) * \nu \tag{3.7}$$

for all $A \in \mathcal{P}_{H\cdot M}$ and $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable functions W such that $W(t,y) * \nu^{H\cdot M}$ exists.

Moreover, if in addition M is orthogonal, then

$$C^{H\cdot M}(\mathrm{d}t) = \int_E H^2(t,x)\, C(\mathrm{d}t, \mathrm{d}x). \tag{3.8}$$

Proof. The second part of this theorem is clear as soon as we have proved the first part. Since characteristics are defined locally, we may assume that $H \in L^{1,0}(M)$. We first consider the continuous part $C^{H\cdot M}$: to this end, let $(H_n)_{n\in\mathbb{N}}$ be a sequence of simple integrands with $|H_n| \leq |H|$ for all $n \in \mathbb{N}$ and $H_n \to H$ pointwise. Since for simple integrands the claim follows directly from the definition of C and the bimeasure integral, we would like to use the (DCT) and Lemma 3.3(1) on the one hand and the dominated convergence theorem for bimeasure integrals (see [18, Cor. 2.9]) on the other hand to obtain the result. In order to do so, we only have to show that $(H;H)$ is strictly C-integrable, which means by the symmetry of C the following two points: first, that H is integrable w.r.t. the measure $A \mapsto C(A;B) = [(1_A \cdot M)^{\mathrm{c}}, (1_B \cdot M)^{\mathrm{c}}]_\infty$ for all $B \in \tilde{\mathcal{P}}_M$, and second, that H is integrable w.r.t. the measure $A \mapsto \int H(t,x)\,\mathrm{d}C(A;\mathrm{d}t,\mathrm{d}x) = [(1_A \cdot M)^{\mathrm{c}}, (H\cdot M)^{\mathrm{c}}]_\infty$.

Let G be 1_B or H. From [35], Theorem 2 and its Corollary, we know that there exists a probability measure Q equivalent to P such that M is an $L^2(Q)$-random measure with $G, H \in L^{1,2}(M;Q)$. Since the bounded sets in $L^0(P)$ are exactly the bounded sets in $L^0(Q)$, convergence in $\|\cdot\|^{\mathrm{D}}_{M,0;P}$ is equivalent to convergence in $\|\cdot\|^{\mathrm{D}}_{M,0;Q}$. Similarly, stochastic integrals and predictable quadratic covariation remain unchanged under Q (cf. [14, Prop. 3.6.20] and [27, Thm. III.3.13]). Consequently, if we write $\gamma(A) := [1_A \cdot M^{\mathrm{c}}, G\cdot M^{\mathrm{c}}]_\infty$ for $A \in \tilde{\mathcal{P}}_M$, it suffices to show that

$$\sup_{S\in\mathcal{S}_M, |S|\leq|rH|} \left\| \int S\,\mathrm{d}\gamma \right\|_{L^0(Q)} = \sup_{S\in\mathcal{S}_M, |S|\leq|rH|} \left\| [(S\cdot M)^{\mathrm{c}}, (G\cdot M)^{\mathrm{c}}]_\infty \right\|_{L^0(Q)} \to 0 \quad \text{as } r\to 0.$$

Indeed, using Fefferman's inequality (cf. [14, Thm. 4.2.7]), we can find a constant $R > 0$, which only depends on G, such that

$$\begin{aligned} &\sup_{S\in\mathcal{S}_M, |S|\leq|rH|} \left\| [(S\cdot M)^{\mathrm{c}}, (G\cdot M)^{\mathrm{c}}]_\infty \right\|_{L^0(Q)} \leq R \sup_{S\in\mathcal{S}_M, |S|\leq|rH|} \mathbb{E}_Q\left[[(S\cdot M)^{\mathrm{c}}]_\infty\right]^{1/2} \\ &= R \sup_{S\in\mathcal{S}_M, |S|\leq|rH|} \|(S\cdot M)^{\mathrm{c}}_\infty\|_{L^2(Q)} = R\|rH\|^{\mathrm{D}}_{M^{\mathrm{c}},2;Q} \to 0 \end{aligned}$$

as $r \to 0$, which finishes the proof for $C^{H\cdot M}$.

For $B^{H\cdot M}$ and $\nu^{H\cdot M}$, we first take some $D \in \mathcal{P}\otimes\mathcal{B}_0(\mathbb{R})$ and claim that

$$1_D(s,y) * \mu^{H\cdot M} = 1_D(s, H(s,x)y) * \mu. \tag{3.9}$$

This identity immediately extends to finite linear combinations of such indicators and thus, by (DCT), also to all functions $W(\omega,t,y)$ for which $W * \mu^{H\cdot M}$ exists. By the definition of the predictable compensator, this statement also passes to the case where μ is replaced by ν.

In order to prove (3.9), first observe that the jump process of the semimartingale $H\cdot M$ up to infinity is given by $\Delta(H\cdot M)_t = (H\cdot M)(\Omega\times\{t\}\times E)$. Furthermore, we can assume that D does not contain any points in $\bar{\Omega}\times\{0\}$. Hence, in the case where $H = 1_A$ with $A\in\tilde{\mathcal{P}}_M$, we have for all $t\in\mathbb{R}$

$$1_D(s,y) * \mu^{H\cdot M}_t = 1_D(s,y) * \mu^{1_A\cdot M}_t = 1_D(s,y)1_A(s,x) * \mu_t = 1_D(s, 1_A(s,x)y) * \mu_t.$$

Now a similar calculation yields that (3.9) remains true for all functions $H \in \mathcal{S}_M$. Finally, let $H \in L^{1,0}(M)$. By decomposing $H = H^+ - H^-$ into its positive and negative part, we may assume that $H \geq 0$ and choose a sequence $(H_n)_{n\in\mathbb{N}}$ of simple functions with $H_n \uparrow H$ as $n \to \infty$. As we have already seen in the proof of Theorem 3.2, we have $1_D(s,y) * \mu^{H_n \cdot M} \to 1_D(s,y) * \mu^{H\cdot M}$ in $\mathcal{SM}$. On the other hand, if D is of the form $R \times (a,b]$ with $R \in \mathcal{P}$ and $(a,b] \subseteq (0,\infty)$ or of the form $R \times [a,b)$ with $[a,b) \subseteq (-\infty,0)$, then $1_D(\omega,s,H_n(\omega,s,x)y) \to 1_D(\omega,s,H(\omega,s,x)y)$ as $n \to \infty$ for every $(\omega,s,x,y) \in \tilde{\Omega} \times \mathbb{R}$, which shows that (3.9) holds up to indistinguishability. For general D, use Dynkin's π-λ-lemma [16, Thm. 3.2].

Finally, we compute $B^{H\cdot M}$. The results up to now yield that for all $t \in \mathbb{R}$,

$$(H\cdot M)_t - (y-\tau(y)) * \mu_t^{H\cdot M} = (H\cdot B)_t + (H\cdot M^{\mathrm{c}})_t + H(s,x)(y-\tau(y)) * \mu_t + \\ + H(s,x)\tau(y) * (\mu-\nu)_t - [H(s,x)y - \tau(H(s,x)y)] * \mu_t.$$

By definition, $B^{H\cdot M}$ is the finite variation part in the canonical decomposition of this special semimartingale, which exactly equals $H\cdot B + [\tau(H(t,x)y) - H(t,x)\tau(y)] * \nu$. □

Finally, we show a Girsanov-type theorem comparable to [27, Thm. III.3.24] for semimartingales. First, let us introduce some notation. We consider another probability measure P' on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}})$ such that $P'_t := P'|_{\mathcal{F}_t}$ is absolutely continuous w.r.t. $P_t := P|_{\mathcal{F}_t}$ for all $t \in \mathbb{R}$. Then denote by Z the unique P-martingale such that $Z \geq 0$ identically and Z_t is a version of the Radon-Nikodym derivative $\mathrm{d}P'_t/\mathrm{d}P_t$ for all $t \in \mathbb{R}$, cf. [27, Thm. III.3.4].

Now let M be a random measure with different times of discontinuity under the probability measure P with characteristics (B,C,ν) w.r.t. the truncation function τ. We modify the sequence $(\tilde{O}_k)_{k\in\mathbb{N}}$ of Definition 2.1(1) by setting $\tilde{O}'_k := \tilde{O}_k \cap (\Omega \times (-k,k] \times E)$ for $k \in \mathbb{N}$ and $\tilde{\mathcal{P}}'_M := \bigcup_{k=1}^\infty \tilde{\mathcal{P}}|_{\tilde{O}'_k}$. Next, we denote the jump measure of M by μ and set $M^P_\mu(W) := \mathbb{E}_P[W * \mu_\infty]$ for all non-negative $\mathcal{F}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{E}\otimes\mathcal{B}(\mathbb{R})$-measurable functions W. Furthermore, for every such W, there exists an M^P_μ-a.e. unique $\tilde{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R})$-measurable function $M^P_\mu(W|\tilde{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R}))$ such that

$$M^P_\mu(WU) = M^P_\mu(M^P_\mu(W|\tilde{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R}))U) \quad \text{for all } \tilde{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R})\text{-measurable } U \geq 0.$$

Finally, we set

$$Y(t,x,y) := M^P_\mu(Z/Z_- 1_{\{Z_->0\}}|\tilde{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R}))(t,x,y), \quad t\in\mathbb{R}, x\in E, y\in\mathbb{R},$$
$$C^Z(A) := [(Z_-^{-1}\cdot Z)^{\mathrm{c}}, (1_A\cdot M)^{\mathrm{c}}]_\infty, \quad A \in \tilde{\mathcal{P}}'_M.$$

In the last line, the stochastic integral process $Z_-^{-1}\cdot Z$ is meant to start at t_0, where $t_0 \in \mathbb{R}$ is chosen such that $(1_A\cdot M)^{\mathrm{c}} = 0$ on $(-\infty,t_0]$. Then $C^Z(A)$ is well-defined by [27, Prop. III.3.5a] and does not depend on the choice of t_0. Moreover, as in Theorem 3.2, one shows that C^Z can be chosen as a positive predictable strict random measure.

The following theorem extends [27, Thm. III.3.24] to the space-time framework.

Theorem 3.6. *Under P', M is also a random measure with different times of discontinuity (w.r.t. $(\tilde{O}'_k)_{k\in\mathbb{N}}$). Its P'-characteristics (B',C',ν') w.r.t. τ are versions of*

$$B'(\mathrm{d}t,\mathrm{d}x) := B(\mathrm{d}t,\mathrm{d}x) + C^Z(\mathrm{d}t,\mathrm{d}x) + \tau(y)(Y(t,x,y)-1)\,\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y),$$
$$C'(\mathrm{d}t,\mathrm{d}x) := C(\mathrm{d}t,\mathrm{d}x),$$
$$\nu'(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y) := Y(t,x,y)\,\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y).$$

Proof. Since each set in $\tilde{\mathcal{P}}'_M$ is $\mathcal{F}_t$-measurable for some $t \in \mathbb{R}$, properties (a), (b) and (d) of Definition 2.1(1) still hold under P'. Since (c) does not depend on the underlying probability measure, M is also a random measure under $\tilde{P}$. To show that M still has different times of discontinuity under P', it suffices to notice the following: using the notation of Definition 3.1, the event that $1_{O_k\times U_1}\cdot M$ and $1_{O_k\times U_2}\cdot M$ have a common jump in $\mathbb{R}$ is the union over $n\in\mathbb{N}$ of the events that they have a common jump in $(-\infty,n]$. Since these latter events are $\mathcal{F}_n$-measurable, their P'-probability is 0, as desired. Finally, the characteristics under P' can be derived, up to obvious changes, exactly as in [27, Thm. III.3.24]. □

4 An integrability criterion

The canonical decomposition of M in Theorem 3.2 together with Theorem 3.5 enables us to reformulate (2.8) in terms of conditions only depending on the characteristics of M. This result extends the null-spatial case as found in [27, Thm. III.6.30], [19, Thm. 4.5], [12, Thm. 4.5] or [34, Thm. 9.4.1]. It also generalizes the results of [46, Thm. 2.7] to predictable integrands and also to random measures which are not necessarily Lévy bases. Our proof mimics the approach in [12, Thm. 4.5] and takes care of the additional spatial structure.

Theorem 4.1. *Let M be a random measure with different times of discontinuity whose characteristics w.r.t. some truncation function τ are given by Theorem 3.2. Furthermore, let $H\in\tilde{\mathcal{P}}$ satisfy* (2.6)*. Then $H\in L^0(M)$ if and only if each of the following conditions is satisfied a.s.:*

$$\int_{\mathbb{R}\times E}\left|H(t,x)b(t,x)+\int_{\mathbb{R}}[\tau(H(t,x)y)-H(t,x)\tau(y)]\;K(t,x,\mathrm{d}y)\right|\,A(\mathrm{d}t,\mathrm{d}x)<\infty, \tag{4.1}$$

$$\int_{\mathbb{R}}K_t^{-2}\,\mathrm{d}\left(\int_{((-\infty,t]\times E;(-\infty,t]\times E)}(HK;HK)\,\mathrm{d}C\right)<\infty, \tag{4.2}$$

$$\int_{\mathbb{R}\times E}\int_{\mathbb{R}}\left(1\wedge(H(t,x)y)^2\right)K(t,x,\mathrm{d}y)\,A(\mathrm{d}t,\mathrm{d}x)<\infty. \tag{4.3}$$

If M is additionally orthogonal, the spaces $L^0(M)$ and $L^{1,0}(M)$ are equal and condition (4.2) *is equivalent to*

$$\int_{\mathbb{R}\times E}H^2(t,x)\,C(\mathrm{d}t,\mathrm{d}x)<\infty. \tag{4.4}$$

The following lemma is a straightforward extension of [46, Lemma 2.8]. We omit its proof:

Lemma 4.2. *For $t\in\mathbb{R}$, $x\in E$ and $a\in\mathbb{R}$ define*

$$U(t,x,a):=\left|ab(t,x)+\int_{\mathbb{R}}\left(\tau(ay)-a\tau(y)\right)K(t,x,\mathrm{d}y)\right|,\quad \tilde{U}(t,x,a):=\sup_{-1\le c\le 1}U(t,x,ca).$$

Then there exists a constant $\kappa>0$ such that

$$\tilde{U}(t,x,a)\le U(t,x,a)+\kappa\int_{\mathbb{R}}\left(1\wedge(ay)^2\right)K(t,x,\mathrm{d}y).$$

Proof of Theorem 4.1. We first prove that $H\in L^0(M)$ implies (4.1)-(4.3). Since $H\cdot M$ is a semimartingale up to infinity, $B^{H\cdot M}(\mathbb{R})$ and $C^{H\cdot M}(\mathbb{R})$ exist. Thus, Theorem 3.5 gives the first two conditions. For the last condition observe that $(1\wedge y^2)*\nu_\infty^{H\cdot M}<\infty$ a.s. is equivalent to $(1\wedge y^2)*\mu_\infty^{H\cdot M}<\infty$ a.s., which obviously holds since $H\cdot M$ is a semimartingale up to infinity. This completes the first direction of the proof.

For the converse statement, we define $\mathcal{D} := \{G \in \mathcal{P}\colon |G| \leq 1, GH \in L^{1,0}(M)\}$. By (2.8) we have to show that the set $\{\int GH \,\mathrm{d}M \colon G \in \mathcal{D}\}$ is bounded in L^0 (i.e. bounded in probability) whenever H satisfies (4.1)-(4.3). By Theorem 3.5,

$$\int GH \,\mathrm{d}M = \int GH \,\mathrm{d}M^{\mathrm{c}} + \tau(GHy) * (\mu - \nu)_\infty + (GHy - \tau(GHy)) * \mu_\infty + B^{GH\cdot M}(\mathbb{R}).$$

We consider each part of this formula separately and show that each of the sets

$$\{B^{GH\cdot M}(\mathbb{R})\colon G \in \mathcal{D}\}, \tag{4.5}$$

$$\{\textstyle\int GH \,\mathrm{d}M^{\mathrm{c}}\colon G \in \mathcal{D}\}, \tag{4.6}$$

$$\{\tau(GHy) * (\mu - \nu)_\infty \colon G \in \mathcal{D}\}, \tag{4.7}$$

$$\{(GHy - \tau(GHy)) * \mu_\infty \colon G \in \mathcal{D}\} \tag{4.8}$$

is bounded in probability.

If $G \in \mathcal{D}$ and $\kappa > 0$ denotes the constant in Lemma 4.2, (4.1) and (4.3) imply

$$\begin{aligned}
&\int_{\mathbb{R}\times E} U(t,x,G_tH(t,x))\, A(\mathrm{d}t,\mathrm{d}x) \leq \int_{\mathbb{R}\times E} \tilde{U}(t,x,G_tH(t,x))\, A(\mathrm{d}t,\mathrm{d}x)\\
&\leq \int_{\mathbb{R}\times E} \tilde{U}(t,x,H(t,x))\, A(\mathrm{d}t,\mathrm{d}x)\\
&\leq \int_{\mathbb{R}\times E} U(t,x,H(t,x))\, A(\mathrm{d}t,\mathrm{d}x) + \kappa \int_{\mathbb{R}\times E}\int_{\mathbb{R}} \big(1 \wedge (H(t,x)y)\big)\, K(t,x,\mathrm{d}y)\, A(\mathrm{d}t,\mathrm{d}x) < \infty
\end{aligned}$$

a.s., which shows that (4.5) is bounded in probability.

Next consider (4.6) and fix some $G \in \mathcal{D}$ for a moment. Using Lenglart's inequality [27, Lemma I.3.30a], we have for all ϵ, $\eta > 0$

$$\begin{aligned}
&P\left[\left|\int GH \,\mathrm{d}M^{\mathrm{c}}\right| \geq \epsilon\right] \leq P\left[\sup_{t\in\mathbb{R}} |(GH\cdot M^{\mathrm{c}})(\bar{\Omega}_t)| \geq \epsilon\right]\\
&= P\left[\sup_{t\in\mathbb{R}} |(GH\cdot M^{\mathrm{c}})(\bar{\Omega}_t)|^2 \geq \epsilon^2\right] \leq \frac{\eta}{\epsilon^2} + P\big[[GH\cdot M^{\mathrm{c}}]_\infty \geq \eta\big]\\
&= \frac{\eta}{\epsilon^2} + P\big[G^2K^{-2}\cdot[KH\cdot M^{\mathrm{c}}]_\infty \geq \eta\big] \leq \frac{\eta}{\epsilon^2} + P\big[K^{-2}\cdot[KH\cdot M^{\mathrm{c}}]_\infty \geq \eta\big].
\end{aligned}$$

Now (4.2) allows us to make the quantity on the left-hand side arbitrarily small, independently of $G \in \mathcal{D}$, by first choosing $\eta > 0$ and then $\epsilon > 0$ large enough.

For (4.7), we use the abbreviation $W(t,x,y) = \tau(G_tH(t,x)y)$. Lenglart's inequality again yields

$$P[|W*(\mu-\nu)_\infty| \geq \epsilon] \leq P\left[\sup_{t\in\mathbb{R}} |W*(\mu-\nu)_t|^2 \geq \epsilon^2\right] \leq \frac{\eta}{\epsilon^2} + P\big[\langle W*(\mu-\nu)\rangle_\infty \geq \eta\big] \tag{4.9}$$

for every ϵ, $\eta > 0$. Furthermore, by Theorem 3.5 and [27, Prop. II.2.17] we have

$$\langle W*(\mu-\nu)\rangle_\infty = \langle \tau(y) * (\mu^{GH\cdot M} - \nu^{GH\cdot M})\rangle_\infty \leq \tau(y)^2 * \nu_\infty,$$

which is finite by (4.3) yielding the boundedness of (4.7).

Next choose r, $\epsilon > 0$ such that $f(y) := r|y|1_{\{|y|>\epsilon\}}$ satisfies $|y - \tau(y)| \leq f(y)$ for all $y \in \mathbb{R}$. Obviously, f is symmetric and increasing on $\mathbb{R}_+$ so that

$$\big|\big(GHy - \tau(GHy)\big) * \mu_\infty\big| \leq f(GHy) * \mu_\infty \leq f(Hy) * \mu_\infty.$$

Now the third condition and Lemma 3.3(2) imply that

$$\sum_{t\in\mathbb{R}}(1\wedge\epsilon^2)1_{\{|\Delta(H\cdot M)_t|>\epsilon\}} \leq (1\wedge y^2) * \mu_\infty^{H\cdot M} = \big(1\wedge (H(t,x)y)^2\big) * \mu_\infty < \infty$$

a.s. such that $\{|\Delta(H\cdot M)_t| > \epsilon\}$ only happens for finitely many time points. Hence

$$f(Hy) * \mu_\infty = f(y) * \mu_\infty^{H\cdot M} = r\sum_{t\in\mathbb{R}} |\Delta(H\cdot M)_t| 1_{\{|\Delta(H\cdot M)_t|>\epsilon\}} < \infty$$

a.s., which implies that the set in (4.8) is also bounded in probability.

Finally, in the case where M is also orthogonal, we show that (4.1),(4.4) and (4.3) imply $H\in L^{1,0}(M)$. By Theorem 2.3 and the fact that for predictable functions H

$$\|H\|^{\mathrm{D}}_{M,0} = \sup_{S\in\mathcal{S}_M, |S|\leq |H|} \left\|\int S\,\mathrm{d}M\right\|_0 = \sup_{G\in\tilde{\mathcal{P}}, |G|\leq 1, GH\in L^{1,0}(M)} \left\|\int GH\,\mathrm{d}M\right\|_0,$$

we have to show that the set $\{\int GH\,\mathrm{d}M \colon G\in\mathcal{D}'\}$ is bounded in L^0, where $\mathcal{D}'$ consists of all functions $G\in\tilde{\mathcal{P}}$ with $|G|\leq 1$ and $GH\in L^{1,0}(M)$. Obviously, the previously considered set $\mathcal{D}$ is a subset of $\mathcal{D}'$. Intending to verify (4.5)-(4.8) with G taken from $\mathcal{D}'$, we observe that all calculations remain valid except those for (4.6). For (4.6) we argument as follows: for all $\epsilon,\eta>0$, Lenglart's inequality implies

$$\begin{aligned} &P\left[\left|\int GH\,\mathrm{d}M^{\mathrm{c}}\right| \geq \epsilon\right] \leq P\left[\sup_{t\in\mathbb{R}} |(GH\cdot M)^{\mathrm{c}}(\bar{\Omega}_t)|^2 \geq \epsilon^2\right] \\ &\leq \frac{\eta}{\epsilon^2} + P\big[[(GH\cdot M)^{\mathrm{c}}]_\infty \geq \eta\big] = \frac{\eta}{\epsilon^2} + P\left[\int_{\mathbb{R}\times E} G^2(t,x)H^2(t,x)\,C(\mathrm{d}t,\mathrm{d}x) \geq \eta\right] \\ &\leq \frac{\eta}{\epsilon^2} + P\left[\int_{\mathbb{R}\times E} H^2(t,x)\,C(\mathrm{d}t,\mathrm{d}x)\geq\eta\right]. \end{aligned}$$

This finishes the proof of Theorem 4.1. □

The remaining part of this section illustrates Theorem 4.1 by a series of remarks, examples and useful extensions.

Remark 4.3 If M has summable jumps, which means that each of the semimartingales $(M(\tilde{\Omega}_t\cap\tilde{O}_k))_{t\in\mathbb{R}}$, $k\in\mathbb{N}$, has summable jumps over finite intervals, it is often convenient to construct the characteristics w.r.t. $\tau=0$, which is not a proper truncation function. Then one would like to use $\tau=0$ in (4.1) and replace (4.3) by

$$\int_{\mathbb{R}\times E}\int_{\mathbb{R}} \big(1\wedge |H(t,x)y|\big)\,K(t,x,\mathrm{d}y)\,A(\mathrm{d}t,\mathrm{d}x) < \infty. \tag{4.10}$$

We show that (4.1) with $\tau=0$, (4.2) and (4.10) are together sufficient conditions for $H\in L^0(M)$. First note that we can choose $\kappa=0$ in Lemma 4.2(2) since τ is identical 0 and therefore $\tilde{U}=U$. So the calculations done for (4.5) remain valid. Moreover, (4.6) does not depend on τ and the boundedness of (4.7) becomes trivial. For (4.8) observe that

$$|GHy| * \mu_\infty \leq |Hy| * \mu_\infty = |y| * \mu_\infty^{H\cdot M} = |y|1_{\{|y|\leq 1\}} * \mu_\infty^{H\cdot M} + |y|1_{\{|y|>1\}} * \mu_\infty^{H\cdot M}. \tag{4.11}$$

Now (4.10) implies by Lemma 3.3(2) that a.s.,

$$|y|1_{\{|y|\leq 1\}} * \mu_\infty^{H\cdot M} + 1_{\{|y|>1\}} * \mu_\infty^{H\cdot M} = \int_{\mathbb{R}\times E}\int_{\mathbb{R}} \big(1 \wedge |H(t,x)y|\big)\, K(t,x,\mathrm{d}y)\, A(\mathrm{d}t,\mathrm{d}x) < \infty.$$

As a result, on the right-hand side of (4.11), the first summand converges a.s. and the second one is in fact just a finite sum a.s.

The converse statement is not true, already in the null-spatial case: let $(N_t)_{t\geq 0}$ be a standard Poisson process and $\tilde{N}_t = N_t - t$, $t \geq 0$, its compensation. Set $H_t := (1+t)^{-1}$ for $t \geq 0$. Then $H \in L^0(\tilde{N})$ as one can see from (4.1)-(4.3) with the proper truncation function $\tau(y) = y1_{\{|y|<1\}}$; but $\int_0^\infty H_t\,\mathrm{d}t = \infty$ violating both (4.1) with $\tau = 0$ and (4.10).

However, if M is a positive (or negative) random measure, that is, $M(A)$ is a positive (or negative) random variable for all $A \in \tilde{\mathcal{P}}_M$, then $C = 0$ necessarily and (4.1) with $\tau = 0$ and (4.10) also become necessary conditions for $H \in L^0(M) = L^{1,0}(M)$; cf. [15, Ex. 5, p. 7, and Thm. 4.12]. □

Next we compare our results and techniques to the standard literature:

Remark 4.4 (Lévy bases [46]) Lévy bases are originally called infinitely divisible independently scattered random measures in [46]. They are the space-time analogues of processes with independent increments and have attracted interest in several applications in the last few years, see Section 5 for some examples. The precise definition is as follows: Assume that we have $\tilde{O}_k = \Omega \times O'_k$ in the notation of Definition 2.1, where $(O'_k)_{k\in\mathbb{N}}$ is a sequence increasing to $\mathbb{R}\times E$. Set $\mathcal{S} := \bigcup_{k=1}^\infty \mathcal{B}(\mathbb{R}^{1+d})|_{O'_k}$. Then a **Lévy basis** Λ is a random measure on $\mathbb{R}\times E$ with the following additional properties:

(1) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{S}$, then $(\Lambda(\Omega\times A_n))_{n\in\mathbb{N}}$ are independent random variables.

(2) For all $A \in \mathcal{S}$, $\Lambda(\Omega\times A)$ has an infinitely divisible distribution.

Note that we have altered the original definition of [46]: in order to perform stochastic integration, we need to single out one coordinate to be time and introduce a filtration based definition of the integrator Λ. For notational convenience, we will write $\Lambda(A)$ instead of $\Lambda(\Omega\times A)$ in the following. As shown in [46, Prop. 2.1 and Lemma 2.3], Λ induces a characteristic triplet (B, C, ν) w.r.t. some truncation function τ via the Lévy-Khintchine formula:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}u\Lambda(A)}] = \exp\left(\mathrm{i}uB(A) - \frac{u^2}{2}C(A) + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}uy} - 1 - \mathrm{i}u\tau(y))\,\nu(A,\mathrm{d}y)\right), \quad A \in \mathcal{S}, u \in \mathbb{R}.$$

It is natural to ask how this notion of characteristics compares with Theorem 3.2. Obviously, Λ is an orthogonal random measure. In order that Λ has different times of discontinuity, it suffices by independence to assume that Λ has no fixed times of discontinuity. In this case, recalling the construction in the proof of Theorem 3.2 and using [49, Thm. 3.2] together with [27, Thm. II.4.15], one readily sees that the two different definitions of characteristics agree in the natural filtration of Λ. In particular, the canonical decomposition of Λ determines its Lévy-Itô decomposition as derived in [44].

Consequently, the integrability criteria obtained in Theorem 4.1 extend the corresponding result of [46, Thm. 2.7] for deterministic functions (or, as used in [9], for integrands which are independent of Λ) to allow for predictable integrands. □

Remark 4.5 (Martingale measures [51]) In [51] a stochastic integration theory for predictable integrands is developed with so-called worthy martingale measures as integrators. The concept of worthiness is needed since a martingale measure in Walsh's sense does not guarantee that it is a random measure in the sense of Definition 2.1. What is missing is, loosely speaking, a joint σ-additivity condition in space and time; see also the example in [51, pp. 305ff.]. The worthiness of a martingale measure, i.e. the existence of a dominating (σ-additive) measure, turns it into a random measure.

In essence, the integration theory presented in [51] for worthy martingale measures is an L^2-theory similar to [21, 26], where the extension from simple to general integrands is governed by a dominating measure. The latter also determines whether a predictable function is integrable or not in terms of a square-integrability condition; see [51, p. 292]. We see the main advantages of the L^2-theory as follows: it does not require the martingale measure to have different times of discontinuity, works with fairly easy integrability conditions and produces stochastic integrals again belonging to L^2. However, many interesting integrators (e.g. stable noises) are not L^2-random measures. Moreover, even if the integrator M is an L^2-random measure, the class $L^0(M)$ is usually considerably larger than the class $L^2(M)$. Thus, in comparison to [51], it is the compensation of these two shortages of the L^2-theory that constitutes the main advantage of our integrability conditions in Theorem 4.1. We will come back to this point in Section 5.2, where it is shown that in the study of stochastic PDEs, solutions often do not exist in the L^2-sense but in the L^0-sense. □

Remark 4.6 ((Compensated) strict random measures [27]) Chapters I and II of [27] are an established reference for integration theory w.r.t. semimartingales. Moreover, they also cover the integration theory w.r.t. strict random measures or compensated strict random measures as follows: if M is a strict random measure, they define stochastic integrals w.r.t. M path-by-path. More precisely, a measurable function $H\colon \tilde{\Omega} \to \mathbb{R}$ is pathwise integrable w.r.t. M if for a.e. $\omega \in \Omega$

$$\int_{\mathbb{R}\times E} |H|(\omega, t, x)\, |M|(\omega, \mathrm{d}t, \mathrm{d}x) < \infty. \tag{4.12}$$

If $\tilde{M} := M - M^{\mathrm{p}}$ is the compensation of an integer-valued strict random measure M, we have the following situation: let $H \in \tilde{\mathcal{P}}$ and introduce an auxiliary process by

$$\tilde{H}_t(\omega) := \int_E H(\omega, t, x)\, \tilde{M}(\omega, \{t\} \times \mathrm{d}x), \quad (\omega, t) \in \bar{\Omega}, \tag{4.13}$$

hereby setting $\tilde{H}_t(\omega) := +\infty$ whenever (4.13) diverges. Then H is integrable in the sense of [27, Def. II.1.27] if there exists a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ with $T_n \uparrow +\infty$ a.s. and

$$\mathbb{E}\left[\left(\sum_{-T_n \le t \le T_n} \tilde{H}_t^2\right)^{1/2}\right] < \infty. \tag{4.14}$$

How do these integrability conditions compare to those of Theorem 4.1? Obviously, pathwise integrability w.r.t. M does not require the integrand to be predictable. Furthermore, if H is predictable and (4.12) holds, then the pathwise integral coincides with the stochastic integral $H \cdot M$. Still, Theorem 4.1 provides a useful extension in some situations: first, there are examples $H \in L^0(M)$ which fail the condition (4.12) (see the example at the end of Remark 4.3). And second, given some specific H, it may be difficult in general to determine whether (4.12) holds or not (e.g., if M has no finite first moment). The characteristic triplet that is used in Theorem 4.1 is often easier to handle than $|M|$.

As for $\tilde{M}$ we have following situation: first, one should notice that (4.14) ensures integrability on *finite* intervals, whereas Theorem 4.1 is concerned with *global* integrability on $\mathbb{R}$. Second, even on finite intervals, the conditions of Theorem 4.1 are more general than (4.14), see [15, Prop. 3.10]. Finally, whereas (4.14) involves a localizing sequence of stopping times and moment considerations, Theorem 4.1 relates integrability only to the integrand itself and the characteristics of $\tilde{M}$, which is often more convenient. □

In order to illustrate condition (4.2) in Theorem 4.1, we now discuss the example of a Gaussian random measure, which is white in time but coloured in space. Such random measures are often encountered as the driving noise of stochastic PDEs, see [20] and references therein.

Example 4.7 Let $(M(\Omega\times B))_{B\in\mathcal{B}_\mathrm{b}(\mathbb{R}^{1+d})}$ be a mean-zero Gaussian process whose covariance functional for $B, B'\in\mathcal{B}_\mathrm{b}(\mathbb{R}^{1+d})$ is given by

$$C(B;B') := \mathbb{E}[M(\Omega\times B)M(\Omega\times B')] = \int_\mathbb{R}\int_{B(t)\times B'(t)} f(x-x')\,\mathrm{d}(x,x')\,\mathrm{d}t, \tag{4.15}$$

where $B(t) := \{x\in\mathbb{R}^d\colon (t,x)\in B\}$. For the existence of such a process, it is well-known [21, Thm. II.3.1] that $f\colon\mathbb{R}^d\to[0,\infty)$ must be a symmetric and nonnegative definite function for which the integral on the right-hand side of (4.15) exists. Under these conditions, C defines a deterministic bimeasure which is symmetric in $B, B'\in\mathcal{B}_\mathrm{b}(\mathbb{R}^{1+d})$.

For the further procedure let $(\mathcal{F}_t)_{t\in\mathbb{R}}$ be the natural filtration of M and set

$$M(F\times(s,t]\times U) := 1_F M(\Omega\times(s,t]\times U),\quad F\in\mathcal{F}_s.$$

By [15, Thm. 2.25], M can be extended to a random measure on $\mathbb{R}\times\mathbb{R}^d$ provided that

$$S_n\to 0 \text{ pointwise}, |S_n|\le|S| \implies \int S_n\,\mathrm{d}M\to 0\quad\text{in } L^0$$

for all step functions S_n and S over sets of the form $F\times(s,t]\times U$ with $F\in\mathcal{F}_s$, $s<t$ and $U\in\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$. Indeed, using obvious notation and observing that 1_F is independent of $M(\Omega\times(s,t]\times U)$ for $F\in\mathcal{F}_s$ since M is white in time, we have

$$\begin{aligned}\mathbb{E}\left[\left(\int S_n\,\mathrm{d}M\right)^2\right] &= \sum_{i,j=1}^{r_n} a_i^n a_j^n \mathbb{E}[M(A_i^n)M(A_j^n)]\\ &= \sum_{i,j=1}^{r_n} a_i^n a_j^n P[F_i^n]P[F_j^n]\mathrm{Leb}((s_i^n,t_i^n]\cap(s_j^n,t_j^n])\int_{U_i^n\times U_j^n} f(x-x')\,\mathrm{d}(x,x')\\ &= \int_{(\mathbb{R}^{1+d};\mathbb{R}^{1+d})}(\tilde{S}_n,\tilde{S}_n)\,\mathrm{d}C\to 0\end{aligned}$$

by dominated convergence [18, Cor. 2.9]. Here $\tilde{S}_n$ arises from S_n by replacing a_i^n with $a_i^n P[F_i^n]$.

Having established that M is a random measure, let us derive its characteristics. Obviously, B and ν are identically 0. It is also easy to see that C is the second characteristic of M: it is clear for sets of the form $(s,t]\times U$, and extends to general sets in $\mathcal{B}_\mathrm{b}(\mathbb{R}^{1+d})$ by dominated convergence. Therefore, as shown in the proof of Theorem 3.5, $L^{1,0}(M)$ consists of those $H\in\tilde{\mathcal{P}}$ such that $(H;H)$ is strictly C-integrable, or, equivalently,

$$\int_\mathbb{R}\int_{\mathbb{R}^d\times\mathbb{R}^d}|H|(t,x)|H|(t,x')f(x-x')\,\mathrm{d}(x,x')\,\mathrm{d}t<\infty\quad\text{a.s.} \tag{4.16}$$

The class $L^0(M)$, however, is the set of all $H \in \tilde{\mathcal{P}}$ such that a.s. the inner integral in (4.16) is finite for a.e. $t \in \mathbb{R}$, and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}^d} H(t,x)H(t,x')f(x-x')\,\mathrm{d}(x,x')\,\mathrm{d}t < \infty \quad \text{a.s.} \tag{4.17}$$

A (deterministic) function $H \in L^0(M)$ which is not in $L^{1,0}(M)$ is, for instance, given by $H(t,x) := th(x)$ where h is chosen such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} h(x)h(x')f(x-x')\,\mathrm{d}(x,x') = 0.$$

One important example is a fractional correlation structure in space. In this case, we have $f(x_1,\dots,x_d) = \prod_{i=1}^d |x_i|^{2H_i-2}$, where $H_i \in (1/2,1)$ is the Hurst index of the i-th coordinate. Then $L^0(M)$ can be interpreted as the extension of the class $|\Lambda_H|$ studied in [45] to several parameters and stochastic integrands. However, in [45] as well as in [11], stochastic integrals are constructed for even larger classes of integrands. These classes, denoted Λ_H or Λ_X, respectively, are obtained as limits of simple functions under L^2-norms ($\|\cdot\|_{\Lambda_H}$ and $\|\cdot\|_{\Lambda_X}$, respectively), which are defined via fractional derivatives or Fourier transforms. In particular, the stochastic integrals defined via these norms are no longer of Itô type, i.e. no dominated convergence theorem holds for these stochastic integrands. Indeed, $L^{1,0}(M)$ is the largest class of predictable integrands for which a dominated convergence theorem holds (see Theorem 2.3), and $L^0(M)$ is its improper extension to functions for which $H \cdot M$ is a finite measure. □

The investigation of multi-dimensional stochastic processes often involves stochastic integrals where the integrand H is a matrix-valued predictable function and the integrator $M = (M^1,\dots,M^d)$ is a d-dimensional random measure, that is, $M^1,\dots,M^d$ are all random measures in the sense of Definition 2.1 w.r.t. the same underlying filtration and the same sequence $(\tilde{O}_k)_{k\in\mathbb{N}}$. By considering each row of H separately, we can assume for the following that H is an $\mathbb{R}^d$-valued predictable function. It is obvious that the construction of stochastic integrals requires no more techniques than those presented in Section 2. In fact, replacing E by E^d reduces the multivariate case to the univariate one. However, there is a difference when we want to apply the canonical decomposition as in Theorem 3.2 or the integrability conditions in Theorem 4.1: in the multi-dimensional case, it is not reasonable to assume that M^i and M^j for $i \neq j$ have different times of discontinuity. Instead, one would define d-dimensional characteristics (B,C,ν) for M, similar to [27, Chap. II] or [12, Section 2.2], and use these to characterize integrability.

In the next theorem we rephrase 4.1 for the multivariate setting. Since no novel arguments are needed, we omit its proof. We will use the product notation in a self-explanatory way: for instance, if $x,y \in \mathbb{R}^d$, xy denotes their inner product; for $A \in \tilde{\mathcal{P}}_M$, $1_A \cdot M$ denotes the d-dimensional semimartingale $(1_A \cdot M^1,\dots,1_A \cdot M^d)$; $H \cdot M$ denotes $\sum_{i=1}^d H^i \cdot M^i$ for $H \in L^{1,0}(M)$ and is suitably extended to $H \in L^0(M)$, cf. Section 2. Similarly, given a matrix $\beta = (\beta^{ij})_{i,j=1}^d$ of bimeasures from $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathbb{R}$ and $\mathcal{F}_i$-measurable functions $f_i = (f_i^1,\dots,f_i^d)$ for $i=1,2$, we define

$$\int_{(A;B)} (f_1;f_2)\,\mathrm{d}\beta := \sum_{i,j=1}^d \int_{(A;B)} (f_1^i;f_2^j)\,\mathrm{d}\beta^{ij}, \quad A \in \mathcal{F}_1, B \in \mathcal{F}_2,$$

whenever the right-hand side exists.

Assume that M has different times of discontinuity, which means that $1_{O_k \times U_i} \cdot M$, $i=1,2$, a.s. never jump at the same time for all disjoint sets $U_1,U_2 \in \mathcal{E}_M$ and $k \in \mathbb{N}$. Given a truncation function $\tau\colon \mathbb{R}^d \to \mathbb{R}^d$, define for $A, B \in \tilde{\mathcal{P}}_M$ and $V \in \mathcal{B}_0(\mathbb{R}^d)$

$$B(A) := \mathfrak{B}(1_A \cdot M)_\infty, \quad \mu(A,V) := \mu^{1_A \cdot M}(\mathbb{R}, V), \quad \nu(A,V) := \nu^{1_A\cdot M}(\mathbb{R}, V)$$
$$M^{\mathrm{c}}(A) := (1_A \cdot M)^{\mathrm{c}}, \quad C^{ij}(A;B) := [(1_A\cdot M^i)^{\mathrm{c}}, (1_B \cdot M^j)^{\mathrm{c}}]_\infty. \tag{4.18}$$

As in Theorem 3.2 (B, C, ν) can be extended to predictable strict random (bi-)measures and give rise to the following canonical decomposition of M:

$$M(A) = B(A) + M^{\mathrm{c}}(A) + \int_{\mathbb{R}\times E\times\mathbb{R}^d} 1_A(t,x)(y - \tau(y))\,\mu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y) + \\ + \int_{\mathbb{R}\times E\times\mathbb{R}^d} 1_A(t,x)\tau(y)\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y), \quad A \in \tilde{\mathcal{P}}_M. \tag{4.19}$$

Moreover, there exist a positive predictable strict random measure $A(\omega, \mathrm{d}t, \mathrm{d}x)$, a $\tilde{\mathcal{P}}$-measurable $\mathbb{R}^d$-valued function $b(\omega, t, x)$ and a transition kernel $K(\omega, t, x, \mathrm{d}y)$ from $(\tilde{\Omega}, \tilde{\mathcal{P}})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that for all $\omega \in \Omega$,

$$B(\omega, \mathrm{d}t, \mathrm{d}x) = b(\omega, t, x)\,A(\omega, \mathrm{d}t, \mathrm{d}x), \quad \nu(\omega, \mathrm{d}t, \mathrm{d}x, \mathrm{d}y) = K(\omega, t, x, \mathrm{d}y)\,A(\omega, \mathrm{d}t, \mathrm{d}x).$$

The multi-dimensional version of Theorem 4.1 reads as follows:

Theorem 4.8. *Let M be a d-dimensional random measure with different times of discontinuity and $H\colon \tilde{\Omega} \to \mathbb{R}^d$ be a predictable function such that there exists a strictly positive predictable process $K\colon \bar{\Omega} \to \mathbb{R}$ with $HK \in L^{1,0}(M)$. Then $H \in L^0(M)$ if and only if each of the following conditions is satisfied a.s.:*

$$\int_{\mathbb{R}\times E} \left| H(t,x)b(t,x) + \int_{\mathbb{R}^d} [\tau(H(t,x)y) - H(t,x)\tau(y)]\; K(t,x,\mathrm{d}y) \right| A(\mathrm{d}t, \mathrm{d}x) < \infty,$$
$$\int_{\mathbb{R}} K_t^{-2}\mathrm{d}\left(\int_{((-\infty,t]\times E; (-\infty,t]\times E)} (HK; HK)\,\mathrm{d}C \right) < \infty,$$
$$\int_{\mathbb{R}\times E}\int_{\mathbb{R}^d} \left(1 \wedge |H(t,x)y|^2\right) K(t,x,\mathrm{d}y)\,A(\mathrm{d}t,\mathrm{d}x) < \infty.$$

5 Ambit processes

In this section we present various applications, where the integrability conditions of Theorem 4.1 are needed. Given a filtered probability space satisfying the usual assumptions, our examples are processes of the following form:

$$Y(t,x) := \int_{\mathbb{R}\times\mathbb{R}^d} h(t,s;x,y)\,M(\mathrm{d}s, \mathrm{d}y), \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{5.1}$$

where $h\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ is a deterministic measurable function and M a random measure with different times of discontinuity such that the integral in (5.1) exists in the sense of (2.3). If the characteristics of M in the sense of Theorem 3.2 are known, (5.1) exists if and only if the conditions of Theorem 4.1 are satisfied for each pair $(t,x) \in \mathbb{R}\times\mathbb{R}^d$. We call processes of the form (5.1) **ambit processes** although the original definition in [9] requires the random measure to be a volatility modulated Lévy basis, i.e. $M = \sigma.\Lambda$ where Λ is a Lévy basis and $\sigma \in \tilde{\mathcal{P}}$. As already explained in the Introduction, this class of models is relevant in many different areas of applications. In the following subsections, we discuss two applications where interesting choices for h and M will be presented.

5.1 Stochastic PDEs

The connection between stochastic PDEs and ambit processes is exemplified in [9] relying on the integration theory of [46] or [51]. Let U be an open subset of $\mathbb{R}\times\mathbb{R}^d$ with boundary ∂U, P a polynomial in $1+d$ variables with constant coefficients and M a random measure with different times of discontinuity. The goal is to find a solution Z to the stochastic PDE

$$P(\partial_t,\partial_1,\ldots,\partial_d)Z(t,x)=\partial_t\partial_1\ldots\partial_d M(t,x),\quad (t,x)\in U, \tag{5.2}$$

subjected to some boundary conditions on ∂U, where $\partial_t\partial_1\ldots\partial_d M(t,x)$ is the formal derivative of M, its noise. We want to apply the method of Green's function to our random setting: first, we find a solution Y to (5.2) with vanishing boundary conditions, then we find a solution Y' to the homogeneous version of (5.2) which satisfies the prescribed boundary conditions, and finally we obtain a solution Z by the sum of Y and Y'. Since the problem of finding Y' is the same as in ordinary PDE theory, we concentrate on finding Y. However, since the noise of M does not exist formally, there exists no solution Y' in the strong sense. One standard approach based on [51, Section 3] is to interprete (5.2) in weak form and to define

$$Y(t,x):=\int_U G(t,s;x,y)\,M(\mathrm{d}s,\mathrm{d}y),\quad (t,x)\in U, \tag{5.3}$$

as a solution, where G is the Green's function for P in the domain U. Obviously, Y is then an ambit process, where the integrand is determined by the partial differential operator and the domain, and the integrator is the driving noise of the stochastic PDE. Therefore, Theorem 4.1 provides necessary and sufficient conditions for the existence of Y. Let us stress again that, in contrast to [46] and [51], we need no distributional assumptions on M.

Finally, we want to come back to Remark 4.5 and explain why the L^2-approach is too stringent for stochastic PDEs. To this end, we consider the stochastic heat equation in $\mathbb{R}^d$:

Example 5.1 We take $P(t,x)=t-\sum_{i=1}^d x_i$, $U=(0,\infty)\times\mathbb{R}^d$ and $M=\sigma.\Lambda$ where σ is a predictable function and Λ a Lévy basis with characteristics $(0,\Sigma\,\mathrm{d}t\,\mathrm{d}x,\mathrm{d}t\,\nu(\mathrm{d}\xi)\,\mathrm{d}x)$, where $\Sigma\geq 0$ and ν is a symmetric Lévy measure. [51, Section 3] considers a similar equation with $\nu=0$. The Green's function for P and U is the heat kernel

$$G(t,s;x,y)=\frac{\exp(-|x-y|^2/(4(t-s)))}{(4\pi(t-s))^{d/2}}1_{\{0<s\leq t\}},\quad s,t>0,x,y\in\mathbb{R}^d.$$

Since for all $(t,x)\in U$ the kernel $G(t,\cdot;x,\cdot)\in L^p(U)$ if and only if $p<1+2/d$, it is square-integrable only for $d=1$. Therefore, in the L^2-approach function-valued solutions only exist for $d=1$. However, if $\Sigma=0$, a sufficient condition for (4.3) and thus the existence of (5.3) is

$$\int_0^t\int_{\mathbb{R}^d}|G(t,s;x,y)\sigma(s,y)|^p\,\mathrm{d}s\,\mathrm{d}y<\infty\text{ a.s.},\quad (t,x)\in U,\qquad \int_{[-1,1]}|\xi|^p\,\nu(\mathrm{d}\xi)<\infty \tag{5.4}$$

for some $p\in[0,2)$. For instance, if σ is stationary in U with finite p-th moment, (5.4) becomes

$$\int_{[-1,1]}|\xi|^p\,\nu(\mathrm{d}\xi)<\infty\quad\text{for some } p<1+2/d.$$

In particular, we see that function-valued solutions exist in arbitrary dimensions, which cannot be "detected" in the L^2-framework, even for integrators which are L^2-random measures. □

The stochastic heat equation or similar equations driven by non-Gaussian noise have already been studied in a series of papers, e.g. [1, 3, 41, 42, 48], partly also extending Walsh's approach beyond the L^2-framework. Although they do not only consider the linear case (5.2), there are always limitations in dimension (e.g. only $d = 1$) or noise type (e.g. only stable noise without volatility modulation). Thus, in the linear case, Theorem 4.1 provides a unifying extension of the corresponding results in the given references.

5.2 Superposition of stochastic volatility models

In this subsection we give examples of ambit processes, where the spatial component in the stochastic integral has the meaning of a parameter space. First we discuss one possibility of constructing a superposition of COGARCH processes, following [13]. The COGARCH model of [32] itself is designed as a continuous-time version of the celebrated GARCH model and is defined as follows: Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process with Lévy measure ν_L. Given $\beta, \eta > 0$ the COGARCH model (V^φ, G^φ) with parameter $\varphi \geq 0$ is given by the equations

$$\mathrm{d}G_t^\varphi = \sqrt{V_{t-}^\varphi}\,\mathrm{d}L_t, \quad G_0^\varphi = 0, \tag{5.5}$$

$$\mathrm{d}V_t^\varphi = (\beta - \eta V_t^\varphi)\,\mathrm{d}t + \varphi V_{t-}^\varphi\,\mathrm{d}S_t, \quad t \in \mathbb{R}, \tag{5.6}$$

where $S := [L]^{\mathrm{d}}$ denotes the pure-jump part of the quadratic variation of L. By [32, Thm. 3.1], (5.6) has a stationary solution if and only if

$$\int_{\mathbb{R}_+} \log(1 + \varphi y^2)\,\nu_L(\mathrm{d}y) < \eta. \tag{5.7}$$

Let us denote the collection of all $\varphi \geq 0$ satisfying (5.7) by Φ, which by (5.7) must be of the form $[0, \varphi_{\max})$ with some $0 < \varphi_{\max} < \infty$. Although the COGARCH model essentially reproduces the same stylized features as the GARCH model, there are two unsatisfactory aspects:

(1) Right from the definition, the COGARCH shows a deterministic relationship between volatility and price jumps, an effect shared by many continuous-time stochastic volatility models [28]. More precisely, we have

$$\Delta V_t^\varphi = \varphi V_{t-}^\varphi (\Delta L_t)^2 = \varphi(\Delta G_t^\varphi)^2, \quad t \in \mathbb{R}. \tag{5.8}$$

A realistic stochastic volatility model should allow for different scale parameters φ.

(2) The autocovariance function of the COGARCH volatility is, when existent and $\varphi > 0$, always of exponential type: $\mathrm{Cov}[V_t^\varphi, V_{t+h}^\varphi] = C\mathrm{e}^{-ah}$ for $h \geq 0$, $t \in \mathbb{R}$ and some constants $C, a > 0$. A more flexible autocovariance structure is desirable.

In [13], three approaches to construct superpositions of COGARCH processes (supCOGARCH) with different values of φ are suggested in order to obtain a stochastic volatility model keeping the desirable features of the COGARCH but avoiding the two disadvantages mentioned above. One of them is the following: With β and η remaining constant, take a Lévy basis Λ on $\mathbb{R} \times \Phi$ with characteristics $(b\,\mathrm{d}t\,\pi(\mathrm{d}\varphi), \Sigma\,\mathrm{d}t\,\pi(\mathrm{d}\varphi), \mathrm{d}t\,\nu_L(\mathrm{d}y)\,\pi(\mathrm{d}\varphi))$, where $b \in \mathbb{R}$, $\Sigma \geq 0$, π is a probability measure on Φ and ν_L the Lévy measure of the Lévy process given by

$$L_t := \Lambda^L((0,t] \times \Phi), \quad t \geq 0, \quad L_t := -\Lambda^L((-t, 0] \times \Phi), \quad t < 0,$$

Furthermore, define another Lévy basis by $\Lambda^S(\mathrm{d}t, \mathrm{d}\varphi) := \int_{\mathbb{R}} y^2\,\mu^\Lambda(\mathrm{d}t, \mathrm{d}\varphi, \mathrm{d}y)$, where μ^Λ is the jump measure of Λ as in Theorem 3.2. Next define V^φ for each $\varphi \in \Phi$ as the COGARCH volatility process driven by L with parameter φ. Motivated by (5.6), the **supCOGARCH** $\bar{V}$ is now defined by the stochastic differential equation

$$\mathrm{d}\bar{V}_t = (\beta - \eta\bar{V}_t)\,\mathrm{d}t + \int_\Phi \varphi V^\varphi_{t-}\,\Lambda(\mathrm{d}t, \mathrm{d}\varphi), \quad t \in \mathbb{R}. \tag{5.9}$$

As shown in [13, Prop. 3.15], (5.9) has a unique solution given by

$$\bar{V}_t = \frac{\beta}{\eta} + \int_{-\infty}^t \int_\Phi \mathrm{e}^{-\eta(t-s)} \varphi V^\varphi_{s-}\,\Lambda(\mathrm{d}s, \mathrm{d}\varphi), \quad t \in \mathbb{R}, \tag{5.10}$$

such that $\bar{V}$ is an ambit process as in (5.1).

Here the integrability conditions of Section 4 come into play. Immediately from Theorem 4.1 and Remark 4.3 we obtain:

Corollary 5.2. *The supCOGARCH $\bar{V}$ as in* (5.10) *exists if and only if*

$$\int_{\mathbb{R}_+} \int_\Phi \int_{\mathbb{R}_+} 1 \wedge \left(y^2 \varphi \mathrm{e}^{-\eta s} V^\varphi_s\right) \nu_L(\mathrm{d}y)\,\pi(\mathrm{d}\varphi)\,\mathrm{d}s < \infty \quad a.s. \tag{5.11}$$

In particular, the supCOGARCH (5.10) provides an example where the stochastic volatility process $\sigma(s,\varphi) := \varphi V^\varphi_{s-}$ is *not* independent of the underlying Lévy basis Λ. So the conditions of [46, Thm. 2.7] are not applicable. For further properties of the supCOGARCH, in particular regarding its jump behaviour, autocovariance structure etc., we refer to [13].

Finally, let us comment on superpositions of other stochastic volatility models:

Remark 5.3 The usage of Ornstein-Uhlenbeck processes in stochastic volatility modelling has become popular through the Barndorff-Nielsen-Shephard model [6]. A natural extension is given by the CARMA stochastic volatility model [50], which generates a more flexible autocovariance structure. Another generalization of the BNS model is obtained via a superposition of OU processes with different memory parameters leading to the class of supOU processes [4]. This method does not only yield a more general second order structure but can also generate long-memory processes; cf. [4, 24]. A similar technique was used in [7, 37] to construct supCARMA processes, again leading to a possible long-range dependent process.

Note that in all these models the driving noise is assumed to have stationary independent increments, which is certainly a model restriction. Therefore, [8] suggests a volatility modulation of this noise to obtain a greater model flexibility. In this way, it is possible to generate a volatility clustering effect, similar to the behaviour of the (sup)COGARCH. Without volatility modulation, supOU or supCARMA processes are defined as stochastic integrals of deterministic kernel functions w.r.t. a Lévy basis, so the approach of [46] is sufficient. Theorem 4.1 now enables us to replace Λ by a volatility modulated Lévy basis $\sigma.\Lambda$ with a possible dependence structure between σ and Λ. □

Acknowledgment

We take pleasure in thanking Jean Jacod for extremely useful discussions and his advice on this topic. We also thank two referees and an associate editor for their constructive comments which led to a considerable improvement of the paper. The first author acknowledges support from the graduate program TopMath of the Elite Network of Bavaria and the TopMath Graduate Center of TUM Graduate School at Technische Universität München.

References

[1] S. Albeverio, J.-L. Wu, and T.-S. Zhang. Parabolic SPDEs driven by Poisson white noise. *Stoch. Process. Appl.*, 74(1):21–36, 1998.

[2] A. Andresen, F.E. Benth, S. Koekebakker, and V. Zakamulin. The CARMA interest rate model. *Int. J. Theor. Appl. Finan.*, 17(2), 2014. 27 pages.

[3] D. Applebaum and J.-L. Wu. Stochastic partial differential equations driven by Lévy space-time white noise. *Random Oper. Stoch. Equ.*, 8(3):245–259, 2000.

[4] O.E. Barndorff-Nielsen. Superposition of Ornstein-Uhlenbeck type processes. *Theory Probab. Appl.*, 45(2):175–194, 2001.

[5] O.E. Barndorff-Nielsen and J. Schmiegel. Ambit processes; with applications to turbulence and tumour growth. In F.E. Benth, editor, *Stochastic analysis and applications*, pages 93–124. Springer, Berlin, 2007.

[6] O.E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63 (2):167–241, 2001.

[7] O.E. Barndorff-Nielsen and R. Stelzer. Multivariate supOU processes. *Ann. Appl. Probab.*, 21(1):140–182, 2011.

[8] O.E. Barndorff-Nielsen and A.E.D. Veraart. Stochastic volatility of volatility and variance risk premia. *J. Financ. Econom.*, 11(1):1–46, 2012.

[9] O.E. Barndorff-Nielsen, F.E. Benth, and A.E.D. Veraart. Ambit processes and stochastic partial differential equations. In G. Di Nunno and B. Øksendal, editors, *Advanced Mathematical Methods for Finance*, pages 35–74. Springer, Berlin, 2011.

[10] O.E. Barndorff-Nielsen, F.E. Benth, and A.E.D. Veraart. Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli*, 19(3):803–845, 2013.

[11] A. Basse-O'Connor, S.-E. Graversen, and J. Pedersen. Multiparameter processes with stationary increments: spectral representation and integration. *Electron. J. Probab.*, 17(74), 2012. 21 pages.

[12] A. Basse-O'Connor, S.-E. Graversen, and J. Pedersen. Stochastic integration on the real line. *Theory Probab. Appl.*, 2014. To appear.

[13] A. Behme, C. Chong, and C. Klüppelberg. Superposition of COGARCH processes. Preprint available under arXiv:1305.2296 [math.PR], 2013.

[14] K. Bichteler. *Stochastic Integration with Jumps.* Cambridge University Press, Cambridge, 2002.

[15] K. Bichteler and J. Jacod. Random measures and stochastic integration. In G. Kallianpur, editor, *Theory and Application of Random Fields*, pages 1–18. Springer, Berlin, 1983.

[16] P. Billingsley. *Probability and Measure.* Wiley, New York, 3rd edition, 1995.

[17] S. Cambanis, J.P. Nolan, and J. Rosiński. On the oscillation of infinitely divisible and some other processes. *Stoch. Process. Appl.*, 35(1):87–97, 1990.

[18] D.K. Chang and M.M. Rao. Bimeasures and sampling theorems for weakly harmonizable processes. *Stoc. Anal. Appl.*, 1(1):21–55, 1983.

[19] A.S. Cherny and A.N. Shiryaev. On stochastic integrals up to infinity and predictable criteria for integrability. In M. Émery, M. Ledoux, and M. Yor, editors, *Séminaire de Probabilités XXXVIII*, pages 49–66. Springer, Berlin, 2005.

[20] R.C. Dalang. Extending martingale measures stochastic integral with applications to spatially homogeneous S.P.D.E's. *Electron. J. Probab.*, 4(6), 1999. 24 pages.

[21] J.L. Doob. *Stochastic Processes.* Wiley, New York, 1953.

[22] M. Émery. Une topologie sur l'espace des semimartingales. In C. Dellacherie, P.A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XIII*, pages 260–280. Springer, Berlin, 1979.

[23] V. Fasen. Extremes of Lévy driven mixed MA processes with convolution equivalent distributions. *Extremes*, 12(3):265–296, 2009.

[24] V. Fasen and C. Klüppelberg. Extremes of supOU processes. In F.E. Benth, G. Di Nunno, T. Lindstrøm, B. Øksendal, and T. Zhang, editors, *Stochastic Analysis and Applications: The Abel Symposium 2005*, pages 340–359. Springer, Heidelberg, 2007.

[25] F. Fuchs and R. Stelzer. Mixing conditions for multivariate infinitely divisible processes with an application to mixed moving averages and the supOU stochastic volatility model. *ESAIM Probab. Stat.*, 17:455–471, 2013.

[26] K. Itô. Stochastic integral. *Proc. Imp. Acad. Tokyo*, 20(8):519–524, 1944.

[27] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, Berlin, 2nd edition, 2003.

[28] J. Jacod, C. Klüppelberg, and G. Müller. Functional relationships between price and volatility jumps and its consequences for discretely observed data. *J. Appl. Probab.*, 49(4):901–914, 2012.

[29] E.B.V. Jensen, K.Ý. Jónsdóttir, J. Schmiegel, and O.E. Barndorff-Nielsen. Spatio-temporal modelling: with a view to biological growth. In B. Finkenstadt, L. Held, and V. Isham, editors, *Statistical methods for spatio-temporal systems*, pages 47–76. Chapman & Hall, Boca Raton, 2007.

[30] K.Ý. Jónsdóttir, A. Rønn-Nielsen, K. Mouridsen, and E.B.V. Jensen. Lévy-based modelling in brain imaging. *Scand. J. Stat.*, 40(3):511–529, 2013.

[31] O. Kallenberg. *Random Measures.* Akademie-Verlag, Berlin, 4th edition, 1986.

[32] C. Klüppelberg, A. Lindner, and R. Maller. A continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour. *J. Appl. Probab.*, 41(3):601–622, 2004.

[33] T.G. Kurtz and P.E. Protter. Weak convergence of stochastic integrals and differential equations II: Infinite dimensional case. In D. Talay and L. Tubaro, editors, *Probabilistic Models for Nonlinear Partial Differential Equations*, pages 197–285. Springer, Berlin, 1996.

[34] S. Kwapień and W.A. Woyczyński. *Random Series and Stochastic Integrals: Single and Multiple*. Birkhäuser, Boston, 1992.

[35] V.A. Lebedev. Behavior of random measures under filtration change. *Theory Probab. Appl.*, 40(4):645–652, 1995.

[36] M.B. Marcus and J. Rosiński. Continuity and boundedness of infinitely divisible processes: a Poisson point process approach. *J. Theor. Probab.*, 18(1):109–160, 2005.

[37] T. Marquardt and L.F. James. Generating long memory models based on CARMA processes. Technical report, Technische Universität München, 2007.

[38] J. Mémin. Espaces de semi martingales et changement de probabilité. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, 52(1):9–39, 1980.

[39] M. Metivier and J. Pellaumail. Mesures stochastiques à valeurs dans des espaces L_0. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, 40(2):101–114, 1977.

[40] M. Moser and R. Stelzer. Functional regular variation of Lévy-driven multivariate mixed moving average processes. *Extremes*, 16(3):351–382, 2013.

[41] C. Mueller. The heat equation with Lévy noise. *Stoch. Process. Appl.*, 74(1):67–82, 1998.

[42] L. Mytnik. Stochastic partial differential equation driven by stable noise. *Probab. Theory Relat. Fields*, 123(2):157–201, 2002.

[43] M. Nagasawa and H. Tanaka. Stochastic differential equations of pure-jumps in relativistic quantum theory. *Chaos Solitons Fractals*, 10(8):1265–1280, 1999.

[44] J. Pedersen. The Lévy-Itô decomposition of an independently scattered random measure. Technical report, MaPhySto, University of Aarhus, 2003.

[45] V. Pipiras and M.S. Taqqu. Integration questions related to fractional Brownian motion. *Probab. Theory Relat. Fields*, 118(2):251–291, 2000.

[46] B.S. Rajput and J. Rosiński. Spectral representation for infinitely divisible processes. *Probab. Theory Relat. Fields*, 82(3):451–487, 1989.

[47] J. Rosiński. On path properties of certain infinitely divisible processes. *Stoch. Process. Appl.*, 33(1):73–87, 1989.

[48] E. Saint Loubert Bié. Étude d'une EDPS conduite par un bruit poissonnien. *Probab. Theory Relat. Fields*, 111(2):287–321, 1998.

[49] K.-I. Sato. Stochastic integrals in additive processes and applications to semi-Lévy processes. *Osaka J. Math.*, 41(1):211–236, 2004.

[50] V. Todorov and G. Tauchen. Simulation methods for Lévy-driven continuous-time autoregressive moving average (CARMA) stochastic volatility models. *J. Bus. Econ. Stat.*, 24(4): 455–469, 2006.

[51] J.B. Walsh. An introduction to stochastic partial differential equations. In P.L. Hennequin, editor, *École d'Été de Probabilités de Saint Flour XIV - 1984*, pages 265–439. Springer, Berlin, 1986.