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Complexity and Approximation of Fundamental Problems in Computational Convexity

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Abstract

The present thesis investigates several fundamental problems in Computational Convexity, i.e. it studies algorithmic questions on convex sets in unbounded dimensions. Throughout, a particular focus is put on the question how the dimension influences the complexity and approximability of these problems.

To found our work geometrically, we use symmetry coefficients of convex bodies to sharpen classic geometric inequalities and formulate them in a dimension independent way. We employ the theory of Fixed Parameter Tractability to assess the influence of the dimension on the complexity of various geometric optimization problems. Whenever possible, we give variants of Helly-type theorems (sometimes approximative or local), which contribute to the algorithmic solutions of the considered problems.

Zusammenfassung

Die vorliegende Dissertation untersucht eine Reihe grundlegender Probleme des Gebiets der *Computational Convexity*, d.h. des Studiums algorithmischer Fragen auf konvexen Mengen in unbeschränkter Dimension. Besonderes Augenmerk liegt dabei stets auf der Frage, welchen Einfluss die Dimension auf die Komplexität bzw. die Approximierbarkeit dieser Probleme hat. Obwohl die betrachteten Probleme auf den ersten Blick sehr ähnlich erscheinen, zeigt sich in den einzelnen Kapiteln, dass dieser Einfluss die verschiedensten Ausprägungen annehmen kann.

In einem ersten Schritt zeigen wir, dass viele bekannte geometrische Ungleichungen verschärft werden können, indem sie auf Symmetriewerte der beteiligten konvexen Körper gestützt werden. Anders als die Originale machen unsere Verallgemeinerungen deutlich, dass die in diesen Ungleichungen auftretenden Koeffizienten nicht zwangsläufig von der Dimension abhängen. Vielmehr geht die Dimension lediglich als obere Schranke für die größtmögliche Asymmetrie eines Körpers in die Ungleichungen ein.

Wir verwenden die Theorie der *Fixed Parameter Tractability*, um die Komplexität des exakten Lösens und der Approximation des Normmaximierungsproblems mit speziellem Hinblick auf die Dimension zu untersuchen. Die zentrale Rolle dieses Problems in der Computational Convexity ermöglicht es dann, Aussagen über die Komplexität einiger grundlegender Radienberechnungsprobleme und des Problems der Berechnung des Hausdorffabstandes zu treffen, auf denen die nachfolgenden Kapitel aufbauen.

Das Konzept der *Core-Sets* zeigt, dass die Komplexität der Approximation des euklidischen Umkugelradius vollständig unabhängig von der Dimension ist. Wir zeigen scharfe geometrische Ungleichungen zwischen verschiedenen Radien, die scharfe Schranken an die Größe von Core-Sets (sowohl im euklidischen Fall als auch in einem ganz allgemeinen Set-up) implizieren.

Für das Problem, einen konvexen Körper homothetisch so zu transformieren, dass er den Hausdorffabstand zu einem anderen Körper minimiert, charakterisieren wir Optimallösungen und geben einen polynomiellen Algorithmus, der dieses Problem für Polytope in Eckendarstellung in unbeschränkter Dimension löst.

Schließlich geben wir eine Art „lokale Helly-Typ Aussage“ für das Problem, einen kleinsten einschließenden Zylinder im Raum zu berechnen. Da andere übliche Ansätze für dieses Problem beweisbar nicht funktionieren können, zeigt unsere Aussage Möglichkeiten auf, doch verwertbare strukturelle Eigenschaft dieses Problems auszumachen.

Contents

Chapter 1

Introduction

The present thesis investigates several basic problems from the area of Computational Convexity. Its results can be seen as an attempt to identify structures in these problems that allow an efficient solution (or approximation) or to show that no such structure exists. The problems under consideration can all be classified as *geometric* optimization problems in a finite dimensional real vector space. Throughout the individual chapters, the thesis puts a special focus on the question how the dimension of that space influences the complexity of the problem. The obtained results show that this influence can lie in a very broad range: There are problems, the hardness of which is directly linked to the dimension; there are others, where the running time of an approximation algorithm does not depend on the dimension at all; and there is a group of problems somewhere in between, where the hardness of the problem is sensitive to the dimension but also to other parameters of the problem.

After a quick overview of the thesis, this introductory chapter will first illustrate the field of Computational Convexity, highlight some central questions in the field and show how the present thesis can contribute to the answers of these questions. After introducing the notation that is used throughout the thesis, we close this introduction with thanks to the people that contributed to the completion of this thesis.

1.1 Overview and Organization of the Thesis

Besides this introductory chapter, the thesis consists of five chapters, which are designed in the style of scientific papers. All chapters try to be as self-contained as possible, so that they can in principle be read in arbitrary order.
Each chapter also contains a short introduction which presents the respective problem of interest and gives references to background material and related work. Moreover, the introduction of each chapter states the main results of the chapter, which are then elaborated in the subsequent sections. Where necessary the introductions also explain special notation adapted to the particular needs in a chapter. Our general notation, however, is explained in Section 1.3.

In a first step, we show that many well-known geometric inequalities can be sharpened by basing them on symmetry coefficients of convex bodies. Unlike the original theorems, our generalizations demonstrate that the parameters of these inequalities do not intrinsically depend on the dimension. Instead, the dimension only enters these inequalities as a worst case upper bound on the asymmetry of a body.

We employ the theory of Fixed Parameter Tractability to revisit the computational complexity and the complexity of approximation of Norm Maximization with a special focus on the dependence on the dimension. The central role of this problem in Computational Convexity then allows us to deduce corollaries on the complexity of several basic radii computation problems and the problem of computing the Hausdorff distance, which are of interest in later chapters.

The concept of core-sets shows that the complexity of approximating the Euclidean circumradius does not depend on the dimension at all. We derive tight geometric inequalities between certain radii that are turned into tight bounds on the size of these core-sets in the Euclidean and in a very general setting.

For the problem of homothetically transforming a convex body in order to minimize its Hausdorff distance to another one, we characterize optimal solutions and give a polynomial time algorithm that solves the problem for vertex-presented polytopes in unbounded dimension.

Finally, we give a "local Helly-type theorem" for the problem of computing a smallest enclosing cylinder in three-space. The result is quite specific but constitutes a first step towards an exploitable structure theorem of a problem for which other common approaches have been shown to fail.

1.2 Computational Convexity

As the name already indicates, the field of Computational Convexity is concerned with the study of *computational* and algorithmic problems on *convex* sets. The problems of interest are often to evaluate some functional on a convex set in a finite dimensional real vector space. An important aspect in the study of these problems is the fact that they are considered in arbitrary (unbounded) dimension rather than in spaces of a (low) a-priori fixed dimension. This leads to a close interplay of Computational Convexity with Mathematical Programming, where the number of variables of a problem is usually not fixed in advance: On the one hand, it allows for the analysis of problems that arise in those fields; on the other hand, methods from Mathematical Programming are also often basic building blocks for efficient algorithms in Computational Convexity. Moreover, the focus on problems in unbounded dimension distinguishes Computational *Convexity* from the area of Computational *Geometry*, which usually investigates algorithmic versions of geometric problems in the plane or in three-space.

In general, the methods that are applied can usually be located somewhere around the intersection of Convex Geometry, Mathematical Programming and Discrete Mathematics as well as Complexity Theory. For complexity questions, usually the binary (Turing machine) model is applied.

For a rough overview, the following Section 1.2.1 highlights some classic and basic problems in Computational Convexity and shows at which points the present thesis can contribute to the field. We refer to [88, 89, 90] for more complete surveys on Computational Convexity and to [75, 92, 150, 162] for comprehensive background material.

1.2.1 Classic Problems

The Volume of a Polytope

As a first example to illustrate the basic questions in Computational Convexity, we mention the problem of computing the volume of a convex compact set $P \subseteq \mathbb{R}^d$. There is no doubt that the volume of P is a basic and important quantity associated with P. It is very intuitive and comes with a whole theory behind it, which one is familiarized with already during the first courses of calculus. However, it is not quite obvious how to actually compute the volume of P even if P is a polytope. Hence, typical questions that arise in this context are the following:

1. For a polytope P, is there a formula or an algorithm to compute its volume?

2. What is a suitable way to present a polytope P for this purpose? Is it convenient to work with a presentation as convex hull of finitely many points or is it preferable to deal with the intersection of finitely many halfspaces?

3. Is there an efficient algorithm for polytopes in high dimensions?

The answers that have been given to these questions in the literature are:

1. A possible approach is to dissect P into simplices, the volume of which can easily be computed, and to add up these volumes. If a polytope P is presented as intersection of halfspaces, there is a closed formula for its volume [127], but it involves summation over all vertices.

2. Seemingly, no particular presentation of P is more suitable than the others [90].

3. The running time of none of the above algorithms depends polynomially on the dimension. In fact, volume computation is #$\mathbb{P}$-hard for the above-mentioned presentations of P [63].

Hence, it turns out that the seemingly innocent problem of computing the volume of a polytope is (not unfeasible but) intractable for general polytopes in high dimensions under standard complexity theoretic assumptions.

Linear Programming

Although the theory around Linear Programming is an independent field for itself, it has to be mentioned in the present context nevertheless. First of all, the problem of computing the maximum of a linear function over a polytope presented as intersection of halfspaces is probably the most basic problem in Computational Convexity. The question whether this maximum can be computed efficiently in arbitrary dimension and the theory developed around it can be seen as the initial motivation for asking, what other functionals can be computed efficiently on arbitrary dimensional polytopes or convex bodies.

Especially the fact that many practical problems can be formulated as linear programs with a problem dependent and possibly large number of variables makes it desirable to consider the problem and its relatives in Computational Convexity in unbounded dimension. Since Linear Programming has been shown to be solvable in polynomial time in this setting, it may also serve as a basic tool for showing the tractability of other more complex problems:

Consider for $A \in \mathbb{Q}^{n \times d}$, $b \in \mathbb{Q}^n$, $c \in \mathbb{Q}^d$ the linear program

$$\begin{array}{rl} \max & c^T x \\ s.t. & Ax \leq b. \end{array} \tag{1.1}$$

If L denotes the coding length of A, b, c, then [118] shows that (1.1) can be solved in $O(n^{3.5}L^2)$. This result is at the heart of many tractability proofs. The algorithms for Hausdorff matching in Chapter 5 also rely on solving linear programs (or the more general second order cone programs).

On the other hand, concerning Linear Programming in fixed dimension, [132] shows that (1.1) can be solved in time $O(2^{2^d}n)$. Thus, if d is considered as a constant, Linear Programming can be solved in linear time. Using this result, we show in Theorem 3.2.12 that the $\mathbb{NP}$-hard problem of maximizing the 1-norm over a halfspace presented polytope is *fixed parameter tractable*. In addition, we also obtain an FPT approximation algorithm for general norm maximization (Theorem 3.1.3), again by solving suitable linear programs.

Radii of Convex Bodies

A problem that has received a lot of attention is the computation of inner and outer radii of convex bodies. Especially the *circumradius, inradius, diameter* and *width* of a convex body have a long tradition in convex geometry. The references ranging from [30] and [67] over [59] to [34], [103] and [102], radii have been used in order to approximate "complicated" sets by simpler ones of appropriate size or to bound other geometric functionals in terms of radii.

There are several ways to embed these four prominent radii into larger series of inner and outer radii of convex bodies and, in this thesis, we will encounter different generalizations at different places. The ones that we are mainly concerned with may also serve as an introductory example[1].

Definition 1.2.1 (*Outer radii*)
For $P \subseteq \mathbb{R}^d$ compact and $C \subseteq \mathbb{R}^d$ a closed convex set, we let

$$R(P,C) := \inf\{\rho \geq 0 : P \subseteq c + \rho C, c \in \mathbb{R}^d\}$$

be the classic circumradius of P with respect to C. For $k \in \mathbb{N}$, let $\mathcal{L}_k^d$ denote the set of all k-dimensional linear subspaces of $\mathbb{R}^d$ and define

$$\overline{R}_k^\pi(P,C) := \sup_{F \in \mathcal{L}_{d-k}^d} R(P, C+F) \tag{1.2}$$

$$\underline{R}_k^\pi(P,C) := \inf_{F \in \mathcal{L}_{d-k}^d} R(P, C+F) \tag{1.3}$$

$$\overline{R}_k^\sigma(P,C) := \sup_{F \in \mathcal{L}_k^d} \sup_{c \in \mathbb{R}^d} R(P \cap (c+F), C) \tag{1.4}$$

$$\underline{R}_k^\sigma(P,C) := \inf_{F \in \mathcal{L}_k^d} \sup_{c \in \mathbb{R}^d} R(P \cap (c+F), C). \tag{1.5}$$

Choosing $C = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$, we obtain the Euclidean circumradius of P via $R(P,C) = \overline{R}_d^\pi(P,C) = \underline{R}_d^\pi(P,C) = \overline{R}_d^\sigma(P,C) = \underline{R}_d^\sigma(P,C)$, half the width of P equals $\underline{R}_1^\pi(P,C)$, half the diameter is $\overline{R}_1^\pi(P,C) = \overline{R}_1^\sigma(P,C)$ and the inradius can be recovered via $R(C,P)^{-1}$. These relations are also illustrated in Figure 1.1.

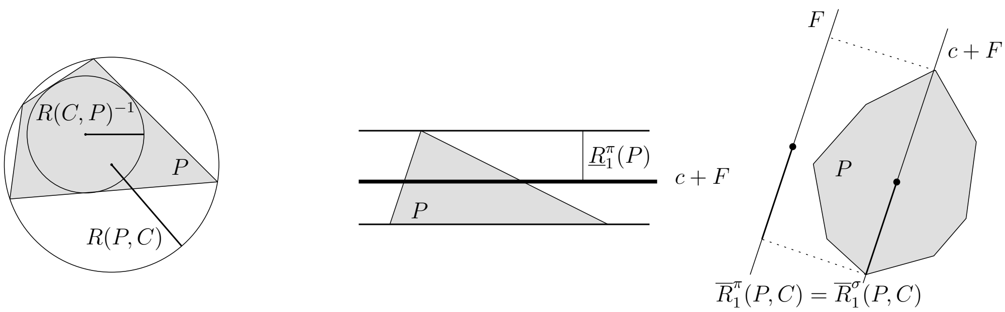

Figure 1.1: Some special radii. Left: A set $P \subseteq \mathbb{R}^2$ with its biggest contained ball of radius $R(C,P)^{-1}$ and circumball of radius $R(P,C)$. Middle: $P \subseteq \mathbb{R}^2$ contained in the slab $c + F + \underline{R}_1^\pi(P,C)C$. Right: $P \subseteq \mathbb{R}^2$ together with the one-dimensional section and projection of biggest radius.

Using some radius of a set to simplify calculations immediately rises the question if and how efficiently the radii of the set can be computed. It is this question that we

[1]We refer to page 15 for an explanation of the notation.

mainly focus on in this thesis. For special polytopes such as simplices, boxes and cross-polytopes, there exist explicit formulae for many radii [35, 68], but in general, it turns out that the presentation of the polytope plays a crucial role for the tractability of radii computation tasks, see [87] and below.

This thesis touches the field of radii of convex bodies at several points:

Basically, the combinatorial properties that make some of the radii computation tasks hard are already present in the problem of maximizing a norm over a polytope presented as intersection of halfspaces. In fact, the $\mathbb{NP}$-hardness proofs for the radii problems rely on the fact of this problem being hard. Since, on the other hand, all these problems are solvable in polynomial time in fixed dimension, Chapter 3 investigates the fixed-parameter-complexity of norm maximization with respect to the dimension. In Theorems 3.1.2 and 3.1.3, we answer the question, to which degree the hardness of computing or approximating the norm maximum over a halfspace presented polytope depends on the dimension. By the well established identities from [86], these results carry over to the radii computation problems and are given in Corollaries 3.4.3 and 3.4.6.

As was observed in [67] and [86], Helly's Theorem plays an important role for the problem of computing $R(P,C)$. The results of Chapter 4 are basically approximative Helly-type theorems that are formulated in terms of inequalities between the radii $\overline{R}_k^\pi(P,C)$ for different values of k. These inequalities between radii can now be translated into sharp bounds on the sizes of so called ε-core-sets, a concept which attracted a lot of interest in recent years: In Euclidean spaces the size of these core-sets does not depend on the dimension of the space (e.g. [17, 18, 19, 55]), which allows approximation of the Euclidean circumradius independently of the dimension, and which in turn leads to efficient approximation algorithms for other, harder problems (e.g. [19]). We are able to give sharp bounds on the size of core-sets in the Euclidean (Theorem 4.1.3) and a more general setting (Theorem 4.1.2) and show that the dimension independence does not carry over from Euclidean to arbitrary normed spaces (Theorem 4.1.2).

However, the same approach fails for the computation of $\underline{R}_{d-1}^\pi(P)$. Already [58] presented a counterexample showing that there is no comparable Helly-property that would make applicable the same techniques as above. A major obstruction on the way to such a property is the non-convexity of the objective function in the optimization problem for the computation of $\underline{R}_{d-1}^\pi(P)$. Hence, a possible approach to overcome this is characterizing *local* optima of the objective function instead of global ones. We follow this approach in Chapter 6, where a first step in this direction is made and a bound on the pinning number (i.e. a local Helly-number) of intersecting balls (Theorem 6.1.1) and ovaloids (Theorem 6.4.5) in $\mathbb{R}^3$ is given.

The Hausdorff distance

As a last example for the type of problems that are considered in this thesis, we mention another indispensable tool in Convex Geometry: the Hausdorff distance of two convex bodies. Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$.

Definition 1.2.2 (*Hausdorff distance*)
For a non-empty, compact and convex set $K \subseteq \mathbb{R}^d$, let

$$\begin{array}{rccl} d(\cdot, K): & \mathbb{R}^d & \to & \mathbb{R} \\ & x & \mapsto & d(x,K) = \min\{\|x-y\| : y \in K\} \end{array}$$

be the distance mapping of K induced by $\|\cdot\|$.
For two compact convex sets $K, L \subseteq \mathbb{R}^d$, the Hausdorff distance between K and L is

$$\delta(K,L) := \max\big\{\max_{x\in K} d(x,L), \max_{x\in L} d(x,K)\big\} \tag{1.6}$$

and measures the maximum distance of a point in K or L to the other body as illustrated in Figure 1.2.

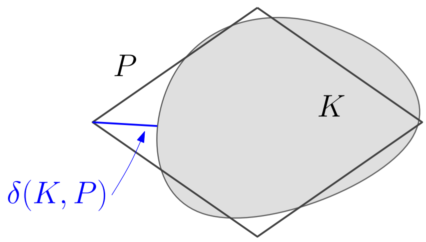

Figure 1.2: The Hausdorff distance of a convex body $K \subseteq \mathbb{R}^2$ and a polytope $P \subseteq \mathbb{R}^2$.

Together with the volume of the symmetric difference of two bodies, the Hausdorff distance is one of the commonly used metrics on the cone of convex bodies that appears whenever the similarity of two convex bodies has to be measured. It is a natural way to express results in the theory of approximations of convex bodies by polytopes and vice versa [93], [150, Section 3.3]. Also the functional itself is very well studied [150, Section 1.8]. In addition, the Hausdorff distance also arises as a natural measure of similarity in shape fitting tasks [10]. Here, also more general matching problems, where one body is allowed to undergo a transformation from a certain family, are considered. For the case of homothetic transformations, Helly-type properties of the Hausdorff distance are shown in [14]. However, the attention in Computational Geometry has focused on the case of two-dimensional polygons and finite point sets in two or three dimensions, which is surveyed in [10]. Polynomial-time algorithms computing the Hausdorff distance between collections of simplices in fixed dimension are presented in [9].

In the context of the present thesis, a natural question is to ask for the complexity of computing the Hausdorff distance of polytopes in arbitrary dimension. To the best of our knowledge, this question has not been answered before and we close this gap in

Theorems 5.2.13 and 5.2.16 building on the results on norm maximization from Chapter 3. Guided by the insights of those theorems, we consider the optimization problem, where a vertex-presented polytope is allowed to be transformed by a homothetic mapping in order to minimize its Hausdorff distance to another vertex-presented polytope. Using the techniques developed in Chapter 4, we characterize optimal solutions in Theorem 5.3.8. We also give an algorithm to solve this matching problem and variants of it in Section 5.3.3. The theoretical running times of our algorithm are comparable to the ones in low dimension cited above and the only ones which also depend polynomially on the dimension.

1.2.2 Applications in Practice

The problems presented in the previous section are interesting mathematical problems that deserve to be investigated in their own right. Additionally, the obtained insights – in particular the ones on the algorithmic side – may be helpful for the solution of many applied problems. The present subsection is intended to support this claim by presenting some applications that are closely related to this thesis: During the time of this thesis, the author has been working actively in the mentioned application areas by contributing to third party projects, supervising students or co-authoring papers in the respective area. The selection is by no means exhaustive; we rather refer to the list of references at the end of this section for further possible applications.

Shape Fitting

The task of quantifying how much two given object resemble each other arises in many application areas, such as pattern recognition, computer vision, computer graphics, etc. Often the task is not only to evaluate the resemblance of two shapes but to transform one shape such that it becomes as close as possible to another one. We give two examples that show how methods from Computational Convexity can be applied for these shape fitting problems:

The paper [37] investigates a problem that arises in operation planning in medical surgery and explores the applicability of approximation by affine subspaces, i.e. radii computations, for shape matching problems. In order to facilitate automation of the planning process of a limb lengthening surgery, the data obtained by a computerized tomography scan of the patient's leg has to be processed to a compact and tractable description. Since the relevant parts of the (deformed) femur basically resemble a cylinder, [37] suggests to use the union of two cylinders instead of the whole bone data. Hence, once the bone data is split into the appropriate parts, the task amounts to computing the outer 2-radius (in the sense of Equation (1.3) in Definition 1.2.1) of the respective part of the bone together with an optimal axis. Figure 1.3 shows a major part of a femur diaphysis and its best approximating 2-cylinder.

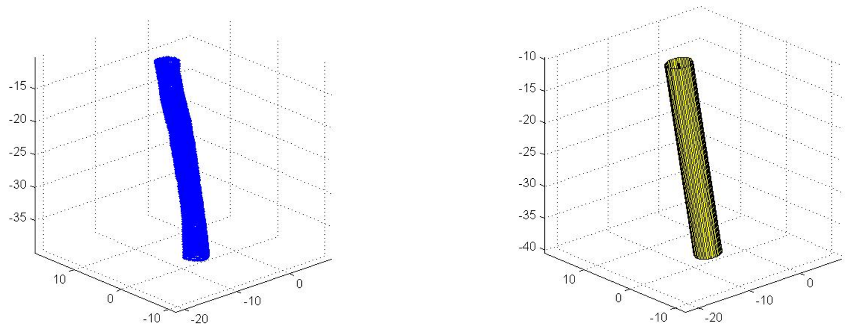

Figure 1.3: A femur part and its best approximating cylinder. Left: 3341 points in 3-space segmented from a CT of a femur. Right: A best approximating 2-cylinder that replaces the femur in computations.

It has already been claimed above that the Hausdorff distance is a natural objective function in shape fitting tasks. In order to support this claim, we mention a few of the possible applications.

In [112], the Hausdorff distance and variants of it are investigated in order to detect objects from one image in another one. The paper [114] develops a face detection algorithm based on the Hausdorff distance as the underlying similarity measure which is used to compare the region around the eyes in images to a geometric description thereof. In the application in Computer Vision in [148], a model set that is to be detected in an image is allowed to be affinely transformed before being matched to some part of the image.

The basic idea behind all these approaches is to define some model set and then find a suitable transformation of that model that minimizes the Hausdorff distance between the transformed model and some object segmented in the (image) data. In all three cases, the authors report good results and especially highlight the fact that the Hausdorff distance is very robust concerning changes in illumination, contrast and other image properties.

The particular application of Hausdorff Matching that we are aiming at in Section 5.3.3 can be interpreted as a shape fitting approach for tomographic reconstruction with prior knowledge and is explained in the next paragraph.

Tomographic Reconstruction by Shape Fitting

In the field of Geometric Tomography, one is concerned with the problem of reconstructing convex sets from tomographic data, e.g. from X-rays (i.e. Radon transforms) or simply projections or sections of the set. The textbook [71] gives a comprehensive overview of the field. Here, we focus on two specific problems at the intersection of Geometric Tomography and Computational Convexity.

As mentioned in the paragraph about Shape Fitting, Section 5.3.3 shows, that if strong prior knowledge about the objects that are to be reconstructed is assumed, shape fitting methods can also be used for tomographic reconstruction. In particular, we will show how the methods for Hausdorff matching developed in Chapter 5 can be applied to the task of “reconstructing” a femur from only two projections.
This approach is again motivated by the planning procedure for the limb lengthening surgery considered in [37] and described above. Since for the actual planning procedure the CT data of the femur of the patient is processed to a cylinder with the essential features of the bone, one might as well think of replacing the whole CT scan by fewer ordinary radiographs and extract the relevant information from these images. The benefits of this approach are self-explanatory: The patient benefits from a lower radiation dose and the two radiographs taken during the initial examination of the patient can be reused for the planning procedure, saving time and money thereby.
A possible way to construct a model of the patient’s femur is as follows. In a first step, we provide a geometric description of a model femur (or of parts of it) which we extract from the CT data of a plastic femur with standard parameters [25]. In order to recover essential parameters of the femur of a specific patient, we segment the bones in each radiograph and solve the following matching problem: Find a transformation of the model femur such that the sum of Hausdorff distances of the bone regions in the radiographs to the projections of the transformed model is minimized. In our application, these radiographs are taken in two a-priori fixed directions which leads to the optimization problem (5.28) in Section 5.3.3.
Since we are working with polytopes instead of curved bones, we are, strictly speaking, matching the convex hull of the model femur to the convex hull of the segmented bones in the radiographs. However, applying the optimal transformation for the convex hull to the model femur itself seems to yield good approximations of the femur of the patient in practice (cf. Section 5.3.3, in particular Figure 5.3).

Another possibility of providing a 3D model of the patient’s femur is to build it directly from geometric primitives that are suitably adjusted according to parameters that can be determined from the two radiographs. Using a standard set of parameters as listed in e.g. [137, 139], this approach is explored in [108]. Figure 1.4 shows a typical result of a femur model together with its projection into the original radiograph.

Tomographic Reconstruction by Quadratic Programming

Another method from Geometric Tomography that can be subsumed in Computational Convexity in the broadest sense is the following algorithm introduced in [73]. The task of this algorithm is to reconstruct an approximating polytope to a convex body $K \subseteq \mathbb{R}^d$ from which noisy measurements of its support function

$$h(K, \cdot) : \mathbb{R}^d \to \mathbb{R}; u \mapsto \max\{u^T x : x \in K\}$$

are available. In cooperation with an interdisciplinary team of material scientists and mathematicians, we applied this algorithm to the task of reconstructing a nanowire from

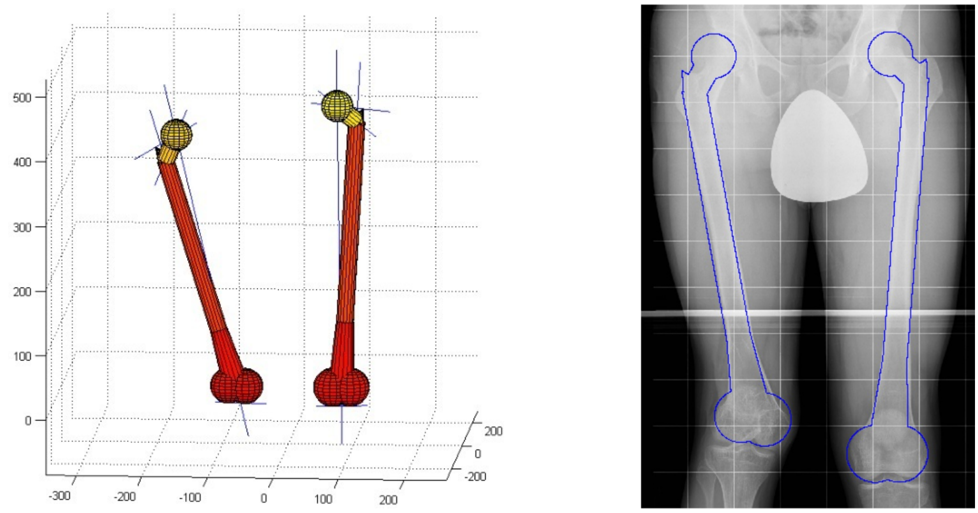

Figure 1.4: Femur reconstruction with geometric primitives. Left: The geometric model of the femora of a patient consisting of cylinders, balls, and truncated cones. Right. Its projection into an X-Ray image of the lower extremities of the respective patient. Both images are taken from [108].

electron microscopy data. We also compared its performance, along with other geometric algorithms, to conventional tomography algorithms such as SIRT [117, Chapter 7], different discrete variants of ART [106, Chapter 11], or filtered backprojection [106, Chapter 8]. For the present purpose, we sketch our main results in the following; for a complete assessment, however, we refer to the publications [7] and [141].

Nanowires, small wires that are tens of nanometers in diameter and micrometers in length, are promising building blocks for future electronic and optical devices; see [129, 135]. They are typically grown from a substrate and much research effort is being focused on understanding and controlling their growth mechanisms [62]. Electron tomography, as in various materials science applications, is rapidly developing into a powerful 3D imaging tool for studying these effects at the nanoscale [22, 134].
With current technology, however, the tomographic data acquisition time for 140 projections of a single nanowire is about 2 hours when performed manually. This is currently a bottleneck preventing many in-situ experiments on short time scales and the imaging of multiple nanowires. Hence, methods that require considerably fewer projection images are of particular interest.

The cross-sections in horizontal (i.e. perpendicular to the growth direction) slices of the nanowire in our application are all convex. In fact, most of the cross-sections are close to regular hexagons [157]. This allows us to reconstruct the whole nanowire by reconstructing one 2D slice after the other from the support function values obtained from the microscopy images.

The support function values in the two directions perpendicular to the projection direction can be easily determined from the data, because they correspond to the minimal and maximal coordinates of the pixels in the data that recorded non-zero intensities, cf. Figure 1.5. As data is available for several tilt angles, we collect support function measurements for different directions. These serve as input to the algorithm.

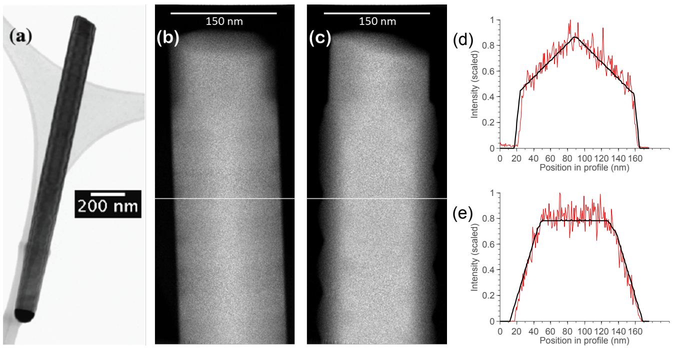

Figure 1.5: Nanowire data: (a) bright-field electron microscopy image of the InAs nanowire specimen used for tomography; (b,c) aligned images taken from the tilt series of the wire at angles of 10° and 40°, respectively (slice 220 is indicated by the white line; the tilt axis is in the vertical direction through the center of the image); (d,e) measured projections of slice 220 (non-linear projection intensities) shown in red and ideal projections of slice 220 (linear projection intensities) shown in black, at angles of 10° and 40°, respectively. (The ideal projections were estimated from our reconstructions. For further details, we refer to [7].)

Unfiltered backprojection (U-FBP) U-FBP is probably the most basic geometric method to reconstruct a convex set K from measurements of its support function. The idea is to "backproject" the slab between two hyperplanes where K is known to be contained in, and to return the intersection of all these slabs. The returned object is necessarily a polyhedron.

Modified Prince-Willsky (MPW) Since the measurements in our application can be very noisy which may lead to inconsistencies in the data (cf. Figure 1.6), we apply the more sophisticated modified Prince-Willsky algorithm from [73], which is a modification of the algorithm in [143].

The MPW algorithm is designed to cope with noise in the measurements of the support function. More precisely, for (Gaussian) noise affected measurements $h_1, \ldots, h_n$ of the support function of K in a finite number of directions $u_1, \ldots, u_n$, the MPW algorithm solves a constrained least-squares problem to obtain values $y_1, \ldots, y_n$, which are the support function values of a best-approximating polyhedron P^*. The set P^* itself is then obtained as intersection of halfspaces $P^* = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d : u_i^T x \leq y_i\}$, for instance

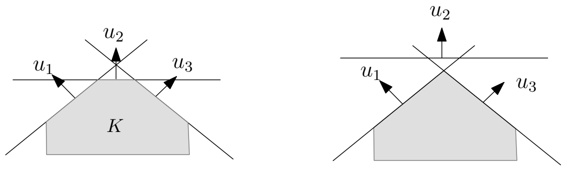

Figure 1.6: Inconsistent support function measurements. Left: A convex set K and three exact measurements of its support function. Middle: A wrong support function measurement leads to inconsistent data. Right: A wrong support function measurement cuts away a big part of K.

via U-FBP.
With mild restrictions, the output of the algorithm converges as the number of shadows, affected by Gaussian noise of fixed variance, approaches infinity [74].

Figure 1.7 shows that the quality of the reconstruction of MPW is comparable to the ones obtained by standard tomography algorithms. Further, it appears superior in settings where the reconstruction has to be performed from very few, very noisy images. A detailed assessment of the results is, however, beyond the scope of this thesis and we refer to [141] and [7] for further details.

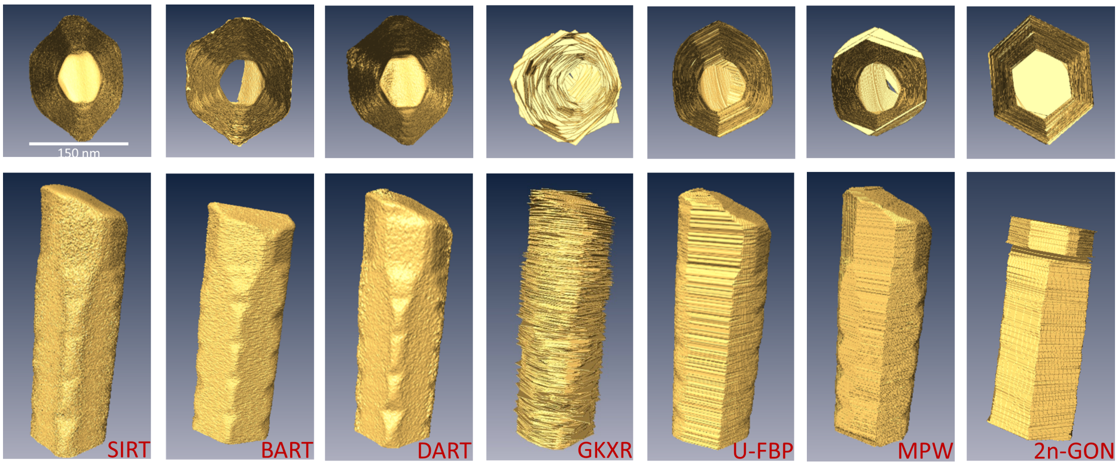

Figure 1.7: Reconstruction of the nanowire using different algorithms. Top-to-bottom and frontal views are shown in the first and second row, respectively. BART is the algorithm introduced in [105]; DART in [24]; for GKXR see [72]; $2n$-GON reconstructs a near-regular $2n$-gon in every cross section and is introduced in [7].

Clustering via k-center

Besides the direct applications that have already been mentioned, an additional combinatorial layer can be added to almost all of the above problems. For instance, a bent

bone is probably better approximated by the union of two (or more) cylinders than by one; the Hausdorff distance could also serve as a similarity measure between unions of polytopes; etc. In these cases the task is not only to evaluate a geometric functional on a single convex set but an additional assignment problem has to be solved in order to determine which functional has to be evaluated for which set.
We make this precise for the well-studied example of the so called k-center problem, which is the combinatorial extension of the outer radii of finite point sets as introduced in Definition 1.2.1.

Problem 1.2.3 (*k-center*)
Let $(C_d)_{d\in\mathbb{N}}$ be a family of convex compact sets with $C_d \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$ and $j \in \{1, \ldots, d\}$. The following problem is called k-center problem and illustrated in Figure 1.8.

Input. A finite point set $P := \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ in arbitrary dimension d.
Task: Find an assignment $I : \{1, \ldots, n\} \to \{1, \ldots, k\}$ that minimizes

$$\max_{v=1,\ldots,k} \underline{R}_j^\pi(\{p_i : I(i) = v\}, C_d).$$

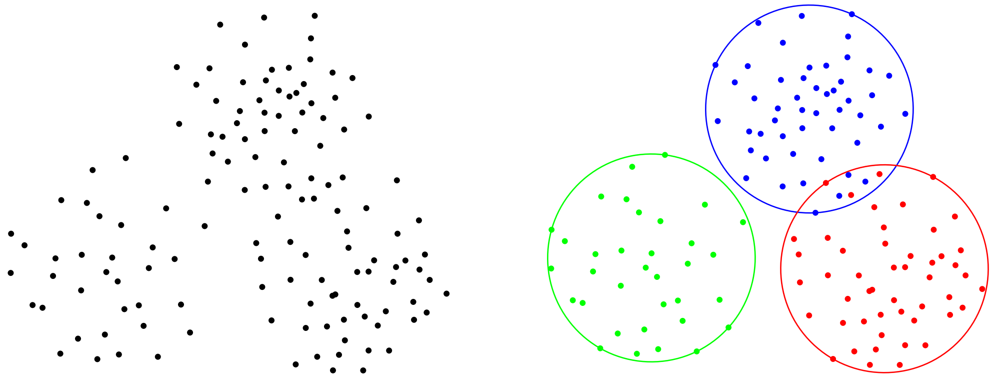

Figure 1.8: The k-center problem. Left: A 2-dimensional input point set. Right: An optimal solution for the 3-center problem with $j = 2$ in the Euclidean norm. The assignment to three clusters is indicated by the three different colors.

Because of the additional combinatorial difficulties, it is not surprising that the k-center problem and similar problems tend to become hard very quickly (see e.g. [49, 133]). This complexity is also made perceptible for a non-mathematical audience in the Java applet developed in [61], which is now available at http://www-m9.ma.tum.de/Allgemeines/KCenterSpielUeberblick. The reason why these problems are nevertheless extensively studied is their apparent application in areas that involve the clustering of data. Hence, applications range from data mining, pattern recognition, preprocessing for the creation of efficient data structures as examples in high dimensional spaces to

problems like facility location or shape fitting with unions of convex sets in low dimensional spaces [3, 41, 98, 144, 158]. Of course, once a solution of Problem 1.2.3 is at hand, it can also be used for classification tasks by classifying a new point as belonging to its nearest cluster.
The paper [41] proposes different solution techniques for the case $j = 1$ of Problem 1.2.3; [2] is mainly concerned with the cases where $j < d$ and the data is spread around several affine subspaces in a high dimensional space. Both papers contain an extensive list of pointers to problems where this clustering technique has been applied successfully. For applications of clustering in general, we refer to the broad references in the introduction of [32].
For the actual solution of the k-center problem, the concept of ε-core-sets allows to derive a very effective polynomial time approximation scheme via a Branch&Bound technique [19, 41]. The present thesis contributes to this study by giving sharp bounds on the size of ε-core-sets in Chapter 4.

We close this section by pointing again to [88, 89, 90, 125], where many other successful applications of methods from Computational Convexity are described.

1.3 General Notation and Background

Before going into details in the following chapters, we fix some general notations that we will make use of throughout the thesis. Notations that only appear locally in certain sections are explained within the respective section. Additionally, the table on page 131 gives a list of all notations used and pointers to the pages where they are introduced.

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are used to denote the set of positive integers, integers, rational numbers and real numbers, respectively.
For a positive integer $n \in \mathbb{N}$, we will abbreviate $[n] := \{1, \ldots, n\}$.
Throughout this thesis, we are working in d-dimensional real space $\mathbb{R}^d$ and for $A \subseteq \mathbb{R}^d$ we write $\mathrm{lin}(A)$, $\mathrm{aff}(A)$, $\mathrm{conv}(A)$, $\mathrm{pos}(A)$, $\mathrm{int}(A)$, $\mathrm{relint}(A)$, and $\mathrm{bd}(A)$ for the linear, affine, convex or positive hull and the interior, relative interior and the boundary of A, respectively.
For a set $A \subseteq \mathbb{R}^d$, its dimension is $\dim(A) := \dim(\mathrm{aff}(A))$. Furthermore, for any two sets $A, B \subset \mathbb{R}^d$ and $\rho \in \mathbb{R}$, let $\rho A := \{\rho a : a \in A\}$ and $A + B := \{a + b : a \in A, b \in B\}$ the *ρ-dilatation* of A and the *Minkowski sum* of A and B, respectively. We abbreviate $A + (-B)$ by $A - B$ and $A + \{c\}$ by $A + c$. A set $K \subseteq \mathbb{R}^d$ is called 0-*symmetric* if $-K = K$. If there is a $c \in \mathbb{R}^d$ such that $-(c + K) = c + K$ we call K *symmetric*. At several points, we will make use of the identity $A + A = 2A$ for $A \subseteq \mathbb{R}^d$ convex.
A set $C \subset \mathbb{R}^d$ which is non-empty, convex and compact will be called a *convex body* or *body* for short. The set of all convex bodies in $\mathbb{R}^d$ will be denoted by $\mathcal{C}^d$. The set of all polytopes in $\mathbb{R}^d$ is denoted by $\mathcal{P}^d$. For $k \in [d]$, a k-simplex is the convex hull of $k + 1$ affinely independent points. For $x, y \in \mathbb{R}^d$, we write $[x, y] = \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$ for the *line segment* joining x and y. If a polytope $P \in \mathcal{P}^d$ is described as a bounded

intersection of halfspaces, we say that P is in $\mathcal{H}$-presentation. If P is given as the convex hull of finitely many points, we call this a $\mathcal{V}$-presentation of P. For a convex set $C \subseteq \mathbb{R}^d$, $\operatorname{ext}(C)$ and $\operatorname{rec}(C)$ denote the set of *extreme points* and the *recession cone* of K, respectively.
Furthermore, $\mathcal{L}_k^d$ and $\mathcal{A}_k^d$ denote the family of all k-dimensional linear and affine subspaces of $\mathbb{R}^d$, respectively, and $A|F$ is used for the orthogonal projection of A onto F for $F \in \mathcal{A}_k^d$.
For $1 \leq p < \infty$, the p-norm of a point $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$ is defined as

$$\|x\|_p := \left(\sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}$$

for $p = \infty$, we let $\|x\|_\infty := \max\{|x_i| : i \in [d]\}$.
For $p \in [1, \infty]$, we write $\mathbb{B}_p^d := \{x \in \mathbb{R}^d : \|x\|_p \leq 1\}$ for the unit ball of $\|\cdot\|_p$ and $\mathbb{S}_p^{d-1} := \{x \in \mathbb{R}^d : \|x\|_p = 1\}$ for the unit sphere in $\mathbb{R}^d$.
For two vectors $x, y \in \mathbb{R}^d$, we use the notation $x^T y := \sum_{i=1}^d x_i y_i$ for the standard scalar/inner/dot product of x and y and by

$$H_{\leq}(a, \beta) := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$$

we denote the half-space induced by $a \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, bounded by the hyperplane $H_=(a, \beta) := \{x \in \mathbb{R}^d : a^T x = \beta\}$. For a vector $a \in \mathbb{R}^d$ and a convex set $K \subseteq \mathbb{R}^d$, we write

$$h(K, a) := \sup\{a^T x : x \in K\}$$

for the *support function* of K in direction a.
For a convex body $C \in \mathcal{C}^d$, we write $C^\circ := \{a \in \mathbb{R}^d : a^T x \leq 1 \ \forall x \in C\}$ for its polar.
If X is a finite set and $k \in \mathbb{N}$, then $\binom{X}{k} := \{Y \subseteq X : |Y| = k\}$ denotes the set of all subsets of X of cardinality k.
The standard basis in $\mathbb{R}^d$ is denoted by $\{e_i : i \in [d]\}$; the all-ones vector by $\mathbb{1} := (1, \ldots, 1)^T \in \mathbb{R}^d$.

In general, we use lower case latin letters for vectors, and upper case latin letters for sets or sometimes matrices. Lower case greek letters are usually used for real scalars whereas the letters n, m, k, d are usually positive integers. We implicitly assume this throughout the thesis in statements such as "Let $p_1, \ldots, p_n \in \mathbb{R}^d$".

Throughout the thesis, the term "Theorem" indicates the major results of a chapter. "Lemmas" are used to prepare the proof of the main theorems or give additional side notes. The term "Proposition" is employed for a theorem which is stated within this thesis but which has been proved elsewhere.

We denote by $\mathbb{P}$ (and $\mathbb{NP}$, respectively) the classes of decision problems that are solvable (verifiable, respectively) in polynomial time. For an account on complexity theory, we refer to [75]. We write FPT for the class of fixed-parameter-tractable problems and W[1] for the problems of the first level of the W-hierarchy in the theory of Fixed Parameter Tractability. For an introduction to Fixed Parameter Tractability, we refer to the textbooks [70, 138].

For comprehensive background material, we refer to [150] for the general theory of convex bodies, to [96, 162] for polytopes in particular, to [147] for Convex Analysis, to [92] for the Algorithmic Theory of Convex Bodies and to [56, 57] for an account on Linear Programming.

1.4 Acknowledgments

First and foremost, I would like to use this occasion to thank the coauthors of the different papers this thesis is based on. Besides the many fruitful and motivating discussions concerning the content of the papers, I could benefit a lot from their knowledge and their scientific experience. The individual authors are acknowledged again within the respective chapters together with a reference to the joint paper. Still, within this group, I would like to point out three persons to whom I feel particularly grateful:

My first thanks go to René Brandenberg who has guided my way since 2006 by supervising my *Projekt-* and *Diplomarbeit* as well as this thesis. I am very grateful to him for introducing me to the scientific world and in particular to the convex geometry community, as well as for his open ear for all kinds of problems concerning research, teaching duties, funding, etc. that arise during the time of a Ph.D. .
I also would like to thank Xavier Goaoc for being a great host during my INRIA internship, which not only culturally but also scientifically broadened my mind and where I was able to learn a lot. Moreover, the invitation to the 10th INRIA-McGill workshop on Computational Geometry in February 2011 is at the origin of the results in Chapter 3.
As well, I would like to thank Andreas Alpers for introducing me to the tomography community, showing interesting real world problems to me and of course for the opportunity to work for his and Peter Gritzmann's DFG grant (AL 1431/1-1, GR 993/10-1), by which I was partially funded.

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Last but by no means least, I would like to express my gratefulness towards my adviser Peter Gritzmann. I am very thankful for the advice and insights that he shared with me and for the freedom that he accorded me in many aspects. I am very glad that I was able to write my dissertation in his group and benefit from his experience in scientific questions and beyond.

Chapter 2

Symmetry Coefficients in Geometric Inequalities

Many classical geometric inequalities on functionals of convex bodies depend on the dimension of the ambient space. In this first chapter, we show that this dimension dependence may often be replaced (totally or partially) by different symmetry measures of the convex body. Since these coefficients are bounded by the dimension but possibly smaller, our inequalities sharpen the original ones. Since they can often be computed efficiently, the improved bounds may also be used to obtain better bounds in approximation algorithms.

This chapter is joint work with René Brandenberg and a joint paper presenting the obtained results is currently in preparation [39].

2.1 Introduction

Since Jung's famous inequality [116] in 1901, geometric inequalities relating different radii of convex bodies form a central area of research in convex geometry. Starting with [30], in many classic treatises on convexity, significant parts are devoted to geometric inequalities among radii (e.g. [33], [59, Section 6], [66, Chapter 6], [97, Section 4.1.3]). In a broader context concerning the relevance of geometric inequalities, also the textbooks [47] and [150] should be mentioned here.

Interesting and beautiful results of their own, geometric inequalities also serve as indispensable tools for many results in convex geometry itself as well as in other application areas. It is therefore not surprising that results such as Jung's Inequality or John's Theorem [115] still are frequently cited in a broad variety of papers. Thus, even more than a century after Jung's seminal inequality, the area of geometric inequalities in general and especially among radii is still a prosperous field of research (see [36, 83, 100, 107, 142, 151] for inequalities among radii of convex bodies and [26, 44, 104] for inequalities involving radii and other geometric functionals).

The kind of inequalities to be considered in the following usually bound a geometric functional (e.g. a certain radius) of a convex body in terms of another one. The statement of the theorem then usually consists of two parts: a general bound on the ratio of these two functionals that holds true for arbitrary convex bodies and an additional statement that the bound can be improved (sometimes to a trivial bound) if the body under investigation is symmetric. In this paper, we propose to use measures of symmetry to strengthen geometric inequalities for convex bodies that are not symmetric but possibly far from the worst case bound in the original theorem.
In particular, we prove sharpened versions of a classic inequality between in- and circumradius (e.g. [69, p. 28]), and of the famous theorems of *Jung* [116], *Steinhagen* [153], *Bohnenblust* [28], *Leichtweiß* [128], and *John* [115].

The symmetry measures that we use for this purpose are variants of Minkowski's measure of asymmetry and have the desirable advantage that they are computable for polytopes via Linear Programming (see Lemmas 2.3.5 and 2.3.9). Hence, the improvement from basing these inequalities on symmetry coefficients is not only of theoretical interest but also allows better bounds in practical applications as in core set algorithms (cf. Chapter 4) or e.g. in [65, 121].
As a first note on the role of the dimension in the present context, we point out that our inequalities show, that in many cases the ratio between two functionals is bound solely to the symmetry coefficients and not intrinsically dependent on the dimension. The dimension dependence, which is known from the original theorems, only enters the inequalities as a worst case bound on the symmetry coefficient.

This chapter is organized as follows. Section 2.2 starts with the definition of the different radii that appear in the course of this chapter along with some basic properties. Then, Section 2.3 introduces two variants of symmetry measures that we use in the subsequent sections. The remainder of the chapter is organized in groups around the individual theorems in the section headings that are generalized.

2.2 Radii Definitions and Preliminaries

2.2.1 Radii Definitions

We start this section by defining the circumradius of a closed convex set $K \subseteq \mathbb{R}^d$ with respect to some gauge body $C \subseteq \mathbb{R}^d$. The circumradius appears at many points throughout this chapter and also serves for the definition of other radii and symmetry coefficients. Note that in all the following definitions C is not necessarily assumed to be symmetric.

Definition 2.2.1 (*C-radius*)
Let $K, C \subseteq \mathbb{R}^d$ non-empty, closed and convex. We denote by $R(K, C)$ the least dilatation factor $\rho \geq 0$ such that a translate of ρC contains K, and call it the *C-radius* of K (cf. Figure 2.1). In mathematical terms,

$$R(K, C) := \inf\{\rho \geq 0 : \exists c \in \mathbb{R}^d \ s.t. \ K \subseteq c + \rho C\}. \tag{2.1}$$

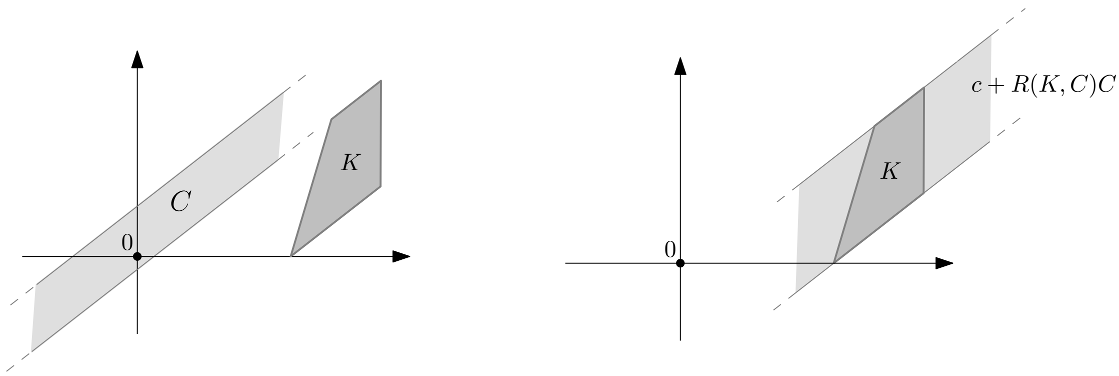

Figure 2.1: The C-radius of a convex body $K \subseteq \mathbb{R}^2$. Left: The convex body K and an unbounded closed convex set C. Right: A minimally scaled copy of C is translated such that it contains K.

If $C = \mathbb{B}_2^d$ is the Euclidean ball, $R(K, \mathbb{B}_2^d)$ is the common Euclidean circumradius of K. If C is 0-symmetric $R(K, C)$ measures the circumradius of K with respect to the norm $\|\cdot\|_C$ induced by the gauge body C. Since Definition 2.2.1 allows unbounded convex sets K and C, one has to be careful with the cases where the infimum in (2.1) is not attained. We treat these cases in the following lemma. Note that by definition $R(K, C)$ is invariant under translations of K and C. Hence, we may assume $0 \in \text{relint}(K) \cap \text{relint}(C)$ without loss of generality, wherever it simplifies the notation.

Lemma 2.2.2
Let $K, C \subseteq \mathbb{R}^d$ non-empty, convex and closed with $0 \in \text{relint}(K) \cap \text{relint}(C)$. Then,

a) $R(K, C) < \infty$ if and only if $K \subseteq \text{lin}(C)$ and $\text{rec}(K) \subseteq \text{rec}(C)$,

b) $R(K, C) = 0$ if and only if $R(K, \text{rec}(C)) < \infty$, and

c) if $R(K, C) \notin \{0, \infty\}$, there exists a center $c \in \mathbb{R}^d$ such that $K \subseteq c + R(K, C)C$.

Proof.
Let $K_1 := \text{conv}(\text{ext}(K))$ and $C_1 := \text{conv}(\text{ext}(C))$ such that K and C can be expressed as $K = K_1 + \text{rec}(K)$ and $C = C_1 + \text{rec}(C)$, respectively.

a) If $R(K, C) < \infty$, there exist $c \in \mathbb{R}^d$, $\rho \geq 0$ such that $K \subseteq c + \rho C$. This implies the right hand side in a). If, on the other hand, $K \subseteq \text{lin}(C)$ and $\text{rec}(K) \subseteq \text{rec}(C)$, we immediately obtain $K_1 \subseteq \text{lin}(C)$ and since K_1 is bounded and $0 \in \text{relint}(C)$, there exists $\rho > 0$ such that $K_1 \subseteq \rho C$. Moreover, since $\text{rec}(K) \subseteq \text{rec}(C) = \rho \text{ rec}(C)$, we obtain $K = K_1 + \text{rec}(K) \subseteq \rho C$.

b) Assume that $R(K, C) = 0$. Then, by a), $\operatorname{rec}(K) \subseteq \operatorname{rec}(C)$. If $R(K, \operatorname{rec}(C)) = \infty$, then a) implies the existence of a point $x \in K$ such that $x \notin \operatorname{lin}(\operatorname{rec}(C))$. Now, assume without loss of generality that $C_1 \subseteq \mathbb{B}_2^d$. Thus, $c + \rho C = c + \rho C_1 + \operatorname{rec}(C) \subseteq c + \rho \mathbb{B}_2^d + \operatorname{lin}(\operatorname{rec}(C))$ for all $c \in \mathbb{R}^d$ and $\rho > 0$. Denote the Euclidean distance of x to $\operatorname{lin}(\operatorname{rec}(C))$ by $\bar{\rho} > 0$. Since $x, 0 \in K$, we conclude that $K \subseteq c + \rho C$ is possible only if $\rho \geq \frac{\bar{\rho}}{2} > 0$, which contradicts the assumption.

   If, on the other hand, there exist $c \in \mathbb{R}^d$ and $\rho^* \geq 0$ such that $K \subseteq c + \rho^* \operatorname{rec}(C)$, then $K \subseteq c + \rho \operatorname{rec}(C) \subseteq c + \rho C$ for all $\rho > 0$ and therefore $R(K, C) = 0$.

c) If $R(K, C) \in (0, \infty)$, Part a) and b) imply $\operatorname{rec}(K) \subseteq \operatorname{rec}(C)$ and $K_1 \not\subseteq \operatorname{lin}(\operatorname{rec}(C))$. Hence, there exists $\rho > 0$ such that $R(K, C) = R(K_1, C) = R(K_1, C \cap \rho \mathbb{B}_2^d)$ and therefore by the Blaschke selection theorem [150, Theorem 1.8.6] some $c \in \mathbb{R}^d$ such that $K_1 \subseteq c + R(K, C)(C \cap \rho_2 \mathbb{B}_2^d) \subseteq c + R(K, C)C$. Because of $\operatorname{rec}(K) \subseteq \operatorname{rec}(C)$, this implies $K \subseteq c + R(K, C)C$.

□

As an immediate corollary of Lemma 2.2.2, we obtain the following if K and C are bounded.

Corollary 2.2.3
Let $K, C \in \mathcal{C}^d$ with $0 \in \operatorname{relint}(K) \cap \operatorname{relint}(C)$. Then,

a) $R(K, C) < \infty$ if and only if $K \subseteq \operatorname{lin}(C)$,

b) $R(K, C) = 0$ if and only if K is a singleton, and

c) if $R(K, C) \neq \infty$, there exists a center $c \in \mathbb{R}^d$ such that $K \subseteq c + R(K, C)C$.

In the same way as the circumradius, we introduce the inradius of a convex set K with respect to a gauge body C.

Definition 2.2.4 (*C-inradius*)
Let $K, C \subseteq \mathbb{R}^d$ non-empty, closed and convex. The *C-inradius* $r(K, C)$ of K is the greatest scaling factor $\rho \geq 0$, such that a translate of ρC is contained in K (cf. Figure 2.2). In other words,

$$r(K, C) := \sup\{\rho \geq 0 : \exists c \in \mathbb{R}^d \ s.t. \ c + \rho C \subseteq K\}. \tag{2.2}$$

Strictly speaking, there is no need to introduce $r(K, C)$, since it can easily be expressed as

$$r(K, C) = R(C, K)^{-1}, \tag{2.3}$$

using the conventions $\infty^{-1} = 0$ and $0^{-1} = \infty$ (cf. e.g. [97, Section 4.1.2]). Nevertheless, we keep the notation, as the little r, reminiscent of *inradius*, will emphasize the resemblance with the theorems being generalized.

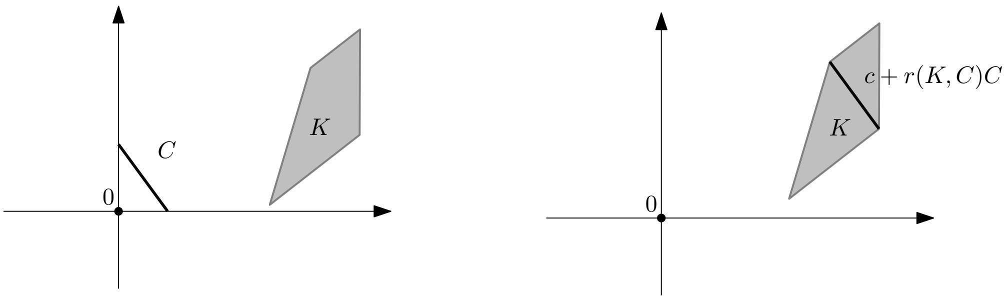

Figure 2.2: The C-inradius of a convex body $K \subseteq \mathbb{R}^2$. Left: The convex body K and a line segment $C \subseteq \mathbb{R}^2$. Right: A maximally scaled copy of C is translated such that it is contained in K.

Whereas the definitions of in- and circumradius are canonical even for asymmetric C, there seem to be at least two natural generalizations of the diameter of K with respect to a non-symmetric C. The first possibility is given in Definition 2.2.5 and uses the maximal C-radius of two-point-subsets of K. In the line of the general definition of core radii in Definition 4.2.1, we denote this radius by $R_1(K, C)$.

Definition 2.2.5 (*C-diameter*)
Let $K, C \subseteq \mathbb{R}^d$ non-empty, closed and convex. We define

$$R_1(K, C) := \sup\big\{R\big([x, y], C\big) : x, y \in K\big\}$$

as the C-radius of the "longest" segment in K and

$$D(K, C) := 2R_1(K, C)$$

as the *C-diameter* of K (cf. Figure 2.3).

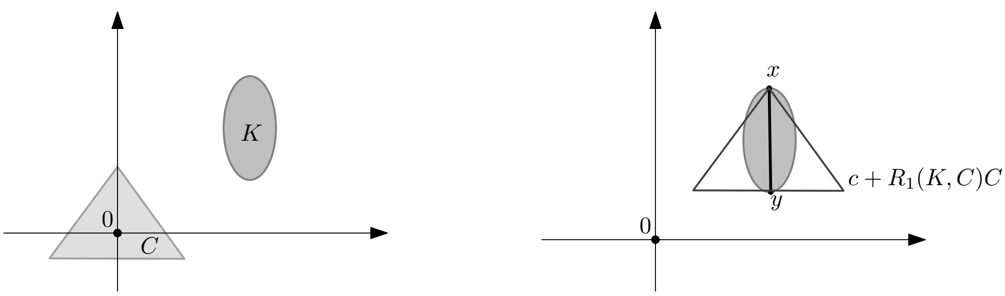

Figure 2.3: The C-diameter of a convex body K. Left: $K, C \subseteq \mathbb{R}^2$. Right: The indicated segment $[x, y]$ has maximal C-radius among all line segments contained in K.

A second way of introducing a diameter with respect to a non-symmetric C is via the (possibly asymmetric) distance measure induced by C. Although Leichtweiß already uses this definition under the name *Minkowski diameter* in [128], we prefer to speak of *asymmetric diameter* to emphasize its asymmetric nature, which is in contrast to the properties of $R_1(K,C)$ shown in Lemma 2.2.10.

Definition 2.2.6 (*Asymmetric diameter*)
Let $C \in \mathcal{C}_0^d$ and denote by

$$\| \cdot \|_C : \mathbb{R}^d \to [0,\infty); \; x \mapsto \|x\|_C := \min\{\lambda \geq 0 : x \in \lambda C\}$$

the (possibly asymmetric) norm/gauge functional induced by C. For $K \subseteq \mathbb{R}^d$ non-empty, closed and convex, we define

$$AD(K,C) := \sup\{\|x-y\|_C : x,y \in K\}$$

as the *asymmetric diameter* of K with respect to C (cf. Figure 2.4).

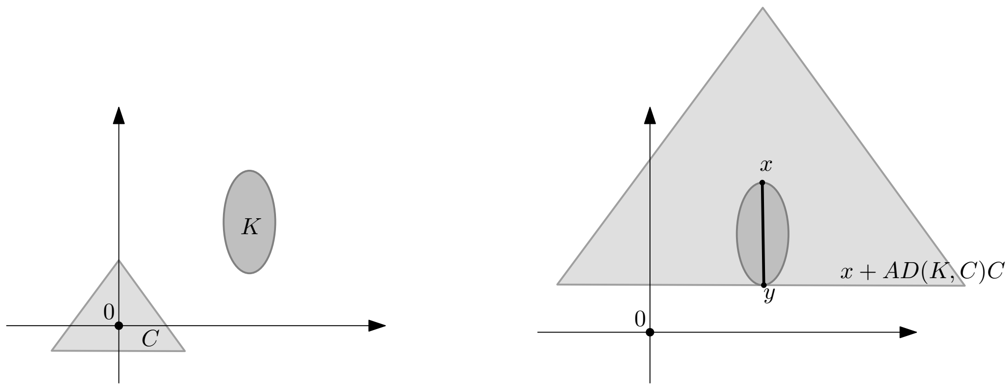

Figure 2.4: The asymmetric C-diameter of convex body $K \subseteq \mathbb{R}^2$. Left: The convex body K and the gauge body $C \subseteq \mathbb{R}^2$. Right: The indicated segment $[x,y]$ has maximal length with respect to the "norm" induced by C.

Comparing the two diameter definitions, we remark that $AD(K,C)$ is strongly dependent on the position of C with respect to the origin (note the assumption $C \in \mathcal{C}_0^d$ in Definition 2.2.6), whereas $D(K,C)$ is invariant under translations of C. Moreover, in view of Theorem 2.4.1 and Corollary 2.4.6, Definition 2.2.5 seems more advantageous for our purposes. However, if C is 0-symmetric, we have $AD(K,C) = D(K,C)$.

Analogously to the diameter, we define the width for a closed and convex set $K \subseteq \mathbb{R}^d$ with respect to a general gauge body $C \subseteq \mathbb{R}^d$ in two ways. The idea of the first definition is to measure the ratio of distances of two parallel hyperplanes that sandwich K and C, respectively (cf. Figure 2.5).

Definition 2.2.7 (*C-width*)
Let $K, C \subseteq \mathbb{R}^d$ non-empty, closed and convex. If $h(K-K,a) = \infty$ or $h(C-C,a) = 0$ for all $a \in \mathbb{R}^d \setminus \{0\}$, we define $r_1(K,C) := \infty$. If $h(C-C,a) = \infty$ or $h(K-K,a) = 0$ for all $a \in \mathbb{R}^d \setminus \{0\}$, let $r_1(K,C) := 0$. Otherwise, we define

$$r_1(K,C) := \inf\left\{\frac{h(K-K,a)}{h(C-C,a)} : a \in \mathbb{R}^d \setminus \{0\}\right\}. \tag{2.4}$$

and denote by

$$w(K,C) := 2r_1(K,C)$$

the *C-width* of K.

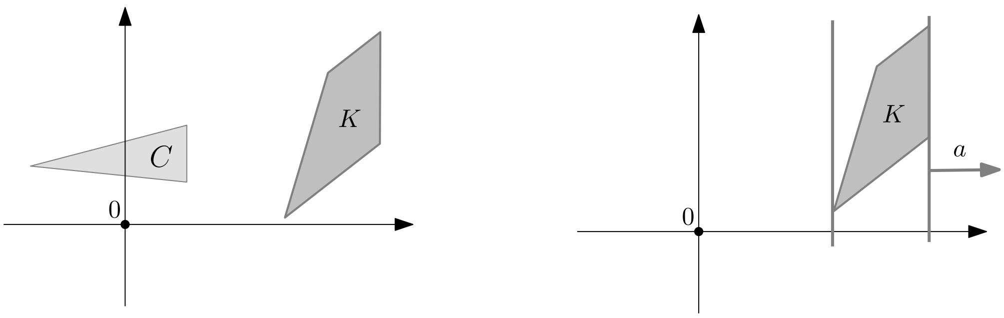

Figure 2.5: The symmetric C-width of K. Left: $K, C \subseteq \mathbb{R}^2$. Right: The symmetric C-width of K is attained for a direction a, for which the ratio $h(K-K,a)/h(C-C,a)$ is minimal.

As for the diameter, the width of K with respect to C may also be defined as follows.

Definition 2.2.8 (*Asymmetric width*)
For $K \subseteq \mathbb{R}^d$ non-empty, closed and convex, and $C \in \mathcal{C}_0^d$, define

$$aw(K,C) := \min\{h(K-K,a) : a \in \mathrm{bd}(C^\circ)\}.$$

Again, in case C is symmetric, $w(K,C) = aw(K,C)$ is the width of K in the usual sense. To avoid confusion, we explicitly remark that the width of K is sometimes also called the *minimal* width or minimal breadth in the literature.

Remark 2.2.9 (*Pathological cases*)
Note that, by the definitions of the radii and Lemma 2.2.2, we immediately obtain: whenever one of the four radii introduced above is 0 or ∞ all of them are. More precisely:

$$\begin{aligned} &\{R(K,C), R_1(K,C), r(C,K), r_1(C,K)\} \cap \{0,\infty\} \neq \emptyset \\ \Longrightarrow\quad & R(K,C) = R_1(K,C) = r(C,K)^{-1} = r_1(C,K)^{-1}. \end{aligned}$$

Our first lemma shows that both the C-width and the C-diameter remain unaffected if the arguments are symmetrized. This allows us to establish a useful identity relating $R_1(K,C)$ to $r_1(C,K)$.

Lemma 2.2.10 (*Invariance under symmetrization*)
Let $K, C \subseteq \mathbb{R}^d$ non-empty, closed and convex. The following three identities hold

a) $r_1(K,C) = r_1\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C)\right)$,

b) $R_1(K,C) = R_1\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C)\right)$, and

c) $r_1(K,C) = R_1(C,K)^{-1}$ (or equivalently, $D(K,C)w(C,K) = 4$).

Proof.
First, observe that for $K \subseteq \mathbb{R}^d$, we have

$$K - K = \frac{1}{2}(K-K) - \frac{1}{2}(K-K). \tag{2.5}$$

Using Identity (2.5), a) follows immediately from the definition of the C-width via Equation (2.4).
For the proof of b), let $A \in \{K, C\}$ and $p, q \in A$. Then, $p - \frac{1}{2}(p+q) = \frac{1}{2}(p-q) \in \frac{1}{2}(A-A)$ and $q - \frac{1}{2}(p+q) = \frac{1}{2}(q-p) \in \frac{1}{2}(A-A)$. Thus, with $A = K$, we obtain that $R_1(K,C) \leq R_1(\frac{1}{2}(K-K), C)$ and, with $A = C$, that $R_1(K,C) \geq R_1(K, \frac{1}{2}(C-C))$.
On the other hand, let $p = \frac{1}{2}(x_p - y_p), q = \frac{1}{2}(x_q - y_q) \in \frac{1}{2}(A-A)$ with $x_p, x_q, y_p, y_q \in A$. Then $p + \frac{1}{2}(y_p + y_q) = \frac{1}{2}(x_p + y_q) \in A$ and $q + \frac{1}{2}(y_p + y_q) = \frac{1}{2}(x_q + y_p) \in A$. Hence it follows $R_1(\frac{1}{2}(K-K), C) \leq R_1(K,C)$ from using $A = K$ and $R_1(K, \frac{1}{2}(C-C)) \geq R_1(K,C)$ from using $A = C$.
For Part c), we use the well known identities $R_1(K,C) = R(K,C)$ and $r_1(K,C) = r(K,C)$ for symmetric K and C (e.g. [86, (1.3)]) and obtain

$$\begin{aligned} R_1(K,C) &\overset{b)}{=} R_1\left(\tfrac{1}{2}(K-K), \tfrac{1}{2}(C-C)\right) = R\left(\tfrac{1}{2}(K-K), \tfrac{1}{2}(C-C)\right) \\ &\overset{(2.3)}{=} r\left(\tfrac{1}{2}(C-C), \tfrac{1}{2}(K-K)\right)^{-1} = r_1\left(\tfrac{1}{2}(C-C), \tfrac{1}{2}(K-K)\right)^{-1} \overset{a)}{=} r_1(C,K)^{-1}. \end{aligned}$$

□

Finally, the following lemma gives an alternative way to express $AD(K,C)$, which will be useful in the sequel.

Lemma 2.2.11 (*Alternative formulation of $AD(K,C)$*)
Let $K \subseteq \mathbb{R}^d$ non-empty, closed and convex, and $C \in \mathcal{C}_0^d$. Then,

$$AD(K,C) = \sup\{h(K-K, a) : a \in C^\circ\}.$$

Proof.
For $x \in \mathbb{R}^d$, $\|x\|_C = \max\{a^T x : a \in C^\circ\}$. Hence,

$$AD(K,C) = \sup\{a^T(p-q) : a \in C^\circ, p, q \in K\} = \sup\{h(K-K,a) : a \in C^\circ\}.$$

□

2.2.2 Some Specific Radii

We conclude this section of preparing lemmas by computing some radii of certain convex bodies that will serve to show the sharpness of several inequalities in the sequel.

Lemma 2.2.12 (*Partial difference bodies of simplices*)
Let $S \subseteq \mathbb{R}^d$ be a d-simplex and $\alpha, \beta \in [0,1]$. Define $C := S - \alpha S$, $K := -S + \beta S$. Then,

$$R(K,C) = \frac{d+\beta}{1+d\alpha} \quad \text{and} \quad R_1(K,C) = \frac{\beta+1}{\alpha+1}. \tag{2.6}$$

Proof.
Since $R(K,C)$ and $R_1(K,C)$ are invariant under translations of K and C, we may assume that there exist $x_1, \ldots, x_{d+1}, a_1, \ldots, a_{d+1} \in \mathbb{R}^d$ such that

$$S = \operatorname{conv}\{x_1, \ldots, x_{d+1}\} = \bigcap_{i=1}^{d+1} H_{\leq}(a_i, 1),$$

where the a_i are numbered such that

$$a_i^T x_j = \begin{cases} 1 & \text{if } i \neq j \\ -d & \text{if } i = j \end{cases}$$

for all $i, j \in [d+1]$.
In a first step, we prove $-S + \beta S \subseteq \frac{d+\beta}{1+d\alpha}(S - \alpha S)$, which implies $R(-S+\beta S, S-\alpha S) \leq \frac{d+\beta}{1+d\alpha}$. For this purpose let $i, j \in [d+1]$ with $i \neq j$ such that $-x_i + \beta x_j$ is a vertex of $-S + \beta S$. Showing that there exists $p \in S$ such that

$$-x_i + \beta x_j = \frac{d+\beta}{1+d\alpha} p - \frac{\alpha(d+\beta)}{1+d\alpha} x_i, \tag{2.7}$$

implies that $-x_i + \beta x_j \in \frac{d+\beta}{1+d\alpha}(S - \alpha S)$. Rearranging (2.7) yields that we need

$$p = \frac{1+d\alpha}{d+\beta}(-x_i + \beta x_j) + \alpha x_i.$$

However, with this expression, it is straightforward to verify that, $a_i^T p = 1$ and $a_k^T p < 1$ for all $k \in [d+1] \setminus \{i\}$ and therefore that $p \in S$.

On the other hand, we have $R(-S+\beta S,S)=d+\beta$ and $h(S-\alpha S,a_i)=1+d\alpha$ for all $i\in[d+1]$, which implies $R(-S+\beta S,S-\alpha S)\geq\frac{d+\beta}{1+d\alpha}$.
Now consider the diameter: Since $K-K=(1+\beta)(S-S)$ and $C-C=(1+\alpha)(S-S)$, Lemma 2.2.10b) yields

$$R_1(K,C)=R_1\left((1+\beta)(S-S),(1+\alpha)(S-S)\right)=\frac{1+\beta}{1+\alpha}.$$

□

Lemma 2.2.13 (*Regular simplex intersected with a ball*)
Let $T\subseteq\mathbb{B}_2^d$ be a regular simplex with all its vertices on the Euclidean unit sphere, $\rho\in\left[\frac{1}{d},1\right]$, and $K:=T\cap\rho\mathbb{B}_2^d$. Then,

$$R(-K,K)=d\rho,\quad r(K,\mathbb{B}_2^d)=\frac{1}{d},\quad\text{and}\quad r_1(K,\mathbb{B}_2^d)=\min\left\{r_1(T,\mathbb{B}_2^d),\rho+\frac{1}{d}\right\}.$$

If further $C=T\cap\rho_2\mathbb{B}_2^d$ with $\rho_2\leq\rho$. Then,

$$R(K,C)=\frac{\rho}{\rho_2}\quad\text{and}\quad R(C,K)=1.$$

Proof.
As $\rho\geq\frac{1}{d}$ and $r(T,\mathbb{B}_2^d)=\frac{1}{d}$,

$$-K\subseteq\rho\mathbb{B}_2^d\subseteq d\rho T\cap d\rho^2\mathbb{B}_2^d=d\rho K.$$

Again, since $r(T,\mathbb{B}_2^d)=\frac{1}{d}$, this scaling is best possible. Hence, $R(-K,K)=d\rho$.
Further, since $\rho\geq\frac{1}{d}$, $r(K,\mathbb{B}_2^d)=r(T,\mathbb{B}_2^d)=\frac{1}{d}$. And, if $r_1(K,\mathbb{B}_2^d)<r_1(T,\mathbb{B}_2^d)$, then, because of $\rho\geq\frac{1}{d}$, the width of K is attained between a pair of hyperplanes supporting T in a point x in the relative interior of a facet of T and $-\rho x$, respectively. Hence,

$$r_1(K,\mathbb{B}_2^d)=\min\left\{r_1(T,\mathbb{B}_2^d),\rho+\frac{1}{d}\right\}.$$

For the second statement, we immediately obtain $R(K,C)=R(\rho\mathbb{B}_2^d,\rho_2\mathbb{B}_2^d)=\frac{\rho}{\rho_2}$ by the definition of K and C. And finally, since $\rho_2\leq\rho$, $C\subseteq K$ and C touches all facets of T. Since these are also facets of K, we obtain $R(C,K)=1$ by Corollary 4.2.4 and Theorem 4.2.3. □

Lemma 2.2.14 (*Convex hull of a regular simplex and a ball*)
Let $T\subseteq\mathbb{B}_2^d$ be a regular simplex with all its vertices on the Euclidean unit sphere, $\rho\in\left[\frac{1}{d},1\right]$ and $K:=\operatorname{conv}\left(T\cup\rho\mathbb{B}_2^d\right)$. Then,

$$R(-K,K)=\frac{1}{\rho},\quad R(K,\mathbb{B}_2^d)=1\quad\text{and}\quad R_1(K,\mathbb{B}_2^d)=\max\left\{R_1(T,\mathbb{B}_2^d),\frac{1+\rho}{2}\right\}.$$

Proof.
We have $\frac{1}{\rho}K = \operatorname{conv}\left(\frac{1}{\rho}T \cup \mathbb{B}_2^d\right)$. Since $-T \subseteq \mathbb{B}_2^d$ and $\rho \leq 1$, it follows that $-K \subseteq \frac{1}{\rho}K$. Optimality of this inclusion is easily verifiable by Theorem 4.2.3, since $\operatorname{ext}(T) \subseteq \mathbb{S}_2^{d-1}$. This shows $R(-K, K) = \frac{1}{\rho}$. Further, by definition of K we have $R(K, \mathbb{B}_2^d) = 1$.
If $R_1(K, \mathbb{B}_2^d) > R_1(T, \mathbb{B}_2^d)$, then, because of $\rho \leq 1$, the diameter of K is attained between a vertex x of T and $-\rho x$. Hence,

$$R_1(K, \mathbb{B}_2^d) = \max\left\{R_1(T, \mathbb{B}_2^d), \frac{1+\rho}{2}\right\}.$$

□

2.3 Asymmetry Measures

There is a rich variety of measurements for the asymmetry of a convex body; see [95, in particular Section 6] for an overview. It is already claimed in [95] that the one which has received most interest is Minkowski's measure of symmetry. Its reciprocal measures the extent to which K needs to be scaled in order to contain a translate of $-K$ (cf. [150, Notes for Section 3.1]), which in our terminology, is the K-radius of $-K$. For short, we call the latter value, which is large for "very asymmetric" sets, the *Minkowski asymmetry.*

Definition 2.3.1 (*Minkowski asymmetry*)
Let $K \subseteq \mathbb{R}^d$ non-empty, closed and convex. We denote by

$$s(K) := R(-K, K) \tag{2.8}$$

the *Minkowski asymmetry* of K.
Further, if $c \in \mathbb{R}^d$ is such that $-(K - c) \subseteq s(K)(K - c)$, we call c a *Minkowski center* of K, and if 0 is a Minkowski center of K, we say that the body K is *Minkowski centered* (cf. Figure 2.6).

In all three examples in Figure 2.6, the Minkowski center of K_i is contained in K_i, $i = 1, 2, 3$, a property which is also true in general as the following lemma shows.

Lemma 2.3.2 (*Minkowski center is inside K*)
Let $K \subseteq \mathbb{R}^d$ non-empty, closed and convex, and $c \in \mathbb{R}^d$ a Minkowski center of K. Then,

$$c \in \operatorname{relint}(K).$$

Proof.
Without loss of generality we may assume $\operatorname{int}(K) \neq \emptyset$ and $c = 0$. For a contradiction suppose $0 \notin \operatorname{int}(K)$. Then there exists $a \in \mathbb{R}^d \setminus \{0\}$ such that $a^T x \leq 0$ for all $x \in K$. Since $-K \subseteq s(K)K$, we obtain $a^T x = 0$ for all $x \in K$, which contradicts $\operatorname{int}(K) \neq \emptyset$. □

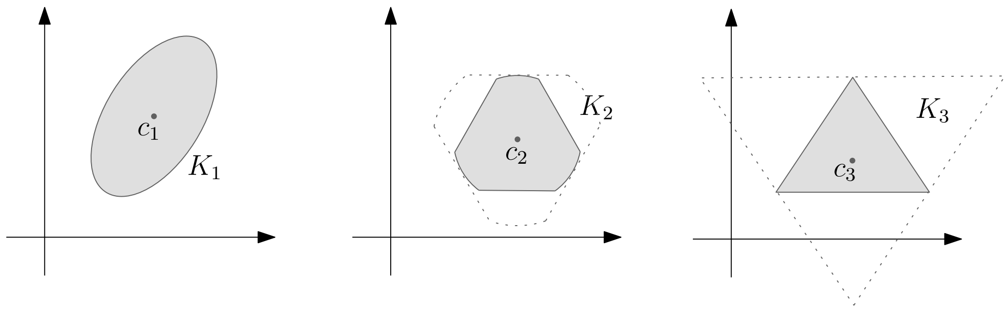

Figure 2.6: Planar examples with different Minkowski asymmetry. Left: K_1 is symmetric, $s(K_1) = 1$ and its Minkowski center is c_1. Middle: K_2 with $s(K_2) = 3/2$ and Minkowski center c_2. Right: A 2-simplex K_3 with $s(K_3) = 2$ and Minkowski center c_3. The suitable homothetics of $-K_2$ and $-K_3$ containing K_2 and K_3, respectively, are indicated in dotted gray.

For unbounded K, the following statement can easily be obtained from Lemma 2.2.2.

Remark 2.3.3 (*Asymmetry for unbounded convex sets*)
$R(-K, K) = 0$ if and only if K is an affine subspace, and $R(-K, K) = \infty$ if and only if $\operatorname{rec}(K)$ is not a linear subspace. The latter means that cylinders $K = K_1 + F$, with F a linear subspace and $K_1 \subseteq F^\perp$ a non-singleton compact convex set, are the only unbounded sets with Minkowski asymmetry different from 0 and ∞ and for them $s(K) = s(K_1)$.

Because of Remark 2.3.3, we henceforth assume $K \in \mathcal{C}^d$. In this case, the following proposition from [95] states the well-known bounds on $s(K)$. With the tools developed in Chapter 4, we also give a transparent proof of this fact in Corollary 4.2.7.

Proposition 2.3.4 (*Bounds on the Minkowski asymmetry and the set of centers*)
For $K \in \mathcal{C}^d$,

$$1 \leq s(K) \leq d,$$

with $s(K) = 1$ if and only if K is symmetric, and $s(K) = d$ if and only if K is a d-simplex.

In contrast to the three examples in Figure 2.6, for an arbitrary $K \in \mathcal{C}^d$, it can happen that the Minkowski center is not unique and even that the set of centers is of dimension up to $d - 2$ as indicated by Figure 2.7 and proved in [122].

Next, we turn to the computability of the Minkowski asymmetry.

Lemma 2.3.5 (*Computability*)
Let $P \in \mathcal{C}^d$ be a rational polytope given in $\mathcal{H}$- or $\mathcal{V}$-presentation Then $s(P)$ and a Minkowski center $c \in \mathbb{R}^d$ such that $-(P - c) \subseteq s(P)(P - c)$ can be computed in polynomial time.

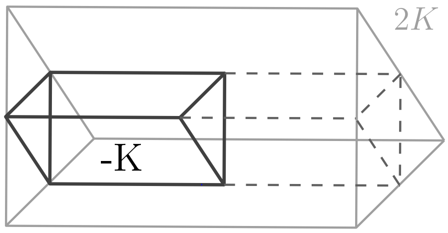

Figure 2.7: Since K is a prism with triangular base area, we enforce $s(K) \geq 2$ by Proposition 2.3.4. However, in direction of the third coordinate the dilatation is twice as much as needed and therefore the Minkowski center of K is not unique.

Proof.
Using (2.1), the computation of $s(P) = R(-P, P)$ requires the solution of the following optimization problem:

$$\begin{aligned} \min \quad & \rho \\ s.t. \quad & -P \subseteq c + \rho P \\ & c \in \mathbb{R}^d \\ & \rho \geq 0. \end{aligned} \tag{2.9}$$

By Proposition 2.3.4, and Lemma 2.2.2c), there exists a solution $(c^*, \rho^*) \in \mathbb{R}^d \times [1, d]$ of (2.9). By definition, $s(P) = \rho^*$ and we have that $c = -\frac{1}{s(P)+1}c^*$ is a Minkowski center of P, as

$$-\left(P + \frac{1}{s(P)+1}c^*\right) \subseteq c^* + s(P)P - \frac{1}{s(P)+1}c^* = s(P)\left(P + \frac{1}{s(P)+1}c^*\right).$$

Now, the proof of [87, Theorem 3.4] demonstrates that the computation of $R(K, C)$ amounts to solving a Linear Program if K and C are both given in $\mathcal{H}$-presentation or both given in $\mathcal{V}$-presentation. Throughout [87], it is assumed that C is 0-symmetric and fixed in advance, i.e. not considered as part of the input. However, this is not needed for the formulation as Linear Program and the complexity of the Linear Program is polynomial in the coding length of C, too. See e.g. [42, Section 2.1.2], where this is made explicit for the case where K is given in $\mathcal{V}$-presentation. Hence, in both cases, $s(P) = R(-P, P)$ and a respective Minkowski center can be computed in polynomial time. □

Note that Lemma 2.3.5 can also be used to decide whether a polytope K in $\mathcal{H}$- or $\mathcal{V}$-presentation is symmetric and to compute its center of symmetry if K is symmetric. This yields an alternative proof for [87, Theorem 2.2].

In the following, we consider centered versions of asymmetry of a convex body K, i.e. we are interested in the minimal dilatation factor needed to cover $-(K - c_0)$ with a copy of $K - c_0$ for some fixed $c_0 \in \mathbb{R}^d$ depending on K, but not free to be chosen for the optimal covering. The choice of c_0 that we focus on here is the center of the maximum

volume inscribed K. Measuring the symmetry of K around this center nicely interacts with John's Theorem [115]: on the one hand, the classic formulation of John's Theorem can be used to bound this centered asymmetry of a body (as in Corollary 2.3.8). On the other hand, we will use the centered asymmetry in Theorem 2.7.1 to strengthen John's Theorem itself. Because of its importance in this context, we give an explicit statement of John's Theorem in Proposition 2.3.6 and refer to [20, 21, 94] for the best readable proofs.
When talking about John's Theorem, we usually assume that K is full dimensional, i.e. without loss of generality $K \in \mathcal{C}_0^d$. One may use the usual identification $\operatorname{aff}(K) \cong \mathbb{R}^{\dim(K)}$ to extend the results to lower-dimensional bodies.

Proposition 2.3.6 (*John's Theorem*)
For every $K \in \mathcal{C}_0^d$, there exists a unique ellipsoid of maximal volume contained in K. This ellipsoid is $\mathbb{B}_2^d$ if and only if

(1) $\mathbb{B}_2^d \subseteq K$, and

(2) for some $k \in \left\{d+1, \dots, \frac{d(d+3)}{2}\right\}$, there are points $p_1, \dots, p_k \in \operatorname{bd}(K) \cap \mathbb{S}_2^{d-1}$ and scalars $\lambda_1, \dots, \lambda_k > 0$ such that

$$0 = \sum_{i=1}^{k} \lambda_i p_i \quad \text{and} \quad I = \sum_{i=1}^{k} \lambda_i p_i p_i^T . \tag{2.10}$$

Moreover, if $\mathbb{B}_2^d$ is the ellipsoid of maximal volume contained in K, then $K \subseteq d\mathbb{B}_2^d$ in general, and $K \subseteq \sqrt{d}\mathbb{B}_2^d$, if K is 0-symmetric.

Definition 2.3.7 (*John asymmetry*)
Let $K \in \mathcal{C}_0^d$, $A \in \mathbb{R}^{d \times d}$ and $b \in \mathbb{R}^d$ such that $\varphi : \mathbb{R}^d \to \mathbb{R}^d; x \mapsto Ax + b$ is the unique affine transformation that brings K into John position (i.e. $\mathbb{B}_2^d$ is the ellipsoid of maximal volume contained in $AK + b$). We define

$$s_0(K) := \min\{\rho \geq 0 : -(K - A^{-1}b) \subseteq \rho(K - A^{-1}b)\}$$

as the asymmetry of K around the center of its maximum volume inscribed ellipsoid and call it the *John asymmetry* of K.

As already mentioned, one may use John's Theorem to obtain the same bounds on $s_0(K)$ as on $s(K)$ (cf. [95, p. 248]).

Corollary 2.3.8 (*Bounds on the John asymmetry*)
Let $K \in \mathcal{C}_0^d$. Then,

$$1 \leq s_0(K) \leq d$$

with equality if and only if K is symmetric in the first case and if and only if K is a d-simplex in the latter case.

As for the Minkowski asymmetry, the John asymmetry is computable for suitably presented polytopes.

Lemma 2.3.9 (*Computability of the John asymmetry*)
If $P \subseteq \mathbb{R}^d$ is a polytope in $\mathcal{H}$-presentation, $s_0(P)$ can be approximated to any accuracy in polynomial time.

Proof.
First, we mention that $\mathrm{aff}(P)$ is efficiently computable for both presentations of P. Hence, we may assume without loss of generality that P is full-dimensional. In [120], it is shown that for a polytope $P \subseteq \mathbb{R}^d$ in $\mathcal{H}$-presentation, the center of the ellipsoid of maximal volume contained in P can be approximated to any accuracy in polynomial time. An approximation of this center at hand, call it c_P, we can compute $\min\{\rho \geq 0 : -(P - c_P) \subseteq \rho(P - c_P)\}$ via Linear Programming analogously to the Linear Program in the proof of Lemma 2.3.5. □

Of course, there are also other natural choices how the asymmetry of K can be measured. One particular choice is mentioned in the following remark, an extensive list of others may be found in [95].

Remark 2.3.10 (*Loewner asymmetry*)
One could also measure the asymmetry of a body K around its Loewner center, i.e. the center of the volume minimal enclosing ellipsoid of K. With the same arguments as for the John center, the values of this asymmetry measure are also contained in the interval $[1, d]$. Moreover, for a $\mathcal{V}$-presented polytope $P \subseteq \mathbb{R}^d$, this center can be approximated to any accuracy in polynomial time [120] and therefore the asymmetry around the Loewner center can be approximated efficiently for $\mathcal{V}$-polytopes by the same argument as in the proof of Lemma 2.3.9.

2.4 The Inequalities of Bohnenblust and Leichtweiß

The present section gives generalizations of the Inequalities of Bohnenblust [28] and Leichtweiß [128] and shows that these generalizations are actually one and the same inequality unifying the two old theorems.
First, we prove a version of Bohnenblust's Inequality for general convex bodies with the ratio of the C-radius and C-diameter bounded in terms of the Minkowski asymmetry of K and C.

A note on pathological cases. For all the geometric inequalities that follow, we assume $K, C \in \mathcal{C}^d$. As a consequence of Proposition 2.3.4, all the right hand sides in the inequalities are therefore well defined. In view of Remark 2.2.9, the only pathological cases that can appear on the left hand side are of the form $0/0$ or ∞/∞. Presuming both ratios to be 1, we tacitly ignore these cases in the remainder.

Theorem 2.4.1 (*Generalizing Bohnenblust's Inequality*)
Let $K, C \in \mathcal{C}^d$. Then,

$$\frac{R(K,C)}{R_1(K,C)} \leq \frac{(s(C)+1)s(K)}{s(K)+1} \tag{2.11}$$

and for every $\sigma_K, \sigma_C \in [1,d]$, there exist bodies K and C with $s(K) = \sigma_K$, $s(C) = \sigma_C$ such that (2.11) is sharp for K and C.

Proof.
Suppose without loss of generality that $R_1(K,C) = 1$. Then, Corollary 2.2.3 ensures that for all $p_1, p_2 \in K$, there is a $c \in \mathbb{R}^d$, such that $p_1, p_2 \in c + C$; explicitly, $p_1 = c + v$ and $p_2 = c + w$ with $v, w \in C$. Hence $p_1 - p_2 = v - w \in C - C$ for all $p_1, p_2 \in K$ and thus $K - K \subseteq C - C$. Using Proposition 2.3.4, there exist $c_K, c_C \in \mathbb{R}^d$, such that

$$\begin{aligned} & c_K + \left(1 + \tfrac{1}{s(K)}\right) K = K + c_K + \tfrac{1}{s(K)} K \subseteq K - K \\ \subseteq \quad & C - C \subseteq C + c_C + s(C)C = c_C + (1 + s(C))C \end{aligned}$$

and therefore

$$\frac{R(K,C)}{R_1(K,C)} \leq \frac{s(C)+1}{1+1/s(K)} = \frac{(s(C)+1)s(K)}{s(K)+1}.$$

For the sharpness of the inequality, let $S \subseteq \mathbb{R}^d$ be a simplex, $\alpha := \frac{\sigma_C - d}{1-\sigma_C d}$ and $\beta := \frac{\sigma_K - d}{1 - \sigma_K d}$, $C := S - \alpha S$ and $K := -S + \beta S$. By Lemma 2.2.12, $R(K,C) = \frac{d+\beta}{1+d\alpha}$, $s(C) = \sigma_C$, $s(K) = \sigma_K$ and $R_1(K,C) = \frac{1+\beta}{1+\alpha}$. Together, we obtain

$$\frac{R(K,C)}{R_1(K,C)} = \frac{(d+\beta)(\alpha+1)}{(1+d\alpha)(\beta+1)} = \frac{(s(C)+1)s(K)}{s(K)+1}.$$

□

Remark 2.4.2 (*Bohnenblust's Inequality with John asymmetry*)
Since $s(K) \leq s_0(K)$ and $s(C) \leq s_0(C)$, a version of Theorem 2.4.1 with $s(K), s(C)$ replaced by $s_0(K), s_0(C)$ would be weaker but still valid and still strengthening Bohnenblust's original inequality. As one may easily deduce from Proposition 2.3.6, it stays sharp for the families of K and C as given in the proof above.

Note that the statement of Theorem 2.4.1 is different from the version proved by Leichtweiß in [128]. In his proof of Bohnenblust's Inequality, Leichtweiß shows an inequality relating $R(K,C)$ and the asymmetric diameter $AD(K,C)$ from Definition 2.2.6 instead of the symmetric C-diameter from Definition 2.2.5. Leichtweiß remarks that the inequality can be generalized to non-symmetric containers and that even the convexity of C can be relaxed to starshapedness. The idea of his proof is as elementary as the proof of Theorem 2.4.1 and we repeat it here in contemporary notation with the slight improvement that the dimension is replaced by $s(K)$. Note, however, that the inequality is independent of the Minkowski asymmetry of C but strongly dependent on the position of C via the dependence of $AD(K,C)$ on the position of C with respect to the origin. We also refer to [59, Section 6], where Bohnenblust's Inequality is discussed for both versions of the diameter definition.

The first observations in Leichtweiß's proof is the following lemma, which, in the context of symmetry measures, might be of independent interest and which we therefore repeat with the constant d replaced by $s(K)$.

Lemma 2.4.3 (*Symmetry coefficients and the support function*)
Let $K \in \mathcal{C}^d$ be Minkowski centered and $a \in \mathbb{R}^d$. Then,

$$h(K,-a) \leq s(K)h(K,a)$$

and therefore

$$h(K-K,a) \leq (1+s(K))h(K,a).$$

Proof.
Since K is Minkowski centered, we have

$$h(K,-a) = \max\{a^Tx : x \in (-K)\} \leq \max\{a^Tx : x \in s(K)K\} = s(K)h(K,a).$$

This immediately yields the second statement via

$$h(K-K,a) = h(K,a) + h(K,-a) \leq (1+s(K))h(K,a).$$

□

This lemma at hand, we easily obtain the following version of Bohnenblust's Inequality.

Lemma 2.4.4 (*Bohnenblust's Inequality, Leichtweiß's version*)
Let $K \in \mathcal{C}^d$ and $C \in \mathcal{C}^d_0$. Then,

$$\frac{R(K,C)}{AD(K,C)} \leq \frac{s(K)}{s(K)+1}.$$

Proof.
We can assume without loss of generality that K is Minkowski centered. Let $a \in C^\circ$ and $x, y \in K$ be such that $a \in N(K,x)$ and $-a \in N(K,y)$. Then, by Lemma 2.4.3, $\frac{a^Tx}{-a^Ty} \leq s(K)$, which via $\frac{a^Tx-a^Ty}{a^Tx} \geq \frac{1}{s(K)}+1$ yields

$$h(K,a) = a^Tx \leq \frac{s(K)}{s(K)+1}a^T(x-y) \leq \frac{s(K)}{s(K)+1}AD(K,C), \tag{2.12}$$

where the second inequality follows from Lemma 2.2.11. Since (2.12) is true for all $a \in C^\circ$, we obtain $K \subseteq \frac{s(K)}{s(K)+1}AD(K,C)C$. □

Leichtweiß observed in [128], that his own inequality relating the width in the inradius of a body K directly follows from $h(K,a) \geq \frac{h(K-K,a)}{d+1}$ for $a \in \mathbb{R}^d$. Replacing this inequality by Lemma 2.4.3, his original inequality can be stated as follows.

Proposition 2.4.5 (*Leichtweiß's Inequality, original version*)
For $K \in \mathcal{C}^d$ and $C \in \mathcal{C}_0^d$, we have

$$\frac{aw(K,C)}{r(K,C)} \leq s(K) + 1.$$

In contrast to Proposition 2.4.5, the use of the symmetric diameter and width definitions reveals that Leichtweiß's Inequality no longer needs a separate proof, but is the direct dual to Bohnenblust's Inequality as demonstrated in the following corollary.

Corollary 2.4.6 (*Generalizing Leichtweiß's Inequality*)
For $K, C \in \mathcal{C}^d$, we have

$$\frac{r_1(K,C)}{r(K,C)} \leq \frac{(s(K)+1)s(C)}{s(C)+1} \tag{2.13}$$

and for every $\sigma_K, \sigma_C \in [1,d]$, there exist bodies K and C with $s(K) = \sigma_K$, $s(C) = \sigma_C$ such that (2.13) is sharp for K and C.

Proof.
The claim follows readily from Theorem 2.4.1 using $r(K,C) = R(C,K)^{-1}$ (Equation (2.3)) and $r_1(K,C) = R_1(C,K)^{-1}$ (Lemma 2.2.10). For the statement about the sharpness of (2.13), we switch the roles of K and C used in the proof of the sharpness of (2.11). □

2.5 The Inequalities of Jung and Steinhagen

In the important special case where $C = \mathbb{B}_2^d$, stronger formulations of the original inequalities of Bohnenblust and Leichtweiß are known in the form of Jung's [116] and Steinhagen's [153] Inequalities. However, for a body $K \in \mathcal{C}^d$ with $s(K) < d$, the bounds of Theorems 2.4.1 and Corollary 2.4.6 become smaller for certain values of $s(K)$ and can therefore be used to improve Jung's and Steinhagen's Inequalities. The two following theorems show that, building on symmetry coefficients, this is already the best one can obtain.

Theorem 2.5.1 (*Strengthening Jung's Inequality*)
Let $K \in \mathcal{C}^d$. Then,

$$\frac{R(K,\mathbb{B}_2^d)}{R_1(K,\mathbb{B}_2^d)} \leq \min\left\{\sqrt{\frac{2d}{d+1}}, \frac{2s(K)}{s(K)+1}\right\}. \tag{2.14}$$

This bound is best possible in the sense that for every value of $\sigma \in [1,d]$, there is a $K \in \mathcal{C}^d$ such that $s(K) = \sigma$ and (2.14) is sharp for K.

Proof.
The inequality in (2.14) follows directly from Jung's original inequality in conjunction with Theorem 2.4.1. For the statement about the sharpness, let $\sigma \in [1,d]$, $T \subseteq \mathbb{B}_2^d$ a regular simplex with all its vertices on the Euclidean unit sphere, and

$$K := \operatorname{conv}\left(T \cup \frac{1}{\sigma}\mathbb{B}_2^d\right).$$

Then, by Lemma 2.2.14, $s(K) = \sigma$, $R(K,\mathbb{B}_2^d) = 1$, and

$$R_1(K,\mathbb{B}_2^d) = \max\left\{R_1(T,\mathbb{B}_2^d), \frac{\sigma+1}{2\sigma}\right\}.$$

Since $R_1(T,\mathbb{B}_2^d) = \sqrt{\frac{d+1}{2d}}$ by Jung's Theorem, K fulfills (2.14) with equality. □

Theorem 2.5.2 (*Strengthening Steinhagen's Inequality*)
Let $K \in \mathcal{C}^d$. Then,

$$\frac{r_1(K,\mathbb{B}_2^d)}{r(K,\mathbb{B}_2^d)} \leq \begin{cases} \min\left\{\sqrt{d},\ \ s(K)+1\right\} & \text{if } d \text{ is odd} \\ \min\left\{\frac{d+1}{\sqrt{d+2}}, s(K)+1\right\} & \text{if } d \text{ is even.} \end{cases} \tag{2.15}$$

This bound is best possible in the sense that for every value of $\sigma \in [1,d]$, there is a $K \in \mathcal{C}^d$ such that $s(K) = \sigma$ and (2.15) is sharp for K.

Proof.
The inequality in (2.15) follows directly from Steinhagen's original theorem in conjunction with Corollary 2.4.6. In order to show that the bound is best possible, let $\sigma \in [1,d]$ and

$$K := T \cap \frac{\sigma}{d}\mathbb{B}_2^d.$$

Then $\frac{\sigma}{d} \in \left[\frac{1}{d}, 1\right]$ and, by Lemma 2.2.13,

$$s(K) = \sigma, \quad r(K,\mathbb{B}_2^d) = \frac{1}{d} \quad \text{and} \quad r_1(K,\mathbb{B}_2^d) = \min\left\{r_1(T,\mathbb{B}_2^d), \frac{\sigma+1}{d}\right\}.$$

Thus, K fulfills (2.15) with equality. □

2.6 An Inequality between the In- and Circumradius

In this section we present a generalization of a classical inequality, stating that the Euclidean circumradius of a simplex is at least d times larger than its inradius. We refer to [69, p. 28] for historical comments on the original authorship of the inequality itself and different proofs thereof. Theorem 2.6.1 generalizes this inequality by lower bounding the ratio of $R(K,C)$ and $r(K,C)$ in terms of $s(K)$ and $s(C)$ for arbitrary $K, C \in \mathcal{C}^d$. The original inequality can be recovered from Theorem 2.6.1 by choosing $C = \mathbb{B}_2^d$ and restricting K to simplices.

Theorem 2.6.1 (*Ratio of in- and circumradius*)
Let $K, C \in \mathcal{C}^d$. Then,

$$\frac{R(K,C)}{r(K,C)} \geq \max\left\{\frac{s(K)}{s(C)}, \frac{s(C)}{s(K)}\right\}. \tag{2.16}$$

This bound is best possible in the sense that for every $\sigma_K, \sigma_C \in [1,d]$, there exist K, C such that $s(K) = \sigma_K$, $s(C) = \sigma_C$, and K and C fulfill (2.16) with equality.

Proof.
Since, by (2.3),

$$\frac{R(K,C)}{r(K,C)} = R(K,C)R(C,K) = \frac{R(C,K)}{r(C,K)},$$

it suffices to show $R(K,C)R(C,K) \geq \frac{s(K)}{s(C)}$ and we may assume without loss of generality that C is Minkowski centered.
Because of Lemma 2.2.2, there exist $c_i \in \mathbb{R}^d$, $i = 1,2$, such that $c_1 + K \subseteq R(K,C)C$ and $-C \subseteq c_2 + R(C,K)(-K)$. Hence,

$$c_1 + K \subseteq R(K,C)s(C)(-C) \subseteq R(K,C)s(C)c_2 + R(K,C)s(C)R(C,K)(-K)$$

and thus $R(K,C)s(C)R(C,K) \geq s(K)$ by definition of $s(K)$.
For the sharpness of (2.16), let $\sigma_K, \sigma_C \in [1,d]$, $T \subseteq \mathbb{B}_2^d$ a regular simplex with all its vertices on the Euclidean unit sphere and let

$$K := T \cap \frac{\sigma_K}{d}\mathbb{B}_2^d \quad \text{and} \quad C := T \cap \frac{\sigma_C}{d}\mathbb{B}_2^d.$$

By Lemma 2.2.13, $s(K) = \sigma_K$ and $s(C) = \sigma_C$. Since the roles of K and C are interchangeable, we can assume without loss of generality that $\sigma_K \geq \sigma_C$. Then, by Lemma 2.2.13, $R(K,C) = \frac{\sigma_K}{\sigma_C}$ and $R(C,K) = 1$. Hence, we obtain

$$R(K,C)R(C,K) = \frac{s(K)}{s(C)} = \max\left\{\frac{s(K)}{s(C)}, \frac{s(C)}{s(K)}\right\}.$$

□

Remark 2.6.2
Let again $T \subseteq \mathbb{B}_2^d$ be a regular simplex with all its vertices on the Euclidean unit sphere and $\sigma \in [1,d]$. With the help of Lemma 2.2.14, it is easy to verify that also the body $K := \mathrm{conv}(\sigma^{-1}\mathbb{B}_2^d \cup T)$ has $s(K) = \sigma$ and fulfills (2.16) with equality.

Remark 2.6.3
Combining Theorem 2.4.1, Corollary 2.4.6 and Theorem 2.6.1, we obtain the following chain of inequalities for $K, C \in \mathcal{C}^d$, with C symmetric. For the sake of shortness the arguments (K,C) as in $R(K,C)$ and (K) in $s(K)$ are omitted.

$$2r \leq w \leq (1+s)r \leq r + R \leq \frac{s+1}{s}R \leq D \leq 2R \tag{2.17}$$

From (2.17), one easily sees that in every normed space all three generalized inequalities (2.11), (2.13), (2.16) are sharp for any set of constant width. However, since $s(K) = d$ is attained only for d-simplices, the equality chain $w = (1+d)r = r + R = \frac{d+1}{d}R = D$ can only hold true if there is a d-simplex K of constant width. This means that, in this case, the unit ball of that space must be a dilatation of $K - K$, see [50]. The fact that, in Euclidean spaces of dimension at least 2, simplices cannot be of constant width also retrospectively explains the two cases in (2.14) and (2.15).

Furthermore, the inequality

$$\frac{w(K,\mathbb{B}_2^d)}{R(K,\mathbb{B}_2^d)} \leq \begin{cases} 2\sqrt{\frac{1}{d}}, & \text{if } d \text{ is odd} \\ \frac{2(d+1)}{d\sqrt{d+2}}, & \text{if } d \text{ is even.} \end{cases}$$

by Alexander [6] (independently found in [91]), relating the width and circumradius of simplices in Euclidean space is an immediate consequence of (2.17). Allowing sets K of arbitrary Minkowski asymmetry, we obtain two new inequalities by combining (2.16), and (2.14) or (2.15), respectively:

Corollary 2.6.4 (*Generalizing and Dualizing Alexander's Inequality*)
Let $K, C \in \mathcal{C}^d$, C symmetric. Then,

a) $$\frac{r_1(K,\mathbb{B}_2^d)}{R(K,\mathbb{B}_2^d)} \leq \begin{cases} \min\left\{\frac{\sqrt{d}}{s(K)}, 1 + \frac{1}{s(K)}\right\} & \text{if } d \text{ is odd} \\ \min\left\{\frac{d+1}{s(K)\sqrt{d+2}}, 1 + \frac{1}{s(K)}\right\} & \text{if } d \text{ is even, and} \end{cases}$$

b) $$\frac{r(K,\mathbb{B}_2^d)}{R_1(K,\mathbb{B}_2^d)} \leq \min\left\{\frac{\sqrt{2d}}{s(K)\sqrt{d+1}}, \frac{2}{s(K)+1}\right\}.$$

The two inequalities are sharp exactly for the sets for which the corresponding inequalities (2.14) or (2.15) are sharp.

2.7 John's Theorem

We now leave the field of radii behind and turn to the probably most famous containment problem under affinity: computing ellipsoids of maximal volume contained in convex bodies. In particular the second part of Proposition 2.3.6, which states that $\mathbb{B}_2^d$ being the ellipsoid of maximal volume in K ensures that $K \subseteq d\mathbb{B}_2^d$, is an indispensable tool when it comes to approximations of convex bodies by simpler geometric objects. We strengthen this part of the theorem by introducing the John asymmetry coefficient in Ball's proof [20] of John's theorem. For a historical account on John's Theorem and its importance, we refer to [101].

Theorem 2.7.1 (*Strengthening John's Theorem*)
Let $K \in \mathcal{C}_0^d$ such that $\mathbb{B}_2^d$ is the ellipsoid of maximal volume enclosed in K. Then,

$$K \subseteq \sqrt{s_0(K)d}\,\mathbb{B}_2^d.$$

Proof.
If $\mathbb{B}_2^d$ is the ellipsoid of maximal volume enclosed in K, by John's Theorem (Proposition 2.3.6), for some $k \in \{d+1, \dots, \frac{d+3}{2}\}$, there exist $u_1, \dots, u_k \in \mathrm{bd}(K) \cap \mathbb{S}_2^{d-1}$ and $\lambda_1, \dots, \lambda_k > 0$ which satisfy

$$\sum_{i=1}^{k} \lambda_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i u_i u_i^T = I. \tag{2.18}$$

First, observe that, because of (2.18), $\sum_{i=1}^k \lambda_i = \mathrm{trace}(I) = d$ and that $-\frac{1}{s_0(K)} K \subseteq K \subseteq C := \{x \in \mathbb{R}^d : u_i^T x \leq 1 \; \forall i \in [k]\}$, which means

$$u_i^T \left(-\frac{1}{s_0(K)} x \right) \leq 1$$

and therefore

$$-s_0(K) \leq u_i^T x \leq 1$$

for all $x \in K$ and all $i \in [k]$. Together with $\lambda_i > 0$ for $i \in [k]$ and the identities in (2.18), this yields for every $x \in K$

$$\begin{aligned} 0 &\leq \sum_{i=1}^{k} \lambda_i (1 - u_i^T x)(s_0(K) + u_i^T x) \\ &= \sum_{i=1}^{k} \lambda_i \left(s_0(K) + u_i^T x - s_0(K) u_i^T x - (u_i^T x)^2 \right) \\ &= \left(\sum_{i=1}^{k} \lambda_i \right) s_0(K) + (1 - s_0(K)) \left(\sum_{i=1}^{k} \lambda_i u_i \right)^T x - \|x\|_2^2 \\ &= d s_0(K) - \|x\|_2^2. \end{aligned}$$

Thus, $\|x\|_2 \leq \sqrt{s_0(K) d}$. □

Note that replacing the John asymmetry by the Loewner asymmetry as suggested in Remark 2.3.10, one can easily derive the same results as above for the latter one.

If a polytope $P \subseteq \mathbb{R}^d$ is given in $\mathcal{H}$-presentation, it is shown in [120] that the ellipsoid of maximal volume inscribed to P can be approximated to arbitrary accuracy in polynomial time. (See also [155] and the extensive list of references therein.)
It is not known, on the other hand, whether the same is true for the minimum volume enclosing ellipsoid of P. In fact, it is conjectured in [120] that approximation to arbitrary accuracy of the minimum volume enclosing ellipsoid of an $\mathcal{H}$-presented polytope is $\mathbb{NP}$-hard.
An approximation with a multiplicative error factor of at most $(1+\varepsilon)d$, however, is readily provided by combining the algorithm mentioned above and John's Theorem.

Depending on the input polytope P, the strengthened inequality in Theorem 2.7.1 allows to improve this bound to $(1+\varepsilon)\sqrt{s_0(P)d}$, where the coefficient $s_0(P)$ can be computed (approximated) via Linear Programming once (an approximation of) the center of the ellipsoid of maximum volume contained in P is known. Taking into account the hardness of approximating the circumradius of an $\mathcal{H}$-presented polytope even around a fixed center (cf. [45] and the results in Chapter 3), the improvement of the bound by the computation of $s_0(P)$ is quasi at no cost.

As a second application, we mention that the the strengthening in Theorem 2.7.1 can also be used to improve the bound that John's Theorem implies on the *Banach-Mazur* distance of two convex bodies.

Corollary 2.7.2 (*A bound on the Banach-Mazur distance*)
Let $K, L \in \mathcal{C}_0^d$. If

$$\delta(K,L) := \min\{\rho \geq 0 : K \subseteq AL + b \subseteq \rho K + c, A \in \mathbb{R}^{d\times d}, b, c \in \mathbb{R}^d\}$$

denotes the Banach-Mazur distance of K and L, then

$$\delta(K,L) \leq \sqrt{s_0(K)s_0(L)}d.$$

As stated in [76, Section 7.2], apart from the bound from John's Theorem the "question of the maximal distance between non-symmetric bodies is open". Corollary 2.7.2 provides at least some partial improvement in this direction.

2.8 Asymmetry and Polarity

Finally, this last section gives bounds on the product of outer radii of K with inner radii of K°. More precisely, we are interested in the following type of radii, which are similar to the ones in Definition 1.2.1.

Definition 2.8.1 (*Successive inner and outer radii*)
For $K, C \in \mathcal{C}^d$, let

$$\underline{R}_j(K,C) := \inf_{F\in\mathcal{L}^d_{d-j}} R(K, C+F)$$

the j-th outer radius of K with respect to C and

$$\overline{r}_j(K,C) := \sup_{F\in\mathcal{L}^d_j} r(K, C\cap F)$$

the j-th inner radius of K with respect to C.

Note that these series of radii are different from the series investigated in Chapter 4 which inspired the nomenclature in the previous sections. In contrast to the radii considered above, for the series $\underline{R}_j(K,C)$, $\overline{r}_j(K,C)$, we can for instance recover the Euclidean width of K via $2\underline{R}_1(K,\mathbb{B}_2^d)$ and the Euclidean diameter via $2\overline{r}_1(K,\mathbb{B}_2^d)$.

It is shown in [86] that, for K and C symmetric, the identity

$$\underline{R}_j(K,C)\overline{r}_j(K^\circ,C^\circ)=1 \tag{2.19}$$

holds for all $j\in[d]$, and [99] shows that for $C=\mathbb{B}_2^d$ and possibly non-symmetric K at least the inequality

$$\underline{R}_j(K,\mathbb{B}_2^d)\overline{r}_j(K^\circ,\mathbb{B}_2^d)\geq 1 \tag{2.20}$$

remains valid. For arbitrary bodies K and C, Theorem 2.8.5 gives lower and upper bounds on the product $\underline{R}_j(K,C)\overline{r}_j(K^\circ,C^\circ)$. Compared to (2.20), the bounds of Theorem 2.8.5 seem somewhat weaker and some of them are indeed provably not tight. Nonetheless, they complement the results above and specialize to (2.19) if K and C are restricted to be symmetric.

We start by collecting some results about the Minkowski asymmetry of the polar of a body and centered inballs and circumcylinders, which make the same technique as for (2.19) applicable for the proof of Theorem 2.8.5.

Remark 2.8.2 (*Minkowski asymmetry of the polar*)
For $K\in\mathcal{C}^d$ Minkowski centered, we have $s(K^\circ)=s(K)$.

Proof.
Let $\rho>0$. Then $-K\subseteq\rho K\Leftrightarrow(-K)^\circ\supseteq\frac{1}{\rho}K^\circ\Leftrightarrow\rho K^\circ\supseteq -K^\circ$. □

Lemma 2.8.3 (*Centered inballs and circumcylinders*)
Let $K,C\in\mathcal{C}^d$ be Minkowski centered. Then the following two inclusions hold:

a) $K\subseteq\dfrac{(s(C)+1)s(K)}{s(K)+1}R(K,C)C$,

b) $\dfrac{s(C)+1}{(s(K)+1)s(C)}r(K,C)C\subseteq K$.

Proof.

a) By definition of $R(K,C)$, $s(K)$, $s(C)$, and the assumptions that K and C are Minkowski centered, there exists $c\in\mathbb{R}^d$ such that

$$K\subseteq c+R(K,C)C \text{ and} \tag{2.21}$$

$$\frac{1}{s(K)}K\subseteq -K\subseteq -c+R(K,C)(-C)\subseteq -c+R(K,C)s(C)C. \tag{2.22}$$

Adding (2.21) and (2.22) yields

$$\frac{s(K)+1}{s(K)}K\subseteq R(K,C)(1+s(C))C,$$

which is equivalent to the claimed inclusion.

b) The claim follows from a) by changing the roles of K and C, and $r(K,C)=R(C,K)^{-1}$. □

The above inclusions imply lower and upper bounds on the product $\underline{R}_j(K,C)\overline{r}_j(K^\circ,C^\circ)$. In the Euclidean case, we can improve the lower bound by using a result from [99]. The following proposition can be extracted from [99, Lemma 2.1].

Proposition 2.8.4
Let $c \in \mathbb{R}^d, \rho \geq 0$ and $B := c + \rho\mathbb{B}_2^d$ such that $0 \in \operatorname{int}(B)$. Then, $\overline{r}_d(B^\circ, \mathbb{B}_2^d) \geq \frac{1}{\rho}$ and $\overline{r}_d(B^\circ, \mathbb{B}_2^d) = \frac{1}{\rho}$ if and only if $c = 0$.

Putting together both parts from Lemmas 2.8.3, we obtain the following inequalities. The theorem assumes both bodies to be Minkowski centered. This assumption is without loss of generality for the value of $R(K,C)$. However, the value $R(K^\circ,C^\circ)$ is heavily dependent on this positioning of K and C since polarization is always performed with respect to the origin. However, the Minkowski center seems a natural generalization of the center of symmetry which is used as polarization center for 0-symmetric bodies.

Theorem 2.8.5 (*Radii of the polar*)
Let $K, C \in \mathcal{C}^d$ be Minkowski centered. Then,

$$\underline{R}_j(K,C)\overline{r}_j(K^\circ,C^\circ) \in \left[\frac{s(K)+1}{(s(C)+1)s(K)}, \frac{(s(K)+1)s(C)}{s(C)+1}\right].$$

In the Euclidean case , the lower bound can be improved to

$$\underline{R}_j(K,\mathbb{B}_2^d)\overline{r}_j(K^\circ,\mathbb{B}_2^d) \geq 1.$$

Proof.
Let $c \in \mathbb{R}^d$ and $F \in \mathcal{L}_{d-j}^d$ be such that $K \subseteq c+F+\underline{R}_j(K,C)C$. Then, by Lemma 2.8.3a), $K \subseteq F + \frac{(s(C)+1)s(K)}{s(K)+1}\underline{R}_j(K,C)C$ and thus

$$K^\circ \supseteq F^\circ \cap \frac{s(K)+1}{(s(C)+1)s(K)}\underline{R}_j(K,C)^{-1}C^\circ.$$

As $F^\circ = F^\perp \in \mathcal{L}_j^d$, we obtain $\underline{R}_j(K,C)\overline{r}_j(K^\circ,C^\circ) \geq \frac{s(K)+1}{(s(C)+1)s(K)}$.
If $C = \mathbb{B}_2^d$, we may use $K \subseteq F + c + \underline{R}_j(K,\mathbb{B}_2^d)\mathbb{B}_2^d$ and

$$(F + c + \underline{R}_j(K,\mathbb{B}_2^d)\mathbb{B}_2^d)^\circ = F^\perp \cap (c + \underline{R}_j(K,\mathbb{B}_2^d)\mathbb{B}_2^d)^\circ$$

to obtain $F^\perp \cap (c+\underline{R}_j(K,\mathbb{B}_2^d)\mathbb{B}_2^d)^\circ \subseteq K^\circ$. As $\overline{r}_d\Big((c+\underline{R}_j(K,\mathbb{B}_2^d)\mathbb{B}_2^d)^\circ, \mathbb{B}_2^d\Big) \geq \underline{R}_j(K,\mathbb{B}_2^d)^{-1}$ by Proposition 2.8.4 and $F^\perp \in \mathcal{L}_j^d$, we obtain

$$\overline{r}_j(K^\circ,\mathbb{B}_2^d) \geq \overline{r}_j(K^\circ \cap F^\perp,\mathbb{B}_2^d) \geq \underline{R}_j(K,\mathbb{B}_2^d)^{-1}.$$

On the other hand, if $c \in \mathbb{R}^d$ and $F \in \mathcal{L}_j^d$ are such that $K^\circ \supseteq c + \overline{r}_j(K^\circ, C^\circ)(C^\circ \cap F)$, then by Lemma 2.8.3 b) $K^\circ \supseteq \frac{s(C^\circ)+1}{(s(K^\circ)+1)s(C^\circ)} \overline{r}_j(K^\circ, C^\circ)(C^\circ \cap F)$ and by Remark 2.8.2,

$$K \subseteq F^\circ + \frac{(s(K)+1)s(C)}{s(C)+1} \underline{R}_j(K^\circ, C^\circ)C.$$

Again, $F^\circ = F^\perp \in \mathcal{L}_j^{d-j}$ yields $\underline{R}_j(K,C)\overline{r}_j(K^\circ, C^\circ) \leq \frac{(s(K)+1)s(C)}{s(C)+1}$.

□

Chapter 3

Fixed Parameter Complexity of Norm Maximization

The problem of maximizing the p-th power of a p-norm over a halfspace-presented polytope in $\mathbb{R}^d$ is a convex maximization problem which plays a fundamental role in computational convexity. It has been shown in [130] that this problem is $\mathbb{NP}$-hard for all values $p \in \mathbb{N}$, if the dimension d of the ambient space is part of the input. In this chapter, we use the theory of parametrized complexity to analyze how heavily the hardness of norm maximization relies on the parameter d.
More precisely, we show that for $p = 1$ the problem is fixed parameter tractable but that for all $p \in \mathbb{N} \setminus \{1\}$ norm maximization is W[1]-hard.
Concerning approximation algorithms for norm maximization, we show that for fixed accuracy, there is a straightforward approximation algorithm for norm maximization in FPT running time, but there is no FPT approximation algorithm, the running time of which depends polynomially on the accuracy.
As with the $\mathbb{NP}$-hardness of norm maximization, the W[1]-hardness immediately carries over to various radius computation tasks in Computational Convexity.

This chapter is joint work with Christian Knauer and Daniel Werner and we hereby gratefully acknowledge that this work has been initiated during the *10th INRIA-McGill workshop on Computational Geometry.* A joint paper with the obtained results is currently in preparation [124].

3.1 Introduction and Preliminaries

The problem of computing geometric functionals of polytopes arises in many applications in mathematical programming, operations research, statistics, physics, chemistry or medicine (see e.g. [90] for an overview). Hence, the question how efficiently these functionals can be computed or approximated has been studied extensively, e.g. in [27, 45, 85, 87, 130].

Of particular interest is the problem of maximizing (the p-th power of) a p-norm over a polytope. Despite its simple formulation, this problem already exhibits the combinatorial properties which are responsible for hardness or tractability of the computation of many important geometric functionals. As for most computational problems on polytopes, the presentation of the input polytope is crucial for the computational complexity of norm maximization: If the input polytope is presented as the convex hull of finitely many points, norm maximization is solvable in polynomial time by the trivial algorithm of computing and comparing the norm of all these points. The situation changes dramatically when the input polytope is presented as the intersection of halfspaces. The present chapter is concerned with the investigation of the parametrized complexity of this problem.
For $p \in \mathbb{N} \cup \{\infty\}$, a precise formulation of the norm maximization problem that we consider is as follows:

Problem 3.1.1 (NORMMAX$_p$)
Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $\max\{\|x\|_p^p : x \in P\} \geq \gamma$?

Here, a rational $\mathcal{H}$-presentation of a polytope is a presentation as intersection of finitely many halfspaces which are defined by inequalities that have only rational coefficients.

As shown in [130], for $p = \infty$ (with the understanding that $\|x\|_\infty^\infty = \|x\|_\infty$), NORMMAX$_\infty$ is solvable in polynomial time via Linear Programming. For all $p \in \mathbb{N}$, on the other hand, NORMMAX$_p$ is $\mathbb{NP}$-complete. (When speaking of $\mathbb{NP}$-hardness of parameterized problems, we mean the same decision problem, simply ignoring the parameter.) Moreover, in [27], it is shown that $\mathbb{NP}$-hardness persists for all $p \in \mathbb{N}$ even when the instances are restricted to full-dimensional parallelotopes presented as a Minkowski sum of d linearly independent line segments. Moreover, by [45], there is no polynomial time approximation algorithm for norm maximization for any constant performance ratio, unless $\mathbb{P} = \mathbb{NP}$.

It is important to note that, as usual in the realm of computational convexity, the dimension d is part of the input and the hardness of NORMMAX$_p$ relies heavily on this fact, especially for the very restricted instances in [27]. Indeed, if d is a constant, the obvious brute force algorithm of converting the presentation of P yields a polynomial time algorithm with running time $O(n^d)$, where n denotes the number of halfspaces in the presentation of P. However, this algorithm quickly becomes impractical as n grows, even for moderate values of d. The main purpose of this chapter is to close the gap between $\mathbb{NP}$-hardness for unbounded dimension and a theoretically polynomial, yet impractical algorithm for fixed dimension.
A suitable tool that allows us to analyze how strongly the hardness of NORMMAX$_p$ depends on the parameter d is the theory of Fixed Parameter Tractability. For an

introduction to Fixed Parameter Tractability, we refer to the textbooks [70, 138]. This theory has already been applied successfully to show the intractability of several problems in Computational Geometry even in low dimensions, see e.g. [48, 49, 77, 78, 79, 123].

Our analysis of NORMMAX_p shows that, although NORMMAX_p is $\mathbb{NP}$-hard for all $p \in \mathbb{N}$, the hardness has a different flavor for different types of norms: Whereas hardness of NORMMAX_1 only comes with the growth of the dimension, NORMMAX_p has to be considered intractable already in small dimensions for all other values of p.
More precisely, we prove the following theorem:

Theorem 3.1.2 (*Fixed-parameter complexity of* NORMMAX)
NORMMAX_1 is in FPT, whereas NORMMAX_p is W[1]-hard for all $p \in \mathbb{N} \setminus \{1\}$.

The presented reduction also shows that in the hard cases no $n^{o(d)}$ algorithm for NORMMAX_p exists, unless the *Exponential Time Hypothesis*[1] is false. Thus, the brute force algorithm for NORMMAX_p mentioned above already has the best achievable complexity, if $p \in \mathbb{N} \setminus \{1\}$.

In this case, one can also ask how strongly the inapproximability result of [45] relies on the fact that NORMMAX_p is a problem in unbounded dimension. For this purpose, call an algorithm that produces an $\bar{x} \in P$ such that, for some $\beta \in \mathbb{N}$,

$$\|\bar{x}\|_p^p \geq \left(\frac{\beta - 1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}$$

a β-approximation-algorithm for NORMMAX_p. The proof of the fact that NORMMAX_1 is in FPT then suggests the following: Replace the unit ball of the p-norm by a suitable symmetric polytope which approximates it sufficiently well and use the maximum of this polytopal norm as an approximation for the maximum of the p-norm. As polytopal norms can be maximized by solving a linear program for every facet of the unit ball and linear programs can be solved in $T_{LP}(d, n) := O(2^{2^d} n)$ (see [132]), which is polynomial in n for fixed d, this yields an FPT-time approximation algorithm for fixed accuracy β.

Theorem 3.1.3 (*Approximation complexity of* NORMMAX)
Let $p \in \mathbb{N} \setminus \{1\}$. For every fixed $\beta \in \mathbb{N}$, there is a β-approximation-algorithm for NORMMAX_p which runs in time $O(\beta^d T_{LP}(d, n))$. Conversely, there is no scheme of β-approximation-algorithms for NORMMAX_p with running time $O(f(d)q(\beta, d, n))$ with a polynomial q and an arbitrary computable function f.

Hence, although the problem is not in APX, approximation of NORMMAX_p is possible for moderate values of β and d. On the other hand, approximation tends to become costly as soon as the dimension or the desired accuracy grows.

[1] The Exponential Time Hypothesis conjectures that n-variable 3-CNFSAT cannot be solved in $2^{o(n)}$-time; cf. [113].

Finally, analogously to the $\mathbb{NP}$-hardness of NORMMAX$_p$, the W[1]-hardness of NORMMAX$_p$ implies the intractability of various problems in Computational Convexity as immediate corollaries. In Section 3.4, we show that for the respective values of p, the problems CIRCUMRADIUS$_p$-$\mathcal{H}$, DIAMETER$_p$-$\mathcal{H}$, INRADIUS$_p$-$\mathcal{V}$ and WIDTH$_p$-$\mathcal{V}$ (all parameterized by the dimension) are W[1]-hard.

This chapter is organized as follows. In Section 3.2, we will analyze the parameterized complexity of NORMMAX$_p$, i.e. we prove Theorem 3.1.2 and prepare some technical lemmas, which we will also use in Section 3.3 where we prove Theorem 3.1.3. Finally, in Section 3.4, we prove the corollaries for the mentioned radius computation tasks.

3.2 Fixed Parameter Complexity of Norm Maximization

3.2.1 Intractability

We will first prove the hardness result for NORMMAX$_p$ for $p \geq 2$ via an FPT reduction of the W[1]-complete problem CLIQUE to NORMMAX$_p$. The formal parametrized decision problem of CLIQUE is given in Problem 3.2.1; a proof of its W[1]-completeness can be found e.g. in [70, Theorem 6.1].

Problem 3.2.1 (CLIQUE)

Input:	$n, k \in \mathbb{N}$, $E \subseteq \binom{[n]}{2}$
Parameter:	k
Question:	Does $G = ([n], E)$ contain a clique of size k?

Moreover, it is shown in [51] that CLIQUE cannot be solved in time $n^{o(k)}$, unless the Exponential Time Hypothesis fails.

In order to show the hardness result, we will first show how to construct a polytope P for a graph $G = ([n], E)$ with the property that

$$\max\{\|x\|_p^p : x \in P\} = k \quad \Longleftrightarrow \quad G \text{ contains a clique of size } k.$$

This "reduction" will be laid out as if irrational numbers were computable with infinite precision. The second part of this section will then show that the numbers can be rounded to a sufficiently rough grid in order to make the reduction suitable for the Turing machine model.

The construction.

Let (n, k, E) be an instance of CLIQUE and $p \in [1, \infty)$. Throughout this chapter, we assume without loss of generality that n is an even number. (If not, we add an isolated vertex to the graph.)

We choose $d := 2k$ and consider

$$\mathbb{R}^{2k} = \mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \times \mathbb{R}^2$$

i.e. we will think of a vector $x \in \mathbb{R}^{2k}$ as k two-dimensional vectors stacked upon each other. Therefore, it will be convenient to use the following notation.

Notation 3.2.2
By indexing a vector $x \in \mathbb{R}^{2k}$, we refer to the k two-dimensional vectors $x_1, \ldots, x_k \in \mathbb{R}^2$ such that $x = (x_1^T, \ldots, x_k^T)^T$. Further, for $a \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$, we let

$$H^i_{\leq}(a, \beta) := \{x \in \mathbb{R}^{2k} : a^T x_i \leq \beta\}.$$

In order to construct an $\mathcal{H}$-presentation of a polytope $P \subseteq \mathbb{B}_p^2 \times \mathbb{B}_p^2 \times \cdots \times \mathbb{B}_p^2$, we will first construct a 2-dimensional polytope $P_1 \subseteq \mathbb{B}_p^2$ as our basic building block by placing vertices on the unit sphere $\mathbb{S}_p^1$ (compare Figure 3.1):
For $v \in [\frac{n}{2}]$, let

$$p'_v := \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2(v-1)}{n} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \{p_v\} := \left(p'_v + [0, \infty) \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \cap \mathbb{S}_p^1; \tag{3.1}$$

for $v \in [n] \setminus [\frac{n}{2}]$ let

$$p'_v := \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2v-(n+2)}{n} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{and} \quad \{p_v\} := \left(p'_v + [0, \infty) \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) \cap \mathbb{S}_p^1. \tag{3.2}$$

For $v \in [2n] \setminus [n]$, let

$$p_v := -p_{v-n}$$

and

$$P_1 := \operatorname{conv}\{p_1, \ldots, p_{2n}\} = \bigcap_{v \in [2n]} H_{\leq}(a_v, \beta_v) \subseteq \mathbb{R}^2. \tag{3.3}$$

Note that P_1 is 0-symmetric by construction and that the required $\mathcal{H}$-presentation of P_1 in (3.3) can be computed in time $O(n \log(n))$, see e.g. [60]. For notational convenience, we also define

$$p_{2n+1} := p_1 \quad \text{and} \quad p_{-1} := p_{2n}.$$

Lemma 3.2.3 (*Distance between the p_v*)
Let $P_1 := \operatorname{conv}\{p_1, \ldots, p_{2n}\}$ be the polytope defined in Equation (3.3) and $v \in [2n]$. The distance between two neighboring points on $\mathbb{S}_p^1$ satisfies

$$\|p_v - p_{v+1}\|_2 \in \left[\frac{2\sqrt{2}}{n}, \frac{4}{n}\right].$$

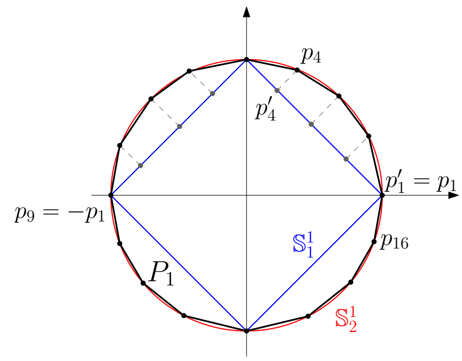

Figure 3.1: Construction of P_1 in the case $p = 2$, $n = 8$.

Proof.
Let $\Pi : \mathbb{R}^2 \to \mathbb{B}_1^2$ denote the projection onto $\mathbb{B}_1^2$. By the definitions in (3.1) and (3.2), we have $\Pi(p_v) = p'_v$. Since Π is contracting, the equidistant placement of $p'_1, \ldots, p'_n$ yields $\|p_v - p_{v+1}\|_2 \geq \|p'_v - p'_{v+1}\|_2 = \frac{2\sqrt{2}}{n}$ for all $v \in [2n]$.
For the other bound, assume that $v \leq \frac{n}{4}$. (The other cases can be handled with the same arguments.) By elementary properties of $\mathbb{B}_p^2$, we have $e_1^T p_{v+1} \leq e_1^T p_v$ and $\mathbb{1}^T p_{v+1} \geq \mathbb{1}^T p_v$ and thus $p_{v+1} \in [q_1, q_2]$ with q_1, q_2 defined as in Figure 3.2.

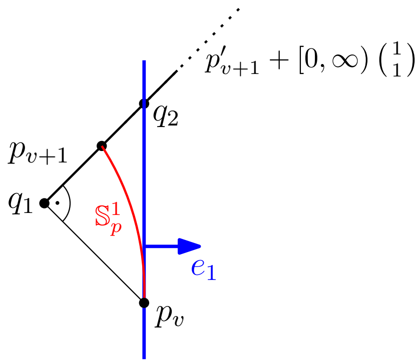

Figure 3.2: The situation in the proof of Lemma 3.2.3.

Inspection of the triangle $\text{conv}\{p_v, q_1, q_2\}$ shows that it is equilateral with a right angle at q_1. Thus, $\|p_v - p_{v+1}\|_2 \leq \|p_v - q_2\|_2 = \sqrt{2\|p_v - q_1\|_2^2} = \frac{4}{n}$. □

Using Notation 3.2.2, we define a polytope $P_2 \subseteq \mathbb{R}^{2k}$ via

$$P_2 := \bigcap_{i \in [k]} \bigcap_{v \in [2n]} H_{\leq}^i(a_v, \beta_v) \subseteq \mathbb{R}^{2k}.$$

Observe that P_2 is 0-symmetric by construction and that any vertex x of P_2 is of the form $x = (p_{v_1}, \ldots, p_{v_k})^T$ for suitable $v_1, \ldots, v_k \in [2n]$.

As for any $x = (x_1^T, \dots, x_k^T)^T \in \mathbb{R}^{2k}$ the identity

$$\|x\|_p^p = \sum_{i=1}^{k} \|x_i\|_p^p$$

holds, and as for $p \in \mathbb{N} \setminus \{1\}$ the unit sphere $\{x \in \mathbb{R}^2 : \|x\|_p^p = 1\}$ contains no straight line segments, it follows that for $x \in P_2$,

$$\|x\|_p^p \geq k \quad \iff \quad x = \begin{pmatrix} p_{v_1} \\ \vdots \\ p_{v_k} \end{pmatrix} \text{ for some } v_1, \dots, v_k \in [2n].$$

For $v \in [2n]$, let $x_v, y_v \in \mathbb{R}$ be the coordinates of $p_v = (x_v, y_v)^T$ and define

$$q_v := \begin{pmatrix} \operatorname{sgn}(x_v)|x_v|^{p-1} \\ \operatorname{sgn}(y_v)|y_v|^{p-1} \end{pmatrix}. \tag{3.4}$$

Noting that for all $x \in P_1$ and $v \in [2n]$, $q_v^T x = 1$ if and only if $x = p_v$, we define

$$\varepsilon := 1 - \max\{q_u^T p_v : u, v \in [2n], u \neq v\} > 0 \tag{3.5}$$

and for $u, v \in [n]$ and $i, j \in [k]$,

$$E_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \varepsilon - 2 \leq q_u^T x_i + q_v^T x_j \leq 2 - \varepsilon\}$$

and

$$F_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \varepsilon - 2 \leq q_u^T x_i - q_v^T x_j \leq 2 - \varepsilon\}.$$

Thus, if x is a vertex of P_2 with $x_i = \pm p_u$ and $x_j = \pm p_v$ for some $u, v \in [n]$, then $x \notin E_{uv}^{ij} \cap F_{uv}^{ij}$, i.e. if $u, v \in [n]$ and $\{u, v\} \notin E$ the constraints of $E_{uv}^{ij} \cap F_{uv}^{ij}$ make sure that P does not contain a vertex with $x_i = \pm p_u$ and $x_j = \pm p_v$.

Finally, to encode the Clique instance, we let $N := \binom{[n]}{2} \setminus E$, define

$$P := P_2 \cap \bigcap_{\substack{\{u,v\} \in N \\ i,j \in [k], i \neq j}} (E_{uv}^{ij} \cap F_{uv}^{ij}) \cap \bigcap_{\substack{v \in [n] \\ i,j \in [k], i \neq j}} (E_{vv}^{ij} \cap F_{vv}^{ij}),$$

and obtain the following lemma.

Lemma 3.2.4 (*Reduction with infinite precision*)
Let (n, k, E) be an instance of Clique, $p \in [1, \infty)$ and $P \subseteq \mathbb{R}^{2k}$ the polytope obtained by the construction above. Then,

$$\max\{\|x\|_p^p : x \in P\} = k \quad \iff \quad G = ([n], E) \text{ contains a clique of size } k.$$

Analysis of the constructed polytope.

We will now investigate how much we can perturb the (possibly irrational) polytope P in order to make it suitable for an FPT-reduction without loosing its ability to decide between Yes- and No-instances of CLIQUE. For this purpose, we define the constant

$$U := \frac{1}{n^{2p}k^2}. \tag{3.6}$$

In the following, we show that rounding the vertices $p_1, \dots, p_{2n}$ of our initial polytope $P_1 \subseteq \mathbb{R}^2$ to the grid $\frac{U}{2}\mathbb{Z}^2$ preserves all important features of our reduction. Since the parameter p is a constant in NORMMAX$_p$, all the necessary computations can be carried out with a precision of $O(\log(nk))$ bits. Since we only need a polynomial number of computations, the whole reduction can be carried out in polynomial time.

Lemma 3.2.5

Let $P_1 = \operatorname{conv}\{p_1, \dots, p_{2n}\} \subseteq \mathbb{R}^2$ with $p_1, \dots, p_{2n} \in \mathbb{S}^1_p$ be the polytope from Equation (3.3). For $\varepsilon := 1 - \max\{q_u^T p_v : u, v \in [2n], u \neq v\}$ with q_u defined as in Equation (3.4), we have

$$\varepsilon \geq \frac{2^{p-1}}{pn^p}.$$

Proof.

Let $x := (x_1, x_2)^T \in \mathbb{S}^1_p$ and $y := (y_1, y_2)^T \in \mathbb{S}^1_1$ with $x, y \geq 0$, $\|x - e_1\|_2 \geq \frac{2\sqrt{2}}{n}$, and $\|y - e_1\|_2 \geq \frac{2\sqrt{2}}{n}$. Since $\mathbb{B}^2_1 \subseteq \mathbb{B}^2_p$, $x_2 \geq y_2 \geq \frac{2}{n}$. Combining this inequality with $x \in \mathbb{S}^1_p$ yields

$$x_1 = (1 - x_2^p)^{\frac{1}{p}} \leq \left(1 - \left(\frac{2}{n}\right)^p\right)^{\frac{1}{p}} \leq 1 - \frac{2^p}{pn^p}, \tag{3.7}$$

where the last inequality follows by bounding the concave function $x \mapsto x^{\frac{1}{p}}$ from above by a linear approximation at $x = 1$.

Now, let $u, v \in [2n]$ with $u \neq v$. Then,

$$q_u^T p_v = q_u^T p_u + q_u^T (p_v - p_u) = 1 + \cos(q_u, p_v - p_u)\, \|q_u\|_2\, \|p_v - p_u\|_2. \tag{3.8}$$

Since the points of lowest curvature on $\mathbb{S}^1_p$ are $\pm e_1$ and $\pm e_2$, and since $e_1 = p_1 = q_1$, we obtain $\cos(q_u, p_v - p_u) \leq \cos(e_1, p_2 - e_1)$, which in turn can be bounded by

$$\cos(e_1, p_2 - e_1) \leq \frac{x_1 - 1}{\|p_2 - e_1\|_2}$$

with $x_1 = e_1^T x$ for the point $x \in \mathbb{S}^p_1$ defined above. Further, $q_u \in \mathbb{S}^1_{\frac{p}{p-1}}$ implies $\|q_u\|_2 \geq \frac{\sqrt{2}}{2}$, and $\|p_v - p_u\|_2 \geq \frac{2\sqrt{2}}{n}$ by Lemma 3.2.3. Using (3.7), we can continue Equation (3.8) to

$$q_u^T p_v \leq 1 - \frac{2^p}{pn^p \|p_2 - e_1\|} \cdot \frac{\sqrt{2}}{2} \cdot \frac{2\sqrt{2}}{n} \leq 1 - \frac{2^{p-1}}{pn^p},$$

where the last inequality follows again from Lemma 3.2.3. □

For $v \in [n]$, let $\bar{p}_v$ be the rounding of p_v to the grid $\frac{U}{2}\mathbb{Z}^2$ and define $\bar{p}_v = -\bar{p}_{v-n}$ for $v \in [2n] \setminus [n]$ and further

$$\bar{P}_1 := \operatorname{conv}\{\bar{p}_1, \ldots, \bar{p}_{2n}\}. \tag{3.9}$$

For $\bar{p}_v = (\bar{x}_v, \bar{y}_v)^T \in \mathbb{R}^2$, define

$$\bar{q}_v := \begin{pmatrix} \operatorname{sgn}(\bar{x}_v)\,|\bar{x}_v|^{p-1} \\ \operatorname{sgn}(\bar{y}_v)\,|\bar{y}_v|^{p-1} \end{pmatrix}.$$

By choice of our grid, we get

$$\|p_v - \bar{p}_v\|_{p'} \leq U \quad \forall p' \geq 1. \tag{3.10}$$

Moreover, if $q \in [1, \infty)$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\|q_v\|_q = 1$ for all $v \in [2n]$ and since $x \mapsto x^{p-1}$ is Lipschitz continuous on $[-1, 1]$ with Lipschitz constant $L = p - 1$, we obtain

$$\|q_v - \bar{q}_v\|_1 \leq (p-1)U. \tag{3.11}$$

First, we show that the points $\bar{p}_1, \ldots, \bar{p}_{2n}$ are still in convex position, which is binned into a separate lemma for later use in Chapter 5.

Lemma 3.2.6
Let $\bar{P}_1 = \operatorname{conv}\{\bar{p}_1, \ldots, \bar{p}_{2n}\} \subseteq \mathbb{R}^2$ the polytope from (3.9). Then, $\operatorname{ext}(\bar{P}_1) = \{\bar{p}_1, \ldots, \bar{p}_{2n}\}$ and the coding length of an $\mathcal{H}$-presentation of $\bar{P}_1$ is polynomially bounded in the coding length of $\bar{p}_1, \ldots, \bar{p}_{2n}$.

Proof.
For $v \in [2n]$, we have $q_v^T \bar{p}_v \geq 1 - \|q_v\|_2 U \geq 1 - \|q_v\|_q U = 1 - U$, since $p \geq 2$ and therefore $q \leq 2$. For $u \in [2n] \setminus \{v\}$, we get $q_v^T \bar{p}_u \leq 1 - \varepsilon + U$. Since $1 - \varepsilon + U < 1 - U$, the hyperplane $H_=(q_v, 1 - \varepsilon + U)$ separates $\bar{p}_v$ from $\operatorname{conv}(\{\bar{p}_1, \ldots, \bar{p}_{2n}\} \setminus \{\bar{p}_v\})$ and hence $\bar{p}_v \in \operatorname{ext}(\bar{P}_1)$.
Assume now that $\bar{P}_1 := \{x \in \mathbb{R}^2 : \bar{a}_v^T x \leq 1 \ \forall v \in [2n]\}$ is an $\mathcal{H}$-presentation of $\bar{P}_1$. Applying Cramer's Rule, we see that, for all $v \in [2n]$, the entries of $\bar{a}_v$ are quotients of polynomials in $\bar{p}_1, \ldots, \bar{p}_{2n}$ and so the coding length of the $\mathcal{H}$-presentation of $\bar{P}_1$ is bounded by a polynomial in the coding length of $\bar{p}_1, \ldots, \bar{p}_{2n}$. □

Since the coding length of $\bar{P}_1$ is polynomially bounded, we also get that the coding length of

$$\bar{P}_2 := \bigcap_{i \in [k]} \bigcap_{v \in [2n]} H^i_{\leq}(\bar{a}_v, \bar{\beta}_v) \subseteq \mathbb{R}^{2k}.$$

is polynomially bounded.

Now, let $\bar{\varepsilon} := 1 - \max\{\bar{q}_u^T \bar{p}_v : u, v \in [2n], u \neq v\}$. By expanding the expression $\bar{q}_u^T \bar{p}_v = \big(q_u + (\bar{q}_u - q_u)\big)^T \big(p_v + (\bar{p}_v - p_v)\big)$ and using (3.10) and (3.11), we obtain

$$\bar{\varepsilon} \geq \varepsilon - 3pU > 0. \tag{3.12}$$

Finally, define

$$\bar{E}_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \bar{\varepsilon} - 2 \leq \bar{p}_u^T x_i + \bar{p}_v^T x_j \leq 2 - \bar{\varepsilon}\},$$

and

$$\bar{F}_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \bar{\varepsilon} - 2 \leq \bar{p}_u^T x_i -, \bar{p}_v^T x_j \leq 2 - \bar{\varepsilon}\},$$

and, for $N := \binom{[n]}{2} \setminus E$, let

$$\bar{P} := \bar{P}_2 \cap \bigcap_{\substack{\{u,v\} \in N \\ i,j \in [k], i \neq j}} (\bar{E}_{uv}^{ij} \cap \bar{F}_{uv}^{ij}) \cap \bigcap_{\substack{v \in [n] \\ i,j \in [k], i \neq j}} (\bar{E}_{vv}^{ij} \cap \bar{F}_{vv}^{ij}). \tag{3.13}$$

The following two lemmas will now prepare the proof that we can still reduce Clique to norm maximization over $\bar{P}$. To be able to state them in a concise way, we introduce the following notation.

Notation 3.2.7
Let $\bar{P} \subseteq \mathbb{R}^{2k}$ be the polytope from Equation (3.13) and $x = (x_1^T, \ldots, x_k^T)^T \in \bar{P}$. By letting

$$m_i(x) \in \arg\max\{\bar{q}_v^T x_i : v \in [2n]\},$$

we can refer to the index of a vertex which is "closest" to x in the sense that $\bar{q}_{m_i(x)}^T x \geq \bar{q}_v^T x$ for all $v \in [2n]$. This is illustrated in Figure 3.3.

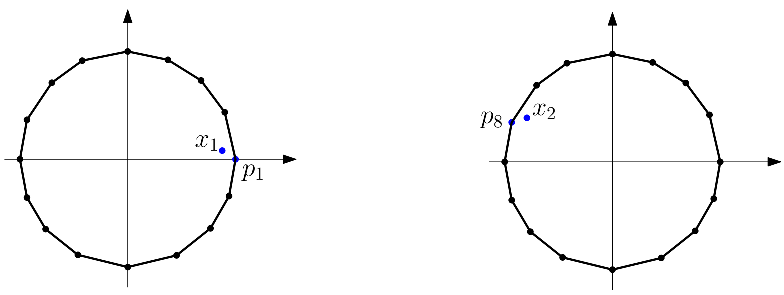

Figure 3.3: Illustration of Notation 3.2.7. The figure shows a point $x = (x_1^T, x_2^T)^T \in \mathbb{R}^4$ with $m_1(x) = 1$ and $m_2(x) = 8$.

First, we show that if $\bar{P}$ contains a point which is "close" (in the sense specified in Notation 3.2.7) to a clique vertex, then $\bar{P}$ contains the clique vertex itself.

Lemma 3.2.8
Let $\bar{P} \subseteq \mathbb{R}^{2k}$ be the polytope constructed above in Equation (3.13). If there exists $\bar{x} \in \bar{P}$ such that $\bar{q}_{m_i(\bar{x})}^T \bar{x} > 1 - \frac{\bar{\varepsilon}}{2}$ for all $i \in [k]$, then $(\bar{p}_{m_1(\bar{x})}^T, \ldots, \bar{p}_{m_k(\bar{x})}^T)^T \in \bar{P}$.

Proof.
Since $\bar{q}_{m_i(\bar{x})}^T \bar{x}_i > 1 - \frac{\bar{\varepsilon}}{2}$ for all $i \in [k]$, for no pair $(i,j) \in [k]^2$ the inequalities $\bar{q}_{m_i(\bar{x})}^T x_i + \bar{q}_{m_j(\bar{x})}^T x_j \leq 2 - \bar{\varepsilon}$ can be present in the description of $\bar{P}$. Since, by definition of $\bar{\varepsilon}$, we have $\bar{q}_v^T \bar{p}_{m_i(\bar{x})} \leq 1 - \bar{\varepsilon}$ for all $v \in [2n] \setminus \{m_i(\bar{x})\}$ and $i \in [k]$, we can conclude that

$$(\bar{p}_{m_1(\bar{x})}^T, \ldots, \bar{p}_{m_k(\bar{x})}^T)^T \in \bar{P}.$$

□

In view of Lemma 3.2.8, it remains to show that the norm of a vertex which is "far" from a clique vertex is sufficiently small:

Lemma 3.2.9
Let $v \in [2n]$ and $Q := \text{conv}\{0, \bar{p}_v, \bar{p}_{v+1}\} \cap H_{\leq}(\bar{q}_v, 1 - \frac{\bar{\varepsilon}}{2}) \cap H_{\leq}(\bar{q}_{v+1}, 1 - \frac{\bar{\varepsilon}}{2})$. Then, for n sufficiently large,

$$\max\{\|x\|_p^p : x \in Q\} \leq 1 - \frac{2^{p-3}}{pn^p}.$$

Proof.
Let $Q' := \text{conv}\{0, e_1, \bar{p}_2\} \cap H_{\leq}\left(e_1, 1 - \frac{\bar{\varepsilon}}{2}\right) \cap H_{\leq}\left(\bar{q}_2, 1 - \frac{\bar{\varepsilon}}{2}\right)$. Since e_1 is a point of lowest curvature on the boundary of $\mathbb{B}_p^2$, we have $\max\{\|x\|_p^p : x \in Q\} \leq \max\{\|x\|_p^p : x \in Q'\} = \|x^*\|_p^p$, where x^* fulfills $e_1^T x^* = 1 - \frac{\bar{\varepsilon}}{2}$ and $x^* = \lambda e_1 + (1-\lambda)\bar{p}_2$ for some $\lambda \in [0,1]$. From the first property, we can deduce $\lambda = 1 - \frac{\bar{\varepsilon}}{2}$, which implies $e_2^T x^* = \frac{\bar{\varepsilon}}{2} e_2^T \bar{p}_2$. By Lemma 3.2.3, $e_2^T \bar{p}_2 \leq \frac{2}{n} + U$. Putting things together, we obtain

$$\|x^*\| \leq \left(1 - \frac{\bar{\varepsilon}}{2}\right)^p + \left(\frac{\bar{\varepsilon}}{2}\left(\frac{2}{n} + U\right)\right)^p \leq \left(1 - \frac{\bar{\varepsilon}}{2}\right) + \left(\frac{\bar{\varepsilon}}{2}\left(\frac{2}{n} + U\right)\right)^p. \tag{3.14}$$

By Lemma 3.2.5 and Equation (3.12), $\bar{\varepsilon} \geq \frac{2^{p-1}}{pn^p} - 3pU$. By the choice of U and the assumption that n is sufficiently large, we can therefore continue (3.14) and obtain

$$\left(1 - \frac{\bar{\varepsilon}}{2}\right) + \left(\frac{\bar{\varepsilon}}{2}\left(\frac{2}{n} + U\right)\right)^p \leq 1 - \frac{2^{p-3}}{pn^p}.$$

□

Hardness part of Theorem 3.1.2.
The following lemma shows that it is sufficient to carry out the reduction described by Lemma 3.2.4 with finite precision as described in this subsection. It completes the proof of the hardness part of Theorem 3.1.2. For notational convenience, we use the clique number $\omega(G)$ to denote the size of the biggest clique in a graph $G = ([n], E)$.

Lemma 3.2.10 (*Reduction with finite precision*)
Let (n, k, E) be an instance of Clique, $G = ([n], E)$ and $\bar{P} \subseteq \mathbb{R}^{2k}$ the polytope with rounded coordinates constructed above in (3.13). Then,

$$\omega(G) \geq k \quad \Longleftrightarrow \quad \max\{\|x\|_p^p : x \in \bar{P}\} \geq k(1-U)^p \tag{3.15}$$

and

$$\omega(G) < k \quad \Longleftrightarrow \quad \max\{\|x\|_p^p : x \in \bar{P}\} \leq (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}. \tag{3.16}$$

Proof.
Since $(k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p} < k(1-U)^p$, if suffices to show the "forward" direction in both (3.15) and (3.16).
If $\omega(G) \geq k$ and $\{v_1, \ldots, v_k\} \subseteq [n]$ is the vertex set of a k-clique in G, then $\bar{P}$ contains the vertex $x^* = (\bar{p}_{v_1}^T, \ldots, \bar{p}_{v_k}^T)^T$ and $\|x^*\|_p^p \geq k(1-U)^p$ by (3.10).
Assume now that $\omega(G) < k$ and let $x^* \in \bar{P}$ be a vertex of maximal norm in $\bar{P}$. If $\bar{q}_{m_i(x^*)}^T x^* > 1 - \frac{\bar{\varepsilon}}{2}$ for all $i \in [k]$, Lemma 3.2.8 would imply that $(\bar{p}_{m_1(x^*)}^T, \ldots, \bar{p}_{m_k(x^*)}^T)^T$ is a vertex of $\bar{P}$ and therefore contradict $\omega(G) < k$. Hence, there is some $i \in [k]$ such that $\bar{q}_{m_i(x^*)}^T x^* \leq 1 - \frac{\bar{\varepsilon}}{2}$. By adding a constant number of vertices to G, we can assume that n is sufficiently large and apply Lemma 3.2.9 in order to obtain $\|x_i^*\|_p^p \leq 1 - \frac{2^{p-3}}{pn^p}$. As $\|x_j^*\|_p^p \leq (1+U)^p$ for all $j \in [k] \setminus \{i\}$, the right hand side of (3.16) follows. □

The construction of the polytope P (or $\bar{P}$) relies on the fact that, for $p \geq 2$, the boundary of the unit ball of a p-norm contains no straight line segment. This is not the case for $p = 1$ and we show in the next subsection that Normmax$_1$ is indeed in FPT.

3.2.2 Tractability

This subsection completes the proof of Theorem 3.1.2 by showing that Normmax$_1$ is fixed parameter tractable.
The statement of Theorem 3.2.12 is slightly more general than needed for Theorem 3.1.2 but will be of use in Section 3.3. The result for Normmax$_1$ can be obtained from Theorem 3.2.12 by choosing $\varphi_d : \mathbb{R}^d \to \mathbb{R}; x \mapsto \|x\|_1$ in Problem 3.2.11.

Problem 3.2.11 (Max-Φ)
Suppose that for each $d \in \mathbb{N}$, $\varphi_d : \mathbb{R}^d \to \mathbb{R}$ is positive homogeneous of degree 1 and let $\Phi := (\varphi_d)_{d \in \mathbb{N}}$. The problem Max-$\Phi$ is defined as follows:

Input:	$d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a polytope $P \subseteq \mathbb{R}^d$
Parameter:	d
Question:	Is $\max\{\varphi_d(x) : x \in P\} \geq \gamma$?

Theorem 3.2.12 (*Tractability of* MAX-Φ)
For each $d \in \mathbb{N}$, let $\varphi_d : \mathbb{R}^d \to \mathbb{R}$ be positive homogeneous of degree 1 and $\Phi := (\varphi_d)_{d\in\mathbb{N}}$. Suppose that, for $d \in \mathbb{N}$, the set $\mathbb{B}^d := \{x \in \mathbb{R}^d : \varphi_d(x) \leq 1\}$ is a full-dimensional polytope, a rational $\mathcal{H}$-presentation of which can be computed in time $f(d)$ for a computable function $f : \mathbb{N} \to \mathbb{N}$. Then, MAX-$\Phi$ is in FPT and can be solved in time $O(f(d)T_{LP}(d,n))$.

Proof.
Let $\mathbb{B}^d = \bigcap_{i=1}^m H_{\leq}(a_i, 1)$ be an $\mathcal{H}$-presentation of $\mathbb{B}^d$. Then, $m \in O(f(d))$. Because of the homogeneity of φ_d, $\{x \in \mathbb{R}^d : \varphi_d(x) \leq \lambda\} = \lambda\mathbb{B}^d$ and $\varphi_d(x) = \max_{i\in[m]} a_i^T x$. Hence,

$$\max\{\varphi_d(x) : x \in P\} = \max_{i\in[m]} \max\{a_i^T x : x \in P\}.$$

Thus, MAX-Φ can be decided by the following algorithm:
(1) Compute an $\mathcal{H}$-presentation of $\mathbb{B}^d$ in time $f(d)$.
(2) Solve m linear programs $\max\{a_i^T x : x \in P\}$ in time $T_{LP}(d,n)$.
(3) Compare the biggest objective value to γ.
As $T_{LP}(d,n) \in O(2^{2^d} n)$, the above algorithm has FPT running time $O(f(d)2^{2^d} n)$. □

We can also establish fixed parameter tractability for the two problems $[-1,1]$-PARMAX$_p$ and $[0,1]$-PARMAX$_p$ as considered in [27].

Problem 3.2.13 ($[0,1]$-PARMAX$_p$)

Input:	$d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, $v_1, \ldots, v_n \in \mathbb{Q}^d$ linearly independent
Parameter:	d
Question:	Is $\max\{\|x\|_p^p : x \in \sum_{i=1}^d [0,1]v_i\} \geq \gamma$?

Problem 3.2.14 ($[-1,1]$-PARMAX$_p$)

Input:	$d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, $v_1, \ldots, v_n \in \mathbb{Q}^d$ linearly independent
Parameter:	d
Question:	Is $\max\{\|x\|_p^p : x \in \sum_{i=1}^d [-1,1]v_i\} \geq \gamma$?

In [27], it was shown that Problem 3.2.13 and 3.2.14 are both $\mathbb{NP}$-hard, so that the $\mathbb{NP}$-hardness of NORMMAX$_p$ persists even on very restricted instances. However, the following theorem shows that these problems are fixed parameter tractable, when parametrized by the dimension. So in this case, the hardness of PARMAX$_p$ is really a phenomenon of high dimensions.

Theorem 3.2.15 (*Tractability of* PARMAX$_p$)
For all $p \in \mathbb{N}$, Problems 3.2.13 and 3.2.14 are in FPT.

Proof.
We only consider Problem 3.2.13; the argument for Problem 3.2.14 is exactly the same. The vertices of the polytope $P := \sum_{i=1}^{d}[0,1]v_i$ are all of the form $\sum_{i=1}^{d} \lambda_i v_i$ for some vector $\lambda = (\lambda_1, \ldots, \lambda_d)^T \in \{0,1\}^d$. As the the maximum of $\| \cdot \|_p^p$ is attained at a vertex of P, it suffices to compute the norm of all 2^d possible choices of $\lambda \in \{0,1\}^d$. This clearly is an FPT-algorithm for Problem 3.2.13. □

3.3 Approximation

3.3.1 FPT-Approximation for Fixed Accuracy

In [45], it is shown that, for all $p \in \mathbb{N}$, NORMMAX_p is not contained in APX (i.e. there is no polynomial time approximation algorithm with a fixed performance guarantee). As norm maximization with a polytopal unit ball is in FPT, we can give a straightforward approximation algorithm that has FPT running time for any fixed accuracy by replacing the unit ball $\mathbb{B}_p^d$ by an approximating polytope. The following proposition concerning the complexity of such a polytope can be obtained from [46, Lemmas 3.7 and 3.8].

Proposition 3.3.1 (*Approximation of balls by polytopes*)
Let $p \in \mathbb{N}$ and $\beta \in \mathbb{N}$ be fixed. There is a symmetric polytope $B \subseteq \mathbb{R}^d$ with a rational $\mathcal{H}$-presentation and at most $O(\beta^d)$ facets such that

$$\mathbb{B}_p^d \subseteq B \subseteq \frac{\beta}{\beta - 1}\mathbb{B}_p^d, \tag{3.17}$$

and B can be computed in time $O(\beta^d)$.

Lemma 3.3.2 (*FPT-Approximation algorithm for fixed accuracy*)
Let $p \in \mathbb{N}$ and $\beta \in \mathbb{N}$ be fixed. There is an algorithm which for every $\mathcal{H}$-presented polytope $P \subseteq \mathbb{R}^d$ runs in time $O(\beta^d T_{LP}(d,n))$ and produces an $\bar{x} \in P$ such that

$$\|\bar{x}\|_p^p \geq \left(\frac{\beta - 1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}.$$

Proof.
The following algorithm has the desired properties:

(1) Compute an $\mathcal{H}$-presentation of a symmetric polytope $B \subseteq \mathbb{R}^d$ with the properties of Proposition 3.3.1 and let $\| \cdot \|_B : \mathbb{R}^d \to \mathbb{R}; x \mapsto \|x\|_B := \min\{\lambda \geq 0 : x \in \lambda B\}$

(2) Choose $\bar{x} \in \arg\max\{\|x\|_B : x \in P\}$.

It follows from Proposition 3.3.1 that step (1) can be accomplished in time $O(\beta^d)$. As the number of facets of B is in $O(\beta^d)$, it follows from Theorem 3.2.12 that the maximization of $\|\cdot\|_B$ over P can be done in time $O(\beta^d T_{LP}(d,n))$.
In order to show the performance ratio of the above algorithm, observe that Property (3.17) of B implies that $\frac{\beta-1}{\beta}\|x\|_p \leq \|x\|_B \leq \|x\|_p$ for all $x \in \mathbb{R}^d$. Hence, if $x^* \in \arg\max\{\|x\|_p^p : x \in P\}$, we get

$$\|\bar{x}\|_p^p \geq \|\bar{x}\|_B^p \geq \|x^*\|_B^p \geq \left(\frac{\beta-1}{\beta}\right)^p \|x^*\|_p^p = \left(\frac{\beta-1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}.$$

□

3.3.2 No FPT-approximation for Variable Accuracy

Finally, we will show that the straightforward approximation of the previous subsection is already best possible in the sense that there is no algorithm with polynomial dependence on the approximation quality and exponential dependence only on the dimension. Hence, combined with Lemma 3.3.2, Lemma 3.3.3 completes the proof of Theorem 3.1.3. In fact, the basis for this has already been established in Lemma 3.2.10 and we can give the result right away.

Lemma 3.3.3 (*No polynomial dependence on β*)
Let $f : \mathbb{N} \to \mathbb{R}$ be a computable function and $q : \mathbb{R}^3 \to \mathbb{R}$ a polynomial function. If W[1]≠FPT, there is no algorithm which for every $\mathcal{H}$-presented polytope $P \subseteq \mathbb{R}^d$ runs in time $O(f(d)q(\beta,d,n))$ and produces an $\bar{x} \in P$ such that

$$\|\bar{x}\|_p^p \geq \left(\frac{\beta-1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}.$$

Proof.
Let (n,k,E) be an instance of the W[1]-hard problem Clique and $\bar{P} \subseteq \mathbb{R}^{2k}$ the polytope constructed in Equation (3.13). By Lemma 3.2.10, it can be decided if $G = ([n], E)$ has a clique of size k by determining, whether

$$\begin{array}{rl} \textbf{either} & \max\{\|x\|_p^p : x \in \bar{P}\} \geq k(1-U)^p \\ \textbf{or} & \max\{\|x\|_p^p : x \in \bar{P}\} \leq (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p} \end{array} \tag{3.18}$$

Assume that an algorithm with the claimed properties exists and call it $\mathcal{A}$. One easily checks that there is a suitable constant $C > 0$ such that it suffices to choose $\beta \geq \frac{pn^p k}{C}$ in order to fulfill

$$\left(\frac{\beta}{\beta-1}\right)^p \left((k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}\right) < k(1-U)^p.$$

Hence, we can run the following algorithm $\mathcal{A}'$ in order to decide (3.18):

1) Choose $\beta := \left\lceil \frac{pn^p k}{C} \right\rceil$.

2) Run $\mathcal{A}$ on the polytope $\bar{P}$ and obtain an approximate normmaximal vertex $\bar{x} \in \bar{P}$.

3) If $\|\bar{x}\|_p^p > (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}$, decide $\max\{\|x\|_p^p : x \in \bar{P}\} \geq k(1-U)^p$.

   Else, decide $\max\{\|x\|_p^p : x \in P\} \leq (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}$.

By the properties of $\mathcal{A}$, the running time of the algorithm $\mathcal{A}'$ is $O(f(d)q(n^p k, d, n))$ and by Lemma 3.2.10 and the choice of β, $\mathcal{A}'$ decides (3.18) correctly. $\mathcal{A}'$ is thus an FPT algorithm for Clique. Unless FPT=W[1], this is a contradiction to the fact that Clique is W[1]-hard. □

3.4 Some Implications

As stated in the introduction of this chapter, norm maximization over polytopes plays a fundamental role in Computational Convexity. This section gives corollaries concerning the hardness of determining four important geometric functionals on polytopes.
If $P \subseteq \mathbb{R}^d$ is a polytope, we denote by $R(P, \mathbb{B}_p^d)$ ($r(P, \mathbb{B}_p^d)$, respectively) the circumradius (inradius) of P with respect to the p-norm, as introduced in Definitions 2.2.1 and 2.2.4. Further, as introduced in Defintion 2.8.1, we write $\underline{R}_1(P, \mathbb{B}_p^d)$ ($\overline{r}_1(P, \mathbb{B}_p^d)$) for half of the width (diameter) of P, i.e. half the radius of a smallest slab containing P (half the length of the longest line segment contained in P).
For $p \in \mathbb{N} \cup \{\infty\}$, we consider the following problems:

Problem 3.4.1 (Circumdadius$_p$-$\mathcal{H}$)

Input:	$d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter:	d
Question:	Is $R(P, \mathbb{B}_p^d)^p \geq \gamma$?

Problem 3.4.2 (Diameter$_p$-$\mathcal{H}$)

Input:	$d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter:	d
Question:	Is $\overline{r}_1(P, \mathbb{B}_p^d)^p \geq \gamma$?

It has been shown in [87] that Problems 3.4.1 and 3.4.2 are solvable in polynomial time if $p = \infty$ and, by using an identity for symmetric polytopes from [86], that both problems are $\mathbb{NP}$-hard when $p \in \mathbb{N}$. Using the same identity, we can establish (in-)tractability for both problems when parameterized by the dimension:

Corollary 3.4.3 (*Circumradius & Diameter*)
For $p = 1$, Problems 3.4.1 and 3.4.2 are in FPT. For $p \in \mathbb{N} \setminus \{1\}$, both problems are W[1]-hard.

Proof.
As shown in [86, (1.3)], for a 0-symmetric polytope $P \subseteq \mathbb{R}^d$, we have

$$R(P, \mathbb{B}_p^d)^p = \overline{r}_1(P, \mathbb{B}_p^d)^p = \max\{\|x\|_p^p : x \in P\}.$$

Thus tractability or hardness of Problems 3.4.1 and 3.4.2 follow from Theorem 3.1.2. □

Additionally, let $q \in [1, \infty]$ be such that $1/p + 1/q = 1$ (with $1/\infty = 0$).

Problem 3.4.4 (INRADIUS$_p$-$\mathcal{V}$)
Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{V}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $r(P, \mathbb{B}_q^d)^p \leq \gamma$?

Problem 3.4.5 (WIDTH$_p$-$\mathcal{V}$)
Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{V}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $\underline{R}_1(P, \mathbb{B}_q^d)^p \leq \gamma$?

As for the previous two problems, the question of $\mathbb{NP}$-hardness of INRADIUS$_p$-$\mathcal{V}$ and WIDTH$_p$-$\mathcal{V}$ has been studied in [87]. It is shown that Problems 3.4.4 and 3.4.5 are solvable in polynomial time if $p = 1$ and by using an identity for symmetric polytopes from [86] that both problems are $\mathbb{NP}$-hard when $p \in \mathbb{N}$. Here again, we can use the same identity to establish (in-)tractability for both problems when parameterized by the dimension:

Corollary 3.4.6 (*Inradius & Width*)
For $p = 1$, Problems 3.4.4 and 3.4.5 are in FPT. For $p \in \mathbb{N} \setminus \{1\}$, both problems are W[1]-hard.

Proof.
It is shown in [86] and follows from Theorem 2.8.5 of this thesis that if $P \subseteq \mathbb{R}^d$ is a 0-symmetric polytope and P° is its polar the identities

$$\underline{R}_j(P, \mathbb{B}_q^d)\overline{r}_j(P^\circ, \mathbb{B}_p^d) = 1$$

hold for all $j \in [d]$. As an $\mathcal{H}$-presentation of P° is readily translated into a $\mathcal{V}$-presentation of P, tractability or hardness of Problems 3.4.4 and 3.4.5 follow from Corollary 3.4.3. □

The reductions of Corollaries 3.4.3 and 3.4.6 also show that the algorithm in the proof of Lemma 3.3.2 can be used to compute the respective radii of a symmetric polytope $P \subseteq \mathbb{R}^d$ in the respective presentation. Lemma 3.3.3, in turn, shows that in these cases the given running time is also best possible.

Chapter 4

Core-Sets for Containment under Homothetics

This chapter deals with the containment problem under homothetics which has the minimal enclosing ball (MEB) problem as a prominent representative. We connect the problem to results in classic convex geometry and introduce a new series of radii, which we call core-radii. For the MEB problem, these radii have already been considered from a different point of view and sharp inequalities between them are known. In this chapter, sharp inequalities between core-radii for general containment under homothetics are obtained.
Moreover, the presented inequalities are used to derive sharp upper bounds on the size of core-sets for containment under homothetics. In the MEB case, this yields a tight (dimension independent) bound for the size of such core-sets. In the general case, we show that there are core-sets of size linear in the dimension and that this bound stays sharp even if the container is required to be symmetric.

This chapter is joint work with René Brandenberg. Preliminary results already appear in [125]. Its main results have been published in [40] at the *27th Annual Symposium on Computational Geometry* in June 2011 and in [38] in a special issue of *Discrete & Computational Geometry* on the occasion of this symposium.

4.1 Introduction

Many well-known problems in computational geometry can be classified as some type of optimal containment problem, where the objective is to find an extremal representative C^* of a given class of convex bodies, such that C^* contains a given point set P (or vice versa). These problems arise in many different applications, e.g. facility location, shape fitting and packing problems, clustering, pattern recognition or statistical data reduction. Typical representatives are the minimal enclosing ball (MEB) problem, smallest enclosing cylinders, slabs, boxes, or ellipsoids; see [86] for a survey. Also the

well-known k-center problem (cf. Page 14 in Section 1.2.2), where P is to be covered by k homothetic copies of a given container C, has to be mentioned in this context.
Because of its simple description and the multitude of both theoretical and practical applications there is vast literature concerning the MEB problem. In recent years, a main focus has been on so called core-sets, i.e. small subsets S of P requiring balls of (almost) the same radius to be enclosed as P itself. For the Euclidean MEB problem algorithms constructing core-sets of sizes only depending on the approximation quality but neither on the number of points to be enclosed nor the dimension have been developed in [17, 19, 55, 161]. This yields not only another fully polynomial time approximation scheme (FPTAS) for MEB, but also a polynomial time approximation scheme (PTAS) for the harder Euclidean k-center problem which also works very well in practice [41].
However, all variants of core-set algorithms for MEB are based on the so called half-space lemma [17, 82] or equivalent optimality conditions, a property characterizing the Euclidean ball [86], thus not allowing immediate generalization to the superordinate *Containment under Homothetics* that we consider here:

Problem 4.1.1
For $P \subseteq \mathbb{R}^d$ non-empty and compact and $C \subseteq \mathbb{R}^d$ a full-dimensional compact convex set (called container) the *minimal containment problem under homothetics* is to find the least dilatation factor $\rho \geq 0$, such that a translate of ρC contains P. In other words, we are looking for a solution to the following optimization problem (cf. also Figure 4.1):

$$\begin{array}{rl} \min & \rho \\ s.t. & P \subseteq c + \rho C \\ & c \in \mathbb{R}^d \\ & \rho \geq 0. \end{array} \tag{4.1}$$

The assumption that C be full-dimensional ensures that Problem 4.1.1 has a feasible solution for every P; cf. also Lemma 2.2.2, which shows that in this case the minimum in (4.1) is attained for every P and C. As in Chapter 2, we write $R(P, C)$ for the optimal value of (4.1) and call it the *C-radius* of P. Hence, if C is a Euclidean ball and P is finite this specializes to the MEB problem. If C is 0-symmetric this is the problem of computing the outer radius of P with respect to the norm $\|\cdot\|_C$ induced by the gauge body C as already considered e.g. in [30].

Besides direct applications Problem 4.1.1 is often the basis for solving much harder containment problems (e.g. containment under similarities), which already gives a reason for an intensive search for good (approximation) algorithms. Compared to the approach in [149], approximation via core-sets has the additional advantage that it may be turned into a PTAS for the k-center problem as demonstrated in [19]. Whereas there is a rich literature on the Euclidean MEB problem (and its core-sets) that exhibit many nice properties, only little is known about the general case and how much of the Euclidean properties carry over to Problem 4.1.1. (For an overview of possible solution strategies depending on given container classes, we refer to [42].)

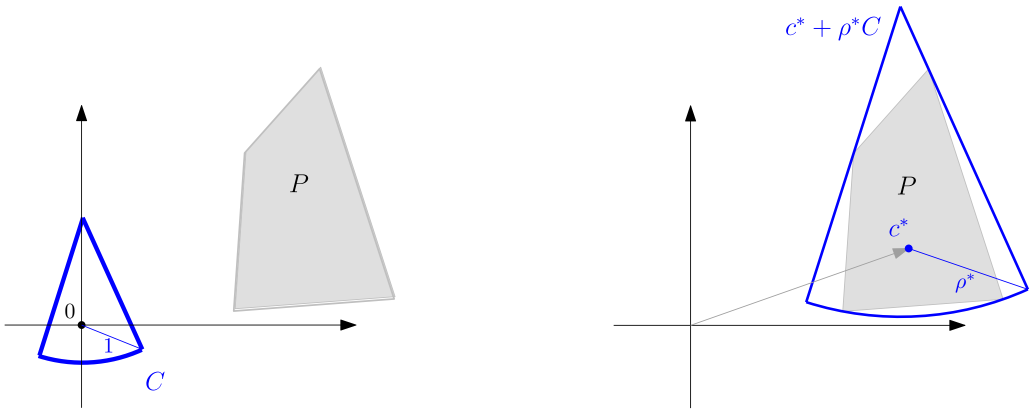

Figure 4.1: Left: Possible input for Problem 4.1.1. Right: An optimal solution for the input on the left.

For P and C as in Problem 4.1.1, we call a subset $S \subseteq P$ an ε-core-set for some $\varepsilon \geq 0$ if

$$R(P, C) \leq (1 + \varepsilon) R(S, C).$$

Answering the questions for the size of core-sets for Problem 4.1.1, we prove the following result:

Theorem 4.1.2 (*No sublinear core-sets for containment under homothetics*)
For every non-empty, compact set $P \subseteq \mathbb{R}^d$, every container $C \subseteq \mathbb{R}^d$, and $\varepsilon \geq 0$ there exists an ε-core-set of P of size at most $\left\lceil \frac{d}{1+\varepsilon} \right\rceil + 1$. Moreover, for any $\varepsilon < 1$ there exists a body $P \subseteq \mathbb{R}^d$ and a 0-symmetric container C such that no smaller subset of P suffices.

In order to prove the positive part of Theorem 4.1.2, we will state several new geometric identities and inequalities between radii of convex sets, which connect Problem 4.1.1 to results in classic convex geometry. The negative part of the theorem (i.e. that the bound cannot be improved even for 0-symmetric containers) then follows by proving that these inequalities (and so the resulting bounds on core-set sizes) are best possible.
Moreover, the connection between core-sets and a series of radii from convex geometry will enable us to give a sharp upper bound for the size of core-sets for the MEB problem:

Theorem 4.1.3 (*Size of ε-core-sets for MEB*)
Let $P \subseteq \mathbb{R}^d$ be compact and $\varepsilon > 0$. If $C = \mathbb{B}_2^d$, then there exists an ε-core-set of P of size at most

$$\left\lceil \frac{1}{2\varepsilon + \varepsilon^2} \right\rceil + 1,$$

and this is the best possible d-independent bound.

In the following section, we will shortly explain notational particularities of this chapter, state the basic definitions and collect the tools that we need in order to prove Theorems 4.1.2 and 4.1.3. Section 4.3 then proves the mentioned radius identities that will lead to Theorem 4.1.3. Finally, Section 4.4 is dedicated to the derivation of Theorem 4.1.2.

4.2 Geometric Foundations

Recall that $\mathcal{C}^d$ denotes the set of convex bodies in $\mathbb{R}^d$, i.e. non-empty, compact and convex sets in $\mathbb{R}^d$. In this chapter we will call a convex body $C \in \mathcal{C}^d$ with the property that $0 \in \operatorname{int}(C)$ a *container*. The set of all containers in $\mathbb{R}^d$ will therefore be $\mathcal{C}_0^d$.[1]
For a fixed container $C \in \mathcal{C}_0^d$, we denote by c_P a possible center for P, i.e. a point such that $P \subseteq c_P + R(P,C)C$. Notice, that for general C, the center c_P might not be unique. Additionally, we denote by $T^d \in \mathcal{C}^d$ some regular d-simplex. This simplex will play an important role in many of the inequalities that we derive. We do not specify its orientation or edge length since the obtained results hold for all possible choices of these values.

4.2.1 Core-Sets and Core-Radii

As already pointed out in the introduction the concept of ε-core-sets has proved very useful for the special case of the Euclidean MEB problem. Here, we introduce two slightly different definitions for the more general Problem 4.1.1: core-sets and center-conform core-sets together with a series of radii closely connected to them. The explicit distinction between the two types of core-sets is intended to help to overcome possible confusion founded in the use of the term core-set for both variants in earlier publications.

Definition 4.2.1 (*Core-radii and ε-core-sets*)
For $P \subseteq \mathbb{R}^d$, $C \in \mathcal{C}_0^d$, and $k \in [d]$, we call

$$R_k(P,C) := \max\{R(S,C) : S \subseteq P,\ |S| \le k+1\}$$

the *k-th core-radius* of P.
Let $\varepsilon \ge 0$. A subset $S \subseteq P$ such that

$$R(S,C) \le R(P,C) \le (1+\varepsilon)R(S,C) \tag{4.2}$$

will be called an *ε-core-set* of P (with respect to C).
An ε-core-set $S \subseteq P$ which has the additional property, that there exists a center c_S of S, such that

$$P \subseteq c_S + (1+\varepsilon)R(S,C)C, \tag{4.3}$$

[1] Usually, we consider Problem 4.1.1 as being parametrized by the container and having varying sets $P \in \mathcal{C}^d$ as input. By Lemma 2.2.2, it would suffice for the feasibility of Problem 4.1.1 to impose the condition that P be contained in some affine subspace parallel to $\operatorname{aff}(C)$. As this condition is rather technical and yields no further insight, we restrict to full-dimensional containers; and, as the problem is invariant under translation of the container, we simply assume that $0 \in \operatorname{int}(C)$ for convenience.

will be called a *center-conform ε-core-set* of P (with respect to C).
All three notions are illustrated in Figure 4.2.

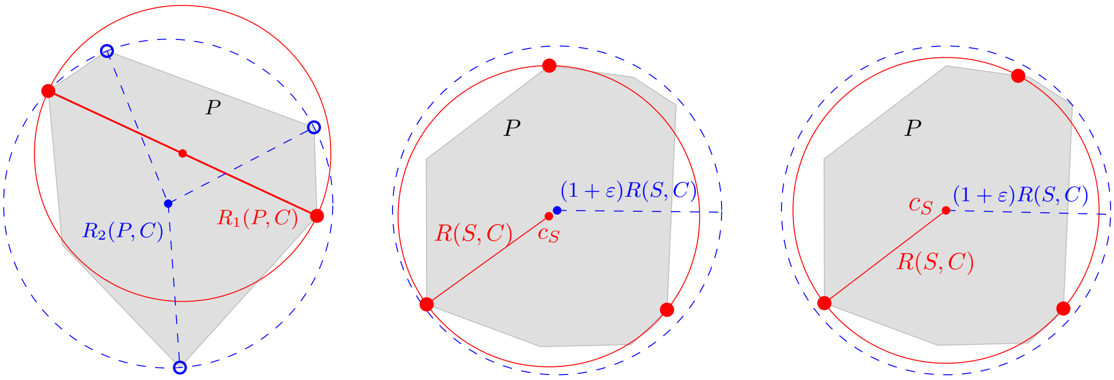

Figure 4.2: Illustration of Core-Radii and Core-Sets. Left: The red filled and the blue empty points define $R_1(P,C)$ and $R_2(P,C)$, respectively. Middle (Right): For some $\varepsilon > 0$, the red filled point set S forms a (center-conform) ε-core-set of P of size 3. (In all three cases $C = \mathbb{B}_2^2$.)

By definition, every center-conform ε-core-set is also an ε-core-set. It will be shown in Lemma 4.2.6 that an ε-core-set is a center-conform ε'-core-set for an ε' slightly greater than ε, if $C = \mathbb{B}_2^d$.
Surely, if one is only interested in an approximation of $R(P,C)$ the knowledge of a good core-set suffices. A center-conform core-set S carries the additional information of a center c_S of S that can be used to cover P. However, if the center of S is not unique, it may not be possible to actually determine which of the centers of S is suitable, when S is the only information about P to be considered.
We present lower bounds on the sizes of core-sets (these are also lower bounds on the size of center-conform core-sets), and we note that most existing positive results (via construction algorithms) already hold for center-conform core-sets. When searching for lower bounds, we use the fact that there exist ε-core-sets of size at most $k+1$ if and only if the ratio $R(P,C)/R_k(P,C)$ is less than or equal to $1+\varepsilon$. This allows us to transfer the size-of-core-sets problem to bounding the ratio between the core-radii of P.

As already observed e.g. in [67] and [86], the reason for restricting the core-radii to $k \leq d$ follows directly from Helly's Theorem (see e.g. [59]). We need a slightly more general statement here, which we prove in the following lemma for completeness. However, the main part of the proof is parallel to the ones in [67] and [86] for balls. Figure 4.3 also illustrates the situation.

Lemma 4.2.2 (*0-core-sets*)
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $\dim(P) \leq k \leq d$. Then $R_k(P,C) = R(P,C)$, i.e. there exist

(center-conform) 0-core-sets of size at most $\dim(P)+1$ for all P and C.
Furthermore, for $k \leq \dim(P)$, there always exists a simplex $S \subseteq P$ such that $\dim(S) = k$ and $R(S,C) = R_k(P,C)$.

Proof.
Clearly, $R_k(P,C) \leq R(P,C)$. To show $R_k(P,C) \geq R(P,C)$ for $k \geq \dim(P)$, observe that by definition of $R_k(P,C)$, every $S \subseteq P$ with $|S| \leq k+1$ can be covered by a copy of $R_k(P,C)C$. This means $\bigcap_{p\in S}(p - R_k(P,C)C) \neq \emptyset$ for all such S. Now, as the sets $p - R_k(P,C)C$ are compact, Helly's Theorem applied within $\operatorname{aff}(P)$ yields $\bigcap_{p\in P}(p - R_k(P,C)C) \neq \emptyset$. Thus the whole set P can be covered by a single copy of $R_k(P,C)C$. Moreover, by applying Helly's Theorem within $\operatorname{aff}(S)$ one may always assume that the finite set S with $R(S,C) = R_k(P,C)$ is affinely independent. Hence, if $|S| \leq k \leq \dim(P)$ one may complete S to the vertex set of a k-dimensional simplex within P. □

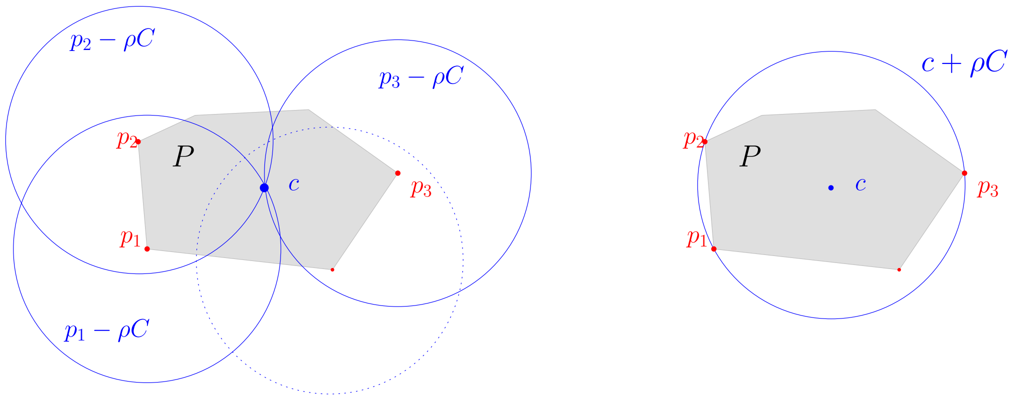

Figure 4.3: The duality argument that makes Helly's Theorem applicable for Containment under Homothetics: $\bigcap_{p\in S}(p - \rho C) \neq \emptyset \Leftrightarrow R(S,C) \leq \rho$.

4.2.2 Optimality Conditions

A characterization of optimal solutions for the MEB case of Problem 4.1.1 can already be found in [30]. A corollary, known as "half-space lemma", proved very useful in the construction of fast algorithms for MEB (see, e.g. [17, 19, 82]). However, to our knowledge, the literature does not contain any explicit optimality conditions for Problem 4.1.1 in its general form.
For brevity, P is said to be *optimally contained* in C, if $P \subseteq C$ but there is no $c \in \mathbb{R}^d$ and $\rho < 1$ such that $P \subseteq c + \rho C$.

Theorem 4.2.3 (*Optimality condition for Problem 4.1.1*)
Let $P \in \mathcal{C}^d$ and $C \in \mathcal{C}^d_0$. Then P is optimally contained in C if and only if

(i) $P \subseteq C$ and

(ii) for some $2 \leq k \leq d+1$, there exist $p_1, \dots, p_k \in P$ and hyperplanes $H_=(a_i, 1)$ supporting P and C in p_i, $i = 1, \dots, k$ such that $0 \in \operatorname{conv}\{a_1, \dots, a_k\}$.

The theorem stays valid even if one allows C to be unbounded.

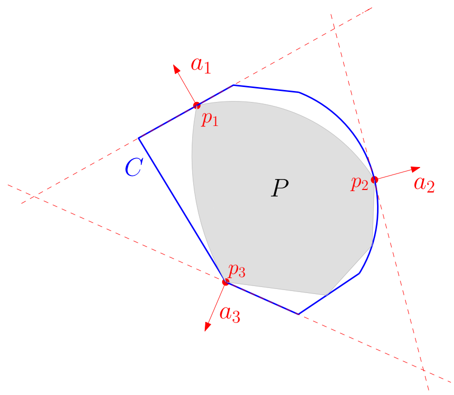

Figure 4.4: The necessary and sufficient conditions from Theorem 4.2.3: Condition (ii) is fulfilled by the three points p_1, p_2, p_3 and the three hyperplanes with outer normals a_1, a_2, a_3, all highlighted in dashed red. Note that (in general) $\operatorname{conv}\{p_1, \dots, p_k\}$ is optimally contained in $\bigcap_{i=1}^{k} H_{\leq}(a_i, 1)$.

Proof.
Let $C \in \mathcal{C}_0^d$ be given as $C = \bigcap_{a \in N} H_{\leq}(a, 1)$ where $N = \operatorname{bd}(C^{\circ})$ is the set of outer normals of C.
First, assume (i) and (ii) hold. By (i), $R(P, C) \leq 1$. Now suppose $R(P, C) < 1$. Then there exists $c \in \mathbb{R}^d$ and $0 < \rho < 1$ such that $c + P \subseteq \rho C$. From (ii) follows $P \cap \operatorname{bd}(C) \neq \emptyset$ and therefore $c \neq 0$. Moreover, as $c + P \subseteq \rho C$, it follows $\sup_{a \in N} a^T(c + p_i) \leq \rho$ and in particular, $a_i^T(c + p_i) \leq \rho < 1$ for all i. Now, as $0 \in \operatorname{conv}\{a_1, \dots, a_k\}$, there exist $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ such that $\sum_i \lambda_i a_i = 0$ and $\sum_i \lambda_i a_i^T(c + p_i) < 1$. Using $a_i^T p_i = 1$ one obtains $\sum_i \lambda_i a_i^T c < 0$, an obvious contradiction. Thus, conditions (i) and (ii) imply optimality.
Now, let P be optimally contained in C. The following part of the proof is illustrated in Figure 4.5. As C is compact, we can apply Lemma 4.2.2 which yields $k \leq d+1$ points $p_i \in P \cap \operatorname{bd}(C)$ for $i = 1, \dots, k$ such that

$$R(\operatorname{conv}\{p_1, \dots, p_k\}, C) = 1. \tag{4.4}$$

Let $A = \{a \in N : \exists i \in [k] \text{ } s.t. \text{ } a^T p_i = 1\}$. Since $P \subseteq C$, for $a \in A$, we have that $a^T p \leq 1$ for all $p \in P$, and $a^T p_i = 1$ for at least one i by definition of A. We will show that $0 \in \operatorname{conv}(A)$. The statement that there exists a set of at most $d+1$ outer normals with 0 in their convex hull then follows from Carathéodory's Theorem (see [59]). Assume, for a contradiction, that $0 \notin \operatorname{conv}(A)$. Then 0 can be strictly separated from $\operatorname{conv}(A)$, i.e. there exists $y \in \mathbb{R}^d$ with $a^T y \geq 1$ for all $a \in A$. Now, for $A' = \{a \in N : a^T y \leq 0\}$ there exists $\varepsilon > 0$ such that $(A' + \varepsilon \mathbb{B}_2^d) \cap A = \emptyset$, i.e. $a^T p_i < 1 - \varepsilon$ for all $a \in A'$ and therefore

$$a^T (p_i - \frac{\varepsilon}{\|a\|\|y\|} y) = a^T p_i + \varepsilon \frac{-a^T y}{\|a\|\|y\|} < 1.$$

Moreover, if $a \in N \setminus A'$ then

$$a^T (p_i - \frac{\varepsilon}{\|a\|\|y\|} y) = a^T p_i - \varepsilon \frac{a^T y}{\|a\|\|y\|} < 1.$$

As $0 \in \mathrm{int}(C)$, we know that $N \subseteq C^\circ$ is bounded and therefore there exists $\alpha > 0$ such that $\|a\| \leq \alpha$ for all $a \in N$. Thus, altogether, $p_i - \frac{\varepsilon}{\alpha\|y\|} y \in \mathrm{int}(C)$ for all i, which contradicts (4.4). Finally, observe that the last statement about a possibly unbounded

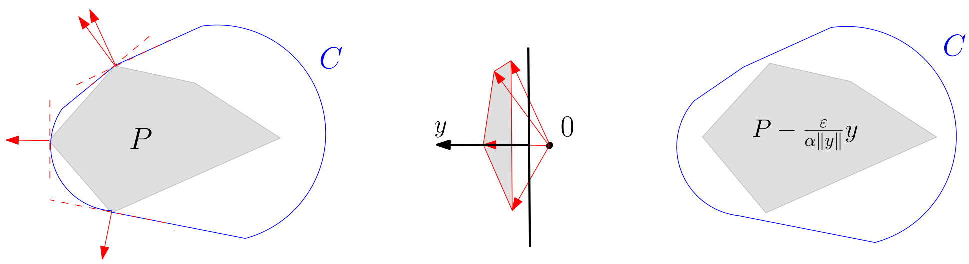

Figure 4.5: The idea of the proof of Theorem 4.2.3: If the outer normals in the points where P touches C do not contain the origin in their convex hull, the separation theorem yield a direction $y \in \mathbb{R}^d \setminus \{0\}$ such that $P - \lambda y \subseteq \mathrm{int}(C)$ for a sufficiently small $\lambda \geq 0$.

C can be obtained from the one for bounded containers by considering a new container $C' = C \cap C''$ where $C'' \in \mathcal{C}_0^d$ such that $P \subseteq C''$ and $P \cap \mathrm{bd}(C'') = \emptyset$. □

Remark. Besides the direct geometric proof of Theorem 4.2.3 as stated above, it is also possible to derive the result from the Karush-Kuhn-Tucker conditions (see e.g. [147, Corollary 28.3.1]) in convex optimization in conjunction with Lemma 4.2.2; see also [125, p. 19].
As we assume that C has non-empty interior, "P optimally contained in C" implicitly implies $|P| > 1$. So, in case $P = \{p\}$, Theorem 4.2.3 is not applicable and we note for completeness, that in this case, P is optimally contained in $p + 0 \cdot C$.

Corollary 4.2.4
Let $P \in \mathcal{C}^d$ and C a polytope in $\mathbb{R}^d$. If $P \subseteq C$ and P touches every facet of C, then P is optimally contained in C.

Proof.
If C is a polytope with facets $F_i = C \cap H_=(a_i, 1)$, $i = 1, \dots, m$, it is well known [30] that, with the choice $\lambda_i = \mathrm{vol}_{d-1}(F_i)$, one has $\sum_{i=1}^m \lambda_i a_i = 0$. □

Corollary 4.2.5 (*Optimality condition for the MEB problem / Half-space lemma*)
Let $P \in \mathcal{C}^d$. If $P \subseteq \mathbb{B}_2^d$, then the following are equivalent:

(i) $R(P, \mathbb{B}_2^d) = 1$.

(ii) For some $k \leq d+1$, there exist $p_1, \dots, p_k \in P \cap \mathrm{bd}(\mathbb{B}_2^d)$ such that $0 \in \mathrm{conv}\{p_1, \dots, p_k\}$.

(iii) 0 can not be strictly separated from $P \cap \mathrm{bd}(\mathbb{B}_2^d)$.

(iv) $P \cap \mathrm{bd}(\mathbb{B}_2^d) \cap H \neq \emptyset$ for every half-space H containing the origin in its boundary. (*Half-space lemma*)

4.2.3 Side Notes

Lemma 4.2.6 (*Center-conformity for MEB*)
If $P \in \mathcal{C}^d$, $\varepsilon > 0$ and $S \subseteq P$ is an ε-core-set of P with respect to $\mathbb{B}_2^d$, then S is also a center-conform $(\varepsilon + \sqrt{2\varepsilon + \varepsilon^2})$-core-set of P.

Proof.
Let $p \in P$ such that $\max_{x \in P} \|c_S - x\|_2 = \|c_S - p\|_2$. Further let H be a hyperplane perpendicular to $\mathrm{aff}\{c_S, c_P\}$ passing through c_S. Denote by H^- the halfspace which is bounded by H and does not contain c_P. Then by Corollary 4.2.5, there is a point $q \in S \cap H^-$ at distance $R(S, \mathbb{B}_2^d)$ of c_S. Hence

$$\|c_P - c_S\|_2^2 \leq \|c_P - q\|_2^2 - \|q - c_S\|_2^2 \leq R(P, \mathbb{B}_2^d)^2 - R(S, \mathbb{B}_2^d)^2 \leq (2\varepsilon + \varepsilon^2) R(S, \mathbb{B}_2^d)^2$$

and

$$\|c_S - p\| \leq \|c_S - c_P\| + \|c_p - p\| \leq \sqrt{2\varepsilon + \varepsilon^2} R(S, \mathbb{B}_2^d) + R(P, \mathbb{B}_2^d) = (1 + \varepsilon + \sqrt{2\varepsilon + \varepsilon^2}) R(S, \mathbb{B}_2^d).$$

□

We recall that choosing $P = -C$ yields the symmetry measure $s(C)$ from Definition 2.3.1 on which we built our results in Chapter 2. As observed in Chapter 2, the Minkowski asymmetry $s(C)$

$$s(C) = R(-C, C) \tag{4.5}$$

can also be expressed as a special case of containment under homothetics.

As a further corollary of Theorem 4.2.3, we now present a very transparent proof (due to [43]) of Proposition 2.3.4, which states the well-known fact that the Minkowski asymmetry of a body C is bounded from above by $\dim(C)$. We will make use of the sharpness condition of Proposition 2.3.4/Corollary 4.2.7 to show the sharpness of the inequality in Theorem 4.4.1.

Corollary 4.2.7 (*Maximal asymmetry*)
For every $C \in \mathcal{C}^d$, the inequalities $1 \leq s(C) \leq \dim(C)$ hold, with equality, if C is 0-symmetric in the first and if C is a d-simplex in the latter case.

Proof.
Clearly, the Minkowski asymmetry is bounded from below by 1 and $s(C) = 1$ if $C = -C$. For the upper bound we suppose (without loss of generality) that C is full-dimensional. Then Lemma 4.2.2 yields a d-simplex $S \subseteq C$ such that $s(C) = R(-C, C) = R(-S, C) \leq R(-S, S) = s(S)$. Thus, it suffices to show $s(S) = \dim(S)$ for every simplex S. Suppose $S = \mathrm{conv}\{x_1, \ldots, x_{d+1}\} \subseteq \mathbb{R}^d$ is a d-simplex, without loss of generality such that $\sum_{i=1}^{d+1} x_i = 0$. For all $i \in [d+1]$, the center of the facet $F_j = d \cdot \mathrm{conv}\{x_i, i \neq j\}$ of dS is $c_j = \sum_{i \neq j} x_i = -x_j$. Hence $-S \subseteq dS$ and $-S$ touches every facet of dS, showing the optimality of the containment by Corollary 4.2.4. □

Remark. In [95] also the "only if" direction for the sharpness of the bounds in Corollary 4.2.7 is shown.

Finally, note that Lemma 4.2.2 can also be seen as a result bounding the combinatorial dimension of Problem 4.1.1 interpreted as a Generalized Linear Program (GLP). As it is not our main focus here, we simply mention the connection and refer to [131, 152] for a treatise on GLPs and to [13] for their relation to Helly-type theorems.

4.3 Radii Identities and Small Core-Sets

4.3.1 Identities between Different Radii

In this section, we show the identity of the core-radii from Definition 4.2.1 to two series of intersection- and cylinder/projection-radii in convex geometry, similar to the ones defined in [100] and to the more often considered ones in [86] and [145]. This identity will help us to use a set of known geometric inequalities on these radii to obtain bounds on core-set sizes.

Definition 4.3.1 (*Intersection- and cylinder-radii*)
For $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $k \in [d]$, let

$$R_k^\sigma(P, C) := \max\{R(P \cap E, C) : E \in \mathcal{A}_k^d\} \tag{4.6}$$

and

$$R_k^\pi(P,C) := \max\{R(P,C+F) : F \in \mathcal{L}_{d-k}^d\} \tag{4.7}$$

It follows from Blaschke's Selection Theorem [150], that the maxima in (4.6) and (4.7) exist.

Remark 4.3.2 (*Cylinder-radii in Euclidean spaces*)
When dealing with the Euclidean unit ball $\mathbb{B}_2^d$, the observation that, for $F \in \mathcal{L}_{d-k}^d$, $R(P,\mathbb{B}_2^d+F) = R(P|F^\perp,\mathbb{B}_2^d)$ shows that the cylinder-radii can be interpreted as projection-radii, i.e.

$$R_k^\pi(P,\mathbb{B}_2^d) = \max\{R(P|E,\mathbb{B}_2^d) : E \in \mathcal{L}_k^d\}\ .$$

The following theorem states the identity of these series of radii. To the best of our knowledge, even the equality between the intersection- and projection-radii in the Euclidean case has not been shown before.

Theorem 4.3.3 (*Identity of intersection-, cylinder- and core-radii*)
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $k \in [d]$. Then,

$$R_k(P,C) = R_k^\sigma(P,C) = R_k^\pi(P,C).$$

Proof.
We show $R_k(P,C) \le R_k^\sigma(P,C) \le R_k^\pi(P,C) \le R_k(P,C)$.
First, $R_k(P,C) \le R_k^\sigma(P,C)$: By definition of the core-radii, there exists $S \subseteq P$ with $|S| = k+1$ and $R(S,C) = R_k(P,C)$. Since $\dim(\mathrm{aff}(S)) \le k$, one obtains

$$R_k(P,C) = R(S,C) \le R(P \cap \mathrm{aff}(S),C) \le R_k^\sigma(P,C).$$

Now, $R_k^\sigma(P,C) \le R_k^\pi(P,C)$: Let $E \in \mathcal{L}_k^d$ such that $R_k^\sigma(P,C) = R(P\cap E,C)$. As $\dim(P\cap E) \le k$, Lemma 4.2.2 and Theorem 4.2.3 show that, for $m \le k+1$, there are points $p_1,\dots,p_m \in P\cap E$ and hyperplanes $H_=(a_1,1),\dots,H_=(a_m,1)$ such that $H_=(a_i,1)$ supports C in p_i and $0 \in \mathrm{conv}\{a_1,\dots,a_m\}$. As $0 \in \mathrm{conv}\{a_1,\dots,a_m\}$, we get that $\dim\{a_1,\dots,a_m\}^\perp \ge d-k$ and we may choose $F \in \mathcal{L}_{d-k}^k$ such that $F \subseteq \{a_1,\dots,a_m\}^\perp$. Again by Theorem 4.2.3, if follows that

$$R_k^\sigma(P,C) = R(P\cap E,C) = R(P\cap E,C+F) \le R(P,C+F) \le R_k^\pi(P,C).$$

Finally, $R_k^\pi(P,C) \le R_k(P,C)$: Let $F \in \mathcal{L}_{d-k}^d$ such that $R_k^\pi(P,C) = R(P,C+F)$ and suppose without loss of generality that P is optimally contained in $C+F$ (i.e. the optimal radius and center are $\rho^* = 1$ and $c^* = 0$, respectively). Then it follows from the statement for unbounded containers in Theorem 4.2.3 that there exist $m \le d+1$ points $p_1,\dots,p_m \in P$ and hyperplanes $H_=(a_i,1)$, $i=1,\dots,m$ such that $H_=(a_i,1)$ supports $C+F$ in p_i and $0 \in \mathrm{conv}\{a_1,\dots,a_m\}$. Since every direction in F is an unbounded direction in $C+F$, one obtains $a_i \in F^\perp$ for all $i \in [m]$. Now, by Carathéodory's

Theorem, there exists a subset $I \subseteq [m]$ with $|I| \le \dim(F^\perp) + 1 = k + 1$ such that $0 \in \mathrm{conv}\{a_i : i \in I\}$. Applying again Theorem 4.2.3,

$$R_k^\pi(P, C) = R(P, C{+}F) = R(\mathrm{conv}\{p_i : i \in I\}, C{+}F) \le R(\mathrm{conv}\{p_i : i \in I\}, C) \le R_k(P, C).$$

□

4.3.2 Dimension Independence for Special Container Classes

The most evident (non-trivial) example for a restricted class of containers allowing small core-sets may be parallelotopes. E.g. in [29, Chapter 25], the following proposition is shown:

Proposition 4.3.4 (*Core-radii for parallelotopes*)
The identity

$$R_1(P, C) = R(P, C)$$

holds true for all $P \in \mathcal{C}^d$ if and only if $C \in \mathcal{C}_0^d$ is a parallelotope.

In terms of core-sets, this means that there is a 0-core-set of size two for all $P \in \mathcal{C}^d$, if C is a parallelotope and that these are the only containers with this property.

Surely, a more important restricted class of containers is the class of ellipsoids. In [100], geometric inequalities are derived which relate the radii of Definition 4.3.1 within each series. Using Theorem 4.3.3, these inequalities can be presented in a unified way in terms of core-radii:

Proposition 4.3.5 (*Henk's Inequality*)
Let $P \in \mathcal{C}^d$ and $k, l \in \mathbb{N}$ where $l \le k \le d$. Then

$$\frac{R_k(P, \mathbb{B}_2^d)}{R_l(P, \mathbb{B}_2^d)} \le \sqrt{\frac{k(l+1)}{l(k+1)}} \tag{4.8}$$

with equality if $P = T^d$.

Remark. Because of the affine invariance of (4.8) one may replace $\mathbb{B}_2^d$ by any d-dimensional ellipsoid.

This inequality can now directly be turned into a sharp bound on the size of ε-core-sets for the MEB problem and Theorem 4.1.3 follows:

Proof of Theorem 4.1.3
Let $\varepsilon > 0, k = \left\lceil \frac{1}{2\varepsilon+\varepsilon^2} \right\rceil$, and $S \subseteq P$ such that $R(S, \mathbb{B}_2^d) = R_k(P, \mathbb{B}_2^d)$. Then $|S| \leq k+1$ and by Proposition 4.3.5 and Lemma 4.2.2:

$$R(P, \mathbb{B}_2^d) \leq \sqrt{\frac{d(k+1)}{k(d+1)}} \cdot R(S, \mathbb{B}_2^d)$$

where k is chosen such that $\sqrt{\frac{d(k+1)}{k(d+1)}} \leq 1+\varepsilon$ independently of $d \in \mathbb{N}$.
Now, we show the sharpness of the bound: Let $d \in \mathbb{N}$ such that $\frac{d}{d+1} > (1+\varepsilon)^2 \frac{k}{k+1}$ and choose $P = T^d$. Now, for $k < \frac{1}{2\varepsilon+\varepsilon^2}$ if $S' \subseteq P$ consists of no more than $k+1$ points then

$$R(P, \mathbb{B}_2^d) = \sqrt{\frac{d(k+1)}{k(d+1)}} R_k(T^d, \mathbb{B}_2^d) > (1+\varepsilon) R_k(T^d, \mathbb{B}_2^d) \geq (1+\varepsilon) R(S', \mathbb{B}_2^d).$$

Hence S' is not an ε-core-set of P. □

Remark. Jung's well-known inequality (see [116] and the sharpened version in Theorem 2.5.1), relating the diameter and the outer radius of P, can be obtained from Proposition 4.3.5 just by choosing $k = d$ and $l = 1$. As Proposition 4.3.5, it can be turned into a core-set result saying that, for the Euclidean ball in every dimension, a diametral pair of points in P is already a $(\sqrt{2}-1)$-core-set.

A very easy and intuitive algorithm to actually find ε-core-sets of a finite set P was first introduced in [19]. Roughly speaking, it starts with a subset $S \subseteq P$ of two (good) points and computes (or approximates) the minimum enclosing ball B_S for S. Whenever a dilatation by $(1+\varepsilon)$ of B_S centered at c_S does not cover the whole set P, an uncovered point is added to S and the process is iterated. The analysis in [19] shows that this algorithm produces ε-core-sets of size $O(1/\varepsilon^2)$, and, by construction, these are even center-conform.
In [18], the existence of center-conform ε-core-sets of size $1/\varepsilon$ and the sharpness of this bound are shown. Theorem 4.1.3 now complements this result and gives a tight upper bound on the size of (general) core-sets, which is roughly half the center-conform bound.

4.4 No Sublinear ε-Core-Sets

In this section, several geometric inequalities between core-radii are collected and then used to derive positive and negative results on possible ε-core-set sizes. We recall that, because of Lemma 4.2.2, we already know the existence of 0-core-sets of size $d+1$, i.e. not depending on the size of P (nor C) and only linearly depending on d.

4.4.1 General (non-symmetric) Containers

Theorem 4.4.1 (*Inequality relating core-radii*)
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $k, l \in \mathbb{N}$ such that $l \le k \le d$. Then

$$\frac{R_k(P,C)}{R_l(P,C)} \le \frac{k}{l}$$

with equality if $P = -C = T^d$.

Proof.
It suffices to show

$$\frac{R_k(P,C)}{R_{k-1}(P,C)} \le \frac{k}{k-1}. \tag{4.9}$$

as for $l < k-1$ the claim follows by repeatedly applying (4.9). Without loss of generality one may assume the existence of a k-simplex $S = \operatorname{conv}\{x_1, \ldots, x_{k+1}\} \subseteq P$ satisfying $R(S,C) = R_k(P,C)$, as (4.9) is certainly fulfilled if $R_k(P,C) = R_{k-1}(P,C)$. Moreover, it can also be supposed that $\sum_{i=1}^{k+1} x_i = 0$ and $R_{k-1}(S,C) = 1$. Now, let $S_j = \operatorname{conv}\{x_i : i \neq j\}$, $j = 1, \ldots, k+1$ denote the facets of S.
Since $\sum_{i=1}^{k+1} x_i = 0$, it follows $-1/k \cdot x_j = 1/k \sum_{i \neq j} x_i \in \operatorname{conv}\{x_i : i \neq j\} = S_j$ for all j and surely $x_j \in S_i$ for all i, j, $i \neq j$.
Since $R_{k-1}(S,C) = 1$, there exist translation vectors $c_j \in \mathbb{R}^d$ such that $S_j \subseteq c_j + C$ for all $j \in [k+1]$ which implies

$$\left(k - \frac{1}{k}\right) x_j \in \sum_{i=1}^{k+1} S_i \subseteq \sum_{i=1}^{k+1} c_i + (k+1)C$$

for all j and thus $R(S,C) \le (k+1)/(k - \frac{1}{k})$. However, since $R_{k-1}(S,C) = 1$ we obtain

$$R_k(P,C) = R(S,C) \le \frac{k}{k-1} R_{k-1}(S,C) \le \frac{k}{k-1} R_{k-1}(P,C)$$

proving (4.9).
The sharpness of the inequality for $-P = C = T^d$ follows directly from showing $R_k(T^d, -T^d) = k$ for $k \in [d]$:
Since every k-face F of T^d can be covered by the k-face of $-T^d$ parallel to F and since these k-faces are regular k-simplices, it follows from Corollary 4.2.7 that $R(F, -F) = k$ and, thus, $R_k(T^d, -T^d) \le k$ for all $k \in [d]$.
Finally, for every face F of $-T^d$, it is true that $-T^d | \operatorname{aff}(F) = F$. Thus, if $S_k \subseteq T^d$ is a k-face of T^d and $S_k \subseteq c + \rho(-T^d)$ for some $c \in \mathbb{R}^d$ and $\rho \ge 0$, then $S_k | \operatorname{aff}(c + \rho(-S_k)) \subseteq (c + \rho(-S_k))$. However, $\operatorname{aff}(c + \rho(-S_k))$ is parallel to S_k, and therefore the above projection is just a translation, which means there exists $c' \in \mathbb{R}^d$ such that $S_k \subseteq c' + \rho(-S_k)$. Using Corollary 4.2.7 again, it follows that $\rho \ge k$. □

Corollary 4.4.2 (*No sublinear core-sets for general containers*)
For every $P \in \mathcal{C}^d, C \in \mathcal{C}_0^d$ and $\varepsilon \ge 0$, there exists an ε-core-set of P of size at most $\left\lceil \frac{d}{1+\varepsilon} \right\rceil + 1$ and for $P = -C = T^d$ no smaller subset of P will suffice.

Proof.
The case $\varepsilon = 0$ equates to Lemma 4.2.2. So, let $\varepsilon > 0$ and $k = \left\lceil \frac{d}{1+\varepsilon} \right\rceil$. If $S \subseteq P$ such that $R(S, C) = R_k(P, C)$ then $|S| \leq k + 1$ and by Theorem 4.4.1:

$$R(P, C) = R_d(P, C) \leq \frac{d}{k} R_k(P, C) = \frac{d}{k} R(S, C).$$

By the choice of k, $d/k \leq 1 + \varepsilon$.
In order to show the sharpness of the bound, choose $P = T^d$ and $C = -T^d$. Now, for $k < \frac{d}{1+\varepsilon}$, if $S' \subseteq P$ consists of no more than $k + 1$ points, then it follows from the sharpness condition in Theorem 4.4.1, that

$$R(T^d, -T^d) = \frac{d}{k} R_k(T^d, -T^d) > (1 + \varepsilon) R_k(T^d, -T^d) \geq (1 + \varepsilon) R(S', -T^d).$$

Hence S' is no ε-core-set of P. □

Remarks.
(1) Note that, by Lemma 4.2.2, the minimal size of a 0-core-set depends linearly on d and Corollary 4.4.2 now shows that allowing $\varepsilon > 0$ does not improve this situation. Thus, Corollary 4.4.2 already proves Theorem 4.1.2 for general containers.
(2) Additionally, we mention that, whenever C is a polytope presented as $C = \{x \in \mathbb{R}^d : a_k^T x \leq 1 \ \forall k \in [m]\}$ and $P = \text{conv}\{p_1, \dots, p_n\}$, Problem 4.1.1 can be rewritten as a Linear Program [42, 87], with the help of which a 0-core-set of P of at most $d + 1$ points can be computed in polynomial time.

Remark 4.4.3 (*Center-conformity*)
Choosing $P = -C = T^d$, every subset S of d vertices of P yields $R(S, C) = d - 1$ with a unique center c_S. But to cover P by $c_S + \rho C$, we need $\rho \geq \frac{2d}{d-1} R(S, C)$. So, for $\varepsilon \in (0, 1)$, a center-conform ε-core-set may need to be of size $d + 1$.

Moreover, as much as we understood it, [140, Theorem 5] asserts (in particular) that for every $\varepsilon > 0$ there is a subset $S \subseteq T^d$ of size $O(1/\varepsilon^2)$ such that every point in T^d has Euclidean distance at most ε to $c_S + R(S, -T^d)(-T^d)$. Again, taking any subset $S \subseteq T^d$ of d vertices and the fact that the distance of the remaining vertex to $c_S + (d-1)(-T^d)$ is strictly greater than $1/\sqrt{2}$, shows that this theorem cannot be true for $\varepsilon < 1/\sqrt{2}$.

4.4.2 Symmetric Containers/Normed Spaces

As mentioned several times, every 0-symmetric container $C \in \mathcal{C}_0^d$ induces a norm $\|\cdot\|_C$ and vice versa. We will always talk about symmetric containers here, but one may as well reformulate all results in terms of Minkowski spaces.

The results in section 4.3.2 may motivate the hope that symmetry of the container is the key for positive results on dimension-independence. Indeed, in [28], Bohnenblust proved an equivalent to Jung's Inequality (see the remark after the proof of Theorem 4.1.3) for general normed spaces. Taking into account the Minkowski asymmetry $s(C)$ of a possibly asymmetric container C, we showed a generalized version of this inequality in Theorem 2.4.1; adapted to our purposes here and expressed in terms of core-radii, it reads as follows:

Proposition 4.4.4 (*Generalized Bohnenblust*)
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}^d_0$. Then

$$\frac{R(P,C)}{R_1(P,C)} \leq \frac{(1+s(C))d}{d+1}$$

with equality, if $P = T^d = -C$ or $P = T^d$ and $C = T^d - T^d$.

One might hope that, for the class of symmetric containers, Bohnenblust's Inequality can be generalized in the same way as Henk's Inequality generalizes Jung's in the Euclidean case (giving a bound on the core-radii ratio as in (4.10) at the end of this section). This inequality would be tight for $P = T^d$ and $C = T^d - T^d$. However, the remainder of this section will show that the bound on the ratio of core-radii with symmetric containers does not improve the general bound from Theorem 4.4.1.

Lemma 4.4.5
With $C^d = T^d \cap (-T^d)$,

$$R_k(T^d, C^d) = \begin{cases} \frac{d+1}{2} & \text{if } k \leq \frac{d+1}{2} \\ k & \text{if } k \geq \frac{d+1}{2} \end{cases} .$$

Proof.
Let $T^d = \operatorname{conv}\{x_1, \ldots, x_{d+1}\}$ for suitable $x_1, \ldots, x_{d+1} \in \mathbb{R}^d$. Independently of which coordinates we choose for $x_1, \ldots, x_{d+1}$, we can index the normals $a_1, \ldots, a_{d+1} \in \mathbb{R}^d$ of a halfspace presentation of T^d such that $T^d = \bigcap_{i=1}^{d+1} H_{\leq}(a_i, 1)$ and

$$a_j^T x_i = \begin{cases} 1 & \text{if } j \neq i \\ -d & \text{if } j = i \end{cases} \quad \text{for } i,j \in [d+1]$$

Let $k \in [d+1]$ and consider an arbitrary k-face F of T^d, without loss of generality, $F = \operatorname{conv}\{x_1, \ldots, x_{k+1}\}$.
For $k \leq \frac{d+1}{2}$, let

$$\gamma = -\frac{(k-1)(d+1)}{2(d-k)} \quad \text{and} \quad c = \frac{1}{k+1}\left(\sum_{l=1}^{k+1} x_l + \gamma \sum_{l=k+2}^{d+1} x_l\right).$$

Then $\gamma \geq -\frac{d+1}{2}$ and for $i \in [k+1]$ and $j \in [d+1]$

$$a_j^T(x_i - c) = \begin{cases} \gamma & \text{if } j > k+1 \\ \frac{d+1}{2} & \text{if } j \leq k+1, j \neq i \\ -\frac{d+1}{2} & \text{if } j = i\ . \end{cases}$$

Hence $F - c \subseteq \frac{d+1}{2}C^d$. Moreover, these equalities show that $T^d - c$ touches the facets of $\frac{d+1}{2}C^d$ induced by the hyperplanes $H_=(a_i, (d+1)/2)$, $H_=(a_i, -(d+1)/2)$ for $i = 1, \ldots, k+1$ and therefore it follows by Theorem 4.2.3 that $R_k(T^d, C^d) = (d+1)/2$.
For $k \geq \frac{d+1}{2}$, let $c = \sum_{i=1}^{k+1} x_i$. Then $1 - k + d \leq k$ and for $i \in [k+1]$ and $j \in [d+1]$

$$a_j^T(x_i - c) = \begin{cases} -k & \text{if } j > k+1 \\ 1-k+d & \text{if } j \leq k+1, j \neq i \\ -k & \text{if } j = i \end{cases}$$

showing $F - c \subseteq kC^d$. Here again, the equalities show that $T^d - c$ touches every facet of $-kT^d$ and $R_k(T^d, C^d) = k$ follows by Theorem 4.2.3. □

Theorem 4.4.6 (*Inequality relating core-radii for 0-symmetric containers*)
Let $k, l \in \mathbb{N}$ such that $l \leq k \leq d$, $P \in \mathcal{C}^d$ and $C \in \mathcal{C}_0^d$ a 0-symmetric container. Then

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \begin{cases} \frac{2k}{k+1} & \text{for } l \leq \frac{k+1}{2} \\ \frac{k}{l} & \text{for } l \geq \frac{k+1}{2}\ . \end{cases}$$

Moreover, let T^k be a k-simplex embedded in the first k coordinates of $\mathbb{R}^d$ and $C^k = (T^k \cap (-T^k)) + (\{0\}^k \times [-1,1]^{d-k})$. Then

$$\frac{R_k(T^k,C^k)}{R_l(T^k,C^k)} = \begin{cases} \frac{2k}{k+1} & \text{for } l \leq \frac{k+1}{2} \\ \frac{k}{l} & \text{for } l \geq \frac{k+1}{2}\ . \end{cases}$$

Proof.
Let $S \subseteq P$ be a k-simplex such that $R_k(P,C) = R(S,C)$ and assume without loss of generality that $R(S,C) = k$. By Bohnenblust's Inequality, we get that $R_1(S,C) \geq (k+1)/2$ and thus $R_l(P,C) \geq R_1(P,C) \geq (k+1)/2$. Thus $R_k(P,C)/R_l(P,C) \leq 2k/(k+1)$. On the other hand

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \frac{k}{l}$$

by Theorem 4.4.1. Together this yields

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \min\left\{\frac{2k}{k+1}, \frac{k}{l}\right\},$$

which splits into the two cases claimed above. The second statement follows from Lemma 4.4.5 and the observation that the computation of $R(T^k, C^k)$ is in fact the k-dimensional containment problem of containing T^k in $T^k \cap (-T^k)$. □

With Theorem 4.4.6 at hand, Theorem 4.1.2 follows as a simple corollary:

Proof of Theorem 4.1.2
For $k = d$ and $l \geq (d+1)/2$ the inequalities in Theorem 4.4.1 and 4.4.6 coincide. Hence the proof of Corollary 4.4.2 can simply be copied up to the additional condition that $\varepsilon < 1$ and the change from $C = T^d$ to $C = T^d \cap (-T^d)$ to show that the bound is best possible. □

On the other hand diametral pairs of points in P are 1-core-sets for every 0-symmetric container C as Bohnenblust's result already shows. Theorem 4.4.6 then shows that no choice of up to $\lfloor (d-3)/2 \rfloor$ points to add to the core-set improves the approximation quality.

Remark. Theorem 4.1.2 also implies the non-existence of sublinear center-conform ε-core-sets for $\varepsilon < 1$. On the other hand, we know from Lemma 4.2.2 that there are linear ones, even if $\varepsilon = 0$.

Theorem 4.1.2 shows that the class of symmetric containers is too large for an extension of Bohnenblust's Inequality to other core-radii than R_1. A question that remains open is, whether there is a sensible class of containers $\mathcal{C} \subseteq \mathcal{C}_0^d$ such that the following inequality holds for $1 \leq l \leq k \leq d$:

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \frac{k(l+1)}{l(k+1)} \tag{4.10}$$

If true, (4.10) would yield dimension independent ε-core-sets for all $\varepsilon > 0$ in the same way as shown for the MEB problem in the proof of Theorem 4.1.3. In the remainder of this section, we elaborate this idea and show that the phenomenon of dimension independent core-sets is not restricted to Euclidean balls. In fact, the container does not even have to be symmetric in order to allow dimension independent core-sets. However, the result below is sill quite weak since the containers for which we establish this property all converge to the Euclidean ball as d tends to infinity.

Lemma 4.4.7 (*Proposition 4.3.5 applied to almost-balls*)
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ such that for $\lambda \in \left[\sqrt{\frac{d^2-1}{d^2}}, 1\right]$ the inclusions $\lambda \mathbb{B}_2^d \subseteq C \subseteq \mathbb{B}_2^d$ hold. If $k, l \in \mathbb{N}$ and $l \leq k \leq d$, then,

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \frac{k(l+1)}{l(k+1)}.$$

Proof.
As in Theorem 4.4.1, it suffices to show

$$\frac{R_k(P,C)}{R_{k-1}(P,C)} \leq \frac{k^2}{k^2-1}. \tag{4.11}$$

as for $l < k-1$ the claim follows by repeatedly applying (4.11). Without loss of generality, we may assume the existence of a k-simplex $S = \text{conv}\{x_1, \dots, x_{k+1}\} \subseteq P$ satisfying $R(S,C) = R_k(P,C)$, as (4.11) is certainly fulfilled if $R_k(P,C) = R_{k-1}(P,C)$.
By Proposition 4.3.5, we get $R(S, \mathbb{B}_2^d) \leq \sqrt{\frac{d^2}{d^2-1}} \bar{R}_{d-1}(S, \mathbb{B}_2^d)$. Hence, there is a $c \in \mathbb{R}^d$ such that

$$\begin{aligned} S &\subseteq c + \sqrt{\tfrac{d^2}{d^2-1}} R_{d-1}(S, \mathbb{B}_2^d)\mathbb{B}_2^d \\ &\subseteq c + \sqrt{\tfrac{d^2}{d^2-1}} R_{d-1}(P, \mathbb{B}_2^d)\mathbb{B}_2^d \\ &\subseteq c + \sqrt{\tfrac{d^2}{d^2-1}} R_{d-1}(P, \mathbb{B}_2^d)\tfrac{1}{\lambda}C \\ &\subseteq c + \sqrt{\tfrac{d^2}{d^2-1}} R_{d-1}(P, C)\tfrac{1}{\lambda}C \\ &\subseteq c + \tfrac{d^2}{d^2-1} R_{d-1}(P, C)C, \end{aligned}$$

where the last inclusion follows from $\lambda \geq \sqrt{\frac{k^2}{k^2-1}}$. Thus, $R_k(P,C) = R(T,C) \leq \frac{k^2}{k^2-1} R_{k-1}(P,C)$. □

Theorem 4.4.8 (*Dimension independent core-sets for almost-balls*)
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ such that for $\lambda \in \left[\sqrt{\frac{d^2-1}{d^2}}, 1\right]$ the inclusions $\lambda \mathbb{B}_2^d \subseteq C \subseteq \mathbb{B}_2^d$ hold and $\varepsilon > 0$. There exists an ε-core-set $S \subseteq P$ of size

$$\left\lceil \frac{1}{\varepsilon} \right\rceil + 1.$$

Proof.
Let $k := \lceil \frac{1}{\varepsilon} \rceil$ and $S \subseteq P$ such that $R(S,C) = R_k(P,C)$ and $|S| \leq k+1$. By Lemmas 4.4.7 and 4.2.2,

$$R(P,C) \leq \frac{d(k+1)}{k(d+1)} \cdot R(S,C),$$

where k is chosen such that $\frac{d(k+1)}{k(d+1)} \leq 1 + \varepsilon$ independently of $d \in \mathbb{N}$. □

We close this chapter by summarizing the different inequalities between core-radii in Table 4.1.

$C = \mathbb{B}_2^d$	$\frac{R(P,\mathbb{B}_2^d)}{R_1(P,\mathbb{B}_2^d)} \leq \sqrt{\frac{2d}{d+1}}$ Jung's Inequality	$\frac{R_k(P,\mathbb{B}_2^d)}{R_l(P,\mathbb{B}_2^d)} \leq \sqrt{\frac{k(l+1)}{l(k+1)}}$ Henk's Inequality
$C \in \mathcal{C}_0^d$ 0-symm.	$\frac{R(P,C)}{R_1(P,C)} \leq \frac{2d}{d+1}$ Bohnenblust's Inequality	$\frac{R_k(P,C)}{R_l(P,C)} \leq \begin{cases} \frac{2k}{k+1} & \text{for } l \leq \frac{k+1}{2} \\ \frac{k}{l} & \text{for } l \geq \frac{k+1}{2} \end{cases}$ Theorem 4.4.6
$C \in \mathcal{C}_0^d$	$\frac{R(P,C)}{R_1(P,C)} \leq d$ Theorem 4.4.1	$\frac{R_k(P,C)}{R_l(P,C)} \leq \frac{k}{l}$

Table 4.1: Synopsis of inequalities between core-radii as collected in this chapter.

Chapter 5

Hausdorff Matching

Whereas the previous chapters were concerned with the problem of computing functionals of a single polytope, the present chapter will investigate different similarity measures of two polytopes based on the Hausdorff distance of two compact sets in $\mathbb{R}^d$. We study the computational complexity of determining the Hausdorff distance of two polytopes and a more general matching problem. Here, one polytope is allowed to be homothetically transformed in order to minimize its Hausdorff distance to the other one. For the matching problem, we characterize optimality, deduce a Helly-type theorem and give polynomial time algorithms. We also demonstrate that a variant of our matching algorithm can be applied for a particular tomographic reconstruction problem.

5.1 Introduction

The problem of comparing two geometric objects and evaluating their similarity or dissimilarity arises naturally in many applications such as shape fitting, pattern recognition or computer vision. A suitable means that has been widely applied to evaluate the resemblance of two compact sets is the *Hausdorff distance*, see e.g. [112, 114, 148] or the survey [10]. Consequently, there is an extensive amount of mathematical literature that is concerned with the question of how to compute the Hausdorff distance of geometric objects such as finite point sets or triangulations [8, 9, 16]. Not only the static evaluation problem, but also problems where one object is allowed to undergo a transformation from a certain class in order to match the other one are of particular interest, see e.g. [4, 8, 14, 54].

The geometric objects that are considered in the papers mentioned above are usually finite point sets or finite unions of simple geometric objects such as line segments or simplices and are usually considered in (low) fixed dimension. The tables in [160, Chapter 3] give a detailed list of problems that have been considered, together with the respective references and show that the term “in low fixed dimension” can be replaced by “in the plane” in most cases. For higher dimensions, the problem of exact point pattern matching has been investigated in [11], while [160] considers the task of minimizing the Hausdorff distance of two sets of points or simplices under translations. Apart from that, very little seems to be known.

In this chapter, we investigate the problem of evaluating the Hausdorff distance and the matching problem under homothetics for polytopes in arbitrary dimension. More precisely, we are interested in the following two problems, where $\mathbb{R}^d$ is equipped with an arbitrary but fixed norm.

Problem 5.1.1 (*Hausdorff evaluation*)
For two polytopes $P, Q \subseteq \mathbb{R}^d$ in arbitrary dimension, compute their Hausdorff distance $\delta(P, Q)$.

Problem 5.1.2 (*Hausdorff matching*)
For two polytopes $P, Q \subseteq \mathbb{R}^d$ in arbitrary dimension, compute $\delta_H(P, Q)$, i.e. the minimal Hausdorff distance of a homothetic transformation of P to Q. In other words,

$$\begin{array}{rcl} \delta_H(P,Q) = & \min & \delta(\alpha P + c, Q) \\ & s.t & c \in \mathbb{R}^d \\ & & \alpha > 0. \end{array} \tag{5.1}$$

Of course, the computational complexity of these problems depends on the presentation of the polytopes P and Q. We show that if both polytopes are in $\mathcal{V}$-presentation, both problems can be solved efficiently in arbitrary dimension for most important norms (Theorem 5.2.13 and Section 5.3.3). If, on the other hand, at least one polytope is given in $\mathcal{H}$-presentation, already the evaluation problem has to be considered intractable in low dimensions (Theorem 5.2.16).
Hence, the class of $\mathcal{V}$-polytopes represents a first step towards a large class of geometric objects for which the matching problem can be solved efficiently in arbitrary dimension. Moreover, in view of the running times of our algorithms in Section 5.3.3, replacing dense finite point clouds in high dimensions by their respective convex hulls might also be an interesting approximation in many practical applications (cf. Remark 5.2.4).

This chapter is organized as follows. In Section 5.2, we state basic properties of the Hausdorff distance that will be useful throughout this chapter and investigate the evaluation problem. In Section 5.3, we study the matching problem, for which we prove an optimality condition and a Helly-type theorem, and give (approximation) algorithms.

5.2 Computing the Hausdorff Distance of Two Polytopes

5.2.1 Problem Statement and General Properties

Throughout this chapter, we work in $(\mathbb{R}^d, \|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$. Besides our standard notation $\mathbb{B}_p^d$ for the unit ball of the p-norm, we will employ $\mathbb{B} := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ if the norm is not explicitly specified.

For a convex function $f : C \to \mathbb{R}$ on some convex set $C \subseteq \mathbb{R}^d$ and $x \in C$, we write $\partial f(x)$ for the *subdifferential* of f in x and, if f is continuously differentiable in x, we denote by $\nabla f(x)$ the *gradient* of f in x.

We start by defining the functional of interest in this chapter which is based on the distance function of convex bodies.

Definition 5.2.1 (*Distance mapping*)
For $P \subseteq \mathbb{R}^d$ non-empty and compact, define

$$\begin{array}{rccl} d(\cdot, P): & \mathbb{R}^d & \to & [0, \infty) \\ & x & \mapsto & d(x, P) = \min\{\|x - p\| : p \in P\} \end{array} \tag{5.2}$$

the distance mapping of P induced by $\|\cdot\|$. (The minimum in (5.2) is attained, as $P \neq \emptyset$ is compact and $\|\cdot\|$ is continuous.)

Since it is the key for the tractability result in Theorem 5.2.13, we give an explicit proof that the distance function of a convex body is convex.

Lemma 5.2.2 (*Convexity of* $d(\cdot, P)$)
Let $P \in \mathcal{C}^d$. Then, $d(\cdot, P)$ is convex.

Proof.
Let $x_1, x_2 \in \mathbb{R}^d$ and $\lambda \in [0, 1]$. For $i = 1, 2$ let $p_i \in P$ be a point in P such that $d(x_i, P) = \|x_i - p_i\|$. We have $d(\lambda x_1 + (1-\lambda)x_2, P) \leq \|\lambda x_1 + (1-\lambda)x_2 - (\lambda p_1 + (1-\lambda)p_2)\|$ $\leq \lambda\|x_1 - p_1\| + (1-\lambda)\|x_2 - p_2\| = \lambda d(x_1, P) + (1-\lambda)d(x_2, P)$ by convexity of P and $\|\cdot\|$. □

Definition 5.2.3 (*Hausdorff distance*)
Let $P, Q \subseteq \mathbb{R}^d$ non-empty and compact. The Hausdorff distance induced by $\|\cdot\|$ between P and Q is denoted by

$$\delta(P, Q) := \max\{\max_{p \in P} d(p, Q), \max_{q \in Q} d(q, P)\}. \tag{5.3}$$

Since P, Q are non-empty and compact, and $d(\cdot, P)$, $d(\cdot, Q)$ are continuous by Lemma 5.2.2, the maximum in (5.3) is attained. It is easy to see that it can also be expressed by

$$\delta(P, Q) = \min\{\rho \geq 0 : P \subseteq Q + \rho\mathbb{B}, Q \subseteq P + \rho\mathbb{B}\}. \tag{5.4}$$

As an immediate consequence of Lemma 5.2.2, we can bound the Hausdorff distance of finite point clouds in terms of the Hausdorff distance of their respective convex hulls.

Remark 5.2.4 (*Relation to finite point sets*)
Let $p_1, \dots, p_n, q_1, \dots, q_m \in \mathbb{R}^d$, $P := \operatorname{conv}\{p_1, \dots, p_n\}$ and $Q := \operatorname{conv}\{q_1, \dots, q_m\}$. Further, let $R := \max\Big(\{d(x, \{p_1, \dots, p_n\}) : x \in \operatorname{bd}(P)\} \cup \{d(x, \{q_1, \dots, q_m\}) : x \in \operatorname{bd}(Q)\}\Big)$. Then,

$$\delta(P, Q) \leq \delta\big(\{p_1, \dots, p_n\}, \{q_1, \dots, q_m\}\big) \leq \delta(P, Q) + R,$$

and both bounds are best possible.

Proof.
By Lemma 5.2.2, $d(\cdot, P)$ and $d(\cdot, Q)$ are convex and hence $\max_{x \in P} d(x, Q)$ is attained within $\{p_1, \dots, p_n\}$ as well as $\max_{x \in Q} d(x, P)$ is attained within $\{q_1, \dots, q_m\}$. As we have $\{p_1, \dots, p_n\} \subseteq P$ and $\{q_1, \dots, q_m\} \subseteq Q$, the first inequality follows. On the other hand, for all $i \in [n]$, there is an $x_i \in Q$ such that $\|p_i - x_i\| \leq \delta(P, Q)$. By the definition of R and the triangle inequality, we get $d(p_i, \{q_1, \dots, q_m\}) \leq \delta(P, Q) + R$. Applying this argument also to $q_1, \dots, q_m$ yields the second inequality.
It is obvious that the first bound can be fulfilled with equality. By the definition of R, the second bound cannot be improved, either, as the point sets $\{(0,0)^T, (0,-1)^T, (0,1)^T\}$ and $\{(0,-1)^T, (0,1)^T\}$ in $\mathbb{R}^2$ show. $\square$

Of course, the quality of this bound strongly depends on the "density" of the point clouds under investigation, which is reflected in the value of R. If the point clouds are rather coarse, the approximation via convex hulls may be of no interest. If, on the other hand, the point clouds are dense samples from convex regions (such as pixels of a segmented region in an image), it might be possible to bound R by a small constant.

Besides the two equivalent definitions of the Hausdorff distance in Definition 5.2.3, there is also a third formulation based on the support functions of P and Q. This formulation will be of great use for the hardness proof of Theorem 5.2.16 below. Since it can also be used as an equivalent definition of the Hausdorff distance, we already state it at this point. Before doing so, we fix some notation, which we make use of at several points in this chapter.

Notation 5.2.5 (*Tangential and normal cone*)
Let $P \in \mathcal{C}^d$ and $x \in P$. We denote by

$$T(P, x) := \operatorname{cl}\{v \in \mathbb{R}^d : \exists \lambda > 0 \ s.t. \ x + \lambda v \in P\}$$

the tangential cone of P at x and by

$$N(P, x) := \{a \in \mathbb{R}^d : h(P, a) = a^T x\} = T(P, x)^\circ$$

the normal cone of P at x.
Here, $h(P, a)$ denotes the *support function* of P in direction a.

Lemma 5.2.6 (*Hausdorff distance via support functions*)
Let $P, Q \in \mathcal{C}^d$. Then,

$$\delta(P,Q) = \max_{u \in \mathbb{B}^\circ} |h(P,u) - h(Q,u)|. \tag{5.5}$$

Proof.
Let $\rho := \delta(P,Q)$ and $u \in \mathbb{B}^\circ$. As $P \subseteq Q + \rho\mathbb{B}$, it follows that $h(P,u) \leq h(Q + \rho\mathbb{B}, u) = h(Q,u) + h(\rho\mathbb{B}, u) \leq h(Q,u) + \rho$. In the same way, $h(Q,u) \leq h(P,u) + \rho$. As $u \in \mathbb{B}^\circ$ was arbitrary, this yields $\max_{u\in\mathbb{B}^\circ} |h(P,u) - h(Q,u)| \leq \rho$.
For the other inequality, let $p^* \in P$ and $q^* \in Q$ such that $\rho = \|p^* - q^*\|$. Let $f(q) := \|q - p^*\|$. As $f(q^*) \leq f(q)$ for all $q \in Q$, there exists

$$u' \in \partial f(q^*) = \{u \in \mathbb{B}^\circ : u^T(q^* - p^*) = f(q^*)\}$$

such that $-u' \in N(Q, q^*)$. Thus, for $u^* = -u'$,

$$\rho = f(q^*) = (u^*)^T(p^* - q^*) = (u^*)^T p^* - \max_{q \in Q} (u^*)^T q \leq h(P,u^*) - h(Q,u^*)$$
$$\leq |h(P,u^*) - h(Q,u^*)| \leq \max_{u\in\mathbb{B}^\circ} |h(P,u) - h(Q,u)|.$$

□

Remark. Due to the homogeneity of $h(P,\cdot), h(Q,\cdot)$ and $|\cdot|$, the maximum in (5.5) is attained for some vector $u^* \in \text{bd}(\mathbb{B}^\circ)$. But since the function $f(u) := |h(P,u) - h(Q,u)|$ is not convex in general, the maximization in (5.5) cannot be restricted to $\text{ext}(\mathbb{B}^\circ)$.

Before turning to the matching problem in Section 5.3, we first ask for the complexity of computing the Hausdorff distance of two fixed sets. For the case of convex polygons in the plane, this question has first been studied in [16]. The paper [8] investigates the case of finite point sets or sets of line segments in the plane. Both papers settle the issue by giving efficient algorithms that solve the respective problems. In [9], these algorithms are extended in order to compute the Hausdorff distance between a union of n and one of m simplices of dimension k in $\mathbb{R}^d$ in time $O(nm^{k+2})$.
In the context of the present thesis, however, it is natural to ask for the computational complexity of computing the Hausdorff distance of two polytopes in unbounded dimension. As with NormMax$_p$ (Problem 3.1.1) and the radii computation tasks in Chapter 3, we investigate the problem for a fixed p-norm and different presentations of the polytopes. Again, the problems are parameterized by the dimension, which allows a refined analysis of the role of the dimension in the $\mathbb{NP}$-hard cases.

For the statement of the following three problems, let $p \in \mathbb{N} \cup \{\infty\}$ be fixed.

Notation 5.2.7 (*Specific norms*)
When working in $(\mathbb{R}^d, \|\cdot\|_p)$ for some $p \in \mathbb{N} \cup \{\infty\}$, we write

$$\delta_p(P,Q) = \max\{\max_{x\in P} d_p(x,Q), \max_{q\in Q} d_p(q,P)\},$$

where the subscript p explicitly indicates the norm with respect to which the Hausdorff distance is measured.

Depending on the presentation of the input polytopes, we distinguish three different decision problems for the problem of computing the Hausdorff distance of two polytopes.

Problem 5.2.8 (Hausdorff$_p$-$\mathcal{V}$-$\mathcal{V}$)

Input: $d \in \mathbb{N}, n, m \in \mathbb{N}, p_1, \dots, p_n, q_1, \dots, q_m \in \mathbb{Q}^d, \rho \in \mathbb{Q}$

Parameter: d

Question: Is $\delta_p\Big(\mathrm{conv}\{p_1, \dots, p_n\}, \mathrm{conv}\{q_1, \dots, q_m\}\Big)^p \geq \rho$?

Problem 5.2.9 (Hausdorff$_p$-$\mathcal{V}$-$\mathcal{H}$)

Input: $d \in \mathbb{N}, n, m \in \mathbb{N}, p_1, \dots, p_n, a_1, \dots, a_m \in \mathbb{Q}^d, \beta_1, \dots, \beta_m \in \mathbb{Q}, \rho \in \mathbb{Q}$

Parameter: d

Question: Is $\delta_p\left(\mathrm{conv}\{p_1, \dots, p_n\}, \bigcap_{i=1}^{m} H_{\leq}(a_i, \beta_i)\right)^p \geq \rho$?

Problem 5.2.10 (Hausdorff$_p$-$\mathcal{H}$-$\mathcal{H}$)

Input: $d \in \mathbb{N}, n, m \in \mathbb{N}, a_1, \dots, a_n, d_1, \dots, d_m \in \mathbb{Q}^d$,
$\beta_1, \dots, \beta_n, \varepsilon_1, \dots, \varepsilon_m \in \mathbb{Q}, \rho \in \mathbb{Q}$

Parameter: d

Question: Is $\delta_p\left(\bigcap_{i=1}^{n} H_{\leq}(a_i, \beta_i), \bigcap_{i=1}^{m} H_{\leq}(d_i, \varepsilon_i)\right)^p \geq \rho$?

5.2.2 Tractability Results

In this subsection, we show that the Hausdorff distance of two polytopes in $\mathcal{V}$-presentation can in principle be computed efficiently (Lemma 5.2.11, Theorem 5.2.13). The restriction "in principle" is due to the fact, that for $p \in \mathbb{N} \setminus \{1, 2\}$ the (power of the) distance $d_p(x, P)^p$ of a point x to a polytope P might not be rational and therefore not computable in polynomial time.

Lemma 5.2.11 (*Restriction to vertices*)
In an arbitrary normed space $(\mathbb{R}^d, \|\cdot\|)$, let $\mathcal{O}$ be an oracle that on input $x, p_1, \dots, p_n \in \mathbb{Q}^d$ returns $d(x, \mathrm{conv}\{p_1, \dots, p_n\})$. Then for any set of points $p_1, \dots, p_n, q_1, \dots, q_m \in \mathbb{Q}^d$, $\delta(\mathrm{conv}\{p_1, \dots, p_n\}, \mathrm{conv}\{q_1, \dots, q_m\})$ can be computed using at most $m+n$ calls to $\mathcal{O}$.

Proof.
Let $P := \mathrm{conv}\{p_1, \dots, p_n\}$ and $Q := \mathrm{conv}\{q_1, \dots, q_m\}$. Since $d(\cdot, Q)$ is convex by Lemma 5.2.2, $\max_{x \in P} d(x, Q)$ is attained at a vertex of P as well as $\max_{x \in Q} d(x, P)$ is attained at a vertex of Q. Hence, $\delta(P, Q) = \max\big\{\max\{d(p, Q) : p \in P\}, \max\{d(q, P) : q \in Q\}\big\}$ can be computed by determining $d(p_i, Q)$ and $d(q_j, P)$ for all $i \in [n]$ and $j \in [m]$ via calls to $\mathcal{O}$. □

Lemma 5.2.12 (*Computing* $d(\cdot, P)$)
If the unit ball $\mathbb{B} \subseteq \mathbb{R}^d$ is a polytope given in $\mathcal{V}$-presentation or in $\mathcal{H}$-presentation, then for any point $x \in \mathbb{Q}^d$ and any rational polytope $P \subseteq \mathbb{R}^d$ in $\mathcal{V}$- or $\mathcal{H}$-presentation, the distance $d(x, P)$ can be computed in polynomial time.
If $\mathbb{B} = \mathbb{B}_2^d$, for any point $x \in \mathbb{Q}^d$ and any rational polytope $P \subseteq \mathbb{R}^d$ in $\mathcal{V}$- or $\mathcal{H}$-presentation, $d(x, P)^2$ can be computed exactly in polynomial time.

Proof.
The cases where a $\mathcal{V}$- or $\mathcal{H}$-presentation of $\mathbb{B}$ is available can all be solved by Linear Programming:
First, let $\mathbb{B} = \{x \in \mathbb{R}^d : u_j^T x \leq 1 \ \forall j \in [m]\}$. If $P = \{x \in \mathbb{R}^d : Ax \leq b\}$, then

$$\begin{array}{rll} d(x,P) = & \min & t \\ & s.t. & t \geq u_j^T(x-y) \quad \forall j \in [m] \\ & & Ay \leq b \\ & & y \in \mathbb{R}^d \\ & & t \in \mathbb{R}. \end{array}$$

If $P = \text{conv}\{p_1, \ldots, p_n\}$, then

$$\begin{array}{rlll} d(x,P) = & \min & t & \\ & s.t. & t \geq u_j^T(x - \sum_{i=1}^n \lambda_i p_i) & \forall j \in [m] \\ & & \sum_{i=1}^n \lambda_i = 1 & \\ & & \lambda_i \geq 0 & \forall i \in [n] \\ & & t \in \mathbb{R}. & \end{array}$$

Now, let $\mathbb{B} = \text{conv}\{v_1, \ldots, v_m\}$. If $P = \{x \in \mathbb{R}^d : Ax \leq b\}$, then

$$\begin{array}{rlll} d(x,P) = & \min & \alpha & \\ & s.t. & x - y = \sum_{j=1}^m \mu_j v_j & \\ & & Ay \leq b & \\ & & \sum_{j=1}^m \mu_j = \alpha & \\ & & \mu_j \geq 0 & \forall j \in [m] \\ & & y \in \mathbb{R}^d & \\ & & \alpha \geq 0. & \end{array}$$

If $P = \text{conv}\{p_1, \ldots, p_n\}$, then

$$\begin{array}{rlll} d(x,P) = & \min & \alpha & \\ & s.t. & x - \sum_{i=1}^n \lambda_i p_i = \sum_{j=1}^m \mu_j v_j & \\ & & \sum_{i=1}^n \lambda_i = 1 & \\ & & \sum_{j=1}^m \mu_j = \alpha & \\ & & \mu_j \geq 0 & \forall j \in [m] \\ & & \lambda_i \geq 0 & \forall i \in [n] \\ & & \alpha \geq 0. & \end{array}$$

Finally, let $\mathbb{B} = \mathbb{B}_2^d$. If $P = \{x \in \mathbb{R}^d : Ax \leq b\}$, then

$$d(x,P)^2 = \begin{array}{ll} \min & (p-x)^T(p-x) \\ s.t. & Ap \leq b. \\ & p \in \mathbb{R}^d. \end{array} \tag{5.6}$$

If $P = \mathrm{conv}\{p_1, \ldots, p_n\}$, then

$$d(x,P)^2 = \begin{array}{lll} \min & (p-x)^T(p-x) & \\ s.t. & p = \sum_{i=1}^n \lambda_i p_i & \\ & \sum_{i=1}^n \lambda_i = 1 & \\ & \lambda_i \geq 0 & \forall i \in [n]. \end{array} \tag{5.7}$$

By [126], the optimal solutions of (5.6) and (5.7) are rational and can be found in polynomial time. □

Together, Lemmas 5.2.11 and 5.2.12 yield the following:

Theorem 5.2.13 (*Tractability of* Hausdorff-$\mathcal{V}$-$\mathcal{V}$)
Hausdorff$_1$-$\mathcal{V}$-$\mathcal{V}$, Hausdorff$_2$-$\mathcal{V}$-$\mathcal{V}$ and Hausdorff$_\infty$-$\mathcal{V}$-$\mathcal{V}$ are in $\mathbb{P}$. More generally, if a $\mathcal{V}$- or $\mathcal{H}$-presentation of the unit ball $\mathbb{B}$ can be computed in polynomial time, the Hausdorff distance of two rational polytopes $P, Q \subseteq \mathbb{R}^d$ can be computed in polynomial time.

Remark 5.2.14 (*Direct approximation of* $\delta_2(P,Q)$)
For two rational $\mathcal{V}$-polytopes $P, Q \subseteq \mathbb{R}^d$, the Hausdorff distance $\delta_2(P,Q)$ can also be approximated to any accuracy in polynomial time by solving a Second Order Cone Program (cf. the SOCP in Lemma 5.3.12 with $\alpha = 1$ and $c = 0$).

Remark 5.2.15 (*Approximation for* Hausdorff$_p$-$\mathcal{V}$-$\mathcal{V}$)
For $p \in \mathbb{N}$ and rational polytopes $P = \mathrm{conv}\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ and $Q = \mathrm{conv}\{q_1, \ldots, q_m\} \subseteq \mathbb{R}^d$, the Hausdorff distance $\delta_p(P,Q)$ can be approximated to any accuracy in polynomial time by combining Lemma 5.2.11 and the Ellipsoid Method [92] for the approximation of $d(p_i, Q)$ and $d(q_j, P)$ for $i \in [n]$ and $j \in [m]$.

5.2.3 Hardness Results

The main result of this section is Theorem 5.2.16, which states the hardness of Hausdorff$_p$-$\mathcal{H}$-$\mathcal{H}$ for $p \in \mathbb{N}$. After the proof of Theorem 5.2.16, the section concludes with remarks about the case $p = \infty$, the complexity of Hausdorff$_p$-$\mathcal{V}$-$\mathcal{H}$ and an interesting connection to the VertexEnumeration problem (Problem 5.2.22).

Theorem 5.2.16 (*Hardness of* Hausdorff$_p$-$\mathcal{H}$-$\mathcal{H}$)
For $p \in \mathbb{N}$, Hausdorff$_p$-$\mathcal{H}$-$\mathcal{H}$ is W[1]-hard, even if both $\mathcal{H}$-polytopes are required to be 0-symmetric.

The proof of Theorem 5.2.16 is given by Lemma 5.2.17 and Lemma 5.2.19, where the key observation that enables the latter result is given in Lemma 5.2.18.

Lemma 5.2.17 (*Reduction of* NORMMAX$_p$)
For all $p \in \mathbb{N} \cup \{\infty\}$, NORMMAX$_p$ (Problem 3.1.1) can be reduced in polynomial and FPT (with respect to the dimension) time to HAUSDORFF$_p$-$\mathcal{H}$-$\mathcal{H}$.

Proof.
If (d, P, λ) is an instance of NORMMAX$_p$ with an $\mathcal{H}$-presented rational polytope $P \subseteq \mathbb{R}^d$, let $Q := \{x \in \mathbb{R}^d : \pm e_i^T x \leq 0 \ \forall i \in [d]\} = \{0\}$. Then,

$$\max\{\|x\|_p^p : x \in P\} \geq \lambda \Leftrightarrow \delta_p(P, Q)^p \geq \lambda$$

□

Since Theorem 3.1.2 shows that NORMMAX$_p$ is W[1]-hard for $p \in \mathbb{N} \setminus \{1\}$ on 0-symmetric polytopes, the statement of Theorem 5.2.16 follows in all cases except $p = 1$. Since NORMMAX$_p$ is $\mathbb{NP}$-hard for all $p \in \mathbb{N}$ (e.g. [27, 130]), the same reduction can be used to show $\mathbb{NP}$-hardness for all $p \in \mathbb{N}$, too.

Even though NORMMAX$_1$ is in FPT (Theorem 3.1.2), HAUSDORFF$_1$-$\mathcal{H}$-$\mathcal{H}$ is also W[1]-hard, as the following shows. Together with Lemma 5.2.17, this completes the proof of Theorem 5.2.16.
As in Chapter 3, we present an FPT reduction of CLIQUE (Problem 3.2.1) to HAUSDORFF$_1$-$\mathcal{H}$-$\mathcal{H}$. The reduction uses a similar technique as the one used for Theorem 3.1.2 and considers $\mathbb{R}^{2k}$ as the direct product of k two-dimensional spaces. We start by investigating the Hausdorff distance induced by the 1-norm for polytopes that are products of k two-dimensional polytopes.

Lemma 5.2.18 (*Hausdorff distance of direct products*)
For $k \in \mathbb{N}$, let $P_1, \ldots, P_k \subseteq \mathbb{R}^2$ and $Q_1, \ldots, Q_k \subseteq \mathbb{R}^2$ be polytopes with $Q_i \subseteq P_i$ for all $i \in [k]$ and further $P := P_1 \times \cdots \times P_k \subseteq \mathbb{R}^{2k}$ and $Q := Q_1 \times \cdots \times Q_k \subseteq \mathbb{R}^{2k}$. Then,

$$\delta_1(P, Q) = \sum_{i=1}^{k} \delta_1(P_i, Q_i).$$

Proof.
Since $Q_i \subseteq P_i$ for all $i \in [k]$ and hence $Q \subseteq P$, we have $h(P_i, u) \geq h(Q_i, u)$ for all $i \in [k]$ and $u \in \mathbb{R}^2$, and $h(P, u) \geq h(Q, u)$ for all $u \in \mathbb{R}^{2k}$. Decomposing a vector $u \in \mathbb{R}^{2k}$ as

$u = (u_1^T, \dots, u_k^T)^T$ with $u_1, \dots, u_k \in \mathbb{R}^2$, we obtain by Lemma 5.2.6,

$$\delta_1(P,Q) = \max_{u\in\mathbb{B}_\infty^{2k}} (h(P,u) - h(Q,u)) = \max_{u\in\mathbb{B}_\infty^{2k}} \sum_{i=1}^{k} (h(P_i,u_i) - h(Q_i,u_i)) =$$
$$= \sum_{i=1}^{k} \left(\max_{u_i\in\mathbb{B}_\infty^{2}} (h(P_i,u_i) - h(Q_i,u_i)) \right) = \sum_{i=1}^{k} \delta_1(P_i,Q_i).$$

□

Lemma 5.2.19 (*W[1]-Hardness of* HAUSDORFF$_1$-$\mathcal{H}$-$\mathcal{H}$)
HAUSDORFF$_1$-$\mathcal{H}$-$\mathcal{H}$ is W[1]-hard, even when restricted to 0-symmetric polytopes.

Proof.
Let (m,k,E) be an instance of CLIQUE with m vertices, define $n := 2m$ and let $\bar{P}_1 = \text{conv}\{\bar{p}_1, \dots, \bar{p}_{2n}\} \subseteq \mathbb{R}^2$ be the polytope given in Equation (3.9) for $p = 2$, i.e. $\bar{p}_1, \dots, \bar{p}_{2n}$ are the roundings to a grid of evenly spread points on the Euclidean unit sphere. By Lemma 3.2.6,

$$P_1 := \bar{P}_1^\circ = \{x \in \mathbb{R}^2 : \bar{p}_v^T x \le 1 \ \forall v \in [2n]\}$$

is an irredundant $\mathcal{H}$-presentation of P_1, the coding length of which is polynomial in m and k. By relabeling and scaling, we can achieve that

$$P_1 = \bigcap_{v\in[n]} \big(H_\le(a_v,\beta_v) \cap H_\le(c_v,\gamma_v)\big),$$

where the facets induced by a_v and c_v alternate along the boundary of P_1 (compare Figure 5.1), $\|a_v\|_\infty = \|c_v\|_\infty = 1$ and $\beta_v, \gamma_v > 0$ for all $v \in [n]$. Observe that also the sets $\{c_1, \dots, c_n\}$ and $\{a_1, \dots, a_n\}$ are 0-symmetric.
Now, compute all vertices of P_1 in time $O(n\log(n))$ (e.g. [60]), such that $P_1 = \text{conv}\{p_1, \dots, p_{2n}\}$ and define, for $v \in [n]$,

$$\varepsilon_v := \gamma_v - \max\{c_v^T p_w : c_v^T p_w \ne \gamma_v,\ w \in [2n]\} > 0,$$
$$\varepsilon := \frac{1}{10} \min\{\varepsilon_v : v \in [n]\} > 0$$

and

$$Q_1 := \{x \in \mathbb{R}^2 : a_v^T x \le \beta_v,\ c_v^T x \le \gamma_v - \varepsilon \ \forall v \in [n]\}.$$

For any $u \in \text{bd}(\mathbb{B}_\infty^2)$, let $p(u) \in P_1$ and $q(u) \in Q_1$ be vertices of the two polytopes with $u \in N(P_1, p(u)) = N(Q_1, q(u)) = \text{pos}\{c_v, a_w\}$ for some $v \in [n]$ and $w \in \{v-1, v\}$. Then, we can express $u = \lambda c_v + \mu a_w$ with $\lambda, \mu \ge 0$. By adding isolated vertices to the graph, we can assume without loss of generality that $n \ge 32$. Hence, we have $\|c_v - a_w\|_2 \le \frac{1}{4}$, by Lemma 3.2.3. Since $\|c_v\|_\infty = 1$, there exists $e \in \{\pm e_1, \pm e_2\}$ such that $e^T c_v = 1$. Since $\|c_v - a_w\|_2 \le \frac{1}{4}$ and $\|a_w\|_\infty = 1$, we can conclude that $e^T a_w > 0$. Thus,

$$1 \ge e^T u = \lambda e^T c_v + \mu e^T a_w \ge \lambda$$

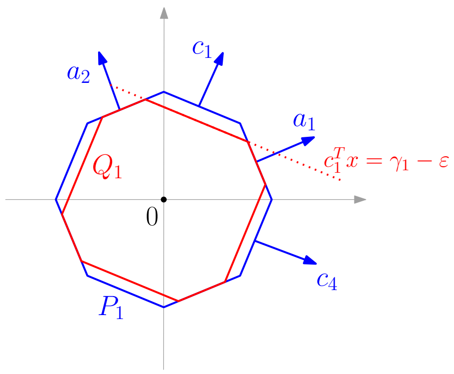

Figure 5.1: Illustration of the two polytopes $P_1, Q_1 \subseteq \mathbb{R}^2$ from the reduction in the proof of Lemma 5.2.19.

and equality holds if and only if $\lambda = 1$ and $\mu = 0$.
Therefore,

$$h(P_1, u) - h(Q_1, u) = u^T(p(u) - q(u)) = \lambda c_v^T(p(u) - q(u)) + \mu a_w^T(p(u) - q(u)) = \lambda\varepsilon \in [0, \varepsilon].$$

Hence,

$$\delta_1(P_1, Q_1) = \max_{u \in \mathbb{B}_\infty^2} (h(P_1, u) - h(Q_1, u)) = \varepsilon, \tag{5.8}$$

where the maximum in (5.8) is attained if and only if $u \in \{c_1, \dots, c_n\}$.
Using again Notation 3.2.2 from Chapter 3, let

$$P_2 := \bigcap_{i \in [k]} \bigcap_{v \in [n]} \left(H_\leq^i(a_v, \beta_v) \cap H_\leq^i(c_v, \gamma_v) \right) \subseteq \mathbb{R}^{2k}$$

and

$$Q := \bigcap_{i \in [k]} \bigcap_{v \in [n]} \left(H_\leq^i(a_v, \beta_v) \cap H_\leq^i(c_v, \gamma_v - \varepsilon) \right) \subseteq \mathbb{R}^{2k}.$$

By Lemma 5.2.18 and (5.8), we obtain $\delta_1(P_2, Q) = k\varepsilon$.
For $i, j \in [k]$ and $v, w \in [m]$ define

$$E_{vw}^{ij} := \{x \in \mathbb{R}^{2k} : -(\gamma_v + \gamma_w - \varepsilon) \leq c_v^T x_i + c_w^T x_j \leq \gamma_v + \gamma_w - \varepsilon\},$$

$$F_{vw}^{ij} := \{x \in \mathbb{R}^{2k} : -(\gamma_v + \gamma_w - \varepsilon) \leq c_v^T x_i - c_w^T x_j \leq \gamma_v + \gamma_w - \varepsilon\}$$

and, for $N := \binom{[m]}{2} \setminus E$, let

$$P := P_2 \cap \bigcap_{\substack{\{v,w\} \in N \\ i,j \in [k], i \neq j}} (E_{vw}^{ij} \cap F_{vw}^{ij}) \cap \bigcap_{\substack{v \in [m] \\ i,j \in [k], i \neq j}} (E_{vv}^{ij} \cap F_{vv}^{ij})$$

Now, by definition, P and Q are 0-symmetric and we claim that $\delta_1(P,Q) = k\varepsilon$ if and only if $G = ([m], E)$ has a clique of size k. Since CLIQUE is W[1]-complete [70, Theorem 6.1], this completes the hardness proof for HAUSDORFF$_1$-$\mathcal{H}$-$\mathcal{H}$.
Assume $\delta_1(P,Q) = k\varepsilon$. Since $Q \subseteq P$, there exists $u^* \in \mathbb{B}_\infty^{2k}$ such that $h(P,u^*) = h(Q,u^*) + k\varepsilon$. Since $P \subseteq P_2$, we obtain via the sharpness condition in (5.8), that $u^* = (c_{v_1}^T, \ldots, c_{v_k}^T)^T$ for some $v_1, \ldots, v_k \in [n]$. Hence, there is a vector $x^* \in P$ such that

$$c_{v_1}^T x_1^* = \gamma_{v_1}, \quad \ldots, \quad c_{v_k}^T x_k^* = \gamma_{v_k}.$$

If $c_{v_j} = \pm c_{v_i}$ for some $i \neq j \in [k]$, this would imply $c_{v_i}^T x_i^* \pm c_{v_i}^T x_j^* = 2\gamma_{v_i}$ which contradicts $x^* \in P \subseteq E_{v_i v_i}^{ij} \cap F_{v_i v_i}^{ij}$. Hence, defining for $i \in [k]$,

$$u_i := \begin{cases} v_i & \text{if } v_i \leq m \\ v_i - m & \text{else} \end{cases}$$

yields $|\{u_1, \ldots, u_k\}| = k$. In the same way, $c_{v_i}^T x_i^* + c_{v_j}^T x_j^* = \gamma_i + \gamma_j$ implies that $\{u_i, u_j\} \in E$. Thus, $\{u_1, \ldots, u_k\}$ is the set of vertices of a clique of size k in G.
If on the other hand $\{v_1, \ldots, v_k\} \subseteq [m]$ is the vertex set of a k-clique, then $u^* := (c_{v_1}^T, \ldots, c_{v_k}^T)^T$ satisfies $h(P,u^*) = h(Q,u^*) + k\varepsilon$ and therefore $\delta_1(P,Q) = k\varepsilon$. □

As an immediate corollary of Lemma 5.2.19, we obtain the hardness of computing the Hausdorff distance of two $\mathcal{H}$-polytopes in an arbitrary polytopal norm which is part of the input. Hence, the approximation of the unit ball of a p-norm by a polytopal norm as in Theorem 3.2.12 and Lemma 3.3.2 is not possible for the computation of the Hausdorff distance of two $\mathcal{H}$-presented polytopes.

Corollary 5.2.20 (*W[1]-hardness for polytopal norms*)
For 0-symmetric $\mathbb{B} \in \mathcal{C}_0^d$ and $P, Q \in \mathcal{C}^d$, let $\delta_{\mathbb{B}}(P,Q)$ denote the Hausdorff distance induced by the norm with unit ball $\mathbb{B}$. Then the problems

Input:	$d \in \mathbb{N}$, $\rho \in \mathbb{Q}$, $P, Q \subseteq \mathbb{R}^d$ rational polytopes in $\mathcal{H}$-presentation, polytopal unit ball $\mathbb{B} \subseteq \mathbb{R}^d$ in $\mathcal{H}$-presentation ($\mathcal{V}$-presentation, resp.)
Parameter:	d
Question:	Is $\delta_{\mathbb{B}}(P,Q) \geq \rho$?

are both W[1]-hard.

Proof.
Since we can construct an $\mathcal{H}$- and $\mathcal{V}$-presentation of $\mathbb{B}_1^d$ in FPT-time, we can reduce HAUSDORFF$_1$-$\mathcal{H}$-$\mathcal{H}$ to either of the above problems. □

Concerning mixed presentations, we can use the same reduction as in Lemma 5.2.17 and obtain the following hardness result.

Theorem 5.2.21 (*W[1]-Hardness of* HAUSDORFF$_p$-$\mathcal{V}$-$\mathcal{H}$)
For $p \in \mathbb{N} \setminus \{1\}$, HAUSDORFF$_p$-$\mathcal{V}$-$\mathcal{H}$ is W[1]-hard.

Finally, we remark that, regardless the norm used to measure the Hausdorff distance, HAUSDORFF-$\mathcal{V}$-$\mathcal{H}$ is closely linked to the following vertex enumeration problem.

Problem 5.2.22 (VERTEXENUMERATION)
Input: $d \in \mathbb{N}, n, m \in \mathbb{N}, a_1, \ldots, a_m \in \mathbb{Q}^d, \beta_1, \ldots, \beta_m \in \mathbb{Q}, p_1, \ldots, p_n \in \mathbb{Q}^d$
Question: Is $\operatorname{ext}\{x \in \mathbb{R}^d : a_i^T x \leq \beta_i \ \forall i \in [m]\} \setminus \{p_1, \ldots, p_n\} \neq \emptyset$?

Clearly, if $Q := \{x \in \mathbb{R}^d : a_i^T x \leq \beta_i \ \forall i \in [m]\}$ is bounded and $P := \operatorname{conv}\{p_1, \ldots, p_n\}$, then the question in VERTEXENUMERATION is equivalent to the question, whether $\delta(P, Q) > 0$ in any norm.
In [119], VERTEXENUMERATION is shown to be $\mathbb{NP}$-hard for a general (possibly unbounded) polyhedron Q. The question for the complexity of the same decision problem where Q is required to be bounded is open [119] and we refer to [154] for a recent account on this topic and related results.

5.3 Hausdorff Matching under Homothetics

The remaining part of this chapter is concerned with the Hausdorff matching problem under homothetics. As stated in the introduction, many variants of this problem have received considerable attention which mainly turns around the matching problem for finite point sets or sets of line segments in low dimensions. For an overview, we refer to [10]. For the problem of matching two convex bodies in arbitrary dimension, we first characterize optimal solutions in Theorem 5.3.8 in Section 5.3.1, which in turn implies a Helly-type theorem in Section 5.3.2. We combine these results with the insights of the previous section in order to derive polynomial time algorithms for different matching problems in Section 5.3.3.

In order to avoid degeneracies, we assume both bodies to have an interior point (the origin without loss of generality, i.e. $P, Q \in \mathcal{C}_0^d$). We state the problem and the results only for the Euclidean norm since it is probably the case of most interest and it offers some notational conveniences. Arbitrary norms can be handled with exactly the same ideas at the expense of a slightly more bulky notation.

Problem 5.3.1 (*Hausdorff matching under homothetics*)
For $P, Q \in \mathcal{C}_0^d$, find a scaled translate of P such that its Hausdorff distance to Q is minimized. In other words, solve the problem

$$\begin{aligned} \min \quad & \delta_2(\alpha P + c, Q) \\ s.t. \quad & c \in \mathbb{R}^d \\ & \alpha > 0. \end{aligned} \tag{5.9}$$

The optimal value of (5.9) will be denoted by

$$\delta_H(P, Q),$$

where the subscript "H" indicates that the Hausdorff distance (in Euclidean norm) of P and Q is measured up to homothetics.

A first glance at the objective function shows that the problem at hand is a convex optimization problem, so that (for a suitable presentation of the input) efficient algorithms do not seem out of reach.

Lemma 5.3.2 (*Convexity of the objective function*)
Let $P, Q \in \mathcal{C}^d$. The function

$$\begin{array}{rccl} f: & (0,\infty) \times \mathbb{R}^d & \to & [0,\infty) \\ & (\alpha, c) & \mapsto & \delta(\alpha P + c, Q) \end{array}$$

is convex.

Proof.
Let $\alpha_1, \alpha_2 \geq 0$ and $c_1, c_2 \in \mathbb{R}^d$, $\lambda \in [0,1]$ and for brevity $\rho_1 := f(\alpha_1, c_1), \rho_2 := f(\alpha_2, c_2)$. We have

$$\alpha_1 P + c_1 \subseteq Q + \rho_1 \mathbb{B} \quad \wedge \quad Q \subseteq \alpha_1 P + c_1 + \rho_1 \mathbb{B} \tag{5.10}$$

and

$$\alpha_2 P + c_2 \subseteq Q + \rho_2 \mathbb{B} \quad \wedge \quad Q \subseteq \alpha_2 P + c_2 + \rho_2 \mathbb{B}. \tag{5.11}$$

By combining (5.10) and (5.11), we obtain $(\lambda\alpha_1 + (1-\lambda)\alpha_2)P + \lambda c_1 + (1-\lambda)c_2 \subseteq Q + (\lambda\rho_1 + (1-\lambda)\rho_2)\mathbb{B}$ and $Q \subseteq (\lambda\alpha_1 + (1-\lambda)\alpha_2)P + \lambda c_1 + (1-\lambda)c_2 + (\lambda\rho_1 + (1-\lambda)\rho_2)\mathbb{B}$, yielding $f(\lambda\alpha_1 + (1-\lambda)\alpha_2, \lambda c_1 + (1-\lambda)c_2) \leq \lambda\rho_1 + (1-\lambda)\rho_2$. □

5.3.1 Optimality Criterion

The goal of this subsection is to give an optimality criterion for Problem 5.3.1 with two arbitrary convex bodies in the spirit of John's Theorem (Proposition 2.3.6) or Theorem 4.2.3.
Let $P, Q \in \mathcal{C}_0^d$. We say, that P is in *optimal homothetic position* with respect to Q if

$$\delta_H(P, Q) = \delta(P, Q). \tag{5.12}$$

In preparation of Theorem 5.3.8, we first prove a series of technical lemmas that provide required details such as normal cones of outer parallel bodies, several derivatives and a statement about subgradients of certain convex functions. The first proposition (see e.g. [150, Section 1.2]) allows us to express things in terms of projections on convex bodies in the Euclidean case and alleviates the notation.

Proposition 5.3.3 (*Projection onto convex bodies*)
Let $P \in \mathcal{C}^d$ and $x \in \mathbb{R}^d$. Then, there is a unique point $\Pi_P(x) \in P$ such that $\{p \in P : \|x - p\|_2 = d(x, P)\} = \{\Pi_P(x)\}$. Further, for all $p \in P$, we have

$$(x - \Pi_P(x))^T (p - \Pi_P(x)) \leq 0 \tag{5.13}$$

and the mapping

$$\Pi_P : \mathbb{R}^d \mapsto P;\ x \mapsto \Pi_P(x)$$

is Lipschitz-continuous with constant $L = 1$.

Lemma 5.3.4 (*Normal cone of outer parallel bodies*)
Let $P \in \mathcal{C}^d$ and $\rho > 0$. For $x \in \operatorname{bd}(P + \rho\mathbb{B}_2^d)$, the normal cone of $P + \rho\mathbb{B}_2^d$ in x is given by

$$N(P + \rho\mathbb{B}_2^d, x) = \operatorname{pos}\big\{x - \Pi_P(x)\big\}.$$

If further $\rho' > 0$, then

$$P + \rho'\mathbb{B}_2^d = \bigcap_{x \in \operatorname{bd}(P+\rho\mathbb{B}_2^d)} H_{\leq}\Big(x - \Pi_P(x),\ (x - \Pi_P(x))^T \Pi_P(x) + \rho\rho'\Big).$$

Proof.
Since $\Pi_P(x) + \rho\mathbb{B}_2^d \subseteq P + \rho\mathbb{B}_2^d$, we have

$$H_{\leq}(x - \Pi_P(x), 0) = T(\Pi_P(x) + \rho\mathbb{B}_2^d, x) \subseteq T(P + \rho\mathbb{B}_2^d, x). \tag{5.14}$$

Since $P + \rho\mathbb{B}_2^d$ is convex and hence the tangential cone $T(P + \rho\mathbb{B}_2^d, x)$ is at most a half-space, we conclude that the inclusion in (5.14) is actually an equality and that

$$N(P + \rho\mathbb{B}_2^d, x) = T(\Pi_P(x) + \rho\mathbb{B}_2^d, x)^\circ = \operatorname{pos}\{x - \Pi_P(x)\}.$$

For $\rho' > 0$, the above yields

$$\begin{aligned}
P + \rho'\mathbb{B}_2^d &= \bigcap_{x' \in \operatorname{bd}(P+\rho'\mathbb{B}_2^d)} H_{\leq}\Big(x' - \Pi_P(x'), \big(x' - \Pi_P(x')\big)^T x'\Big) \\
&= \bigcap_{x \in \operatorname{bd}(P+\rho\mathbb{B}_2^d)} H_{\leq}\Big(x - \Pi_P(x), \big(x - \Pi_P(x)\big)^T \Big(\Pi_P(x) + \frac{\rho'}{\rho}(x - \Pi_P(x))\Big)\Big) \\
&= \bigcap_{x \in \operatorname{bd}(P+\rho\mathbb{B}_2^d)} H_{\leq}\Big(x - \Pi_P(x), \big(x - \Pi_P(x)\big)^T \Pi_P(x) + \rho\rho'\Big).
\end{aligned}$$

□

Lemma 5.3.5 (*Differentiating* $d_2(\cdot, P)$)
Let $P \in \mathcal{C}^d$ and define

$$g_P : \mathbb{R}^d \to [0, \infty); \quad x \mapsto d_2(x, P) = \min\{\|x - p\|_2 : p \in P\}.$$

The mapping g_P is convex and, for $x \in \mathbb{R}^d \setminus P$, it is continuously differentiable in x with

$$\nabla g_P(x) = \frac{x - \Pi_P(x)}{\|x - \Pi_P(x)\|_2}.$$

Proof.
By Lemma 5.2.2, g_P is convex regardless of the norm. Since, for $x \in \mathbb{R}^d \setminus P$, the function g_P does not attain its minimum in x, we can apply Theorem 23.7 from [147] in order to obtain

$$\operatorname{pos}(\partial g_P(x)) = N(P + d_2(x,P)\mathbb{B}_2^d, x).$$

Hence, by Lemma 5.3.4, $\partial g(x) = \{\lambda(x - \Pi_P(x))\}$ for some $\lambda > 0$. Using the linearity of $\|\cdot\|_2$ on rays emanating from the origin, we obtain $\lambda = \|x - \Pi_P(x)\|_2^{-1}$.
As $|\partial g(x)| = 1$ for all $x \in \mathbb{R}^n \setminus P$, Theorem 25.1 and Corollary 25.5.1 in [147] yield that g is continuously differentiable in x and that $\nabla g_P(x) = \|x - \Pi_P(x)\|_2{}^{-1}(x - \Pi_P(x))$. □

Lemma 5.3.6 (*Differentiating the objective*)
Let $P, Q \in \mathcal{C}^d$ and define, for $p \in P$,

$$f_p : (0,\infty) \times \mathbb{R}^d \to [0,\infty); \quad (\alpha, c) \mapsto d_2(\alpha p + c, Q)$$

and, for $q \in Q$,

$$f_q : (0,\infty) \times \mathbb{R}^d \to [0,\infty); \quad (\alpha, c) \mapsto d_2(q, \alpha P + c).$$

Then f_p and f_q are convex.
Further, if $p \in \mathbb{R}^d \setminus Q$,

$$\nabla f_p(1,0) = \frac{1}{\|p - \Pi_Q(p)\|_2} \begin{pmatrix} (p - \Pi_Q(p))^T p \\ p - \Pi_Q(p) \end{pmatrix} \tag{5.15}$$

and, if $q \in \mathbb{R}^d \setminus P$,

$$\nabla f_q(1,0) = -\frac{1}{\|q - \Pi_P(q)\|_2} \begin{pmatrix} (q - \Pi_P(q))^T \Pi_P(q) \\ q - \Pi_P(q) \end{pmatrix} \tag{5.16}$$

Proof.
First, observe that for $x, y \in \mathbb{R}^d$ and $K, L \in \mathcal{C}^d$

$$d(x + y, K + L) \leq d(x, K) + d(y, L),$$

as $\Pi_K(x) + \Pi_L(y) \in K + L$. Now, for the convexity of f_q for $q \in Q$, let $\alpha_1, \alpha_2 > 0$, $c_1, c_2 \in \mathbb{R}^d$ and $\lambda \in [0,1]$. We have

$$f_q\big(\lambda\alpha_1 + (1-\lambda)\alpha_2, \lambda c_1 + (1-\lambda)c_2\big) = d\big(\lambda q + (1-\lambda)q, \lambda(\alpha_1 P + c_1) + (1-\lambda)(\alpha_2 P + c_2)\big)$$

$$\leq d\big(\lambda q, \lambda(\alpha_1 P + c_1)\big) + d\big((1-\lambda)q, (1-\lambda)(\alpha_2 P + c_2)\big) = \lambda f_q(\alpha_1, c_1) + (1-\lambda) f_q(\alpha_2, c_2),$$

which shows the convexity of f_q. For $p \in P$, the function f_p is a composition of a linear function and a convex function and hence convex.

A direct application of the chain rule together with Lemma 5.3.5 yields (5.15). In order to differentiate f_q, we express $f_q(\alpha, c) = d(q, \alpha P + c) = \alpha g_P(\frac{1}{\alpha}(q - c))$ with g_P defined as in Lemma 5.3.5. Differentiating this expression yields

$$\frac{\partial f_q}{\partial c}(\alpha, c) = -\alpha \nabla g_P\left(\frac{1}{\alpha}(q - c)\right)$$

and

$$\frac{\partial f_q}{\partial \alpha}(\alpha, c) = g_P\left(\frac{1}{\alpha}(q - c)\right) - \frac{1}{\alpha}\nabla g_P\left(\frac{1}{\alpha}(q - c)\right)^T (q - c).$$

Plugging in $\alpha = 1$ and $c = 0$ and using $g_P(q)^2 = (q - \Pi_P(q))^T(q - \Pi_P(q))$, we obtain (5.16). □

We remark that the convexity of f_p and f_q for all $p \in P$ and $q \in Q$ can be used as an alternative proof for Lemma 5.3.2 using the definition as maximum of convex functions in (5.3) instead of the containment formulation in (5.4). The next lemma now investigates the subdifferential of such a supremum of uncountably many convex functions.

Lemma 5.3.7 (*Subgradient of the supremum of convex functions*)
Let I be a (possibly uncountable) index set and $f_i : \mathbb{R}^d \to \mathbb{R}$ convex for $i \in I$. Let

$$f : \mathbb{R}^d \to \mathbb{R}; \quad f(x) := \sup\{f_i(x) : i \in I\}$$

and for $x \in \mathbb{R}^d$ define $\mathcal{A}(x) := \{i \in I : f_i(x) = f(x)\}$. Then, for all $x \in \mathbb{R}^d$,

$$\mathrm{cl}\Big(\mathrm{conv}\Big(\bigcup_{i \in \mathcal{A}(x)} \partial f_i(x)\Big)\Big) \subseteq \partial f(x). \tag{5.17}$$

Proof.
Let $x \in \mathbb{R}^d$, $i \in \mathcal{A}(x)$ and $a \in \partial f_i(x)$. Let further $\bar{a} := \binom{a}{-1}$ and $\beta := \bar{a}^T \binom{x}{f(x)}$. Then, $H_{=}(\bar{a}, \beta)$ supports $\mathrm{epi}(f_i)$ and $\mathrm{epi}(f_i) \subseteq H_{\leq}(\bar{a}, \beta)$ (cf. [147, Section 23]).
Since $\mathrm{epi}(f) = \bigcap_{i \in I} \mathrm{epi}(f_i)$, we have $\mathrm{epi}(f) \subseteq H_{\leq}(\bar{a}, \beta)$; since $i \in \mathcal{A}(x)$, we also have that $H_{=}(\bar{a}, \beta)$ supports $\mathrm{epi}(f)$ in $\binom{x}{f(x)}$. Thus, $a \in \partial f(x)$. Since $\partial f(x)$ is convex and closed by [147, Theorem 23.4], the inclusion in (5.17) follows. □

We are now ready to prove an optimality criterion for Problem 5.3.1 for convex bodies $P, Q \in \mathcal{C}_0^d$. The conditions of Theorem 5.3.8 are also illustrated in Figure 5.2.

Theorem 5.3.8 (*Optimality criterion for Hausdorff matching*)
Let $P, Q \in \mathcal{C}_0^d$. Then, P is in optimal homothetic position with respect to Q, if and only if there are $\rho \geq 0$ and $R \subseteq P, S \subseteq Q$ with $|R| + |S| \leq d + 2$ such that the following three conditions hold:

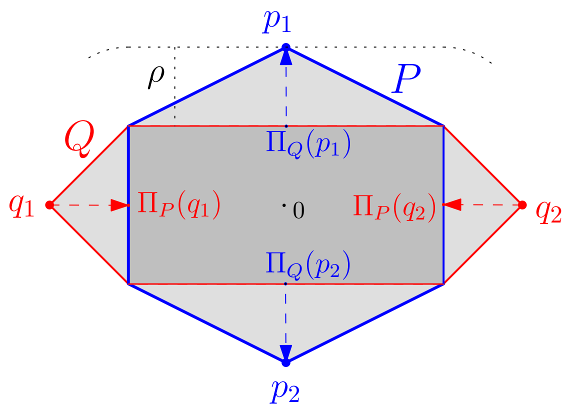

Figure 5.2: The conditions of Theorem 5.3.8. The blue polytope $P \subseteq \mathbb{R}^2$ is in optimal position with respect to the red polytope $Q \subseteq \mathbb{R}^2$. Conditions (1) – (3) of Theorem 5.3.8 are verified with $R = \{p_1, p_2\}$, $S = \{q_1, q_2\}$ and ρ as indicated.

(1) $P \subseteq Q + \rho\mathbb{B}_2^d$ and $Q \subseteq P + \rho\mathbb{B}_2^d$

(2) $d_2(p, Q) = \rho \; \forall p \in R$ and $d_2(q, P) = \rho \; \forall q \in S$

(3) $0 \in \operatorname{conv}\left(\left\{\begin{pmatrix}(p - \Pi_Q(p))^T p \\ p - \Pi_Q(p)\end{pmatrix} : p \in R\right\} \cup \left\{\begin{pmatrix}(\Pi_P(q) - q)^T \Pi_P(q) \\ \Pi_P(q) - q\end{pmatrix} : q \in S\right\}\right).$

Proof.
If $P = Q$, then conditions (1) – (3) are trivially necessary and sufficient with the choice $\rho = 0$ and any $R, S \subseteq P$ with $|R| + |S| \leq d + 2$. We will henceforth assume that $P \neq Q$ and consequently $\rho > 0$.
We first show the sufficiency of the conditions: Assume that conditions (1) – (3) hold for some $\rho > 0$, $R \subseteq P$ and $S \subseteq Q$ with $|R| + |S| \leq d + 2$. Let

$$f : (0, \infty) \times \mathbb{R}^d \to [0, \infty); \quad (\alpha, c) \mapsto \delta_2(\alpha P + c, Q),$$

and, as in Lemma 5.3.6, let $f_p(\alpha, c) = d_2(\alpha p + c, Q)$ for $p \in P$, and $f_q(\alpha, c) = d_2(q, \alpha P + c)$ for $q \in Q$. Then, (1) and (2) imply

$$\rho = f(1, 0) = \max\big(\{f_p(1, 0) : p \in P\} \cup \{f_q(1, 0) : q \in Q\}\big)$$

with $(R \cup S) \subseteq \mathcal{A}((1, 0))$ in the notation of Lemma 5.3.7. Hence, by Lemmas 5.3.6 and 5.3.7, condition (3) yields $0 \in \partial f(1, 0)$ which in turn implies that $\delta_2(P, Q) \leq \delta_2(\alpha P + c, Q)$ for all $\alpha > 0$ and $c \in \mathbb{R}^d$.
It remains to show that the conditions are also necessary. Although the procedure is slightly more involved here, we will argue in the same way as in the respective part of the proof of Theorem 4.2.3.

Let P be in optimal position with respect to Q. Choose $\rho := \delta(P,Q) > 0$ and define

$R' := \{p \in P : d(p,Q) = \rho\} \subseteq \mathrm{bd}(Q+\rho\mathbb{B}_2^d)$ and $S' := \{q \in Q : d(q,P) = \rho\} \subseteq \mathrm{bd}(P+\rho\mathbb{B}_2^d)$.

We have $R' \neq \emptyset$ and $S' \neq \emptyset$, because one of these sets being empty would imply that P is not optimally scaled.
For $x \in \mathbb{R}^d$, define further

$$a_Q(x) := x - \Pi_Q(x) \quad \text{and} \quad a_P(x) := x - \Pi_P(x)$$

and let

$$A := \left\{ \begin{pmatrix} a_Q(p)^T p \\ a_Q(p) \end{pmatrix} : p \in R' \right\} \quad B := \left\{ \begin{pmatrix} a_P(q)^T \Pi_P(q) \\ a_P(q) \end{pmatrix} : q \in S' \right\}.$$

We show $0 \in \mathrm{conv}(A \cup (-B))$. That R' and S' can be reduced to subsets $R \subseteq R'$, $S \subseteq S'$ with $|R| + |S| \leq d+2$ then follows from Carathéodory's Theorem (see e.g. [59]).
For a contradiction, assume that $0 \notin \mathrm{conv}(A \cup (-B))$. Then, 0 can be strictly separated from $\mathrm{conv}(A \cup (-B))$, i.e. there exists $(\eta, y^T)^T \in \mathbb{R}^{d+1}$ such that

$$\eta a_Q(p)^T p + a_Q(p)^T y \leq -1 \ \forall p \in R' \quad \text{and} \quad \eta a_P(q)^T \Pi_P(q) + a_P(q)^T y \geq 1 \ \forall q \in S'. \tag{5.18}$$

We will show that there exists $\lambda > 0$ such that $\delta((1+\lambda\eta)P + \lambda y, Q) < \delta(P,Q)$, which contradicts the optimality of $(1,0)$.
For this purpose, let

$$Q_\rho := Q + \rho\mathbb{B}_2^d = \bigcap_{x \in \mathrm{bd}(Q_\rho)} H_{\leq}(a_Q(x), a_Q(x)^T x),$$

where the second presentation is obtained via Lemma 5.3.4.
Define $N := \left\{(x,p) \in \mathrm{bd}(Q_\rho) \times P : \eta a_Q(x)^T p + a_Q(x)^T y \geq 0\right\}$ and further

$$f : N \to \mathbb{R}; \quad (x,p) \mapsto a_Q(x)^T x - a_Q(x)^T p$$

For $(x,p) \in N$, (5.18) yields $\left(a_Q(x)^T p, a_Q(x)^T\right)^T \notin A$, which implies $p \neq x$. Together with $p \in Q + \rho\mathbb{B}_2^d$, this yields $a_Q(x)^T p < a_Q(x)^T x$. Thus, for all $(x,p) \in N$, $f(x,p) > 0$. Since f is continuous and N is compact, there exists $\varepsilon_1 > 0$ such that

$$f(x,p) \geq \varepsilon_1 \ \forall (x,p) \in N \tag{5.19}$$

Now, for $(x,p) \in \left(\mathrm{bd}(Q_\rho) \times P\right) \setminus N$ and $\lambda > 0$, we have

$$a_Q(x)^T((1+\lambda\eta)p + \lambda y) = \underbrace{a_Q(x)^T p}_{\leq a_Q(x)^T x} + \lambda \underbrace{(\eta a_Q(x)^T p + a_Q(x)^T y)}_{<0} < a_Q(x)^T x.$$

By (5.19) and the boundedness of Q_ρ and P, we can choose $\lambda > 0$ sufficiently small such that for all $(x,p) \in N$

$$a_Q(x)^T((1+\lambda\eta)p + \lambda y) = \underbrace{a_Q(x)^T p}_{\leq a_Q(x)^T x - \varepsilon_1} + \lambda(\eta a_Q(x)^T p + a_Q(x)^T y) < a_Q(x)^T x.$$

Thus,

$$(1+\lambda\eta)P + \lambda y \subseteq \operatorname{int}(Q_\rho) \quad \text{for all } \lambda > 0 \text{ sufficiently small.} \tag{5.20}$$

Assume further that $\lambda > 0$ is sufficiently small such that $\lambda\eta > -1$ and let

$$P + \frac{\rho}{1+\lambda\eta}\mathbb{B}_2^d = \bigcap_{x \in \operatorname{bd}(P+\rho\mathbb{B}_2^d)} H_\leq \left(a_P(x), a_P(x)^T \Pi_P(x) + \frac{\rho^2}{1+\lambda\eta} \right),$$

where again the $\mathcal{H}$-presentation is obtained via Lemma 5.3.4.
Define $M := \left\{ x \in \operatorname{bd}(P+\rho\mathbb{B}_2^d) : \eta a_P(x)^T \Pi_P(x) + a_P(x)^T y \leq 0 \right\}$ and further

$$g : M \times Q \to \mathbb{R}; \quad (x,q) \mapsto a_P(x)^T \Pi_P(x) + \rho^2 - a_P(x)^T q.$$

For $x \in M$, (5.18) yields $\left(a_P(x)^T \Pi_P(x), a_P(x)^T\right)^T \notin B$, which implies that there is no $q \in Q$ such that $a_P(x)^T q = a_P(x)^T \Pi_P(x) + \rho^2$. Together with $Q \subseteq P + \rho\mathbb{B}_2^d$, we obtain $g(x,q) > 0$ for all $x \in M$ and $q \in Q$. Since $M \times Q$ is compact and g continuous, there exists $\varepsilon_2 > 0$ such that $g(x,q) \geq \varepsilon_2$ for all $x \in M$ and $q \in Q$.
Hence, we can again choose $\lambda > 0$ sufficiently small such that for all $x \in M$ and $q \in Q$, we have

$$a_P(x)^T q < a_P(x)^T \Pi_P(x) + \rho^2 + \lambda(\eta a_P(x)^T \Pi_P(x) + a_P(x)^T y) \tag{5.21}$$

and (5.21) is also fulfilled for $x \in \operatorname{bd}(P+\rho\mathbb{B}_2^d) \setminus M$, $\lambda > 0$ and $q \in Q$. Rearranging (5.21) shows that it is equivalent to

$$\frac{1}{1+\lambda\eta}(Q - \lambda y) \subseteq \operatorname{int}\left(P + \frac{\rho}{1+\lambda\eta}\mathbb{B}_2^d\right) \Longleftrightarrow Q \subseteq \operatorname{int}\left((1+\lambda\eta)P + \lambda y + \rho\mathbb{B}_2^d\right) \tag{5.22}$$

Together, (5.20) and (5.22) show that for $\lambda > 0$ sufficiently small, $\delta((1+\lambda\eta)P + \lambda y, Q) < \delta(P,Q)$, which is the desired contradiction.

□

5.3.2 Helly-Type Properties

As in Chapter 4, we investigate Helly-properties of the Hausdorff matching problems as a first step towards algorithmic and approximation questions. For the case of $\mathcal{V}$-polytopes, this question has already received attention and it was shown in [14] that Hausdorff matching with $\mathcal{V}$-polytopes can be formulated as a convex program in order to show that it is a Generalized Linear Program [131, 152] of combinatorial dimension $\delta = d + 2$.
With Theorem 5.3.8 at hand, we can improve this result and state a generalized version of this Helly-type theorem which holds true for arbitrary convex bodies.

Corollary 5.3.9 (*Helly-type theorem for Problem 5.3.1*)
Let $P, Q \in \mathcal{C}_0^d$ and for $R \subseteq P$, $S \subseteq Q$, define

$$\begin{array}{rlllll} \rho(R,S) := & \min & \rho & & & \\ & s.t. & \alpha p + c & \in & Q + \rho\mathbb{B}_2^d & \forall p \in R \\ & & q & \in & \alpha P + c + \rho\mathbb{B}_2^d & \forall q \in S \\ & & c \in \mathbb{R}^d & & & \\ & & \alpha, \rho \geq 0. & & & \end{array} \tag{5.23}$$

Then for any $\rho^* \geq 0$,

$$\rho(P,Q) \leq \rho^* \quad \Longleftrightarrow \quad \rho(R,S) \leq \rho^* \ \ \forall R \subseteq P, S \subseteq Q, |R| + |S| \leq d+2.$$

In addition to restricting the number of constraints, we can also state a Helly-type theorem which restricts the number of vertices of a polytope that need to be considered:

Theorem 5.3.10 (*0-core-sets for Problem 5.3.1*)
Let $P, Q \in \mathcal{C}_0^d$. There are subsets $R \subseteq \mathrm{ext}(P)$ and $S \subseteq \mathrm{ext}(Q)$ such that $|R| + |S| \leq d(d+2)$ and

$$\delta_H(\mathrm{conv}(R), \mathrm{conv}(S)) = \delta_H(P, Q).$$

Proof.
Assume without loss of generality that $\delta_H(P,Q) = \delta(P,Q)$. Corollary 5.3.9 implies that there are subsets $R' \subseteq P$ and $S' \subseteq Q$ with $|R'| + |S'| \leq d+2$ such that in (5.23) only the containment constraints for $p \in R'$ and $q \in S'$ are necessary.
For $p \in R'$, we have that $Q \subseteq H_{\leq}(p - \Pi_Q(p), (p - \Pi_Q(p))^T \Pi_Q(p))$ and $\Pi_Q(p) \in H_{=}(p - \Pi_Q(p), (p - \Pi_Q(p))^T \Pi_Q(p))$ by Proposition 5.3.3. Now, Carathéodory's Theorem (see e.g. [59]) applied to $Q \cap H_{=}(p - \Pi_Q(p), (p - \Pi_Q(p))^T \Pi_Q(p))$ yields the existence of $q_1^p, \ldots, q_d^p \in \mathrm{ext}(Q \cap H_{=}(p - \Pi_Q(p), (p - \Pi_Q(p))^T \Pi_Q(p))) \subseteq \mathrm{ext}(Q)$ such that $\Pi_Q(p) \in \mathrm{conv}\{q_1^p, \ldots, q_d^p\}$. Defining

$$S := S' \cup \bigcup_{p \in R'} \{q_1^p, \ldots, q_d^p\}$$

assures that $d(p, \mathrm{conv}(S)) = d(p, Q)$ for all $p \in R'$. Applying the same argument to R' shows that $|R| + |S| \leq d(d+2)$. □

Remark. The restriction to extreme points in Theorem 5.3.10 aims at the algorithmic application, where P and Q are specified as $\mathcal{V}$-polytopes and $\mathrm{ext}(P)$ and $\mathrm{ext}(Q)$ are easily available. If one drops this restriction, the bound $|S| + |R| \leq d(d+2)$ can be improved to $|S| + |R| \leq 2(d+2)$.

5.3.3 Exact Algorithms and Approximations

The following two lemmas show that, in case the input polytopes are specified as $\mathcal{V}$-polytopes, Problem 5.3.1 can be formulated as a Linear or Second Order Cone Program depending on the choice of the norm. These cases can thus be approximated to any accuracy or even solved exactly in polynomial time.

Lemma 5.3.11
Let $P := \text{conv}\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ and $Q := \text{conv}\{q_1, \ldots, q_m\} \subseteq \mathbb{R}^d$. If $\mathbb{B} = \{x \in \mathbb{R}^d : u_k^T x \leq 1,\ \forall k \in [r]\}$, then $\delta_H(P, Q)$ can be computed by solving the following Linear Program:

$$\begin{array}{lllll} \min & \rho & & & \\ s.t. & u_k^T(\alpha p_i + c - \sum\limits_{j=1}^{m} \lambda_{ij} q_j) & \leq & \rho & \forall i \in [n] \\ & u_k^T(q_j - \sum\limits_{i=1}^{n} \mu_{ij} p_i - c) & \leq & \rho & \forall j \in [m] \\ & \sum\limits_{j=1}^{m} \lambda_{ij} = 1 & & & \forall i \in [n] \\ & \sum\limits_{i=1}^{n} \mu_{ji} = \alpha & & & \forall j \in [m] \\ & \lambda_{ij}, \mu_{ji} \geq 0 & & & \forall i \in [n], j \in [m]. \end{array} \tag{5.24}$$

Lemma 5.3.12
Let $P := \text{conv}\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ and $Q := \text{conv}\{q_1, \ldots, q_m\} \subseteq \mathbb{R}^d$. If $\mathbb{B} = \mathbb{B}_2^d$, then $\delta_H(P, Q)$ can be approximated to any accuracy by solving the following SOCP:

$$\begin{array}{lllll} \min & \rho & & & \\ s.t. & \|\alpha p_i + c - \sum\limits_{j=1}^{m} \lambda_{ij} q_j\|_2 & \leq & \rho & \forall i \in [n] \\ & \|q_j - \sum\limits_{i=1}^{n} \mu_{ij} p_i - c\|_2 & \leq & \rho & \forall j \in [m] \\ & \sum\limits_{j=1}^{m} \lambda_{ij} = 1 & & & \forall i \in [n] \\ & \sum\limits_{i=1}^{n} \mu_{ji} = \alpha & & & \forall j \in [m] \\ & \lambda_{ij}, \mu_{ji} \geq 0 & & & \forall i \in [n], j \in [m]. \end{array} \tag{5.25}$$

Theorems 5.2.16 and 5.2.21 show that simple evaluation of the Hausdorff distance is W[1]-hard, if at least one $\mathcal{H}$-presented polytope is involved. Nonetheless, we can at least give a polynomial time approximation algorithm in these cases. This can be done via the so-called "Reference Point Method" developed in [5]. The general framework in [5] was devised to produce approximate solutions for the Hausdorff matching problem under similarities for compact sets in $\mathbb{R}^d$. However, since it requires the computation of the diameter of the involved sets, and diameter computation of $\mathcal{H}$-polytopes is also W[1]-hard (Corollary 3.4.3), we can not apply it directly. Hence, in the following, we describe a version of Reference Point Matching which is specially tailored to our needs: It is only formulated for matching under homothetics and replaces the diameter of a polytope by the diameter of its bounding box. The performance guarantee of this version is slightly worse but in the same order of magnitude as the one in [5].

The general idea of the algorithm is to replace the polytopes that are to be matched by reference points of the polytopes and to compute a transformation that is only based on the knowledge of these points (see [5] for a comprehensive study of the method). For the present purpose, we choose two opposite vertices of the bounding boxes of the two input polytopes P and Q.

Definition 5.3.13 (*Reference points*)
For $P \in \mathcal{C}_0^d$, let

$$r(P) := (h(P,-e_1),\dots,h(P,-e_d)) \in \mathbb{R}^d, \quad s(P) := (h(P,e_1),\dots,h(P,e_d))^T \in \mathbb{R}^d$$

the "lower left" and "upper right" corner of the bounding box of P and

$$D(P) := \|s(P) - r(P)\|_2$$

the diameter of this box.

We first state some elementary properties of the interplay between the reference points of P and Q and the Hausdorff distance $\delta_2(P,Q)$.

Lemma 5.3.14 (*Properties of the reference points*)

a) Let $P \in \mathcal{C}_0^d$ and $\alpha_1, \alpha_2 > 0$. Then, $\delta_2\big(\alpha_1(P - r(P)), \alpha_2(P - r(P))\big) \le |\alpha_1 - \alpha_2| D(P)$.

b) Let $P, Q \in \mathcal{C}_0^d$. Then $|D(P) - D(Q)| \le 2\sqrt{d}\delta_2(P,Q)$.

c) Let $P, Q \in \mathcal{C}_0^d$. Then $\|r(P) - r(Q)\|_2 \le \sqrt{d}\delta_2(P,Q)$.

Proof.

a) Let without loss of generality $r(P) = 0$, and $p \in P$. Then, $d(\alpha_1 p, \alpha_2 P) \le \|\alpha_1 p - \alpha_2 p\|_2 \le |\alpha_1 - \alpha_2| D(P)$ and the same argument shows that also $\max\{d(x, \alpha_1 P) : x \in \alpha_2 P\} \le |\alpha_1 - \alpha_2| D(P)$.

b) By Lemma 5.2.6, for $i \in [d]$, we have $|h(P, \pm e_i) - h(Q, \pm e_i)| \le \delta_2(P,Q)$ and hence $|D(P) - D(Q)| \le 2\sqrt{d}\delta_2(P,Q)$.

c) The statement also follows from $|h(P, -e_i) - h(Q, -e_i)| \le \delta_2(P,Q)$ for all $i \in [d]$.

□

If we compute a homothetic transformation which only relies on our two reference points, we can give the following performance guarantee.

Lemma 5.3.15 (*Approximation by reference points*)
Let $P, Q \in \mathcal{C}_0^d$ and

$$\bar{\alpha} := \frac{D(Q)}{D(P)} \quad \text{and} \quad \bar{c} := r(Q) - \bar{\alpha} r(P)$$

Then,

$$\delta_2\big(\bar{\alpha}P + \bar{c},\ Q\big) \leq \left(3\sqrt{d} + 1\right) \delta_H(P, Q)$$

Proof.
Let $\alpha^* > 0$, $c^* \in \mathbb{R}^d$ and $\rho^* \geq 0$ such that $\rho^* = \delta_H(P, Q) = \delta_2(\alpha^* P + c^*, Q)$. Then, by Lemma 5.3.14c),

$$\delta_2\Big(\alpha^* P + c^* + r(Q) - r(\alpha^* P + c^*),\ Q\Big) \leq \left(1 + \sqrt{d}\right) \rho^*. \tag{5.26}$$

Moreover, by using Lemma 5.3.14a) and b),

$$\begin{aligned} & \delta_2\Big(\bar{\alpha}(P - r(P)) + r(Q), \alpha^*(P - r(P)) + r(Q)\Big) \\ \leq \quad & |\alpha^* - \bar{\alpha}| D(P) = |\alpha^* D(P + c^*) - D(Q)| \leq 2\sqrt{d}\rho^*. \end{aligned} \tag{5.27}$$

By combining (5.26) and (5.27), we obtain

$$\begin{aligned} \delta_2(\bar{\alpha}P + \bar{c}, Q) \quad \leq \quad & \delta_2\Big(\bar{\alpha}(P - r(P)) + r(Q),\ \alpha^*(P - r(P)) + r(Q)\Big) \\ & + \delta_2\Big(\alpha^* P + c^* + r(Q) - r(\alpha^* P + c^*),\ Q\Big) \\ \leq \quad & (3\sqrt{d} + 1)\rho^*. \end{aligned}$$

□

We close this chapter by showing that the above algorithms can in principle also be applied for special variants of tomographic reconstruction problems. This is motivated by the application problem in Section 1.2.2 in the introduction, where for one of the two convex bodies only two projections in fixed directions are accessible. We therefore formulate the following variant of Problem 5.3.1.

Problem 5.3.16 (*Hausdorff matching for projections*)
Let $P \in \mathcal{C}_0^d$, and $v_1, v_2 \in \mathbb{S}_2^{d-1}$ two fixed directions. For $i = 1, 2$, let $Q_i \subseteq v_i^\perp$ be a convex body and denote by

$$\Pi_i : \mathbb{R}^d \to v_i^\perp$$

the orthogonal projection along direction v_i onto $v_i^\perp$. The Hausdorff matching problem for projections consists of finding a homothetic transformation of the body P such that its projections along v_1 and v_2 best fit Q_1 and Q_2. More precisely, the task is to solve the following optimization problem

$$\begin{array}{ll} \min & \max_{i=1,2} \delta_2(\Pi_i(\alpha P + c), Q_i) \\ s.t. & c \in \mathbb{R}^d \\ & \alpha > 0, \end{array} \tag{5.28}$$

where the optimal solution of (5.28) will be denoted by $\delta_P(P, Q_1, Q_2)$.

Since the projection mappings Π_1, Π_2 are linear operators that are fixed in advance and not subject to optimization, we immediately obtain the following lemma as an adaption of Lemma 5.3.12.

Lemma 5.3.17 (*Solving Problem 5.3.16 for $\mathcal{V}$-polytopes*)
Let $P := \text{conv}\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$, and $v_1, v_2 \in \mathbb{S}_2^{d-1}$ two fixed rational directions. Further, for $i = 1, 2$, let $Q_i := \text{conv}\{q_1^{(i)}, \ldots, q_{m_i}^{(i)}\} \subseteq v_i^\perp$ polytopes in rational $\mathcal{V}$-presentation, let $I \in \mathbb{R}^{d \times d}$ denote the d-dimensional identity matrix and $\Pi_i := (I - v_i v_i^T) \in \mathbb{R}^{d \times d}$. Then $\delta_P(P, Q_1, Q_2)$ can be approximated to any accuracy by solving the following SOCP:

$$\begin{array}{llll} \min & \rho & & \\ s.t. & \left\| \alpha \Pi_1 p_i + \Pi_1 c - \sum_{j=1}^{m_1} \lambda_{ij}^{(1)} q_j^{(1)} \right\|_2 & \leq \rho & \forall i \in [n] \\ & \left\| \alpha \Pi_2 p_i + \Pi_2 c - \sum_{j=1}^{m_2} \lambda_{ij}^{(2)} q_j^{(2)} \right\|_2 & \leq \rho & \forall i \in [n] \\ & \left\| q_j^{(1)} - \sum_{i=1}^{n} \mu_{ij}^{(1)} \Pi_1 p_i - \Pi_1 c \right\|_2 & \leq \rho & \forall j \in [m_1] \\ & \left\| q_j - \sum_{i=1}^{n} \mu_{ij}^{(2)} \Pi_2 p_i - \Pi_2 c \right\|_2 & \leq \rho & \forall j \in [m_2] \\ & \sum_{j=1}^{m_1} \lambda_{ij}^{(1)} = \sum_{j=1}^{m_2} \lambda_{ij}^{(2)} = 1 & & \forall i \in [n] \\ & \sum_{i=1}^{n} \mu_{ji}^{(1)} = \alpha & & \forall j \in [m_1] \\ & \sum_{i=1}^{n} \mu_{ji}^{(2)} = \alpha & & \forall j \in [m_2] \\ & \lambda^{(1)}, \lambda^{(2)} \mu^{(1)}, \mu^{(2)} \geq 0. & & \end{array} \tag{5.29}$$

Remark 5.3.18 (*Possible generalizations*)
Of course, Problem 5.3.16 and the SOCP (5.29) can immediately be generalized to cope with more than two projections, projections of lower dimensions, other polytopal norms, or other objective functions such as the average of the Hausdorff distances in the projections. However, we restrict the above presentation to the simplest case, since its main purpose is to point out the general feasibility of this approach.

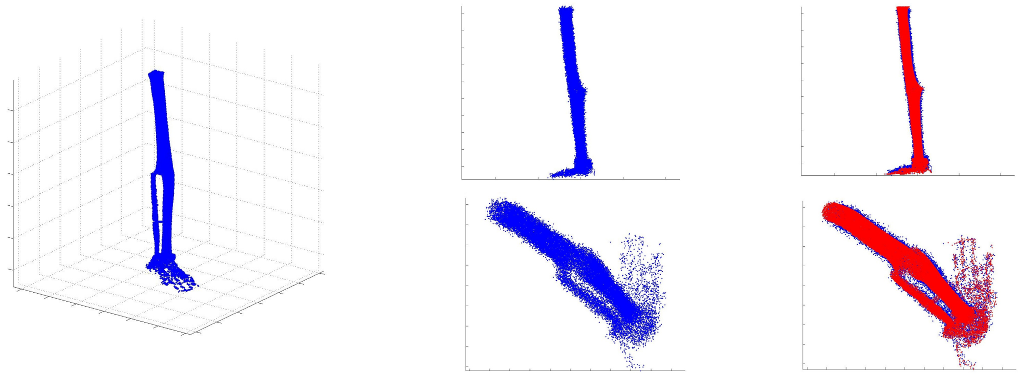

Figure 5.3: Hausdorff matching for limb reconstruction. Left: 3D model leg presented as a finite point cloud in $\mathbb{R}^3$. Middle: Two projections of the transformed leg with noise added to each point. Right: The optimal solution of (5.29) applied to the model set and projected into the input projections.

Concerning the application mentioned in the introduction, we refer to Figure 5.3: It shows that, although the model set is not convex in this case, matching its convex hull to the convex hulls of the two projections via (5.29) yields acceptable results.
Certainly, the assumption of a simple homothetic transformation is unrealistic for radiographs of a (deformed) limb. However, Lemma 5.3.17 shows that for the presumably more complicated problem of matching (parts of) a limb under affinities, the matching problem under homothetics can serve as a subproblem for which an efficient algorithm is available.

Chapter 6

Bounding the Pinning Number of Intersecting Balls

In this chapter, we show that if a line ℓ is an isolated line transversal to a finite family $\mathcal{F}$ of (possibly intersecting) balls in $\mathbb{R}^3$ and no two balls are externally tangent on ℓ then there is a subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size at most 12 such that ℓ is an isolated line transversal to $\mathcal{G}$. We generalize this result to families of semi-algebraic ovaloids.
As already motivated in the introduction, the special interest for isolated line transversals in the context of this thesis is due to the fact that statements about isolated line transversals are statements about locally minimal cylinder axes: "A line ℓ is a transversal to a family of balls of equal radius" can readily be translated to "ℓ is the axis of a cylinder of the same radius that encloses the centers of the balls". If, in addition, ℓ is an isolated line transversal, every slight perturbation of the axis ℓ needs a bigger cylinder radius to enclose the centers. This application in mind, the important feature of our result is, that it allows the balls (or ovaloids) to intersect so that it does not need the centers to be somewhat dispersed when formulated as a result on cylinder axes of finite point clouds.

This chapter is joint work with Xavier Goaoc and Sylvain Petitjean. Part of it was achieved during the author's stay at INRIA/LORIA supported by the INRIA Internship Program. Preliminary results of Sections 6.2 and 6.3 already appear in [125]; the main results of this chapter are published in [81] in a special issue of *Discrete & Computational Geometry* on geometric transversals and Helly-type theorems.

6.1 Introduction

A straight line that intersects every member of a family $\mathcal{F}$ of convex compact subsets of $\mathbb{R}^d$ is called a *line transversal* to $\mathcal{F}$. A line transversal to a family $\mathcal{F}$ that cannot be moved without missing some member of $\mathcal{F}$ is said to be *pinned* by $\mathcal{F}$ (we also say that $\mathcal{F}$ is a *pinning* of that line). In other words, a line is pinned by $\mathcal{F}$ if it is an isolated point of the space of line transversals to $\mathcal{F}$.

The study of sufficient conditions for the existence of a line transversal also plays an important rule in *geometric transversal theory.* Of particular interest are again conditions that can be stated in the elegant form of a *Helly-type theorem* as e.g. in Sections 4.2 and 5.3.2. One of the earliest examples is the following theorem proven by Danzer [58] in 1957: *If every* 5 *members in a finite family of disjoint unit disks have a line transversal then the whole family has a line transversal.* Danzer conjectured that this statement generalizes to families of disjoint unit balls in arbitrary dimension. This conjecture was recently settled in the positive in [53], and one of the main ingredients in the proof is the following *pinning theorem* [53, Proposition 13]: *If a finite family $\mathcal{F}$ of disjoint unit balls in $\mathbb{R}^d$ pins a line ℓ then some subset of $\mathcal{F}$ of size at most $2d-1$ pins ℓ.* This pinning theorem can be understood as a Helly-type theorem for the existence of a line transversal *locally* near a pinned line: if no other line than the pinned line exists locally, this can be witnessed by $2d-1$ of the balls.

Pinning theorems seem more “robust” than Helly-type theorems for the existence of a line transversal. For instance, Danzer's theorem is best possible in the sense that it becomes false if the disks are allowed to intersect or have arbitrary radii, whereas the pinning theorem remains valid for disjoint balls of arbitrary radii in $\mathbb{R}^d$ [31]. Similarly, polytopes in three dimensions have a pinning theorem under certain conditions [15] but admit no global Helly-type theorem [111].
In this chapter, we show that the pinning theorem for disjoint balls also extends, in the three-dimensional case, to families of intersecting balls, provided no two balls are externally tangent on the line. More precisely, we prove:

Theorem 6.1.1 (*Bounding the pinning number of intersecting balls*)
Let $\mathcal{F}$ be a finite family of balls in $\mathbb{R}^3$ that pin a line ℓ. If no two balls are externally tangent in a point of ℓ, then a subset of $\mathcal{F}$ of size at most 12 pins ℓ.

While disjointness of the objects is crucial for global Helly-type theorems, its relevance for the existence of a pinning theorem is not clear. On the one hand, whether disjointness alone guarantees the existence of a pinning theorem for convex sets is a natural question, and we do not know of any minimal pinning of a line by more than 6 pairwise disjoint convex sets in $\mathbb{R}^3$ (see [53, Section 6]). On the other hand, Theorem 6.1.1 suggests that disjointness may be relevant for pinning only insofar as it prevents certain singularities from happening (a line tangent to two balls at their external tangency point is a singular point of the space of their line transversals, cf. the remark at the end of Section 6.3) and, in the polyhedral setting, such singularities can indeed lead to arbitrarily large minimal families of (intersecting) convex polytopes pinning a line [15, Theorem 3].

The proof of the pinning theorem for disjoint balls is based on properties of the so-called cones of directions, which have been studied since Vincensini's original paper [156] that initiated geometric transversal theory. The *cone of directions* of a family of objects is the set of directions of its line transversals. The proof of the pinning theorem in [53]

is based on the observation that for families of disjoint balls, this set is surprisingly well-behaved [12, 31, 53, 110]: its connected components are strictly convex, and are in one-to-one correspondence with the orders in which a line can intersect the family (the *geometric permutations* of the family, see e.g. [80]). Interestingly, this approach fails to extend to situations where the balls intersect outside of the immediate vicinity of the pinned line: the cone of directions of a family of intersecting balls can be locally non-convex at directions of transversals meeting the balls in distinct points (see e.g. Figure 2d in [31]). In some sense, the fact that the cone of directions be convex *locally* near a particular direction somehow requires that the balls be *globally* disjoint.
Our proof of Theorem 6.1.1, on the other hand, uses essentially *local* arguments and extends to solids bounded by *ovaloids*, a class of "locally sphere-like" surfaces (cf. Theorem 6.4.5). Let us sketch our proof briefly. It is well-known that a family $\mathcal{F}$ pins a line ℓ if and only if the direction of ℓ is an isolated point of the cone of directions of $\mathcal{F}$ and that the cone of directions of $\mathcal{F}$ is the intersection of the cones of directions of the triples of balls in $\mathcal{F}$ (Lemma 6.2.2). We first prove that the cone of directions of three balls is "nice" in the sense that it is a manifold with boundary, and that this boundary is smooth except in directions of lines tangent to the three balls and passing through a point of tangency of two of the balls (Lemma 6.2.7). This allows to recast the intersection of the $\binom{n}{3}$ cones of triples as a sandwich region defined by semi-algebraic functions in the plane, and Theorem 6.1.1 follows. We then give a geometric interpretation of the "first-order approximation" of the cone of directions at a smooth point (Lemma 6.4.3). A consequence of that interpretation is that our smoothness condition extends to ovaloids: the cone of directions of three ovaloids is smooth, except possibly at (points representing) directions of lines tangent to the three ovaloids and passing through a point of tangency of two of them (Lemma 6.4.4). The same argument on sandwich regions then yields a pinning theorem for ovaloids of "bounded description complexity" (Theorem 6.4.5).
The main idea leading to our interpretation of the "first-order approximation" of the cone of directions to three balls is to associate to a configuration of a ball and a line tangent to that ball a particular halfplane, which we call a screen; this construction was previously introduced in [52] to analyze the stability of pinning configurations.

For a more general discussion of geometric transversal theory, we refer to the classic survey [59] and to the more recent [64], [84] and [159]. More specific discussions of recent progress on line transversals can be found in the survey [109] for the case of families of translated ovals in the plane and in [80] for the case of families of disjoint balls in arbitrary dimension.

Preliminaries and notation.
In the present section, the space of directions in $\mathbb{R}^3$ is the sphere $\mathbb{S}^2_2$ and we assume that lines are oriented. Given a direction $u \in \mathbb{S}^2_2$, we denote by $u^\perp := \{v \in \mathbb{S}^2_2 : u^T v = 0\}$ the set of directions (in $\mathbb{S}^2_2$) orthogonal to u. In some cases it will be more convenient to identify opposite directions, and to work in the real projective space $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{R})$ in view of the identification $\mathbb{P}^2 = \mathbb{S}^2_2/\mathbb{Z}_2$.

We call a set *strictly convex* if any supporting hyperplane intersects it in a single point and that a convex set is *smooth* if through any boundary point there is a unique supporting hyperplane. We say that two smooth surfaces are *internally tangent* (resp. *externally tangent*) at a point p if they are tangent at p and locally lie on the same side (resp. on opposite sides) of their common tangent plane.
We use the term *family* to denote a finite unordered set; in particular, in a family of balls we assume that the balls are *pairwise distinct*.

6.2 Cones of Directions

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^3$. The directions of line transversals to $\mathcal{F}$ make up a subset $\mathcal{K}(\mathcal{F}) \subseteq \mathbb{S}_2^2$ called the *cone of directions* of $\mathcal{F}$. Here, we extend properties of the cone of directions previously known for disjoint balls [12, 31, 53, 110] to arbitrary balls. Note that in these papers, which deal with disjoint objects, $\mathcal{K}(\mathcal{F})$ stands for the directions of lines piercing the sets of $\mathcal{F}$ *in a specific order* $\prec$.

6.2.1 Arbitrary Families of Balls

We now assume that we are given an arbitrary family of balls $\mathcal{F}$. The cone of directions of $\mathcal{F}$ can be seen as the image of the set of line transversals to $\mathcal{F}$ under the projection that maps a line to its direction. It is clear that the image of a connected component of transversals in this projection is connected. We can prove a stronger result:

Lemma 6.2.1 (*Connected components of* $\mathcal{K}(\mathcal{F})$)
If $\mathcal{F}$ is a family of balls in $\mathbb{R}^d$ then there is a one-to-one correspondence between the connected components of line transversals to $\mathcal{F}$ and the connected components of $\mathcal{K}(\mathcal{F})$.

Proof.
Let ϕ be the map associating to a line its direction. Let $u \in \mathcal{K}(\mathcal{F})$, and for $\Pi := H_=(u, 0)$, let $I := \bigcap_{i=1}^n (B_i|\Pi)$. Since each member of $\mathcal{F}$ is convex, I is also convex and therefore connected. Since the line transversals to $\mathcal{F}$ with direction u are exactly the lines with direction u that intersect I, we get that $\phi^{-1}(u)$ is connected. Now, let $T_1, \dots, T_k$ denote the connected components of line transversals to $\mathcal{F}$. Since ϕ is continuous, each $\phi(T_i)$ is connected. Since $\phi^{-1}(u)$ is connected for any $u \in \mathcal{K}(\mathcal{F})$, the $\phi(T_i)$ are pairwise disjoint. Since each of them is closed, it implies that each $\phi(T_i)$ is a connected component of $\mathcal{K}(\mathcal{F})$, and that each connected component of $\mathcal{K}(\mathcal{F})$ is the image of a single connected component of line transversals to $\mathcal{F}$. □

We now describe various properties of the boundary of $\mathcal{K}(\mathcal{F})$. In what follows, a line is said to intersect a ball *transversally* if it intersects its interior. A line is an *inner special bitangent* if it is tangent to two elements of $\mathcal{F}$ and lies in a common tangent plane that separates them. In particular, two balls with intersecting interiors have no inner special bitangent and the inner special bitangents to two externally tangent balls are the tangents through the point of tangency. A line is a *tritangent* if it is tangent to three elements of $\mathcal{F}$; a tritangent is called *strict* if it is not at the same time an inner special bitangent.

Lemma 6.2.2 (*Basic properties*)
Let $\mathcal{F}$ be a family of balls in $\mathbb{R}^3$. We have:

(i) $\mathcal{K}(\mathcal{F}) = \bigcap_{X \in \binom{\mathcal{F}}{3}} \mathcal{K}(X)$.

(ii) We have $u \in \mathrm{bd}\mathcal{K}(\mathcal{F})$ only if the projections of the members of $\mathcal{F}$ on a plane orthogonal to u intersect with empty interior.

(iii) The boundary of $\mathcal{K}(\mathcal{F})$ consists of directions of tritangents and inner special bitangents.

(iv) $\mathcal{F}$ pins a line ℓ if and only if the direction of ℓ is an isolated point of $\mathcal{K}(\mathcal{F})$.

Proof.
Let u be a direction and let $\mathcal{F}_u$ denote the family of the orthogonal projections of the elements of $\mathcal{F}$ on a plane orthogonal to a direction u.
Helly's theorem in the plane yields that $\mathcal{F}_u$ has non-empty intersection if and only if every triple has non-empty intersection. Thus, $u \in \mathcal{K}(\mathcal{F})$ if and only if $u \in \bigcap_{X \in \binom{\mathcal{F}}{3}} \mathcal{K}(X)$, which proves statement (i).
If the intersection of the elements in $\mathcal{F}_u$ has non-empty interior, then there exists a line that intersects every member of $\mathcal{F}$ transversally. Since any sufficiently small perturbation of that line remains a line transversal to $\mathcal{F}$, u is in the interior of $\mathcal{K}(\mathcal{F})$. This proves assertion (ii).
Let $u \in \mathrm{bd}\mathcal{K}(\mathcal{F})$. Then there exists a triple $X \in \binom{\mathcal{F}}{3}$ such that $u \in \mathrm{bd}\mathcal{K}(X)$. By (ii), this implies that the orthogonal projections of the members of X on a plane orthogonal to u intersect in a single point. This point is either on the boundary of the three projections or an external tangency point of two of them. In the former case u is a direction of tritangent and in the latter a direction of inner special bitangent. This implies statement (iii).
Assume that u is the only point of $\mathcal{K}(\mathcal{F})$ in some open set $R \subseteq \mathbb{P}^2$. Property (ii) implies that $\mathcal{F}$ has a unique line transversal ℓ with direction u. Thus, ℓ is the only line transversal to $\mathcal{F}$ with direction in R. Since the set of all lines with direction in R forms a neighborhood of ℓ, it follows that $\mathcal{F}$ pins ℓ. The reverse implication follows from Lemma 6.2.1. □

Remark 6.2.3
The proofs of Lemmas 6.2.1 and 6.2.2 hold for general closed convex sets, with the understanding that (a) a line is "tangent" to a convex set if it intersects the set and lies in some supporting plane and (b) a line intersects a convex set transversally if it intersects its relative interior but is not included in a plane containing the object. As we shall see in the next section, there are cases where the necessary condition (ii) of Lemma 6.2.2 is not sufficient.

6.2.2 Arbitrary Triple of Balls

Let us now turn our attention to a triple $T = \{B_0, B_1, B_2\}$ of possibly intersecting balls in $\mathbb{R}^3$. The main result of this section is Lemma 6.2.7, which shows that $\mathcal{K}(T)$ has a "nice" structure. The proof is split across several lemmas:
Lemma 6.2.4 characterizes what it means for a direction to be on the boundary of $\mathcal{K}(T)$; Lemma 6.2.5 describes the topology of $\mathcal{K}(T)$; and Lemma 6.2.6 shows that $\mathrm{bd}\mathcal{K}(T)$ is "almost always" smooth.

We first characterize the directions of transversals of T appearing on the boundary of $\mathcal{K}(T)$ (extending [53, Lemma 9]) and those, among the directions of tritangents, that appear on the boundary of the cone of directions (extending [31, Proposition 3]).

Lemma 6.2.4 (*Characterization of* $\mathrm{bd}\mathcal{K}(T)$)
The direction of a line transversal ℓ to T belongs to $\mathrm{bd}\mathcal{K}(T)$ if and only if the following three conditions hold:

(i) The three balls have no point in common.

(ii) The line ℓ is not tangent to two externally tangent balls at their tangency point while meeting the third ball in its interior.

(iii) There is no other line transversal to T parallel to ℓ.

If ℓ is a (strict) tritangent then condition (iii) can be replaced in the above equivalence by:

(iv) The line ℓ intersects the (interior of the) triangle formed by the centers of the balls.

Proof.
Let $u \in \mathbb{S}^2_2$ denote the direction of ℓ. If the three balls have a point in common then $\mathcal{K}(T) = \mathbb{S}^2_2$ has no boundary, so condition (i) is necessary. If ℓ is tangent to two externally tangent balls at their point p of tangency and meets the interior of the third ball then any line through p with direction sufficiently close to u is a transversal to T, and $u \in \mathrm{int}\mathcal{K}(T)$; condition (ii) is therefore necessary. If T has another line transversal parallel to ℓ then the projections of the balls of T along u intersect with non-empty interior, and Lemma 6.2.2 (ii) ensures that $u \in \mathrm{int}\mathcal{K}(T)$; condition (iii) is thus also necessary.
Before we show that the conditions are sufficient, let us first remark that for v close enough to u, T has a line transversal with direction v if and only if T has a line transversal with direction v close to ℓ. Indeed, let $\Pi_v := H_=(v, 0)$ denote the plane through the origin with normal v. The set of transversals to T with direction v are precisely those that meet Π_v in a point of the intersection of the orthogonal projections of the balls on Π_v. Since the orthogonal projection of a fixed ball on Π_v depends continuously on v, it follows that for v close enough to u, T has a line transversal with direction v if and only if T has a line transversal with direction v close to ℓ.

Now, assume that (i), (ii) and (iii) hold. First consider the case where ℓ is not tritangent to T. Then, by Lemma 6.2.2 (iii) ℓ is an inner special bitangent to two of the balls and condition (ii) implies that these balls meet ℓ in distinct points, and are thus disjoint. The perturbation argument used in [53, Lemma 9] guarantees that there are directions v arbitrarily close to u such that these two balls, and therefore T, have no line transversal and therefore that $u \in \mathrm{bd}\mathcal{K}(T)$. In the case where ℓ is not tritangent, (i), (ii) and (iii) are thus sufficient.
Consider now the case where ℓ is tritangent to T. If the balls meet ℓ in distinct points then, again, the same perturbation argument ([53, Lemma 9]) guarantees that $u \in \mathrm{bd}\mathcal{K}(T)$. Condition (i) requires that at least two tangency points be distinct, so it remains to consider the case where two balls, say B_0 and B_1, are tangent to ℓ at the same point and the third ball is tangent at a different point. If B_0 and B_1 are externally tangent, then there is a direction v in the plane of tangency of B_0 and B_1 and arbitrarily close to u such that the line with direction v passing through the point of tangency of B_0 and B_1 misses B_2; clearly, T has no transversal with such a direction v, and $u \in \mathrm{bd}\mathcal{K}(T)$. If B_0 and B_1 intersect properly, their bounding spheres intersect in a circle Λ. Observe that ℓ is tangent to Λ in its plane and that there is a direction v in this plane arbitrarily close to u such that the tangent to Λ with direction v close to ℓ misses B_2; clearly, T has no transversal with such a direction v, and $u \in \mathrm{bd}\mathcal{K}(T)$. This proves that in the case where ℓ is tritangent, conditions (i), (ii) and (iii) are also sufficient.
Property (iv) was observed for tritangents to triples of disjoint balls in [31, Proposition 3]. Their proof easily extends to intersecting balls. □

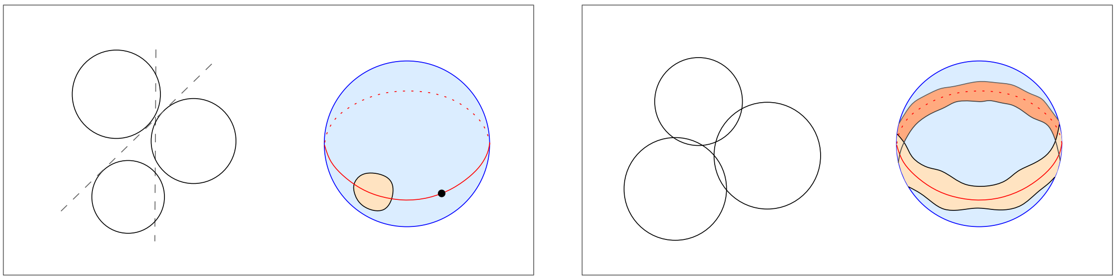

Figure 6.1: Possible topologies of the cone of directions of three balls. Left: Contractible connected components, possibly reduced to a point. Right: A strip containing the $\mathbb{S}^1_2$ of directions in the plane of centers in its interior. Note that, in all cases, the figure is symmetric with respect to the $\mathbb{S}^1_2$ of directions in the plane of centers.

Next, we describe the topology of the cone of directions (extending [31, Proposition 4]).

Lemma 6.2.5 (*Possible topologies of* $\mathcal{K}(T)$)
Let C be a connected component of $\mathcal{K}(T)$. The following holds:

(i) C is a single point if and only if there is a line with that direction that is pinned by T.

(ii) C is all of $\mathbb{S}_2^2$ if and only if the three balls have a point in common.

(iii) C is a strip that contains the $\mathbb{S}_2^1$ of directions of the plane of centers in its interior if and only if the balls in T intersect pairwise but not triplewise. In that case, C is the only connected component of $\mathcal{K}(T)$.

(iv) In all other cases, C is contractible and is the closure of its interior.

Proof.
Let Π denote the plane containing the centers of the balls in T (or any such plane, if the centers are aligned). Statement (i) follows from Lemma 6.2.2 (iv). If there is a point common to the three balls in T then $\mathcal{K}(T) = \mathbb{S}_2^2$. If the intersection of the three balls is empty then T has no line transversal with direction orthogonal to Π, and $\mathcal{K}(T) \neq \mathbb{S}_2^2$. This proves statement (ii).

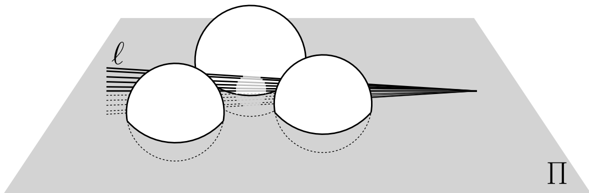

Figure 6.2: Line transversals and mirror images. All lines between ℓ and its mirror image with respect to Π also intersect the three balls.

As observed in [31, Proposition 4], all the lines between a line transversal ℓ to T and its mirror image with respect to Π are also line transversals to T (cf. also Figure 6.2). When the three balls have no point in common, T has no line transversal orthogonal to Π and the set of line transversals to T can be retracted onto the set of line transversals to T contained in Π. This induces a retraction from $\mathcal{K}(T)$ onto the cone of directions of the disks $T' = \{B_i \cap \Pi : i \in \{0, 1, 2\}\}$.
Now onto statement (iii). Observe that $\mathcal{K}(T') = \mathbb{S}_2^1$ if and only if the disks, and hence the balls, intersect pairwise. One direction follows from Helly's theorem in one dimension, the other from the observation that if two disks are disjoint, then T' has no transversal in the direction orthogonal to the vector joining their centers.
We now show that, assuming every two balls in T intersect but the three balls have empty intersection, $\mathcal{K}(T)$ is a strip containing $\mathcal{K}(T') = \mathbb{S}_2^1$ in its interior. Observe first that under those assumptions no direction of bd$\mathcal{K}(T)$ is parallel to Π. Indeed, assume for a contradiction that u is such a direction and let ℓ be the (unique by Lemma 6.2.4) line transversal to T with direction u. If ℓ is an inner special bitangent to two balls, these balls must be tangent and ℓ must meet them in their point of tangency; the interior of the third ball must intersect ℓ (otherwise, as ℓ is in the plane of centers, the third ball would not intersect one of the first two), and $u \notin$ bd$\mathcal{K}(T)$ by Lemma 6.2.4. If ℓ is not an inner special bitangent to any pair in T then it is tangent to all three balls; in the

plane Π, the three disks must be on the same side of ℓ, and u is therefore clearly not on $\mathrm{bd}\mathcal{K}(T)$.
Now, if every pair in T intersects with the three balls having empty intersection, then we can retract $\mathcal{K}(T)$ onto the $\mathbb{S}^1_2$ of directions parallel to Π, and no direction from this $\mathbb{S}^1_2$ lies in $\mathrm{bd}\mathcal{K}(T)$; $\mathcal{K}(T)$ is thus a strip that contains the $\mathbb{S}^1_2$ of directions of the plane of centers in its interior. Conversely, if $\mathcal{K}(T)$ has this geometry, then the cone of directions of T' is a $\mathbb{S}^1_2$ (since the projection of any line transversal to T onto Π gives a line transversal to T'). It follows that the disks in T' intersect pairwise but not triplewise, and so do the balls in T, proving (iii).
Finally, in all other cases, a connected component of $\mathcal{K}(T')$ is an interval, and (iv) follows. □

The cone of directions of a triple of disjoint balls is strictly convex, but this property fails for intersecting balls [31]. However, we can still show that the boundary of the cone is "almost always" smooth.

Lemma 6.2.6 (*Singular points on* $\mathrm{bd}\mathcal{K}(T)$)
A direction u is a singular point of $\mathrm{bd}\mathcal{K}(T)$ if and only if the intersection of the three balls is empty and there exists a line with direction u that is:

(i) pinned by the three balls, or

(ii) tangent to all three balls, meeting two of them in the same point in which they are externally tangent.

Proof.
Let u be a point of $\mathrm{bd}\mathcal{K}(T)$. By Lemma 6.2.2 (ii) there exists a unique line transversal, say ℓ, to the three balls having direction u. We argue that if u is a singular point then ℓ must satisfy condition (i) or (ii).
By Lemma 6.2.2 (iii), the boundary of $\mathcal{K}(T)$ consists of two types of arcs, arcs of directions of inner special bitangents to some pair $\{B_i, B_j\}$ and arcs of directions of tritangents to $\{B_0, B_1, B_2\}$. The directions of inner special bitangents to two distinct balls is either empty or a smooth conic (cf. [31]). Note that Lemma 6.2.2 (ii) implies that for directions on the boundary of $\mathcal{K}(T)$, two such arcs meet in a tritangent direction. Therefore, if u is a singularity of $\mathrm{bd}\mathcal{K}(T)$ it must be a direction of tritangent.
Let c_i and s_i denote the center and squared radius of B_i, respectively. The directions of tritangents to $\{B_0, B_1, B_2\}$ make up an algebraic curve of degree 6 in $\mathbb{P}^2$, the *direction-sextic* $\sigma_{B_0B_1B_2}(u)$ of these three balls. Letting $e_{ij} = c_j - c_i$ and $\delta_{ij} = e_{ij}^T e_{ij}$ and writing, for a given direction $u \in \mathbb{P}^2$,

$$q = q(u) = u^T u \quad \text{and} \quad t_{ij} = t_{ji} = (e_{ij} \times u)^T (e_{ij} \times u) = \delta_{ij} q - (e_{ij}^T u)^2,$$

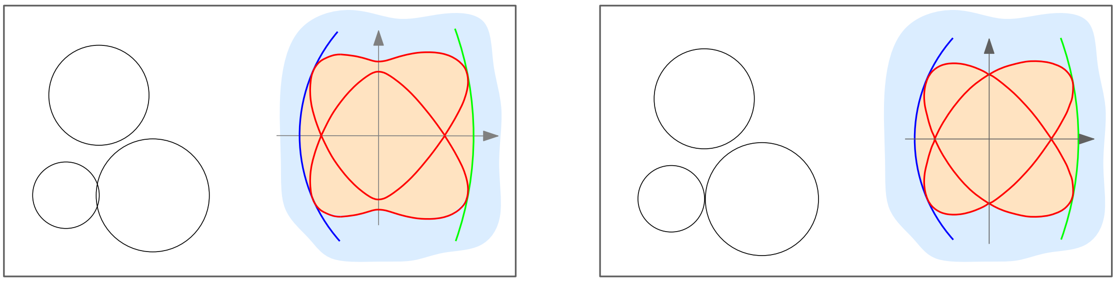

Figure 6.3: Two triples of balls (represented by their trace on their plane of centers) and a planar depiction of their cone of directions. The direction-sextic is drawn in red, conics of directions of inner special bitangents are drawn in blue and green; the orange region is the cone of directions (a connected set in this example) and the horizontal axis corresponds to directions in the plane of the centers of the balls. Observe that when the balls intersect properly (left) the cone of directions is smooth as when the balls are disjoint, whereas when some of the balls are tangent (right) the cone of directions exhibits a singularity.

the direction-sextic of B_0, B_1, B_2 rewrites as a Cayley determinant [31, Proposition 2]:

$$\sigma_{B_0B_1B_2}(u) = \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & qs_0 & qs_1 & qs_2 \\ 1 & qs_0 & 0 & t_{01} & t_{02} \\ 1 & qs_1 & t_{01} & 0 & t_{12} \\ 1 & qs_2 & t_{02} & t_{12} & 0 \end{pmatrix} = 0. \qquad (6.1)$$

Figure 6.3 illustrates typical situations.

We now assume that u lies on the boundary of $\mathcal{K}(T)$ and is a singular point of the direction-sextic, i.e. that the gradient of $\sigma_{B_0B_1B_2}$ in u vanishes. To analyze the gradient of the direction-sextic in a direction on bd$\mathcal{K}(T)$, we proceed as in [31]. Consider the projection along a direction u of the three balls and a tritangent ℓ with that direction; by Lemma 6.2.4, u is on the boundary of the cone of directions if and only if the projection of ℓ lies in the projection Δ of the triangle of centers. We equip $\mathbb{R}^3$ with a coordinate frame with axes x, y, z such that this projected triangle lies in the plane $z = 0$ (i.e., we consider tritangent lines having $e_3 = (0, 0, 1)$ as direction) and has its vertices at $\tilde{c}_0 = (0, 0, 0), \tilde{c}_1 = (a, 0, 0), \tilde{c}_2 = (b, c, 0)$, with the understanding that there is a point p inside,

$$p = \sum p_i \tilde{c}_i, \quad \text{with} \quad p_0, p_1, p_2 \geq 0, \sum p_i = 1.$$

Note that the squared distance from p to vertex $\tilde{c}_i$ is the squared radius of B_i, i.e. $s_i = (p - \tilde{c}_i)^T (p - \tilde{c}_i)$. Then, we use three real parameters x_0, x_1, x_2 to describe the possible positions of the three centers of B_0, B_1, B_2:

$$c_0 = \tilde{c}_0 + x_0 e_3, \quad c_1 = \tilde{c}_1 + x_1 e_3, \quad c_2 = \tilde{c}_2 + x_2 e_3,$$

where we assume without loss of generality that $x_2 \geq x_1 \geq x_0$. Substituting in (6.1) we can express the direction-sextic $\sigma_{B_0B_1B_2}$ and its derivatives in the direction $e_3 = (0,0,1)$ as a function of x_0, x_1, x_2 depending on the parameters a, b, c, p_0, p_1, p_2. Denoting $u = (u_0, u_1, u_2)$, the computation gives:

$$\begin{aligned}
\frac{\partial\sigma}{\partial u_0}(0,0,1) = & \quad 16a^2c^2 & \Big[ap_1(x_1p_0 - x_0p_0 - x_2p_2 + x_1p_2) \\
& & -bp_2(-x_2p_0 - x_2p_1 + x_0p_0 + x_1p_1)\Big], \\
\frac{\partial\sigma}{\partial u_1}(0,0,1) = & -16a^2c^3p_2 & \Big[p_0(x_0 - x_2) + p_1(x_1 - x_2)\Big], \\
\frac{\partial\sigma}{\partial u_2}(0,0,1) = & \quad 0. &
\end{aligned} \tag{6.2}$$

Note that the partial derivatives all vanish if $a = 0$ or $c = 0$. If $a = 0$ then the centers of two balls coincide and these balls must have equal radii to allow for a common tangent, a situation our assumptions rule out. If $c = 0$ then the three centers are aligned; by symmetry of revolution around the line of the centers, the cone of directions is a circle, and therefore smooth. We can also rule this case out, and assume from now on that $ac \neq 0$.

First assume Δ is degenerate, i.e. it collapses to a segment s. Then ℓ is a tritangent line contained in a plane tangent to all three balls and is also contained in the plane of centers P. Since ℓ hits s by Lemma 6.2.5 (iv), those centers must lie on both sides of ℓ in P, say c_i on one side and c_j, c_k on the other. Since there are no strict tritangent directions, either u is isolated in $\mathcal{K}(T)$, in which case the three balls pin ℓ and we are in case (i); or u is not isolated in which case ℓ is the intersection of the sets of inner special bitangents to $\{B_i, B_j\}$ and $\{B_i, B_k\}$. These two sets correspond, in the space of directions, to two conics, one being internally tangent to the other at u. In other words, $\mathcal{K}(T)$ is not singular at u.

Now assume Δ is non-degenerate. Suppose u is a strict tritangent direction, i.e. it is a singular point of σ and the point p is strictly inside Δ by Lemma 6.2.5 (iv). This implies that the three derivatives above vanish and $p_i > 0, i = 0,1,2$. From $\left(\frac{\partial\sigma}{\partial u_1}\right)(0,0,1) = 0$ we conclude that $x_0 = x_1 = x_2$. Geometrically, this means that the three balls intersect in a common point and every line through this point is a transversal to the three balls. Therefore $\mathcal{K}(T) = \mathbb{P}^2$ which has no singular point, a contradiction.

Thus, u must also be a direction of inner special bitangent and exactly one p_i is zero (if two of them vanish, p is a vertex of Δ, i.e. ℓ goes through the center of one ball, implying that this ball has radius 0, which we rule out). Let j and k denote the other two indices; notice then that ℓ is an inner special bitangent to B_j and B_k. We then obtain from the vanishing of the partial derivatives above that $x_j = x_k$. Thus, B_j and B_k meet ℓ in the same point, and are externally tangent at that point; we are then in case (ii).

Altogether, we have that if u is a singularity then ℓ satisfies condition (i) or (ii). Conversely, if ℓ satisfies (i) then $\mathcal{K}(T)$ is, locally around u, reduced to a point and u is a singularity. If ℓ satisfies (ii), then the gradient of σ in u vanishes and u is therefore a singularity. □

As a consequence, we obtain that the cone of directions of three balls has a nice structure.

Lemma 6.2.7 (*Topology of* $\mathcal{K}(T)$)
If T is a triple of balls, no two externally tangent, then every connected component of $\mathcal{K}(T)$ is either a single point, $\mathbb{S}^2_2$ or a semi-algebraic 2-manifold with a smooth boundary.

Proof.
As observed in [1], the set of line transversals to a family of semi-algebraic objects of “bounded description complexity” (i.e. semi-algebraic sets defined using a bounded number of polynomial equalities and inequalities of bounded degrees), seen as a 4-dimensional subset of line space, is a semi-algebraic set. For T a triple of balls, the set of transversals to T is therefore a semi-algebraic set, and so is its projection $\mathcal{K}(T)$ onto the space of directions. The fact that each connected component is a single point or a 2-manifold with boundary (except when $\mathcal{K}(T) = \mathbb{S}^2_2$) follows from Lemma 6.2.5. The fact that the boundary is smooth when $\mathcal{K}(T)$ is not reduced to a single point and no two balls are externally tangent follows from Lemma 6.2.6. □

6.3 Pinning Theorem for Intersecting Balls

We can now prove Theorem 6.1.1, which states that if n balls in $\mathbb{R}^3$ pin a line and no two balls are externally tangent on the line, then a subset of at most 12 of these balls pins that line.

Proof of Theorem 6.1.1
Let $\mathcal{F}$ be a finite family of balls in $\mathbb{R}^3$ that pins a line ℓ, no two balls being externally tangent on ℓ. Assume no triple of balls of $\mathcal{F}$ already pins ℓ, as otherwise the statement is trivially true. Let u denote the direction of ℓ. By Lemma 6.2.2 (iv), u is an isolated point of $\mathcal{K}(\mathcal{F})$, and it suffices to find a subfamily $Y \subseteq \mathcal{F}$ of size at most 12 such that u is an isolated point of $\mathcal{K}(Y)$ to prove the statement.
By Lemma 6.2.2 (i), $\mathcal{K}(\mathcal{F})$ is the intersection of the cones $\mathcal{K}(T)$ for all triples $T \subseteq \mathcal{F}$ and dropping any triple T such that $u \in \mathrm{int}\mathcal{K}(T)$ keeps u isolated in the intersection. Thus, if we denote by $\mathcal{N}$ the set of triples $T \subseteq \mathcal{F}$ such that u is on the boundary of $\mathcal{K}(T)$, we have that u is an isolated point of $\bigcap_{T\in\mathcal{N}} \mathcal{K}(T)$.
By Lemma 6.2.7, for every $T \in \mathcal{N}$ there exists an arbitrarily small neighborhood U_T of u such that $\mathcal{K}(T) \cap U$ is homeomorphic to a halfplane. Let U denote a neighborhood of u such that $U \cap \mathcal{K}(T)$ is homeomorphic to a halfplane for all $T \in \mathcal{N}$. Let a_T denote the normal to $\mathrm{bd}\mathcal{K}(T)$ in u that points outward of $\mathcal{K}(T)$, and consider an orthogonal coordinate system (u, x, y) in U such that $a_T^T y \neq 0$ for all $T \in \mathcal{N}$. We split $\mathcal{N}$ into two

subsets:

$$\mathcal{N}^+ = \{T \in \mathcal{N} : a_T^T y > 0\} \quad \text{and} \quad \mathcal{N}^- = \{T \in \mathcal{N} : a_T^T y < 0\}.$$

By the semi-algebraic implicit function theorem [23, p. 97], for U small enough bd$\mathcal{K}(T)$ can be written in the form $y = f_T(x)$, where f_T is a semi-algebraic function. Since $\mathcal{K}(T) \cap U$ is homeomorphic to a halfplane, it follows that in U, $\mathcal{K}(T)$ can be written as $\{(x, y) : y \leq f_T(x)\}$ if $T \in \mathcal{N}^+$ and as $\{(x, y) : y \geq f_T(x)\}$ if $T \in \mathcal{N}^-$.
Now, observe that in U, $\bigcap_{T\in\mathcal{N}} \mathcal{K}(T)$ is exactly the set of points that are below all curves in $\{f_T : T \in \mathcal{N}^+\}$ and above all curves in $\{f_T : T \in \mathcal{N}^-\}$. Since the functions f_T are semi-algebraic, near u they either coincide or there is a neighborhood of u in which they only meet in u. It follows that there exists $\varepsilon > 0$ and four subsets $A, B \in \mathcal{N}^+$ and $C, D \in \mathcal{N}^-$ such that on the interval $[-\varepsilon, 0]$, all functions in $\{f_T : T \in \mathcal{N}^+\}$ are above f_A and all functions in $\{f_T : T \in \mathcal{N}^-\}$ are below f_C, and similarly on $[0, \varepsilon]$ all functions in $\{f_T : T \in \mathcal{N}^+\}$ are above f_B and all functions in $\{f_T : T \in \mathcal{N}^-\}$ are below f_D. As a consequence,

$$\mathcal{K}(A) \cap \mathcal{K}(B) \cap \mathcal{K}(C) \cap \mathcal{K}(D) \cap U = \{u\}$$

and $Y := A \cup B \cup C \cup D$ is a subset of $\mathcal{F}$ of size at most 12 that pins ℓ. □

Remark 6.3.1
It is not clear whether the condition that no two balls be externally tangent on the line is really needed for the pinning theorem to hold. However, we do note that these configurations are indeed particular, in the sense that the space of oriented line transversals to two externally tangent balls is singular at any line through their tangency point. Indeed, the set of lines through the tangency point is 2-dimensional, and removing that set from the set of lines intersecting the two balls creates two connected components, each being 4-dimensional.

Remark 6.3.2
Extending the proof of Theorem 6.1.1 to pinnings in higher dimension seems difficult. First, generalizing Lemma 6.2.2 (i), one would have to work with cones of directions of d balls in $\mathbb{R}^d$ and identifying the singularities of such cones may not be an easy task. Second, and more importantly, our proof exploits the fact that in the plane, a lower/upper envelope of semi-algebraic functions is defined, near one of its vertices, by a constant number of the functions (2 in this case). Already in dimension 3 this is not true for general semi-algebraic sets (consider the lower envelope of several copies of the paraboloid $z = x^2 + 2y^2$, rotated around the z axis, in the neighborhood of $(0, 0, 0)$), and it is not clear why it would be true for cones of directions.

6.4 Extension to Ovaloids

An *ovaloid* is a smooth closed surface in $\mathbb{R}^3$ with strictly positive Gauss curvature everywhere. According to a classical theorem of Hadamard, ovaloids are topologically spheres (cf. [136, Chap. 4 & 6]). More precisely, Hadamard's theorem asserts that an

ovaloid is the boundary of a bounded, open, strictly convex set. By abuse of language, we use the term ovaloid both for the surface and for the bounded solid it encloses.
In this section, we extend our pinning theorems to families of ovaloids of bounded description complexity. A key idea is to represent a first-order approximation of the cone of directions of three balls in a given direction as the cone of directions of three well-chosen parallel halfplanes. Our proof is split across several lemmas:
Lemma 6.4.1 extends our characterization of the boundary of the cone of directions from triples of balls to triples of ovaloids; Lemma 6.4.2 describes the cone of directions of three parallel halfplanes; Lemma 6.4.3 shows that the first-order approximation of the cone of directions of three balls in a neighborhood of a given boundary direction can be represented by the cone of directions of three well-chosen halfplanes; Lemma 6.4.4 then extends our sufficient condition for the smoothness of the cone of directions to triples of ovaloids, and our pinning theorem for ovaloids (Theorem 6.4.5) follows.

6.4.1 Boundary of the Cone of Directions of three Ovaloids

An ovaloid has positive Gauss curvature everywhere. Its second fundamental form relative to the inward normal is positive definite at every point. It follows that its principal curvatures κ_1 and κ_2 are strictly positive everywhere. Assume κ_1 is the largest of the two principal curvatures. As is well known, any ball internally tangent to a smooth convex surface at p of radius less than the principal radius of curvature $1/\kappa_1(p)$ is inside the surface locally around p. Actually, picking a radius "small enough" will ensure that the ball is not just locally but globally inside the ovaloid[1]. Similarly, any ball internally tangent at p of radius more than the principal radius of curvature $1/\kappa_2(p)$ is outside the surface locally around p and picking a radius large enough will ensure the ovaloid is globally inside the ball. One can therefore "sandwich" an ovaloid at any of its points between two balls. Note that this is not true for all smooth, strictly convex sets: for instance the set defined by $f \leq 0$ where $f = x^4 + y^4 + z^4 - 1$ is smooth and strictly convex but the two principal curvatures of the zero-set of f at all extreme points along the x, y and z axes vanish.
This "sandwich" property allows to extend the characterization of the boundary of the cone of directions of Lemma 6.2.4 to ovaloids:

Lemma 6.4.1 (*Characterization of* $\mathrm{bd}\mathcal{K}(T)$)
Let T be a triple of ovaloids. The direction of a line transversal ℓ to T belongs to $\mathrm{bd}\mathcal{K}(T)$ if and only if the following three conditions hold:

(i) The three ovaloids have no point in common.

(ii) The line ℓ is not tangent to two externally tangent ovaloids at their tangency point while meeting the third ovaloid in its interior.

(iii) There is no other line transversal to T parallel to ℓ.

[1] We do not need to be more precise here, but note that, by Blaschke's Rolling Theorem (cf. [146]), any ball of radius less than the infimum of $1/\kappa_1(p)$ can roll along the surface of the ovaloid while always staying inside.

Proof.
The three conditions are clearly necessary, so we prove the converse. Let u denote the direction of ℓ. By (iii), the orthogonal projections of the ovaloids on a plane orthogonal to u intersect in a single point, so ℓ is either an inner special bitangent or a tritangent. In the former case, the same argument as in Lemma 6.2.4 shows that $u \in \mathrm{bd}\mathcal{K}(T)$ if (i) and (iii) hold. In the latter case, let T^+ denote a triple of balls, where each ball is tangent to an ovaloid of T at its tangency point with ℓ and contains that ovaloid. Now, observe that $u \in \mathrm{bd}\mathcal{K}(T^+)$ by Lemma 6.2.4 and that $\mathcal{K}(T) \subseteq \mathcal{K}(T^+)$. It follows that u is in the boundary of $\mathcal{K}(T)$, and the statement follows. □

6.4.2 Screens

We associate to a pair (ℓ, B) of a line ℓ intersecting a ball B an object, which we call a *screen*, as follows. If ℓ meets the interior of B then the screen of (ℓ, B) is the plane orthogonal to ℓ that passes through the center of B. If ℓ is tangent to B, we let t denote that tangency point and v the outward normal to B at t, and define the screen of (ℓ, B) as the (closed) halfplane

$$\mathcal{S}_\ell(B) := \{x \in \mathbb{R}^3 : u^T(x-t) = 0 \text{ and } v^T(x-t) \leq 0\},$$

that is the intersection of the plane perpendicular to ℓ through t with the (closed) halfspace "tangent" to B at t and containing B (cf. Figure 6.4). When a screen is a halfplane, we call its boundary line in its affine hull its *boundary*.

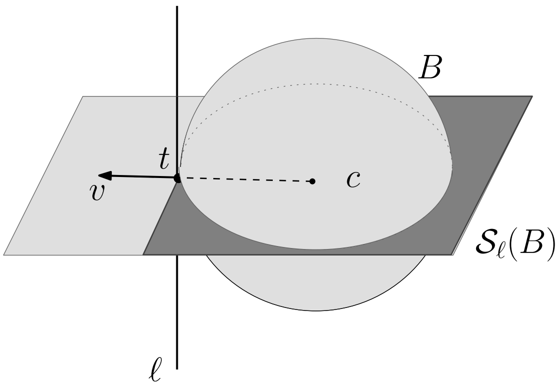

Figure 6.4: Illustration of the screen $\mathcal{S}_\ell(B)$ defined by a line ℓ tangent to a ball B.

To simplify the presentation, we assume here that a line parallel to a halfplane intersects it at infinity (and thus $u^\perp$ is in the cone of directions of any triple of screens lying in planes orthogonal to u). Note that this convention has no effect on pinning problems.

Lemma 6.4.2 (*Balls and screens*)
Let T be a triple of balls, ℓ a line transversal to T with direction u and S the triple of screens defined by ℓ and the balls of T. Let Γ denote the great circle of directions of transversals to the boundaries of the screens in S. The following holds:

(i) If T pins ℓ then $\mathcal{K}(S) = \Gamma$.

(ii) If $u \in \mathrm{bd}\mathcal{K}(T)$ and T does not pin ℓ then $\mathcal{K}(S)$ is the union of $A \cap B$ and its symmetric, where A and B are closed hemispheres bounded by Γ and $u^\perp$ respectively.

(iii) If $u \in \mathrm{int}\mathcal{K}(T)$ then $\mathcal{K}(S) = \mathbb{S}^2_2$.

Proof.
If T pins ℓ then T consists of three balls tangent to ℓ with a common tangent plane Π and whose positions with respect to Π alternate. Thus, the screens of S are bounded by lines contained in Π and their positions with respect to Π also alternate. Since the set of directions contained in Π is exactly Γ, statement (i) follows.

We now assume that T does not pin ℓ and that $u \in \mathrm{bd}\mathcal{K}(T)$. By Lemma 6.2.2 (iii), ℓ is an inner special bitangent to two of the balls or is tangent to all three balls in T. Let S_1, S_2 and S_3 be the three screens in S.

We first consider the case where ℓ is an inner special bitangent to the first two balls. Then, the boundaries of S_1 and S_2 are parallel and span a plane Π that contains ℓ. Moreover, Π separates S_1 and S_2. Since T does not pin ℓ, S_3 is a plane, a halfplane whose boundary intersects Π in a single point, or a halfplane whose boundary is contained in Π and that lies on the same side as S_2 with respect to Π. In each of these cases, it can easily be checked that $\mathcal{K}(S) = \mathcal{K}(\{S_1, S_2\})$. Consider a direction v. If a direction v is orthogonal to u then by our convention v is in $\mathcal{K}(S)$. If v makes a positive dot product with u then $v \in \mathcal{K}(\{S_1, S_2\})$ if and only if v is parallel to Π or crosses it from the side of S_1 to that of S_2. If v makes a negative dot product with u them we are in the symmetric case, and the result follows. The case where ℓ is an inner special bitangent to another pair is handled similarly.

Now, assume that ℓ is tangent to all three balls in T but is not an inner special bitangent to any two of them. In particular, this implies that no two screens in S have parallel boundaries. The orthogonal projections of the S_i's on a plane orthogonal to u intersect in a single point. Now, if we consider a direction v moving on Γ starting in u, the orthogonal projections of the S_i's on a plane orthogonal to v change continuously, and the boundaries of these three halfplanes keep intersecting in a point. The intersection thus remains a single point unless two of the projected halfplanes become equal or opposite; this cannot happen, as it requires the boundaries of the corresponding screens to be parallel. Thus, the intersection of the projections of the screens in S along any direction of Γ is a single point. Conversely, the intersection of the projections of the screens in S along any direction not in $\Gamma \cup u^\perp$ is either empty or has non-empty interior. Thus, the boundary of $\mathcal{K}(S)$ consists of $\Gamma \cup u^\perp$. A perturbation argument similar to [53, Lemma 9] shows that $\mathcal{K}(S)$ is, locally, on one side of Γ, and statement (ii) follows.

If $u \in \mathrm{int}\mathcal{K}(T)$ then for any direction v the projections of the screens in S along v intersect with non-empty interior. Statement (iii) follows. □

6.4.3 First-order Approximation of Cones of Directions

The *tangential cone* $\mathcal{T}_p(X)$ of a closed non-empty set $X \subseteq \mathbb{R}^d$ at a point p of its boundary is the set of all directions d such that $d = \lim_{k\to\infty} \lambda_k(p_k - p)$, where $\lambda_k > 0, p_k \in X$ for each k and $p_k \to p$. From the above definition, it is clear that d belongs to the tangential cone if there is a sequence $(p_k)_{k\in\mathbb{N}}$ of points in X converging to p such that the direction of the chords $p_k - p$ converges to d. In particular, if bd$X$ is smooth at p then $\mathcal{T}_p(X)$ is a closed halfspace whose outward normal is the outward normal of X at p.

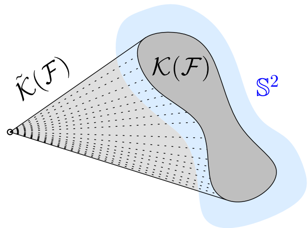

Figure 6.5: The cone of directions on the sphere and as a solid cone in $\mathbb{R}^3$.

The next lemma gives a simple local geometric interpretation of the tangential cone of a cone of directions of three balls as a cone of directions of at most three attached screens. Since cones of directions live in $\mathbb{S}^2_2$, where defining the tangential cone is awkward, we first pull these objects back to $\mathbb{R}^3$ by defining $\tilde{\mathcal{K}}(\mathcal{F})$ as the solid cone in $\mathbb{R}^3$ formed by all rays originating from the origin 0 and with directions in $\mathcal{K}(\mathcal{F})$ (cf. Figure 6.5). Notice that a direction u is on the boundary of $\mathcal{K}(\mathcal{F})$ if and only if the ray $0 + \mathbb{R}^+u$ is on the boundary of $\tilde{\mathcal{K}}(\mathcal{F})$, and that $\mathcal{K}(\mathcal{F})$ is smooth at u if and only if $\tilde{\mathcal{K}}(\mathcal{F})$ is smooth at any point of the ray $0 + \mathbb{R}^+u$, except the origin.

Lemma 6.4.3 (*Screens and linear approximation of* $\mathcal{K}(T)$)
Let T be a triple of balls in $\mathbb{R}^3$, u a smooth point of bd$\mathcal{K}(T)$ and ℓ the line transversal to T with direction u. Let S be the set of screens defined by ℓ and the balls of T. Let p be any point in $0+\mathbb{R}^+u$ other than 0. Then locally near p, $\tilde{\mathcal{K}}(S)$ coincides with $\mathcal{T}_p\tilde{\mathcal{K}}(T)$, the tangential cone of $\tilde{\mathcal{K}}(T)$ at p.

Proof.
Since u is on the boundary of $\mathcal{K}(T)$, Lemma 6.2.2 (ii) implies that T has a unique line transversal with direction u, which we denote by ℓ. We equip $\mathbb{R}^3$ with a frame such that ℓ is the z-axis and let $\mathcal{L}$ denote the set of lines not orthogonal to ℓ. The map ψ that associates to $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ the line through the points $(x_1, x_2, 0)$ and $(x_3, x_4, 1)$ is an homeomorphism from $\mathbb{R}^4$ to $\mathcal{L}$, that is, it defines a proper parametrization of $\mathcal{L}$ and $\psi(0,\ldots,0) = \ell$.
Let s be a screen defined by ℓ. If s is a plane then any line in $\mathcal{L}$ intersects it. Otherwise, let δ denote the boundary of s. Let p and p' be two points on δ. The condition that

a line $\tilde{\ell}(x) = \psi(x_1, x_2, x_3, x_4)$ intersects s amounts to evaluating the orientation of a tetrahedron formed by two points from $\tilde{\ell}(x)$ and two points from δ. Since δ is orthogonal to the z axis, the z coordinates of p and p' are equal and this orientation test recasts as a sign condition on the 4×4 determinant

$$\begin{vmatrix} x_{p'} - x_p & x_p & x_1 & x_3 \\ y_{p'} - y_p & y_p & x_2 & x_4 \\ 0 & z_p & 0 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

which is linear in $x_1, \ldots, x_4$. Thus, if L is the set of lines in $\mathcal{L}$ that intersect a screen defined by ℓ, $\psi^{-1}(L)$ is thus a halfspace in $\mathbb{R}^4$ or all of $\mathbb{R}^4$. In other words, ψ^{-1} maps the set of line transversals to any screen defined by ℓ to $\mathbb{R}^4$ or one of its halfspaces.
Now, let p be a point in $0 + \mathbb{R}^+ u$ other than 0. Let S denote the set of screens defined by (ℓ, B) where B is a ball of T tangent to ℓ. By Lemma 6.2.2 (iii), $|S| \leq 3$. Since $u \in \text{bd}\mathcal{K}(T)$, Lemma 6.4.2 (ii) implies that $\mathcal{K}(S)$ is, near u, bounded by a great circle. It follows that $\tilde{\mathcal{K}}(S)$ coincides with a halfspace near p. Since $\text{bd}\mathcal{K}(T)$ is smooth at u, the tangential cone to $\tilde{\mathcal{K}}(T)$ at p is also a halfspace. To prove the statement, it suffices to show that in a neighborhood of p, the interior of $\tilde{\mathcal{K}}(S)$ is contained in $\mathcal{T}_p\tilde{\mathcal{K}}(T)$.
Let v be a direction in the interior of $\mathcal{K}(S)$. By Lemma 6.4.2 (ii), the relative interiors of the screens of S have a line transversal γ. Consider the family of lines $\bar{\ell}(t) = \psi(t\psi^{-1}(\gamma))$ that interpolates linearly (in our parametrization of lines) between ℓ and γ. Let s be a screen in S, b the corresponding ball in T and let Π denote the plane perpendicular to ℓ that contains s. We note that the trace of $\bar{\ell}(t)$ on Π forms a line passing through $b \cap \ell$. Since Π intersects B in a disk tangent to the halfplane s in $b \cap \ell$, it follows that there exists $\varepsilon_s > 0$ such that for any $t \in [0, \varepsilon_s]$ the line $\bar{\ell}(t)$ intersects b. If b is a ball in T to which ℓ is not tangent, the same holds trivially. Thus, there exists $\varepsilon > 0$ such that if $0 \leq t < \varepsilon$ then $\bar{\ell}(t)$ is a line transversal to T. The set of directions of the lines $\{\bar{\ell}(t) : t \geq 0\}$ forms, in $\mathbb{S}^2_2$, a great circle arc with endpoints u and v. It follows that $0 + \mathbb{R}^+ v$ belongs to the tangential cone of $\tilde{\mathcal{K}}(T)$ at p. □

6.4.4 Pinning Theorem for Ovaloids

The proof of Theorem 6.1.1 uses one property of cones of triples of balls that may not hold, in general, for cones of triples of ovaloids: that if the boundaries of two such cones of directions intersect in a direction u, there is a neighborhood of u in which the curves either coincide or intersect only at u. In other words, we use the property that the lower (or upper) envelope of boundaries of cones of directions near one of its vertices is defined by at most 2 curves[2]. To ensure that a similar property holds here, we now assume we deal with *semi-algebraic ovaloids* with bounded description complexity.
We first extend the smoothness condition of Lemma 6.2.6 to cones of directions of ovaloids, in view of Lemma 6.4.3.

[2] The constant 2 is not crucial: any constant bound k would imply a pinning theorem with constant $6k$.

Lemma 6.4.4 (*Singular points of* $\mathrm{bd}\mathcal{K}(T)$)
Let T be a triple of semi-algebraic ovaloids in $\mathbb{R}^3$. A direction u is a singular point of $\mathrm{bd}\mathcal{K}(T)$ only if the intersection of the three solid ovaloids is empty and there exists a line with direction u that is

(i) pinned by T, or

(ii) tangent to all three ovaloids, meeting two of them in the same point in which they are externally tangent.

Proof.
We prove the statement by contraposition. Let u be a direction of $\mathrm{bd}\mathcal{K}(T)$, ℓ the line transversal to T with direction u and let p be a point distinct from 0 on the ray $0+\mathbb{R}^+u$. For each ovaloid $C \in T$ we consider a ball $B^-(C)$ contained in (resp. $B^+(C)$ containing) C such that if ℓ is tangent to C then C and $B^-(C)$ (resp. $B^+(C)$) are internally tangent at $C \cap \ell$, and if ℓ intersects the interior of C then ℓ also intersects the interior of $B^-(C)$ (resp. $B^+(C)$). We let $T^- := \{B^-(C) : C \in T\}$ and $T^+ := \{B^+(C) : C \in T\}$. Observe that ℓ and T^- define the same triple of screens as ℓ and T^+; we call S that triple of screens.
We now make two observations. First, note that u belongs to $\mathrm{bd}\mathcal{K}(T^-)$ and $\mathrm{bd}\mathcal{K}(T^+)$, and thus p belongs to $\mathrm{bd}\tilde{\mathcal{K}}(T^-)$ and $\mathrm{bd}\tilde{\mathcal{K}}(T^+)$. Second, the inclusions $B^-(C) \subseteq C \subseteq B^+(C)$ imply that $\mathcal{K}(T^-) \subseteq \mathcal{K}(T) \subseteq \mathcal{K}(T^+)$, and similarly $\tilde{\mathcal{K}}(T^-) \subseteq \tilde{\mathcal{K}}(T) \subseteq \tilde{\mathcal{K}}(T^+)$. As a consequence, $\mathcal{T}_p\tilde{\mathcal{K}}(T^-) \subseteq \mathcal{T}_p\tilde{\mathcal{K}}(T) \subseteq \mathcal{T}_p\tilde{\mathcal{K}}(T^+)$.
Now, by Lemma 6.4.3, we have that near p the tangential cones $\mathcal{T}_p\tilde{\mathcal{K}}(T^-)$ and $\mathcal{T}_p\tilde{\mathcal{K}}(T^+)$ both agree with $\tilde{\mathcal{K}}(S)$. By Lemma 6.4.2, we have that near p the cone $\tilde{\mathcal{K}}(S)$ is a halfspace. Altogether, we get that near p, the tangential cone $\mathcal{T}_p\tilde{\mathcal{K}}(T)$ is a halfspace. Since $\mathcal{K}(T)$ (and therefore $\tilde{\mathcal{K}}(T)$) is semi-algebraic, this implies that p is a smooth point of $\mathrm{bd}\tilde{\mathcal{K}}(T)$. Therefore u is a smooth point of $\mathrm{bd}\mathcal{K}(T)$. □

We are now ready to prove our extension of Theorem 6.1.1 to semi-algebraic ovaloids:

Theorem 6.4.5 (*Pinning theorem for semi-algebraic ovaloids*)
Let $\mathcal{F}$ be a finite family of semi-algebraic ovaloids in $\mathbb{R}^3$ that pin a line ℓ. If no two members of $\mathcal{F}$ are externally tangent on ℓ then there is a subfamily of $\mathcal{F}$ of size at most 12 that pins ℓ.

Proof.
Assume that no triple of $\mathcal{F}$ pins ℓ, as otherwise the statement is trivially true. Let u denote the direction of ℓ. As noted in Section 6.2.2, Lemma 6.2.1 and Lemma 6.2.2 (ii) immediately extend to ovaloids. It follows that a family $\mathcal{F}$ of ovaloids pins a line ℓ if and only if the direction of ℓ is an isolated point of $\mathcal{K}(\mathcal{F}) = \bigcap_{T \in \binom{\mathcal{F}}{3}} \mathcal{K}(T)$. Now, for every triple $T \subseteq \mathcal{F}$ such that u is on the boundary of $\mathcal{K}(\mathcal{F})$ Lemma 6.4.4 ensures that $\mathrm{bd}\mathcal{K}(T)$ is smooth at u. We can then, as in the proof of Theorem 6.1.1, recast $\mathcal{K}(\mathcal{F})$ near u as the region above the lower envelope and below the upper envelope of a family

of semi-algebraic functions. Locally, these upper and lower envelopes are defined by two curves each, and the statement follows. □

List of Figures

Symbols and Notions

Symbols

N

O

P

R

S

T

U

V

W

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