Control with Exponentially Decaying Lyapunov Functions and Its Use for Systems with Input Saturation

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Abstract— **This paper presents a design method for feedback controls which leads to an exponentially decaying Lyapunov function for the closed-loop system. The rate of decay is the only and thus central parameter of the proposed design. The tight connection between the control law and the Lyapunov function is particularly favorable once the input saturation becomes active. In that case an analytic estimate for the domain of attraction is derived by using the Lyapunov function. Increasing the rate of decay decreases the size of the estimate. Hence the online variation of the rate of decay turns out to be a natural way for designing a variable-structure controller.**

Typically, the aim of linear state-feedback control is to transfer the state of a dynamical system to a desired equilibrium point within short time on the one hand and with moderate amplitude of the control input signal on the other hand. This compromise is to be found by suitably choosing the parameters of the control design (which, for instance, are the weighting matrices in a LQR design). If the allowable range of the control input signal is limited, the design is often done in a way that no input signal greater than the saturation limit is generated. In consequence the stability proof simplifies to the determination of invariant subspaces where the system behaves in a linear way (see [1] or the work related to low-gain feedback design e.g. [12]).

In this contribution, in order to achieve highly dynamic control, we do not only consider the standard linear state-space representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{1}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{b}, \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R}$, but will also take the input saturation explicitly into account, resulting in the nonlinear model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\,\mathrm{sat}(u) \tag{2}$$

with the saturation function

$$\mathrm{sat}(u) = \begin{cases} -u_{max} & \text{for} \quad u \le -u_{max} \\ u & \text{for} \quad -u_{max} < u < u_{max} \\ u_{max} & \text{for} \quad u \ge u_{max} \end{cases}.$$

Starting from system (1), we will first present a control design which results from constructing an exponentially decaying Lyponov function for the closed-loop system. The decay rate α can be seen as a natural indicator for the speed of the control system. Hence, using α as the only parameter of the control design makes the design process transparent.

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An important benefit of the suggested α-controller comes into play once the amplitude of u touches the input limitations of the systems u_{max} leading to a nonlinear system dynamic. In that case, other methods often require a separate stability analysis and the control design is done by extensive numerical optimization [7]. However, with the α-controller and based on the Lyapunov function coming with it, it is possible to *analytically* determine an estimate for the domain of attraction of the closed-loop, nonlinear system. Especially for a small rate of decay the achieved estimates seem to be quite good approximations of the full domain of attraction.

Furthermore, the estimates confirm an intuitively expected result, namely that a high rate of decay leads to a small domain of attraction and vice versa. Hence, increasing the speed of the controller (by increasing α) *while* $\mathbf{x}$ is approaching the origin, turns out to be a natural concept for improving the speed of the α-controller. Increasing the feedback gains and thereby the speed of the controller once the state vector approaches the origin is a well-known concept (see survey paper [1]) and is mainly referred to as variable-structure controls or switching controls [3]. However, apart from [3] and the presented controller no work is known to the authors, where the *variation* is done among controls, which may command input values u larger than u_{max}, leading to input saturation over long time intervals.

I. Design of the α-controller

In this section the design of a controller leading to an exponentially decaying Lyapunov function $V(\mathbf{x})$ is presented. The fundamental equation for the design of the controller is therefore

$$\dot{V} = -\alpha V \tag{3}$$

with $\alpha > 0$ being the rate of decay. If the input saturation is neglected it is possible to find a control law, such that a quadratic Lyapunov function $V = \mathbf{x}^T\mathbf{P}\mathbf{x}$ will fulfill (3). For this purpose the time derivative of V is computed by using (1):

$$\begin{aligned} \dot{V} &= \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \dot{\mathbf{x}} = 2\mathbf{x}^T\mathbf{P}\left(\mathbf{A}\mathbf{x} + \mathbf{b}u\right) \\ &= \mathbf{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\mathbf{b}u = -\alpha\mathbf{x}^T\mathbf{P}\mathbf{x}. \end{aligned} \tag{4}$$

In view of (4) the most natural control law for making $\dot{V}$ negative clearly is

$$u = -\mathbf{b}^T\mathbf{P}\mathbf{x}. \tag{5}$$

Inserting the control law (5) back into (4) one gets

$$\mathbf{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x} - 2\mathbf{x}^T\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}\mathbf{x} = -\alpha\mathbf{x}^T\mathbf{P}\mathbf{x}. \tag{6}$$

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From (6) an algebraic Riccati equation can be derived for determining the matrix $\mathbf{P}$:

$$\left(\mathbf{A}+\tfrac{\alpha}{2}\mathbf{I}\right)^T\mathbf{P}+\mathbf{P}\left(\mathbf{A}+\tfrac{\alpha}{2}\mathbf{I}\right)-2\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}=\mathbf{0} \quad (7)$$

Thus the algebraic Riccati equation (7) together with the control law (5) result in a closed-loop system, for which an exponentially decaying Lyapunov function can be found. The rate of decay α is the only and hence central parameter of the control design. We would therefore like to name the controller α-controller. In the following sections the properties of the α-controller will be investigated closer. With respect to (1) and (2) we thereby distinguish between the linear properties and the properties in the presence of input saturation.

II. Linear properties of the α-controller

The properties of $\mathbf{P}$, the solution to the Riccati equation (7), are the key for understanding the linear properties of the α-controller. The required results are mainly taken from [11] and briefly presented in the next section.

A. Properties of $\mathbf{P}$

Applying the theorems 13.5, 13.6 and 13.7 in [11] to the Riccati equation (7) results in the following theorem:

Theorem 1: Suppose $\mathbf{A}+\frac{\alpha}{2}\mathbf{I}=\mathbf{A}_\alpha$ has no imaginary eigenvalues and $(\mathbf{A}_\alpha,-2\mathbf{b}\mathbf{b}^T)$ is stabilizable then the solution to the Riccati equation (7) has the following properties:

- $\mathbf{P}$ is real and symmetric.
- $\mathbf{P}$ is positive semi-definite ($\mathbf{P}\geq\mathbf{0}$).
- $\mathbf{P}$ is positive definite, if $\mathbf{A}_\alpha$ has no stable modes.
- $\mathbf{A}_\alpha-2\mathbf{b}\mathbf{b}^T\mathbf{P}$ is stable.

Furthermore by following the proof of theorem 13.7 in [11] the next theorem can be formulated.

Theorem 2: If $\mathbf{P}$ is positive semi-definite, then the kernel of $\mathbf{P}$ is spanned by the stable eigenvectors of $\mathbf{A}_\alpha$.

Proof: Let $\mathbf{x}\in\text{Ker}(\mathbf{P})$, then $\mathbf{P}\mathbf{x}=\mathbf{0}$. Post-multiplying (7) by $\mathbf{x}$ leads to $\mathbf{P}\mathbf{A}_\alpha\mathbf{x}=\mathbf{0}$. Hence $\text{Ker}(\mathbf{P})$ is an $\mathbf{A}_\alpha$-invariant subspace and thus has to be spanned by the eigenvectors $\boldsymbol{v}$ of $\mathbf{A}_\alpha$. For those eigenvectors one can write $\lambda\boldsymbol{v}=\mathbf{A}_\alpha\boldsymbol{v}=(\mathbf{A}_\alpha-2\mathbf{b}\mathbf{b}^T\mathbf{P})\boldsymbol{v}$. As $\mathbf{A}_\alpha-2\mathbf{b}\mathbf{b}^T\mathbf{P}$ is stable, $Re(\lambda)$ has to be negative.
Conversely if $\mathbf{A}_\alpha$ has a stable eigenvalue λ one gets by pre- and post-multiplying (7) by the corresponding vector $\boldsymbol{v}^H$ and $\boldsymbol{v}$

$$2Re(\lambda)\boldsymbol{v}^H\mathbf{P}\boldsymbol{v}-2\boldsymbol{v}^H\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}\boldsymbol{v}=\mathbf{0}. \quad (8)$$

As $\mathbf{P}\geq\mathbf{0}$ and $Re(\lambda)<0$ one gets $\boldsymbol{v}^H\mathbf{P}\boldsymbol{v}=\mathbf{0}$. Thus if $\text{Ker}(\mathbf{P})\neq\mathbf{0}$ it is spanned by the stable eigenvectors of $\mathbf{A}_\alpha$ and if $\mathbf{A}_\alpha$ has stable eigenvectors $\text{Ker}(\mathbf{P})\neq\mathbf{0}$. □

B. Influence of α on the closed loop dynamics

For the choice of α the conditions of theorem 1 are important. Due to them the existence of a positive semi-definite solution $\mathbf{P}$ requires the pair $(\mathbf{A}_\alpha,-2\mathbf{b}\mathbf{b}^T)$ to be stabilizable. As $\mathbf{A}$ and $\mathbf{A}_\alpha$ have the same eigenvectors this is always fulfilled if $(\mathbf{A},\mathbf{b}\mathbf{b}^T)$ is controllable. If $(\mathbf{A},\mathbf{b}\mathbf{b}^T)$ is only stabilizable the parameter α has to be chosen in a way such that no uncontrollable eigenvector of $\mathbf{A}_\alpha$ becomes unstable. Furthermore, it is claimed that $\mathbf{A}_\alpha$ must not have any imaginary eigenvalues. This condition is always made when the solution of the Riccati equation is discussed. However, it is not important for the presented controller design, see Remark VI-A in the appendix.
If the parameter α is chosen in a reasonable way, then the most important relation between α and the dynamics of the closed-loop system is given by the following theorem.

Theorem 3: Let the matrix $\mathbf{A}$ have q eigenvalues with $Re(\lambda_{1\ldots q})>-\frac{\alpha}{2}$ and $p=n-q$ eigenvalues with $Re(\lambda_{q+1\ldots n})\leq-\frac{\alpha}{2}$. If there exists a positive semi-definite solution $\mathbf{P}$ to the corresponding Riccati equation (7), then the matrix $\mathbf{A}-\mathbf{b}\mathbf{b}^T\mathbf{P}$ of the closed-loop system has q eigenvalues with $Re(\lambda_{1\ldots q})=-\frac{\alpha}{2}$ and the p eigenvalues $\lambda_{q+1\ldots n}$ of $\mathbf{A}$ with $Re(\lambda_{q+1\ldots n})\leq-\frac{\alpha}{2}$.

Proof: The p eigenvalues of $\mathbf{A}$ with $Re(\lambda_{q+1\ldots n})\leq-\frac{\alpha}{2}$ are the stable eigenvalues of $\mathbf{A}_\alpha$. Hence the corresponding eigenvectors span the kernel of $\mathbf{P}$ (Theorem 2). Therefore they are not influenced by the control law $u=-\mathbf{b}^T\mathbf{P}\mathbf{x}$ and remain eigenvalues of the closed-loop system.

Let $\boldsymbol{v}$ be an eigenvector of $\mathbf{A}-\mathbf{b}\mathbf{b}^T\mathbf{P}$, which is not in the kernel of $\mathbf{P}$. By multiplying the Riccati equation (7) by $\boldsymbol{v}^H$ and $\boldsymbol{v}$ one gets

$$\begin{aligned}\underbrace{\boldsymbol{v}^H(\mathbf{A}-\mathbf{b}\mathbf{b}^T\mathbf{P})^T}_{\bar{\lambda}\boldsymbol{v}^H}\mathbf{P}\boldsymbol{v}+\boldsymbol{v}^H\mathbf{P}\underbrace{(\mathbf{A}-\mathbf{b}\mathbf{b}^T\mathbf{P})\boldsymbol{v}}_{\lambda\boldsymbol{v}} &= -\alpha\boldsymbol{v}^H\mathbf{P}\boldsymbol{v}\\ 2Re(\lambda)\boldsymbol{v}^H\mathbf{P}\boldsymbol{v} &= -\alpha\boldsymbol{v}^H\mathbf{P}\boldsymbol{v}\\ \Rightarrow Re(\lambda) &= -\tfrac{\alpha}{2}.\end{aligned}$$

Thus it is shown that the eigenvalues of the closed-loop matrix have either a real part of $-\frac{\alpha}{2}$ or the corresponding eigenvectors have to be in the kernel of $\mathbf{P}$. □

1) Example: For illustrating the meaning of theorem 1 an α-controller is designed for the system:

$$\mathbf{A}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-4&0&5&0\end{bmatrix}\qquad \mathbf{b}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad (9)$$

having eigenvalues at $-2,-1,1,2$. The real parts of the eigenvalues of the closed-loop system are shown in Fig. 1 as a function of α. In Fig. 2 the eigenvalues of the closed-loop and the open-loop system are shown in the complex plane. As stated by theorem 1 for small α the controller influences only the unstable eigenvalues of $\mathbf{A}$. The stable eigenvalues of $\mathbf{A}$ are only affected by the controller if their rate of decay is less than $\frac{\alpha}{2}$.

III. Properties of the α-controller in the presence of saturation

Neglecting the input saturation the α-controller can be just seen as a special kind of pole-placement controller. The importance of a Lyapunov function fulfilling (3) mainly comes into play once the input saturation is considered. In this case the Lyapunov function allows a good and *analytical*

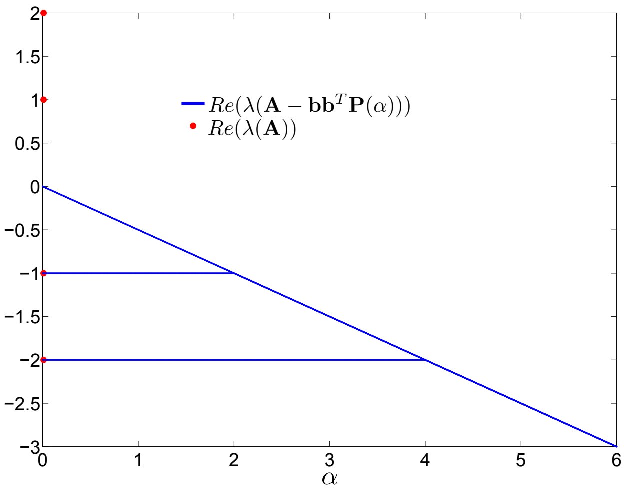

Fig. 1. Real part of the eigenvalues for the open-loop system (9) and the corresponding α-controller applied closed-loop system

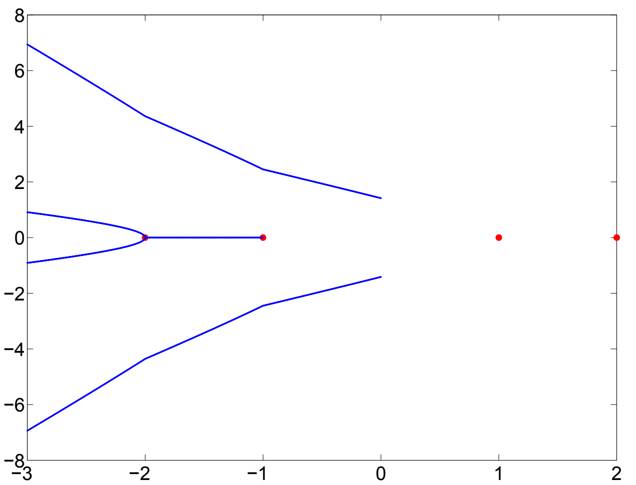

Fig. 2. Eigenvalues in the complex plane for the open-loop system (9) and the corresponding α-controller applied closed-loop system

estimation for the domain of attraction of the equilibrium point.

Definition 1: The domain of attraction of an equilibrium point is formed by all points asymptotically approaching the equilibrium point.

In general the domain of attraction of the nonlinear system (2) can only be determined by simulations. However, based on Lyapunov's theory, an analytical estimation $\mathbb{S}$ can be derived. ([8])

Theorem 4: Consider the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Let $V(\mathbf{x})$ be a scalar function with continuous partial derivatives. If in a neighborhood $\mathbb{S}(\eta) = \{\mathbf{x} \mid V(\mathbf{x}) < \eta\}$ of the origin

- $V(\mathbf{x})$ is positive definite
- and $\dot{V}$ is negative definite,

then $\mathbb{S}(\eta)$ is part of the domain of attraction.

As stated in theorem 4 $\mathbb{S}(\eta)$ is only *part* of the domain of attraction. Whether $\mathbb{S}(\eta)$ is a big or small part and hence a good or a very conservative estimation of the complete domain of attraction mainly depends on the choice of $V(\mathbf{x})$.

For linear systems with input saturation quadratic functions $V = \mathbf{x}^T\mathbf{P}\mathbf{x}$ seem appropriate and are used in general. Instead of choosing the matrix $\mathbf{P}$ by numerical optimization, as this is often done, equation (3) serves as a criterion for determining $\mathbf{P}$. Besides the exponential decay of V a constant value of $\dot{V}$ on the level sets of V is achieved by (3). In the presence of input saturation (for $|u| > u_{max}$) one can easily check that $\dot{V} > -\alpha V$. Thus by choosing α small, the shape of $\dot{V}$ on the boundary $\partial\mathbb{S}(\eta) = \{\mathbf{x} \mid V(\mathbf{x}) = \eta\}$ has to stay in a small interval, namely $[0; -\alpha\eta]$. If $\partial\mathbb{S}$ exactly separated the domains with $\dot{V} > 0$ and $\dot{V} < 0$, $\mathbb{S}(\eta)$ would be the complete domain of attraction. Therefore the possibility of limiting $\dot{V}$ on the boundary of $\mathbb{S}(\eta)$ to an arbitrarily small interval is expected to result in a good estimation for the domain of attraction.

A second benefit of the Lyapunov function coming with the α-controller is that η can be calculated *analytically*. Based on Theorem 4 η is the solution of the constraint optimization problem:

$$\max\ \eta \tag{10}$$
$$\dot{V} \leq 0\ \forall \mathbf{x} \in \mathbb{S}(\eta) \tag{11}$$

Instead of solving the optimization problem with the inequality constraint (11), one can look for the smallest level set where the equality $\dot{V} = 0$ holds:

$$\min_{\mathbf{x}} \eta = \mathbf{x}^T\mathbf{P}\mathbf{x} \tag{12}$$
$$\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}, \mathrm{Ker}(\mathbf{P})\} \tag{13}$$
$$\dot{V}(\mathbf{x}) = \mathbf{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\mathbf{b}\,\mathrm{sat}(u) = 0 \tag{14}$$

The solution of this optimization problem is among the stationary points of the related Lagrange function

$$L(\mathbf{x}, \mu) = \mathbf{x}^T\mathbf{P}\mathbf{x} + \mu\dot{V}. \tag{15}$$

Replacing $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}$ by $-\alpha\mathbf{P} + 2\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}$ and noticing that (14) can only be fulfilled where $|u| > u_{max}$ (otherwise $\dot{V} = -\alpha V < 0$), one gets

$$L(\mathbf{x}, \mu) = \mathbf{x}^T\mathbf{P}\mathbf{x} + \mu\left(\mathbf{x}^T\left(-\alpha\mathbf{P} + 2\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}\right)\mathbf{x} + +2\mathbf{x}^T\mathbf{P}\mathbf{b}u_{max}\mathrm{sgn}(u)\right). \tag{16}$$

The stationary points of L are given by

$$\frac{\partial L}{\partial \mu} = \mathbf{x}^T\left(-\alpha\mathbf{P} + 2\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}\right)\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\mathbf{b}u_{max}\mathrm{sgn}(u) = 0 \tag{17}$$
$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{P}\left((2 - \mu\alpha)\mathbf{x} + \mu(4\mathbf{b}^T\mathbf{P}\mathbf{x} + 2u_{max}\mathrm{sgn}(u))\mathbf{b}\right) = \mathbf{0}. \tag{18}$$

The derivative of $\mathrm{sgn}(u)$ has been neglected in (18) because the solution $\mathbf{x}^*$ has to be where $|u| > u_{max}$. As the expressions in the two brackets in (18) are scalars and $\mathbf{x}^* \notin \mathrm{Ker}(\mathbf{P})$, it can be concluded that

$$\mathbf{x}^* = a\mathbf{b}. \tag{19}$$

The factor $a \in \mathbb{R}\backslash\{0\}$ can be computed by inserting (19) into (17):

$$a = \pm\frac{2}{2\mathbf{b}^T\mathbf{P}\mathbf{b} - \alpha}u_{max} \tag{20}$$

Thus one gets the two solutions for $\mathbf{x}^*$

$$\mathbf{x}^* = \pm\frac{2}{2\mathbf{b}^T\mathbf{P}\mathbf{b} - \alpha}u_{max}\mathbf{b} \tag{21}$$

which are both situated on the level set $\mathbb{S}(\eta)$

$$\eta = \mathbf{x}^{*T}\mathbf{P}\mathbf{x}^* = \frac{4\mathbf{b}^T\mathbf{P}\mathbf{b}}{(2\mathbf{b}^T\mathbf{P}\mathbf{b} - \alpha)^2}u_{max}^2. \tag{22}$$

Hence an analytical estimation for the domain of attraction is given by

$$\mathbb{S}(\mathbf{P}, \eta) = \{\mathbf{x} \mid \mathbf{x}^T\mathbf{P}\mathbf{x} < \eta\} \tag{23}$$

with η according to (22). As shown in [5] it is possible to express $\mathbf{b}^T\mathbf{P}\mathbf{b}$ by α and the eigenvalues of $\mathbf{A}$.

Theorem 5: Let $\mathbf{A}$ have exactly q eigenvalues with $Re(\lambda_{1\ldots q}(\mathbf{A})) > -\frac{\alpha}{2}$ and denote the sum of these eigenvalues by γ

$$\gamma = \sum_{i=1}^{q}\lambda_i(\mathbf{A}).$$

Then the following equation holds

$$\mathbf{b}^T\mathbf{P}\mathbf{b} = \gamma + q\frac{\alpha}{2}. \tag{24}$$

Using (24) in (22) one gets

$$\eta = 2\frac{2\gamma + q\alpha}{(2\gamma + (q-1)\alpha)^2}u_{max}^2. \tag{25}$$

From (25) some interesting results concerning the size of $\mathbb{S}(\mathbf{P}, \eta)$ can be derived. The proofs for the theorems 6 - 8 are given in [5].

Theorem 6: As long as $\alpha > 0$ is chosen in a way such that only one eigenvalue of $\mathbf{A}$ is shifted by the controller, the size of $\mathbb{S}(\mathbf{P}, \eta)$ doesn't change.

If the shifted eigenvalue has originally been in the open right-half complex plane, $\mathbb{S}(\mathbf{P}, \eta)$ exactly describes the maximal domain of attraction of the equilibrium point.

Theorem 6 is mainly of interest for systems with one unstable eigenvalue. In that case a reasonable lower bound for α is given by $\alpha_{min} = -2Re(\lambda_2)$ with $\lambda_2(\mathbf{A})$ being the eigenvalue of $\mathbf{A}$ with the second biggest real part. Choosing α smaller than α_{min} results in a slower controller without enlarging the domain of attraction. Furthermore $\mathbb{S}(\mathbf{P}(\alpha_{min}), \eta_{min})$ describes the domain of attraction exactly and cannot be enlarged by any other control law.

Theorem 7: If all eigenvalues of $\mathbf{A}$ are in the closed left-half plane, then there exists an $\hat{\alpha}$ such that for $\alpha \to \hat{\alpha}$ (from above) η tends towards infinity and hence $\mathbb{S}(\mathbf{P}, \eta)$ contains the whole state space. The value of $\hat{\alpha}$ is given by

$$\hat{\alpha} = \frac{-2\gamma}{q-1} \tag{26}$$

with q denoting the number of all eigenvalues of $\mathbf{A}$ with $Re(\lambda(\mathbf{A})) \geq -\frac{\hat{\alpha}}{2}$ and γ denoting the sum of those eigenvalues.

Theorem 7 is in accordance with the expectation that it is always possible to find a globally stabilizing controller for a stable system. The interesting point is the rate of decay, which can be achieved by the globally stabilizing controller. Table I shows the resulting $\hat{\alpha}$ for some exemplary pole configurations. It can be seen there, that although the sum of $\lambda(\mathbf{A})$ is for all examples -12, the globally stabilizing rate of decay $\hat{\alpha}$ differs quite a lot depending on the distribution of the open-loop eigenvalues. In any case, however, $\hat{\alpha}$ is obviously a reasonable lower bound for α.

TABLE I

$\hat{\alpha}$ FOR DIFFERENT POLE CONFIGURATIONS

$\lambda(\mathbf{A})$	$\hat{\alpha}$
$-1, -2, -9$	6
$-2, -2, -8$	8
$-3, -4, -5$	12

For the special case of an asymptotically stable scalar system q becomes 1 and hence $\hat{\alpha}$ becomes ∞. Although this very simple case is not of real interest, the resulting $\hat{\alpha}$ confirms that arbitrarily fast controllers can be applied to an asymptotically stable scalar plant.

Theorem 8: If $\alpha > \hat{\alpha}$ (for open-loop stable systems) or chosen such that more than one eigenvalue of $\mathbf{A}$ is shifted by the control law ($q \geq 2$), the following property holds:

$$\begin{aligned} \eta' &= \frac{d\eta}{d\alpha} < 0 \\ \mathbf{P}' &= \frac{d\mathbf{P}}{d\alpha} \geq 0 \end{aligned}$$

The consequence of theorem 8 is that the size of $\mathbb{S}(\mathbf{P}, \eta)$ is decreasing with increasing values of α. Hence theorem 8 partially confirms some considerations of the control design, namely that small values of α result in a good or respectively large estimation for the domain of attraction.

2) Example: For illustrating the meaning of theorem 6 and theorem 8 the estimations $\mathbb{S}(\alpha) = \mathbb{S}(\mathbf{P}(\alpha), \eta(\alpha))$ are shown in Fig. 3 for different values of α. The dynamics of the therefore considered system are

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\text{sat}(u) \tag{27}$$

with $u_{max} = 5$. Besides the decreasing of $\mathbb{S}(\alpha)$ with increasing α, the situation for $\alpha = 2$ is of special interest. In that case $\mathbf{P}$ becomes singular and $\mathbb{S}(\alpha)$ becomes unbounded in the direction of the stable eigenvector of $\mathbf{A}$. However, because of the input saturation neither a further decrease of α nor the use of any other controller can enlarge the size of $\mathbb{S}$ in the direction of the unstable left eigenvector.

IV. VARIABLE-STRUCTURE CONTROLLER

In summary the α-controller reveals the well-known trade-off between the closed-loop dynamics and the domain of attraction, which can be achieved by a linear controller. A possible way to overcome this trade-off can be seen in a variable-structure control design [1], [3]. The basic idea of such a control design is to vary the feedback gains of a linear controller in dependency on the state vector thus resulting in a nonlinear controller. The main goal of such a variation is a better utilization of the actuator and hence approaching the time optimal controller.

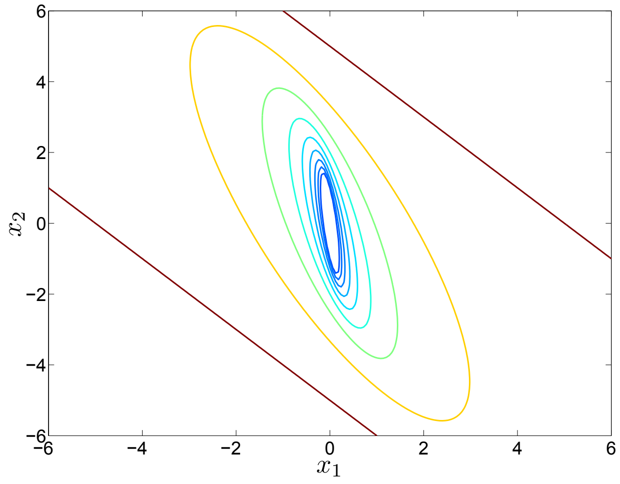

Fig. 3. $\mathbb{S}(\alpha)$ for $\alpha = 2, 4, ..., 10$ (from outer to inner) for the system (27)

Considering the properties of the α-controller the variation of α turns out to be a natural way for the design of a variable-structure controller. The variation of α is therefore done by the following selection strategy:

Chose the largest α *inside the interval* $[\alpha_{min}; \alpha_{max}]$ *for which* $\mathbf{x} \in \mathbb{S}(\alpha)$ *is true.*

If this strategy is applied continuously, $\mathbf{x}$ will always be on the boundary of $\mathbb{S}(\alpha)$. Therefore the *continuous* selection strategy can be expressed as

$$\mathbf{x}^T \mathbf{P}(\alpha)\mathbf{x} = \beta\eta(\alpha) \tag{28}$$

with $\beta \rightarrow 1$. Unfortunately (28) cannot explicitly be solved for α. Thus one way is to solve (28) at every time step by a numerical solver. This requires some numerical effort, however resulting in a smooth variation of α and u.

Besides the continuous variation also a *discrete* variation of α is possible. In the discrete case l matrices $\mathbf{P}_{1...l}$ and values $\eta_{1...l}$ are computed and stored for the corresponding values $\alpha_{1...l}$ in advance. During runtime only the largest α_i fulfilling (28) has to be selected among the l values $\alpha_{1...l}$, which requires much less computational power. The disadvantage of the discrete variation is the occurrence of jumps in α and u. By storing $\mathbf{P}$ and η for a sufficient large number of values α these jumps can however be made arbitrarily small. In dependence on the available computational power, the storage on the machine and the required smoothness of u either the continuous variation, the discrete variation or a combination of both variation strategies can be chosen.

If α is varied continuously, the stability of the variable-structure controller can be shown with the Lyapunov function

$$V = \mathbf{x}^T \mathbf{P}(\alpha)\mathbf{x} \tag{29}$$

with α and hence $\mathbf{P}$ chosen by (28). Computing the time derivative of (29) one gets

$$\dot{V} = \underbrace{\dot{\mathbf{x}}^T \mathbf{P}\mathbf{x} + \mathbf{x}^T \mathbf{P}\dot{\mathbf{x}}}_{\dot{V}_\alpha} + \mathbf{x}^T \mathbf{P}'\mathbf{x}\dot{\alpha} \tag{30}$$

with $\dot{V}_\alpha$ denoting the time derivative of V for a fixed α. As long as $\alpha < \alpha_{max}$ the selection strategy (28) is active and hence one can write

$$\dot{V} = \dot{V}_\alpha + \mathbf{x}^T \mathbf{P}'\mathbf{x}\dot{\alpha} = \beta\eta'\dot{\alpha}. \tag{31}$$

The time derivative $\dot{V}_\alpha$ has to be negative definite because $\mathbf{x} \in \mathbb{S}(\alpha)$. Furthermore, because of theorem 8 one has $\mathbf{P}' \geq \mathbf{0}$ and $\eta' < 0$. Therefore one can conclude from (31) that $\dot{\alpha} > 0$ and thus $\dot{V} = \beta\eta'\dot{\alpha} < 0$.

Once α has reached α_{max} the time derivative $\dot{\alpha}$ vanishes and $\dot{V} = \dot{V}_\alpha < 0$.

If α is varied discretely, $\mathbf{P}' \geq \mathbf{0}$ and $\eta' < 0$ are again the key properties for proving the stability of the controller. Due to these two properties the switching surfaces $\partial\mathbb{S}(\alpha_i)$ (where α is increased from α_{i-1} to α_i) are nested (see e.g. Fig. 3). While $\alpha = \alpha_i$ it holds that $\mathbf{x} \in \mathbb{S}(\alpha_i)$ and therefore $V_i = \mathbf{x}^T \mathbf{P}(\alpha_i)\mathbf{x}$ is a valid Lyapunov function and the corresponding closed-loop system performs asymptotically stable. In consequence $\mathbf{x}$ approaches the origin thereby crossing the switching surface $\mathbb{S}(\alpha_{i+1})$. Thus the same conditions as in the continuous case ensure also in the discrete case the increase of α. Once $\alpha = \alpha_l = \alpha_{max}$ asymptotic stability results by using $V = \mathbf{x}^T \mathbf{P}(\alpha_{max})\mathbf{x}$.

3) Example: As an example the variable-structure controller is applied to the model of a submarine, which was firstly described by [6]

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -0.005 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{sat}(u) \tag{32}$$

with $\mathbf{x}_1$ being the dive depth of the submarine. The input saturation of the linearized model is given by $u_{max} = 2.5e - 5$. For testing the controller an initial disturbance of $\mathbf{x}_0 = [0\ 0\ -0.004]^T$ shall be compensated. In Fig. 4-6 the results are shown for the time optimal controller, a fixed α-controller and a variable-structure α-controller. The latter varies the parameter α inside the interval $[0.01; 0.58]$. The α for the fixed controller is set to 0.011 ensuring $\mathbf{x}_0 \in \mathbb{S}$.

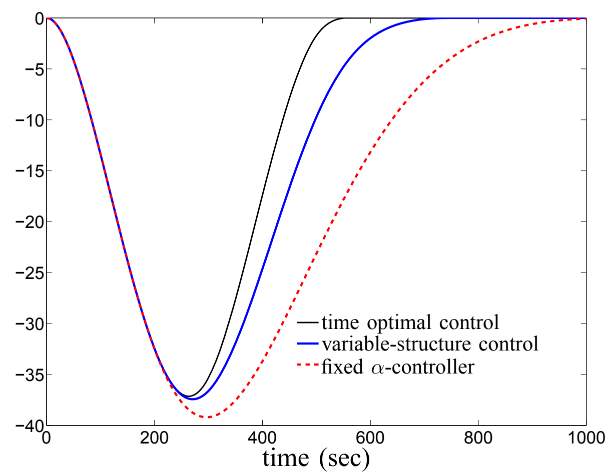

Fig. 4. Dive depth of the submarine x_1 in m

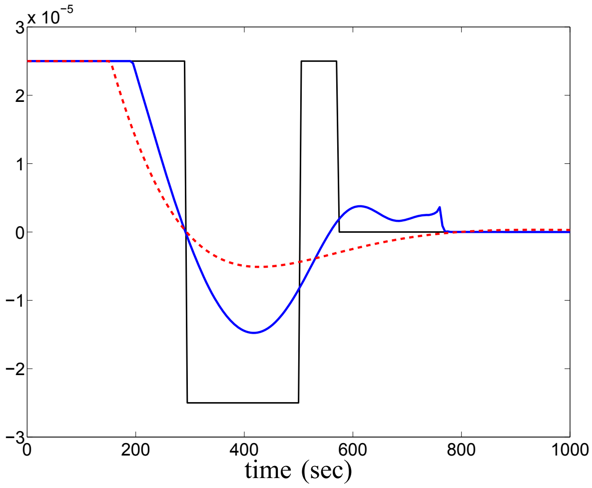

Fig. 5. Input to the submarine system sat(u)

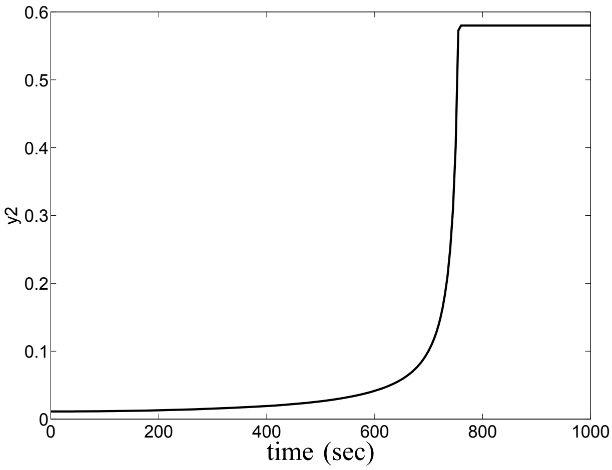

Fig. 6. Variation of α

For the evaluation of the controller performance Table II shows the time which is achieved by another structure-variable controller [1] or saturating fixed controller [6] for reaching a dive depth of $0.1m$. In comparison to the fixed but saturating controller [6] or to the variable-structure controller [1] the results of the corresponding α-controllers can compete very well.

TABLE II

TIME TO REACH A DIVE DEPTH OF $0.1m$

time optimal	var.-str. [1]	var.-str. α	fixed sat. [6]	fixed α
541	792	703	2340	989

V. SUMMARY AND CONCLUSION

The paper presents a control design, which allows to compute an exponentially decreasing Lyapunov function for the closed loop system. The rate of decay α is the *only* parameter of the control design. Hence the complexity of the design process is independent of the order of the system. Furthermore, α is a very meaningful parameter. Therefore a reasonable choice of α should be an easy task for the control engineer.

In the context of actuator saturation the suggested control design provides an analytic estimate for the domain of attraction. Based on that the incorporation of the α-controller in a variable-structure control design turns out to be a natural way for achieving a fast controller with a very large estimate for the domain of attraction (resulting from α_{min}). Besides the limits α_{min} and α_{max} no further parameters have to be set by the control engineer. A numerical example shows that the resulting controller comes rather close to the time optimal trajectory confirming the performance of the controller.

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VI. APPENDIX

A. Remark

If $\mathbf{A}_\alpha$ has imaginary eigenvalues, which are not observable by $\mathbf{Q}$ (this is always the case, because $\mathbf{Q} = \mathbf{0}$), it can be easily shown that the corresponding eigenvectors are in the kernel of $\mathbf{P}$. Therefore $\mathbf{A}_\alpha - 2\mathbf{b}\mathbf{b}^T\mathbf{P}$ still has imaginary eigenvalues and hence is not asymptotically stable. In order to avoid a solution $\mathbf{P}$, which is not asymptotically stabilizing, $\mathbf{A}_\alpha$ is in general assumed to have no imaginary eigenvalues. However as the eigenvalues of the real system $\mathbf{A}$ are placed $\frac{\alpha}{2}$ left of the eigenvalues of $\mathbf{A}_\alpha$, the corresponding eigenvalues of the real closed-loop system $\mathbf{A}-\mathbf{b}\mathbf{b}^T\mathbf{P}$ have a real part of $-\frac{\alpha}{2}$. Therefore the occurrence of imaginary eigenvalues in $\mathbf{A}_\alpha$ does not cause any problems for the α-controller.