Parametric Model Order Reduction by Matrix Interpolation

Parametrische Ordnungsreduktion mittels Matrixinterpolation

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Summary In this paper, a new framework for model order reduction of LTI parametric systems is introduced. After generating and reducing several local original models in the parameter space, a parametric reduced-order model is calculated by interpolating the system matrices of the local reduced models. The main task is to find compatible system representations with optimal interpolation properties. Two approaches for this purpose are presented together with several numerical simulations.
►►► **Zusammenfassung** In diesem Beitrag wird ein neuer Rahmen zur Modellordnungsreduktion parametrischer LZI-Systeme vorgestellt. Er sieht zunächst die Reduktion des Originalmodells in einigen Stützstellen des Parameterraums vor. Anschließend wird ein reduziertes parametrisches System generiert, indem die Systemmatrizen der lokalen Reduktionen durch geeignete Transformationen kompatibel gemacht und interpoliert werden. Hierfür werden zwei alternative Verfahren beschrieben, deren Eigenschaften an drei numerischen Testfällen dargestellt werden.

Keywords Order reduction, parametric model order reduction, interpolation, LTI systems ►►►
Schlagwörter Ordnungsreduktion, parametrische Modellordnungsreduktion, Interpolation, LZI-Systeme

1 Introduction and Problem Formulation

Mathematical modelling of dynamical systems for the sake of optimization, simulation or control often yields systems of ordinary differential equations (ODE) whose complexity grows dramatically with increasing demands on the accuracy of the FE model. Model Order Reduction (MOR) techniques like Truncated Balanced Realization (TBR) [2; 11], Krylov Subspace Methods (also known as Moment Matching) [2; 7] or Modal Reduction Methods e. g. [8; 14] seek to replace such large-scale systems by considerably smaller ones which approximate the transfer behaviour of the original system as accurately as possible, preserve some of its properties like stability or passivity and offer an error bound for the approximation.

Once the considered large-scale systems depend on one or several parameters, for instance geometry, material properties or mode of operation, the already existing methods are not suitable anymore to conduct the reduction step while preserving the parameter-dependency in the reduced-order model. Hence, there is a need to adapt and further improve these reduction methods or possibly create new ones that allow the reduction of such classes of dynamical systems. This new and emerging branch of MOR is known as *Parametric Model Order Reduction* (pMOR).

Consider a parametric MIMO state-space model in descriptor form

$$\begin{aligned} \mathbf{E}(\mathbf{p})\dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{p})\,\mathbf{x}(t) + \mathbf{B}(\mathbf{p})\,\mathbf{u}(t)\,, \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p})\,\mathbf{x}(t)\,, \end{aligned} \tag{1}$$

where $\mathbf{E}(\mathbf{p})$, $\mathbf{A}(\mathbf{p}) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(\mathbf{p}) \in \mathbb{R}^{n\times m}$, $\mathbf{C}(\mathbf{p}) \in \mathbb{R}^{p\times n}$ are parameter-dependent matrices with constant coefficients, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, and $\mathbf{x}(t) \in \mathbb{R}^n$ are, respectively, the inputs, outputs and states of the system, while $\mathbf{p} \in \mathbf{\Pi} \subseteq \mathbb{R}^d$ is the vector of parameters. The main goal of parametric model order reduction is to find a reduced model that preserves the parameter-dependency, thus al-

lowing a variation of any of the parameters without the need to repeat the reduction step. Thereby, the reduction method should ideally be able to cope with any number of parameters and with systems where no analytical expression of the parameter-dependency in the matrices is available. In addition, it should be numerically efficient to be suitable for the reduction of large-scale systems and at the same time its computational cost should be low enough to keep the reduction step numerically justified.

The first work dealing with Krylov-based parametric order reduction of linear systems was presented in [15], where the moment matching approach has been generalized to a parametric system with a matrix $\mathbf{A}$ linearly depending on one parameter. It was shown how a projection matrix $\mathbf{V}$ can be calculated, such that the reduced model not only matches some of the first moments of the transfer function $\mathbf{G}(s, p)$ with respect to s, but also with respect to the parameter p. This work has been generalized in [5] to the multiple parameter case by deriving suitable Krylov subspaces guaranteeing matching of the coefficients of the multivariate Taylor series having s and all the parameters p_i as variables. The method suffers from the *curse of dimensionality* where the order of the reduced system grows very rapidly even for a low number of parameters. In addition, it turned out that it is often difficult in practice to generate parametric models with an analytically expressed parameter dependency.

Another well-known technique [9; 13] consists of calculating local projection matrices from several local models in the parametric space, merge these matrices together, and then apply a common order reducing projection to the original parametric model. The main advantage of this method is its simple and direct way of calculating the projection matrices. However, in order to result in a parametric reduced order model, the parameter dependency needs to be affine. Moreover, the order of the reduced system tends to become very large once many local models are considered, and no moment matching property can be guaranteed or proven for the obtained reduced models.

In [4], a TBR-based method using interpolation and a soft switching between weighted linear reduced order transfer functions calculated at different points in the parameter space is presented. For the choice of the parameter points where the local reduced model are to be calculated and reduced, it was suggested to employ the sparse grids method which is an efficient tool once the parameter dependency of the system matrices can be analytically expressed. The weighting functions play a major role in this method and need to be carefully chosen to minimize the interpolation error at different parameter values. The order of the resulting model depends on the order and number of the involved local models and the method faces difficulties when the involved systems have weakly damped modes. In [3], this approach has been further developed and coupled to the interpolatory $\mathcal{H}_2$ optimal model reduction method.

In this article, a novel framework for the reduction of parameter-dependent linear dynamical systems of the form (1) is introduced. In a first step, the original model is generated and independently reduced at k different parameter values with individual projection matrices [1]. The reduced parametric model is then calculated by a weighted interpolation between the matrices of these k reduced models. Prior to that, however, suitable transformations are applied to each of these reduced systems, in order to make this interpolation meaningful.

The main features of the new approach are that the parametric matrices $\mathbf{E}(\mathbf{p}), \mathbf{A}(\mathbf{p}), \mathbf{B}(\mathbf{p})$, and $\mathbf{C}(\mathbf{p})$ need to be known and evaluated only at k discrete values of the vector $\mathbf{p}$. This significantly simplifies the modeling process in many practical cases, where it is often impossible to assume or obtain an affine or analytical parameter dependency. Also, the order of the reduced model will equal q, independently of the number k of local models considered. This allows increasing the number of the local models in order to better capture the generally unknown parameter dependency without increasing the complexity of the resulting reduced model. In addition, a main feature of this new framework is that the reduction method to be applied to compute the local reduced models can be freely chosen.

The rest of this paper is organized as follows: In the following section, a short overview of projection-based order reduction is given. The new interpolation-based framework for parametric model reduction is introduced in Sect. 3. The different approaches for adjusting the local reduced order models to allow their interpolation are presented together with their main features in Sect. 4. In Sect. 5, reduced parametric models of three technical systems are calculated to illustrate the suitability and the main features of the proposed methods.

2 Projection-based Order Reduction

Consider the Linear Time Invariant (LTI) dynamical system in descriptor form

$$\begin{aligned} \mathbf{E}\,\dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)\,, \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t)\,, \end{aligned} \tag{2}$$

where $\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ are matrices with constant coefficients, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, and $\mathbf{x}(t) \in \mathbb{R}^n$ are, respectively, the input, output and state vectors of the system. $n \in \mathbb{N}$ is called the order of the system and is considered here to be very large.

Projection-based Model Order Reduction seeks to approximate the state vector $\mathbf{x}(t) \in \mathbb{R}^n$ as

$$\mathbf{x}(t) \approx \mathbf{V}\mathbf{x}_r(t) \tag{3}$$

where $\mathbf{x}_r(t) \in \mathbb{R}^q$, $\mathbf{V} \in \mathbb{R}^{n\times q}$ and $q \ll n$. Inserting (3) in (9) leads to an overdetermined system

$$\mathbf{E}\,\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{A}\,\mathbf{V}\mathbf{x}_r(t) + \mathbf{B}\,\mathbf{u}(t) + \boldsymbol{\epsilon}(t) \tag{4}$$

with q unknowns but n equations. In general, this system can not be exactly solved without a residual $\boldsymbol{\epsilon}(t)$. In order to obtain a well-determined system of equations, the state equation above is multiplied from the left by the transpose of a matrix $\mathbf{W} \in \mathbb{R}^{n\times q}$ leading to

$$\begin{aligned} \mathbf{W}^T\mathbf{E}\mathbf{V}\,\dot{\mathbf{x}}_r(t) &= \mathbf{W}^T\mathbf{A}\mathbf{V}\,\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{B}\,\mathbf{u}(t)\,, \\ \mathbf{y}_r(t) &= \mathbf{C}\mathbf{V}\,\mathbf{x}_r(t)\,. \end{aligned} \tag{5}$$

Note that the resulting residual $\boldsymbol{\epsilon}(t)$ is then orthogonal to the subspace spanned by the columns of $\mathbf{W}$, i. e. $\mathbf{W}^T\boldsymbol{\epsilon}(t) \equiv \mathbf{0}$.

Accordingly, the matrices of the reduced system can be calculated as:

$$\begin{aligned} \mathbf{E}_r &= \mathbf{W}^T\mathbf{E}\mathbf{V}\,, \qquad & \mathbf{A}_r &= \mathbf{W}^T\mathbf{A}\mathbf{V}\,, \\ \mathbf{B}_r &= \mathbf{W}^T\mathbf{B}\,, & \mathbf{C}_r &= \mathbf{C}\mathbf{V}\,. \end{aligned} \tag{6}$$

This procedure can be seen in fact as a projection of the original state equation onto a rank-q subspace spanned by the columns of $\mathbf{V}$ along the orthogonal complement of $\mathbf{W}$ using the projector

$$\mathcal{P} = \mathbf{V}(\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\,, \tag{7}$$

leading to

$$\begin{aligned} &\mathbf{V}(\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \\ &\mathbf{V}(\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{V}(\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B}\,\mathbf{u}(t)\,. \end{aligned} \tag{8}$$

The equality above is described with respect to the basis $\mathbf{V}$. Its solution $\mathbf{x}_r(t)$ is not affected when multiplying (8) from the left by $\mathbf{W}^T$, which then leads to the commonly used description of the reduced order system (5).

Finding suitable projection matrices $\mathbf{V}$ and $\mathbf{W}$ under the restriction $\det(\mathbf{W}^T\mathbf{V}) \neq 0$, is in fact the main task of a MOR method such as the Modal Reduction approach, the Truncated Balanced Realization method, the Krylov-Subpace method (also known as Moment Matching) or Proper Orthogonal Decomposition.

3 Interpolation of Locally Reduced Systems

Consider the parametric LTI dynamical system (1) given at the discrete parameter values $\mathbf{p}_i$, for $i = 1\ldots k$, as:

$$\begin{aligned} \mathbf{E}_i\,\dot{\mathbf{x}}(t) &= \mathbf{A}_i\,\mathbf{x}(t) + \mathbf{B}_i\,\mathbf{u}(t)\,, \\ \mathbf{y}(t) &= \mathbf{C}_i\,\mathbf{x}(t)\,. \end{aligned} \tag{9}$$

Each of the k models is reduced using separate subspaces represented by the different projection matrices $\mathbf{V}_i, \mathbf{W}_i \in \mathbb{R}^{n\times q}$. This makes it possible to focus on the approximation of each of the local models which, in general, do not have similar dynamics. Also, it should be noted that each of these projection matrices can be calculated according to a given order reduction method, e. g. by Truncated Balanced Realization, by a Modal reduction approach, by Proper Orthogonal Decomposition, or by Krylov subspace methods.

The resulting reduced-order models are then the following:

$$\begin{aligned} \overbrace{\mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i}^{\mathbf{E}_{r,i}}\,\dot{\mathbf{x}}_{r,i}(t) &= \overbrace{\mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i}^{\mathbf{A}_{r,i}}\,\mathbf{x}_{r,i}(t) + \overbrace{\mathbf{W}_i^T\mathbf{B}_i}^{\mathbf{B}_{r,i}}\,\mathbf{u}(t)\,, \\ y_{r,i}(t) &= \underbrace{\mathbf{C}_i\mathbf{V}_i}_{\mathbf{C}_{r,i}}\,\mathbf{x}_{r,i}(t) \end{aligned} \tag{10}$$

Once all the local reduced models are obtained, the resulting parametric reduced model is calculated by a weighted interpolation of the matrices of these models as:

$$\begin{aligned} \mathbf{E}_r\,\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\,\mathbf{x}_r(t) + \mathbf{B}_r\,\mathbf{u}(t)\,, \\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\mathbf{x}_r(t)\,, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathbf{A}_r &= \textstyle\sum_{i=1}^k \omega_i(\mathbf{p})\,\mathbf{A}_{r,i}\,, \qquad & \mathbf{E}_r &= \textstyle\sum_{i=1}^k \omega_i(\mathbf{p})\,\mathbf{E}_{r,i}\,, \\ \mathbf{B}_r &= \textstyle\sum_{i=1}^k \omega_i(\mathbf{p})\,\mathbf{B}_{r,i}\,, & \mathbf{C}_r &= \textstyle\sum_{i=1}^k \omega_i(\mathbf{p})\,\mathbf{C}_{r,i}\,, \end{aligned} \tag{12}$$

and $\sum_{i=1}^k \omega_i(\mathbf{p}) = 1$, $\omega_i(\mathbf{p}_j) = \delta_{ij}$ for $i, j = 1\ldots k$, $\mathbf{p} \in \boldsymbol{\Pi}$.

So far, however, it is not clear if the interpolation (12) is in fact meaningful, since the reduced state vectors $\mathbf{x}_{r,i}$ do not have the same physical interpretation due to the fact that the projection matrices $\mathbf{V}_i$ are generally not equal. Clearly, applying a state transformation $\mathbf{T}_i$ and pre-multiplying the models (10) by $\mathbf{M}_i$ from the left,

$$\begin{aligned} \overbrace{\mathbf{M}_i\mathbf{E}_{r,i}\mathbf{T}_i^{-1}}^{\mathbf{E}^*_{r,i}}\,\dot{\mathbf{x}}^*_{r,i}(t) &= \overbrace{\mathbf{M}_i\mathbf{A}_{r,i}\mathbf{T}_i^{-1}}^{\mathbf{A}^*_{r,i}}\,\mathbf{x}^*_{r,i}(t) + \overbrace{\mathbf{M}_i\mathbf{B}_{r,i}}^{\mathbf{B}^*_{r,i}}\,u(t)\,, \\ y_{r,i}(t) &= \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i^{-1}}_{\mathbf{C}_{r,i}{}^*}\,\mathbf{x}^*_{r,i}(t)\,, \end{aligned} \tag{13}$$

where $\mathbf{M}_i, \mathbf{T}_i \in \mathbb{R}^{q\times q}$ are regular matrices, leaves their input-output behaviour unchanged. However, using the models (13) instead of those in (10) strongly affects the dynamics of the resulting parametric reduced model (11) as illustrated by the following example:

Consider a parametric undamped spring-mass-system in second-order form,

$$m\,\ddot{x} + x = 0\,. \tag{14}$$

Its transformation to state-space representation can be carried out in many ways, e. g.

$$\underbrace{\begin{bmatrix} 0 & m \\ 1 & 0 \end{bmatrix}}_{\mathbf{E}_1}\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}_1}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \tag{15}$$

Choosing the state vector as two linearly independent combinations of x and $\dot{x}$, leads to different matrices $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{A}}_1$ which are connected to $\mathbf{E}_1$ and $\mathbf{A}_1$ by a transformation $\mathbf{T}$ as in (13); thereby, the *columns* of $\mathbf{E}_1$ and $\mathbf{A}_1$, respectively, are linearly recombined, but the system dynamics unchanged.

In addition, the two equations of the first-order system can be exchanged, resulting in the equivalent system

$$\underbrace{\begin{bmatrix}1 & 0\\ 0 & m\end{bmatrix}}_{\mathbf{E}_2}\begin{bmatrix}\dot{x}\\ \ddot{x}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}}_{\mathbf{A}_2}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}, \tag{16}$$

where the state vector is the same as above, but the *rows* of the matrices are linearly recombined. Accordingly, this change corresponds to a multiplication from the left by a matrix $\mathbf{M}$ which again does not affect the system dynamics.

Suppose $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{A}_1$, $\mathbf{A}_2$ are interpolated both for $m = 1$ according to (12) using $\omega_i = 0.5$, then a singular matrix $\mathbf{E}_r$ results, although $\mathbf{E}_r$, $\mathbf{A}_r$ would have been expected to again describe the same dynamics as (15) and (16).

Hence, the interpolation between the locally reduced systems may only be performed after modifying them using appropriate $\mathbf{M}_i$ and $\mathbf{T}_i$ to make them compatible to each other in a certain sense[1].

This choice will be discussed in detail in the next section.

4 Methods for Adjusting the Local Reduced Models

Starting from the local reduced order model (10), that has been obtained using a given reduction approach, several methods for the choice of the matrices $\mathbf{T}_i$ and $\mathbf{M}_i$ are presented in this section. Each of them adjusts those local models in a different way so that after their weighted interpolation according to (12), properties of interest are attained in the resulting reduced order model (11), in addition to a good approximation for parameter values within the considered set $\mathbf{\Pi}$.

4.1 Reprojection into a Common Subspace

Compatibility of Local Coordinate Systems

The first approach is based on the fact that interpolating the matrices of several dynamical systems, which is equivalent to summing up their underlying systems of ODEs, is generally not advisable if their respective state variables represent different physical quantities. This is however the case for the vectors $\mathbf{x}_{r,i}$ which represent different linear combinations of the original, high-dimensional state space given by the columns of the respective matrices $\mathbf{V}_i$, which generally span different subspaces.

As seen in Sect. 2, one degree of freedom in the new pMOR framework is represented by the transformation matrices $\mathbf{T}_i$ which allow the transition from the local reduced coordinates $\mathbf{x}_{r,i}$ to a modified coordinate system $\mathbf{x}^*_{r,i}$, i. e. $\mathbf{x}_{r,i} = \mathbf{T}_i^{-1}\mathbf{x}^*_{r,i}$. Thereby, it is desired to find a common basis $\mathbf{x}^*_r = \mathbf{x}^*_{r,1} = \mathbf{x}^*_{r,2} = \ldots = \mathbf{x}^*_{r,k}$ such that all the reduced local models are described using the same set of state variables.

In the original state-space, however, the state vectors $\mathbf{x}^*_{r,i}$ correspond to $\hat{\mathbf{x}}_i = \mathbf{V}_i\mathbf{x}_{r,i} = \mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}^*_{r,i}$. Consequently, the backprojected states $\hat{\mathbf{x}}_i$ will still lie in the subspaces spanned by the corresponding matrices $\mathbf{V}_i$, independently of the choice of the matrices $\mathbf{T}_i$. Hence, the states $\mathbf{x}^*_{r,i}$ cannot be given a common meaning when projected back to the original state-space (except for the unlikely case that the matrices $\mathbf{V}_i$ of all involved local reduced models span the same subspace).

However, choosing the $\mathbf{T}_i$ appropriately, it is possible to make the state vectors $\mathbf{x}^*_{r,i}$ compatible with respect to a subspace spanned by the columns of a matrix $\mathbf{R} \in \mathbb{R}^{n\times q}$. This means, that starting from a given reduced state vector

$$\mathbf{x}^*_{r,i},$$

transforming it to a local reduced coordinate system,

$$\mathbf{x}_{r,i} = \mathbf{T}_i^{-1}\mathbf{x}^*_{r,i},$$

projecting it back to the original subspace using the associated projection matrix $\mathbf{V}_i$,

$$\hat{\mathbf{x}}_i = \mathbf{V}_i\mathbf{x}_{r,i} = \mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}^*_{r,i},$$

and reprojecting it to the subspace spanned by the columns of $\mathbf{R}$,

$$\mathbf{R}^T\hat{\mathbf{x}}_i = \mathbf{R}^T\mathbf{V}_i\mathbf{x}_{r,i} = \mathbf{R}^T\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}^*_{r,i}, \tag{17}$$

the same vector is obtained for all the involved reduced systems, as illustrated in Fig. 1.

From (17), it is clear that this can be achieved by choosing:

$$\mathbf{T}_i = \mathbf{R}^T\mathbf{V}_i. \tag{18}$$

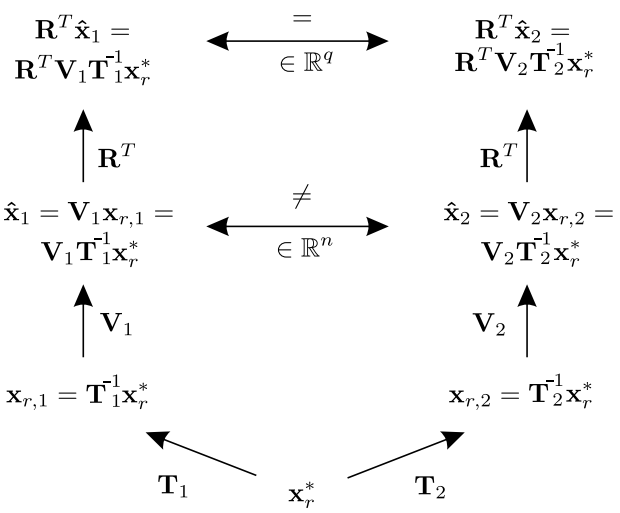

Figure 1 Transformation, backprojection, reprojection.

[1] Note that if $\mathbf{E}_{r,i} = \mathbf{I}$, $\mathbf{M}_i = \mathbf{T}_i$ leads to $\mathbf{E}^*_{r,i} = \mathbf{I}$.

which leads to:

$$\underbrace{\mathbf{R}^T\mathbf{V}_1}_{\mathbf{T}_1}\,\mathbf{x}_{r,1} = \underbrace{\mathbf{R}^T\mathbf{V}_2}_{\mathbf{T}_2}\,\mathbf{x}_{r,2} = \ldots =: \mathbf{x}_r^* \tag{19}$$

Definition 1. *The coordinate systems of the state vectors* $\mathbf{x}_{r,i}^* \in \mathbb{R}^{q\times q}$ *are called compatible w.r.t. a matrix* $\mathbf{R} \in \mathbb{R}^{n\times q}$*, if the images of their basis vectors under a transformation* $\mathbf{T}_i \in \mathbb{R}^{q\times q}$*, backprojection using the matrices* $\mathbf{V}_i \in \mathbb{R}^{n\times q}$ *and reprojection into the subspace spanned by the columns of the matrix* $\mathbf{R}$*, are identical.*

The matrix $\mathbf{R}$ is supposed to be orthogonal, i. e. $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, so that it is possible to project the common state vectors $\mathbf{x}_r^* \in \mathbb{R}^q$ back to the original high dimensional state-space by $\mathbf{R}\mathbf{x}_r^*$. In addition, this matrix has to be chosen so that none of the transformation matrices $\mathbf{T}_i$ becomes singular.

Recall that the columns of $\mathbf{R}$ span the subspace with respect to which a common state description $\mathbf{x}_r^*$ is guaranteed. Hence, $\mathbf{R}$ is chosen in a way to contain q directions in the state-space that are most important to approximate the dominant dynamics of the involved local models.

The Choice of the Matrix R

Assuming that k models are involved during the interpolation (12), up to $k \cdot q$ linearly independent columns may be obtained when building the matrix

$$\mathbf{V}_{all} := [\mathbf{V}_1 \;\; \mathbf{V}_2 \;\; \ldots \;\; \mathbf{V}_k]\,. \tag{20}$$

This matrix constitutes a "pool" from which q directions can be chosen to form the matrix $\mathbf{R}$.

The first approach is based on choosing the q "most important" directions of the subspace spanned by the columns of $\mathbf{V}_{all}$. This can be achieved by a Singular Value Decomposition (SVD) of this matrix

$$\mathbf{V}_{all} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{N}^T\,, \tag{21}$$

which yields $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times(kq)}$ and $\mathbf{U} \in \mathbb{R}^{n\times n}$ where the first $(k \cdot q)$ columns of the orthogonal matrix $\mathbf{U}$ form a basis for the subspace spanned by $\mathbf{V}_{all}$. Thereby, the basis vectors are sorted with respect to their relative importance reflected by the corresponding singular value. In fact, common directions of most of the subspaces $\mathbf{V}_i$ will outvalue those appearing in only few of them. Accordingly, choosing for $\mathbf{R}$ the first q columns of $\mathbf{U}$ guarantees to capture the most important directions in $\mathbf{V}_{all}$.

This SVD needs to be calculated once, after all the projection matrices $\mathbf{V}_i$ have been obtained, and can be used to generate any reduced-order model at any point $\mathbf{p}$ in the parameter space. Note that the SVD results are only useful if all the matrices $\mathbf{V}_i$ are orthonormalized, which is for instance the case when they are calculated to span a Krylov subspace using the Arnoldi algorithm. The numerical cost of the SVD step can be minimized when its economy version is used. This is possible because the matrix $\mathbf{V}_{all} \in \mathbb{R}^{n\times(kq)}$, and only the first q columns of the matrix $\mathbf{U}$ are of interest.

The second approach follows the main scheme of the previous one, however with a different matrix $\mathbf{V}_{all}$ in the SVD step. The weights $\omega_i(\mathbf{p})$ involved in the interpolation of the matrices in (12) are also employed to weight their corresponding projections matrices $\mathbf{V}_i$ when building the new matrix $\mathbf{V}_{all}^{\omega}$, i. e.

$$\mathbf{V}_{all}^{\omega} := [\omega_1(\mathbf{p})\mathbf{V}_1 \;\; \omega_2(\mathbf{p})\mathbf{V}_2 \;\; \ldots \;\; \omega_k(\mathbf{p})\mathbf{V}_k]\,. \tag{22}$$

The SVD of $\mathbf{V}_{all}^{\omega}$, also here in its economy version, has to be repeated every time a reduced-order model at a given parameter $\mathbf{p}$ is to be calculated. This is numerically more expensive than the first approach, however, it allows to favor the subspaces spanned by the projection matrices corresponding to the local models that are closest to the interpolation point $\mathbf{p}$ in question. In other words, the subspace spanned by the columns of $\mathbf{R}$ is fit to best approximate the subspace that would have resulted from the reduction of the original model at the parameter $\mathbf{p}$.

It has to be noted that only the first approach for the choice of $\mathbf{R}$ allows the generation of a parametric reduced-order model according to (11), where only the weights are parameter-dependent. In the second approach, both the interpolated matrices and the weights change with $\mathbf{p}$ and thus they need to be recalculated for every parameter.

The Choice of the Matrix M

To motivate the choice of the transformation matrices $\mathbf{M}_i$ in (13), it is necessary to investigate their effect on the projection-based model reduction procedure presented in Sect. 2. As seen before, the state equation (5) of the reduced model can be derived from the intermediate equation (4) by multiplying the latter by a matrix projector $\mathcal{P}$ from the left. Thereby, different choices for $\mathcal{P}$ lead to different state equations which however all have the same solution $\mathbf{x}_r(t)$.

Let $\mathbf{S} \in \mathbb{R}^{n\times q}$ be a full column rank matrix with $\det(\mathbf{W}^T\mathbf{S}) \neq 0$. Then, by $\mathcal{P}_{\mathbf{S}} := \mathbf{S}(\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T$, a projector is defined which maps vectors along the orthogonal complement of $\mathbf{W}$ onto the subspace spanned by $\mathbf{S}$. Multiplying (4) from the left by $\mathcal{P}_{\mathbf{S}}$ instead of $\mathcal{P}$ leads to

$$\begin{aligned}&\mathbf{S}(\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \\ &\mathbf{S}(\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{S}(\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T\mathbf{B}\,\mathbf{u}(t)\,.\end{aligned} \tag{23}$$

Unlike (8), the description of the reduced system's dynamics is now enrooted in the subspace spanned by $\mathbf{S}$, although the solution $\mathbf{x}_r(t)$ to (23) is the same as the solution to (8). This can be seen by equivalently rewriting the above equation as

$$\begin{aligned}&(\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \\ &(\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + (\mathbf{W}^T\mathbf{S})^{-1}\mathbf{W}^T\mathbf{B}\,\mathbf{u}(t)\,,\end{aligned} \tag{24}$$

and multiplying it from the left by $(\mathbf{W}^T\mathbf{V})^{-1}(\mathbf{W}^T\mathbf{S})$. Thereby, nothing changes but the representation of the

reduced system. Note that (5) describes the same reduced order model just with respect to another basis.

Accordingly, the choice $\mathbf{M}_i := (\mathbf{W}_i^T\mathbf{S})^{-1}$ in (13) guarantees that all the locally reduced models are modified such that their state equations are given with respect to the same basis $\mathbf{S}$.

Based on the results from Sect. 4.1, it is then reasonable to choose $\mathbf{S} := \mathbf{R}$, leading to

$$\mathbf{M}_i := (\mathbf{W}_i^T\mathbf{R})^{-1}. \tag{25}$$

Thereby, the state equation in (13) changes using (18) and (25) to (for ease of presentation, we only consider one of the terms)

$$\begin{aligned}&\mathbf{A}^*_{r,i}\,\mathbf{x}^*_{r,i}(t) = \mathbf{M}_i\mathbf{A}_{r,i}\mathbf{T}_i^{-1}\,\mathbf{x}^*_{r,i}(t) = \\ &= (\mathbf{W}_i^T\mathbf{R})^{-1}\mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i(\mathbf{R}^T\mathbf{V}_i)^{-1}\,\mathbf{x}^*_{r,i}(t) = \\ &= \mathbf{R}^T\underbrace{\mathbf{R}(\mathbf{W}_i^T\mathbf{R})^{-1}\mathbf{W}_i^T}_{③}\,\mathbf{A}_i\underbrace{\mathbf{V}_i(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T}_{②}\underbrace{\mathbf{R}\,\mathbf{x}^*_{r,i}(t)}_{①}\end{aligned} \tag{26}$$

Evaluating this term from right to left shows the following:

① The reduced state $\mathbf{x}^*_{r,i}$ is firstly backprojected to the original state space using $\mathbf{R}$.

② Afterwards, it is projected onto $\mathbf{V}_i$ orthogonally to the subspace spanned by $\mathbf{R}$ and mapped by $\mathbf{A}_i$.

③ Remembering that $\mathbf{R}$ is the underlying basis of (26), one can see that the resulting vector is finally projected onto $\mathbf{R}$ orthogonally to the subspace spanned by $\mathbf{W}_i$.

Remark 1. For the case where $\mathbf{R}^T\mathbf{R} \neq \mathbf{I}$, Eq. (26) becomes:

$$\begin{aligned}\mathbf{A}^*_{r,i}\,\mathbf{x}^*_{r,i}(t) = (\mathbf{R}^T\mathbf{R})^{-1}\mathbf{R}^T\mathbf{R}(\mathbf{W}_i^T\mathbf{R})^{-1}\mathbf{W}_i^T\mathbf{A}_i \\ \mathbf{V}_i(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T\mathbf{R}(\mathbf{R}^T\mathbf{R})^{-1}\mathbf{x}^*_{r,i}(t)\end{aligned}$$

which however does not affect the compatibility of the local reduced models.

Accordingly, for calculating the reduced parametric model (11) and (12) we use the local reduced models (13) instead of (10), with $\mathbf{T}_i = \mathbf{R}^T\mathbf{V}_i$ and $\mathbf{M}_i = (\mathbf{W}_i^T\mathbf{R})^{-1}$.

With this choice, the local reduced models have compatible state vectors with respect to $\mathbf{R}$ and are, in addition, all described in terms of $\mathbf{R}$ as basis.

4.2 Optimization-Based Matrix Matching

In this section, we assume that the reduced systems have already been transformed into some standard form, e. g. balanced or real modal canonical form. In particular, we suppose $\mathbf{E} = \mathbf{I}$. Note that the reduced systems are usually small enough to invert the matrix $\mathbf{E}$ explicitly, while this might not at all be true for the original systems. Moreover, if the reduced matrix $\mathbf{E}_r$ is still singular, this indicates algebraic states, which may be taken from the dynamic system and added to a direct feedthrough matrix $\mathbf{D}$. Therefore, assuming normal forms for the reduced systems does usually not imply any loss of generality.

Matrix matching modifies the system realizations at different parameters in such a way that corresponding matrices become as similar as possible. In the following, we motivate why this induces good interpolation properties. For ease of presentation, we assume linear interpolation in a single parameter $p \in [0,1]$. From [4] we know that interpolating transfer functions gives good results away from poles. For $p \in \{0,1\}$ let

$$\mathbf{G}_p(s) = \mathbf{C}_p\,\mathbf{X}_p\,\mathbf{B}_p \quad \text{with} \quad \mathbf{X}_p = \mathbf{X}_p(s) = \left(s\,\mathbf{I} - \mathbf{A}_p\right)^{-1} \tag{27}$$

be the transfer functions of the known reduced systems and

$$\tilde{\mathbf{G}}_p(s) = \left(1-p\right)\,\mathbf{G}_0(s) + p\,\mathbf{G}_1(s)\,,\ p \in [0,1] \tag{28}$$

linearly interpolated transfer functions at intermediate parameters. This interpolation is compared to the transfer function of the interpolated matrices. With $\Delta\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_0$ etc., it reads

$$\mathbf{G'}_p(s) = \left(\mathbf{C}_0 + p\,\Delta\mathbf{C}\right)\left(\mathbf{X}_0 + p\,\Delta\mathbf{X}\right)\left(\mathbf{B}_0 + p\,\Delta\mathbf{B}\right)\,. \tag{29}$$

Up to third order terms in the Δ-matrices, the difference between these two interpolations proves to be

$$\begin{aligned}&\tilde{\mathbf{G}}_p(s) - \mathbf{G'}_p(s) \approx p\,\left(1-p\right)\times \\ &\underbrace{\left(\mathbf{C}_0\,\Delta\mathbf{X}\,\Delta\mathbf{B} + \Delta\mathbf{C}\,\mathbf{X}_0\,\Delta\mathbf{B} + \Delta\mathbf{C}\,\Delta\mathbf{X}\,\mathbf{B}_0\right)}_{\Delta}\,.\end{aligned} \tag{30}$$

For a sub-multiplicative matrix-norm, e. g. the Frobenius norm, this difference can be estimated as follows:

$$\begin{aligned}&\|\Delta\| = \|\mathbf{C}_0\|\,\|\mathbf{X}_0\|\,\|\mathbf{B}_0\|\times \\ &\qquad\left\|\frac{\mathbf{C}_0\,\Delta\mathbf{X}\,\Delta\mathbf{B} + \Delta\mathbf{C}\,\mathbf{X}_0\,\Delta\mathbf{B} + \Delta\mathbf{C}\,\Delta\mathbf{X}\,\mathbf{B}_0}{\|\mathbf{C}_0\|\,\|\mathbf{X}_0\|\,\|\mathbf{B}_0\|}\right\| \\ &\le \|\mathbf{C}_0\|\,\|\mathbf{X}_0\|\,\|\mathbf{B}_0\|\times \\ &\left[\frac{\|\Delta\mathbf{X}\|}{\|\mathbf{X}_0\|}\frac{\|\Delta\mathbf{B}\|}{\|\mathbf{B}_0\|} + \frac{\|\Delta\mathbf{C}\|}{\|\mathbf{C}_0\|}\frac{\|\Delta\mathbf{B}\|}{\|\mathbf{B}_0\|} + \frac{\|\Delta\mathbf{C}\|}{\|\mathbf{C}_0\|}\frac{\|\Delta\mathbf{X}\|}{\|\mathbf{X}_0\|}\right] \\ &\le \|\mathbf{C}_0\|\,\|\mathbf{X}_0\|\,\|\mathbf{B}_0\|\times \\ &\left[\left(\frac{\|\Delta\mathbf{X}\|}{\|\mathbf{X}_0\|}\right)^2 + \left(\frac{\|\Delta\mathbf{B}\|}{\|\mathbf{B}_0\|}\right)^2 + \left(\frac{\|\Delta\mathbf{C}\|}{\|\mathbf{C}_0\|}\right)^2\right] \\ &\le \alpha\,\|\mathbf{X}_1 - \mathbf{X}_0\|^2 + \beta\,\|\mathbf{B}_1 - \mathbf{B}_0\|^2 + \gamma\,\|\mathbf{C}_1 - \mathbf{C}_0\|^2\end{aligned} \tag{31}$$

with

$$\alpha = \frac{\|\mathbf{B}_0\|\,\|\mathbf{C}_0\|}{\|\mathbf{X}_0\|}\,,\ \beta = \frac{\|\mathbf{C}_0\|\,\|\mathbf{X}_0\|}{\|\mathbf{B}_0\|}\,,\ \gamma = \frac{\|\mathbf{X}_0\|\,\|\mathbf{B}_0\|}{\|\mathbf{C}_0\|}\,. \tag{32}$$

In the second inequality of (31), we have used the general relation $ab + bc + ca \le a^2 + b^2 + c^2$.

Now we keep fixed the realization of the system at $p = 0$, i. e. $\mathbf{M}_0 = \mathbf{T}_0 = \mathbf{I}$, but apply a state transformation $\mathbf{M}_1 = \mathbf{T}_1 = \mathbf{P}^{-1}$ to the system at $p = 1$. Although this keeps the I/O behaviour of the latter system unchanged, it does

affect the transfer function of the interpolated matrices. The idea of matrix matching is to choose $\mathbf{P}$ such that the interpolation of the transfer function and the transfer function of the interpolated matrices differ as little as possible at a test frequency s away from poles. More precisely, we minimize the bound (31) on Δ:

$$\tilde{J}(\mathbf{P}, s) = \alpha \left\| \mathbf{P}^{-1}\mathbf{X}_1(s)\,\mathbf{P} - \mathbf{X}_0(s) \right\|^2 + \beta \left\| \mathbf{P}^{-1}\mathbf{B}_1 - \mathbf{B}_0 \right\|^2 + \gamma \left\| \mathbf{C}_1\,\mathbf{P} - \mathbf{C}_0 \right\|^2 . \quad (33)$$

As the reduced systems are given in some normal from, it turns out that the $\mathbf{P}$ minimizing (33) is usually close to a unitary matrix and we can minimize a more simple functional instead:

$$J(\mathbf{P}, s) = \alpha \left\| \mathbf{X}_1(s)\,\mathbf{P} - \mathbf{P}\,\mathbf{X}_0(s) \right\|^2 + \beta \left\| \mathbf{B}_1 - \mathbf{P}\,\mathbf{B}_0 \right\|^2 + \gamma \left\| \mathbf{C}_1\,\mathbf{P} - \mathbf{C}_0 \right\|^2 . \quad (34)$$

$J(\cdot, s)$ becomes minimal, if the Fréchet-derivative with respect to the matrix $\mathbf{P}$ equals 0. Choosing the Frobenius norm, i. e. $\|\mathbf{A}\| = \mathrm{tr}\left(\mathbf{A}^*\mathbf{A}\right)$ and using $\|\mathbf{A}\,\mathbf{B}\| = \|\mathbf{B}\,\mathbf{A}\|$, $\|\mathbf{A}^*\| = \|\mathbf{A}\|$ finally yields the following generalized Sylvester equation for $\mathbf{P}$:

$$\left(\alpha\mathbf{X}_1^*\mathbf{X}_1 + \gamma\,\mathbf{C}_1^*\mathbf{C}_1\right)\,\mathbf{P} + \mathbf{P}\left(\alpha\mathbf{X}_0\mathbf{X}_0^* + \beta\mathbf{B}_0\mathbf{B}_0^*\right) - \alpha\left(\mathbf{X}_1\mathbf{P}\,\mathbf{X}_0^* + \mathbf{X}_1^*\,\mathbf{P}\,\mathbf{X}_0\right) = \beta\mathbf{B}_1\mathbf{B}_0^* + \gamma\,\mathbf{C}_1^*\mathbf{C}_0 . \quad (35)$$

Note that this is not a standard Sylvester equation, as $\mathbf{P}$ appears also in the third term of the above equation. However, as long as the dimension of the reduced systems stays moderate, say $n < 200$, Eq. (35) can be solved efficiently in Matlab using the `kron` command.

Now we turn to the case of more than one parameter and more than two reduced models to be interpolated. As before, one reduced model is chosen as reference and takes the role of system 0. All the other models are transformed with respect to this reference, i. e. for each of them we solve an equation of type (35), where the model takes the role of system 1. Once all transformed realizations are available, the matrices are interpolated for the new parameters as shown in (12). More precisely, the $\mathbf{X}_{p_i}$ are interpolated to get some $\mathbf{X}_p$ and we set $\mathbf{A}_p = s\,\mathbf{I} - \mathbf{X}_p^{-1}$, $\mathbf{E}_p = \mathbf{I}$.

The most important advantage of *matrix matching* is the fact that it relies only on the reduced models and does not need any link to the full model, e. g. in form of the projection matrices $\mathbf{V}_i$ used for reduction. This implies that matrix matching can even be used in cases where the original meshes have different topology. This becomes relevant, for instance, if parameters describe geometry and the systems result from FE-models with automatically created meshes. For the same reason, matrix matching is also applicable to models resulting from system identification, i. e. from measurements rather than FE-models.

5 Numerical Results

5.1 Plate

This example demonstrates that both, SVD-based method and matrix matching, can handle eigenvalue crossing, i. e. the effect that the ordering of corresponding eigenvalues changes with the model parameters. In order to illustrate this effect, we consider a rectangular steel plate clamped at its boundary and excited by a point force close to the center. The dimension of the plate is $L \times 500 \times 0.2\ \mathrm{mm}^3$, where the parameter L varies between 450 and 550 mm. The excitation point is located 83.5 mm from the center in both, x and y direction. A parametric Ansys model is made from $15 \times 15 = 225$ shell elements and contains 1452 degrees of freedom.

In the numerical experiment, models for $L = 450$ mm and $L = 550$ mm are created, transferred to Matlab and reduced by modal truncation, i. e. the columns of $\mathbf{V} = \mathbf{W}$ are the eigenvectors corresponding to the $q = 12$ eigenvalues with smallest modulus. Three alternative approaches are used to generate a model of the squared plate. First, a full Ansys model is created and reduced for $L = 500$ mm, which is accurate, but expensive. This model is compared to those found by interpolation using the SVD-based method and matrix matching, respectively. In both cases, the two models created before enter with weight 0.5.

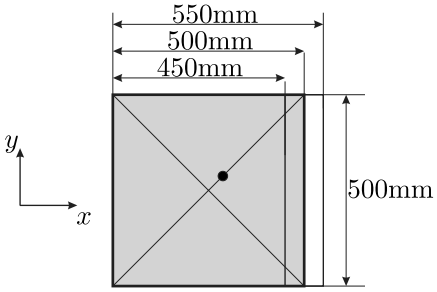

Figure 2 The plate model.

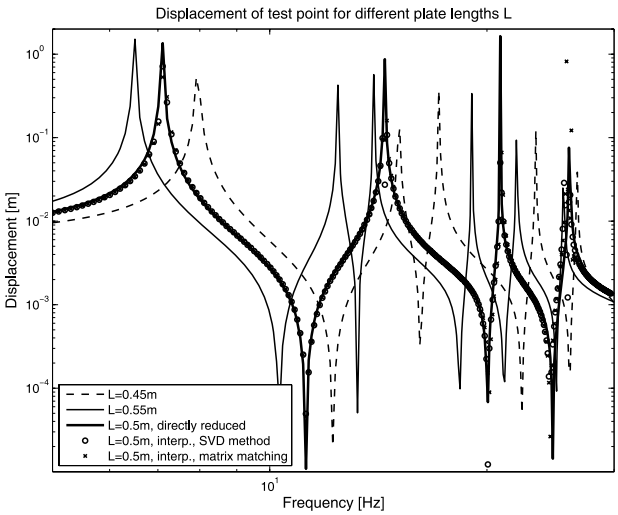

Figure 3 Interpolation of the plate model.

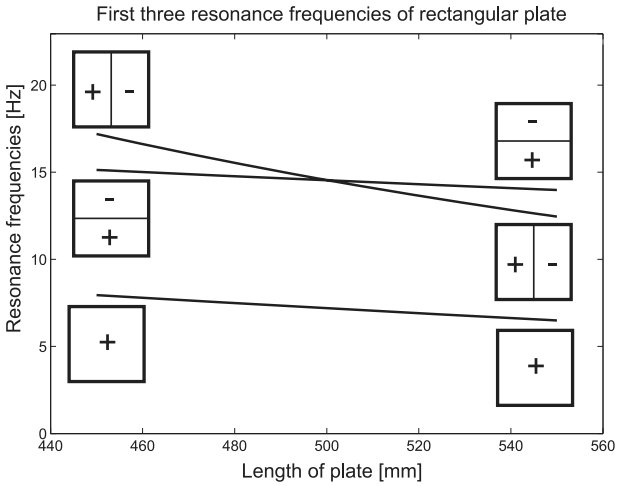

Figure 4 Eigenvalue crossing for rectangular plate. ± indicate areas of associated eigen-shapes moving up/down.

Obviously, the interpolated models approximate well the directly reduced one. In particular, the degeneration of the second and third eigenmodes is captured. The SVD-based method is slightly more accurate close to the resonance peaks, if $s = 0$ is used as test frequency for matrix matching. Moreover, the SVD-based method gives already good results, if it is provided with the SISO-system, while matrix matching only works, if more inputs and outputs are added. This is due to the fact that matrix matching does not depend on the state vectors.

5.2 Beam

The following FE model describes the motion of a 3D cantilever Timoshenko beam and is generated in Matlab according to [12]. The length L of the beam is the free parameter which is varied between $L = 800$ mm and $L = 1200$ mm. The model input is the vertical force $F(t)$ applied at its tip as shown in Fig. 5.

The order of the LTI system for both considered lengths is $n = 1200$ resulting from taking 100 nodes along the beam, each having six degrees of freedom: Three translational displacements u, v, w and three rotational degrees of freedom α, β, γ with respect to the x-, y- and z-axis, respectively. Two models for lengths $L = 800$ mm and $L = 1200$ mm have been generated and reduced using a two-sided Krylov subspace method with $q = 10$ and $s_0 = 0$. Then, they were suitably adjusted using the SVD-based method from Sect. 4.1 to generate a parametric reduced order model with linear weights. In Fig. 6, the frequency response of the two local models, the reduced system obtained by a direct reduction of the original beam model at $L = 1000$ mm and the generated parametric model for the same length are shown.

The suitability of the new approach for the reduction of this model is evident, as the frequency response of the parametric reduced order model almost equals that of the local reduced order model at $L = 1000$ mm.

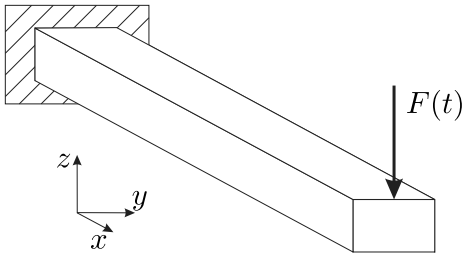

Figure 5 The cantilever beam.

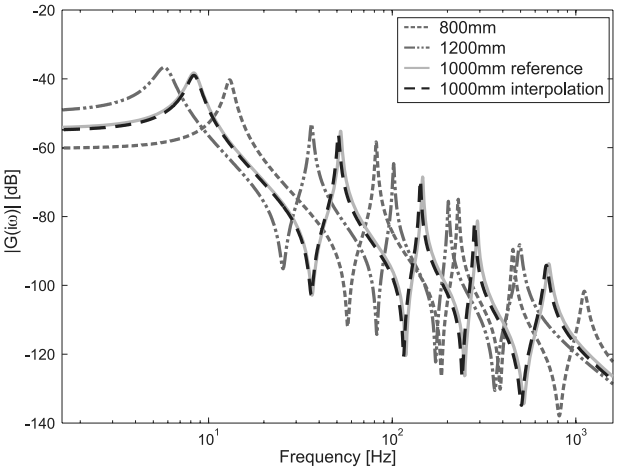

Figure 6 Frequency Response for the cantilever beam.

5.3 Concrete Car Benchmark

The concrete car is an installation at LMS International (Belgium) which has been used in several European research projects to investigate active noise reduction [6]. It consists of two cavities enclosed by concrete and separated by a steel plate. The installation models sound transmission from the motor compartment to the cabin of a car in a rudimentary way. Motor noise produced by a loudspeaker is reduced by attaching a piezo patch to the plate and applying a voltage of suitable amplitude and phase. Here, we use parametric model reduction for optimal actuator placement. More precisely, we look for a piezo position within an admissible region of the plate which allows most efficient attenuation of resonances at 50 Hz. For this purpose, we set up a parametric multi-physics Ansys model taking into account fluid structure interaction and the piezo electric effect [10]. The model contains 93 702 state variables, two inputs (volume velocity of loudspeaker, voltage of actuator), one output (sound pressure level at microphone), and two parameters (x- and y-position of patch). Individual models have been created, exported to Matlab, and reduced for the 12 positions shown in Fig. 7 (grey fields). Finally, matrix matching is used to generate models for intermediate patch positions. The weights are those of a polynomial of order 3 in x, order 2 in y and mixed orders ≤ 5. Note that matrix matching is the only method applicable, as mesh topology changes with patch positions.

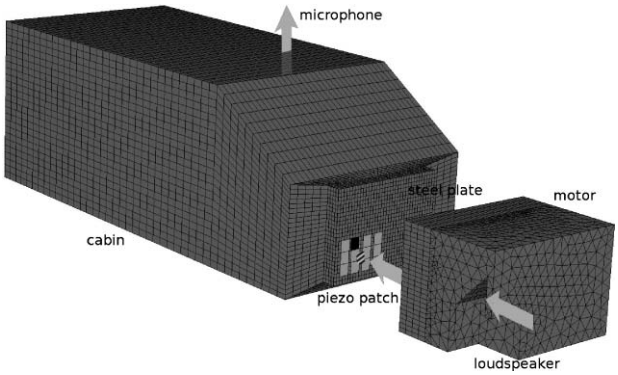

Figure 7 Concrete car model with different piezo positions.

In order to solve the optimization problem, $50 \times 50 = 2500$ models are generated, evaluated, and the best one is chosen. In this simple case, the goal function is just the sound pressure level at the microphone, if the piezo is excited with 100 V at 50 Hz. Once the 12 reduced, transformed models are available, this optimization takes 24 s on an Intel Xeon CPU (2.5 GHz), compared to 120 s per Ansys analysis for a single patch position. If the goal function required more expensive evaluations, e. g. system eigenvalues, the difference would be even more pronounced.

Figure 8 shows the frequency response of the 12 models used for interpolation (thin) and of the optimal patch position (bold). This position is not compatible with the meshes used in Ansys, i. e. we cannot generate an Ansys model to compare with. Therefore, we choose another position, which is compatible with the mesh, but does not belong to the 12 precalculated positions. For this position, a full Ansys model is created, reduced (grey), and compared to the interpolated model (dashed). Obviously, the interpolation error is considerably smaller than the variation between different patch positions.

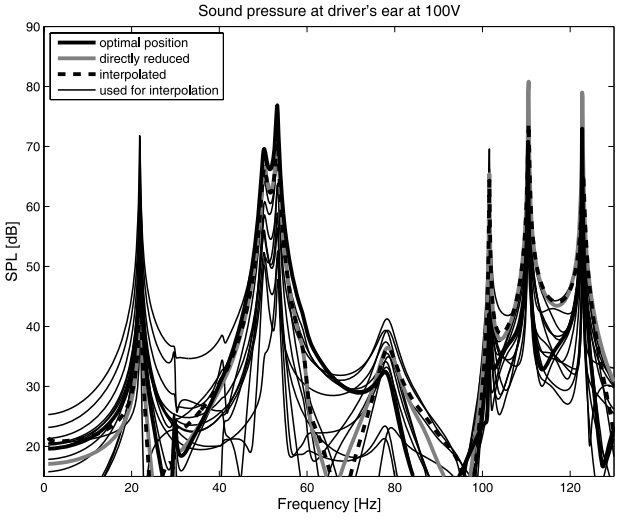

Figure 8 Frequency response for different piezo positions.

6 Conclusions

A new framework for the reduction of parametric LTI systems, based on a weighted interpolation between the matrices of several local reduced models, has been introduced. The new approach is independent of the choice of the MOR method, but the quality of the generated parametric reduced system is strongly affected by the similarity transformations that each of the involved local reduced models undergoes before being interpolated.

To adjust the local reduced-order models involved in the interpolation, two main approaches have been presented, namely the SVD-based and the matrix matching methods, which delivered very reliable reduced models within the parameter interval of interest.

However, a number of open questions has still to be solved to optimize the obtained results, including the choice of the local original systems in the parameter space, the choice of the weighting functions, error estimates to judge the quality of the obtained models, and the stability of the generated parametric reduced-order model.

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