Technische Universität München
Fakultät für Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Particles and populations in random media

Christian Bartsch

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Contents

Preface

This thesis deals with three different models of certain stochastic processes in discrete time. The processes of the considered models have in common that all three are derived from time-homogeneous *Markov processes* in discrete time with the purpose of creating more realistic stochastic models and thus weakening the 'memorylessness' property which is characteristic for Markov processes. Here memorylessness means that the random movement of the memoryless process to the next state at some point in time depends only on the present state but not on the entire path which has led to that state. This property is called Markov property. Since Markov published first results on those processes in 1906, the theory of Markov processes has been developed further with considerable success. Hence, today there are numerous tools and techniques available in the literature which in many cases allow a far-reaching and detailed analysis of such processes. The application of those techniques usually strongly relies on the above mentioned memorylessness. Hence, extensions and generalisations which are aiming at weakening this property are a natural consequent step in the progress of this field.

Processes which do not satisfy the Markov property show a more complex dependence structure. More precisely, the random displacements of the process at some point in time can depend on the entire history of the process and not only on its current state. Generally, such processes can be derived from ordinary Markov processes by assuming that the evolution of the process takes place in a more complex medium which affects the evolution in a certain way and which can also be affected by the evolution. Here the medium is thought of as comprising both the state space of the process and its transition probabilities which can depend on the present state.

One possible approach is to think of the medium itself as a realization of a random mechanism and thus as something which is a priori unknown. In this case the (random) medium is usually called *random environment*. The crucial idea of this approach is to assume that two different random mechanisms are applied. On the one hand the process obeys certain parameters which determine the random transitions of the process in its state space. On the other hand – before the actual evolution takes place – the controlling parameters and/or the explicit shape of the state space are generated as a realization of a random mechanism. Hence, the randomness occurs, in a certain sense, in two consecutive steps. As already mentioned, this construction implies that those processes do not satisfy the Markov property in general. At first glance, this fact might seem surprising. But the ongoing evolution of the process allows more and more inferences to be drawn about the a priori unknown (since random) mechanisms which control the evolution. Hence, the random displacement at each point in time does not only depend on the current state but also on the path on which the state was reached. Models of this type have been investigated intensively over the last decades. Their study goes back to first papers published by Chernov [14] and Temkin [56] in 1962 and 1972, respectively. These authors considered nearest-neighbour random walks on the integers with random transition probabilities. At that time their work was motivated by models for the replication of DNA chains. In 1975 Solomon published his famous paper "Random Walks in a Random Environment" [54]. Therein he provides a mathematically rigorous construction of the model as well as answers to questions on recurrence and the asymptotic speed of the random walk in the case of independent and identically distributed transition probabilities. Since then for the one-dimensional case numerous questions have been solved and today the understanding of these processes has reached a very high level. Thus, it has become clear that these processes show considerably richer phenomena (such as slowdown and aging effects and traps), which differ from those of the original models (without a random environment), and which make their study particularly interesting and rewarding.

Another extension of Markov random walks is a generalisation of both the concepts of Markov processes and of *branching processes*. Instead of considering just one single random walker, it is assumed that this random walker produces offspring according to a (position-dependent) offspring distribution and that the offspring random walkers move independently of one another and identically to their progenitor. Thus, a single initial random walker or particle can result in a cloud or population of particles which moves and/or expands as an entire cloud. Usually the reproduction is assumed to follow a Galton-Watson branching process. Hence, the single particles of a population reproduce independently of each other and they share common mechanisms which determine the reproduction. In a model of this kind new phenomena related to recurrence and transience can be observed. Moreover, as always in the context of branching processes, questions on survival – both locally and globally – arise immediately. Such models are called *branching random walks* or branching Markov processes. A general theory of branching Markov processes has been developed in a series of papers by Ikeda, Nagasawa, and Watanabe in 1968 and 1969 [39]. Similar to ordinary random walks, also a branching random walk can be embedded into a random environment. In this case both the random movement of the particles as well as their random reproduction depend on the underlying medium which is again random. This construction yields a *branching random walk in a random environment*.

In Chapter 1 of this thesis we study a branching random walk in a random environment, described as above, on the non-negative integers. The random movement of this branching random walk can be described as a movement to the right with a location-dependent random delay. Also the reproduction mechanisms depend on the location. Both the parameters which determine the movement and those which determine the reproduction are part of the random environment and thus a priori unknown. In this chapter we answer questions on the local as well as global survival. Besides, we prove a theorem on the asymptotic shape or contour of the cloud of particles.

A different variant of random media, which has attracted a lot of interest in the literature, are *random graphs*. In additon to the analysis of the properties of random graphs itself, the graph can be regarded as the state space of a random walk. Thus, the entire structure of the state space of this random walk is a realization of a random mechanism. The study of random graphs goes back to the famous work by Erdős and Rényi from 1959 [25]. Since then numerous different types of random graphs have been the object of research. A special case of a random graph which is of importance for the present thesis is the genealogical tree of a Galton-Watson branching process. For any graph the easiest way to define a random walk on that graph is the so-called simple random walk. At each point in time the random walker of a simple random walk chooses one of the states which neighbour the current state of the process at random. In general, for each vertex of the graph, a different transition distribution on the set of its neighbours can be given as a part of the random graph, which makes the transition probabilities also random. One important special case of this approach are *random conductance models*. For these models a non-negative random weight is assigned to every edge of the underlying graph. For a fixed configuration of edge weights, the transition probability to move from a given vertex to one of its neighbouring vertices is proportional to the weight of the edge connecting those two vertices. The edge weights are also called conductances referring to physical electric networks, which serve as a good source of intuition in this context. This model comprises, as the simplest special case, the simple random walk for which all conductances share the same deterministic value. A valuable advantage for the analysis of processes defined via edge weights arises from the fact that all such processes show – in a certain sense – a time reversibility. More precisely, they possess a canonical reversible measure depending on the edge weights.

In Chapter 2 of this thesis we study a random walk in a random medium which is given by the genealogical tree of a Galton-Watson branching process with independent and identically distributed edge weights. In papers by Lyons, Pemantle, and Peres [44, 45] it is proved that the simple random walk on infinite Galton-Watson trees is almost surely transient with positive asymptotic speed. Independent and identically distributed edge weights can be regarded as a kind of blurring or smudging of the simple random walk. Consequently, in a paper of Gantert, Müller, Popov, and Vachkovskaia from 2012 [30] it was proved that this generalised model, too, shows transience with a positive speed. As a natural next step after analysing the speed of a random walk, in Chapter 2 of this thesis we derive a central limit theorem for the graph distance as well as for the range of the random walk in this model.

In addition to random media, another approach to increase the complexity of Markov processes is to assume that the medium interacts with the random walker. Thus, it is possible that the mechanisms which determine random transitions of the process are altered in the course of time. In general, this approach can be pursued in various ways. A famous example is the *reinforced random walk* first considered by Diaconis and Coppersmith in 1987 [22]. In this model the probability for the random walker to perform a certain movement is increased (or decreased) if the same movement has been performed previously. Subsequently, Davis [20] and Pemantle [50] have derived far-reaching results on these models. Another way to obtain manipulable media is to consider *excited or cookie random walks*. In these models the transition probabilities of a state are altered after the first visit (or after a certain number of visits) of this state. Here it can be pictured that at a certain state the random walker undergoes an excitement by some kind of cookie which has been placed on the state as part of an initial configuration of cookies, and which is consumed by the random walker after having reached this state. Hence, after the initial cookie storage of a state is depleted, the random walker will no longer undergo an excitement and behave according to different transition probabilities. Excited random walks were introduced by Benjamini and Wilson in 2003 [11].

Obviously, the resulting processes do not satisfy the Markov property. The underlying medium is altered in the course of the evolution of the process by its explicit path. This fact causes a more complicated dependence structure since visiting a certain state can affect the evolution of the process arbitrarily many time steps later. So the transition mechanisms do not only depend on the current position of the random walker but also on the entire history of the process.

Similarly, the cookie random walk model can be combined with the concept of branching processes. In this case, instead of considering only a single random walker, the object of study is a population of random walkers, which move and reproduce independently of one another according to certain transition and offspring distributions. But here, as the available cookies are consumed gradually, the transition probabilities can be altered depending on whether a respective state has been visited (often enough). Similarly, the reproduction mechanisms can change in the course of the evolution of the process.

In Chapter 3 of this work a *cookie branching random walk* as described above is studied. The questions of interest are the same as those in the context of branching random walks without cookies. In Chapter 3 we answer in detail all questions on recurrence/transience phenomena of the considered process.

The thesis is divided into three self-contained chapters which can be read independently of one another. The notation which is made use of within each of the chapters is introduced at the beginning of each chapter. Whenever it is possible and seems reasonable we use consistent notation also across different chapters. In each chapter we use both of the symbols $\blacksquare$ and $\square$ to signal the completion of a proof. $\blacksquare$ is used at the end of the proofs of the major results; whereas $\square$ is used for the proofs of auxiliary results which are part of another proof.

Chapter 1

Survival and growth of a branching random walk in a random environment

1.1 Introduction

In this chapter we consider a particular *branching random walk in a random environment* (BRWRE) on $\mathbb{N}_0$ started with one particle at the origin. The underlying environment is an i.i.d. collection of offspring distributions and transition probabilities. In our model particles can either move one step to the right or they can stay where they are. Given a realization of the environment, we consider a random cloud of particles which evolves as described below. The process is started with one particle at the origin and then the following two steps are iterated indefinitely:

- Each particle produces offspring independently of the other particles and according to the offspring distribution at its location (and then it dies).
- Then all particles move independently of each other. Each particle either moves to the right (with probability h_x, where x is the location of the particle), or it stays at its position (with probability $1 - h_x$).

We are interested in the question whether the BRWRE survives or eventually becomes extinct. Moreover, we analyse the connection between survival/extinction and the growth rate of the (expected) number of particles, and we characterize the asymptotic profile of the expected number of particles on $\mathbb{N}_0$.

The question on survival/extinction is considered for particles moving to the left or to the right in a paper by Gantert, Müller, Popov, and Vachkovskaia [29]. Our model is excluded by the assumptions in [29] (Condition E). The questions on the growth rates are motivated by a series of papers by Baillon, Clement, Greven and den Hollander. In their papers [5, 6, 31, 32, 33] the authors study a similar model which is started with one particle at each location. Since in such a model the global population size is always infinite, the authors introduce different quantities to describe the local and global behaviour of the system. They apply a variational approach to analyse different growth rates.

In this chapter we provide a different (and easier) characterization of the global survival regime by means of an embedded Galton-Watson branching process in a random environment. For a connection between the model considered in this chapter and the model in [31] we refer to Remark 1.4.2.

In order to obtain results on the growth of the global population (Theorem 1.3.4 and Theorem 1.3.6), it is useful to analyse the local behaviour of the process which is carried out in Theorem 1.3.3 and its proof. This theorem involves a function β which describes the asymptotic profile of the expected number of particles. However, the definition of β is not very explicit: Its existence is derived from the subadditive ergodic theorem.

An important difference to the model considered in [17] is that in our model particles can have no offspring, in which case it is possible that the entire process eventually becomes extinct. Thus, it is neccessary to condition on the event of survival in order to determine the growth rate of the population.

If we choose $h \equiv 1$, the spatial component of the BRWRE is trivial (in this case, all particles at time n are located at position n), and the model reduces to the well-known branching process in a random environment, which is comprehensively studied by Tanny in [55]. Our model can be interpreted as an extension of the model considered in [55] towards a process in time and space.

Chapter 1 is organized as follows: In section 1.2 we give a formal description of our model. Section 1.3 contains the results, section 1.4 some remarks and section 1.5 the proofs. At the end of this chapter, in section 1.6, examples and pictures are provided.

The results presented in this chapter have been published in [7] in collaboration with Nina Gantert and Michael Kochler.

1.2 Formal description of the model

The considered BRWRE is constructed in two steps. First, we define the space of environments Ω and the associated environmental measure. Subsequently, for a fixed environment ω we define the mechanisms of reproduction and movement of the particles within this environment.

1.2.1 The environmental law

First, we define

$$\mathcal{M} := \Big\{ (p_i)_{i\in\mathbb{N}_0} : p_i \geq 0, \sum_{i=0}^{\infty} p_i = 1 \Big\}$$

as the set of all offspring distributions (i.e. probability measures on $\mathbb{N}_0$). Then, we define

$$\tilde{\Omega} := \mathcal{M} \times (0,1]$$

as the set of all possible choices for the local environment which now also includes the local drift parameter. Let α be a probability measure on $\tilde{\Omega}$ satisfying

$$\begin{aligned} \alpha\Big(\Big\{ \big((p_i)_{i\in\mathbb{N}_0}, h\big) \in \tilde{\Omega} : p_1 = 1 \Big\}\Big) &< 1, \\ \alpha\Big(\Big\{ \big((p_i)_{i\in\mathbb{N}_0}, h\big) \in \tilde{\Omega} : p_0 \leq 1-\delta,\ h \in [\delta, 1] \Big\}\Big) &= 1 \end{aligned} \tag{1.1}$$

for some $\delta > 0$. The first property ensures that the branching mechanism is non-trivial and the second property is a common ellipticity condition which usually comes up in the context of survival of branching processes in a random environment.

Now we define the space of environments Ω as the product space

$$\Omega := \bigotimes_{x\in\mathbb{N}_0} \tilde{\Omega}.$$

For a suitable (product) σ-algebra $\mathcal{F}$ on Ω, we define the probability measure P on $(\Omega, \mathcal{F})$ as the infinite product measure of α, i.e. $\mathsf{P} := \alpha^{\mathbb{N}_0}$. Hence, if we choose $\omega \in \Omega$ according to the distribution P the sequence $\omega = (\omega_x)_{x\in\mathbb{N}_0} = (\mu_x, h_x)_{x\in\mathbb{N}_0}$ is an i.i.d. sequence in $\tilde{\Omega}$ with marginal distribution α. The measure P is called the *environmental measure* on Ω and we write E for the expectation operator corresponding to P. In the following ω is referred to as the random environment containing the offspring distributions μ_x and the drift parameters h_x. The mean offspring at location $x \in \mathbb{N}_0$ is denoted by

$$m_x = m_x(\omega) := \sum_{k=0}^{\infty} k\mu_x\big(\{k\}\big)$$

and the essential supremum of m_0 by

$$M := \operatorname{ess\,sup} m_0.$$

Furthermore, we define

$$\Lambda := \operatorname{ess\,sup}\big(m_0(1-h_0)\big).$$

1.2.2 The evolution of the cloud of particles – The quenched law

Given a randomly chosen environment $(\omega_x)_{x\in\mathbb{N}_0} = (\mu_x, h_x)_{x\in\mathbb{N}_0}$, the BRWRE is constructed as a discrete-time Markov process. At every point in time $n \in \mathbb{N}_0$ each existing particle at some position $x \in \mathbb{N}_0$ produces offspring according to the distribution μ_x independently of all other particles and dies. Afterwards, the newly produced particles move independently according to an underlying Markov chain starting at position x. The transition probabilities of this Markov chain are also determined by the environment. We only consider a particular type of Markov chain on $\mathbb{N}_0$, which might be called *movement to the right with (random) delay.* This Markov chain is determined by the following transition probabilities:

$$p_\omega(x,y) = \begin{cases} h_x & y = x+1 \\ 1-h_x & y = x \\ 0 & \text{otherwise} \end{cases}. \tag{1.2}$$

We note that the local drift parameter h_x is bounded away from 0 by some positive δ due to the ellipticity condition in (1.1). In Theorem 1.3.7 we consider the case that $\mathsf{P}(h_0 = h) = 1$ holds for some $h \in (0,1]$ (i.e. a constant drift parameter). In this special case, we identify a phase transition for the drift parameter h and different survival regimes depending on h.

For $n \in \mathbb{N}_0$ and $x \in \mathbb{N}_0$, the number of particles at location x at time n is denoted by $\eta_n(x)$ and, moreover, the total number of particles at time n by

$$Z_n := \sum_{x\in\mathbb{N}_0} \eta_n(x).$$

For a fixed environment ω, the probability and the expectation such that the processes $(\eta_n)_{n\in\mathbb{N}_0}$ and $(Z_n)_{n\in\mathbb{N}_0}$ have the properties described as above and such that the population is started with one particle at x is denoted by P_ω^x and E_ω^x, respectively. P_ω^x and E_ω^x are called the *quenched probability and expectation.*

1.2.3 Survival regimes

Now we define two different survival regimes which naturally result from the local and the global point of view, respectively.

Definition 1.2.1. Given an environment $\omega \in \Omega$, we say that

(i) there is *Global Survival* (GS) if

$$P_\omega^0\big(Z_n \to 0\big) < 1.$$

(ii) there is *Local Survival* (LS) if

$$P_\omega^0\big(\eta_n(x) \to 0\big) < 1$$

for some $x \in \mathbb{N}_0$.

Remarks 1.2.2. (i) For fixed ω LS is equivalent to

$$P_\omega^0\big(\eta_n(x) \to 0 \;\forall\, x \in \mathbb{N}_0\big) < 1.$$

(ii) Since the drift parameter is always positive, it is easy to see that for fixed ω LS and GS do not depend on the starting point in Definition 1.2.1. Thus, we will always assume that our process starts at 0. For notational convenience we will omit the superscript 0 and use P_ω and E_ω instead.

1.3 Results

The following results characterize the different survival regimes. As in [29], local and global survival do not depend on the realization of the environment but only on its law.

Theorem 1.3.1. *There is either LS for* P*-a.e.* ω *or there is no LS for* P*-a.e.* ω*. There is LS for* P*-a.e.* ω *iff*

$$\Lambda > 1.$$

Theorem 1.3.2. *We suppose that we have* $\Lambda \leq 1$*. There is either GS for* P*-a.e.* ω *or there is no GS for* P*-a.e.* ω*. There is GS for* P*-a.e.* ω *iff*

$$\mathsf{E}\left[\log\left(\frac{m_0 h_0}{1 - m_0(1-h_0)}\right)\right] > 0.$$

Next, we consider the local and the global growth in terms of the asymptotic behaviour of the moments $E_\omega[\eta_n(x)]$ and $E_\omega[Z_n]$ as n tends to infinity. For the Theorems 1.3.3 to 1.3.6, we need the following stronger condition

$$\alpha\Big(\Big\{\big((p_i)_{i\in\mathbb{N}_0},h\big)\in\tilde{\Omega}:p_1=1\Big\}\Big)<1,$$
$$\alpha\Big(\Big\{\big((p_i)_{i\in\mathbb{N}_0},h\big)\in\tilde{\Omega}:p_0\le 1-\delta,\ h\in[\delta,1-\delta]\Big\}\Big)=1 \tag{1.3}$$

for some $\delta > 0$. In addition, for those theorems we assume $M < \infty$.

Theorem 1.3.3. *There exists a unique, deterministic, continuous and concave function $\beta : [0, 1] \longrightarrow \mathbb{R}$ such that for every $\gamma > 0$ we have*

$$\lim_{n\to\infty}\max_{x\in n[\gamma,1]\cap\mathbb{N}}\Big|\tfrac{1}{n}\log E_\omega\big[\eta_n(x)\big]-\beta(\tfrac{x}{n})\Big|=0$$

for P-a.e. $\omega\in\Omega$. Additionally, we have $\beta(0) = \log\big(\Lambda\big)$ and $\beta(1) = \mathsf{E}[\log(m_0h_0)]$.

Theorem 1.3.4. *We have*

$$\lim_{n\to\infty}\tfrac{1}{n}\log E_\omega\big[Z_n\big]=\max_{x\in[0,1]}\beta(x)$$

for P-a.e.ω.

Theorem 1.3.5 shows that GS is equivalent to exponential growth of the expected global population size $E_\omega[Z_n]$:

Theorem 1.3.5. *The following assertions are equivalent:*

(i) $\lim_{n\to\infty}\frac{1}{n}\log E_\omega\big[Z_n\big]>0$ *holds for P-a.e. ω.*

(ii) There is GS for P-a.e. ω.

In Theorem 1.3.6 we consider the growth of the population Z_n without taking expectation but conditioned on the event of survival:

Theorem 1.3.6. *If there is GS, we have*

$$\lim_{n\to\infty}\tfrac{1}{n}\log Z_n=\max_{x\in[0,1]}\beta(x)>0\quad P_\omega\text{-a.s.}\quad on\quad \{Z_n\not\to 0\}$$

for P-a.e. ω.

As already announced above we now analyse the case of a constant drift parameter, i.e., we have $\mathsf{P}(h_0 = h) = 1$ for some $h \in (0,1]$. We easily conclude from Theorem 1.3.1 that in this case we have LS iff

$$h < h_{LS} := \begin{cases} 1 - \frac{1}{M} & \text{if } M \in (1,\infty] \\ 0 & \text{if } M \in (0,1] \end{cases} .$$

In order to analyse the dependence of GS on the drift parameter h, we define

$$\varphi(h) := \mathsf{E}\left[\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right)\right].$$

Theorem 1.3.7. *We suppose that we have $h \geq h_{LS}$.*

(i) If $M \leq 1$, then we have $\varphi(h) \leq 0$ for all $h \in (0,1]$ and thus there is a.s. no GS.

(ii) We assume that $M > 1$ holds.

(a) If $\varphi(h_{LS}) \geq 0$ and $\varphi(1) \leq 0$, then there is a unique $h_{GS} \in [h_{LS}, 1]$ with $\varphi(h_{GS}) = 0$. In this case we have a.s. GS for $h \in (0, h_{GS})$ and a.s. no GS for $h \in [h_{GS}, 1]$.

(b) If $\varphi(h_{LS}) < 0$, then $\varphi(h) < 0$ for all $h \in [h_{LS}, 1]$. Thus, we have a.s. GS for $h \in (0, h_{LS})$ and a.s. no GS for $h \in [h_{LS}, 1]$. In this case we define $h_{GS} := h_{LS}$.

(c) If $\varphi(1) > 0$, then $\varphi(h) > 0$ for all $h \in [h_{LS}, 1]$. Thus, there is a.s. GS for all $h \in (0,1]$. In this case we define $h_{GS} := \infty$.

Hence, we have a unique $h_{GS} \in [h_{LS}, 1] \cup \{\infty\}$ such that there is a.s. GS for $h < h_{GS}$ and a.s. no GS for $h \geq h_{GS}$.

1.4 Remarks

The following remarks apply to the case of constant drift.

Remarks 1.4.1. (i) Since we have $\varphi(1) = \mathsf{E}[\log m_0]$, our results can be regarded as an extension of the well-known condition for a non-certain extinction of a Galton-Watson branching process in a random environment (cf. Theorem 5.5 and Corollary 6.3 in [55], we recall that we assume that condition (1.1) holds). In fact, our proofs rely on this result.

(ii) If M is finite and $\varphi(h_{LS}) \in (0, \infty]$ holds true, then, by virtue of the continuity of φ, there exists $z > 0$ such that there is a.s. GS but a.s. no LS for every $h \in [h_{LS}, h_{LS} + z)$. In particular, this is the case if $\mathsf{P}(m_0 = M) > 0$ holds true, since this implies $\varphi(h_{LS}) = \infty$.

(iii) In Section 1.6 we provide an example for a choice of the parameters so that the condition of Theorem 1.3.7 *(ii)(b)* holds true. In this case there is a.s. LS for $h \in (0, h_{LS})$ and a.s. no GS for $h \in [h_{LS}, 1]$ for some $h_{LS} \in (0, 1)$.

Remark 1.4.2. The expected global population size $E_\omega[Z_n]$ corresponds to $d_n^{\mathrm{I}}(0, F)$ in the notation of [31]. In Theorem 2 I. the authors of [31] describe the limit

$$\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] = \lim_{n\to\infty} \tfrac{1}{n} \log d_n^{\mathrm{I}}(0, F) =: \lambda(h)$$

as a function of the drift parameter h by an implicit formula.

In order to clarify this correspondence, we consider a random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{N}_0$ started in 0 with (non-random) transition probabilities $(p_h(x, y))_{x,y\in\mathbb{N}_0}$. The transition probabilities are defined by

$$p_h(x, y) := \begin{cases} h & y = x + 1 \\ 1 - h & y = x \\ 0 & \text{otherwise} \end{cases}$$

and let E_h be the associated expectation operator. The local times of the random walk $(S_n)_{n\in\mathbb{N}_0}$ are denoted by $l_n(x)$, i.e.

$$l_n(x) := |\{0 \le i \le n : S_i = x\}| \quad \text{for } x \ge 0, n \ge 0.$$

Then for $x = 0$ we obtain

$$E_\omega[\eta_n(0)] = (1 - h)^n \cdot m_0(\omega)^n = E_h\left[\prod_{i=0}^{n-1} m_{S_i}(\omega) \cdot \mathbb{1}_{\{S_n=0\}}\right].$$

Moreover, for $x \ge 1$ we have

$$E_\omega[\eta_n(x)] = h \cdot m_{x-1}(\omega) \cdot E_\omega\big[\eta_{n-1}(x - 1)\big] + (1 - h) \cdot m_x(\omega) \cdot E_\omega\big[\eta_{n-1}(x)\big],$$

which yields

$$E_\omega[\eta_n(x)] = E_h\left[\prod_{i=0}^{n-1} m_{S_i}(\omega) \cdot \mathbb{1}_{\{S_n=x\}}\right]$$

for all $x \ge 1$ by induction. Finally, we get

$$E_\omega[Z_n] = \sum_{x=0}^{\infty} E_\omega[\eta_n(x)] = E_h\left[\prod_{i=0}^{n-1} m_{S_i}(\omega)\right] = E_h\left[\prod_{x=0}^{n-1} m_x(\omega)^{l_n(x)}\right].$$

Since the environment $\omega = (\omega_x)_{x\in\mathbb{N}_0}$ can be extended to an i.i.d. environment $(\omega_x)_{x\in\mathbb{Z}}$ and since $(\omega_x)_{x\in\mathbb{Z}}$ and $(\omega_{-x})_{x\in\mathbb{Z}}$ have the same distribution with respect to P, formula (1.8) and Theorem 1 in [31] show that there exists a deterministic $c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] = c$$

holds for P-a.e. ω. In our notation this limit coincides with $\max_{x\in[0,1]} \beta(x)$.

The connection between the two models enables us to characterize the critical drift parameter at which the function $h \mapsto \lambda(h)$ in [31] changes its sign using an easier criterion, which can be directly derived from Theorem 1.3.7.

1.5 Proofs

Proof of Theorem 1.3.1. First, we observe that the descendants of a particle at location x which stay at x form a Galton-Watson process with mean offspring $m_x(1 - h_x)$. For a fixed $\omega \in \Omega$, we therefore have

$$P_\omega^x\big(\eta_n(x) \to 0\big) < 1 \qquad \Leftrightarrow \qquad m_x(\omega)(1 - h_x(\omega)) > 1.$$

Now we assume that we have $\Lambda > 1$. Thus, there is some $\lambda > 1$ such that

$$\mathsf{P}(m_0(1 - h_0) \geq \lambda) > \varepsilon > 0$$

holds true for some $\varepsilon > 0$. Using the Borel-Cantelli lemma, we obtain that P-a.s. for infinitely many locations x we have

$$m_x(1 - h_x) > 1.$$

Let $x_0 = x_0(\omega)$ denote a location satisfying $m_{x_0}(1 - h_{x_0}) > 1$. Then, for P-a.e. ω, we have

$$\begin{aligned} &P_\omega(\eta_{x_0}(x_0) \geq 1) \\ &\geq \big(1 - \mu_0(\{0\})\big)h_0 \cdot \big(1 - \mu_1(\{0\})\big)h_1 \cdot \ldots \cdot \big(1 - \mu_{x_0-1}(\{0\})\big)h_{x_0-1} > 0, \end{aligned}$$

where we make use of condition (1.1) for the second inequality. For P-a.e. ω this implies

$$\begin{aligned} &P_\omega(\eta_n(x_0) \to \infty) \\ &\geq P_\omega(\eta_{x_0}(x_0) \geq 1) \cdot P_\omega^{x_0}(\eta_n(x_0) \to \infty) > 0 \end{aligned}$$

and thus LS.

Now we assume that we have $\Lambda \leq 1$. As already mentioned above, for every $x \in \mathbb{N}_0$ and P-a.e. ω the descendants of a particle at location x that stay at x constitue a subcritical or critical Galton-Watson process. Thus, for a given ω, we have

$$\eta_n(0) \xrightarrow[n\to\infty]{P_\omega\text{-a.s.}} 0,$$

which yields that the total number of particles that move from 0 to 1 is P_ω-a.s. finite. Inductively we conclude for every $x \in \mathbb{N}_0$ that the total number of particles that reach location x from $x-1$ is finite. By assumption each of those particles starts a subcritical or critical Galton-Watson process at location x, which P_ω-a.s. dies out. This implies

$$P_\omega\left(\eta_n(x) \xrightarrow[n\to\infty]{} 0\right) = 1 \quad \forall\, x \in \mathbb{N}_0$$

and therefore completes the proof of Theorem 1.3.1. ∎

Proof of Theorem 1.3.2. Since we have $\Lambda \leq 1$ by assumption, there is P-a.s. no LS according to Theorem 1.3.1. This means we have

$$P_\omega(\eta_n(x) \to 0) = 1$$

for P-a.e. ω and for all $x \in \mathbb{N}_0$. We now define a Galton-Watson branching process in a random environment $(\xi_n)_{n\in\mathbb{N}_0}$ which is embedded in the considered BRWRE. After starting with one particle at 0 we freeze all particles that reach position 1 and keep these particles frozen until all existing particles have reached 1. This will happen a.s. after a finite time because the number of particles at 0 constitutes a subcritical or critical Galton-Watson process that dies out with probability 1. The total number of particles frozen in 1 is now denoted by ξ_1. Then we release all particles, let them reproduce and move according to the BRWRE and freeze all particles that hit position 2. As before, the total number of particles frozen at 2 is denoted by ξ_2. We repeat this procedure and with $\xi_0 := 1$ we obtain the process $(\xi_n)_{n\in\mathbb{N}_0}$ which is a branching process in an i.i.d. environment.

Another way to construct $(\xi_n)_{n\in\mathbb{N}_0}$ is to think of ancestral lines. Each particle has a unique ancestral line leading back to the first particle starting from the origin. In this manner of speaking ξ_k is the total number of particles which are the first particles that reach position k among the particles in their particular ancestral lines.

We observe that GS of $(Z_n)_{n\in\mathbb{N}_0}$ is equivalent to survival of $(\xi_n)_{n\in\mathbb{N}_0}$. Due to Theorem 5.5 and Corollary 6.3 in [55] (taking into account condition (1.1)), the process $(\xi_n)_{n\in\mathbb{N}_0}$ survives with positive probability for P-a.e. environment ω if and only if we have

$$\int \log\left(E_\omega[\xi_1]\right) \mathsf{P}(d\omega) > 0.$$

Computing the expectation $E_\omega[\xi_1]$ completes our proof. First, we define $\xi_1^{(k)}$ as the number of particles which move from position 0 to 1 at time k. Using this notation we can write

$$\xi_1 = \sum_{k=0}^{\infty} \xi_1^{(k)}$$

and obtain

$$E_\omega[\xi_1] = \sum_{k=0}^{\infty} E_\omega\left[\xi_1^{(k)}\right].$$

In order to calculate $E_\omega\left[\xi_1^{(k)}\right]$, we observe that (w.r.t. P_ω) the expected number of particles at position 0 at time k equals $\big(m_0(\omega)\cdot(1-h_0(\omega))\big)^k$. Each of those particles contributes $m_0(\omega)\cdot h_0(\omega)$ to $E_\omega\left[\xi_1^{(k)}\right]$. This yields

$$\begin{aligned} E_\omega[\xi_1] &= \sum_{k=0}^{\infty} \big(m_0(\omega)\cdot(1-h_0(\omega))\big)^k \cdot m_0(\omega)\cdot h_0(\omega) \\ &= \frac{m_0(\omega)\cdot h_0(\omega)}{1-m_0(\omega)\cdot(1-h_0(\omega))} \end{aligned} \tag{1.4}$$

which is defined as ∞ if $m_0(\omega)\cdot(1-h_0(\omega)) = 1$. This completes the proof of Theorem 1.3.2. ■

Remark 1.5.1. Alternatively to the computations in the proof of Theorem 1.3.2, equation (1.4) can be obtained using generating functions. The crucial observation is that the generating function $f_x(s) := E_\omega[s^{\xi_{x+1}}|\xi_x = 1]$ is a solution of the equation

$$f_x(s) = g_x\big((1-h_x)f_x(s) + h_x s\big) \tag{1.5}$$

where we write $g_x(s) := \sum_{k=0}^{\infty} \mu_x\big(\{k\}\big)s^k$. Since we have $E_\omega[\xi_1] = f_0'(1)$, we can easily derive (1.4) from equation (1.5).

Proof of Theorem 1.3.3. Following the ideas from [17], we introduce the function β to analyse the local growth rates.

(i) First, we show that β can be defined as a concave function on $(0,1]\cap\mathbb{Q}$ such that

$$\lim_{n\to\infty} \tfrac{1}{sn} \log E_\omega\big[\eta_{sn}(rn)\big] = \beta\left(\tfrac{r}{s}\right) \tag{1.6}$$

holds for all $r,s\in\mathbb{N}$ with $r\le s$ and for P-a.e. ω.

In order to establish this, we fix $r,s\in\mathbb{N}$ with $r\le s$ and define

$$S_{m,n}(\omega) := \tfrac{1}{s}\log E_\omega^{rm}\big[\eta_{s(n-m)}(rn)\big]$$

for $0 \le m \le n$, which is integrable due to (1.3) and $M < \infty$. Using this definition we obtain

$$S_{m+1,n+1}(\omega) = S_{m,n} \circ \Theta(\omega). \tag{1.7}$$

Here Θ is defined by $\Theta(\omega) := \theta^r(\omega)$ with θ denoting the shift operator as usual, i.e. $(\theta\,\omega)_i = \omega_{i+1}$. Furthermore, we have

$$S_{0,n}(\omega) \ge S_{0,m}(\omega) + S_{m,n}(\omega) \tag{1.8}$$

since

$$E^0_\omega\big[\eta_{sn}(rn)\big] \ge E^0_\omega\big[\eta_{sm}(rm)\big] \cdot E^{rm}_\omega\big[\eta_{s(n-m)}(rn)\big].$$

Due to the properties (1.7) and (1.8) we are able to apply the subadditive ergodic theorem to $(S_{m,n})$. We cite Chapter 7.4 of [24] for a textbook reference of the subadditive ergodic theorem. However, we conclude that the limit

$$\lim_{n\to\infty} \tfrac{1}{n} S_{0,n}(\omega) = \lim_{n\to\infty} \tfrac{1}{sn} \log E_\omega\big[\eta_{sn}(rn)\big] =: \beta\left(\tfrac{r}{s}\right)$$

exists for P-a.e. ω. Clearly, the limit only depends on $\frac{r}{s}$ and it is P-a.s. constant since P is a product measure.

(ii) We now show that β is concave on $(0,1] \cap \mathbb{Q}$. We fix $a, b, t \in (0,1] \cap \mathbb{Q}$ with $t \neq 1$ and define $s := a' \cdot b' \cdot t'$ as the product of the denominators of the reduced fractions of the rationals a, b, t. By virtue of (1.8) we have

$$\begin{aligned}
&\tfrac{1}{sn} \log E_\omega\Big[\eta_{sn}\big(s(ta + (1-t)b)n\big)\Big] \\
\ge& t \tfrac{1}{stn} \log E_\omega\Big[\eta_{stn}\big(stan\big)\Big] \\
&\quad + (1-t)\tfrac{1}{s(1-t)n} \log E^{stan}_\omega\Big[\eta_{s(1-t)n}\big(s(ta + (1-t)b)n\big)\Big] \\
=& t \tfrac{1}{stn} \log E_\omega\Big[\eta_{stn}\big(stan\big)\Big] \\
&\quad + (1-t)\tfrac{1}{s(1-t)n} \log E_{\theta^{stan}\omega}\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big].
\end{aligned} \tag{1.9}$$

We observe that for all $n \in \mathbb{N}_0$ we have

$$E_{\theta^{stan}\omega}\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big] \overset{d}{=} E_\omega\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big]$$

w.r.t. the enironmental measure P. Due to (1.6) and since β is P-a.s. constant, this implies that we have

$$(1-t)\tfrac{1}{s(1-t)n} \log E_{\theta^{stan}\omega}\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big] \xrightarrow[n\to\infty]{\mathsf{P}} (1-t)\beta(b).$$

Therefore, there exists a subsequence of the expression in (1.9) which converges P-a.s. as n tends to infinity and this yields that we have

$$\beta(ta + (1-t)b) \ge t\beta(a) + (1-t)\beta(b).$$

We observe that β is bounded with $2\log\delta + \log(1-\delta) \leq \beta(x) \leq \log M$ and thus it can be uniquely extended to a continuous and concave function $\beta : (0,1) \longrightarrow \mathbb{R}$.

(iii) We now investigate the behaviour of $\beta(x)$ if x tends to 0 from above, and we show that we have

$$\lim_{x\downarrow 0} \beta(x) = \log(\Lambda).$$

We fix $\varepsilon > 0$ and $a \in \mathbb{Q} \cap (0, \varepsilon]$. Let a' be the denominator of the reduced fraction of a. For P-a.e. ω there exists $y = y(\omega)$ satisfying

$$m_{y(\omega)}(1 - h_{y(\omega)}) > \Lambda - \varepsilon.$$

Using the definition

$$k := \max\{l \in \mathbb{N} : l \leq (1-\varepsilon)a'n\},$$

for large n such that $k \geq y(\omega)$ we obtain

$$\begin{aligned} E_\omega\big[\eta_{a'n}(a'an)\big] &\geq E_\omega\big[\eta_k(y(\omega))\big] \cdot E_\omega^{y(\omega)}\big[\eta_{a'n-k}(a'an)\big] \\ &\geq \delta_0^{y(\omega)} \cdot (\Lambda - \varepsilon)^{k-y(\omega)} \cdot \delta_0^{a'n-k} \end{aligned}$$

for P-a.e. ω, where we write $\delta_0 := \delta^2 \cdot (1-\delta)$. If we let n tend to infinity and ε to 0, we can conclude that we have

$$\liminf_{x\downarrow 0} \beta(x) \geq \log(\Lambda).$$

For the remaining inequality, we observe that

$$E_\omega\big[\eta_{n_1\cdot n_2}(n_2)\big] \leq \tbinom{n_1\cdot n_2}{n_2} \cdot \Lambda^{(n_1-1)\cdot n_2} \cdot M^{n_2} \tag{1.10}$$

holds true for $n_1, n_2 \in \mathbb{N}$ and for P-a.e. ω. Since we have

$$\tfrac{1}{n_1\cdot n_2} \log \tbinom{n_1\cdot n_2}{n_2} \xrightarrow[n_2\to\infty]{} \tfrac{n_1-1}{n_1} \log\left(\tfrac{n_1}{n_1-1}\right) + \tfrac{1}{n_1}\log(n_1) \xrightarrow[n_1\to\infty]{} 0,$$

the estimate in (1.10) yields

$$\begin{aligned} \tfrac{1}{n_1\cdot n_2} \log E_\omega\big[\eta_{n_1\cdot n_2}(n_2)\big] &\leq (o(n_2) + o(n_1)) + \tfrac{n_1-1}{n_1}\log(\Lambda) + \tfrac{1}{n_1}\log(M) \\ &\xrightarrow[n_2\to\infty]{} \tfrac{n_1-1}{n_1}\log(\Lambda) + o(n_1) \end{aligned}$$

for P-a.e. ω. This implies that we have

$$\limsup_{n\to\infty} \beta\left(\tfrac{1}{n}\right) \leq \log(\Lambda)$$

and due to the continuity of β on $(0,1)$ we conclude

$$\limsup_{x\downarrow 0} \beta(x) \leq \log(\Lambda).$$

(iv) Since the process $(\eta_n(n))_{n\in\mathbb{N}_0}$ is a branching process in an i.i.d. environment satisfying $E_\omega[\eta_1(1)] = m_0h_0$, we have

$$\beta(1) = \mathsf{E}\big[\log(m_0h_0)\big].$$

The continuity of β in 1 can be established with similar arguments as in part (iii).

(v) We fix $\gamma > 0$ and $\varepsilon > 0$ and show that we have

$$\liminf_{n\to\infty} \min_{x\in n[\gamma,1]\cap\mathbb{N}} \Big(\tfrac{1}{n}\log E_\omega[\eta_n(x)] - \beta(\tfrac{x}{n})\Big) \geq 0 \tag{1.11}$$

for P-a.e. ω. In order to show this, we observe that there is a finite set

$$\{a_1,\ldots,a_l\} \subset (0,1)\cap\mathbb{Q}$$

satisfying the following condition:

$$\forall\ b\in[\gamma,1]\ \exists\ i,j\in\{1,\ldots,l\}:\ |b-a_i|<\varepsilon\,,\ a_i\leq b \text{ and } |b-a_j|<\varepsilon\,,\ a_j\geq b.$$

Let a_i' be the denominator of the reduced fraction of a_i. We define

$$k_i := \max\{l\in\mathbb{N} : a_i'l \leq (1-\varepsilon)n\}.$$

By definition of k_i, for large n we have

$$(1-2\varepsilon)n < (1-\varepsilon)n - a_i' < a_i'k_i \leq (1-\varepsilon)n. \tag{1.12}$$

Furthermore, for large n and for all $i\in\{1,\ldots,l\}$ we have

$$\tfrac{1}{a_i'k_i}\log E_\omega\big[\eta_{a_i'k_i}(a_i'a_ik_i)\big] \geq \beta(a_i) - \varepsilon \tag{1.13}$$

for P-a.e. ω as a consequence of (1.6).

Now we fix $y\in n[\gamma,1]\cap\mathbb{N}$. Then, there is $a_i\leq\frac{y}{n}$ with $|\frac{y}{n}-a_i|<\varepsilon$ and we have

$$a_i'a_ik_i \leq (1-\varepsilon)na_i \leq (1-\varepsilon)y \leq y. \tag{1.14}$$

If $\beta(a_i)-\varepsilon\geq 0$ holds true, by virtue of (1.12), (1.13) and (1.14) we have

$$\begin{aligned}
&E_\omega\big[\eta_n(y)\big]\\
\geq\ & E_\omega\big[\eta_{a_i'k_i}(a_i'a_ik_i)\big]\cdot E_\omega^{a_i'a_ik_i}\big[\eta_{n-a_i'k_i}(y)\big]\\
\geq\ & \exp\big(a_i'k_i\cdot(\beta(a_i)-\varepsilon)\big)\cdot\delta_0^{n-a_i'k_i}\\
=\ & \exp\big(\underbrace{a_i'k_i}_{\geq(1-2\varepsilon)n}\cdot(\beta(a_i)-\varepsilon) - \underbrace{(n-a_i'k_i)}_{\leq 2\varepsilon n}\cdot\log(\delta_0^{-1})\big)\\
\geq\ & \exp\Big(n\big((1-2\varepsilon)\cdot(\beta(a_i)-\varepsilon) - 2\varepsilon\cdot\log(\delta_0^{-1})\big)\Big)
\end{aligned}$$

for P-a.e. ω and for all large n, again with $\delta_0 := \delta^2 \cdot (1-\delta)$. This yields

$$\begin{aligned}&\tfrac{1}{n}\log E_\omega\big[\eta_n(y)\big]\\&\geq (1-2\varepsilon)\cdot(\beta(a_i)-\varepsilon) - 2\varepsilon\cdot\log(\delta_0^{-1})\end{aligned} \tag{1.15}$$

for P-a.e. ω. If $\beta(a_i)-\varepsilon<0$ holds true, we conclude in the same way that we have

$$\begin{aligned}&E_\omega\big[\eta_n(y)\big]\\&\geq\ \exp\Big(n\big((1-\varepsilon)\cdot(\beta(a_i)-\varepsilon) - 2\varepsilon\cdot\log(\delta_0^{-1})\big)\Big)\end{aligned} \tag{1.16}$$

for P-a.e. ω. Since we have $|a_i - \frac{y}{n}| < \varepsilon$ and since β is uniformly continuous on $[\gamma, 1]$, the estimates (1.15) and (1.16) imply (1.11) as $n\to\infty$ and $\varepsilon\to 0$.

(vi) In order to complete the proof, it remains to prove that

$$\limsup_{n\to\infty}\ \max_{x\in n[\gamma,1]\cap\mathbb{N}}\Big(\tfrac{1}{n}\log E_\omega\big[\eta_n(x)\big] - \beta(\tfrac{x}{n})\Big)\leq 0 \tag{1.17}$$

holds for P-a.e. ω. So we assume that (1.17) does not hold and, as a consequence, for infinitely many $n\in\mathbb{N}$ there exists $y\in n[\gamma,1]\cap\mathbb{N}$ such that

$$\tfrac{1}{n}\log E_\omega\big[\eta_n(y)\big] \geq \beta(\tfrac{y}{n}) + \varepsilon \tag{1.18}$$

holds with positive probability. As in (v), there exists $a_j \geq \frac{y}{n}$ such that $|\frac{y}{n} - a_j| < \varepsilon$ holds true. Now we define

$$k'_j := \max\{l\in\mathbb{N} : a'_j l \leq (1+\varepsilon)n\}$$

and then (1.6) implies

$$E_\omega\big[\eta_{a'_j k'_j}(a'_j a_j k'_j)\big] < \exp\big(a'_j k'_j\cdot(\beta(a_j)+\varepsilon)\big) \tag{1.19}$$

for P-a.e. ω and for all large n. Moreover, due to (1.18), we have

$$\begin{aligned}&E_\omega\big[\eta_{a'_j k'_j}(a'_j a_j k'_j)\big]\\&\geq E_\omega\big[\eta_n(y)\big]\cdot E^y_\omega\big[\eta_{a'_j k'_j - n}(a'_j a_j k'_j)\big]\\&\geq\ \exp\big(n(\beta(\tfrac{y}{n})+\varepsilon)\big)\cdot\delta_0^{a'_j k'_j - n}\end{aligned}$$

with positive probability since we have

$$a'_j k'_j - n > 0$$

and

$$a'_j a_j k'_j \geq (n+\varepsilon n - a'_j)a_j \geq n a_j \geq y$$

for large n. This yields a contradiction to (1.19) and hence completes the proof of Theorem 1.3.3. ∎

Proof of Theorem 1.3.4. For all $\varepsilon > 0$ there exists $x_0 \in \mathbb{Q} \cap (0,1]$ such that we have

$$\beta(x_0) \geq \max_{x \in [0,1]} \beta(x) - \varepsilon.$$

Let $x_0' \in \mathbb{N}$ denote the denominator of the reduced fraction of x_0. Then we have

$$\begin{aligned}
&\liminf_{n\to\infty} \tfrac{1}{nx_0'} \log E_\omega\big[Z_{nx_0'}\big] \\
\geq\ &\liminf_{n\to\infty} \tfrac{1}{nx_0'} \log E_\omega\big[\eta_{nx_0'}(nx_0' \cdot x_0)\big] \\
=\ &\beta(x_0) \geq \max_{x\in[0,1]} \beta(x) - \varepsilon
\end{aligned}$$

for P-a.e. ω. Moreover, using the ellipticity condition (1.3), we have

$$E_\omega\big[Z_{nx_0'+r}\big] \geq \delta_0^r \cdot E_\omega\big[Z_{nx_0'}\big]$$

for $r \in \{0, 1, \ldots, x_0' - 1\}$ and for P-a.e. ω. If we let $\varepsilon \to 0$, we can conclude that we have

$$\liminf_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[Z_n\big] \geq \max_{x\in[0,1]} \beta(x) \tag{1.20}$$

for P-a.e. ω. In order to establish the remaining estimate, we first state the following lemma.

Lemma 1.5.2. *For $\varepsilon > 0$ there is $\gamma > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\tfrac{1}{n} \log E_\omega\big[\eta_n(y)\big] \leq \log(\Lambda + \varepsilon)$$

for P-a.e. ω and for all $y \in n[0,\gamma] \cap \mathbb{N}_0$.

Proof of Lemma 1.5.2. We fix $\frac{1}{2} > \gamma > 0$ and $y < \gamma n$ and observe that we have

$$E_\omega[\eta_n(y)] \leq \tbinom{n}{y} \cdot \Lambda^{n-y} \cdot M^y$$

for P-a.e. ω. Since we have

$$\tfrac{1}{n} \log \tbinom{n}{y} \leq \tfrac{1}{n} \log \tbinom{n}{\lfloor \gamma n \rfloor} \xrightarrow[\gamma\to 0]{} 0$$

uniformly in n, we can conclude that we have

$$\tfrac{1}{n} \log E_\omega[\eta_n(y)] \leq o(\gamma) + \tfrac{n-y}{n} \log(\Lambda) + \tfrac{y}{n} \log(M) \leq \log(\Lambda + \varepsilon)$$

for P-a.e. ω if $\gamma > 0$ is small enough. ☐

For an arbitrary $\varepsilon > 0$ we now choose $\gamma > 0$ according to Lemma 1.5.2. Then, from Theorem 1.3.3 and Lemma 1.5.2, we can derive that we have

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] \\
&= \limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega \left(\sum_{y=0}^{\lfloor \gamma n \rfloor -1} \eta_n(y) + \sum_{y=\lfloor \gamma n \rfloor}^{n} \eta_n(y) \right) \\
&\leq \limsup_{n\to\infty} \tfrac{1}{n} \log \left(\gamma n \cdot \big(\Lambda + \varepsilon\big)^n + n \cdot \exp\Big(n \cdot \big(\max_{x\in[0,1]} \beta(x) + o(n) \big) \Big) \right) \\
&\leq \max_{x\in[0,1]} \beta(x) + \varepsilon
\end{aligned}$$

for P-a.e. ω since $\beta(0) = \log(\Lambda)$ holds true. For $\varepsilon \to 0$ this yields

$$\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] \leq \max_{x\in[0,1]} \beta(x)$$

for P-a.e. ω. This, together with (1.20), proves Theorem 1.3.4. ∎

Proof of Theorem 1.3.5. First we prove that *(ii)* implies *(i)*. We assume that there is P-a.s. LS. As shown in the proof of Theorem 1.3.1, for P-a.e. ω, there is a location x such that the descendants of a particle at x that stay at x constitute a supercritical Galton-Watson process. Let $x = x(\omega)$ be such a location, i.e. $m_x(1-h_x) > 1$. Then we have

$$\begin{aligned}
&E_\omega[Z_n] \\
&\geq E_\omega[\eta_n(x)] \\
&\geq \big(1-\mu_0(\{0\})\big)h_0 \cdot \ldots \cdot \big(1-\mu_{x-1}(\{0\})\big)h_{x-1} \cdot \big(m_x(1-h_x)\big)^{n-x} \\
&\geq (\delta^{2x} \cdot \big(m_x(1-h_x)\big)^{n-x}
\end{aligned}$$

for P-a.e. ω and for $n \geq x$. Here we use condition (1.1) for the last inequality. Due to Theorem 1.3.4, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] \\
&\geq \limsup_{n\to\infty} \tfrac{1}{n} \log \left(\delta^{2x} \cdot (m_x(1-h_x))^{n-x} \right) \\
&= \log\left(m_x(1-h_x)\right) \\
&> 0
\end{aligned}$$

for P-a.e. ω.

Now let us assume that there is P-a.s. no LS, which is, according to Theorem 1.3.1, equivalent to $\Lambda \leq 1$. Again, we use the process $(\xi_n)_{n\in\mathbb{N}_0}$ defined in the proof of

Theorem 1.3.2. Since there is GS for P-a.e. ω, the process $(\xi_n)_{n\in\mathbb{N}_0}$ has a positive survival probability for P-a.e. ω. Thus, we have

$$\int \log\big(E_\omega[\xi_1]\big)\mathsf{P}(d\omega) > 0 \tag{1.21}$$

by Theorem 5.5 in [55]. For $T \in \mathbb{N}$ we now introduce a slightly modified embedded branching process $(\xi_n^T)_{n\in\mathbb{N}_0}$. For $k \in \mathbb{N}$ we define ξ_k^T as the total number of all particles that move from position $k-1$ to k within T time units after they have been released at position $k-1$. The leftover particles are no longer considered. With $\xi_0^T := 1$ we observe that $(\xi_n^T)_{n\in\mathbb{N}_0}$ is a branching process in an i.i.d. environment. By virtue of the monotone convergence theorem and (1.21) there exists some T such that

$$\int \log\big(E_\omega\big[\xi_1^T\big]\big)\,\mathsf{P}(d\omega) > 0. \tag{1.22}$$

By construction of $(\xi_n^T)_{n\in\mathbb{N}_0}$ we obtain

$$\xi_n^T \;\le\; Z_n + Z_{n+1} + \ldots + Z_{nT}. \tag{1.23}$$

Using the strong law of large numbers and taking into account that ω is an i.i.d. sequence, we have

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[\xi_n^T\big] \\
&= \lim_{n\to\infty} \tfrac{1}{n} \log \prod_{i=0}^{n} E_{\theta^n\omega}\big[\xi_1^T\big] \\
&= \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n} \log E_{\theta^n\omega}\big[\xi_1^T\big] \\
&= \int \log\big(E_\omega\big[\xi_1^T\big]\big)\,\mathsf{P}(d\omega)
\end{aligned} \tag{1.24}$$

for P-a.e. ω. Here again θ denotes the shift operator as usual, i.e. $(\theta\,\omega)_i = \omega_{i+1}$. Together with (1.22) and (1.23) this yields

$$\liminf_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[Z_n + Z_{n+1} + \ldots + Z_{nT}\big] > 0 \tag{1.25}$$

for P-a.e. ω. Using Theorem 1.3.4, we conclude that we have

$$\max_{x\in[0,1]} \beta(x) = \lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] > 0$$

for P-a.e. ω because otherwise there would be a contradiction to (1.25). This shows that *(ii)* implies *(i)*.

In order to prove that *(i)* implies *(ii)*, we first notice that *(ii)* obviously holds true if there is LS for P-a.e. ω. Therefore, we may assume $\Lambda \leq 1$ for the rest of the proof. We now want to consider the branching process focusing on its genealogical structure. We define Γ as the set of all particles produced in the entire process and write $\sigma \prec \tau$ for two particles $\sigma \neq \tau$ if σ is an ancestor of τ. Moreover, $|\sigma|$ denotes the generation which the particle σ belongs to. Furthermore, for every $\sigma \in \Gamma$ let X_σ be the random location of the particle σ. Using these notations, we define

$$G_i := \{\tau \in \Gamma :\, X_\tau = i,\, X_\sigma < i \text{ for all } \sigma \in \Gamma,\, \sigma \prec \tau\} \tag{1.26}$$

for every $i \in \mathbb{N}_0$. Therefore, G_i is for $i \neq 0$ the set of all the particles τ that move from position $i-1$ to position i and hence the particles in G_i are the first particles at position i in their particular ancestral lines. We observe that the process $(|G_n|)_{n\in\mathbb{N}_0}$ coincides with $(\xi_n)_{n\in\mathbb{N}_0}$. Further, for every $\sigma \in \Gamma$ and $n \in \mathbb{N}_0$, we define

$$H_n^\sigma := |\{\tau \in \Gamma :\, \sigma \preceq \tau,\, |\tau| = n,\, X_\tau = X_\sigma\}|$$

and observe that H_n^σ denotes the number of descendants of the particle σ in generation n which are still at the same location as the particle σ. This enables us to decompose Z_n in the following way:

$$Z_n = \sum_{i=1}^{n} \sum_{\sigma \in G_i} H^\sigma_{n-|\sigma|}. \tag{1.27}$$

Since by assumption there is no LS, we have

$$E_\omega[H_n^\sigma \,|\, \sigma \in \Gamma,\, X_\sigma = i] \leq 1 \tag{1.28}$$

for P-a.e. ω because for any existing particle σ its progeny which stays at the location of σ forms a Galton-Watson process which eventually dies out. By (1.27) and (1.28), we conclude that we have

$$E_\omega[Z_n] \leq \sum_{i=1}^{n} E_\omega[|G_i|]$$

for P-a.e. ω. Therefore, we get

$$\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega[|G_n|] > 0$$

for P-a.e. ω as a consequence of (i). Since $(|G_n|)_{n\in\mathbb{N}_0}$ coincides with the branching process in a random environment $(\xi_n)_{n\in\mathbb{N}_0}$, we obtain

$$\int \log\big(E_\omega[\xi_1]\big)\, \mathsf{P}(d\omega) = \lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[|G_n|] > 0$$

for P-a.e. ω as in (1.24). But then again, we have GS for P-a.e. ω since $(\xi_n)_{n\in\mathbb{N}_0}$ survives with positive probability for P-a.e. ω. This completes the proof of Theorem 1.3.5. ■

Proof of Theorem 1.3.6. In this proof we use the abbreviation "a.s." in the sense of "P_ω-a.s. for P-a.e. ω".

Part 1. In the first part of the proof we show in three steps that we a.s. have

$$\limsup_{n\to\infty} \tfrac{1}{n} \log Z_n \le \max_{x\in[0,1]} \beta(x). \tag{1.29}$$

(i) In order to obtain (1.29), we start by showing that for all $\gamma > 0$ we a.s. have

$$\limsup_{n\to\infty} \max_{x\in n[\gamma,1]\cap\mathbb{N}} \left(\tfrac{1}{n}\log \eta_n(x) - \beta(\tfrac{x}{n})\right) \le 0. \tag{1.30}$$

To establish this we fix $\gamma > 0$ and $\varepsilon > 0$. Then, Theorem 1.3.3 implies that for P-a.e. ω there exists $N = N(\omega, \gamma, \varepsilon)$ such that we have

$$E_\omega[\eta_n(y)] \le \exp\left(n \cdot (\beta(\tfrac{y}{n}) + \varepsilon)\right)$$

for all $n \ge N$ and for all $y \in n[\gamma, 1] \cap \mathbb{N}$. Thus, for P-a.e. ω, we obtain

$$P_\omega\Big(\eta_n(y) \ge \exp\left(n \cdot (\beta(\tfrac{y}{n}) + 2\varepsilon)\right)\Big) \le \frac{E_\omega[\eta_n(y)]}{\exp(n \cdot (\beta(\frac{y}{n}) + 2\varepsilon))} = \exp(-\varepsilon n)$$

for large n and all $y \in n[\gamma, 1] \cap \mathbb{N}$. Using the Borel-Cantelli lemma and taking into account that $\left|n[\gamma, 1] \cap \mathbb{N}\right| \le n$ holds, this yields that we a.s. have

$$\limsup_{n\to\infty} \max_{x\in n[\gamma,1]\cap\mathbb{N}} \left(\tfrac{1}{n}\log \eta_n(y) - \beta(\tfrac{y}{n})\right) < 2\varepsilon.$$

Since ε is arbitrarily small, this proves (1.30).

(ii) For the second step of part 1 of this proof we show that for every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 0$ such that we a.s. have

$$\limsup_{n\to\infty} \max_{x\in n[0,\gamma]\cap\mathbb{N}} \left(\tfrac{1}{n}\log \eta_n(x) - \beta(0) - \varepsilon\right) \le 0. \tag{1.31}$$

In order to prove this, we observe that, according to Lemma 1.5.2, for every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 0$ such that we have

$$\tfrac{1}{n}\log E_\omega\big[\eta_n(y)\big] \le \log(\Lambda + \varepsilon) \overset{(\Lambda>1)}{\le} \log(\Lambda) + \varepsilon = \beta(0) + \varepsilon$$

for P-a.e. ω and for $0 \le y \le \gamma n$. Therefore, the same argument as in (i) yields (1.31).

(iii) We now combine (i) and (ii) in order to obtain (1.29). For an arbitrary $\varepsilon > 0$ we choose $\gamma > 0$ as in (ii). Then, (1.30) and (1.31) imply that we a.s. have

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n}\log Z_n \\
= \ &\limsup_{n\to\infty} \tfrac{1}{n}\log \left(\sum_{y=0}^{\lfloor\gamma n\rfloor-1} \eta_n(y) + \sum_{y=\lfloor\gamma n\rfloor}^{n} \eta_n(y)\right)
\end{aligned}$$

$$\le \limsup_{n\to\infty} \tfrac{1}{n} \log \left(\gamma n \cdot \exp\left(n \cdot (\beta(0)+\varepsilon)\right) + n \cdot \exp\left(n \cdot \left(\max_{x\in[0,1]} \beta(x) + o(n)\right)\right)\right)$$
$$\le \max_{x\in[0,1]} \beta(x) + \varepsilon.$$

For $\varepsilon \to 0$ this implies (1.29). Thus, the first part of the proof is complete.

Part 2. In the second part of the proof we show that we have

$$P_\omega \left(\liminf_{n\to\infty} \tfrac{1}{n} \log Z_n \ge \max_{x\in[0,1]} \beta(x) \;\middle|\; Z_n \not\to 0 \right) = 1 \tag{1.32}$$

for P-a.e. ω. We start by stating the following lemma:

Lemma 1.5.3. *For all $\varepsilon > 0$ and $r, s \in \mathbb{N}$ with $r \le s$ and $\beta(\frac{r}{s}) - \varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that we have*

$$P_\omega \left(\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta_{nsN_0}(nrN_0) \ge \beta(\tfrac{r}{s}) - \varepsilon \right) > 0$$

for P-a.e. ω.

Proof. We define

$$M_N := \left\{ \omega \in \tilde{\Omega} : \tfrac{1}{sN} \log E_\omega[\eta_{sN}(rN)] \ge \beta(\tfrac{r}{s}) - \tfrac{\varepsilon}{2} \right\}.$$

Then, for every $\varepsilon_0 > 0$ there exists $N_0 = N_0(\varepsilon_0)$ such that we have

$$\mathsf{P}\big(M_{N_0}\big) \ge 1 - \varepsilon_0.$$

Thus, for sufficiently small ε_0 and the corresponding $N_0(\varepsilon_0)$, we have

$$\begin{aligned} &\int \log E_\omega\big[\eta_{sN_0}(rN_0)\big] \mathsf{P}(d\omega) \\ \ge & sN_0(\beta(\tfrac{r}{s}) - \tfrac{\varepsilon}{2})(1-\varepsilon_0) \\ \ge & sN_0(\beta(\tfrac{r}{s}) - \varepsilon) + sN_0(\tfrac{\varepsilon}{2} - \beta(\tfrac{r}{s})\varepsilon_0 + \tfrac{\varepsilon}{2}\varepsilon_0) \\ \ge & sN_0(\beta(\tfrac{r}{s}) - \varepsilon) \;>\; 0. \end{aligned} \tag{1.33}$$

We now construct a branching process in a random environment $(\psi_n)_{n\in\mathbb{N}_0}$ which is dominated by $\big(\eta_{nsN_0}(nrN_0)\big)_{n\in\mathbb{N}_0}$. After starting with one particle at 0, we count all the particles that are at time sN_0 at position rN_0. This number is denoted by ψ_1. The remaining particles are removed from the system and no longer considered. Next, we count the number of particles at time $2sN_0$ at position $2rN_0$. This number is denoted by ψ_2. An iteration of this procedure yields the process $(\psi_n)_{n\in\mathbb{N}_0}$, which is supercritical due to (1.33). In fact, (1.33) and Theorem 5.5 in [55] imply that, for sufficiently small ε_0, we a.s. have

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \eta_{nsN_0}(nrN_0) \geq sN_0(\beta(\tfrac{r}{s}) - \varepsilon) \tag{1.34}$$

on $\{\psi_n \not\to 0\}$. Since we assume condition (1.3), Corollary 6.3 in [55] implies

$$P_\omega(\psi_n \to 0) < 1 \tag{1.35}$$

for P-a.e. ω. Combining (1.34) and (1.35) finally completes the proof of Lemma 1.5.3. □

Lemma 1.5.3 yields the following corollary.

Corollary 1.5.4. *We fix ε, r, s and N_0 as in Lemma 1.5.3. Then, there exists $\nu > 0$ such that for P-a.e. ω there is an increasing sequence $(x_l)_{l\in\mathbb{N}_0} = (x_l(\omega))_{l\in\mathbb{N}_0}$ in $\mathbb{N}_0$ such that for all $l \in \mathbb{N}_0$ we have*

$$P_\omega^{x_l}\Big(\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta_{nsN_0}(nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon\Big) > \nu.$$

Proof. Due to Lemma 1.5.3 there exists $\nu > 0$ such that we have

$$\mathsf{P}\left(\Big\{\omega:\ P_\omega\Big(\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta_{nsN_0}(nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon\Big) > \nu\Big\}\right) > 0.$$

Since the sequence

$$\begin{aligned} &\left(P_\omega^{x}\Big(\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta_{nsN_0}(nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon\Big)\right)_{x\in\mathbb{N}_0} \\ =&\left(P_{\theta^x\omega}\Big(\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta_{nsN_0}(nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon\Big)\right)_{x\in\mathbb{N}_0} \end{aligned}$$

is ergodic w.r.t. P, Birkhoff's ergodic theorem (e.g. Theorem 20.14 in [43]) yields

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}\Big\{P_{\theta^x\omega}\big(\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta_{nsN_0}(nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon\big) > \nu\Big\} > 0 \tag{1.36}$$

for P-a.e. ω and this completes the proof of Corollary 1.5.4. □

Let $(x_l)_{l\in\mathbb{N}_0}$ be an increasing sequence of positions as in Corollary 1.5.4. Now we show in two steps that on the event of non-extinction there a.s. is a particle at one of the positions x_l such that the process of the descendants of this particle exhibits the desired growth.

(i) As a first step, we show that on the event of survival $(Z_n)_{n\in\mathbb{N}_0}$ a.s. grows as desired along some subsequence $(j + nsN_0)_{n\in\mathbb{N}_0}$ for some $j \in \{0, \ldots, sN_0 - 1\}$. In order to obtain this, as in the proof of Theorem 1.3.5, let Γ again denote the set of all existing particles and for $\sigma \in \Gamma$ let $\eta_n^\sigma(y)$ denote the number of descendants of σ among the particles which belong to $\eta_n(y)$. Using the definition of the sets $(G_l)_{l\in\mathbb{N}_0}$ given in (1.26) and the sequence $(x_l)_{l\in\mathbb{N}_0}$ as in Corollary 1.5.4, we define:

$$\begin{aligned} A_{x_l} &:= \Big\{\exists\, \sigma \in G_{x_l} : \liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta^\sigma_{|\sigma|+nsN_0}(x_l + nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon\Big\}, \\ B_{x_l} &:= \Big\{|G_{x_l}| \geq l\Big\}. \end{aligned}$$

Due to Corollary 1.5.4 and since the descendants of all particles belonging to G_{x_l} evolve independently we get

$$P_\omega\left(A^c_{x_l} \cap B_{x_l}\right) \leq (1-\nu)^l$$

for P-a.e. ω. Hence, by virtue of the Borel-Cantelli lemma, we have

$$P_\omega\left(\limsup_{l\to\infty}\left(A^c_{x_l} \cap B_{x_l}\right)\right) = 0 \tag{1.37}$$

for P-a.e. ω. According to Theorem 5.5 of [55], the process $\big(|G_l|\big)_{l\in\mathbb{N}_0}$ a.s. grows exponentially fast on the event of survival. Therefore, we a.s. have

$$\liminf_{l\to\infty} B_{x_l} = \big\{Z_n \not\to 0\big\}.$$

Together with (1.37) this yields

$$P_\omega\left(\limsup_{l\to\infty} A^c_{x_l} \;\middle|\; Z_n \not\to 0\right) = 0$$

for P-a.e. ω. Thus, on $\{Z_n \not\to 0\}$, there a.s. exists $l \in \mathbb{N}_0$ and $\sigma \in G_{x_l}$ such that we have

$$\liminf_{n\to\infty} \tfrac{1}{nsN_0} \log \eta^\sigma_{|\sigma|+nsN_0}(x_l + nrN_0) \geq \beta(\tfrac{r}{s}) - \varepsilon$$

and hence we have

$$\begin{aligned} &P_\omega\left(\bigcup_{\sigma\in\Gamma}\Big\{\liminf_{n\to\infty} \tfrac{1}{nsN_0}\log Z_{|\sigma|+nsN_0} \geq \beta(\tfrac{r}{s}) - \varepsilon\Big\} \;\middle|\; Z_n \not\to 0\right) \\ &= P_\omega\left(\bigcup_{j\in\mathbb{N}_0}\Big\{\liminf_{n\to\infty} \tfrac{1}{nsN_0}\log Z_{j+nsN_0} \geq \beta(\tfrac{r}{s}) - \varepsilon\Big\} \;\middle|\; Z_n \not\to 0\right) \\ &= P_\omega\left(\bigcup_{j=1}^{sN_0}\Big\{\liminf_{n\to\infty} \tfrac{1}{nsN_0}\log Z_{j+nsN_0} \geq \beta(\tfrac{r}{s}) - \varepsilon\Big\} \;\middle|\; Z_n \not\to 0\right) \\ &= 1 \end{aligned} \tag{1.38}$$

for P-a.e. ω.

(ii) The last step of this part of the proof is to show that the growth along some subsequence $(j + nsN_0)_{n\in\mathbb{N}_0}$ already implies sufficiently strong growth of $(Z_n)_{n\in\mathbb{N}_0}$. Due to the ellipticity condition (1.3) we have

$$P_\omega^x\big(\eta_i(x) \geq 1\big) \geq \delta_0^i$$

for all $i, x \in \mathbb{N}_0$. (We recall that we have $\delta_0 = \delta^2(1-\delta)$.) A large deviation bound for the binomial distribution therefore implies that we have

$$P_\omega\left(Z_{n+i} \leq Z_n \cdot \tfrac{\delta_0^i}{2} \;\middle|\; Z_n = m\right) \leq \exp(-m \cdot \lambda_0) \tag{1.39}$$

for all $m \in \mathbb{N}$ and $i \in \{1, ..., sN_0\}$ and for some constant $\lambda_0 = \lambda_0(N_0) > 0$; cf. Chapter 2 in [21] for the involved large deviation techniques. We now define:

$$\begin{aligned} C_{j,n} &:= \bigcup_{i=1}^{sN_0} \left\{ Z_{j+nsN_0+i} \leq \tfrac{\delta_0^{sN_0}}{2} \exp\left(nsN_0 \cdot (\beta(\tfrac{r}{s}) - \varepsilon)\right)\right\}, \\ D_{j,n} &:= \left\{ \tfrac{1}{nsN_0} \log Z_{j+nsN_0} \geq \beta(\tfrac{r}{s}) - \varepsilon \right\}. \end{aligned}$$

Then, due to (1.39), we have

$$\begin{aligned} &P_\omega\left(C_{j,n} \cap D_{j,n}\right) \\ \leq & sN_0 \cdot \exp\big(-\lambda_0 \exp(n \cdot \lambda_1)\big) \end{aligned} \tag{1.40}$$

for P-a.e. ω and for all $j \in \{1, \dots, sN_0\}$. Here we write $\lambda_1 := sN_0 \cdot (\beta(\frac{r}{s}) - \varepsilon)$. Since the upper bound in (1.40) is summable in $n \in \mathbb{N}_0$, we can apply the Borel-Cantelli lemma and conclude that we have

$$\begin{aligned} &P_\omega\left(\limsup_{n\to\infty} C_{j,n} \;\middle|\; \liminf_{n\to\infty} D_{j,n}\right) \\ \leq\; & P_\omega\left(\liminf_{n\to\infty} D_{j,n}\right)^{-1} \cdot\; P_\omega\left(\limsup_{n\to\infty}(C_{j,n} \cap D_{j,n})\right) \;=\; 0 \end{aligned}$$

for P-a.e. ω and for all $j \in \{1, \dots, sN_0\}$. Thus, we have

$$P_\omega\left(\liminf_{n\to\infty} \tfrac{1}{n} \log Z_n \leq \beta(\tfrac{r}{s}) - 2\varepsilon \;\middle|\; \liminf_{n\to\infty} D_{j,n}\right) = 0$$

for P-a.e. ω and for all $j \in \{1, \dots, sN_0\}$, which implies

$$P_\omega\left(\liminf_{n\to\infty} \tfrac{1}{n} \log Z_n \leq \beta(\tfrac{r}{s}) - 2\varepsilon \;\middle|\; \bigcup_{j=1}^{sN_0} \liminf_{n\to\infty} D_{j,n}\right) = 0. \tag{1.41}$$

Using (1.38) and (1.41), we obtain

$$P_\omega\left(\liminf_{n\to\infty} \tfrac{1}{n} \log Z_n \leq \beta(\tfrac{r}{s}) - 2\varepsilon \;\middle|\; Z_n \not\to 0\right)$$

$$\leq P_\omega \left(Z_n \not\to 0\right)^{-1} \cdot P_\omega \left(\left\{ \liminf_{n\to\infty} \tfrac{1}{n} \log Z_n \leq \beta(\tfrac{r}{s}) - 2\varepsilon \right\} \cap \bigcup_{j=1}^{sN_0} \liminf_{n\to\infty} D_{j,n} \right)$$
$$= 0,$$

which yields that we have

$$P_\omega \left(\liminf_{n\to\infty} \tfrac{1}{n} \log Z_n > \beta(\tfrac{r}{s}) - 2\varepsilon \;\middle|\; Z_n \not\to 0 \right) = 1 \tag{1.42}$$

for P-a.e. ω. Since r and s can be chosen in such a way that $\beta(\frac{r}{s})$ is arbitrarily close to $\max_{x\in[0,1]} \beta(x)$, (1.42) implies (1.32) as $\varepsilon \to 0$ and the proof of Theorem 1.3.6 is complete. ∎

Proof of Theorem 1.3.7. If we assume that $M \leq 1$ holds true, then we P-a.s. have

$$\log \left(\frac{m_0 h}{1 - m_0(1-h)} \right) \leq 0.$$

Therefore, Theorem 1.3.2 implies *(i)*.

We continue with proving *(ii)* and assume that we have $M > 1$. If m_0 is deterministic, i.e. we have $\mathsf{P}(m_0 = M) = 1$, then we P-a.s. have

$$\log \left(\frac{m_0 h}{1 - m_0(1-h)} \right) > 0$$

and thus $\varphi(h) > 0$ for all $h \in (h_{LS}, 1]$. This case is included in *(c)*.

In the following we assume that m_0 is not deterministic. We notice that φ is finite and continuously differentiable for $h \in (h_{LS}, 1]$ since

$$\frac{\partial}{\partial h} \log \left(\frac{m_0 h}{1 - m_0(1-h)} \right) = \frac{1}{h} - \frac{m_0}{1 - m_0(1-h)}$$

is a.s. uniformly bounded for $h \in [h_{LS} + \varepsilon, 1]$ with $\varepsilon > 0$. Thus, we have

$$\frac{\partial}{\partial h} \varphi(h) = \mathsf{E} \left[\frac{1}{h} - \frac{m_0}{1 - m_0(1-h)} \right]. \tag{1.43}$$

Now assume that there exists $h^* \in (h_{LS}, 1]$ satisfying $\varphi(h^*) = 0$, i.e. we have

$$\mathsf{E} \left[\log \left(\frac{m_0}{1 - m_0(1-h^*)} \right) \right] = \log \left(\frac{1}{h^*} \right). \tag{1.44}$$

By virtue of the strict concavity of the function $y \longmapsto \log y$, Jensen's inequality, and equation (1.44), we have

$$\log \left(\mathsf{E} \left[\frac{m_0}{1 - m_0(1-h^*)} \right] \right) > \log \left(\frac{1}{h^*} \right). \tag{1.45}$$

Thus, we obtain that φ is strictly decreasing in $h = h^*$ by (1.43) and (1.45).

Now we assume that $\varphi(h_{LS}) = 0$ holds true. As above, Jensen's inequality implies that (1.45) holds true for h_{LS} instead of h^*. Since the function

$$h \longmapsto \frac{m_0}{1 - m_0(1-h)}$$

is decreasing in $h > 1 - \frac{1}{m_0}$, we have

$$\lim_{\varepsilon \downarrow 0} \mathsf{E}\left[\frac{m_0}{1 - m_0(1 - h_{LS} + \varepsilon)}\right] = \mathsf{E}\left[\frac{m_0}{1 - m_0(1 - h_{LS})}\right] > \frac{1}{h_{LS}}$$

by the monotone convergence theorem. Thus, φ is strictly decreasing and therefore negative in $h \in (h_{LS}, h_{LS} + \varepsilon)$ for some sufficiently small $\varepsilon > 0$.

Finally we obtain *(a)* – *(c)* as a consequence of the continuity of φ and the fact that φ is strictly decreasing in every zero in $[h_{LS}, 1]$. ■

1.6 Examples

1. A basic and natural example to illustrate our results is provided by a choice of the parameters as follows: Let $\mu^{(+)}$ and $\mu^{(-)}$ be two different non-trivial offspring distributions. We define

$$m^{(+)} := \sum_{k=0}^{\infty} k\, \mu^{(+)}(k) \quad \text{and} \quad m^{(-)} := \sum_{k=0}^{\infty} k\, \mu^{(-)}(k)$$

and suppose that we have

$$0 < m^{(-)} < m^{(+)} \leq \infty.$$

Furthermore, we assume that

$$\mathsf{P}\left(\mu_0 = \mu^{(+)}\right) = 1 - \mathsf{P}\left(\mu_0 = \mu^{(-)}\right) = q \in (0,1)$$

holds true. This setting obviously satisfies condition (1.1). For figures 1 and 2 we have chosen

$$\begin{aligned} q &= \frac{3}{4}, \qquad m^{(+)} = \frac{10}{9}, \qquad m^{(-)} = \frac{2}{5}, \\ q &= \frac{1}{2}, \qquad m^{(+)} = \ 2, \qquad m^{(-)} = \frac{2}{3}, \end{aligned}$$

respectively.

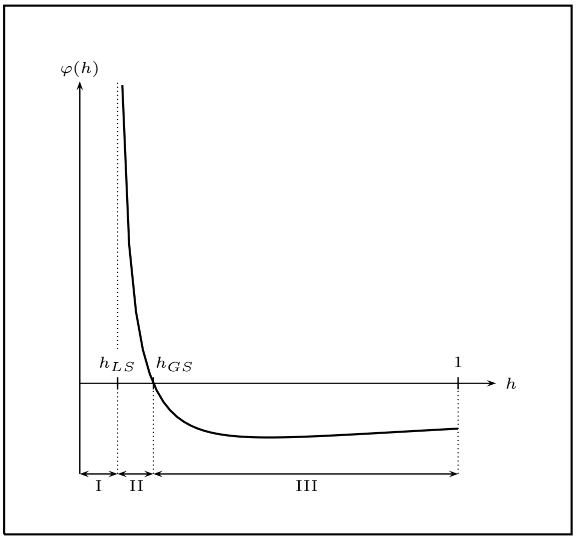

Figure 1.1: *There are three regimes: I: LS, II: GS but no LS, III: no GS*

2. As announced above, we also provide an example for a specific choice of the parameters so that we have $h_{GS} = h_{LS} < 1$. Let the law P^{m_0} of the mean offspring m_0 be given by

$$\frac{d\mathsf{P}^{m_0}}{d\lambda}(x) := 1.6 \cdot \mathbb{1}_{[0.5,1]}(x) + 0.2 \cdot \mathbb{1}_{(1,2]}(x),$$

where λ denotes the Lebesgue measure. Obviously we have $h_{LS} = 0.5$ and a simple computation yields

$$\varphi(h_{LS}) = 0.2 \cdot \Big(2 \cdot \log(2)\Big) + 1.6 \cdot \Big(2 \cdot \log(2) - 1.5 \cdot \log(3)\Big) < 0.$$

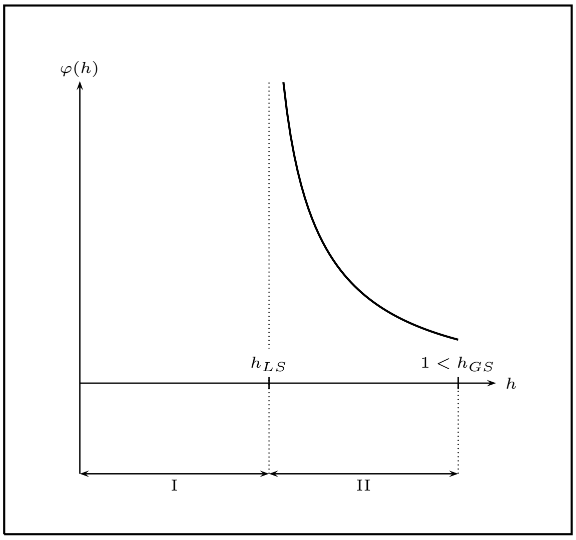

Figure 1.2: *There are two regimes: I: LS, II: GS but no LS*

Chapter 2

A central limit theorem for a random walk on Galton-Watson trees with random conductances

2.1 Introduction

In this chapter we consider a particular model of a random walk in a random environment. The environment of this random walk is a random network which consists of the graph of the genealogical tree of a Galton-Watson branching process and a family of non-negative i.i.d. weights which are assigned to the edges of the graph. In order to guarantee that the considered graph is infinite, we assume that in the corresponding branching process the number of offspring is at least one. For every given realization of the environment, the considered random walk is the Markov chain on the tree whose transition probabilities are given by the weight configuration. More precisely, for each vertex of the underlying graph, the probability for the Markov chain to move to an adjacent vertex is proportional to the weight of the edge connecting those two vertices. This Markov chain is reversible and its reversible measure is given by the sum of the weights of all edges which are incident to some vertex. As a source of intuition, weighted graphs can be regarded as an electric network in the physical sense by thinking of the edges of the graph as electric conductors (or wires) with conductances given by the respective weights. Following this idea, various connections between the concepts of electric current and voltage and the corresponding properties of random walks on those networks can be observed. We refer to the textbooks by Doyle and Snell [23] and by Lyons and Peres [47] for comprehensive introductions to random walks and electric networks.

It has already been shown that in the model described above the random walk is transient for almost every realization of the environment. The transience is a consequence of Proposition 4.10 in [2] if the mean conductance is finite and it is also derived by Gantert, Müller, Popov, and Vachkovskaia in their paper [30] for a more general setting in which the mean conductance can be infinite and in which, in addition, the distribution of the conductances may depend on the degree of the adjacent vertices. Moreover, the authors of [30] also show that the random walk has a deterministic and (strictly) positive speed if the mean conductance is finite. Here the speed of a random walk on a tree is defined as the limit of the graph distance from the root of the tree at time n divided by n as $n \to \infty$ whenever this limit exists. Thus, we almost surely have

$$\nu := \lim_{n\to\infty} \frac{|X_n|}{n} > 0.$$

In [30] the authors also provide a semi-explicit formula for the speed which depends only on the law of certain effective conductances in the underlying electric network. Also, they show that the random walk exhibits a slowdown effect if the constant weight configuration is replaced by i.i.d. random conductances which share the same mean. In our setting the conductances are assumed to be in the bounded interval $[\kappa_1, \kappa_2]$ for $\kappa_1 \leq \kappa_2$. In this case, the positivity of the speed of the random walk can also be derived from Theorem 1.1 in [57] and the fact that the graph of a supercritical Galton-Watson process (conditioned on non-extinction) satisfies the anchored expansion property. The latter result is for example proved in [16] (Corollary 1.3). For further details we refer to the considerations in those papers and to the references therein.

A natural question following up the analysis of the speed of the random walk is whether the random walk satisfies a central limit theorem. In this chapter we provide the proof of such a central limit theorem for bounded conductances and under an additional condition. More precisely, we show that, for a suitable variance $\sigma^2 > 0$, the random variable

$$\frac{|X_n| - \nu \cdot n}{\sqrt{\sigma^2 n}} \tag{2.1}$$

converges in distribution to the standard normal distribution as n tends to infinity. Here the convergence is considered with respect to a measure which averages over the random environment as well as over the random evolution of the random walk. This kind of central limit theorem is often referred to as *annealed*. Also, we derive an analogous result for the range of the random walk. Here the range of the random walk is defined as the number of different vertices visited by time n.

There are various models which have been considered in recent years and which are related to ours. The simple random walk (which corresponds to a constant weight configuration) is considered in [44] and [45]. In these papers the authors prove that the simple random walk is transient and that its speed is almost surely positive, using the concept of the environment observed by the particle. We refer to Chapter 16 of [47] for a textbook treatment of the topic. Basically, there are two important generalizations of the simple random walk, one of which is called λ-biased random walk. In this model the conductances are assigned to the edges on the tree in a way that the resulting probability for the random walk to move one step towards the root is proportional to λ while the probability to move away from the root is proportional to the number of edges leading away from the root. In this setting there are two competing effects: A higher number of offspring in the underlying tree pushes the random walk further away from the root while a higher value of λ keeps it closer to the root. Hence, it is plausible that this model shows a phase transition. It was proved in [44] that the random walk is positive recurrent if $\lambda > m$, null recurrent if $\lambda = m$, and transient if $\lambda < m$. Here m denotes the mean offspring of the corresponding Galton-Watson process. For the transient regime, i.e. for $\lambda < m$, it was shown in [45] that the λ-biased random walk on a Galton-Watson tree has a deterministic and positive speed which depends on the bias parameter λ. An explicit formula for the speed of the random walk is known in the case $\lambda = 1$ which coincides with the simple random walk. More precisely, from Theorem 3.2 of [45] we know that the speed of the simple random walk is given by $\nu_{SRW} = \sum_{k=0}^{\infty} p_k \frac{k-1}{k+1}$ if $(p_j)_{j\in\mathbb{N}_0}$ denotes the offspring distribution of the corresponding Galton-Watson process.

Another model related to ours are random walks on marked Galton-Watson trees which are for example studied in [1] and [26]. In this model, for every vertex of an underlying Galton-Watson tree, random weights are assigned to each of this vertex's direct descendants. Then the probability for the random walker to move to one of those descendants is proportional to the respective weight; whereas the probability to move one step towards the root is proportional to one. For this model the authors of the cited papers prove results on the speed of the random walk and a central limit theorem.

A central limit theorem for the simple random walk on Galton-Watson trees was first proved by Piau [53]. A more general central limit theorem is obtained in [52]. Therein the authors show that if the λ-biased random walk is transient, the random variable in (2.1) converges in distribution to a standard normal distribution for almost every Galton-Watson tree. This version of a central limit theorem is called *quenched*. The quenched result is stronger as it implies its annealed analogue.

This chapter is organized as follows: In Section 2.2 we provide a formal description of the considered model starting with some preliminaries on trees. Next, we introduce different probability measures which we want to study. Moreover, the concept of the environment observed by the random walk is introduced. Section 2.3 contains the results and Section 2.4 the proofs.

2.2 Formal description of the model

In our setting the considered random walk evolves in a random environment which is a weighted Galton-Watson tree. Thus, the formal description of the model begins with some notational preliminaries on trees.

2.2.1 Notational preliminaries on trees

Let $\mathcal{T}$ denote the set of all rooted trees. A rooted tree $\mathbf{T} = (\mathbf{T}, \mathbf{o}) \in \mathcal{T}$ is a non-oriented, connected, and locally finite graph containing no loops. In addition, one of the vertices of $\mathbf{T}$ is defined to be the root and it is denoted by $\mathbf{o}$. For notational convenience, we often omit to explicitly indicate which vertex of $\mathbf{T}$ is its root when it is clear form the context.

As we mostly want to emphasise the genealogical structure of a tree, we make use of the so-called Ulam-Harris labelling for trees. Every rooted tree can be interpreted as a suitable subset of the Ulam-Harris tree

$$\mathbb{V} := \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n,$$

where $\mathbb{N}^0 := \{\varnothing\}$ consists only of the root. Then, a tree $\mathbf{T}$ is a subset of $\mathbb{V}$ which satisfies the following three properties:

(i) $\varnothing \in \mathbf{T}$,

(ii) $v = (v_1, \ldots, v_n) \in \mathbf{T}$ implies $(v_1, \ldots, v_k) \in \mathbf{T}$ for each $k \in \{1, \ldots, n-1\}$,

(iii) $v = (v_1, \ldots, v_n) \in \mathbf{T}$ implies $(v_1, \ldots, v_{n-1}, j) \in \mathbf{T}$ for each $j \in \{1, \ldots, v_n\}$.

Having in mind the genealogy of a tree, we speak of *ancestors* and *descendants* in the canonical way. Besides, we frequently make use of the expression *direct descendant* (child) which is not only a descendant but also a neighbour in the corresponding graph. Moreover, for two vertices v, w in a tree, we refer to the unique self-avoiding path connecting v and w simply as the *path* connecting v and w. Thus, v is a descendant of w if and only if the path connecting v and $\mathbf{o}$ runs through w and in this case we write $w \preceq v$. We also write $w \prec v$ if we have $w \preceq v$ and $w \neq v$. The set of vertices of a tree $\mathbf{T}$ which belong to the n-th level or generation is denoted by $\mathbf{T}_n$. Moreover, for a vertex $v \in \mathbf{T}$ we write $\mathbf{T}^v$ for the subtree of $\mathbf{T}$ rooted in the vertex v, i.e. $\mathbf{T}^v := \{w \in \mathbf{T} : v \preceq w\}$.

For each tree $\mathbf{T}$, the set of its edges is denoted by $\mathcal{E}(\mathbf{T})$ and the edge connecting two neighbouring vertices v and w is denoted by (v, w). We also write $v \sim w$ if v and w are connected by an edge, i.e., $(v, w) \in \mathcal{E}(\mathbf{T})$. In order to obtain weighted trees, we want to assign a non-negative real number $\xi(e)$ to each edge $e \in \mathcal{E}(\mathbf{T})$, which is called the *weight* or *conductance* of the edge e. Thus, we obtain the weight configuration $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{T}) := (\xi(e))_{e\in\mathcal{E}(\mathbf{T})}$ of the tree $\mathbf{T}$. In our setting we only want to allow weight configurations with bounded weights. To be precise, for fixed positive constants $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 \leq \kappa_2$, the set of weight configurations for the tree $\mathbf{T}$ is denoted by $\boldsymbol{\Xi} = \boldsymbol{\Xi}(\mathbf{T}) := \{(\xi(e))_{e\in\mathcal{E}(\mathbf{T})} : \xi(e) \in [\kappa_1, \kappa_2]\,\forall\, e \in \mathcal{E}(\mathbf{T})\}$, and the space of weighted and rooted trees is denoted by

$$\Omega := \{(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) : (\mathbf{T}, \mathbf{o}) \in \mathcal{T}, \boldsymbol{\xi} \in \boldsymbol{\Xi}\}.$$

The set Ω is referred to as the *space of environments*. When there is no risk of confusion, we write only $\mathbf{T}$ for a weighted and rooted tree because it is often clear from the context which root and which weight configuration we are referring to.

2.2.2 Environment measures

We now introduce different probability measures (and their respective expectations) on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a suitable σ-algebra on Ω. To be more precise, the space of environments Ω can be canonically embedded into the set $\mathcal{T} \times [\kappa_1, \kappa_2]^{\mathbb{V}\times\mathbb{V}}$ and $\mathcal{F}$ can be chosen as the product of standard σ-algebras on $\mathcal{T}$ and $[\kappa_1, \kappa_2]^{\mathbb{V}\times\mathbb{V}}$. We will always refer to probability measures on $(\Omega, \mathcal{F})$ as *environment measures*.

The main ingredient of the random environments which we want to consider is a Galton-Watson branching process. So let $(p_j)_{j\in\mathbb{N}_0}$ be the *offspring distribution* of a standard Galton-Watson process, i.e., p_j is the probability that a vertex belonging to the corresponding Galton-Watson tree has j direct descendants. Since we only want to consider infinite trees, we assume that we have $p_0 = 0$, which means that each vertex has at least one direct descendant and thus the branching process cannot become extinct. Moreover, the smallest index j such that p_j is positive is denoted by $d_0 := \inf\{j \in \mathbb{N}_0 : p_j > 0\}$. By definition we have $d_0 \in [1, \infty)$. Furthermore, we assume that we have

$$m := \sum_{j=1}^{\infty} j\, p_j \in (1, \infty].$$

This already implies that there exists $j \geq 2$ such that $p_j > 0$ and thus the tree has infinitely many ends. The genealogical tree of a branching process given by these parameters will yield the underlying (random) graph for the random walk which we want to study. However, one important feature of a standard Galton-Watson tree is the fact that its root $\mathbf{o}$ is a vertex which is singular in the sense that it has no ancestor. In other words, in distribution the root has one neighbour less than all the other vertices of a Galton-Watson tree.

In order to prevent this difficulty, we introduce augmented Galton-Watson trees. An augmented Galton-Watson tree is a random tree which is defined just like the standard Galton-Watson tree except that the root has stochastically one more direct descendant than all other vertices, i.e., the root has $j+1$ direct ancestors with probability p_j and all those $j+1$ vertices have independent standard Galton-Watson descendant trees. Another equivalent way to define augmented Galton-Watson trees is to consider two independent copies of a standard Galton-Watson tree whose roots are connected by one additional edge.

Besides the mere graph structure, which is denoted by $\mathbf{T}$ and given by the branching process, the random environment also includes a family of i.i.d. random conductances $(\xi(e))_{e\in\mathcal{E}(\mathbf{T})}$. Let μ be the distribution of such a random conductance $\xi(e)$. By definition, μ is a probability measure which is concentrated on $[\kappa_1, \kappa_2]$.

We are now ready to introduce two environment measures $\widehat{\mathsf{P}}$ and P on the set of environments Ω. We define $\widehat{\mathsf{P}}$ as the probability measure on Ω such that under $\widehat{\mathsf{P}}$ the random variable $\mathbf{T}$ is the genealogical tree of a standard Galton-Watson process with the above offspring distribution $(p_j)_{j\in\mathbb{N}_0}$. In addtion, under $\widehat{\mathsf{P}}$ all edges of the tree $\mathbf{T}$ are labeled with the random conductances $(\xi(e))_{e\in\mathcal{E}(\mathbf{T})}$ which are i.i.d. with distribution μ and independent of $\mathbf{T}$. Analogously, we define P as the probability measure on Ω such that $\mathbf{T}$ is an augmented Galton-Watson tree with i.i.d. conductances $(\xi(e))_{e\in\mathcal{E}(\mathbf{T})}$. Accordingly, we use $\widehat{\mathsf{E}}$ and E for the respective expectation operators.

2.2.3 The quenched and the annealed law

For each $\omega = (\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) \in \Omega$, we consider the discrete-time random walk $(X_n)_{n\in\mathbb{N}_0}$ on the electric network defined by ω. This electric network is the graph given by $\mathbf{T}$ with the conductances given by $\boldsymbol{\xi}$. More precisely, $(X_n)_{n\in\mathbb{N}_0}$ is the Markov chain on $\mathbf{T}$ with transition probabilities $(q_\omega(u,v))_{u,v\in\mathbf{T}}$ defined by

$$q_\omega(u,v) := \begin{cases} \frac{\xi(u,v)}{\pi(u)} & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}$$

for vertices $u, v \in \mathbf{T}$. Here π denotes the associated reversible measure which is defined by

$$\pi(u) := \pi_\omega(u) := \sum_{v: v \sim u} \xi(u, v) \tag{2.2}$$

for $u \in \mathbf{T}$. For each fixed realization of the environment $\omega = (\mathbf{T}, \mathbf{o}, \boldsymbol{\xi})$, the distribution of the random walk $(X_n)_{n \in \mathbb{N}_0}$ starting from $\mathbf{o}$ is denoted by P_ω, i.e., P_ω is the probability measure on the space of (infinite) trajectories in the weighted and rooted tree $\mathbf{T}$ satisfying

$$P_\omega(X_{n+1} = u | X_n = v) = q_\omega(u, v)$$

and

$$P_\omega(X_0 = \mathbf{o}) = 1.$$

In some situations it is also useful to assume that the random walk starts from a vertex $v \in \mathbf{T}$ which is not the root of the tree. In those cases the associated distribution is denoted by P_ω^v. The probability measures P_ω and P_ω^v are called *quenched probabilities* in the literature and their expectations are denoted by E_ω and E_ω^v, respectively.

In addition to the quenched measure we introduce the so-called *annealed probability* which is obtained by averaging the quenched measure w.r.t. a probability measure on the space of environments. For this reason, annealed measures are also called *averaged measures* which is perhaps a more suggestive name. Formally, for every $\omega \in \Omega$, P_ω can be regarded as a probability measure on the space of trajectories $(\mathbb{V}^{\mathbb{N}_0}, \mathcal{G})$, where $\mathcal{G}$ is the σ-algebra which is generated by the cylinder sets of $\mathbb{V}^{\mathbb{N}_0}$. Moreover, for every $A \in \mathcal{G}$, the function $P_\omega(A) : (\Omega, \mathcal{F}) \to [0, 1]$ is measurable as a function of ω. Therefore, the probability measure $\mathbb{P} := \mathsf{P} \otimes P_\omega$ on $(\Omega \times \mathbb{V}^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G})$ is well-defined by

$$\mathbb{P}(F \times G) := \int_F P_\omega(G)\, \mathsf{P}(\mathrm{d}\omega) \tag{2.3}$$

for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Mostly, the events in which we are interested only concern the trajectory of the random walk and not the specific environment chosen (i.e. events of the form $\Omega \times G$). Thus, with a slight abuse of notation, $\mathbb{P}$ is also used to denote its marginal on $(\mathbb{V}^{\mathbb{N}_0}, \mathcal{G})$, which means that for $G \in \mathcal{G}$ we write

$$\mathbb{P}(G) = \int P_\omega(G)\, \mathsf{P}(\mathrm{d}\omega).$$

In general, the above construction of $\mathbb{P}$ works for each environment measure on $(\Omega, \mathcal{F})$. In particular, the averaged measure w.r.t. the environment measure $\widehat{\mathsf{P}}$ is denoted by $\widehat{\mathbb{P}} := \widehat{\mathsf{P}} \otimes P_\omega$.

2.2.4 The environment observed by the random walk

The crucial advantage of considering augmented Galton-Watson trees instead of standard Galton-Watson trees is that they give rise to a reversible process on the space of weighted and rooted trees (cf. Chapter 3 in [30]). This process is frequently called *the environment observed by the random walk* in the literature and it is the Markov process on Ω with transition operator K defined by

$$\begin{aligned} Kf(\omega) = Kf(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) \;&:= \sum_{v: v\sim \mathbf{o}} q_\omega(\mathbf{o}, v) \cdot f(\mathbf{T}, v, \boldsymbol{\xi}) \\ &= \frac{1}{\pi(\mathbf{o})} \sum_{v: v\sim \mathbf{o}} \xi(\mathbf{o}, v) \cdot f(\mathbf{T}, v, \boldsymbol{\xi}) \end{aligned}$$

for a function $f : \Omega \to \mathbb{R}$. Now let $\widetilde{\mathsf{P}}$ denote the probability measure on Ω defined by its Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}}(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) := \frac{\pi(\mathbf{o})}{\mathsf{E}[\xi] \cdot |\mathbf{T}_1|}$$

and, as usual, let $\widetilde{\mathsf{E}}$ denote its expectation. It is not difficult to show that the transition operator K is reversible w.r.t. $\widetilde{\mathsf{P}}$, i.e., we have

$$\widetilde{\mathsf{E}}\big[f(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) \cdot Kg(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi})\big] = \widetilde{\mathsf{E}}\big[Kf(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) \cdot g(\mathbf{T}, \mathbf{o}, \boldsymbol{\xi})\big] \tag{2.4}$$

for two $\widetilde{\mathsf{P}}$-square-integrale functions f, g on the space of weighted and rooted trees. For a complete proof of equation (2.4) we refer to Lemma 3.1 in [30].

Analogously to the definition in (2.3), we also consider the measure P_ω averaged over all environments w.r.t. $\widetilde{\mathsf{P}}$. This yields the probability measure $\widetilde{\mathbb{P}}$ which is defined by

$$\widetilde{\mathbb{P}} := \widetilde{\mathsf{P}} \otimes P_\omega.$$

2.3 Main results

The main results of this chapter are two central limit theorems with respect to the averaged measure $\mathbb{P}$, one of which involves the distance to the root and one of which involves the range of the random walk $(X_n)_{n\in\mathbb{N}_0}$. As usual, for a vertex v in a rooted tree, we write $|v|$ for the the graph distance between v and the root, which is the same as the length of the path connecting v and $\mathbf{o}$. Thus, the distance between the root and the random walk at time n is denoted by $|X_n|$. Moreover, let $r(n)$ denote the range of the random walk at time n, i.e., $r(n) := \#\{X_1, \ldots, X_n\}$. From Theorem 4.1 and

Theorem 4.2 in [30] we know that the random walk $(X_n)_{n\in\mathbb{N}_0}$ is a.s. transient and that there is even a positive speed $\nu > 0$ such that we have

$$\frac{|X_n|}{n} \xrightarrow[n\to\infty]{P_\omega\text{-a.s.}} \nu \tag{2.5}$$

for P-a.e. environment ω. Using the very same argument as in [30], it is possible to obtain an analogous result for the range of the random walk. Besides, we have $r(n) \geq |X_n|$, and hence there is a positive $\bar{\nu} \geq \nu$ such that we have

$$\frac{r(n)}{n} \xrightarrow[n\to\infty]{P_\omega\text{-a.s.}} \bar{\nu} \tag{2.6}$$

for P-a.e. environment ω.

Remark 2.3.1. As a consequence of the transience of the random walk, the quenched escape probability is P-a.s. and $\widehat{\mathsf{P}}$-a.s. positive. Thus, we have

$$\mathcal{C} = \mathcal{C}(\omega) := \pi(\mathbf{o}) \cdot P_\omega\big(X_n \neq \mathbf{o}\,\forall\, n \in \mathbb{N}\big) = \pi(\mathbf{o}) \cdot P_\omega\Big(\inf\{n \in \mathbb{N} :\, X_n = \mathbf{o}\} = \infty\Big) > 0. \tag{2.7}$$

for P-a.e. and $\widehat{\mathsf{P}}$-a.e. ω. The random variable $\mathcal{C}$ is often called the conductance between $\mathbf{o}$ and infinity within the electric network given by ω.

Theorem 2.3.2. *We assume that we have $d_0 \geq \frac{\kappa_2}{\kappa_1}$. There are constants $\sigma^2, \bar{\sigma}^2 \in (0, \infty)$ such that we have*

(i) $\dfrac{|X_n| - \nu \cdot n}{\sqrt{\sigma^2 n}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0,1)$ *w.r.t. the averaged measure $\mathbb{P}$.*

(ii) $\dfrac{r(n) - \bar{\nu} \cdot n}{\sqrt{\bar{\sigma}^2 n}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}(0,1)$ *w.r.t. the averaged measure $\mathbb{P}$.*

Remark 2.3.3. (i) It is possible to describe the constants σ^2 and $\bar{\sigma}^2$ more explicitly in terms of the averaged variance of certain random variables. More precisely, using the notation introduced in (2.8), (2.9), (2.11), and (2.10), we have

$$\sigma^2 = \frac{\mathbb{E}\big[(V_1 - \mathbb{E}[V_1])^2\big]}{\mathbb{E}[\tau_2 - \tau_1]} \qquad \text{and} \qquad \bar{\sigma}^2 = \frac{\mathbb{E}\big[(\bar{V}_1 - \mathbb{E}[\bar{V}_1])^2\big]}{\mathbb{E}[\sigma_2 - \sigma_1]}.$$

(ii) It is intuitively clear that Theorem 2.3.2 also holds w.r.t. the measures $\widehat{\mathbb{P}}$ and $\widetilde{\mathbb{P}}$ since trees generated according to the three measures only differ in the neighbourhood of the root and since the random walk is transient w.r.t. all of those measures. We will see that this statement holds rigorously true since the first summand on the right hand side of (2.28) does not depend on whether we choose P, $\widehat{\mathsf{P}}$ or $\widetilde{\mathsf{P}}$ as the environment measure (cf. the proof of Theorem 2.3.2 (*i*)).

(iii) The assumption $d_0 \geq \frac{\kappa_2}{\kappa_1}$ implies that the random graph given by ω contains a regular tree with a degree which is at least $\frac{\kappa_2}{\kappa_1} + 1$. For higher values of d_0 more fluctuation for the random conductances is allowed. However, we believe that the presented results do not need this assumption, what may motivate future analysis of the model.

2.4 Proofs

The proof of the above theorem strongly relies on the fact that the process possesses a certain regeneration structure. In order to specify this, we need some further notation: As usual, we define $\inf \emptyset := \infty$. For a vertex v of a tree, let $\zeta(v)$ be the *vertex hitting time* of v defined by

$$\zeta(v) := \inf\{k \geq 1 : X_k = v\}.$$

Analogously, we define the *level hitting time* of level $n \in \mathbb{N}$ by

$$\eta(n) := \inf\{k \geq 1 : |X_k| = n\}.$$

Moreover, we inductively define the sequence of *vertex regeneration times* $(\sigma_n)_{n\in\mathbb{N}}$ by

$$\begin{aligned} \sigma_1 &:= \inf\{k \geq 1 : X_i \neq X_k \,\forall\, i < k \text{ and } X_j \neq X_{k-1} \,\forall\, j \geq k\}, \\ \sigma_{n+1} &:= \inf\{k \geq \sigma_n : X_i \neq X_k \,\forall\, i < k \text{ and } X_j \neq X_{k-1} \,\forall\, j \geq k\} \quad \text{for } n \geq 2, \end{aligned} \tag{2.8}$$

and, analogously, the sequence of *level regeneration times* $(\tau_n)_{n\in\mathbb{N}}$ by

$$\begin{aligned} \tau_1 &:= \inf\{k \geq 1 : |X_i| < |X_k| \,\forall\, i < k \text{ and } |X_j| \geq |X_k| \,\forall\, j \geq k\}, \\ \tau_{n+1} &:= \inf\{k \geq \tau_n : |X_i| < |X_k| \,\forall\, i < k \text{ and } |X_j| \geq |X_k| \,\forall\, j \geq k\} \quad \text{for } n \geq 2. \end{aligned} \tag{2.9}$$

We immediately observe that we have $\sigma_1 \leq \tau_1$ and $\sigma_{n+1} - \sigma_n \leq \tau_{n+1} - \tau_n$ for all $n \in \mathbb{N}$ since at each level regeneration time the process also exhibits a vertex regeneration.

Whenever the $(n+1)$-th regeneration time τ_{n+1} is finite, the level increment between the regeneration times τ_n and τ_{n+1} is denoted by

$$U_n := |X_{\tau_{n+1}}| - |X_{\tau_n}| \tag{2.10}$$

and the centred level increment by

$$V_n := U_n - \nu \cdot (\tau_{n+1} - \tau_n). \tag{2.11}$$

The centring $\nu \cdot (\tau_{n+1} - \tau_n)$ is suitable for the level increments because, asymptotically, with each time step the distance from the root of the random walk increases by ν; cf. (2.5). Analogously, we define the vertex increments and the centred vertex increments by

$$\bar{U}_n := r(\sigma_{n+1}) - r(\sigma_n),$$
$$\bar{V}_n := \bar{U}_n - \bar{\nu} \cdot (\sigma_{n+1} - \sigma_n)$$

whenever σ_{n+1} is finite; cf. (2.6).

Proposition 2.4.1. *(i) For all $n \in \mathbb{N}$ the random variables σ_n and τ_n are P_ω-a.s. finite for* P*-a.e. ω.*

(ii) The sequences $(U_n)_{n\in\mathbb{N}}$, $(\bar{U}_n)_{n\in\mathbb{N}}$, $(\sigma_{n+1} - \sigma_n)_{n\in\mathbb{N}}$, and $(\tau_{n+1} - \tau_n)_{n\in\mathbb{N}}$ are i.i.d. sequences w.r.t. the averaged measure $\mathbb{P}$.

Proof of Proposition 2.4.1. The fact that there are a.s. infinitely many level regeneration points is proved as part of the proof of Theorem 4.1 in [30]. In fact, the authors of [30] prove the ergodicity of a certain dynamical system using techniques presented in Chapter 16 of [47]. Since every level regeneration time is also a vertex regeneration time, we consider part (*i*) of Proposition 2.4.1 as proved.

Part (*ii*) is intuitively clear since the differences between two consecutive regeneration times depend only on mutually disjoint subgraphs of the underlying random tree. This fact yields that the sequences $(\sigma_{n+1} - \sigma_n)_{n\in\mathbb{N}}$ and $(\tau_{n+1} - \tau_n)_{n\in\mathbb{N}}$ are i.i.d. sequences. Obviously, this implies that the same holds true for the increments $(U_n)_{n\in\mathbb{N}}$ and $(\bar{U}_n)_{n\in\mathbb{N}}$. A rigorous proof, which is rather technical, is provided after Proposition 3.4 in [46]. ■

Remark 2.4.2. (i) We note that we have

$$U_n = |X_{\tau_{n+1}}| - |X_{\tau_n}| \leq \tau_{n+1} - \tau_n,$$
$$\bar{U}_n = r(\sigma_{n+1}) - r(\sigma_n) \leq \sigma_{n+1} - \sigma_n$$

for all $n \in \mathbb{N}$ and thus integrability of the right hand side implies the same for U_n and $\bar{U}_n$, respectively.

(ii) We observe that the distribution of σ_1 is different from that of $\sigma_{n+1} - \sigma_n$ for $n \in \mathbb{N}$ and that the same holds true for the distributions of τ_1 and $\tau_{n+1} - \tau_n$ for $n \in \mathbb{N}$. More precisely, we have $\mathbb{P}(\tau_2 - \tau_1 \in \cdot\,) = \mathbb{P}(\tau_1 \in \cdot\,|\zeta(\mathbf{o}) = \infty)$.

In the following proposition we consider the rate of decay of the tail probabilities of the random variables $\sigma_2 - \sigma_1$ and $\tau_2 - \tau_1$.

Proposition 2.4.3. *There are constants $\delta, \bar{\delta} > 0$ such that the following assertions hold true.*

(i) We have

$$\mathbb{P}(\tau_2 - \tau_1 \geq n) \leq \exp(-\delta \cdot n^{1/3}) \tag{2.12}$$

for all $n \in \mathbb{N}$. In particular, the random variables $\tau_2 - \tau_1$, U_1, and V_1 possess all moments w.r.t. $\mathbb{P}$.

(ii) We have

$$\mathbb{P}(\sigma_2 - \sigma_1 \geq n) \leq \exp(-\bar{\delta} \cdot n^{1/3}) \tag{2.13}$$

for all $n \in \mathbb{N}$. In particular, the random variables $\sigma_2 - \sigma_1$, $\bar{U}_1$, and $\bar{V}_1$ possess all moments w.r.t. $\mathbb{P}$.

Remark 2.4.4. Since each level regeneration is also a vertex regeneration, part (ii) of Proposition 2.4.3 is an immediate consequence of part (i). Thus, it suffices to prove part (i).

Proof of part (i) of Proposition 2.4.3. For this proof we adapt techniques already used by Piau in his paper [53]. Throughout the proof, for a non-negative sequence $(a_n)_{n\in\mathbb{N}}$, we say that a_n decays exponentially fast for $n \to \infty$ if there is a positive constant $\delta > 0$ such that we have $a_n \leq \exp(-\delta \cdot n)$ for all $n \in \mathbb{N}$.

The crucial idea of this proof is to decompose the trajectory of the process $(|X_n|)_{n\in\mathbb{N}_0}$ for all times $n \leq \tau_1$ in a suitable manner. More precisely, we decompose $(|X_n|)_{0\leq n\leq\tau_1}$ into its partial trajectories between a random number of (unsuccessful) trials to flee a newly reached level. For this purpose, we need to introduce some further notation. We start with defining $m_0 := 0$ and $s_0 := 1$. Further, if $|X_n| > 0$ for all $n \geq 1$ (i.e. $\tau_1 = 1$), we define $\alpha := 0$. Otherwise, we write

$$\begin{aligned} r_1 &:= \inf\{k \geq 1 : \, |X_k| = 0\}, \\ m_1 &:= \sup\{|X_k| : \, k \leq r_1\}, \\ s_1 &:= \inf\{k \geq 1 : \, |X_k| = 1 + m_1\}. \end{aligned}$$

If $|X_n| > m_1$ for all $n \geq s_1$ (i.e. $\tau_1 = s_1$), we define $\alpha := 1$. Analogously to the above definition as long as r_n, m_n, and s_n are well-defined with $r_n < \infty$, we introduce

$$\begin{aligned} r_{n+1} &:= \inf\{k \geq s_n : \, |X_k| = |X_{s_n}| - 1 = \textstyle\sum_{i=1}^n m_i\}, \\ m_{n+1} &:= \sup\{|X_k| - |X_{s_n}| + 1 : \, k \leq r_n\}, \\ s_{n+1} &:= \inf\{k \geq 1 : \, |X_k| = |X_{s_n}| + m_{n+1}\}. \end{aligned}$$

We refer to Figure 2.1 which illustrates the above definitions and helps to get a clear picture of the random variables $(r_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(m_n)_{n\in\mathbb{N}}$, and the decomposition of the process.

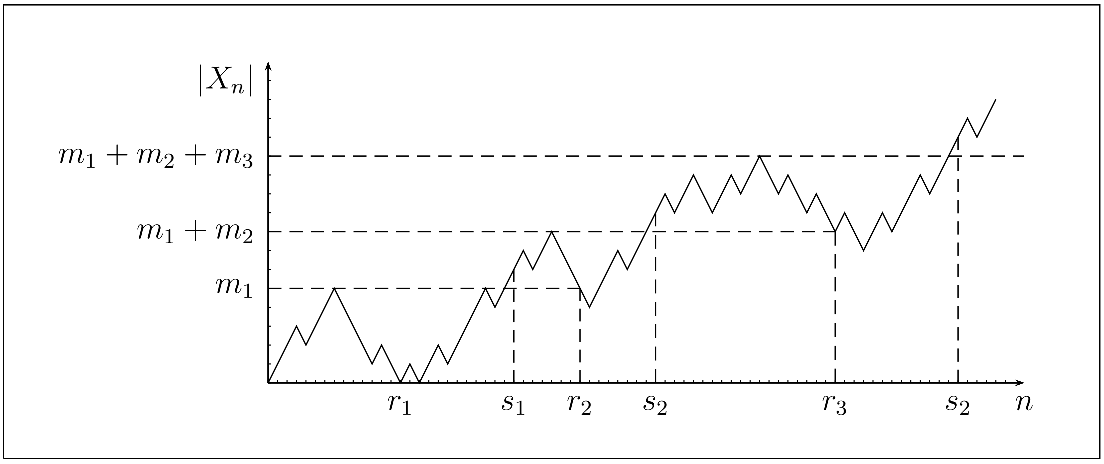

Figure 2.1: *The decomposition of a typical trajectory of the process* $(|X_n|)_{n\in\mathbb{N}_0}$.

We observe that if $r_n < \infty$ and $r_{n+1} = \infty$, we have $\tau_1 = s_n$ and $\alpha = n$, which means that α denotes the index of the last unsuccessful trial to escape a newly reached level, i.e.

$$\alpha = \sup\{n \geq 0 : \, r_n < \infty\} = \inf\{n \geq 0 : \, r_{n+1} = \infty\}.$$

Moreover, we have

$$\{\tau_1 = s_n\} = \{\alpha = n\} = \{r_n < \infty, r_{n+1} = \infty\}$$

and

$$|X_{\tau_1}| \cdot \mathbb{1}_{\{\tau_1 = s_n\}} = (1 + m_1 + \ldots + m_n) \cdot \mathbb{1}_{\{\tau_1 = s_n\}}.$$

The partial trajectories $(|X_n|)_{s_k \leq n \leq s_{k+1}}$ are not independent. However, the excursions

$$(|X_n| - |X_{s_k}|)_{s_k \leq n \leq r_{k+1}}$$

only depend on mutually disjoint subgraphs of the underlying tree ω and therefore, they are i.i.d. for $k \in \{0, \ldots, \alpha - 1\}$ w.r.t. $\mathbb{P}$. In particular, this also holds true for the random variables $(m_n)_{1\leq n\leq\alpha}$.

After having introduced the above notation, we are ready to prove the proposition. We split up the proof into a set of lemmata which finally yield (2.12).

The following lemma is a general lemma on escape probabilities of random walks on electric networks given by graphs (cf. (2.7)).

Lemma 2.4.5. *Let* $(\mathbf{G}, \xi(v,w)_{v,\in\mathbf{G}}))$ *be an infinite weighted graph with finite degree. Further, under* $\mathsf{P}^v_{\boldsymbol{G}}$*, let* $(X_n)_{n\in\mathbb{N}_0}$ *be the reversible random walk started in* v *on the electric network given by* $(\mathbf{G}, \xi(v,w)_{v,\in\mathbf{G}}))$*, whose canonical reversible measure is given by* $\pi_{\mathbf{G}}(v) := \sum_{w\in\mathbf{G}:w\sim v}\xi(v,w)$*. For* $y \in \mathbf{G}$ *we consider the stopping times*

$$\zeta(y) := \inf\{n \geq 0 : X_n = y\} \quad \textit{and} \quad \zeta^+(y) := \inf\{n \geq 1 : X_n = y\}$$

and, moreover, the conductance from y *to infinity which is defined by*

$$\mathcal{C}_{\mathbf{G}}(y) := \pi_{\mathbf{G}}(y) \cdot \mathsf{P}^y_{\mathbf{G}}(\zeta^+(y) = \infty).$$

Let $\mathbf{G}^*$ *denote the graph* $\mathbf{G}$ *plus one additional vertex* x *which is exclusively connected to* y *with edge weight* $\xi(x,y)$*. Then we have*

$$\mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty) = \frac{\xi(x,y)}{\xi(x,y) + \mathcal{C}_{\mathbf{G}}(y)}. \tag{2.14}$$

Proof of Lemma 2.4.5. Due to the Markov property we have

$$\begin{aligned}
&\mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty) \\
&= \mathsf{P}^y_{\mathbf{G}^*}(X_1 = x) + \sum_{z\sim y, z\neq x} \mathsf{P}^y_{\mathbf{G}^*}(X_1 = z) \cdot \mathsf{P}^z_{\mathbf{G}^*}(\zeta(x) < \infty) \\
&= \mathsf{P}^y_{\mathbf{G}^*}(X_1 = x) + \sum_{z\sim y, z\neq x} \mathsf{P}^y_{\mathbf{G}^*}(X_1 = z) \cdot \mathsf{P}^z_{\mathbf{G}^*}(\zeta(y) < \infty) \cdot \mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty) \\
&= \frac{\xi(x,y)}{\pi_{\mathbf{G}^*}(y)} + \sum_{z\sim y, z\neq x} \frac{\xi(y,z)}{\pi_{\mathbf{G}^*}(y)} \cdot \mathsf{P}^z_{\mathbf{G}}(\zeta(y) < \infty) \cdot \mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty)
\end{aligned}$$

and from this we get

$$\begin{aligned}
&\pi_{\mathbf{G}^*}(y) \cdot \mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty) \\
&= \xi(x,y) + \sum_{z\sim y, z\neq x} \xi(y,z) \cdot \mathsf{P}^z_{\mathbf{G}}(\zeta(y) < \infty) \cdot \mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty),
\end{aligned}$$

which is equivalent to

$$\mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty) = \frac{\xi(x,y)}{\pi_{\mathbf{G}^*}(y) - \sum_{z\sim y, z\neq x} \xi(y,z) \cdot \mathsf{P}^z_{\mathbf{G}}(\zeta(y) < \infty)}. \tag{2.15}$$

Furthermore, we have

$$\begin{aligned}
&\sum_{z\sim y, z\neq x} \xi(y,z) \cdot \mathsf{P}^z_{\mathbf{G}}(\zeta(y) < \infty) \\
&= \pi_{\mathbf{G}}(y) \cdot \mathsf{P}^y_{\mathbf{G}}(\zeta^+(y) < \infty)
\end{aligned}$$

$$= \pi_{\mathbf{G}}(y) \cdot \left[1 - \mathsf{P}^y_{\mathbf{G}}(\zeta^+(y) = \infty)\right]$$
$$= \pi_{\mathbf{G}}(y) - \mathcal{C}_{\mathbf{G}}(y)$$

and finally, from this and (2.15) we conclude that we have

$$\mathsf{P}^y_{\mathbf{G}^*}(\zeta(x) < \infty) = \frac{\xi(x,y)}{\pi_{\mathbf{G}^*} - \pi_{\mathbf{G}} + \mathcal{C}_{\mathbf{G}}(y)} = \frac{\xi(x,y)}{\xi(x,y) + \mathcal{C}_{\mathbf{G}}(y)}.$$

□

We make use of a rather technical lemma about weighted Galton-Watson trees which goes back to Grimmett and Kesten.

Lemma 2.4.6. *Let A be a (measurable) subset of all rooted and weighted trees Ω and let x be a vertex of a the n-th generation of a tree $\mathbf{T}$. For $\varepsilon_1 > 0$ we consider the set $A(\varepsilon_1, x)$ of all vertices y with $\boldsymbol{o} \preceq y \prec x$ which satisfy the property that the number of direct descendants z of y such that $x \notin \mathbf{T}^z$ and $\mathbf{T}^z \in A$ is greater or equal to $\varepsilon_1|Z(y)|$. This set is precisely defined by*

$$A(\varepsilon_1, x) := \left\{y : \boldsymbol{o} \preceq y \prec x,\ |\{z \in Z(y) : x \notin \mathbf{T}^z, \mathbf{T}^z \in A\}| \geq \varepsilon_1 |Z(y)|\right\}.$$

Then there are positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that the sequence

$$\widehat{\mathsf{P}}(\exists\, x \in \mathbf{T}_n : |A(\varepsilon_1, x)| \leq \varepsilon_3 n) \tag{2.16}$$

decays exponentially fast as n tends to infinity for all (measurable) $A \subseteq \Omega$ satisfying $\widehat{\mathsf{P}}(A) \geq 1 - \varepsilon_2$. Here we can choose any $\varepsilon_2 > 0$ such that $f'(\varepsilon_2^{1-\varepsilon_1}) < 1$ holds, where f denotes the probability generating function of the offspring distribution corresponding to $\widehat{\mathsf{P}}$. Moreover, the above statement holds also true for augmented Galton-Waltson tree, i.e. w.r.t. the measure P.

Proof of Lemma 2.4.6. The proof of a result which is similar to the above statement is part of an unpublished work of Grimmett and Kesten. We refer to Lemma 1 in [34] (cf. Lemma 1 in [53]). For the sake of completeness, we also provide a proof of Lemma 2.4.6 here, especially because our result differs from that of [34] to some extent.

For the proof we make use of the Ulam-Harris labelling for trees as introduced in section 2.2.1. In order to estimate the probability in (2.16), we first consider the situation in which there is a vertex $x = (x_1, \ldots, x_n)$ in $\mathbf{T}$ such that the vertices $(x_1, \ldots, x_i)$ on the path connecting $\mathbf{o}$ and x have $l_1, \ldots, l_n$ direct descendants, respectively. For a vertex $v \in \mathbf{T}$ we write $Z(v) := \{w \in \mathbf{T} : |w| = |v| + 1, v \preceq w\}$ for the set of its direct

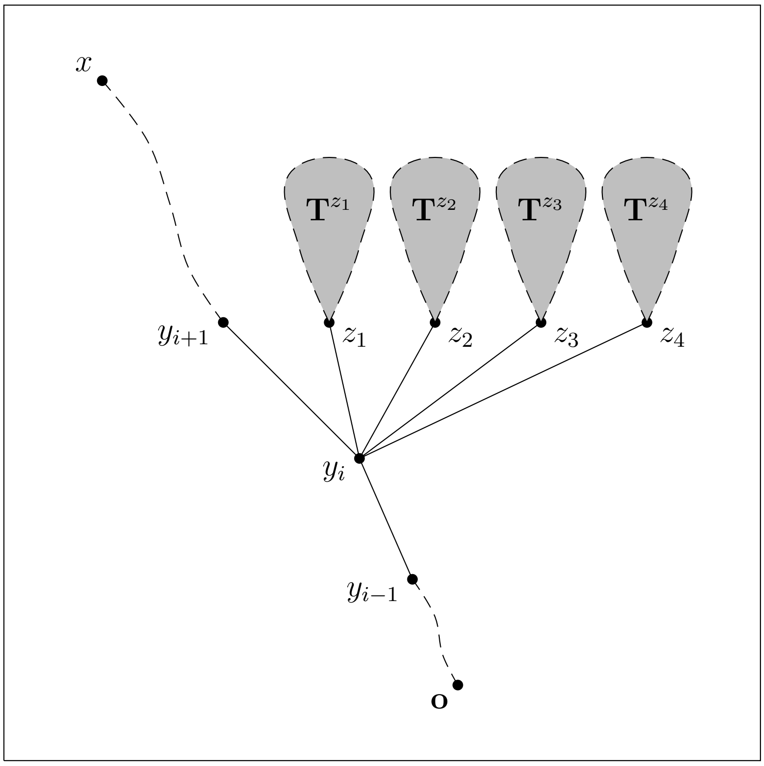

Figure 2.2: *The figure illustrates the subtrees $\mathbf{T}^{z_1}, \ldots, \mathbf{T}^{z_4}$ which are rooted in the direct descendants z_1, z_2, z_3, z_4 of the i-th vertex on the path connecting **o** and $x = (x_1, \ldots, x_n)$.*

descendants, and we observe that for all $k_1, \ldots, k_n \in \mathbb{N}$, $l_1 \geq k_1, \ldots, l_n \geq k_n$, and $\theta > 0$ we have

$$\begin{aligned} &\widehat{\mathsf{P}}\Big(x \in \mathbf{T}_n,\, |Z(x_1, \ldots, x_i)| = l_{i+1} \,\forall\, i \in \{0, \ldots, n-1\},\, |A(\varepsilon_1, x)| \leq \varepsilon_3 n\Big) \\ &= p_{l_1} \cdot \ldots \cdot p_{l_n} \cdot \widehat{\mathsf{P}}\Big(|A(\varepsilon_1, x)| \leq \varepsilon_3 n \,\Big|\, |Z(x_1, \ldots, x_i)| = l_{i+1} \,\forall\, i \in \{0, \ldots, n-1\}\Big) \\ &\leq p_{l_1} \cdot \ldots \cdot p_{l_n} \cdot e^{\theta \varepsilon_3 n} \cdot \widehat{\mathsf{E}}\Big[e^{-\theta |A(\varepsilon_1, x)|} \,\Big|\, |Z(x_1, \ldots, x_i)| = l_{i+1} \,\forall\, i \in \{0, \ldots, n-1\}\Big] \end{aligned} \tag{2.17}$$

as a consequence of Markov's inequality. Moreover, for a vertex $(x_1, \ldots, x_j)$ on the path connecting **o** and x, the probability that this vertex is not an element of the set of vertices $A(\varepsilon_1, x)$ conditioned on $\{|Z(x_1, \ldots, x_j)| = l\}$ is bounded from above by

$$\rho^{(1-\varepsilon_1)(l-1)},$$

where we write $\rho := 1 - \mathsf{P}(A)$. Furthermore, the events $\{(x_1, \ldots, x_i) \in A(\varepsilon_1, x)\}$ are independent for $i \in \{0, \ldots, n-1\}$ (conditioned on the event that x is an element of $\mathbf{T}_n$). Thus, we obtain

$$\widehat{\mathsf{E}}\Big[e^{-\theta |A(\varepsilon_1, x)|} \,\Big|\, |Z(x_1, \ldots, x_i)| = l_{i+1} \,\forall\, i \in \{0, \ldots, n-1\}\Big]$$

$$
\begin{aligned}
&= \prod_{i=0}^{n-1} \widehat{\mathsf{E}}\Big[e^{-\theta\cdot \mathbb{1}_{\{(x_1,\dots,x_i)\in A(\varepsilon_1,x)\}}} \,\Big|\, Z(x_1,\dots,x_i) = l_{i+1}, x\in \mathbf{T}_n\Big] \\
&= \prod_{i=0}^{n-1} \Big[\widehat{\mathsf{P}}\Big((x_1,\dots,x_i)\notin A(\varepsilon_1,x) \,\Big|\, |Z(x_1,\dots,x_i)| = l_{i+1}, x\in \mathbf{T}_n\Big) \\
&\qquad\qquad + e^{-\theta}\cdot \widehat{\mathsf{P}}\Big((x_1,\dots,x_i)\in A(\varepsilon_1,x) \,\Big|\, |Z(x_1,\dots,x_i)| = l_{i+1}, x\in \mathbf{T}_n\Big)\Big] \\
&\le \prod_{i=0}^{n-1} \Big[\rho^{(1-\varepsilon_1)(l_{i+1}-1)} + e^{-\theta}\big(1-\rho^{(1-\varepsilon_1)(l_{i+1}-1)}\big)\Big] \\
&\le \prod_{i=0}^{n-1} \Big[\varepsilon_2^{(1-\varepsilon_1)(l_{i+1}-1)} + e^{-\theta}\big(1-\varepsilon_2^{(1-\varepsilon_1)(l_{i+1}-1)}\big)\Big]. \qquad (2.18)
\end{aligned}
$$

Summing the expression in (2.17) over all $x_1,\dots,x_n \in \mathbb{N}$ and $l_1 \ge x_1,\dots,l_n \ge x_n$ together with (2.18) yields

$$
\begin{aligned}
&\widehat{\mathsf{P}}(\exists\, x\in \mathbf{T}_n : |A(\varepsilon_1,x)| \le \varepsilon_3 n) \\
&\le \sum_{\substack{x_1\ge 1,\dots,x_n\ge 1,\\ l_1\ge x_1,\dots,l_n\ge x_n}} \widehat{\mathsf{P}}\Big(x\in \mathbf{T}_n, |Z(x_1,\dots,x_i)| = l_{i+1}\,\forall\, i\in\{0,\dots,n-1\}, |A(\varepsilon_1,x)|\le \varepsilon_3 n\Big) \\
&\le e^{\theta\varepsilon_3 n}\cdot \sum_{\substack{x_1\ge 1,\dots,x_n\ge 1,\\ l_1\ge x_1,\dots,l_n\ge x_n}} p_{l_1}\cdot\ldots\cdot p_{l_n}\cdot \prod_{i=0}^{n-1}\Big[\varepsilon_2^{(1-\varepsilon_1)(l_{i+1}-1)} + e^{-\theta}\big(1-\varepsilon_2^{(1-\varepsilon_1)(l_{i+1}-1)}\big)\Big]. \quad (2.19)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{l=k}^{\infty} p_l\Big[\varepsilon_2^{(1-\varepsilon_1)(l-1)} + e^{-\theta}\big(1-\varepsilon_2^{(1-\varepsilon_1)(l-1)}\big)\Big] \\
&= \sum_{l=1}^{\infty} l p_l\Big[\varepsilon_2^{(1-\varepsilon_1)(l-1)}(1-e^{-\theta}) + e^{-\theta}\Big] \\
&= (1-e^{-\theta}) f'(\varepsilon_2^{1-\varepsilon_1}) + e^{-\theta} m. \qquad (2.20)
\end{aligned}
$$

We recall that f denotes the probability generating function of the offspring distribution $(p_j)_{j\in\mathbb{N}_0}$ and m the associated mean offspring. If we choose $\theta > 0$ so large that

$$
(1-e^{-\theta}) f'(\varepsilon_2^{1-\varepsilon_1}) + e^{-\theta} m \le f'(\varepsilon_2^{1-\varepsilon_1}) + e^{-\theta} m \le 1-\delta \qquad (2.21)
$$

holds for some $\delta > 0$, we can conclude from (2.19), (2.20), and (2.21) that we have

$$
\begin{aligned}
&\widehat{\mathsf{P}}(\exists\, x\in \mathbf{T}_n : |A(\varepsilon_1,x)| \le \varepsilon_3 n) \\
&\le e^{\theta\varepsilon_3 n}\cdot (1-\delta)^n \\
&= \Big[e^{\theta\varepsilon_3}\cdot (1-\delta)\Big]^n.
\end{aligned}
$$

The proof is complete for the measure $\widehat{\mathsf{P}}$ if we choose $\varepsilon_3 > 0$ sufficiently small. From this we can derive the same statement also for the augmented measure P since the two measures only differ in the number of vertices adjacent to the root. □

Lemma 2.4.7. *The probabilities $\mathbb{P}(\alpha \geq n)$ and $\mathbb{P}(m_1 \geq n)$ decay exponentially fast for $n \to \infty$.*

Proof of Lemma 2.4.7. First, we consider the probability $\mathbb{P}(\alpha \geq n)$. Let $\omega = (\mathbf{T}, \mathbf{o}, \xi)$ be an arbitrary rooted and weighted tree and let $v, w \in \mathbf{T}$ be a pair of vertices of $\mathbf{T}$ such that w is a direct descendant of v, i.e., we have $|w| = |v|+1$ and $v \preceq w$. Moreover, we write

$$B_n(v,w) := \{r_n < \infty,\, X_{s_n - 1} = v,\, X_{s_n} = w\}$$

for $n \in \mathbb{N}$. $B_n(v,w)$ denotes the event that (after n unsuccessful trials) the $(n+1)$-th trial to escape a newly reached level starts at the pair of vertices (v,w). Then, we have

$$\begin{aligned} &P_\omega(\alpha \geq n+1 | \alpha \geq n,\, B_n(v,w)) \\ &= P_\omega(r_{n+1} < \infty | r_n < \infty, X_{s_i} = w) \\ &= P_\omega^w(\zeta(v) < \infty). \end{aligned}$$

From Lemma 2.4.5 we can conclude that we have

$$P_\omega^w(\zeta(v) < \infty) = \frac{\xi(v,w)}{\xi(v,w) + \mathcal{C}(\mathbf{T}^w)}.$$

We recall the $\mathcal{C}(\mathbf{T}^w)$ denotes the effective conductance within the graph T^w from w to infinity, which is the escape probability of the random walk in the electric network given by the subtree of $\mathbf{T}$ rooted in w. Under the averaged measure $\mathbb{P}$ the tree $\mathbf{T}^{X_{s_n}}$ is independent of the remaining part of the entire tree ω since at time s_n the random walk $(X_n)_{n\in\mathbb{N}_0}$ reaches a level which has not been visited before. Thus, under $\mathbb{P}$ the tree $\mathbf{T}^{X_{s_n}}$ has the same distribution as a rooted and weighted tree under $\widehat{\mathsf{P}}$. This implies

$$\begin{aligned} &\mathbb{P}(\alpha \geq n+1 | \alpha \geq n) \\ &= \sum_{v,w \in \mathbb{V}} \mathbb{P}(B_n(v,w)) \cdot \mathbb{P}(\alpha \geq n+1 | \alpha \geq n,\, B_n(v,w)) \\ &= \sum_{v,w \in \mathbb{V}} \mathbb{P}(B_n(v,w)) \cdot \int \frac{\xi(v,w)}{\xi(v,w) + \mathcal{C}(\mathbf{T}^w)} \,\mathsf{P}(\mathrm{d}\omega) \\ &= \sum_{v,w \in \mathbb{V}} \mathbb{P}(B_n(v,w)) \cdot \widehat{\mathsf{E}}\left[\frac{\xi}{\xi + \mathcal{C}}\right] \\ &\leq \widehat{\mathsf{E}}\left[\frac{\kappa_2}{\kappa_2 + \mathcal{C}}\right] =: \delta \in (0,1). \end{aligned}$$

Here the expectation in the last line is in $(0,1)$ since $\mathcal{C}$ is $\widehat{\mathsf{P}}$-a.s. positive (cf. Remark 2.3.1). From this we obtain

$$\begin{aligned}\mathbb{P}(\alpha \geq n+1) &= \mathbb{P}(\alpha \geq n+1,\, \alpha \geq n)\\ &= \mathbb{P}(\alpha \geq n+1 \,|\, \alpha \geq n) \cdot \mathbb{P}(\alpha \geq n)\\ &\leq \delta \cdot \mathbb{P}(\alpha \geq n)\end{aligned}$$

and an iteration of this yields

$$\mathbb{P}(\alpha \leq n) \leq \delta^n.$$

Next, we consider the sequence $(\mathbb{P}(m_1 \geq n))_{n\in\mathbb{N}_0}$. Let us consider the subset A of Ω defined by

$$A = A_\varepsilon := \left\{\omega = (\mathbf{T}, \mathbf{o}, \boldsymbol{\xi}) \in \Omega :\, \frac{\mathcal{C}(\omega)}{\kappa_2 + \mathcal{C}(\omega)} \geq \varepsilon\right\}.$$

By virtue of Lemma 2.4.5 we have

$$P_\omega^y(\zeta(x) = \infty) \geq \frac{\mathcal{C}(\mathbf{T}^y)}{\kappa_2 + \mathcal{C}(\mathbf{T}^y)}$$

for every pair of vertices $x, y \in \mathbf{T}$ such that y is a direct descendant of x. Under the measure P the distribution of $\mathcal{C}(\mathbf{T}^y)$ coincides with the distribution of $\mathcal{C}$ under $\widehat{\mathsf{P}}$. As above, we know that the effective conductance $\mathcal{C}$ is a.s. positive from the transience of the process and hence we have

$$\mathsf{P}(A_\varepsilon) \xrightarrow[\varepsilon \searrow 0]{} 1.$$

Now we want to apply Lemma 2.4.6. For this purpose, we fix $\varepsilon > 0$ such that we have

$$\mathsf{P}(A) = \mathsf{P}(A_\varepsilon) \geq 1 - \varepsilon_2$$

and then Lemma 2.4.6 implies that the sequence

$$\mathsf{P}(\exists\, x \in \mathbf{T}_n :\, |A(\varepsilon_1, x)| \leq \varepsilon_3 n)$$

decays exponentially fast for $n \to \infty$ and with suitable constants $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. By virtue of the Borel-Cantelli lemma, this implies that for large generations $n \in \mathbb{N}$ we P-a.s. have $|A(\varepsilon_1, x)| > \varepsilon_3 n$ for all vertices $x \in \mathbf{T}_n$. In other words, for large $n \in \mathbb{N}$ at least a fraction ε_3 of the vertices $y_0, \dots, y_{n-1}$ on every path connecting $\mathbf{o}$ and level n has at least $\varepsilon_1 |Z(y_i)|$ direct descendants $z \neq y_{i+1}$ such that we have

$$\frac{\mathcal{C}(\mathbf{T}^z)}{\kappa_2 + \mathcal{C}(\mathbf{T}^z)} \geq \varepsilon.$$

This implies that for each $y_i \in A(\varepsilon_1, x)$ we have

$$\begin{aligned}
&P_\omega^{y_i}(\zeta(y_{i-1}) = \infty) \\
&\geq P_\omega^{y_i}\big(X_1 \neq y_{i+1}, \mathbf{T}^{X_1} \in A, X_n \neq y_i \,\forall\, n \geq 1\big) \\
&= P_\omega^{y_i}\big(X_1 \neq y_{i+1}, \mathbf{T}^{X_1} \in A\big) \cdot P_\omega^{y_i}\big(X_n \neq y_i \,\forall\, n \geq 1 \,\big|\, X_1 \neq y_{i+1}, \mathbf{T}^{X_1} \in A\big) \\
&\geq \frac{\varepsilon_1 \kappa_1}{\kappa_2 + (1-\varepsilon_1)\kappa_2 + \varepsilon_1 \kappa_1} \cdot \varepsilon =: \gamma \in (0,1).
\end{aligned}$$

Hence, for large $n \in \mathbb{N}$ and for all $x \in \mathbf{T}_n$ we P-a.s. have

$$P_\omega^x(\zeta(\mathbf{o}) < \infty) \leq (1-\gamma)^{\varepsilon_3 n}.$$

Form this we conclude that for large $n \in \mathbb{N}$ and for P-a.e. ω we have

$$P_\omega(m_1 \geq n) \leq \max_{x \in \mathbf{T}_n} P_\omega^x(\zeta(\mathbf{o}) < \infty) \leq (1-\gamma)^{\varepsilon_3 n}.$$

This completes the proof of Lemma 2.4.7. □

Lemma 2.4.8. *The probability $\mathbb{P}(\tau_1 \geq \eta(n))$ decays exponentially fast for $n \to \infty$.*

Proof of Lemma 2.4.8. In order to estimate the probability of the event $\{\tau_1 \geq \eta(n)\}$, we observe that for all $c > 0$ we have

$$\{\tau_1 \geq \eta(n)\} \subseteq \{\alpha \geq cn\} \cup \left\{ \sum_{i=1}^{\alpha} m_i \geq n,\, \alpha \leq cn \right\}.$$

The probability $\mathbb{P}(\alpha \geq cn)$ decays exponentially fast as n tends to infinity according to Lemma 2.4.7. Thus, it remains to prove that the same holds true for the probability of the event $\{\sum_{i=1}^{\alpha} m_i \geq n,\, \alpha \leq cn\}$. To this end, we introduce a sequence of i.i.d. random variables $(\widetilde{m}_i)_{i \in \mathbb{N}}$ such that the distribution of $\widetilde{m}_1$ (w.r.t. $\mathbb{P}$) coincides with the distribution of m_1 conditioned on $m_1 < \infty$ (or equivalently $\alpha \geq 1$). Then we have

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{\alpha} m_i \geq n,\, \alpha \leq cn \right) \\
&\leq \mathbb{P}\left(\sum_{i=1}^{\lfloor cn \rfloor} \widetilde{m}_i \geq n \right) \\
&= \mathbb{P}\left(\frac{1}{\lfloor cn \rfloor} \sum_{i=1}^{\lfloor cn \rfloor} \widetilde{m}_i \geq \frac{n}{\lfloor cn \rfloor} \right)
\end{aligned} \tag{2.22}$$

and a standard large deviation argument yields that the probability in (2.22) decays exponentially fast for all $c > 0$ such that we have

$$\frac{1}{c} > \mathbb{E}[\widetilde{m}_1] = \mathbb{E}[m_1 \,|\, m_1 < \infty].$$

We refer to Chapter 2 in [21] for a comprehensive textbook treatment of large deviation techniques. Finally, it remains to note that we have $\mathbb{E}[\widetilde{m}_1] \in (1, \infty)$ and that even the moment-generating function of the random variable $\widetilde{m}_1$ is finite for all arguments sufficiently close to 0 due to Lemma 2.4.7. □

Lemma 2.4.9. *The probability $\mathbb{P}(\eta(n) \geq n^3)$ decays exponentially fast for $n \to \infty$. Moreover, if $d_0 > \frac{\kappa_2}{\kappa_1}$ holds true, there is constant $\rho_1 > 0$ such that even the probability $\mathbb{P}(\eta(n) \geq \rho_1 \cdot n)$ decays exponentially fast for $n \to \infty$.*

Proof of Lemma 2.4.9. Since we have $d_0 \geq \frac{\kappa_2}{\kappa_1}$, we are able to couple the distance of the random walk on the tree with a symmetric random walk $(\psi(n))_{n\geq 0}$ on the non-negative integers started at the origin with a reflecting barrier at the origin. This coupling can be constructed in such a way that we have $\psi(n) \leq |X_n|$ for all $n \in \mathbb{N}_0$ (regardless of the environment). Hence, the symmetric random walk $(\psi(n))_{n\in\mathbb{N}_0}$ hits the integer n after time $\eta(n)$. Thus, if $\eta^*(n)$ denotes the hitting time of the integer n of the process $(\psi(n))_{n\in\mathbb{N}_0}$, we have

$$\mathbb{P}\big(\eta^*(n) \geq n^3\big) \geq \mathbb{P}\big(\eta(n) \geq n^3\big). \tag{2.23}$$

This implies that the right hand side of (2.23) decays exponentially fast for $n \to \infty$ as the same holds true for the left hand side, which is e.g. proved in [53] (Lemma 9).

If we have $d_0 > \frac{\kappa_2}{\kappa_1}$, the argument presented above can be altered in such a way that the process involved in the coupling can be chosen as an asymmetric random walk with a positive drift h to the right. Then a large deviation argument yields that, for all $\rho_1 > h^{-1}$, the probability $\mathbb{P}\big(\eta(n) \geq \rho_1 \cdot n\big)$ decays exponentially fast for $n \to \infty$; cf. Chapter 2 in [21] for the involved large deviation techniques.

□

With Lemma 2.4.8 and Lemma 2.4.9 at hand, we turn back to the proof of part *(ii)* of Proposition 2.4.3. We have

$$\mathbb{P}\big(\tau_1 \geq \rho_1 \cdot n^3\big) \leq \mathbb{P}\big(\tau_1 \geq \eta(n)\big) + \mathbb{P}\big(\eta(n) \geq \rho_1 \cdot n^3\big) \tag{2.24}$$

for all $n \in \mathbb{N}_0$ and from Lemma 2.4.8 and Lemma 2.4.9 we know that both of the summands on the right side of (2.24) decay exponentially fast. This already implies that the same holds true for $\mathbb{P}(\tau_1 \geq n^3)$.

Now that we can control the tail probabilities of the random variable τ_1, we have to turn to the problem that the distribution of the first level regeneration time τ_1 differs from that of $\tau_{n+1} - \tau_n$ for $n \in \mathbb{N}$. To fix this problem, we introduce a bi-infite process on trees. Let $\omega = (\mathbf{T}, \mathbf{o}, \boldsymbol{\xi})$ be a weighted and rooted tree. A bi-infinite trajectory $\overset{\leftrightarrow}{x}$ in $\mathbf{T}$

is a family $(x_n)_{n\in\mathbb{Z}}$ of vertices of $\mathbf{T}$ indexed by the integers. Let P^*_ω denote the law of the integer-indexed process $(X_n)_{n\in\mathbb{Z}}$ such that $(X_n)_{n\in\mathbb{N}}$ and $(X_{-n})_{n\in\mathbb{N}}$ are independent and have distribution P_ω. As a consequence of the transience of the random walk, the trajectories of $(X_n)_{n\in\mathbb{N}}$ and $(X_{-n})_{n\in\mathbb{N}}$ a.s. converge to boundary points of $\mathbf{T}$. Thus, we can restrict ourselves to those bi-infinite trajectories which are convergent (in both directions). Let $\overset{\leftrightarrow}{\mathbf{T}}$ denote the set of all bi-infinite and convergent trajectories in $\mathbf{T}$ which start at the root of the tree, i.e. $x_0 = \mathbf{o}$. The set of all bi-infinite trajectories in trees is defined by

$$\mathsf{TrajecInTrees} := \{((\mathbf{T},\mathbf{o},\boldsymbol{\xi}),\overset{\leftrightarrow}{x}) : (\mathbf{T},\mathbf{o},\boldsymbol{\xi}) \in \Omega, \overset{\leftrightarrow}{x} \in \overset{\leftrightarrow}{\mathbf{T}}\}.$$

Now we are able to define the usual shift operator S on $\mathsf{TrajecInTrees}$ by

$$S((\mathbf{T},\mathbf{o},\boldsymbol{\xi}),\overset{\leftrightarrow}{x}) := ((\mathbf{T},x_1,\boldsymbol{\xi}),S\overset{\leftrightarrow}{x})$$

and

$$(S\overset{\leftrightarrow}{x})_n := x_{n+1}$$

for all $n \in \mathbb{Z}$. We note here that with slight abuse of notation $S\overset{\leftrightarrow}{x}$ denotes the shifted path. As usual, we write S^n for the n-fold shift, i.e. the n-th iteration of S. Analogously to the definition in (2.3), we define the probability measure $\widetilde{\mathbb{P}}^*$ on $\mathsf{TrajecInTrees}$ by

$$\widetilde{\mathbb{P}}^* := \widetilde{\mathsf{P}} \otimes P^*_\omega.$$

The crucial observation is that due to the reversibility property given by equation (2.4), the shift operator S is measure preserving w.r.t. $\widetilde{\mathbb{P}}^*$. For further details we refer to the proof of Theorem 4.1 in [30]. In fact, therein it is even proved that the system $(\mathsf{TrajecInTrees}, \widetilde{\mathbb{P}}^*, S)$ is ergodic.
We continue with defining the event

$$R := \{X_m \neq X_0, X_n \neq X_{-1}\,\forall\, m < 0, n \geq 0\},$$

which is the event that the bi-infinite trajectories $(X_n)_{n\in\mathbb{Z}}$ passes the edge (X_{-1}, X_0) only once between time -1 and time 0. Taking into account that every level regeneration is also a vertex regeneration, we observe that the distribution of the random variable τ_1 under the measure $\widetilde{\mathbb{P}}^*$ conditioned on R is the same as the distribution of $\tau_2 - \tau_1$ under the measure $\widetilde{\mathbb{P}}^*$ (cf. section 4.1 in [53] for further details). Further, we observe that the distribution of $\tau_2 - \tau_1$ is not affected either by the change of measure from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$ or by the change to considering bi-infinite trajectories. Thus, we have $\widetilde{\mathbb{P}}^*(\tau_2 - \tau_1 \in \cdot\,) = \mathbb{P}(\tau_2 - \tau_1 \in \cdot\,)$ and we can conclude that the tail probabilities of $\tau_2 - \tau_1$ w.r.t. $\mathbb{P}$ decay exponentially fast if the same holds true for τ_1 w.r.t. $\widetilde{\mathbb{P}}^*(\,\cdot\,|R)$. Moreover, we have

$$\widetilde{\mathbb{P}}^*\big(\tau_1 \geq n^3\big) = \widetilde{\mathbb{P}}\big(\tau_1 \geq n^3\big) = \int P_\omega\big(\tau_1 \geq n^3\big)\frac{\pi(\mathbf{o})}{\mathsf{E}[\xi]|T_1|}\,\mathsf{P}(\mathrm{d}\omega) \leq \frac{\kappa_2}{\mathsf{E}[\xi]}\cdot \mathbb{P}\big(\tau_1 \geq n^3\big).$$

From this and (2.24) we derive that $\widetilde{\mathbb{P}}^*\big(\tau_1 \geq n^3\big)$ decays exponentially fast. Finally, the below lemma implies that $\widetilde{\mathbb{P}}^*\big(\tau_1 \geq n^3\big)$ decays exponentially fast if and only if the same holds true for $\widetilde{\mathbb{P}}^*\big(\tau_1 \geq n^3\,|\,R\big)$. This proves that $\mathbb{P}\big(\tau_2 - \tau_1 \geq n^3\big)$ decays exponentially fast and hence the proof of Proposition 2.4.3 is complete.

Lemma 2.4.10. *Let be $f : \mathbb{N} \to [0,\infty)$ be a non-negative function and let the function $g : \mathbb{N} \to [0,\infty)$ be defined by*

$$g(n) := \sum_{k=1}^{n} f(k).$$

Then we have

$$\widetilde{\mathbb{E}}^*[f(\tau_1)] = \widetilde{\mathbb{E}}^*[\mathbb{1}_R \cdot g(\tau_1)] = \widetilde{\mathbb{P}}^*(R)\cdot \widetilde{\mathbb{E}}^*[g(\tau_1)|R]$$

In particular, the random variable $f(\tau_1)$ is integrable w.r.t. $\widetilde{\mathbb{P}}^*$ if and only if $g(\tau_1)$ is integrable w.r.t. $\widetilde{\mathbb{P}}^*(\,\cdot\,|R)$.

Proof. We define $R_n := R \cap \{\tau_1 = n\}$ and $R_{n,k} := S^k(R_n)$ for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ and observe that the family of sets $(R_{n,k})_{n\in\mathbb{N},k\in\{0,\dots,n-1\}}$ constitutes a (pairwise disjoint) decomposition of $\mathsf{TrajecInTrees}$. Similarly, $(R_n)_{n\in\mathbb{N}}$ is a decomposition of R. Moreover, we have $\tau_1 = n - k$ on $R_{n,k}$ and the fact that S is measure preserving implies that we have $\widetilde{\mathbb{P}}^*(R_{n,k}) = \widetilde{\mathbb{P}}^*(R_n)$ for all $k \in \mathbb{N}_0$. Thus, we have

$$\begin{aligned}
&\int f(\tau_1)\,\mathrm{d}\widetilde{\mathbb{P}}^* \\
&= \sum_{n\in\mathbb{N}}\sum_{k=0}^{n-1}\int_{R_{n,k}} f(\tau_1)\,\mathrm{d}\widetilde{\mathbb{P}}^* \\
&= \sum_{n\in\mathbb{N}}\sum_{k=0}^{n-1}\widetilde{\mathbb{P}}^*(R_{n,k})\cdot f(n-k) \\
&= \sum_{n\in\mathbb{N}}\widetilde{\mathbb{P}}^*(R_n)\sum_{k=1}^{n} f(k) \\
&= \sum_{n\in\mathbb{N}}\int_{R_n} g(\tau_1)\,\mathrm{d}\widetilde{\mathbb{P}}^* \\
&= \int_R g(\tau_1)\,\mathrm{d}\widetilde{\mathbb{P}}^*
\end{aligned}$$

and this completes the proof of Lemma 2.4.10. □

As mentioned above, this completes the proof of Proposition 2.4.3. ■

In order to formulate a last proposition before we continue with the proof of our main results, we introduce the counting processes $(\mathbf{n}_n^\sigma)_{n\in\mathbb{N}}$ and $(\mathbf{n}_n^\tau)_{n\in\mathbb{N}}$ which are derived from the sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$. More precisely, we define

$$\mathbf{n}_n^\sigma := \inf\{k \geq 1 : \sigma_k \geq n\}, \qquad \mathbf{n}_n^\tau := \inf\{k \geq 1 : \tau_k \geq n\}$$

and we obtain the following proposition.

Proposition 2.4.11. *There are constants $\lambda, \bar{\lambda} \in (0,\infty)$ such that we have*

$$\frac{\mathbf{n}_n^\tau}{n} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} \lambda \qquad \text{and} \qquad \frac{\mathbf{n}_n^\sigma}{n} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} \bar{\lambda}. \tag{2.25}$$

In particular, we have

$$\lambda^{-1} = \mathbb{E}[\tau_2 - \tau_1] \qquad \text{and} \qquad \bar{\lambda}^{-1} = \mathbb{E}[\sigma_2 - \sigma_1].$$

Proof of Proposition 2.4.11. We only prove the statement for $(\mathbf{n}_n^\tau)_{n\in\mathbb{N}}$ since the proof for $(\mathbf{n}_n^\sigma)_{n\in\mathbb{N}}$ is completely identical. From Proposition 2.4.1 we know that τ_n is a sum of i.i.d. random variables which is shifted by the non-negative and finite random variable τ_1. We observe that the associated first passage time of the interval $(-\infty, n-1]$ coincides with $\mathbf{n}_n^\tau$. Thus, we obtain (2.25) as a consequence of elementary renewal theory; cf. Chapter 10.14 in [35] or Theorem 4.4.1 in [24]. ■

Now we are ready to prove the main results of this chapter.

Proof of Theorem 2.3.2. (i) First, we define $\tau_n^* := \tau_{\mathbf{n}_n^\tau}$ and observe that we have

$$\frac{|X_n|}{n} = \frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k}{n} + \frac{|X_{\tau_1}|}{n} + \frac{|X_n| - |X_{\tau_n^*}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} \nu$$

from (2.5). Further, the law of large numbers and Proposition 2.4.11 imply

$$\frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k}{n} = \frac{\mathbf{n}_n^\tau}{n} \cdot \frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k}{\mathbf{n}_n^\tau} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} \lambda \cdot \mathbb{E}[U_1].$$

We note here that we have $\mathbf{n}_n^\tau \geq n$ for all $n \in \mathbb{N}$. Moreover, the finiteness of τ_1 implies

$$\frac{|X_{\tau_1}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} 0.$$

This $\mathbb{P}$-a.s. yields

$$\nu = \lim_{n\to\infty} \frac{|X_n|}{n}$$

$$= \lim_{n\to\infty} \left[\frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k}{n} + \frac{|X_{\tau_1}|}{n} + \frac{|X_n| - |X_{\tau_n^*}|}{n} \right]$$
$$= \lambda \cdot \mathbb{E}[U_1] + \lim_{n\to\infty} \frac{|X_n| - |X_{\tau_n^*}|}{n}.$$

Moreover, we have

$$\frac{|X_{\mathbf{n}_n^\tau}| - |X_{\mathbf{n}_n^\tau - 1}|}{n} \geq \frac{|X_{\tau_n^*}| - |X_n|}{n} \geq 0$$

for all $n \in \mathbb{N}$ and this implies

$$\frac{|X_n| - |X_{\tau_n^*}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

since the sequence $(U_n)_{n\geq 0}$ is an i.i.d. sequence. In fact, we even know that the limit $\lim\limits_{n\to\infty} \frac{|X_n|-|X_{\tau_n^*}|}{n}$ exists $\mathbb{P}$-a.s. since both of the limits $\lim\limits_{n\to\infty} \frac{|X_n|}{n}$ and $\lim\limits_{n\to\infty} \frac{\mathbf{n}_n^\tau}{n} \cdot \frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} Y_k}{\mathbf{n}_n^\tau}$ do so. This $\mathbb{P}$-a.s. yields

$$\lim_{n\to\infty} \frac{|X_n| - |X_{\tau_n^*}|}{n} = 0$$

and thus we have

$$\mathbb{E}[U_1] = \frac{\nu}{\lambda} = \nu \cdot \mathbb{E}[\tau_2 - \tau_1]. \tag{2.26}$$

We recall that the centred level increments are denoted by

$$V_n := U_n - \nu \cdot (\tau_{n+1} - \tau_n) = |X_{\tau_{n+1}}| - |X_{\tau_n}| - \nu \cdot (\tau_{n+1} - \tau_n)$$

and by Proposition 2.4.1 we know that $(V_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence w.r.t. the averaged measure $\mathbb{P}$. Further, (2.26) guarantees that the random variables $(V_n)_{n\in\mathbb{N}}$ are indeed centred (w.r.t. $\mathbb{P}$). Let the averaged variance of the random variable V_1 be denoted by

$$\tilde{\sigma}^2 := \mathbb{E}\left[(V_1 - \mathbb{E}[V_1])^2\right] > 0.$$

Proposition 2.4.3 implies that $\tilde{\sigma}^2$ is finite. Hence, the central limit theorem implies that we have

$$\frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} V_k}{\sqrt{\mathbf{n}_n^\tau}} \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\left(0, \tilde{\sigma}^2\right) \tag{2.27}$$

w.r.t. the averaged measure $\mathbb{P}$. Now we consider

$$\frac{|X_n| - \nu \cdot n}{\sqrt{n}} = \frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k - \nu \cdot n}{\sqrt{n}} + \frac{X_{\tau_1}}{\sqrt{n}} + \frac{|X_n| - |X_{\tau_n^*}|}{\sqrt{n}} \tag{2.28}$$

and observe that, as already argued above, the latter two summands of (2.28) converge $\mathbb{P}$-a.s. towards 0. For the first summand of the right hand side of (2.28) we obtain

$$\frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k - \nu \cdot n}{\sqrt{n}}$$

$$= \sqrt{\frac{\mathbf{n}_n^\tau}{n}} \cdot \frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} U_k - \nu \cdot \sum_{k=1}^{\mathbf{n}_n^\tau - 1} (\tau_{k+1} - \tau_k)}{\sqrt{\mathbf{n}_n^\tau}} + \frac{\nu \cdot (\tau_n^* - \tau_1) - \nu \cdot n}{\sqrt{n}}$$
$$= \sqrt{\frac{\mathbf{n}_n^\tau}{n}} \cdot \frac{\sum_{k=1}^{\mathbf{n}_n^\tau - 1} V_k}{\sqrt{\mathbf{n}_n^\tau}} + \nu \cdot \left[\frac{\tau_n^* - n}{\sqrt{n}} + \frac{\tau_1}{\sqrt{n}} \right].$$

We observe that the sequence

$$\left(\frac{\tau_n^* - n}{\sqrt{n}} + \frac{\tau_1}{\sqrt{n}} \right)_{n \in \mathbb{N}_0}$$

converges in probability (w.r.t. $\mathbb{P}$) towards 0 since the sequence $(\tau_n^* - n)_{n \in \mathbb{N}_0}$ converges in distribution (w.r.t. $\mathbb{P}$) towards a finite random variable. In fact, $\tau_n^* - n$ is the residual waiting time of the associated renewal process $(\tau_n)_{n \in \mathbb{N}_0}$ and thus the convergence in distribution is a consequence of suitable results of classic renewal theory, we refer to Example 4.4.8 in [24]. Finally, from Proposition 2.4.11, Slutzky's theorem, and (2.27) we conclude

$$\frac{|X_n| - \nu \cdot n}{\sqrt{n}} \xrightarrow[n \to \infty]{\mathrm{d}} \sqrt{\lambda} \cdot \mathcal{N}\big(0, \tilde{\sigma}^2\big) = \mathcal{N}(0, \sigma^2).$$

This completes the proof of part (i) of Theorem 2.3.2.

For part (ii) the very same arguments as for part (i) apply. So we can skip the details and restrict ourselves to mentioning that we have

$$\frac{r(n) - \bar{\nu} \cdot n}{\sqrt{n}}$$
$$= \frac{\sum_{k=1}^{\mathbf{n}_n^\sigma - 1} \bar{U}_k - \bar{\nu} \cdot n}{\sqrt{n}} + \frac{r(\sigma_1)}{\sqrt{n}} + \frac{r(n) - r(\sigma_n^*)}{\sqrt{n}}$$
$$= \underbrace{\sqrt{\frac{\mathbf{n}_n^\sigma}{n}} \cdot \frac{\sum_{k=1}^{\mathbf{n}_n^\sigma - 1} \bar{V}_k}{\sqrt{\mathbf{n}_n^\sigma}}}_{\xrightarrow[n \to \infty]{\mathrm{d}} \mathcal{N}(0, \bar{\sigma}^2) \text{ w.r.t. } \mathbb{P}} + \underbrace{\bar{\nu} \cdot \left[\frac{\sigma_n^* - n}{\sqrt{n}} + \frac{\sigma_1}{\sqrt{n}} \right] + \frac{r(\sigma_1)}{\sqrt{n}} + \frac{r(n) - r(\sigma_n^*)}{\sqrt{n}}}_{\xrightarrow[n \to \infty]{\mathbb{P}\text{-a.s.}} 0}.$$

This completes the proof of Theorem 2.3.2. ■

Chapter 3

Cookie branching random walks

3.1 Introduction

In this chapter we study a model which is motivated by what is usually called *exicted random walk* in the literature. An exicted random walk is a discrete-time stochastic process whose future evolution depends on its past through the set of visited sites. The process can be informally described as follows: The random walker's movement in a state space (usually $\mathbb{Z}^d$ for $d \geq 1$) at time $n \in \mathbb{N}_0$ depends on whether the random walker has already visited its current position before time n. Such a model was introduced in [11] and studied in numerous subsequent papers. We refer for example to [9, 42, 59] (for the one-dimensional case, where, as usual, more complete results are available), [10, 12, 37, 48] (for the multi-dimensional case and trees), and the references therein. As a source of intuition, excited random walks are also called *cookie random walk* – the idea being that initially all positions of the state space contain (one or several) cookies, which are consumed by the random walker at the time of its first visit of the respective positions. Whenever the random walker consumes a cookie at some position, this changes the transition probabilities at this position (usually by giving the random walk a drift in some direction).

In the model analysed in this chapter, we adopt the idea of having consumable cookies at certain positions to branching random walks. Hence, we consider not only one single random walker or particle that moves around in a state space, which in our case is $\mathbb{Z}$, but a whole population or cloud of particles which independently produce offspring particles according to given offspring distributions. Thereafter, the newly created particles move independently according to given transition probabilities. The transition and branching parameters depend on whether the position of the respective particle was visited before or not. The branching random walk is started with one particle at the origin and we suppose that in the initial configuration of cookies each position of $\mathbb{Z}$ contains one cookie. We call the process which is considered in this chapter a *cookie branching random walk* (CBRW).

Different kinds of models related to branching random walks have recently appeared in the literature; we refer to [15, 17, 19, 38, 49]. With the CBRW we aim at introducing a model for a branching random walk whose evolutionary mechanisms are changed as the process evolves. Another model which lies in some sense inbetween the excited random walk and the CBRW is the *frog model*. In contrast to the CBRW, in the frog model the particles do not produce offspring at every position which has already been visited but merely move. However, when one or several of the particles visit a position which has not been visited before, *exactly one of them* produces offspring according to a given offspring distribution while the other particles just stay alive. The term *frog model* originates from an alternative interpretation of this model. The cookies can also be regarded as frozen particles or sleeping frogs which are actived by other frogs which jump on top of the sleeping frogs. The process is started by an initial activated frog which moves around. We refer to the papers [3, 18] for further details.

Chapter 3 is organized as follows: In section 3.2 the required notation is introduced and a formal description of the model is given. Besides, general assumptions for the main results are stated and different recurrence regimes are defined. Subsequently, section 3.3 contains the main results. After some auxiliary results and their proofs in section 3.4, section 3.5 provides the proofs of the main results. At the end of the chapter, in section 3.6, we make some final remarks.

The results presented in this chapter have been submitted for publication as a paper which is a collaboration with Michael Kochler, Thomas Kochler, Sebastian Müller, and Serguei Popov; cf. [8]

3.2 Formal description of the model

We now turn to the formal description of the CBRW. First, we have to choose the initial configuration of the cookies. We restrict ourselves to the case in which we have (exactly) one cookie at every non-negative integer and no cookies at the negative integers. Thus, if $c_n(x)$ denotes the number of cookies at position $x \in \mathbb{Z}$ at time $n \in \mathbb{N}_0$, the cookie configuration as described above is given by

$$c_0(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

As it turns out, this configuration is a rather natural choice for an initial configuration which makes it possible to point out the essential differences in the evolution of the process. In particular, further results for the initial configuration ($c_0(x) = 1$ for all $x \in \mathbb{Z}$) can be derived easily (cf. Section 3.6). At time 0 the CBRW starts with one initial particle at the origin. In order to specify the evolution of the population of particles, we need the following ingredients:

- *the cookie offspring distribution*

$$\mu_c = \Big(\mu_c(k)\Big)_{k\in\mathbb{N}_0} \text{ with mean } m_c := \sum_{k=1}^{\infty} k\mu_c(k);$$

- *the cookie transition probabilities*

$$p_c \in (0,1), q_c := 1 - p_c;$$

- *the no-cookie offspring distribution*

$$\mu_0 = \Big(\mu_0(k)\Big)_{k\in\mathbb{N}_0} \text{ with mean } m_0 := \sum_{k=1}^{\infty} k\mu_0(k);$$

- *the no-cookie transition probabilities*

$$p_0 \in (0,1), q_0 := 1 - p_0.$$

We say a particle produces offspring according to a offspring distribution $\mu = (\mu(k))_{k\in\mathbb{N}_0}$ if the probability of having k offspring is $\mu(k)$. Having fixed the above quantities, the population of particles evolves at every discrete time unit $n \in \mathbb{N}_0$ according to the following rules:

(i) First, every existing particle produces offspring independently of the other particles. Each particle either reproduces according to the offspring distribution μ_c if there is a cookie at its position or otherwise according to μ_0. Thereafter, the parent particle dies.

(ii) Secondly, after the branching the newly produced offspring particles move independently of each other either one step to the right or one step to the left. Again the movement depends on whether the particles are at a position with or without a cookie. If there is a cookie, each particle moves to the right (left) with probability p_c (q_c). Otherwise, if there is no cookie, the transition probabilities are given by p_0 and q_0.

(iii) Finally, each cookie which is located at a position where at least one particle has produced offspring is removed. We note that different particles share the same cookie if they are at a position with a cookie at the same time. Moreover, due to the chosen initial configuration of the cookies only the leftmost cookie can be consumed at every time step.

3.2.1 Notational preliminaries and general assumptions

We now introduce some essential notations and assumptions. Since we do not want the process to die out, we assume that

$$\mu_c(0) = \mu_0(0) = 0$$

holds. Further, to avoid additional technical difficulties, we suppose that we have

$$M := \sup\{k \in \mathbb{N}_0 : \mu_c(k) + \mu_0(k) > 0\} < \infty. \tag{3.1}$$

In fact, we believe that the results remain true if we replace (3.1) by the assumption that the cookie and the no-cookie offspring variance is finite. In the following we want to distinguish different particles of the CBRW by using the usual Ulam-Harris labelling. Therefore, we enumerate the offspring of every particle and introduce the set

$$\mathbb{V} := \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$$

as the set of all particles which are potentially produced at any time in the entire process. Here $\mathbb{N}^0$ is defined as the set containing only the root $\mathbf{o}$ which denotes the initial particle. In this setting, $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{V}$ labels the particle which is the ν_n-th offspring of the particle $(\nu_1, \nu_2, \ldots \nu_{n-1})$. By iteration we can trace back the ancestral line of ν to the initial particle $\mathbf{o}$. Further, the generation (length) of the particle $\nu \in \mathbb{V}$ is denoted by $|v|$, and for two particles $\nu, \eta \in \mathbb{V}$ we write $\nu \succ \eta$ (or $\nu \succeq \eta$) if ν is a descendant of the particle η (if ν is a descendant of η or η itself). We use the same notation $\nu \succeq U$ (or $\nu \succ U$) for some set $U \subseteq \mathbb{V}$ if there is a particle $\eta \in U$ with $\nu \succeq \eta$ (or $\nu \succ \eta$). With the above notations, we can consider the actually produced particles in the CBRW. For $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}$ let $Z_n(x) \subset \mathbb{N}^n \subset \mathbb{V}$ denote the random set of particles which are at position x at time n. Thus,

$$Z_n := \bigcup_{x\in\mathbb{Z}} Z_n(x)$$

is the set of all particles which exist at time n. On the basis of this finding we can define $\mathcal{Z} := \bigcup_{n\in\mathbb{N}_0} Z_n$ as the set of all particles ever produced. Then, for every particle $\nu \in \mathcal{Z}$, we write X_ν for its random position in $\mathbb{Z}$ and the collection of all positions of all particles $(X_\nu)_{\nu\in\mathcal{Z}}$ is what we call CBRW. Further, the position of the leftmost cookie is denoted by

$$l(n) := \min\{x \in \mathbb{N} : c_n(x) = 1\}.$$

Now we are able to define the set of particles $\mathcal{L}(n)$ which is crucial for our considerations:

$$\mathcal{L}(n) := Z_n(l(n)).$$

The particles that belong to $\mathcal{L}(n)$ are located at the position of the leftmost cookie and thus they are the only particles which produce offspring according to μ_c. We call the process $\big(\mathcal{L}(n)\big)_{n\in\mathbb{N}_0}$ *leading process* (and use the abbreviation LP) since it contains the rightmost particles if $\mathcal{L}(n) \neq \emptyset$. One key observation for the understanding of the CBRW is that the particles in the LP constitute a Galton-Watson process (GWP) as long as there are particles in the LP. The associated mean offspring is given by $p_c m_c$ and thus we call the LP *supercritical*, or *subcritical*, or *critical* when $p_c m_c$ is greater than 1, or smaller than 1, or equal to 1, respectively.

3.2.2 Recurrence regimes

As it is usually done in the context of branching random walks (BRW), we now define three different regimes:

Definition 3.2.1. A CBRW is called

(i) *strongly recurrent* if a.s. infinitely many particles visit the origin, i.e.

$$\mathsf{P}\left(|Z_n(0)| \xrightarrow[n\to\infty]{} 0\right) = 0,$$

(ii) *weakly recurrent* if

$$\mathsf{P}\left(|Z_n(0)| \xrightarrow[n\to\infty]{} 0\right) \in (0,1),$$

(iii) *transient* if

$$\mathsf{P}\left(|Z_n(0)| \xrightarrow[n\to\infty]{} 0\right) = 1.$$

We note that these regimes may have different names in the literature; for example strong local survival, local survival, and local extinction of [29] correspond to the notion of strong recurrence, recurrence, and transience of Definition 3.2.1. The transient regime can be subdivided into *transient to the left* (or *transient to the right*) if the negative (or positive) integers are visited infinitely many times. Criteria for the recurrence/transience behaviour of BRW are well-known in the literature. In our setting the BRW of interest is the process related to the behaviour of the particles without cookies. In the following we call this process *BRW without cookies*. It is a BRW in the usual sense started with one particle at 0, with offspring distribution μ_0 and transition probabilities p_0, q_0 to the nearest neighbours. For this process we have the following proposition that goes back to fundamental papers by Biggins [13], Hammersley [36], and Kingman [41]; for a proof we refer to Theorem 18.3 in [51] and Theorem 3.2 in [28].

Proposition 3.2.2. *The BRW without cookies is*

(i) transient to the right iff

$$p_0 > \frac{1}{2} \quad \text{and} \quad m_0 \leq \frac{1}{2\sqrt{p_0 q_0}},$$

(ii) transient to the left iff

$$p_0 < \frac{1}{2} \quad \text{and} \quad m_0 \leq \frac{1}{2\sqrt{p_0 q_0}},$$

(iii) and strongly recurrent in the remaining cases.

In the transient cases, we define

$$\varphi_\ell := \frac{1}{2p_0 m_0}\left(1 - \sqrt{1 - 4p_0 q_0 m_0^2}\right).$$

We note that φ_ℓ reduces to $\min\{1, \frac{q_0}{p_0}\}$ if we have $m_0 = 1$. The interpretation of the quantity φ_ℓ is explained in Section 3.4 below.

3.3 Main results

Now we are ready to formulate the main results of Chapter 3, which give the classification of the process with respect to weak/strong recurrence in the sense of Definition 3.2.1.

Theorem 3.3.1. *We suppose that the BRW without cookies is transient to the right.*

(a) If the LP is supercritical, i.e. $p_c m_c > 1$ *holds, then*

  (i) the CBRW is strongly recurrent iff $p_c m_c \varphi_\ell \geq 1$,

  (ii) and the CBRW is transient to the right iff $p_c m_c \varphi_\ell < 1$.

(b) *If the LP is subcritical or critical, i.e. $p_c m_c \leq 1$ holds, then the CBRW is transient to the right.*

Theorem 3.3.2. *We suppose that the BRW without cookies is strongly recurrent. Then the CBRW is strongly recurrent, no matter whether the LP is subcritical, critical or supercritical.*

Theorem 3.3.3. *We suppose that the BRW without cookies is transient to the left.*

(a) *If the LP is supercritical, i.e. $p_c m_c > 1$ holds, then the CBRW is weakly recurrent.*

(b) *If the LP is critical or subcritical, i.e. $p_c m_c \leq 1$ holds, then the CBRW is transient to the left.*

3.4 Auxiliary results

Analogously to the notation which we use for the CBRW let $\big(Y_\nu\big)_{\nu\in\mathcal{Y}}$ denote the BRW without cookies. Here $\mathcal{Y}$ denotes the set of all particles ever produced and if ν is such a particle, Y_ν denotes the random position of the particle ν. We define $\Lambda_0^+ = \Lambda_0^- := 1$, and

$$\begin{aligned} \Lambda_n^+ &:= \left|\{\nu \in \mathcal{Y} : Y_\nu = n,\ Y_\eta < n\ \forall\, \eta \prec \nu\}\right|, \\ \Lambda_n^- &:= \left|\{\nu \in \mathcal{Y} : Y_\nu = -n,\ Y_\eta > -n\ \forall\, \eta \prec \nu\}\right| \end{aligned} \tag{3.2}$$

for $n \in \mathbb{N}$. Here Λ_n^+ (or Λ_n^-) denotes the random number of particles which are the first in their ancestral line to reach the position n (or $-n$). In addition, we define

$$\varphi_r := \mathsf{E}[\Lambda_1^+], \qquad \varphi_\ell := \mathsf{E}[\Lambda_1^-]. \tag{3.3}$$

We note that we have

$$\mathsf{P}\big(\Lambda_1^+ < \infty\big) = \mathsf{P}\big(\Lambda_1^- < \infty\big) = 1$$

if the BRW without cookies $\big(Y_\nu\big)_{\nu\in\mathcal{Y}}$ is transient. In this case the processes $\big(\Lambda_n^+\big)_{n\in\mathbb{N}_0}$ and $\big(\Lambda_n^-\big)_{n\in\mathbb{N}_0}$ are both GWPs. An important observation is that φ_r and φ_ℓ can be expressed using the first visit generating function of the underlying random walk. Thus, denote by X_n the nearest-neighbour random walk defined by

$$\mathsf{P}(X_{n+1} = x+1 \mid X_n = x) = p_0 \quad \text{and} \quad \mathsf{P}(X_{n+1} = x-1 \mid X_n = x) = q_0.$$

The *first visit generating function* is defined by

$$F(x,y|z) = \sum_{n=0}^{\infty} \mathsf{P}(X_n = y, X_k \neq y\ \forall k < n \mid X_0 = x) z^n.$$

A (short) thought reveals that $\varphi_r = F(0,1|m_0)$ and $\varphi_\ell = F(0,-1|m_0)$ and standard calculations yield the following formulas; for both arguments we also refer to Chapter 5 in [58].

Proposition 3.4.1. *If the BRW without cookies is transient, we have*

$$\varphi_r = \frac{1}{2q_0m_0}\left(1-\sqrt{1-4p_0q_0m_0^2}\right), \quad \textit{and} \quad \varphi_\ell = \frac{1}{2p_0m_0}\left(1-\sqrt{1-4p_0q_0m_0^2}\right). \tag{3.4}$$

Remark 3.4.2. A natural special case arises from $\mu_0(1) = 1$ (and thus $m_0 = 1$). In this model particles can only branch at positions with a cookie. In sites without cookies the process reduces to an asymmetric random walk $\big(Y_n\big)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with transition probabilities p_0 and q_0. Here φ_r and φ_ℓ simplify to the probabilities of an asymmetric random walk to ever reach $+1$ or -1, respectively, i.e.

$$\varphi_r = \mathsf{P}\big(\exists n\in\mathbb{N}: \ Y_n = +1\big) = \min\left\{1, \tfrac{p_0}{q_0}\right\}, \tag{3.5}$$

$$\varphi_\ell = \mathsf{P}\big(\exists n\in\mathbb{N}: \ Y_n = -1\big) = \min\left\{1, \tfrac{q_0}{p_0}\right\}. \tag{3.6}$$

Next, we collect some known facts about Galton-Watson processes that will be required in the sequel. An important tool for the proofs is to identify GWPs which are embedded in the CBRW. For the rest of this chapter the processes

$$\big(GW_n^{\text{super}}\big)_{n\in\mathbb{N}_0}, \big(GW_n^{\text{sub}}\big)_{n\in\mathbb{N}_0} \text{ and } \big(GW_n^{\text{cr}}\big)_{n\in\mathbb{N}_0}$$

shall denote a supercritical, subcritical or critical GWP started with $z\in\mathbb{N}$ particles with respect to the probability measure P_z. Furthermore, let $T^{\text{super}}, T^{\text{sub}}$ and T^{cr} denote the time of extinction corresponding to the above GWPs, i.e.

$$T^{\text{super}} := \inf\{n\geq 0: \ GW_n^{\text{super}} = 0\}$$

and analogously for the subcritical and critical case.

Proposition 3.4.3. *For a subcritical GWP* $\big(GW_n^{\text{sub}}\big)_{n\in\mathbb{N}_0}$ *with strictly positive and finite offspring variance there is a constant* $c>0$ *such that*

$$\lim_{n\to\infty}\frac{\mathsf{P}_1\left(GW_n^{\text{sub}}>0\right)}{\mathsf{E}_1\left[GW_1^{\text{sub}}\right]^n} = c.$$

For a proof we refer for example to Theorem 2.6.1 in [40].

Proposition 3.4.4. *For a critical GWP* $\big(GW_n^{\text{cr}}\big)_{n\in\mathbb{N}_0}$ *with strictly positive and finite offspring variance there is a constant* $c>0$ *such that*

$$\lim_{n\to\infty} n\,\mathsf{P}_1\left(GW_n^{\text{cr}}>0\right) = c.$$

For a proof we cite for example Theorem I.9.1 in [4]. By virtue of the estimate $1 - x \leq \exp(-x)$ we obtain the following consequence of Proposition 3.4.4.

Proposition 3.4.5. *For the extinction time T^{cr} of a critical GWP with strictly positive and finite offspring variance there exists a constant $C > 0$ such that*

$$\mathsf{P}_z\big(T^{\mathrm{cr}} \leq n\big) \leq \exp\left(-C\frac{z}{n}\right)$$

for all $n \in \mathbb{N}$ and for all $z \in \mathbb{N}$.

Proposition 3.4.6. *For the extinction time T^{cr} of a critical GWP with strictly positive and finite offspring variance there exists a constant $C > 0$ such that*

$$\mathsf{P}_z\big(T^{\mathrm{cr}} = n\big) \leq C\frac{z}{n^2}$$

for all $n \in \mathbb{N}$ and for all $z \in \mathbb{N}$.

Proof. Due to Corollary I.9.1 in [4] (with $s = 0$), there is a constant $c > 0$ such that

$$\lim_{n\to\infty} n^2 \mathsf{P}_1\big(T^{\mathrm{cr}} = n+1\big) = c.$$

Therefore, we get for $n \in \mathbb{N}$

$$\mathsf{P}_z\big(T^{\mathrm{cr}} = n\big) \leq z\,\mathsf{P}_1\big(T^{\mathrm{cr}} = n\big) = z\frac{1}{(n-1)^2}\big(c + o(1)\big) \leq C\frac{z}{n^2}$$

for a suitable constant $C > 0$. □

Lemma 3.4.7. *We consider a BRW (without cookies) $(Y_\nu)\nu \in \mathcal{Y}$ with branching distribution μ_0 and transition probabilities p_0 and q_0 started with one particle at the origin. Further, we assume that the BRW is transient to the right. Then, for a suitable constant $c > 0$, we have*

$$\mathsf{P}(\exists\, \nu \in \mathcal{Y} :\, Y_\nu = -n) = (c + o(1))(\varphi_\ell)^n$$

as n tends to infinity.

Proof. We consider the process $(\Lambda_n^-)_{n\in\mathbb{N}_0}$, which is introduced in (3.2), and observe that this process is a GWP with mean offspring $\varphi_\ell < 1$. Using condition (3.1), it is not difficult to verify that we have $\mathsf{E}[\Lambda_1^-)^2] < \infty$. Therefore, Proposition 3.4.3 completes the proof. □

3.5 Proofs of the main results

Proof of Theorem 3.3.1

Proof of part (a).
In this part of the proof we suppose $p_c m_c > 1$, i.e. the LP is supercritical. For $n \in \mathbb{N}$ we define inductively the n-th extinction time and the n-th rebirth time of the LP by

$$\begin{aligned}\tau_n &:= \inf\big\{i > \sigma_{n-1} : |\mathcal{L}(i)| = 0\big\},\\ \sigma_n &:= \inf\big\{i > \tau_n : |\mathcal{L}(i)| \geq 1\big\}\end{aligned}$$

with $\sigma_0 := 0$ and $\inf \emptyset := \infty$. Since $p_0 > 1/2$ and the LP is supercritical, we know that we have $\mathsf{P}(\sigma_n < \infty \mid \tau_n < \infty) = 1$ and $\mathsf{P}(\tau_{n+1} = \infty \mid \tau_{n-1} < \infty) \geq \mathsf{P}(\tau_1 = \infty) > 0$ for all $n \geq 0$. Hence, we a.s. have

$$\sigma^* := \inf\{n \in \mathbb{N}_0 : |\mathcal{L}(i)| \geq 1 \ \forall\, i \geq n\} < \infty. \tag{3.7}$$

It is a well-known fact that conditioned on survival a supercritical GWP with finite second moment normalized by its mean converges to a strictly positive random variable (e.g. Theorem I.6.2 in [4]). Considering the LP separately on the events $\{\sigma^* = k\}$ for $k \in \mathbb{N}_0$ yields

$$\lim_{n\to\infty} \frac{|\mathcal{L}(n)|}{(p_c m_c)^n} = W > 0 \tag{3.8}$$

for a strictly positive random variable W.

Now, we prove part (i) of Theorem 3.3.1(a). We suppose that we have $p_c m_c \varphi_\ell \geq 1$. For $n \in \mathbb{N}_0$, let us introduce

$$L_n := \{\nu \in Z_{n+1}(l(n) - 1) : \nu \succ \mathcal{L}(n)\}.$$

The set L_n contains all particles that are produced in the LP at time n and then leave the LP to the left. Thus, they are located at the position $l(n) - 1$ at time $n + 1$. We define the events

$$A_n := \big\{\exists\ \nu \succeq L_n : X_\nu = 0\big\}$$

for $n \in \mathbb{N}_0$. In order to show strong recurrence of the CBRW, it is now sufficient to prove that

$$\mathsf{P}\Big(\limsup_{n\to\infty} A_n\Big) = 1. \tag{3.9}$$

As a first step to achieve this, we consider the events

$$B_n := \big\{|L_n| \geq \big(p_c m_c\big)^n n^{-1},\ n \geq \sigma^*\big\}$$

for $n \in \mathbb{N}_0$ and show that

$$\mathsf{P}\left(\liminf_{n\to\infty} B_n\right) = 1. \tag{3.10}$$

This provides a lower bound for the growth of $|L_n|$ for large n. To see that (3.10) holds, we define $C_n := \left\{|\mathcal{L}(n)| \geq (p_c m_c)^n n^{-1/2}\right\}$ and notice that due to (3.8) we have

$$\mathsf{P}\left(\liminf_{n\to\infty} C_n\right) = 1. \tag{3.11}$$

We observe that, given the event C_n, the random variable $|L_n|$ can be bounded from below by a random sum of $\lceil (p_c m_c)^n n^{-1/2} \rceil$ i.i.d. Bernoulli random variables with success probability q_c. Hence, we can use a standard large deviation bound to see that the probabilities $\mathsf{P}(|L_n| < (p_c m_c)^n n^{-1} \mid C_n)$ decay exponentially fast as n tends to infinity; cf. Chapter 2 in [21] for the involved large deviation techniques. An application of the Borel-Cantelli lemma now yields

$$\mathsf{P}\left(\limsup_{n\to\infty}\left(\left\{|L_n| < (p_c m_c)^n n^{-1}\right\} \cap C_n\right)\right) = 0. \tag{3.12}$$

Since $\sigma^* < \infty$ a.s., (3.12) together with (3.11) yields (3.10). We observe that on the event $\{n \geq \sigma^*\}$, for each particle in L_n, the number of its offspring which is located at the positions $1, 2, \ldots$ steps to the left of $l(n) - 1$ for the first time in their genealogy constitutes an embedded GWP in the CBRW. Its mean is given by φ_ℓ, where $\varphi_\ell < 1$ holds since the BRW without cookie is transient to the right (cf. (3.4) and (3.5)). Therefore, by virtue of Lemma 3.4.7, we get

$$\begin{aligned} \mathsf{P}\big(A_n \mid B_n\big) &\geq 1 - (1 - c(\varphi_\ell)^n)^{(p_c m_c)^n n^{-1}} \\ &\geq 1 - \exp\left(-c(\varphi_\ell)^n (p_c m_c)^n n^{-1}\right) \\ &\geq 1 - \exp\left(-\tfrac{c}{n}\right) \\ &\geq \tfrac{C}{n} \end{aligned} \tag{3.13}$$

for some $c, C > 0$. Here we use that the position of a particle $\nu \in L_n$ is bounded by n (in fact by $n - 1$). Notice also that we have $p_c m_c \varphi_\ell \geq 1$ by assumption. Since $\mathbb{1}_{B_n}$ is measurable with respect to the σ-algebra generated by $|L_n|$ and σ^*, we have

$$\begin{aligned} \mathsf{P}\left(\bigcap_{n=i}^{j}\left(A_n^{\mathrm{c}} \cap B_n\right)\right) &= \mathsf{E}\left[\mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{A_n^{\mathrm{c}} \cap B_n} \,\middle|\, |L_i|, \ldots, |L_j|,\ \sigma^*\right]\right] \\ &= \mathsf{E}\left[\left(\prod_{n=i}^{j} \mathbb{1}_{B_n}\right)\mathbb{1}_{\{i \geq \sigma^*\}}\mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{A_n^{\mathrm{c}}} \,\middle|\, |L_i|, \ldots, |L_j|,\ \sigma^*\right]\right] \end{aligned}$$

for all $i, j \in \mathbb{N}$ with $i < j$. Now we observe that on $\{i \geq \sigma^*\}$ the random variables $\left(\mathbb{1}_{A_n^{\mathrm{c}}}\right)_{i \leq n \leq j}$ are conditionally independent, given $|L_i|, \ldots, |L_j|$ and σ^*. This holds because on $\{i \geq \sigma^*\}$ all the particles in $\bigcup_{n=i}^{j} L_n$ start independent BRWs which cannot reach the cookies anymore. For the same reason on $\{i \geq \sigma^*\}$ each of the random variables $(\mathbb{1}_{A_n^c})_{i\leq n}$ is conditionally independent of $(|L_k|)_{k\neq n}$ given $|L_n|$ and σ^*. Using these two facts, we obtain

$$\begin{aligned}
&\mathsf{E}\left[\left(\prod_{n=i}^{j} \mathbb{1}_{B_n}\right) \mathbb{1}_{\{i\geq\sigma^*\}} \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{A_n^c} \;\middle|\; |L_i|,\dots,|L_j|,\ \sigma^*\right]\right] \\
&= \mathsf{E}\left[\prod_{n=i}^{j} \left(\mathbb{1}_{B_n} \mathbb{1}_{\{i\geq\sigma^*\}} \mathsf{E}\left[\mathbb{1}_{A_n^c} \;\middle|\; |L_i|,\dots,|L_j|,\ \sigma^*\right]\right)\right] \\
&= \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{B_n} \mathsf{E}\left[\mathbb{1}_{A_n^c} \;\middle|\; |L_n|,\ \sigma^*\right]\right]. \qquad (3.14)
\end{aligned}$$

With the help of (3.13) and (3.14) we can now conclude that we have

$$\begin{aligned}
\mathsf{P}\left(\bigcap_{n=i}^{j} (A_n^c \cap B_n)\right) &= \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{B_n} \mathsf{E}\left[\mathbb{1}_{A_n^c} \;\middle|\; |L_n|,\ \sigma^*\right]\right] \\
&\leq \prod_{n=i}^{j} \left(1 - \tfrac{C}{n}\right) \xrightarrow[j\to\infty]{} 0. \qquad (3.15)
\end{aligned}$$

Therefore, for all $i \in \mathbb{N}$, we have $\mathsf{P}\left(\cap_{n=i}^{\infty} (A_n^c \cap B_n)\right) = 0$, which implies

$$\mathsf{P}\left(\liminf_{n\to\infty} \left(A_n^c \cap B_n\right)\right) = 0. \qquad (3.16)$$

Since (3.10) holds, (3.16) yields $\mathsf{P}\left(\liminf_{n\to\infty} A_n^c\right) = 0$. Thus, we have established (3.9) and so *(i)* of Theorem 3.3.1*(a)* is proved.

Next, we prove part (ii) of Theorem 3.3.1(a). We suppose that $p_c m_c \varphi_\ell < 1$. For sake of simplicity we assume $\sigma^* = 0$. The proof is analogous for $\sigma^* = k$ for $k \in \mathbb{N}$. The idea of the proof is to show that the expected number of particles that visit the origin the second time (the first time after time 0) in their genealogy is finite. Since the BRW without cookies is transient this implies transience of the CBRW. We note that no descendant of a particle that visited 0 after time 0 can ever reach a cookie again since $\sigma^* = 0$. (In the case $\sigma^* = k$ only a finite number of particles that have visited 0 up to time k can have descendants which reach a cookie again.) More formally, we define

$$\Gamma_n := |\{\xi \in \mathcal{Z} : \xi \preceq L_n, X_\xi = 0,\ X_\omega \neq 0\, \forall \omega : \xi \succ \omega \succeq \nu\}|.$$

Taking expectation yields

$$\mathsf{E}[\Gamma_n \mathbb{1}_{\{\sigma^*=0\}}] = \mathsf{E}[|L_n| \mathbb{1}_{\{\sigma^*=0\}}] F(n, 0|m_0) \leq (p_c m_c)^n q_c m_c (\varphi_\ell)^n,$$

and thus that $\mathsf{E}\left[\sum_{n\in\mathbb{N}}\Gamma_n \mathbb{1}_{\{\sigma^*=0\}}\right] < \infty$ since $p_c m_c \varphi_\ell < 1$. Therefore, we can finally conclude that a.s. only finitely many particles visit the origin, i.e. the CBRW is transient. This completes the proof of part *(a)*. ■

Proof of part (b)

In this part of the proof we suppose that the LP is subcritical or critical, i.e. $p_c m_c \leq 1$. We start with Lemma 3.5.1, which states that except for finitely many times the particles at a single position $x \in \mathbb{Z}$ produce an amount of offspring which is close to the expected amount as long as there are many particles at this position. To do so, we first split the set of particles $Z_n(x)$ into the following two sets

$$\begin{aligned} Z^+_{n+1}(x) &:= \{\nu \in Z_{n+1}(x) : \nu \succ Z_n(x-1)\}, \\ Z^-_{n+1}(x) &:= \{\nu \in Z_{n+1}(x) : \nu \succ Z_n(x+1)\} \end{aligned}$$

containing the particles which have moved to the right or to the left from time n to time $n+1$. For $\varepsilon > 0$, which we specify later (cf. (3.35) and (3.51)), we introduce the following sets:

$$\begin{aligned} D^+_n(x) &:= \{x < l(n), |Z_n(x)| \geq n\} \cap \left(\left\{ \frac{|Z^+_{n+1}(x+1)|}{|Z_n(x)|} < (p_0 m_0 - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (p_0 m_0 + \varepsilon) < \frac{|Z^+_{n+1}(x+1)|}{|Z_n(x)|} \right\} \right), \\ D^-_n(x) &:= \{x < l(n), |Z_n(x)| \geq n\} \cap \left(\left\{ \frac{|Z^-_{n+1}(x-1)|}{|Z_n(x)|} < (q_0 m_0 - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (q_0 m_0 + \varepsilon) < \frac{|Z^-_{n+1}(x-1)|}{|Z_n(x)|} \right\} \right), \\ E^+_n &:= \{\mathcal{L}(n) \geq n\} \cap \left(\left\{ \frac{|\mathcal{L}(n+1)|}{|\mathcal{L}(n)|} < (p_c m_c - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (p_c m_c + \varepsilon) < \frac{|\mathcal{L}(n+1)|}{|\mathcal{L}(n)|} \right\} \right), \\ E^-_n &:= \{\mathcal{L}(n) \geq n\} \cap \left(\left\{ \frac{|Z^-_{n+1}(l(n)-1)|}{|\mathcal{L}(n)|} < (q_c m_c - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (q_c m_c + \varepsilon) < \frac{|Z^-_{n+1}(l(n)-1)|}{|\mathcal{L}(n)|} \right\} \right), \\ F_n &:= E^+_n \cup E^-_n \cup \bigcup_{x\in\mathbb{Z}} \left(D^+_n(x) \cup D^-_n(x) \right). \end{aligned} \tag{3.17}$$

Lemma 3.5.1. *We have*

$$\mathsf{P}\left(\limsup_{n\to\infty} F_n \right) = 0 \tag{3.18}$$

for all $\varepsilon > 0$.

Proof of Lemma 3.5.1. First, we recall that the number of offspring of a single particle is bounded by M. Hence, a large deviation estimate for the random sum $|Z^+_{n+1}(x+1)|$ of $|Z_n(x)|$ i.i.d. random variables with mean $p_0 m_0$ yields

$$\mathsf{P}\Big(|Z^+_{n+1}(x+1)| > (p_0 m_0 + \varepsilon)|Z_n(x)| \Big| \sigma(|Z_n(x)|)\Big) \leq \exp\big(-|Z_n(x)|C_1\big) \qquad (3.19)$$

for some constant $C_1 > 0$ and

$$\mathsf{P}\Big(|Z^+_{n+1}(x+1)| < (p_0 m_0 - \varepsilon)|Z_n(x)| \Big| \sigma(|Z_n(x)|)\Big) \leq \exp\big(-|Z_n(x)|C_2\big) \qquad (3.20)$$

for some constant $C_2 > 0$; cf. Chapter 2 in [21] for the involved large deviation techniques. From (3.19) and (3.20) we can conclude that

$$\mathsf{P}\Big(D^+_n(x)\Big) \leq \exp(-nC_1) + \exp(-nC_2). \qquad (3.21)$$

The same argument leads to analogue estimates for the sets $D^-_n(x)$, E^+_n and E^-_n with constants $C_i > 0$ for $i = 3, \ldots, 8$. Since at time $n \in \mathbb{N}_0$ particles can only be located at the $n+1$ positions $-n, -n+2, \ldots, n-2, n$, we get

$$\mathsf{P}\left(E^+_n \cup E^-_n \cup \bigcup_{x\in\mathbb{Z}} \Big(D^+_n(x) \cup D^-_n(x)\Big)\right) \leq 2(2 + 2(n+1)) \exp(-nC)$$

for $C := \min_{i=1,\ldots,8} C_i > 0$. Therefore, the Borel-Cantelli lemma implies (3.18). □

In the considered case the CBRW behaves very differently depending on whether we have $p_0 m_0 \leq 1$ or $p_0 m_0 > 1$:

(i) For $p_0 m_0 \leq 1$ the offspring particles of a certain particle which move to the right in every step behave as a critical or subcritical GWP as long as the particles do not reach the cookies. Therefore, we can expect that the amount of particles which reach a cookie at the same time is not very large. More precisely, we will show in Proposition 3.5.2 that the amount of particles in the LP does not grow exponentially.

(ii) For $p_0 m_0 > 1$ the amount of offspring which moves to the right in every time step in the corresponding BRW without cookies constitutes a supercritical GWP. Therefore, the number of particles at the rightmost occupied position in the BRW without cookies a.s. grows with exponential rate $p_0 m_0 > 1$. In this case the following proposition shows that the amount of particles in the LP is essentially bounded by the growth rate of the rightmost occupied position of the corresponding BRW without cookies.

Proposition 3.5.2. *For every* $\alpha > \max\{1, p_0 m_0\} =: m^*$ *we have*

$$\mathsf{P}\left(\liminf_{n\to\infty}\{|\mathcal{L}(n)| < \alpha^n\}\right) = 1. \tag{3.22}$$

Proof of Proposition 3.5.2. For the proof we start with the following lemma which states that a large LP at time n leads to a long survival of the LP afterwards (except for finitely many times). For $\beta > 0$ we define

$$G_n := G_n(\beta) := \{|\mathcal{L}(n)| \geq n,\, \tau(n) \leq \beta \log |\mathcal{L}(n)|\}, \tag{3.23}$$

where

$$\tau(n) := \inf\{\ell \geq n :\, |\mathcal{L}(\ell)| = 0\} \tag{3.24}$$

denotes the time of the next extinction of the LP beginning from time n.

Lemma 3.5.3. *There exists* $\beta > 0$ *such that we have*

$$\mathsf{P}\left(\limsup_{n\to\infty} G_n\right) = 0. \tag{3.25}$$

Proof of Lemma 3.5.3. Let us first look at a subcritical GWP $\left(GW_n^{\text{sub}}\right)_{n\in\mathbb{N}_0}$ with reproduction mean $p_c m_c < 1$ and strictly positive, finite offspring variance and its extinction time T^{sub}. Assuming that we have an initial population of $z \in \mathbb{N}$ particles and using Proposition 3.4.3, we get

$$\begin{aligned}\mathsf{P}_z(T^{\text{sub}} \leq n) &= \left(1 - \mathsf{P}(GW_n^{\text{sub}} > 0)\right)^z \\ &\leq \left(1 - c(p_c m_c)^n\right)^z \\ &\leq \exp\left(-c(p_c m_c)^n z\right),\end{aligned}$$

for a suitable constant $c > 0$. In particular, if the LP is subcritical, we conclude that we have

$$\mathsf{P}\big(|\mathcal{L}(n)| \geq n,\, \tau(n) \leq \beta \log |\mathcal{L}(n)|\big) \leq \exp\left(-C\frac{n}{\beta \log(n)}\right)$$

for all sufficiently small $\beta > 0$ and for all $n \in \mathbb{N}$. Hence, the Borel-Cantelli lemma implies (3.25). If the LP is critical, we can use an analogous argument together with Proposition 3.4.5. ☐

In the following we want to investigate the behaviour of the CBRW on the event

$$H_{n_0} := \bigcap_{n \geq n_0} \left(F_n^c \cap G_n^c\right) \tag{3.26}$$

for fixed $n_0 \in \mathbb{N}_0$. (Later we will choose n_0 sufficiently large such that the assumptions of the upcoming Lemma 3.5.4 and equation (3.53) are satisfied.) On this event we have upper and lower bounds for

$$\frac{|Z^+_{n+1}(x+1)|}{|Z_n(x)|} \quad \text{and} \quad \frac{|Z^-_{n+1}(x-1)|}{|Z_n(x)|}$$

for positions $x \in \mathbb{Z}$ containing at least n particles at time $n \geq n_0$ (cf. (3.17)). Additionally, we have a lower bound for the time for which a LP with at least n particles at time $n \geq n_0$ will stay alive afterwards (cf. (3.23)). We note that we have

$$\mathsf{P}\left(\liminf_{n\to\infty}\left(F^c_n \cap G^c_n\right)\right) = \lim_{n\to\infty} \mathsf{P}(H_n) = 1 \tag{3.27}$$

due to Lemma 3.5.1 and Lemma 3.5.3.

For the next lemma we need some additional notation. We define

$$\sigma_0 := \inf\{n > n_0 : |\mathcal{L}(n-1)| = 0, |\mathcal{L}(n)| \neq 0, l(n) \leq n - 2n_0 - 1\},$$

which is the time of the first rebirth of the LP after time n_0 for which we have

$$l(\sigma_0) - (\sigma_0 - n_0) \leq -2n_0 - 1 + n_0 = -n_0 - 1.$$

This implies

$$|Z_{n_0}\big(l(\sigma_0) - (\sigma_0 - n_0 + k)\big)| = 0, \tag{3.28}$$

for all $k \in \mathbb{N}_0$. This is a crucial fact, which we make use of in the following computations (cf. Figure 3.1). Since the LP is critical or subcritical and the BRW without cookies is transient to the right, we a.s. have $\sigma_0 < \infty$. We now define the random times

$$\tau_n := \inf\{\ell > \sigma_n : |\mathcal{L}(\ell)| = 0\} - \sigma_n, \text{ for } n \geq 0,$$
$$\sigma_n := \inf\{\ell > \sigma_{n-1} + \tau_{n-1} : |\mathcal{L}(\ell)| \neq 0\}, \text{ for } n \geq 1,$$

which denote the time period of survival and the time of the restart of the LP, inductively. Due to the assumptions of the CBRW all of these random times are a.s. finite. Using (3.28) we see that we have

$$|Z_{n_0}\big(l(\sigma_j) - (\sigma_j - n_0 + k)\big)| = 0$$

for all $j, k \in \mathbb{N}_0$. Here we note that the argument leading to equation (3.28) also holds true for all $n \leq n_0$ instead of n_0. Hence, we have

$$|Z_n\big(l(\sigma_j) - (\sigma_j - n + k)\big)| = 0 \tag{3.29}$$

for all $n \leq n_0$ and $j, k \in \mathbb{N}_0$.

As the next step of the proof, we state the following upper bounds for the size of the LP on the event H_{n_0}.

Lemma 3.5.4. *The number of particles in the LP is bounded from above as follows:*

$$\begin{aligned}\mathcal{L}(n+k)| &\leq (p_c m_c + \varepsilon)^k + kM(n+k-1)(1+\delta)^{k-1} \\ &\qquad \text{on } H_{n_0} \cap \{|\mathcal{L}(n)| = z\} \cap \{\tau(n) \geq n+k\}, \end{aligned} \tag{3.30}$$

for $k, n, z \in \mathbb{N}, n \geq n_0$. *Further, we define* $m^* := \max\{1, p_0 m_0\}$. *Then, for all* $\gamma > 0$, *there exists* $n^* = n^*(\gamma)$ *such that we have*

$$\begin{aligned}|\mathcal{L}(\sigma_{j+1})| \leq\ & (m^* + 3\gamma)^{\sigma_{j+1}} \\ & \text{on } H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\} \cap \{|\mathcal{L}(\sigma_j)| \leq (m^* + \gamma)^{\sigma_j}\}, \end{aligned} \tag{3.31}$$

$$\begin{aligned}|\mathcal{L}(\sigma_{j+1})| \leq\ & |\mathcal{L}(\sigma_j)|(m^* + 4\gamma)^{\tau_j} \\ & \text{on } H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\} \cap \{|\mathcal{L}(\sigma_j)| > (m^* + \gamma)^{\sigma_j}\}, \end{aligned} \tag{3.32}$$

$$\begin{aligned}|\mathcal{L}(\sigma_{j+1})| \leq\ & (m^* + 2\gamma)^{\sigma_{j+1}} \\ & \text{on } H_{n_0} \cap \{\sigma_{j+1} > \sigma_j + \tau_j + 2\} \end{aligned} \tag{3.33}$$

for all $j \in \mathbb{N}_0$ *and* $n_0 \geq n^*$.

Proof of Lemma 3.5.4. First we choose $0 < \delta < \gamma$ in such a way that

$$1 + \delta \leq \frac{m^* + 2\gamma}{m^* + \gamma}, \quad 1 + \delta \leq \left(\frac{m^* + 3\gamma}{m^* + 2\gamma}\right)^{\beta \log(m^* + \gamma)}, \tag{3.34}$$

where $\beta > 0$ satisfies Lemma 3.5.3. Then we choose $\varepsilon > 0$ for the definitions of the events $\left(F_n\right)_{n \in \mathbb{N}_0}$ (cf. (3.17)) sufficiently small such that

$$p_c m_c + \varepsilon \leq 1 + \delta, \quad 1 < \frac{p_0 m_0 + \varepsilon}{p_0 m_0 - \varepsilon} \leq 1 + \delta, \quad p_0 m_0 + \varepsilon \leq m^* + \gamma. \tag{3.35}$$

For the upcoming estimates we use the following properties of the set H_{n_0}. For $n > n_0$ we have

$$|Z_{n-1}(x-1)| \leq n - 1 \text{ on } H_{n_0} \cap \{|Z_n(x)| = 0\}, \tag{3.36}$$

which means that there cannot be very many particles at position $x - 1$ one time step before n if we know that the position x stays empty at time n. Similarly, the knowledge of $|Z_n(x)|$ gives us upper estimates for $(|Z_{n-k}(x-k)|)_{k \in \mathbb{N}}$. If we are in the case in which the cookies are always to the right of the considered positions, we have for $n > n_0$

$$\begin{aligned}|Z_{n-1}(x-1)| &\leq z(p_0 m_0 - \varepsilon)^{-1} + n - 1 \\ &\qquad \text{on } H_{n_0} \cap \{|Z_n(x)| = z,\ l(n-1) > (x-1)\}, \\ |Z_{n-k}(x-k)| &\leq z(p_0 m_0 - \varepsilon)^{-k} + (n-1)(1 \vee (1 p_0 m_0 - \varepsilon))^{-k+1} \\ &\qquad \text{on } H_{n_0} \cap \{|Z_n(x)| = z,\ l(n-1) > (x-1)\} \end{aligned} \tag{3.37}$$

for $n-k \geq n_0$. The first estimate is easily obtained using a proof by contradiction and an iteration of it yields the second inequality. We note here that by construction the upper bound is at least equal to n_0. Therefore, if the upper bound is exceeded, at least a ratio of $p_0 m_0 - \varepsilon$ of $|Z_{n-k}(x-k)|$ contributes to $|Z_{n-k+1}(x-k+1)|$ on the considered event because of the definition of H_{n_0}. This yields a contradiction.

For $n \geq n_0$ and $k \in \mathbb{N}$, we obtain similar estimates for the size of the LP before the next extinction at time $\tau(n)$ (for the definition of $\tau(n)$ we refer to (3.24)):

$$\begin{aligned}
|\mathcal{L}(n+1)| &\leq z(p_c m_c + \varepsilon) + Mn \\
&\qquad \text{on } H_{n_0} \cap \{|\mathcal{L}(n)| = z\}, \\
|\mathcal{L}(n+2)| &\leq z(p_c m_c + \varepsilon)^2 + 2M(n+1)(1 \vee p_c m_c + \varepsilon) \\
&\qquad \text{on } H_{n_0} \cap \{|\mathcal{L}(n)| = z\} \cap \{\tau(n) \geq n+2\}, \\
|\mathcal{L}(n+k)| &\leq z(p_c m_c + \varepsilon)^k + kM(n+k-1)(1+\delta)^{k-1} \\
&\qquad \text{on } H_{n_0} \cap \{|\mathcal{L}(n)| = z\} \cap \{\tau(n) \geq n+k\}.
\end{aligned} \tag{3.38}$$

Concerning the last estimate, we note that we can distinguish between two cases as follows: If $|\mathcal{L}(n+k)| \leq n+k-1$ holds true, then we have $|\mathcal{L}(n+k-1)| \leq M(n+k-1)$ as a consequence of assumption (3.1). Otherwise, we can use the definition of H_{n_0} to get $|\mathcal{L}(n+k+1)| \leq (p_c m_c + \varepsilon)|\mathcal{L}(n+k)|$ on that event. In particular, we have to show (3.30).

Now, we introduce two processes $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$, which help us – together with the estimates (3.30), (3.36), and (3.37) – to control the number of particles that restart the LP at time σ_{j+1} (cf. Figure 3.1 and 3.2). For $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we define

$$\Phi_n^{(j)} := Z_n(l(\sigma_{j+1}) - \sigma_{j+1} + n) \text{ and } \Psi_n^{(j)} := Z_n(l(\sigma_{j+1}) - \sigma_{j+1} + 2 + n).$$

For sake of a better presentation we drop the superscript j and write just Φ_n and Ψ_n if there is no room for confusion.

We observe that we have

$$|\Phi_{n+1}| = |\Psi_n| = 0 \tag{3.39}$$

for all $n \leq n_0$ due to (3.29). Furthermore, by definition we have $\Phi_{\sigma_{j+1}} = \mathcal{L}(\sigma_{j+1})$ and

$$|\Psi_{\sigma_{j+1}}| = |\Psi_{\sigma_{j+1}-1}| = |\Psi_{\sigma_{j+1}-2}| = 0. \tag{3.40}$$

Again, we split the set of particles Φ_n into the particles which have moved one step to the right from time $n-1$ to time n and the particles which have moved to the left:

$$\Phi_n^+ := Z_n^+(l(\sigma_{j+1}) - \sigma_{j+1} + n),$$

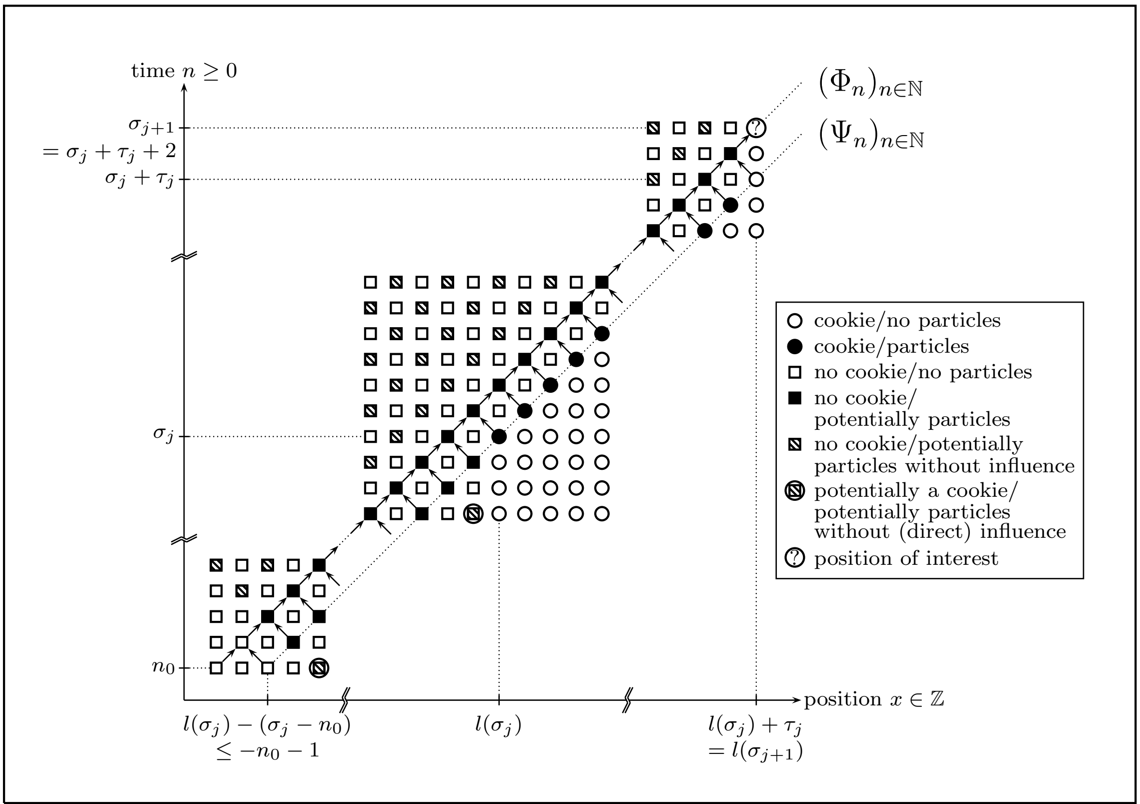

Figure 3.1: *The LP is restarted at time σ_{j+1}, two time steps after the last extinction at time $\sigma_j + \tau_j$. The two diagonals represent the processes $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$.*

$$\Phi_n^- := Z_n^-(l(\sigma_{j+1}) - \sigma_{j+1} + n)$$

To obtain an upper bound for $|\Phi_{\sigma_{j+1}}| = |\mathcal{L}(\sigma_{j+1})|$, we use the following recursive structure. We have $|\Phi_n^-| \leq M|\Psi_{n-1}|$ for $n \in \mathbb{N}$ due to assumption (3.1). Moreover, on H_{n_0} we have $|\Phi_n^+| \leq |\Phi_{n-1}|(p_0 m_0 + \varepsilon) + M\sigma_{j+1}$ for $n_0 + 2 \leq n \leq \sigma_{j+1}$ (since the particles reproduce and move without cookies) and these two facts yield

$$|\Phi_n| = |\Phi_n^+| + |\Phi_n^-| \leq |\Phi_{n-1}|(p_0 m_0 + \varepsilon) + M\sigma_{j+1} + M|\Psi_{n-1}| \tag{3.41}$$

for $n_0 + 2 \leq n \leq \sigma_{j+1}$. Using (3.39), (3.40), and $\sigma_{j+1} - n_0 - 1$ iterations of the recursion in (3.41), we obtain the following upper bound for the particles which start the LP at time σ_{j+1} on H_{n_0}:

$$\begin{aligned} |\Phi_{\sigma_{j+1}}| &\leq M \sum_{k=3}^{\sigma_{j+1}-n_0-1} |\Psi_{\sigma_{j+1}-k}|(p_0 m_0 + \varepsilon)^{k-1} + M\sigma_{j+1} \sum_{k=1}^{\sigma_{j+1}-n_0-1} (p_0 m_0 + \varepsilon)^{k-1} \\ &\leq M \sum_{k=1}^{\sigma_{j+1}-n_0-3} |\Psi_{\sigma_{j+1}-k-2}|(p_0 m_0 + \varepsilon)^{k+1} + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}}. \end{aligned} \tag{3.42}$$

We note that this bound just depends on σ_{j+1} and the process $(\Psi_n)_{n\in\mathbb{N}}$. For this reason we now take a closer look at $(\Psi_n)_{n\in\mathbb{N}}$ and distinguish between the following two cases:

- In the first case we assume that the LP restarts right after it has died out and we therefore have $\sigma_{j+1} = \sigma_j + \tau_j + 2$. In this case the process $(\Psi_n)_{n\in\mathbb{N}}$ coincides with the LP between time σ_j and $\sigma_j + \tau_j$ (cf. Figure 3.1).
- In the second case we assume that we have $\sigma_{j+1} > \sigma_j + \tau_j + 2$. From this we know that there are no particles in the LP at time $\sigma_{j+1} - 2$ and thus the process $(\Psi_n)_{n\in\mathbb{N}}$ is always to the left of the cookies (cf. Figure 3.2).

In both cases the crucial observation is that the amount of particles in $(\Psi_n)_{n\in\mathbb{N}}$ does not exceed a certain level since none of its offspring reaches the leftmost cookie at time $\sigma_{j+1} - 2$.

First, we consider the case $H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\}$. We apply the estimates (3.30) and (3.37) to establish upper bounds for $|\Psi_{\sigma_{j+1}-k}| = |\Psi_{\sigma_j+\tau_j+2-k}|$ for $1 \le k \le \sigma_{j+1} - n_0$. We know by definition of σ_j that we have $l(\sigma_j - 1) = l(\sigma_j) > l(\sigma_j) - 1$. Thus, we can apply (3.37) and conclude that on the event H_{n_0} for $1 \le k \le \sigma_j - n_0$ we have

$$|\Psi_{\sigma_j-k}| = |Z_{\sigma_j-k}(l(\sigma_j) - k)| \le |\mathcal{L}(\sigma_j)|(p_0 m_0 - \varepsilon)^{-k} + \sigma_j(1 \vee (p_0 m_0 - \varepsilon)^{-k+1})$$

and by using (3.38) for $0 \le k \le \tau_j - 1$ we get

$$|\Psi_{\sigma_j+k}| = |\mathcal{L}(\sigma_j + k)| \le |\mathcal{L}(\sigma_j)|(p_c m_c + \varepsilon)^k + kM(\sigma_j + k - 1)(1 + \delta)^{k-1}.$$

Applying these two estimates to (3.42) yields

$$\begin{aligned}
|\Phi_{\sigma_{j+1}}| &\le M \sum_{k=1}^{\tau_j} |\Psi_{\sigma_j+(\tau_j-k)}|(p_0 m_0 + \varepsilon)^{k+1} \\
&\quad + M \sum_{k=\tau_j+1}^{\sigma_j+\tau_j-n_0-1} |\Psi_{\sigma_j-(k-\tau_j)}|(p_0 m_0 + \varepsilon)^{k+1} \\
&\quad + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\le M \sum_{k=1}^{\tau_j} \Big(|\mathcal{L}(\sigma_j)|(p_c m_c + \varepsilon)^{\tau_j-k} \\
&\qquad\qquad + (\tau_j - k)M(\sigma_j + \tau_j - k - 1)(1 + \delta)^{\tau_j-k-1} \Big)(p_0 m_0 + \varepsilon)^{k+1} \\
&\quad + M \sum_{k=\tau_j+1}^{\sigma_j+\tau_j-n_0-1} \Big(|\mathcal{L}(\sigma_j)|(p_0 m_0 - \varepsilon)^{-k+\tau_j} \\
&\qquad\qquad + \sigma_j(1 \vee (p_0 m_0 - \varepsilon)^{-k+\tau_j+1}) \Big)(p_0 m_0 + \varepsilon)^{k+1} \\
&\quad + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\le \tau_j^2 M^2 (|\mathcal{L}(\sigma_j)| + \sigma_{j+1})(1 + \delta)^{\tau_j-1}(m^* + \gamma)^{\tau_j+1}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sigma_{j+1} M |\mathcal{L}(\sigma_j)| \left(\frac{p_0 m_0 + \varepsilon}{p_0 m_0 - \varepsilon}\right)^{\sigma_j} (p_0 m_0 + \varepsilon)^{\tau_j + 1} \\
&\quad + 2M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq 2M^2 \sigma_{j+1}^2 (|\mathcal{L}(\sigma_j)| + \sigma_{j+1})(1+\delta)^{\sigma_j + \tau_j - 1}(m^* + \gamma)^{\tau_j + 1} \\
&\quad + 2M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}}.
\end{aligned} \tag{3.43}
$$

Here we use (3.35) in the last two steps.

If we first investigate $|\mathcal{L}(\sigma_{j+1})|$ on the subset

$$\{|\mathcal{L}(\sigma_j)| \leq (m^* + \gamma)^{\sigma_j}\} \cap H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\},$$

on which we have a limited amount of particles in $\mathcal{L}(\sigma_j)$, we can conclude, by using (3.43) and (3.34), that we have

$$
\begin{aligned}
&|\mathcal{L}(\sigma_{j+1})| = |\Phi_{\sigma_{j+1}}| \\
&\leq 2M^2 \sigma_{j+1}^2 \big((m^* + \gamma)^{\sigma_j} + \sigma_{j+1}\big)(1+\delta)^{\sigma_j + \tau_j - 1}(m^* + \gamma)^{\tau_j + 1} + 2M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq 4M^2 (\sigma_{j+1} + 1)^3 (1+\delta)^{\sigma_{j+1}} (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq (m^* + 4\gamma)^{\sigma_{j+1}}
\end{aligned}
$$

for n_0 (and thus also σ_{j+1}) large enough. This shows (3.31) in Lemma 3.5.4.

On the other hand, if we consider the remaining subset

$$\{|\mathcal{L}(\sigma_j)| > (m^* + \gamma)^{\sigma_j}\} \cap H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\},$$

(3.43) yields

$$
\begin{aligned}
&|\mathcal{L}(\sigma_j)|^{-1} \cdot |\mathcal{L}(\sigma_{j+1})| \\
&= |\mathcal{L}(\sigma_j)|^{-1} \cdot |\Phi_{\sigma_{j+1}}| \\
&\leq 2M^2 \sigma_{j+1}^2 (1 + \sigma_{j+1})(1+\delta)^{\sigma_j + \tau_j - 1}(m^* + \gamma)^{\tau_j + 1} + M\sigma_{j+1}^2 (m^* + \gamma)^{\tau_j + 2} \\
&\leq 4M^2 (\sigma_j + \tau_j + 3)^3 (1+\delta)^{\sigma_j} (m^* + 2\gamma)^{\tau_j + 2} \\
&\leq 4M^2 (\sigma_j + \tau_j + 3)^3 (1+\delta)^{\frac{1}{\beta \log(m^* + \gamma)}\tau_j} (m^* + 2\gamma)^{\tau_j + 2} \\
&\leq (m^* + 4\gamma)^{\tau_j}
\end{aligned}
$$

for n_0 (and thus also σ_j) large enough. Here we use (3.34) and the fact that, on the considered set, we have $\{\tau_j > \beta \log\big((m^* + \gamma)^{\sigma_j}\big)\}$ (cf. Lemma 3.5.3). This shows (3.32) in Lemma 3.5.4.

We now consider the event $H_{n_0} \cap \{\sigma_{j+1} > \sigma_j + \tau_j + 2\}$. First, we observe that on this set, due to (3.36), we have

$$|\Psi_{\sigma_{j+1}-2-1}| = |Z_{\sigma_{j+1}-2-1}(l(\sigma_{j+1}) - 1)| \leq \ \sigma_{j+1} - 2 - 1 \leq \sigma_{j+1} \tag{3.44}$$

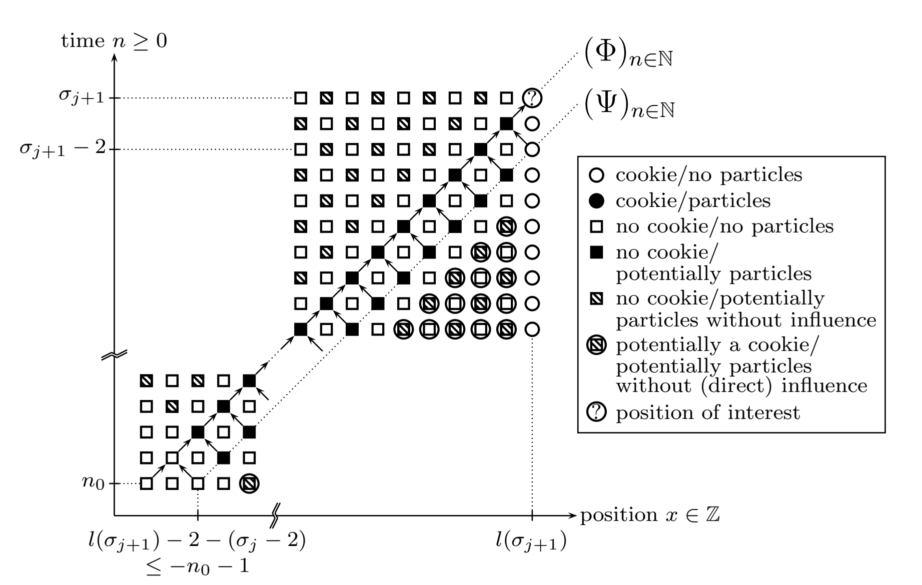

Figure 3.2: *The LP is restarted at time σ_{j+1}, more than two time steps after the last extinction at time $\sigma_j + \tau_j$. The two diagonals represent the processes $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$.*

since $|\Psi_{\sigma_{j+1}-2}| = |Z_{\sigma_{j+1}-2}(l(\sigma_{j+1}))| = 0$ holds. Further, we observe that the particles which belong to $(\Psi_n)_{n\in\mathbb{N}}$ are always to the left of the cookies. In particular, we have $l(\sigma_{j+1} - 2 - 1) = l(\sigma_{j+1}) > l(\sigma_{j+1}) - 1$. Therefore, we can apply (3.37) and, by using (3.44), conclude

$$\begin{aligned}
|\Psi_{\sigma_{j+1}-2-k}| &= |Z_{\sigma_{j+1}-2-k}(l(\sigma_{j+1}) - k)| \\
&\leq \sigma_{j+1}(p_0 m_0 - \varepsilon)^{-k} + (\sigma_{j+1} - 2 - 1)(1 \vee (p_0 m_0 - \varepsilon)^{-k+1}) \\
&\leq 2\sigma_{j+1}(1 \vee (p_0 m_0 - \varepsilon)^{-k}) \qquad (3.45)
\end{aligned}$$

for $2 \leq k \leq \sigma_{j+1} - 2 - n_0$.

With the help of (3.42) and (3.45) on the event $H_{n_0} \cap \{\sigma_{j+1} > \sigma_j + \tau_j + 2\}$ we get

$$\begin{aligned}
|\Phi_{\sigma_{j+1}}| &\leq M \sum_{k=1}^{\sigma_{j+1}-n_0-3} |\Psi_{\sigma_{j+1}-2-k}|(p_0 m_0 + \varepsilon)^{k+1} + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq M \sum_{k=1}^{\sigma_{j+1}-n_0-3} 2\sigma_{j+1}(1 \vee (p_0 m_0 - \varepsilon)^{-k})(p_0 m_0 + \varepsilon)^{k+1} + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq 2M\sigma_{j+1}^2 \left((p_0 m_0 + \varepsilon)^{\sigma_{j+1}-n_0-3} \vee \frac{p_0 m_0 + \varepsilon}{p_0 m_0 - \varepsilon} \right)^{\sigma_{j+1}-n_0-3} \\
&\qquad \cdot (p_0 m_0 + \varepsilon) + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq 3M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\leq (m^* + 2\gamma)^{\sigma_{j+1}}
\end{aligned}$$

for n_0 (and thus also σ_{j+1}) large enough. Here we use (3.34) and (3.35) in the last two steps. This shows (3.33) in Lemma 3.5.4. □

We now return to the proof of Proposition 3.5.2. First, we choose $\gamma \in \mathbb{R}$ such that $0 < 6\gamma < \alpha - m^*$ and n_0 large enough such that the estimations (3.31), (3.32) and (3.33) from Lemma 3.5.4 hold. Using these estimations, we can conclude that on H_{n_0} we a.s. have

$$\eta := \inf\{n \geq n_0 : |\mathcal{L}(\sigma_n)| < (m^* + 5\gamma)^{\sigma_n}\} < \infty. \tag{3.46}$$

To see this, we just have to see what happens on the event

$$H_{n_0} \cap \bigcap_{j=1}^{k} \Big(\big\{ |\mathcal{L}(\sigma_j)| > (m^* + \gamma)^{\sigma_j} \big\} \cap \big\{ \sigma_{j+1} = \sigma_j + \tau_j + 2 \big\} \Big).$$

On this event we can use (3.32) k times in a row and we get

$$|\mathcal{L}(\sigma_k)| \leq |\mathcal{L}(\sigma_0)| \prod_{j=1}^{k} (m^* + 4\gamma)^{\tau_j} \leq |\mathcal{L}(\sigma_0)| (m^* + 4\gamma)^{\sigma_k},$$

from which we conclude that (3.46) indeed holds on H_{n_0}.

Again by using the three estimations (3.31), (3.32), and (3.33) of Lemma 3.5.4, we inductively conclude that on the event H_{n_0} we have $|\mathcal{L}(\sigma_n)| \leq (m^* + 5\gamma)^{\sigma_n}$ for all $n \geq \eta$. Additionally, if we assume $|\mathcal{L}(\sigma_n + i - 1)| \leq (m^* + 5\gamma)^{\sigma_n + i - 1}$, we see inductively by using (3.38) that on the event H_{n_0} we have for all $n \geq \eta$ and for all $1 \leq i \leq \tau_n - 1$

$$\begin{aligned} |\mathcal{L}(\sigma_n + i)| &\leq |\mathcal{L}(\sigma_n + i - 1)|(p_c m_c + \varepsilon) + (\sigma_n + i - 1)M \\ &\leq (m^* + 5\gamma)^{\sigma_n + i - 1}(p_c m_c + \varepsilon) + (\sigma_n + i - 1)M \\ &\leq (m^* + 5\gamma)^{\sigma_n + i - 1}(m^* + \gamma) + (\sigma_n + i - 1)M \\ &\leq (m^* + 6\gamma)^{\sigma_n + i} < \alpha^{\sigma_n + i} \end{aligned}$$

for n_0 (and thus also $\sigma_n \geq n_0$) large enough. Since by the definitions of $(\sigma_n)_{n \in \mathbb{N}_0}$ and $(\tau_n)_{n \in \mathbb{N}_0}$ the LP is empty at the remaining times, we conclude that we have

$$\mathsf{P}\left(\liminf_{n \to \infty} \big(H_n \cap \{ |\mathcal{L}(n)| < \alpha^n \} \big) \right) = 1. \tag{3.47}$$

Finally, this and the fact that we have $\mathsf{P}(\liminf_{n \to \infty} H_n) = 1$ imply (3.22). □

After having investigated the growth of the LP, we are now interested in the speed at which the cookies are consumed:

Proposition 3.5.5. *(a) There exists $\lambda > 0$ such that we a.s. have*

$$\liminf_{n\to\infty} \frac{l(n)}{n} > \lambda. \tag{3.48}$$

(b) In fact, for $p_0 m_0 > 1$ we a.s. have

$$\lim_{n\to\infty} \frac{l(n)}{n} = 1. \tag{3.49}$$

Proof of Proposition 3.5.5. (a) We compare the CBRW with a process $(W_n)_{n\in\mathbb{N}_0}$ which is similar to an excited random walk on the integers. More precisely, it is started at the origin (i.e. $W_0 := 0$) and its transition probabilities are given by

$$\mathsf{P}\big(W_{n+1} = W_n + 1 \mid (W_j)_{1\le j\le n}\big) = \begin{cases} 0 & \text{on } \left\{\max\limits_{j=0,1,\ldots,n} W_j = W_n\right\} \\ p_0 & \text{on } \left\{\max\limits_{j=0,1,\ldots,n} W_j > W_n\right\} \end{cases}$$

and

$$\mathsf{P}\big(W_{n+1} = W_n - 1 \mid (W_j)_{1\le j\le n}\big) = \begin{cases} 1 & \text{on } \left\{\max\limits_{j=0,1,\ldots,n} W_j = W_n\right\} \\ q_0 & \text{on } \left\{\max\limits_{j=0,1,\ldots,n} W_j > W_n\right\} \end{cases}$$

for $n \in \mathbb{N}_0$. The process $(W_n)_{n\in\mathbb{N}_0}$ moves to the left with probability 1 every time it reaches a position $x \in \mathbb{N}_0$ for the first time and otherwise it behaves as an asymmetric random walk on $\mathbb{Z}$ with transition probabilities p_0 and q_0. For all $x \in \mathbb{N}_0$ we consider the random times given by $T_x := \inf\{n \in \mathbb{N}_0 : W_n = x\}$ and observe that $\big(T_{x+1} - T_x\big)_{x\in\mathbb{N}_0}$ is a sequence of i.i.d. random variables with

$$\mathsf{E}[T_1 - T_0] = \mathsf{E}[T_1] = 1 + \frac{2}{2p_0 - 1}.$$

Therefore, the strong law of large numbers implies that we a.s. have

$$\lim_{x\to\infty} \frac{T_x}{x} = \lim_{x\to\infty} \frac{1}{x}\sum_{i=0}^{x-1}(T_{i+1} - T_i) = \mathsf{E}[T_1 - T_0] = 1 + \frac{2}{2p_0 - 1} < \infty.$$

Since we can couple the CBRW and the process $(W_n)_{n\in\mathbb{N}_0}$ in a natural way such that we have $\max_{\nu\in Z_n} X_\nu \ge W_n$ for all $n \in \mathbb{N}_0$, we can conclude that (3.48) holds for $0 < \lambda < \left(1 + \frac{2}{2p_0-1}\right)^{-1}$.

(b) We start this part of the proof with the following lemma:

Lemma 3.5.6. *For a CBRW with $m_0 > 1$, there exists $\gamma > 1$ such that we a.s. have*

$$\lim_{n\to\infty} \frac{|Z_n|}{\gamma^n} = \infty. \tag{3.50}$$

Proof of Lemma 3.5.6. Let us treat the case where $m_c > 1$ first. Let $(V_{n,k})_{n,k\in\mathbb{N}}$ be i.i.d. random variables with

$$1 - \mathsf{P}(V_{1,1} = 1) = \mathsf{P}(V_{1,1} = 2) = \min\left\{\sum_{i=2}^{\infty} \mu_0(i), \sum_{i=2}^{\infty} \mu_c(i)\right\},$$

and we define the corresponding GWP $(\widetilde{Z}_n)_{n\in\mathbb{N}_0}$ by $\widetilde{Z}_0 := 0$, $\widetilde{Z}_{n+1} := \sum_{i=1}^{\widetilde{Z}_n} V_{n+1,i}$. We note that we have $\mathsf{E}[V_{1,1}] > 1$. A standard coupling argument reveals that $\widetilde{Z}_n \leq |Z_n|$ holds true. Now (3.50) follows since $\widetilde{Z}_n$ grows exponentially, e.g. by virtue of Theorem I.10.3 on page 30 in [4].

The other case is similar: We consider now the i.i.d. random variables $(V_{n,k})_{n,k\in\mathbb{N}}$ with

$$1 - \mathsf{P}(V_{1,1} = 1) = \mathsf{P}(V_{1,1} = 2) = \min\{q_0, q_c\} \cdot \sum_{i=2}^{\infty} \mu_0(i),$$

and define the corresponding GWP $(\widetilde{Z}_n)_{n\in\mathbb{N}_0}$ as above. For the coupling we observe that the probability of every particle in the CBRW to produce a particle which moves to the left is bounded from below by $\min\{q_0, q_c\}$. Such a particle cannot be at a position with a cookie and therefore its offspring distribution is given by $(\mu_0(i))_{i\in\mathbb{N}_0}$. Eventually, the corresponding coupling yields $\widetilde{Z}_n \leq |Z_{2n}|$ and (3.50) follows as above. ■

We now return to the proof of Proposition 3.5.5*(b)*. Let us choose $\varepsilon > 0$ such that we have

$$p_0 m_0 - \varepsilon > 1, \quad \text{and} \quad q_c m_c - \varepsilon > 0. \tag{3.51}$$

We use this ε for the definition of the sets $(F_n)_{n\in\mathbb{N}_0}$ and $(H_n)_{n\in\mathbb{N}_0}$, cf. (3.17) and (3.26). Due to Lemma 3.5.6 we can choose $\gamma > 1$ such that we a.s. have

$$\lim_{n\to\infty} \frac{|Z_n|}{\gamma^{2n}} = \infty \quad \text{and} \quad \gamma < p_0 m_0 - \varepsilon. \tag{3.52}$$

In addition, we choose n_0 sufficiently large such that we have for all $n \geq n_0$

$$\gamma^n > n, \quad \gamma^n(q_c m_c - \varepsilon) > (n+1), \quad \gamma^{\beta \log(\gamma^n)}(q_c m_c - \varepsilon) \geq 1 \tag{3.53}$$

for some $\beta > 0$ which satisfies the assumptions of Lemma 3.5.3. In the following we again investigate the behaviour of the CBRW on the event H_{n_0} on which the process does not show certain unlikely behaviour after time n_0 (cf. (3.17) and (3.23)). We prove that already the offspring of one position with "many" particles cause the leftmost cookie to move to the right with speed 1. For this, we introduce the random time

$$\eta := \inf\{n \geq n_0 : \exists x \in \mathbb{Z} \text{ such that } |Z_n(x)| \geq \gamma^n\}.$$

At time η we know that there are sufficiently many particles at the random position $x_0 := \sup\{x \in \mathbb{Z} : Z_\eta(x) \geq \gamma^\eta\}$. Due to (3.52) we a.s. have $\eta < \infty$ since at time n only $n+1$ positions can be occupied. Additionally, we introduce the random time

$$\sigma_0 := \inf\{n \geq \eta : l(n) = x_0 + n - \eta\}$$

at which offspring of the particles belonging to $Z_\eta(x_0)$ can potentially reach the LP for the first time after time η. Since $p_c m_c \leq 1$, the LP dies out infinitely often and therefore we a.s. have $\sigma_0 < \infty$. Then, we inductively define the random times

$$\begin{aligned} \tau_j &:= \inf\{n \geq \sigma_j : |\mathcal{L}(n)| = 0\} - \sigma_j, \text{ for } j \geq 0, \\ \sigma_j &:= \inf\{n \geq \sigma_{j-1} + \tau_{j-1} : |\mathcal{L}(n)| \neq 0\}, \text{ for } j \geq 1, \end{aligned}$$

denoting the time period of survival and the time of the restart of the LP after time σ_0. Due to (3.53) we have

$$|Z_\eta(x_0)| \geq \gamma^\eta \geq \eta \tag{3.54}$$

which allows us to use the lower bound for $|Z^+_{\eta+1}(x_0+1)|$ on H_{n_0}. By using (3.52) and (3.54) we get on the event $H_{n_0} \cap \{l(\eta) > x_0\}$

$$|Z_{\eta+1}(x_0+1)| \geq |Z^+_{\eta+1}(x_0+1)| \geq \gamma^\eta(p_0 m_0 - \varepsilon) \geq \gamma^{\eta+1}.$$

By iteration of the last step, we see that on the event

$$H_{n_0} \cap \bigcap_{i=0}^{k-1} \{l(\eta + k - 1) > x_0 + k - 1\} = H_{n_0} \cap \{l(\eta + k) - 1 > x_0 + k - 1\}$$

we have $|Z_{\eta+k}(x_0+k)| \geq \gamma^{\eta+k}$ and therefore we conclude that

$$|\mathcal{L}(\sigma_0)| = |Z_{\eta+\sigma_0-\eta}(x_0 + \sigma_0 - \eta)| \geq \gamma^{\eta+\sigma_0-\eta} = \gamma^{\sigma_0}$$

holds on H_{n_0}. In the following we see that already the offspring particles of $\mathcal{L}(\sigma_0)$ which move to the left at time σ_0 and afterwards move to the right in every step lead to a very large LP at the next restart at time σ_1. To see this, we first notice that (3.53) implies on the event H_{n_0}

$$|Z_{\sigma_0+1}(l(\sigma_0) - 1)| \geq |Z^-_{\sigma_0+1}(l(\sigma_0) - 1)| \geq \gamma^{\sigma_0}(q_c m_c - \varepsilon) \geq (\sigma_0 + 1)$$

since we have $|Z_{\sigma_0}(l(\sigma_0))| \geq \gamma^{\sigma_0} > \sigma_0$. An iteration of this together with (3.52) and (3.53) yield for $k \in \mathbb{N}$

$$\begin{aligned} |Z_{\sigma_0+1+k}(l(\sigma_0) - 1 + k)| &\geq |Z^+_{\sigma_0+1+k}(l(\sigma_0) - 1 + k)| \\ &\geq \gamma^{\sigma_0}(q_c m_c - \varepsilon)(p_0 m_0 - \varepsilon)^k \end{aligned}$$

$$
\begin{aligned}
&\geq \gamma^{\sigma_0+k}(q_c m_c - \varepsilon) \\
&\geq \sigma_0 + k + 1
\end{aligned}
$$

on the event $H_{n_0} \cap \{\tau_0 \geq k-1\}$. In particular, this implies

$$
\begin{aligned}
|\mathcal{L}(\sigma_0+\tau_0+2)| &= |Z_{\sigma_0+\tau_0+2}(l(\sigma_0)+\tau_0)| \\
&\geq \gamma^{\sigma_0+2(\tau_0+1)}(q_c m_c - \varepsilon) \\
&\geq \gamma^{\sigma_0+\tau_0+2}\gamma^{\beta\log(\gamma^{\sigma_0})}(q_c m_c - \varepsilon) \\
&\geq \gamma^{\sigma_0+\tau_0+2} > 0
\end{aligned}
$$

on the event H_{n_0}. Here we used that, due to Lemma 3.5.3, we have $\tau_0 \geq \beta\log(\gamma^{\sigma_0})$ and recalled (3.53) for the last inequality. Further, we conclude that we have $\sigma_1 = \sigma_0+\tau_0+2$ on H_{n_0}, which implies that the LP is restarted two time steps after it has died out at time $\sigma_0+\tau_0$. Iterating this argument finally implies

$$
|\mathcal{L}(\sigma_{j+1})| \geq \gamma^{\sigma_{j+1}} \quad \text{and} \quad \sigma_{j+1} = \sigma_j + \tau_j + 2 \tag{3.55}
$$

for all $j \in \mathbb{N}_0$ on the event H_{n_0}. For $\beta^* := \beta\log(\gamma) > 0$ we further conclude from (3.55) and Lemma 3.5.3 by induction that on H_{n_0} we have

$$
\tau_j \geq \beta\sigma_j\log(\gamma) \geq \beta^*(1+\beta^*)^j\sigma_0 \tag{3.56}
$$

for all $j \in \mathbb{N}_0$. This implies

$$
\sigma_{j+1} = \sigma_j + \tau_j + 2 \geq (1+\beta^*)^j\sigma_0 + \beta^*(1+\beta^*)^j\sigma_0 = (1+\beta^*)^{j+1}\sigma_0. \tag{3.57}
$$

Hence, on the event H_{n_0} we have for $n \geq \sigma_0$

$$
\begin{aligned}
\frac{l(n)}{n} &\geq \frac{l(\sigma_0)+n-\sigma_0-2|\{j\geq 0:\ \sigma_j+\tau_j \leq n\}|}{n} \\
&\geq \frac{l(\sigma_0)+n-\sigma_0-2\frac{\log(n)-\log(\sigma_0)}{\log(1+\beta^*)}}{n} \xrightarrow[n\to\infty]{} 1.
\end{aligned}
$$

Here we use (3.55) in the first step and in the second step we use the fact that due to (3.56) and (3.57) we have $\sigma_j + \tau_j \geq (1+\beta^*)^{j+1}\sigma_0$ for $j \in \mathbb{N}_0$. This yields that on the event H_{n_0} we have $\lim_{n\to\infty}\frac{l(n)}{n} = 1$. Since by (3.27) we have $\lim_{n\to\infty}\mathsf{P}(H_n) = 1$, we finally established (3.49). □

With Proposition 3.5.2 and Proposition 3.5.5 we are now prepared to prove Theorem 3.3.1*(b)*. Similarly to the proof of Theorem 3.3.1*(a)*, we introduce the event

$$
A_n := \{\exists\nu \succeq L_n:\ X_\nu = 0,\ X_\eta < l(|\eta|)\,\forall\, L_n \prec \eta \prec \nu\}
$$

with $L_n = \{\nu \in Z_{n+1}(l(n)-1) : \nu \succ \mathcal{L}(n)\}$ for $n \in \mathbb{N}$. On A_n, there exists a particle ν which returns to the origin after time n and additionally the last ancestor of ν which has been at a position containing a cookie was the ancestor at time n. For $\lambda_0, \gamma > 0$, which we will specify later (cf. (3.59) and (3.61)), we get the following estimate with $m^* = \max\{1, p_0 m_0\}$:

$$\begin{aligned}
&\mathsf{P}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{|\mathcal{L}(n)| \leq (m^*+\gamma)^n\}\Big) \\
&= 1 - \mathsf{P}\Big(A_n^c \cap \{l(n) \geq n\lambda_0\} \cap \{|\mathcal{L}(n)| \leq (m^*+\gamma)^n\}\Big) \\
&\leq 1 - \mathsf{P}\Big(\Lambda^-_{\lceil n\lambda_0 - 1\rceil} = 0\Big)^{M(m^*+\gamma)^n}.
\end{aligned}$$

Here we use the fact that the number of offspring of every particle belonging to L_n which return to the origin is bounded by the amount of offspring in $\Lambda^-_{l(n)-1}$. Additionally, we have $|L_n| \leq M|\mathcal{L}(n)|$ due to assumption (3.1). Since the GWP $\left(\Lambda_n^-\right)_{n\in\mathbb{N}_0}$ with mean offspring φ_ℓ is subcritical, we can use Proposition 3.4.3 to obtain

$$\begin{aligned}
&\mathsf{P}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{|\mathcal{L}(n)| \leq (m^*+\gamma)^n\}\Big) \\
&\leq 1 - \left(1 - c(\varphi_\ell)^{\lceil n\lambda_0 - 1\rceil}\right)^{M(m^*+\gamma)^n} \\
&\leq 1 - \exp\left(-2c(\varphi_\ell)^{\lceil n\lambda_0 - 1\rceil} M(m^*+\gamma)^n\right) \\
&\leq 2c(\varphi_\ell)^{n\lambda_0 - 1} M(m^*+\gamma)^n \\
&= C(\varphi_\ell)^{n\lambda_0}(m^*+\gamma)^n
\end{aligned} \tag{3.58}$$

for some constants $c, C > 0$ and for large n. In the above display we make use of the estimates $1 - x \geq \exp(-2x)$, which holds for $x \in [0, \frac{1}{2}]$, and $1 - \exp(-x) \leq x$, which holds for all $x \in \mathbb{R}$. We also note that we have $\varphi_\ell < 1$.

Let us first assume that we have $m^* = \max\{1, p_0 m_0\} = 1$. We choose $\lambda_0 = \lambda/2$ for some $\lambda > 0$ which satisfies the assumptions of Proposition 3.5.5*(a)*. We know that we have $\varphi_\ell \leq 2q_0 m_0 < 1$ and therefore can choose $\gamma > 0$ such that

$$(\varphi_\ell)^{\lambda_0}(m^*+\gamma) \leq (2q_0 m_0)^{\lambda_0}(1+\gamma) \leq (1-\gamma). \tag{3.59}$$

By applying (3.59) to (3.58), we get

$$\mathsf{P}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{|\mathcal{L}(n)| \leq (m^*+\gamma)^n\}\Big) \leq o(1)(1-\gamma)^n.$$

Therefore, the Borel-Cantelli lemma implies

$$\mathsf{P}\left(\limsup_{n\to\infty}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{|\mathcal{L}(n)| \leq (m^*+\gamma)^n\}\Big)\right) = 0. \tag{3.60}$$

Moreover, Proposition 3.5.2 and Proposition 3.5.5 together with the choices of λ_0 and γ yield

$$\mathsf{P}\left(\liminf_{n\to\infty}\Big(\{l(n)\geq n\lambda_0\}\cap\{|\mathcal{L}(n)|\leq (m^*+\gamma)^n\}\Big)\right)=1.$$

Finally, we can conclude from (3.60) that we have $\mathsf{P}\left(\limsup_{n\to\infty} A_n\right)=0$, which implies the transience of the CBRW in this case.

We now assume that we have $m^*=p_0m_0>1$. Due the assumption of the transience of the BRW without cookies, we have

$$\varphi_\ell p_0 m_0 \leq 2q_0m_0\cdot p_0m_0\leq \frac{1}{2}.$$

Therefore, we can choose $0<\gamma<1$ such that

$$(\varphi_\ell)^{1-\gamma}(p_0m_0+\gamma)\leq \frac{3}{4}. \tag{3.61}$$

For $\lambda_0:=1-\gamma$, (3.58) and (3.61) imply

$$\mathsf{P}\Big(A_n\cap\{l(n)\geq n\lambda_0\}\cap\{|\mathcal{L}(n)|\leq (m^*+\gamma)^n\}\Big)\leq o(1)\left(\frac{3}{4}\right)^n.$$

Again, by applying the Borel-Cantelli lemma, we get

$$\mathsf{P}\left(\limsup_{n\to\infty}\left(A_n\cap\{l(n)\geq n\lambda_0\}\cap\{|\mathcal{L}(n)|\leq (m^*+\gamma)^n\}\right)\right)=0.$$

Additionally, Proposition 3.5.2 and Proposition 3.5.5 together with the choices of λ_0 and γ yield

$$\mathsf{P}\left(\liminf_{n\to\infty}\left(\{l(n)\geq n\lambda_0\}\cap\{|\mathcal{L}(n)|\leq (m^*+\gamma)^n\}\right)\right)=1.$$

Therefore, we conclude that we have $\mathsf{P}\left(\limsup_{n\to\infty} A_n\right)=0$, which implies the transience of the CBRW in the case $p_0m_0>1$. ■

Proof of Theorem 3.3.2

For this theorem we only have to make sure that the cookies cannot displace the cloud of particles too far to the right. It turns out that, somewhat similarly to the case of a cookie (or excited) random walk (cf. Theorem 12 in [59]) one single cookie at every position $x\in\mathbb{N}_0$ is not enough for a behaviour of such kind.

We divide the proof of the theorem into two cases. At first we consider the case $m_0=1$, i.e. particles can only branch at positions with a cookie, and in the second part we consider the case $m_0>1$.

Let us first assume that $m_0 = 1$ holds true. In this case the BRW without cookies reduces to a nearest-neighbour random walk on $\mathbb{Z}$ and is therefore strongly recurrent iff we have $p_0 = \frac{1}{2}$. Further, it is enough to only take into account the very first offspring particle in each time step since already those particles visit the origin infinitely often with probability 1. For $p_c \leq \frac{1}{2}$, the strong recurrence is obvious since we can bound the trajectories of the considered particles from above by the trajectory of a symmetric random walk on $\mathbb{Z}$ with a standard coupling argument. For $\frac{1}{2} < p_c < 1$ we can couple the random movement of the considered particles to a symmetric random walk and an excited random walk in the sense of [11] (with excitement $\varepsilon = 2p_c - 1$) in such a way that the positions of the considered particles lie in between the symmetric random walk (to the left) and the excited random walk (to the right). Since both random walks are recurrent (cf. Section 2 in [11] for the excited random walk), we can again conclude that the CBRW is strongly recurrent.

Now we suppose that we have $m_0 > 1$. From Proposition 3.2.2 we know that we have $\log(m_0) > -\frac{1}{2}\log(4p_0q_0) = I(0)$, where $I(\cdot)$ denotes the rate function of the nearest-neighbour random walk on $\mathbb{Z}$ with transition probabilities p_0 and q_0. Since the rate I is continuous on $(-1, 1)$, there exist $\varepsilon, \delta \in (0, 1)$ such that $\log(m_0) > I(-\varepsilon) + \delta$. Let $(S_n)_{n\in\mathbb{N}_0}$ denote a nearest-neighbour random walk of this kind started in 0 and with transition probabilities p_0 and q_0. We have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathsf{P}\big(S_n \leq -n\varepsilon\big) = \begin{Bmatrix} -I(-\varepsilon) & \text{for } 2p_0 - 1 > -\varepsilon \\ 0 & \text{for } 2p_0 - 1 \leq -\varepsilon \end{Bmatrix} \geq -I(-\varepsilon).$$

In particular, there exists k_0 such that $\mathsf{P}\big(S_{k_0} \leq -k_0\varepsilon\big) \geq \exp\big(-k_0(I(-\varepsilon)+\delta)\big)$. This yields for the BRW without cookies $(Y_\nu)_{\nu\in\mathcal{Y}}$ that

$$\mathsf{E}\Big[\big|\{\nu \in \mathcal{Y} :\, |\nu| = k_0, Y_\nu \leq -k_0\varepsilon\}\big|\Big] \geq (m_0)^{k_0} \exp\big(-k_0(I(-\varepsilon)+\delta)\big) > 1 \tag{3.62}$$

for the above choice of ε and δ. Therefore, we can conclude that the embedded GWP of those particles which move at least $k_0\varepsilon$ to the left between time 0 and k_0, between k_0 and $2k_0$ and so on is supercritical and therefore survives with strictly positive probability p_{sur}. Let us now turn back to the CBRW. For every existing particle ν the probability

$$\mathsf{P}\big(\exists \eta \in \mathcal{Z} :\ \eta \succeq \nu,\ |\eta| - |\nu| = k_0,\ X_\eta = X_\nu - k_0 \mid \nu \in \mathcal{Z}\big) \geq q_c q_0^{k_0}$$

to have at least one descendant k_0 generations later which is located k_0 positions to the left of the position of ν is bounded away from 0. From this we conclude that, for every existing particle ν in the CBRW, the probability

$$\mathsf{P}\big(\exists \eta \in \mathcal{Z} :\ \eta \succeq \nu,\ X_\tau \leq l(|\nu|)\ \forall \nu \preceq \tau \preceq \eta,\ X_\eta \leq 0 \mid \nu \in \mathcal{Z}\big) \geq q_c q_0^{k_0} p_{\text{sur}} =: c > 0 \tag{3.63}$$

to have at least one descendant located on the negative semi-axis without any cookie contact of the ancestral line connecting ν and this descendant is also bounded away from 0. Here the lower bound is a lower estimate for the probability for each existing particle ν to have at least one descendant k_0 generations later which is located k_0 positions to the left of the position of ν and then starts a surviving embedded GWP which moves at least $k_0\varepsilon$ to the left between time 0 and k_0, between k_0 and $2k_0$ and so on. Since the particles we consider for this embedded GWP cannot hit the cookies inbetween, this GWP has the same probability for survival p_{sur} as in the case of the BRW without cookies (cf. (3.62)). Using (3.63) we can conclude the strong recurrence of the CBRW since the particles on the negative semi-axis behave as the strongly recurrent BRW without cookies before they can reach a cookie again. ■

Proof of Theorem 3.3.3

Proof of part (a). Here we suppose that the LP is supercritical, i.e. $p_c m_c > 1$. On the one hand the probability that all particles which are produced in the first step move to the left and their offspring then escape to $-\infty$ without returning to 0 is strictly positive since every offspring particle starts an independent BRW without cookies at position -1 as long as the offspring does not return to the origin. We note that the probability for the BRW started at -1 never to return to the origin is strictly positive since the BRW without cookies is transient to the left by assumption.

On the other hand the LP which is started at 0 is a supercritical GWP and therefore survives with positive probability. If it survives, a.s. infinitely many particles leave the LP (to the left) at time $n \geq 1$. Afterwards each of those particles starts a BRW without cookies at position $n-1 \geq 0$ since the offspring cannot reach a cookie again. Each of those BRWs without cookies will a.s. produce at least one offspring which visits the origin since the BRW without cookies is transient to the left by assumption. ■

Proof of part (b). Here we suppose that the LP is critical or subcritical, i.e. $p_c m_c \leq 1$. In the following we want to consider the following three quantities. The first one is the number of particles in the LP. The second one is the number of particles which are descendants of the non-LP particles of generation n (i.e. $Z_n \setminus \mathcal{L}(n)$) and which are the first in their ancestral line to reach the position $l(n)$. By definition, these particles can potentially change the position $l(n)$ of the leftmost cookie in the future. The third quantity is the number of particles belonging to $Z_n \setminus \mathcal{L}(n)$ whose descendants will not reach the position $l(n)$ in the future. More precisely, for all $n \in \mathbb{N}_0$ we define (observe that $X_\nu = l(n)$ implies that $|\nu| = n$):

$$\zeta_1(n) := |\mathcal{L}(n)|,$$

$$\zeta_2(n) := |\{\nu \preceq Z_n \setminus \mathcal{L}(n) : \, X_\nu = l(n), \, X_\eta < l(n) \, \forall \, \eta \succ \nu\}|$$
$$\zeta_3(n) := |\{\nu \in Z_n \setminus \mathcal{L}(n) : \, X_\nu < l(n) \, \forall \, \eta \preceq \nu\}|.$$

We note that for the definition of $\zeta_2(n)$ we count the number of descendants of the non-LP particles at time n which will reach the position $l(n)$ in the future. Thus, the type-2 particles belong to a generation larger than n.

In the following we want to allow arbitrary starting configurations from the set

$$\mathcal{S} := \left\{ (a,b) \in \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0 : \, \sum_{k \in \mathbb{Z}} a(k) < \infty, \, \max\{k \in \mathbb{Z} : \, a(k) > 0\} \le b \right\}.$$

Here a contains the information about the number of particles at each position $k \in \mathbb{Z}$ and b is the position of the leftmost cookie. In particular, every configuration of the CBRW which can be reached within finite time is contained in the set $\mathcal{S}$. For each $(a,b) \in \mathcal{S}$ we consider the probability measure $\mathsf{P}_{(a,b)}$ under which the CBRW starts in the configuration (a,b) and then evolves in the usual way.

The main idea of the proof is the following. We show that there is a critical level for the total amount of the type-1 and type-2 particles. Once this level is exceeded the total amount has the tendency to fall back below this level. There are two reasons which cause this behaviour. First, the expected amount of type-2 particles which stay type-2 particles for another time step decreases every time the leftmost cookie is consumed by a type-1 particle. Second, if there are many type-1 particles, the LP survives for a long time with high probability and meanwhile the remaining particles have time to escape to the left.

For the proof we have to analyse the relation between the type-1 and type-2 particles and to distinguish between two distinct situations. In the first situation, there are type-1 particles at time n and therefore the leftmost cookie is consumed. In the second case there are no type-1 particles and therefore the position of the leftmost cookie does not change. Let us first assume that there are type-1 particles at time n. Then, on the event $\{\zeta_1(n) \neq 0\}$ we a.s. have

$$\mathsf{E}_{(a,b)}\big[\zeta_1(n+1) \mid \zeta_1(n), \zeta_2(n)\big] = \zeta_1(n) p_c m_c,$$
$$\mathsf{E}_{(a,b)}\big[\zeta_2(n+1) \mid \zeta_1(n), \zeta_2(n)\big] = \zeta_1(n) q_c m_c (\varphi_r)^2 + \zeta_2(n)\varphi_r. \tag{3.64}$$

Here the last equality holds since each type-1 particle produces an expected number of $q_c m_c$ particles which leave the LP to the left. In order to decide how large their expected contribution to the type-2 particles at time $n+1$ is, we have to count the number of their offspring which will reach position $l(n+1) = l(n)+1$ in the future. For each of these particles the distribution of this random number coincides with the distribution of Λ_2^+ whose expectation is given by $(\varphi_r)^2$. Additionally, since one cookie is consumed the amount of type-2 particles, which are still type-2 particles at time $n+1$, decreases in expectation by φ_r. Observe that due to the transience to the left of the BRW without cookies, the process $\big(\Lambda_n^+\big)_{n\in\mathbb{N}_0}$ is a GWP with mean $\varphi_r < 1$ (cf. (3.4) and (3.5)).

Let us now assume that the LP is empty. Then, on $\{\zeta_1(n) = 0\}$ we a.s. have

$$\mathsf{E}_{(a,b)}\big[\zeta_1(n+1) + \zeta_2(n+1) \mid \zeta_1(n), \zeta_2(n)\big] = \zeta_2(n), \tag{3.65}$$

since the position of the leftmost cookie does not change, i.e. $l(n+1) = l(n)$. Therefore, each type-2 particle of time n either still is a type-2 particle at time $n+1$ or becomes a type-1 particle.

First, we deal with the subcritical case, i.e. $p_c m_c < 1$. For fixed $h \in \mathbb{N}$ (which will be specified later, cf. (3.67)) we define the following random times

$$\eta_{n+1} := \begin{cases} (\eta_n + h) \wedge \inf\{i > \eta_n : \zeta_1(i) = 0\}, & \text{if } \zeta_1(\eta_n) > 0, \\ (\eta_n + h) \wedge \inf\{i > \eta_n : \zeta_1(i) > 0\}, & \text{if } \zeta_1(\eta_n) = 0, \end{cases}$$

for $n \in \mathbb{N}_0$ and $\eta_0 := 0$. We note that we have $\eta_{n+1} - \eta_n \le h$. For $n \in \mathbb{N}_0$ we define

$$\xi_1(n) := \zeta_1(\eta_n), \quad \xi_2(n) := \zeta_2(\eta_n)$$

as the amount of type-1 and type-2 particles along the sequence $(\eta_n)_{n\in\mathbb{N}_0}$ and the associated filtration $\mathcal{F}_n := \sigma\big(\xi_1(i),\ \xi_2(i),\ \eta_i :\ i \le n\big)$. We want to adapt Theorem 2.2.1 of [27] and start with the following lemma:

Lemma 3.5.7. *For suitable (large) $h, u \in \mathbb{N}$ we have*

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \mid \mathcal{F}_n] \le \xi_1(n) + \xi_2(n) \tag{3.66}$$

a.s. on $\{\xi_1(n) + \xi_2(n) \ge u\}$ for all $(a, b) \in \mathcal{S}$.

Proof of Lemma 3.5.7. Let us fix $(a, b) \in \mathcal{S}$. We choose $h \in \mathbb{N}$ large enough such that

$$\big(p_c m_c\big)^h + q_c m_c \sum_{i=0}^{h-1} \big(p_c m_c\big)^i (\varphi_r)^{h-i+1} < \frac{1}{2} \tag{3.67}$$

and

$$(\varphi_r)^h < \frac{1}{2}. \tag{3.68}$$

Such a choice is possible since $p_c m_c < 1$ and $\varphi_r < 1$. Then we fix $c = c(h)$ such that

$$0 < c \le \frac{1}{M^h}(1 - \varphi_r) \tag{3.69}$$

holds true. We recall that the particles in the LP constitute a subcritical GWP. Let $\big(GW_n^{\text{sub}}\big)_{n\in\mathbb{N}_0}$ denote such a GWP (with the same offspring distribution). Then, for every $\delta > 0$, there is $u = u(\delta, h, c) \in \mathbb{N}$ such that

$$\mathsf{P}_{\lfloor c/(c+1)u\rfloor}\left(GW_h^{\text{sub}} \geq 1\right) \geq 1-\delta \tag{3.70}$$

since the probability for each existing particle to have at least one offspring which moves to the right is strictly positive.

We now verify (3.66) separately on the following three events:

$$\begin{aligned}
A_1 &:= \{\xi_1(n)+\xi_2(n) \geq u\} \cap \{\xi_1(n) = 0\},\\
A_2 &:= \{\xi_1(n)+\xi_2(n) \geq u\} \cap \{0 < \xi_1(n) \leq c\xi_2(n)\},\\
A_3 &:= \{\xi_1(n)+\xi_2(n) \geq u\} \cap \{\xi_1(n) > c\xi_2(n)\}.
\end{aligned}$$

We note that $A_1 \cup A_2 \cup A_3 = \{\xi_1(n)+\xi_2(n) \geq u\}$.

On the event A_1 there is no particle in the LP between time η_n and time η_{n+1} by definition. Thus, the position of the leftmost cookie does not change during this period. Hence we a.s. have

$$\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1)\,|\,\mathcal{F}_n]\mathbb{1}_{A_1} = \xi_2(n)\mathbb{1}_{A_1}$$

due to (3.65).

On the event A_2 there is at least one particle in the LP and thus the leftmost cookie is consumed at time η_n. Using $\eta_{n+1}-\eta_n \leq h$ and the fact that the total number of offspring of each particle is bounded by M, we a.s. obtain on the event A_2

$$\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1)\,|\,\mathcal{F}_n] \leq \big(\xi_1(n)M^h + \varphi_r\xi_2(n)\big) \leq \xi_2(n)\big(cM^h+\varphi_r\big) \leq \xi_2(n).$$

Here we use (3.69) in the last step.

Next, we recall that $L_n = \{\nu \in Z_{n+1}(l(n)-1) : \nu \succ \mathcal{L}(n)\}$ denotes the set of particles which leave the leading process to the left at time n. Using (3.64), on the event A_3, we a.s. get

$$\begin{aligned}
&\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1)\,|\,\mathcal{F}_n]\\
&= \mathsf{E}_{(a,b)}\left[\big(\xi_1(n+1)+\xi_2(n+1)\big)\mathbb{1}_{\{\eta_{n+1}-\eta_n<h\}}\mid\mathcal{F}_n\right]\\
&\quad + \mathsf{E}_{(a,b)}\left[\big(\xi_1(n+1)+\xi_2(n+1)\big)\mathbb{1}_{\{\eta_{n+1}-\eta_n=h\}}\mid\mathcal{F}_n\right]\\
&\leq \big(M^{h-1}\xi_1(n)+\varphi_r\xi_2(n)\big)\,\mathsf{E}_{(a,b)}\left[\mathbb{1}_{\{\eta_{n+1}-\eta_n<h\}}\mid\mathcal{F}_n\right]\\
&\quad + (\varphi_r)^h\xi_2(n)\mathsf{E}_{(a,b)}\left[\mathbb{1}_{\{\eta_{n+1}-\eta_n=h\}}\mid\mathcal{F}_n\right]
\end{aligned}$$

$$+ \mathsf{E}_{(a,b)}\left[|\mathcal{L}(\eta_n + h)| \mathbb{1}_{\{\eta_{n+1}-\eta_n = h\}} \,\middle|\, \mathcal{F}_n \right]$$

$$+ \sum_{i=0}^{h-1} \mathsf{E}_{(a,b)}\left[\sum_{\nu \succeq L_{\eta_n+i}} \mathbb{1}_{\{X_\nu = l(\eta_n)+h,\, X_\eta < l(\eta_n)+h\, \forall \eta \prec \nu\}} \mathbb{1}_{\{\eta_{n+1}-\eta_n = h\}} \,\middle|\, \mathcal{F}_n \right].$$

In the second step of this computation we use that on the event $\{\eta_{n+1} - \eta_n < h\}$ (in expectation) the proportion at most φ_r of the type-2 particles does not escape to the left since at least one cookie is consumed. On the event $\{\eta_{n+1} - \eta_n = h\}$ we consider three summands. The first one corresponds to the type-2 particles at time η_n that are still type-2 particles at time η_{n+1}. The second one corresponds to the particles that are still in the LP at time η_{n+1} and the third one to the particles which have left the LP in the meantime. Using (3.64) and the fact that we have at least $\lfloor c/(c+1)u \rfloor$ type-1 particles on the event A_3, we continue the calculation and obtain that on the event A_3 we a.s. have

$$\begin{aligned}
\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \,|\, \mathcal{F}_n] \le & \Bigg[\big(M^{h-1}\xi_1(n) + \varphi_r \xi_2(n) \big) \, \mathsf{P}_{\lfloor c/(c+1)u \rfloor} \big(GW_h^{\text{sub}} = 0 \big) \\
& + (\varphi_r)^h \xi_2(n) + \big(p_c m_c \big)^h \xi_1(n) \\
& + \sum_{i=0}^{h-1} \xi_1(n) (p_c m_c)^i (q_c m_c) (\varphi_r)^{h-i+1} \Bigg] \\
\le & \left(M^{h-1}\delta + \frac{1}{2} \right) \xi_1(n) + \left(\varphi_r \delta + \frac{1}{2} \right) \xi_2(n) \\
\le & \ \xi_1(n) + \xi_2(n)
\end{aligned}$$

for $\delta = \delta(M, h, \varphi_r)$ sufficiently small. Here we use (3.67), (3.68), and (3.70) for the latter estimates. □

We now turn to the case of a critical leading process, i.e., $p_c m_c = 1$. Again, for some $c > 0$, which we specify later (cf. (3.72)), we inductively define the following random times

$$\eta_{n+1} := \begin{cases} \eta_n + 1, & \text{if } \zeta_2(\eta_n) \ge c\zeta_1(\eta_n), \\ \inf\{n > \eta_n :\ \zeta_1(n) = 0\}, & \text{if } \zeta_2(\eta_n) < c\zeta_1(\eta_n), \end{cases}$$

for $n \in \mathbb{N}_0$ and $\eta_0 := 0$. Similarly to above, we define for $n \in \mathbb{N}_0$

$$\xi_1(n) := \zeta_1(\eta_n), \quad \xi_2(n) := \zeta_2(\eta_n)$$

and the associated filtration $\mathcal{F}_n := \sigma\big(\xi_1(i),\ \xi_2(i), \eta_i : i \le n\big)$. We continue with a lemma, which is an analogous statement to Lemma 3.5.7.

Lemma 3.5.8. *For suitable (large) $u \in \mathbb{N}$ we have*

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1)|\mathcal{F}_n] \leq \xi_1(n) + \xi_2(n) \tag{3.71}$$

a.s. on $\{\xi_1(n) + \xi_2(n) \geq u\}$ *for all* $(a,b) \in \mathcal{S}$.

Proof of Lemma 3.5.8. Let us fix $(a,b) \in \mathcal{S}$. Again for some $u = u(c) \in \mathbb{N}$, which we specify later (cf. (3.83)), we introduce the following events

$$A_1 := \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{\xi_2(n) \geq c\xi_1(n)\},$$
$$A_2 := \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{\xi_2(n) < c\xi_1(n)\}.$$

and show (3.71) on the events A_1 and A_2 separately.

On the event A_1 we a.s. have

$$\begin{aligned}
&\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1)|\mathcal{F}_n] \\
\leq\ &\mathbb{1}_{\{\xi_1(n)=0\}}\xi_2(n) + \mathbb{1}_{\{\xi_1(n)>0\}}\left(\varphi_r\xi_2(n) + M\xi_1(n)\right) \\
\leq\ &\mathbb{1}_{\{\xi_1(n)=0\}}\xi_2(n) + \mathbb{1}_{\{\xi_1(n)>0\}}\left(\varphi_r\xi_2(n) + Mc^{-1}\xi_2(n)\right) \\
\leq\ &\left[\mathbb{1}_{\{\xi_1(n)=0\}} + \mathbb{1}_{\{\xi_1(n)>0\}}\left(\varphi_r + Mc^{-1}\right)\right]\xi_2(n) \\
\leq\ &\xi_2(n)
\end{aligned}$$

for any

$$0 < c \leq M\left(1 - \varphi_r\right)^{-1}. \tag{3.72}$$

Here we use that on the event A_1 we have $\eta_{n+1} = \eta_n + 1$. If $\xi_1(n) = 0$ holds, then no cookie is eaten at time η_n and therefore we have $\xi_2(n+1) = \xi_2(n)$. If $\xi_1(n) > 0$ holds, the leftmost cookie is consumed and therefore in expectation the amount of the type-2 particles is reduced by the factor φ_r.

Next, to investigate the behaviour of the process on the event A_2, we first consider the case $(\xi_1(n), \xi_2(n)) = (v, 0)$ for $v \in \mathbb{N}$. From this we can easily derive the general case later on since each time a cookie is consumed the number of type-2 particles is reduced by the factor $\varphi_r < 1$. Therefore, the type-2 particles do not essentially contribute to the growth of the process. We have:

$$\begin{aligned}
&\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \mid \mathcal{F}_n]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\
=\ &\mathsf{E}_{(a,b)}[\xi_2(n+1)|\mathcal{F}_n]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\
=\ &\Bigg(\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\{\eta_{n+1}-\eta_n \leq v^{1/3}\}} \mid \mathcal{F}_n\right] \\
&\quad + \sum_{j>v^{1/3}} \mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\{\eta_{n+1}-\eta_n=j\}} \mid \mathcal{F}_n\right]\Bigg)\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}.
\end{aligned} \tag{3.73}$$

We now consider the first summand in (3.73). For this we define

$$E^0 := \left\{ \max_{\ell=1,\dots,\lfloor v^{1/3} \rfloor} \zeta_1(\eta_n + \ell) \le v^{2/3} \right\},$$
$$E^k := \left\{ \max_{\ell=1,\dots,\lfloor v^{1/3} \rfloor} \zeta_1(\eta_n + \ell) \in \left(2^{k-1} v^{2/3}, 2^k v^{2/3}\right] \right\} \quad \text{for } k \ge 1,$$

in order to control the maximum number of particles in the LP. Using these definitions we write

$$\begin{aligned}
&\mathsf{E}_{(a,b)} \left[\xi_2(n+1) \mathbb{1}_{\left\{\eta_{n+1} - \eta_n \le v^{1/3}\right\}} \;\middle|\; \mathcal{F}_n \right] \\
&= \sum_{k=0}^{\infty} \mathsf{E}_{(a,b)} \left[\xi_2(n+1) \mathbb{1}_{E^k \cap \left\{\eta_{n+1} - \eta_n \le v^{1/3}\right\}} \;\middle|\; \mathcal{F}_n \right] \\
&\le v^{1/3} M v^{2/3} \mathsf{P}_{(a,b)} \left(\eta_{n+1} - \eta_n \le v^{1/3} \mid \mathcal{F}_n \right) \\
&\quad + \sum_{k=1}^{\infty} v^{1/3} M 2^k v^{2/3} \mathsf{P}_{(a,b)} \left(\mathcal{B}_k(n,v) \mid \mathcal{F}_n \right),
\end{aligned} \tag{3.74}$$

where we use the notation

$$\mathcal{B}_k(n,v) := \left\{ \exists \ell \in \{\eta_n + 1, \dots, \eta_{n+1}\} : \zeta_1(\ell) > 2^{k-1} v^{2/3}, \eta_{n+1} - \eta_n \le v^{1/3} \right\}.$$

Here we note that each particle that leaves the LP starts a new BRW without cookies (as long as the offspring particles do not reach a cookie again) which is transient to the left by assumption. Thus, for each of those particles the expected number of descendants which reach the position $l(\eta_{n+1})$ (and therefore are type-2 particles at time η_{n+1}) is less than one since they have to move at least two steps to the right. Now we observe that on $\{(\xi_1(n), \xi_2(n)) = (v, 0)\}$ we a.s. have

$$\mathsf{P}_{(a,b)} \left(\eta_{n+1} - \eta_n \le v^{1/3} \mid \mathcal{F}_n \right) = \mathsf{P}_v \left(T^{\mathrm{cr}} \le v^{1/3} \right) \tag{3.75}$$

and

$$\mathsf{P}_{(a,b)} \left(\mathcal{B}_k(n,v) \mid \mathcal{F}_n \right) \le v^{1/3} \mathsf{P}_{\lceil 2^{k-1} v^{2/3} \rceil} \left(T^{\mathrm{cr}} \le v^{1/3} \right) \tag{3.76}$$

where T^{cr} denotes the extinction time of a critical GWP whose offspring distribution is given by the number of particles produced by a single particle in the LP which stay in the LP. (We note that this coincides with the number of type-1 offspring of a type-1 particle.) Now we apply (3.75), (3.76) and Proposition 3.4.5 to (3.74) and a.s. obtain

$$\begin{aligned}
&\mathsf{E}_{(a,b)} \left[\xi_2(n+1) \mathbb{1}_{\left\{\eta_{n+1} - \eta_n \le v^{1/3}\right\}} \middle| \mathcal{F}_n \right] \cdot \mathbb{1}_{\{(\xi_1(n), \xi_2(n)) = (v,0)\}} \\
&\le \left[M v \exp\left(-C \frac{v}{v^{1/3}} \right) + \sum_{k=1}^{\infty} M 2^k v^{4/3} \exp\left(-C \frac{2^{k-1} v^{2/3}}{v^{1/3}} \right) \right] \cdot \mathbb{1}_{\{(\xi_1(n), \xi_2(n)) = (v,0)\}}
\end{aligned}$$

$$= o(v) \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \tag{3.77}$$

where $C > 0$ is the constant of Proposition 3.4.5.

Now we deal with the second summand in (3.73). For some $\delta \in (0, \frac{1}{3})$ and $j \in \mathbb{N}$ we introduce the events

$$F_j^0 := \left\{ \max_{\ell=1,\dots,\lfloor j^\delta \rfloor} \zeta_1\big(\eta_n + j - \lfloor j^\delta \rfloor + \ell\big) \le j^{2\delta} \right\},$$
$$F_j^k := \left\{ \max_{\ell=1,\dots,\lfloor j^\delta \rfloor} \zeta_1\big(\eta_n + j - \lfloor j^\delta \rfloor + \ell\big) \in \big(2^{k-1} j^{2\delta}, 2^k j^{2\delta}\big] \right\} \quad \text{for } k \ge 1,$$

and

$$G_j^0 := \left\{ \max_{\ell=1,\dots,j} \zeta_1(\eta_n + \ell) \le j^{1+\delta} \right\},$$
$$G_j^k := \left\{ \max_{\ell=1,\dots,j} \zeta_1(\eta_n + \ell) \in \big(2^{k-1} j^{1+\delta}, 2^k j^{1+\delta}\big] \right\} \quad \text{for } k \ge 1.$$

On the events G_j^k we control the maximum number of particles in the LP up to time j, whereas on F_j^k we control the maximum number during the $\lfloor j^\delta \rfloor$ time steps before j. We observe that on the event $F_j^k \cap G_j^\ell$ not more than $M \cdot 2^\ell j^{2+\delta}$ particles leave the LP up to time $\eta_n + j - \lfloor j^\delta \rfloor$ (because of G_j^ℓ). Each of those particles starts a BRW without cookies and in average it contributes not more than $(\varphi_r)^{\lfloor j^\delta \rfloor + 1} \le (\varphi_r)^{j^\delta}$ to the number of type-2 particles at time $\eta_n + j$. Similarly, on $F_j^k \cap G_j^\ell$ not more than $M 2^k j^{3\delta}$ particles leave the LP from time $\eta_n + j - \lfloor j^\delta \rfloor + 1$ to time $\eta_n + j$ (because of F_j^k). Further, it holds that each particle that leaves the LP starts a new BRW without cookies and for each of those particles the expected number of descendants which reach the position $l(\eta_{n+1})$ is less than one since they have to move at least two steps to the right. Thus, we have

$$\begin{aligned} &\mathsf{E}_{(a,b)} \left[\xi_2(n+1) \mathbb{1}_{F_j^k \cap G_j^\ell \cap \{\eta_{n+1} - \eta_n = j\}} \mid \mathcal{F}_n \right] \\ &\le \left(M 2^\ell j^{2+\delta} (\varphi_r)^{j^\delta} + M 2^k j^{3\delta} \right) \mathsf{P}_{(a,b)} \left(F_j^k \cap G_j^\ell \cap \{\eta_{n+1} - \eta_n = j\} \middle| \mathcal{F}_n \right). \end{aligned} \tag{3.78}$$

Now we suppose that $\ell \ge k$ and $(k, \ell) \ne (0, 0)$. Then due to Proposition 3.4.5 we have

$$\begin{aligned} &\mathsf{P}_{(a,b)} \left(F_j^k \cap G_j^\ell \cap \{\eta_{n+1} - \eta_n = j\} \,\middle|\, \mathcal{F}_n \right) \\ &\le \mathsf{P}_{(a,b)} \left(\exists i \in \{1, \dots, j\} : \zeta_1(\eta_n + i) > 2^{\ell-1} j^{1+\delta},\, \zeta_1(\eta_n + j) = 0 \,\middle|\, \mathcal{F}_n \right) \\ &\le j \, \mathsf{P}_{\lceil 2^{\ell-1} j^{1+\delta} \rceil} \left(T^{\text{cr}} \le j \right) \\ &\le j \exp\left(-C \frac{2^{\ell-1} j^{1+\delta}}{j} \right) \end{aligned}$$

$$\leq j \exp\left(-\tfrac{1}{2}C2^{(\ell+k)/2}j^{\delta}\right). \tag{3.79}$$

If otherwise $k \geq \ell$ and $(k,\ell) \neq (0,0)$, then again due to Proposition 3.4.5 we have

$$\begin{aligned}
&\mathsf{P}_{(a,b)}\left(F_j^k \cap G_j^\ell \cap \{\eta_{n+1}-\eta_n = j\} \,\middle|\, \mathcal{F}_n\right) \\
&\leq \mathsf{P}_{(a,b)}\left(\exists i \in \{j - \lfloor j^\delta \rfloor + 1, \dots, j\} : \zeta_1(\eta_n+i) > 2^{k-1}j^{2\delta},\, \zeta_1(\eta_n+j)=0 \,\middle|\, \mathcal{F}_n\right) \\
&\leq j\,\mathsf{P}_{\lceil 2^{k-1}j^{2\delta}\rceil}\left(T^{\mathrm{cr}} \leq j^\delta\right) \\
&\leq j \exp\left(-C\frac{2^{k-1}j^{2\delta}}{j^\delta}\right) \\
&\leq j \exp\left(-\tfrac{1}{2}C2^{(\ell+k)/2}j^{\delta}\right). \qquad (3.80)
\end{aligned}$$

By virtue of (3.79) and (3.80) combined with (3.78) we a.s. obtain

$$\begin{aligned}
&\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\{\eta_{n+1}-\eta_n=j\}} \mid \mathcal{F}_n\right] \\
&= \sum_{k,\ell=0}^{\infty} \mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{F_j^k\cap G_j^\ell \cap\{\eta_{n+1}-\eta_n=j\}} \,\middle|\, \mathcal{F}_n\right] \\
&\leq \left(Mj^{2+\delta}\left(\varphi_r\right)^{j^\delta} + Mj^{3\delta}\right)\mathsf{P}_{(a,b)}\left(\eta_{n+1}-\eta_n = j \mid \mathcal{F}_n\right) \\
&\qquad + \sum_{(k,\ell)\neq(0,0)} \left(M2^\ell j^{2+\delta}(\varphi_r)^{j^\delta} + M2^k j^{3\delta}\right) j \exp\left(-\tfrac{1}{2}C2^{(\ell+k)/2}j^\delta\right) \\
&\leq C_2 j^{3\delta}\,\mathsf{P}_{(a,b)}\left(\eta_{n+1}-\eta_n = j \mid \mathcal{F}_n\right) \\
&\qquad + \sum_{i=1}^{\infty} C_2 j^{1+3\delta}(i+1)2^i \exp\left(-\tfrac{1}{2}C2^{i/2}j^\delta\right) \qquad (3.81)
\end{aligned}$$

for a suitable constant $C_2 > 0$ which does not depend on j. Due to Proposition 3.4.6 and for a suitable choice of a constant $C_3 > 0$, we a.s. have $\mathsf{P}_{(a,b)}\left(\eta_{n+1}-\eta_n = j \mid \mathcal{F}_n\right) \leq C_3\frac{v}{j^2}$ on the event $\{(\xi_1(n),\xi_2(n)) = (v,0)\}$. Therefore, (3.81) yields

$$\begin{aligned}
&\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\cdot\mathbb{1}_{\{\eta_{n+1}-\eta_n=j\}} \mid \mathcal{F}_n\right]\cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\
&\leq \left[C_4 j^{3\delta-2}v + \sum_{i=1}^{\infty} C_2 j^{1+3\delta}(i+1)2^i \exp\left(-\tfrac{1}{2}C2^{i/2}j^\delta\right)\right] \\
&\qquad \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}
\end{aligned}$$

$$\leq \left[C_4 j^{3\delta-2} v + C_2 j^{1+3\delta} \exp\left(-\tfrac{1}{4} C 2^{\frac{1}{2}} j^{\delta}\right) \sum_{i=1}^{\infty} (i+1) 2^i \exp\left(-\tfrac{1}{4} C 2^{i/2} 1\right)\right] \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}$$

$$= C_5 j^{3\delta-2} v \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \tag{3.82}$$

for suitable constants $C_4, C_5 > 0$. Using the estimates (3.77) and (3.82) for the two summands in (3.73), we conclude

$$\begin{aligned} &\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) | \mathcal{F}_n] \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\ &\leq \left[o(v) + v \sum_{j > v^{1/3}} C_5 j^{3\delta-2}\right] \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\ &= v o(v) \cdot \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}, \end{aligned}$$

and therefore there exists $v_0 \in \mathbb{N}$ such that

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \mid \mathcal{F}_n] \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \leq v \mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}$$

for $v \geq v_0$.

For the general case, in which we can also have type-2 particles at time η_n, we notice that for

$$u \geq (1+c) v_0 \tag{3.83}$$

we have

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \mid \mathcal{F}_n] \mathbb{1}_{A_2} \leq \left[\xi_1(n) + \xi_2(n)\right] \mathbb{1}_{A_2}$$

since on A_2 the type-2 particles which exist at time η_n evolve independently of the LP until time η_{n+1}. □

Now we fix $u \in \mathbb{N}$ such that Lemma 3.5.7 and Lemma 3.5.8 hold. Further, we define

$$\tau := \inf\{n \in \mathbb{N}_0 : \xi_1(n) + \xi_2(n) \leq u\}.$$

Due to Lemma 3.5.7 and, respectively, Lemma 3.5.8, we see that in the subcritical (i.e. $p_c m_c < 1$) as well as in the critical (i.e. $p_c m_c = 1$) case the process

$$\left(\xi_1(n \wedge \tau) + \xi_2(n \wedge \tau)\right)_{n \in \mathbb{N}_0}$$

is a non-negative supermartingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ and $\mathsf{P}_{(a,b)}$ for arbitrary $(a,b) \in \mathcal{S}$. Thus, it converges $\mathsf{P}_{(a,b)}$-a.s. to a finite random variable $\mathcal{X}(a,b)$. Moreover, the process is integer-valued and the probability for it to eventually stay at a constant level $v > u$ for all times is equal to 0. Hence, we conclude that $\mathcal{X}(a,b) \leq u$ holds $\mathsf{P}_{(a,b)}$-a.s.

Therefore, for all $(a,b) \in \mathcal{S}$ we have $\mathsf{P}_{(a,b)}\left(\exists n \in \mathbb{N}_0 : \xi_1(n) + \xi_2(n) \le u\right) = 1$, and hence

$$\mathsf{P}_{(a,b)}\left(\exists n \in \mathbb{N}_0 : \zeta_1(n) + \zeta_2(n) \le u\right) = 1. \tag{3.84}$$

We now introduce the random times

$$\begin{aligned} \sigma_i &:= \inf\{n > \tau_i : \ l(n) = l(\tau_i) + 2\}, && \text{for } i \ge 0, \\ \tau_i &:= \inf\{n \ge \sigma_{i-1} : \ \zeta_1(n) + \zeta_2(n) \le u\}, && \text{for } i \ge 1, \end{aligned}$$

with $\tau_0 := 0$. Here σ_i denotes the first time at which two more cookies have been eaten since τ_i. Moreover, we observe that $\big(Y(n)\big)_{n\in\mathbb{N}_0} := \big(\big(Z_n(x)\big)_{x\in\mathbb{Z}}, l(n)\big)_{n\in\mathbb{N}_0}$ is a Markov chain with values in $\mathcal{S}$, which can only reach finitely (thus countably) many states within finite time. Therefore, (3.84) yields for $i \in \mathbb{N}_0$

$$\mathsf{P}_{(e_0,0)}\big(\tau_{i+1} < \infty \mid \sigma_i < \infty\big) = 1 \tag{3.85}$$

where $(e_0, 0)$ denotes the usual starting configuration with one particle and the leftmost cookie at position 0. Finally, we have

$$\mathsf{P}_{(e_0,0)}\big(\sigma_i = \infty \mid \tau_i < \infty\big) \ge \big(q_c \mathsf{P}(\Lambda_1^+ = 0)\big)^{Mu} =: \gamma \in (0,1). \tag{3.86}$$

This inequality holds since at the first time after τ_i, at which any particle reaches the leftmost cookie again, there are not more than u type-1 particles. Each of those type-1 particles cannot produce more than M particles in the next step. Afterwards, the probability for any direct offspring of the type-1 particles to move to the left and then produce offspring which escape to $-\infty$ is given by $q_c \mathsf{P}(\Lambda_1^+ = 0)$. All the remaining type-2 particles escape to the left with probability $\mathsf{P}(\Lambda_1^+ = 0)$ since one more cookie has been eaten. In this case, only one more cookie is consumed after the random time τ_i implying $\sigma_i = \infty$.

Using (3.85) and (3.86) we can conclude

$$\begin{aligned} &\mathsf{P}_{(e_0,0)}\big(\sigma_i < \infty \ \forall\, i \in \mathbb{N}\big) \\ &\le \mathsf{P}_{(e_0,0)}\big(\sigma_k < \infty\big) \\ &= \mathsf{P}_{(e_0,0)}\big(\sigma_0 < \infty\big) \prod_{i=1}^{k} \mathsf{P}_{(e_0,0)}\big(\sigma_i < \infty \mid \tau_i < \infty\big) \mathsf{P}_{(e_0,0)}\big(\tau_i < \infty \mid \sigma_{i-1} < \infty\big) \\ &\le \big(1-\gamma\big)^k \xrightarrow[k\to\infty]{} 0. \end{aligned}$$

In particular, this implies that a.s. only finitely many cookies are consumed and this yields that the CBRW is transient. ■

3.6 Final remarks

At the end of this chapter, let us consider a CBRW with one cookie at every position $x \in \mathbb{Z}$, i.e. $c_0(x) := 1$ for all $x \in \mathbb{Z}$. In this case the leftmost cookie on the positive semi-axis

$$l(n) = \min\{x \geq 0 : c_n(x) = 1\}$$

and the rightmost cookie on the negative semi-axis

$$r(n) := \max\{x \leq 0 : c_n(x) = 1\}$$

are of interest. With the help of these two definitions we are able to introduce the right LP $\mathcal{L}^+(n) := Z_n(l(n))$ as well as the left LP $\mathcal{L}^-(n) := Z_n(r(n))$. Using the symmetry of the CBRW with regard to the origin, the following results can be derived from Theorems 3.3.1, 3.3.2, and 3.3.3:

Theorem 3.6.1. *We suppose that the BRW without cookies is transient to the right.*

(a) If the right LP is supercritical, i.e. $p_c m_c > 1$ holds, then

  (i) the CBRW is strongly recurrent iff $p_c m_c \varphi_\ell \geq 1$;

  (ii) the CBRW is weakly recurrent iff $p_c m_c \varphi_\ell < 1$ and $q_c m_c > 1$;

  (iii) the CBRW is transient to the right iff $p_c m_c \varphi_\ell < 1$ and $q_c m_c \leq 1$.

(b) If the right LP is subcritical or critical, i.e. $p_c m_c \leq 1$ holds, then

  (i) the CBRW is weakly recurrent iff the left LP is supercritical, i.e. $q_c m_c > 1$;

  (ii) the CBRW is transient to the right iff the left LP is subcritical or critical, i.e. $q_c m_c \leq 1$.

Theorem 3.6.2. *We suppose that the BRW without cookies is strongly recurrent. Then the CBRW is strongly recurrent, no matter which kinds of right and left LP we have.*

Theorem 3.6.3. *We suppose that the BRW without cookies is transient to the left. Due to the symmetry of the process we get the same result as in Theorem 3.6.1 if we only replace right LP by left LP, p_c by q_c and φ_ℓ by φ_r.*

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