Optimal portfolios with bounded downside risks

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Abstract

We consider some continuous-time Markowitz type portfolio problems that consist of maximizing expected terminal wealth under the constraint of an upper bound for the Capital-at-Risk of the expected shortfall. In a Black-Scholes setting we obtain bounds for the solutions and compare numerical solutions to those of the mean-quantile problem and of the classical continuous-time mean-variance problem. We also consider portfolio problems in a Black-Scholes setting with jumps which allow only numerical solutions.

Keywords: Portfolio optimization, Black-Scholes model, downside risk measure, Capital-at-Risk, jump diffusion.

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1 Introduction

This paper is a continuation of the paper by Klüppelberg and Korn [5]. In that paper the classical mean-variance portfolio optimization introduced by Markowitz [8] and Sharpe [10] was compared with an alternative procedure based on the Captial-at-Risk replacing the variance as risk measure.

In this paper we make a systematic comparism of risk measures related to so-called *lower partial moments* as introduced by Fishburn [1]

$$LPM_k(x) = \int_{-\infty}^{x} (x-r)^k dF(r), \quad x \in \mathbb{R}, \tag{1.1}$$

for $k \in \mathbb{N}$, where F is the distribution function of the portfolio return, and the corresponding optimal portfolios. Examples are the shortfall probability ($k=0$), the expected shortfall ($k=1$), the target semi-variance ($k=2$) and the target semi-skewness ($k=3$).

2 The Black-Scholes market

In this section, we consider a standard Black-Scholes type market consisting of one *riskless bond* and several *risky stocks*. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$ for $i=1,\ldots,d$ evolve according to the equations

$$dP_0(t) = P_0(t) r dt\,, \quad P_0(0) = 1\,, \tag{2.2}$$

$$dP_i(t) = P_i(t)\left(b_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t)\right), \quad P_i(0) = p_i\,, \quad i = 1,\ldots,d\,. \tag{2.3}$$

Here $W(t) = (W_1(t),\ldots,W_d(t))'$ is a standard d-dimensional Brownian motion, $r \in \mathbb{R}$ is the *riskless interest rate*, $b = (b_1,\ldots,b_d)'$ the vector of *stock-appreciation rates* and $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ is the matrix of *stock-volatilities*. For simplicity, we assume that σ is regular.

Let $\pi(t) = (\pi_1(t),\ldots,\pi_d(t))' \in \mathbb{R}^d$ be an admissible portfolio process, i.e. $\pi_i(t)$ is the fraction of the wealth $X^\pi(t)$, which is invested in asset i (see Korn [5], Section 2.1 for relevant definitions). Denote by $(X^\pi(t))_{t\geq 0}$ the *wealth process*, then it follows the dynamic

$$dX^\pi(t) = X^\pi(t)\left\{((1-\pi(t)'\underline{1})r + \pi(t)'b)dt + \pi(t)'\sigma dW(t)\right\}\,, \quad X^\pi(0) = x\,, \tag{2.4}$$

where $x \in \mathbb{R}$ denotes the initial wealth of the investor and $\underline{1} = (1,\ldots,1)'$ denotes the vector (of appropriate dimension) with unit components. The fraction of the investment into the bond is $\pi_0(t) = 1 - \pi(t)'\underline{1}$.

Let T be a fixed time horizon. Throughout the paper, we restrict ourselves to constant portfolios $\pi(t) = \pi = (\pi_1, \ldots, \pi_d)$ for all $t \in [0, T]$. This means that the fractions into different stocks and the bond remain constant in $[0, T]$ and the positions in the portfolio have to be constantly adapted to the different dynamics of the price processes. This restriction allows us to derive explicit formulae for the wealth process and its moments.

$$X^\pi(t) = x \exp\{(\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)t + \pi'\sigma W(t)\}, \tag{2.5}$$

$$EX^\pi(t) = x \exp\{(\pi'(b - r\underline{1}) + r)t\}, \tag{2.6}$$

$$\mathrm{var}X^\pi(t) = x^2 \exp\{2(\pi'(b - r\underline{1}) + r)t\}(\exp\{\|\pi'\sigma\|^2 t\} - 1). \tag{2.7}$$

The norm $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Lower partial moments as in (1.1) describe the downside risk of a portfolio, where the concept has to be adapted to our situation and the benchmark has to be chosen appropriately. In [5] we considered the lower partial moment of order 0, more precisely, a low quantile of the terminal wealth $X^\pi(T)$ to define the risk of a portfolio by its Capital-at-Risk (CaR). In this paper we shall also consider lower partial moments of higher order. We start with the risk measures to be investigated in this paper ((a) has been derived in [5]).

Definition 2.1 (Risk measures)

For a portfolio π, initial capital x and time horizon T we define the following risk measures.

(a) The *α-quantile* of $X^\pi(T)$:

$$\rho_0(x, \pi, T) = x \exp\{(\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)T + z_\alpha \|\pi'\sigma\| \sqrt{T}\},$$

i.e. $P(X^\pi(T) \le \rho_0(x, \pi, T)) = \alpha$, where z_α is the α-quantile of the standard normal distribution, i.e. $\Phi(z_\alpha) = \alpha$.

(b) The *expected shortfall* of $X^\pi(T)$:

$$\rho_1(x, \pi, T) = E(X^\pi(T) | X^\pi(T) \le \rho_0(x, \pi, T)).$$

(c) The *semi-standard deviation* of $X^\pi(T)$:

$$\rho_2(x, \pi, T) = \sqrt{E((X^\pi(T))^2 | X^\pi(T) \le \rho_0(x, \pi, T))}. \qquad \square$$

Next we define the Capital-at-Risk (CaR) with respect to the different risk measures ρ_0, ρ_1, ρ_2 as their difference to the pure bond strategy. This is different to some authors who take the difference to the mean terminal wealth $EX^\pi(T)$ of exactly this portfolio, a quantity which is called *Earnings at Risk*. Our definition has the advantage that different portfolios can be compared with respect to their market risks.

Definition 2.2 (Capital-at-Risk)
For $k = 0, 1, 2$ we define the difference between the terminal wealth of the pure bond strategy and the risk measure ρ_k of $X^\pi(T)$ as the *Capital-at-Risk (CaR) of the portfolio π with respect to ρ_k (with initial capital x and time horizon T)*. It is given by

$$\widetilde{\rho}_k(x, \pi, T) = xe^{rT} - \rho_k(x, \pi, T) . \qquad \square$$

Next we calculate the risk measures and their corresponding CaR explicitly.

Proposition 2.3 *Let $(X^\pi(t))$ be the wealth process of a portfolio π in the Black-Scholes market and $\rho_0 = \rho_0(x, \pi, t)$ be defined as in Definition 2.1. Denote by φ the density and by Φ the distribution function of a standard normal random variable $N(0,1)$. Let T be a fixed time horizon. Set*

$$\alpha^* = \Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T}) \quad \text{and} \quad \alpha^{**} = \Phi(z_\alpha - 2\|\pi'\sigma\|\sqrt{T}) . \tag{2.8}$$

and

$$a(x, \pi, T) = x \exp\{(\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)T\} .$$

Then

$$\alpha^{**} < \alpha^* < \alpha \tag{2.9}$$

and

$$\rho_1(x, \pi, T) = a(x, \pi, T)\frac{\alpha^*}{\alpha} \exp\left\{\frac{\|\pi'\sigma\|^2}{2}T\right\} , \tag{2.10}$$

$$\rho_2(x, \pi, T) = a(x, \pi, T)\sqrt{\frac{\alpha^{**}}{\alpha}} \exp\{\|\pi'\sigma\|^2 T\}. \tag{2.11}$$

Proof Recall the following identity in law

$$\frac{\pi'\sigma}{\|\pi'\sigma\|}\frac{W(t)}{\sqrt{t}} \stackrel{d}{=} N(0,1), \quad t > 0 , \tag{2.12}$$

which implies

$$\begin{aligned} X^\pi(T) &= a(x, \pi, T) \exp\left\{\pi'\sigma W(T)\right\} \\ &\stackrel{d}{=} a(x, \pi, T) \exp\{N(0,1)\|\pi'\sigma\|\sqrt{T}\} . \end{aligned} \tag{2.13}$$

Furthermore, by definition, $P(X^\pi(T) \le \rho_0) = P(N(0,1) \le z_\alpha) = \alpha$. Hence, for the shortfall we obtain

$$\begin{aligned}\rho_1(x,\pi,T) &= \frac{E(X^\pi(T) I(X^\pi(T) \le \rho_0(x,\pi,T)))}{P(X^\pi(T) \le \rho_0(x,\pi,T))} \\ &= \frac{a(x,\pi,T)}{\alpha} \int_{-\infty}^{z_\alpha} \exp\{x\|\pi'\sigma\|\sqrt{T}\}\varphi(x)dx,\end{aligned}$$

where $I(A)$ is the indicator function of the set A. We calculate the integral by change of variables and obtain:

$$\rho_1(x,\pi,T) = \frac{a(x,\pi,T)}{\alpha} \exp\{\|\pi'\sigma\|^2 T/2\}\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T}).$$

For the semi-standard deviation we obtain

$$\begin{aligned}\rho_2(x,\pi,T) &= \sqrt{\frac{E((X^\pi(T))^2 I(X^\pi(T) \le \rho_0(x,\pi,T)))}{P(X^\pi(T) \le \rho_0(x,\pi,T))}} \\ &= \sqrt{\frac{a^2(x,\pi,T)}{\alpha} \int_{-\infty}^{z_\alpha} \exp\{2x\|\pi'\sigma\|\sqrt{T}\}\varphi(x)dx} \\ &= \sqrt{\frac{a^2(x,\pi,T)}{\alpha} \exp\{2\|\pi'\sigma\|^2 T\}\Phi(z_\alpha - 2\|\pi'\sigma\|\sqrt{T})} \\ &= a(x,\pi,T)\sqrt{\frac{\alpha^{**}}{\alpha}} \exp\{\|\pi'\sigma\|^2 T\}\end{aligned}$$

□

Corollary 2.4 $\rho_1(x,\pi,T) \le \rho_2(x,\pi,T) \le \rho_0(x,\pi,T)$.

Proof

$$\rho_2(x,\pi,T)^2 = E((X^\pi(T))^2 | X^\pi(T) \le \rho_0(x,\pi,T)) \le \rho_0(x,\pi,T)^2,$$

which implies $\rho_2(x,\pi,T) \le \rho_0(x,\pi,T)$, since $\rho_0(x,\pi,T) > 0$ and $\rho_2(x,\pi,T) > 0$.

$$\begin{aligned}&\rho_2(x,\pi,T)^2 - \rho_1(x,\pi,T)^2 \\ &= E((X^\pi(T))^2 | X^\pi(T) \le \rho_0(x,\pi,T)) - (E(X^\pi(T) | X^\pi(T) \le \rho_0(x,\pi,T)))^2 \\ &= E((X^\pi(T) - E(X^\pi(T) | X^\pi(T) \le \rho_0(x,\pi,T)))^2 | X^\pi(T) \le \rho_0(x,\pi,T)) \\ &\ge 0,\end{aligned}$$

which implies $\rho_2(x,\pi,T) \ge \rho_1(x,\pi,T)$, since $\rho_1(x,\pi,T) > 0$ and $\rho_2(x,\pi,T) > 0$. □

Now we want to analyse the behaviour of $\widetilde{\rho}_1$ depending on the strategy π. Therefore it will be convenient to introduce the function

$$f(\pi) = \pi'(b - r\underline{1})T + \ln(\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})/\alpha)\,, \tag{2.14}$$

i.e. $\widetilde{\rho}_1(x, \pi, T) = xe^{rT}(1 - e^{f(\pi)})$. Notice that

$$\lim_{\|\pi'\sigma\| \to \infty} f(\pi) = -\infty\,,$$

hence the use of extremely risky strategies can lead to a risk which is close to the total capital. The same is true for the measure ρ_0 as was shown in [5].

We shall frequently use the following estimate for the standard normal distribution; see e.g. [3].

Lemma 2.5 *Let $x > 0$. Then*

$$(x^{-1} - x^{-3})(2\pi)^{-1/2}\exp\{-x^2/2\} \le 1 - \Phi(x) \le x^{-1}(2\pi)^{-1/2}\exp\{-x^2/2\}$$

and

$$\frac{x\overline{\Phi}(x)}{\varphi(x)} \to 1, \quad x \to \infty$$

Proposition 2.6 *Set $\theta = \|\sigma^{-1}(b - r\underline{1})\|, \varepsilon = \|\pi'\sigma\|$ and $\alpha^* = \Phi(z_\alpha - \varepsilon\sqrt{T})$.*
(a) If $b_i = r$ for all $i = 1, \ldots, d$, then $f(\pi)$ attains its unique maximum for $\pi^* = 0$, i.e. $\varepsilon = 0$ and $\widetilde{\rho}_1(x, 0, T) = 0$. Moreover, for arbitrary $\varepsilon > 0$ and all π with

$$\|\pi'\sigma\| = \varepsilon \tag{2.15}$$

we have

$$f(\pi) = \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha) = \ln(\alpha^*/\alpha) \tag{2.16}$$

and

$$0 < \widetilde{\rho}_1(x, \pi, T) = xe^{rT}(1 - \alpha^*/\alpha) < xe^{rT}\,.$$

(b) If $b_i \neq r$ for some $i \in \{1, \ldots, d\}$ and if $\sqrt{T} \le \frac{\varphi(z_\alpha)}{\alpha\theta}$, then $f(\pi)$ attains its unique maximum only for $\pi^* = 0$, i.e. $\varepsilon = 0$ and $\widetilde{\rho}_1(x, 0, T) = 0$.
(c) If $b_i \neq r$ for some $i \in \{1, \ldots, d\}$ and if $\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha\theta}$ and $\alpha < 0.15$, i.e. $z_\alpha < -1.1$, then $f(\pi)$ attains its unique maximum for a strategy

$$\pi^* = \varepsilon\frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}$$

such that

$$\left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ < \varepsilon < \theta + z_\alpha/\sqrt{T}\,. \tag{2.17}$$

Denote by $a \vee b = \max\{a, b\}$ *and by* $a \wedge b = \min\{a, b\}$*. Then*

$$\begin{aligned} &\left(\left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ \theta T + \ln\left(\Phi(-\frac{2}{3}\theta\sqrt{T} \wedge z_\alpha)/\alpha\right)\right) \vee \left(\left(\theta + z_\alpha/\sqrt{T}\right)\theta T + \ln\left(\Phi(-\theta\sqrt{T}))/\alpha\right)\right) \\ \leq\ & f(\pi^*) \qquad (2.18) \\ \leq\ & (\theta + z_\alpha/\sqrt{T})\theta T + \ln\left(\Phi(-\frac{2}{3}\theta\sqrt{T} \wedge z_\alpha)\sqrt{T})/\alpha\right) \qquad (2.19) \end{aligned}$$

Let $\pi^*_\varepsilon = \mathrm{argmax}_{\{\pi \in \mathbb{R}: \|\pi'\sigma\| = \varepsilon\}} f(\pi)$*.*
If $\varepsilon = 0$*, then* $f(\pi^*_0) = 0$ *and hence* $\widetilde{\rho}_1(x, 0, T) = 0$*.*
If $\varepsilon > 0$*, then*

$$\widetilde{\rho}_1(x, \pi^*_\varepsilon, T) \quad \begin{cases} > 0 & T < \dfrac{\ln(\alpha/\alpha^*)}{\varepsilon\theta} \\ < 0 & T > \dfrac{\ln(\alpha/\alpha^*)}{\varepsilon\theta} \end{cases} \tag{2.20}$$

Proof (a) If $b_i = r$ for all $i = 1, \dots, d$, then

$$f(\pi) = \ln \Phi((z_\alpha - \varepsilon\sqrt{T})/\alpha)$$

with $\varepsilon = \|\pi'\sigma\| \geq 0$. Then the maximum over all non-negative ε is attained for $\varepsilon = 0$. Due to the regularity of σ this is equivalent to π equalling zero.
(b)(c) Consider the problem of maximizing $f(\pi)$ over all π which satisfy the requirement (2.15) for a fixed positive ε. Over the (boundary of the) ellipsoid defined by (2.15) $f(\pi)$ equals

$$f(\pi) = \pi'(b - r\underline{1})T + \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha)$$

Thus the problem is just to maximise a linear function (in π) over the boundary of an ellipsoid. This problem has the explicit solution

$$\pi^*_\varepsilon = \varepsilon\frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{2.21}$$

with

$$f(\pi^*_\varepsilon) = \varepsilon\theta T + \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha)\,.$$

As every $\pi \in \mathbb{R}^d$ satisfies relation (2.15) with a suitable value of ε (due to the fact that σ is regular), we obtain the minimum strategy π^* by maximising $f(\pi^*_\varepsilon)$ over all non-negative ε. Since

$$\frac{df(\pi^*_\varepsilon)}{d\varepsilon} = \theta T - \sqrt{T}\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{\Phi(z_\alpha - \varepsilon\sqrt{T})}$$

$\frac{df(\pi_\varepsilon^*)}{d\varepsilon}(0) < 0$ if and only if $\sqrt{T} < \frac{\varphi(z_\alpha)}{\alpha\theta}$. Furthermore, using Lemma 2.5 we obtain

$$\begin{aligned}
\frac{d^2 f(\pi_\varepsilon^*)}{d\varepsilon^2} &= -\sqrt{T}\frac{\Phi(z_\alpha - \varepsilon\sqrt{T})\varphi(z_\alpha - \varepsilon\sqrt{T})(\varepsilon\sqrt{T} - z_\alpha)(-\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T})\varphi(z_\alpha - \varepsilon\sqrt{T})(-\sqrt{T})}{(\Phi(z_\alpha - \varepsilon\sqrt{T}))^2} \\
&= T\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{(\Phi(z_\alpha - \varepsilon\sqrt{T}))^2}(\Phi(z_\alpha - \varepsilon\sqrt{T})(\varepsilon\sqrt{T} - z_\alpha) - \varphi(z_\alpha - \varepsilon\sqrt{T})) \\
&\leq T\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{(\Phi(z_\alpha - \varepsilon\sqrt{T}))^2}(\varphi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T})) = 0\,. \qquad (2.22)
\end{aligned}$$

This implies that $\frac{df(\pi_\varepsilon^*)}{d\varepsilon}$ decreases in ε on $(0,\infty)$. Then the optimal ε is positive if and only if $\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha\theta}$. Thus, $\sqrt{T} \leq \frac{\varphi(z_\alpha)}{\alpha\theta}$ implies assertion (b).

Now take $\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha\theta}$. Then $\frac{df(\pi_\varepsilon^*)}{d\varepsilon}(0) > 0$ and $\frac{d^2 f(\pi_\varepsilon^*)}{d\varepsilon^2} < 0$ implies the uniqueness of an optimal ε. We shall derive bounds for this optimal ε. Notice that

$$\begin{aligned}
f \text{ increases in } \varepsilon \quad &\Leftrightarrow \quad \frac{df(\pi_\varepsilon^*)}{d\varepsilon} = \theta T - \sqrt{T}\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{\Phi(z_\alpha - \varepsilon\sqrt{T})} \geq 0 \\
&\Leftrightarrow \quad \theta\sqrt{T}\Phi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T}) \geq 0.
\end{aligned}$$

Set $\varepsilon_1 = \frac{2}{3}\theta + z_\alpha/\sqrt{T}$, then

$$\theta\sqrt{T}\Phi(z_\alpha - \varepsilon_1\sqrt{T}) - \varphi(z_\alpha - \varepsilon_1\sqrt{T}) = \theta\sqrt{T}\Phi(-\frac{2}{3}\theta\sqrt{T}) - \varphi(-\frac{2}{3}\theta\sqrt{T})\,.$$

Now define

$$P(y) = \frac{3}{2}y\Phi(-y) - \varphi(-y) = \frac{3}{2}y\overline{\Phi}(y) - \varphi(y)\,, \quad y > 0\,,$$

where we used the symmetry of the standard normal distribution. Taking the first derivative and using the fact that $\varphi'(y) = -y\varphi(y)$ we find that $P(y)$ is increasing if and only if $y\varphi(y)/\overline{\Phi}(y) < 3$. Since the hazard rate $\varphi(y)/\overline{\Phi}(y)$ of the standardnormal distribution is increasing (see e.g. [2]), $y\varphi(y)/\overline{\Phi}(y)$ is increasing in $y > 0$. Thus $P(y)$ is increasing till its unique maximum (where $3 = y\varphi(y)/\overline{\Phi}(y)$) and then always decreasing. Furthermore, by l'Hopital, $P(y)$ converges to 0 for $y \to \infty$. Therefore, if $P(y_0) \geq 0$ for some $y_0 > 0$, then $P(y) > 0$ for all $y > y_0$. But $P(y) = 0$ for $y = 1,04$. This implies that

$$P(\frac{2}{3}\theta\sqrt{T}) = \theta\sqrt{T}\Phi(-\frac{2}{3}\theta\sqrt{T}) - \varphi(-\frac{2}{3}\theta\sqrt{T}) > 0 \quad \text{for} \quad \theta\sqrt{T} > 1.5 \cdot 1,04 = 1.56\,.$$

But $\theta\sqrt{T} \geq 1.56$ is fullfilled by condition $\theta\sqrt{T} \geq \frac{\varphi(z_\alpha)}{\alpha}$ for $\alpha < 0.15$, i.e. $z_\alpha < -1.1$. This gives a lower bound ε_1^+ for the optimal ε.

If $\alpha \geq 0.15$, i.e. $z_\alpha \geq -1.1$, then $P(\frac{2}{3}\theta\sqrt{T}) < 0$ for $\theta\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha}$. Thus in this case the optimal

ε is smaller than ε_1.

Next we derive an upper bound. We know that

$$f \text{ decreases in } \varepsilon \quad \Leftrightarrow \quad \theta\sqrt{T}\Phi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T}) \le 0. \tag{2.23}$$

Since by Lemma 2.5

$$\theta\sqrt{T}\Phi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T}) \le \varphi(z_\alpha - \varepsilon\sqrt{T}) \left(\frac{\theta\sqrt{T}}{\varepsilon\sqrt{T} - z_\alpha} - 1 \right)$$

and $\varphi(z_\alpha - \varepsilon\sqrt{T}) > 0$, f decreases in ε if

$$\frac{\theta\sqrt{T}}{\varepsilon\sqrt{T} - z_\alpha} - 1 \le 0\,.$$

Thus f decreases for $\varepsilon \ge \varepsilon_2 := \theta + \dfrac{z_\alpha}{\sqrt{T}}$. Then

$$f(\pi_{\varepsilon_1}) \vee f(\pi_{\varepsilon_2}) \le f(\pi^*) \le \varepsilon_2\theta T + \ln(\Phi(z_\alpha - \varepsilon_1\sqrt{T})/\alpha)\,,$$

since

$$\max_{[\varepsilon_1,\varepsilon_2]} \varepsilon\theta T = \varepsilon_2\theta T \quad \text{ and } \quad \max_{[\varepsilon_1,\varepsilon_2]} \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha) = \ln(\Phi(z_\alpha - \varepsilon_1\sqrt{T})/\alpha)\,.$$

The estimate (2.20) for the CaR follows from the fact that $f(\pi_\varepsilon^*) < 0$ or $f(\pi_\varepsilon^*) > 0$ according as $T > \ln(\alpha/\alpha^*)/(\varepsilon\theta)$ or $T < \ln(\alpha/\alpha^*)/(\varepsilon\theta)$. □

Now we look at the problem

$$\max_{\pi\in\mathbb{R}^d} E(X^\pi(T)) \text{ subject to } \widetilde{\rho}_1 \le C. \tag{2.24}$$

Proposition 2.7 *Assume that C satisfies*

$$0 \le C \le x\exp\{rT\}.$$

If $b_i \neq r$ for some $i \in \{1,\dots,d\}$ then problem (2.24) will be solved by

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{2.25}$$

with ε^* between

$$\frac{z_\alpha - \Phi^{-1}(\alpha\exp(c - (\frac{z_\alpha}{\sqrt{T}} + \frac{2}{3}\theta)^+\theta T))}{\sqrt{T}} \vee \left(\frac{2}{3}\theta + \frac{z_\alpha}{\sqrt{T}}\right)^+ \tag{2.26}$$

and

$$\theta + \frac{z_\alpha}{\sqrt{T}} + \sqrt{(\theta + \frac{z_\alpha}{\sqrt{T}})^2 - \frac{1}{T}(z_\alpha^2 + 2c + 2\ln(\theta\sqrt{2\pi T}\alpha))}, \tag{2.27}$$

where $\theta = \|\sigma^{-1}(b - r\underline{1})\|$ *and* $c = \ln(1 - \frac{C}{x}e^{-rT})$.

The corresponding maximal expected terminal wealth under the $\widetilde{\rho}_1$ *constraint (2.24) equals*

$$E(X^\pi(T)) = x\exp\{(r + \varepsilon^*\|\sigma^{-1}(b - r\underline{1})\|)T\} \tag{2.28}$$

Proof Every admissible π for problem (2.24) with $\|\pi'\sigma\| = \varepsilon$ satisfies the relation

$$\tilde{\rho}_1(x, \pi, T) = xe^{rT}(1 - e^{f(\pi)}) \leq C \tag{2.29}$$

which is equivalent to

$$f(\pi) \geq c$$

with $c = \ln\left(1 - \frac{C}{x}\exp(-rT)\right)$. On the set of portfolios given by $\|\pi'\sigma\| = \varepsilon$ the linear function $(b - r\underline{1})'\pi T$ is maximised by

$$\pi_\varepsilon = \varepsilon\frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}. \tag{2.30}$$

Hence, if there is an admissible π for problem (2.24) with $\|\pi'\sigma\| = \varepsilon$ then π_ε must also be admissible. Further, due to the explicit form (2.6) of the expected terminal wealth, π_ε also maximizes the expected terminal wealth over the ellipsoid. Consequently, to obtain an optimal π for problem (2.24) it is enough to consider all vectors of the form π_ε for all positive ε such that requirement (2.29) is satisfied. Inserting (2.30) into the left-hand side of inequality (2.29) results in

$$(b - r\underline{1})'\pi_\varepsilon T = \varepsilon\|\sigma^{-1}(b - r\underline{1})\|T \tag{2.31}$$

which is an increasing linear function in ε equalling zero in $\varepsilon = 0$. Therefore, we obtain the solution of problem (2.24) by determining the biggest positive ε such that (2.29) is still valid. We shall derive bounds for this optimal ε.

Notice that for $\pi = \pi_\varepsilon$ by (2.31)

$$(2.29) \Leftrightarrow f(\pi_\varepsilon^*) = \varepsilon\theta T + \ln\left(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha\right) \geq c\,.$$

Since $c < \max_{\varepsilon > o} f(\pi_\varepsilon^*)$, by (2.17) we have

$$\varepsilon > \mathrm{argmax}_{\varepsilon > 0} f(\pi_\varepsilon^*) > \left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+.$$

By (2.17) $f(\pi^*_\varepsilon) \geq c$ is fulfilled, when

$$\left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ \theta T + \ln\left(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha\right) \geq c\,.$$

But this is equivalent to

$$\varepsilon \leq \left(z_\alpha - \Phi^{-1}\left(\alpha \exp\left(c - \left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ \theta T\right)\right)\right)/\sqrt{T}.$$

Thus $f(\pi^*_\varepsilon) \geq c$ holds for all ε with

$$\mathrm{argmax}_{\varepsilon>0} f(\pi^*_\varepsilon) < \varepsilon \leq \left(z_\alpha - \Phi^{-1}\left(\alpha \exp\left(c - \left(z_\alpha/\sqrt{T} + \frac{2}{3}\theta\right)^+ \theta T\right)\right)\right)/\sqrt{T}.$$

In (2.22) we have shown that $f(\pi_\varepsilon)$ is increasing till its unique maximum and then decreasing. Hence we have to determine an $\varepsilon > (z_\alpha - \Phi^{-1}(\alpha \exp(c - (z_\alpha/\sqrt{T} + \frac{2}{3}\theta)^+\theta T)))/\sqrt{T}$ as small as possible such that $f(\pi^*_\varepsilon) < c$ to find an upper bound for the optimal ε.
Since $\varepsilon\theta T + \ln\left(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha\right)$ is decreasing for all ε greater than the optimal ε, we know that

$$\Phi(z_\alpha - \varepsilon\sqrt{T}) \leq \varphi(z_\alpha - \varepsilon\sqrt{T})/(\theta\sqrt{T})$$

by (2.23). Notice that

$$f(\pi^*_\varepsilon) < c \Leftrightarrow e^{\varepsilon\theta T}\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha < e^c.$$

Since this implies that

$$e^{\varepsilon\theta T}\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha \leq e^{\varepsilon\theta T}\varphi(z_\alpha - \varepsilon\sqrt{T})/(\theta\sqrt{T}\alpha),$$

we need to determine an ε with

$$\exp(\varepsilon\theta T - \frac{1}{2}(z_\alpha - \varepsilon\sqrt{T})^2)/(\theta\sqrt{2\pi T}\alpha) \leq e^c.$$

But this is equivalent to

$$-\varepsilon^2 T/2 + \varepsilon(\theta T + z_\alpha\sqrt{T}) - z_\alpha^2/2 - c - \ln(\theta\sqrt{2\pi T}\alpha) \leq 0$$

This inequality is fulfilled for all

$$\varepsilon \geq \theta + z_\alpha/\sqrt{T} + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - (z_\alpha^2 + 2c + 2\ln(\theta\sqrt{2\pi T}\alpha))/T}.$$

Thus the optimal $\varepsilon < \theta + z_\alpha/\sqrt{T} + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - (z_\alpha^2 + 2c + 2\ln(\theta\sqrt{2\pi T}\alpha))/T}$. $\square$

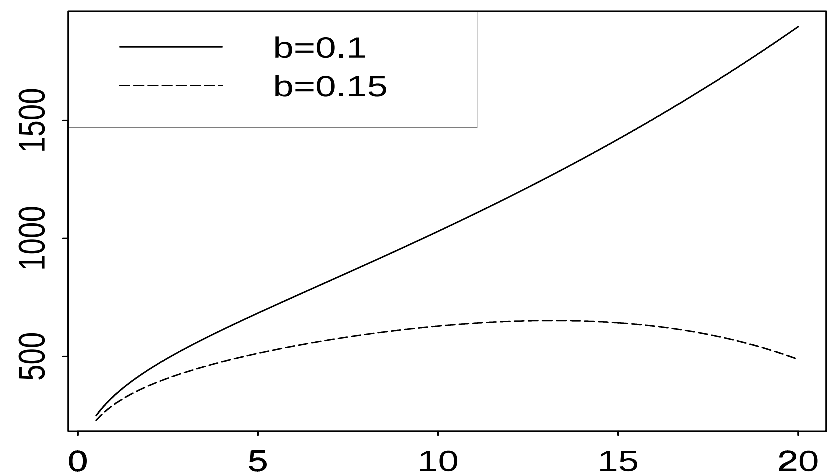

Figure 1: $\widetilde{\rho}_1(1000, 1, T)$ of the pure stock portfolio for different stock appreciation rates for $0 \le T \le 20$. The parameters are $d = 1$, $r = 0.05$, $\sigma = 0.2, \alpha = 0.05$.

Example 2.8 Figure 1 describes the dependence of $\widetilde{\rho}_1(x, \pi, T)$ on time as illustrated by $\tilde{\rho}_1(1000, 1, T)$ for $b = 0.1$ and $b = 0.15$. Note that for $b = 0.15$ the CaR first increases and then decreases with time, while for $b = 0.1$ the CaR increases with time for $T < 20$ and decreases only for very large T. The following figures illustrate the behaviour of the optimal strategy and the maximal expected terminal wealth for varying planning horizon T. In Figures 3 and 4 we have plotted the expected terminal wealth corresponding to the different strategies as functions of the planning horizon T. For a planning horizon $T < 5$ the expected terminal wealth of the optimal portfolio even exceeds the pure stock investment. The reason for this becomes clear if we look at the corresponding portfolios. The optimal portfolio always contains a short position in the bond as long as this is tolerated by the CaR constraint (see Figure 2). After 5 years the optimal portfolio contains a long position in both bond and stock for $b = 0.10$. For $b = 0.15$ the optimal portfolio contains a short position in the bond for all planning horizons. This is due to the behaviour of $\widetilde{\rho}_1$ of the stock price. For $b = 0.10$ $\widetilde{\rho}_1$ is always much larger than for $b = 0.15$ (see Figure 1). This leads to a smaller strategy for $b = 0.10$. Figure 5 shows the mean-CaR efficient frontier for the above parameters with fixed tim $T = 5$. As expected it has a similar form as a mean-variance efficient frontier.

We will now compare the behaviour of the optimal portfolios for the mean-$\tilde{\rho}_1$ problem with solutions of a corresponding mean-variance problem and with solutions of a corresponding mean-

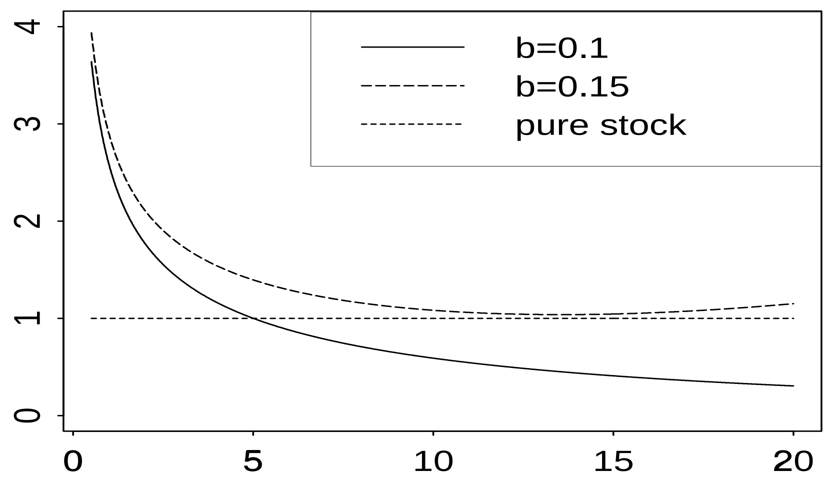

Figure 2: Optimal portfolios and pure stock portfolio for different stock appreciation rates. As upper bound of the CaR $\tilde{\rho}_1(x, \pi, T)$ we took $\tilde{\rho}_1(1000, 1, 5, b = 0.1)$, the CaR of the pure stock strategy with time horizon T=5. All other parameters are chosen as in Figure 1.

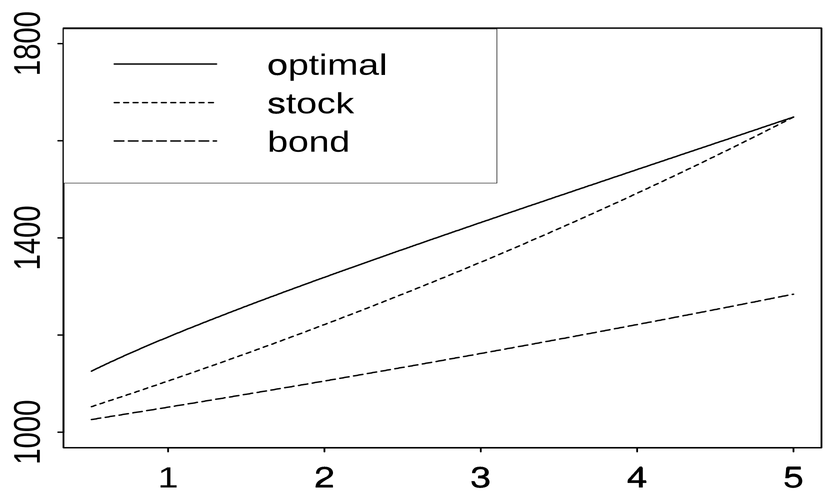

Figure 3: Expected terminal wealth of the optimal portfolio for $b = 0.1$ in comparison to the wealth of a pure bond and a pure stock portfolio depending on the time horizon T, $0 < T \leq 5$. All other parameters are chosen as in Figure 2.

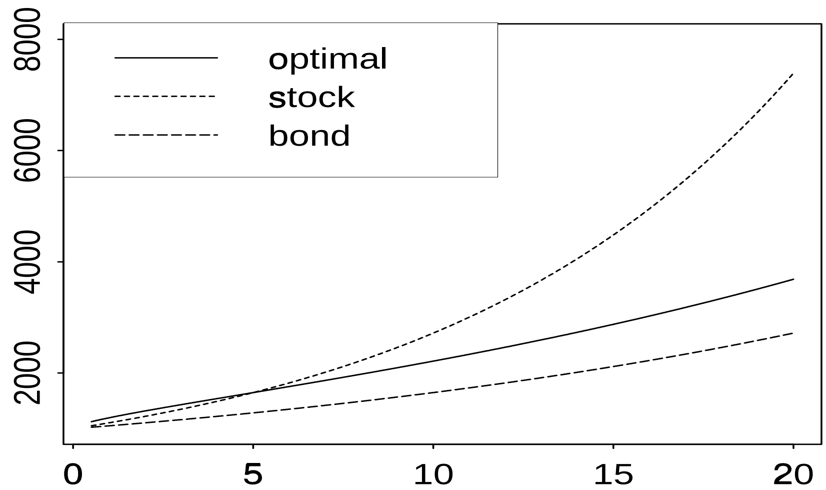

Figure 4: Expected terminal wealth of the optimal portfolio for $b = 0.1$ in comparison to the wealth of a pure bond and a pure stock portfolio depending on the time horizon T, $0 < T \leq 20$. All other parameters are chosen as in Figure 2.

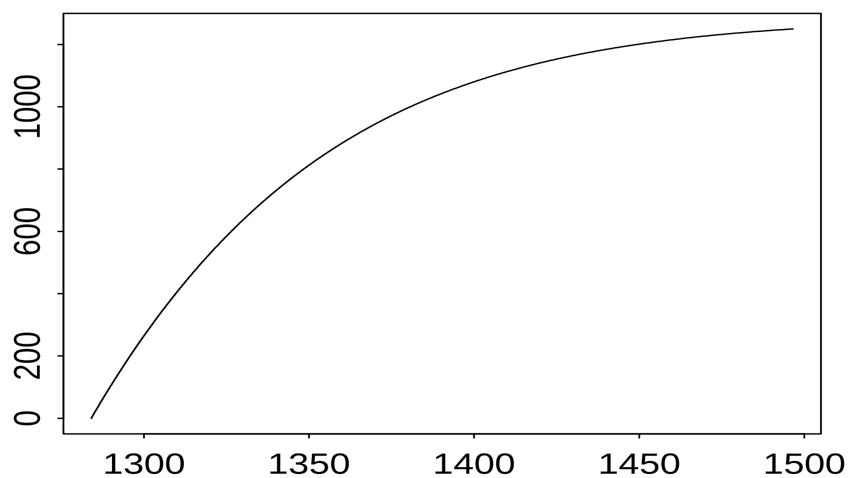

Figure 5: Mean-$\tilde{\rho}_1$ efficient frontier. The parameters are the same as in Figure 3.

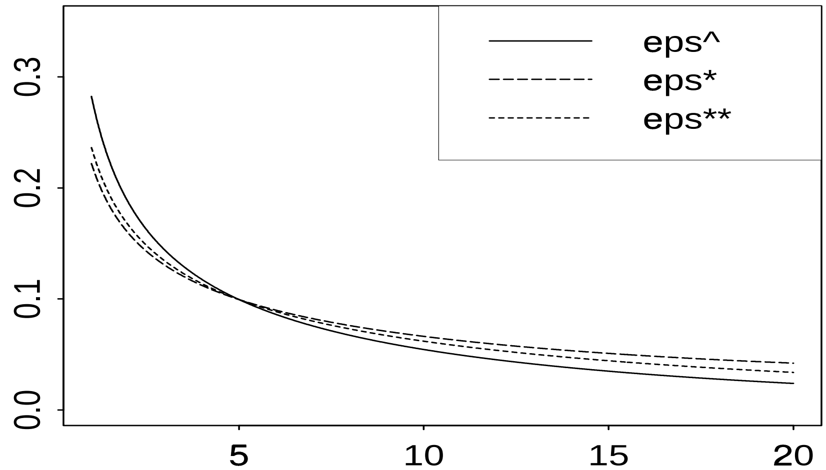

Figure 6: $\hat{\varepsilon}$, ε^* and ε^{**} as functions of the time horizon for $0 \leq T \leq 20$ and $\widehat{C} = 107100$, $C^{**} = 300$ and $C^* = 384$.

$\tilde{\rho}_0$ problem. These two corresponding problems are discussed in the paper by Klüppelberg and Korn [5].

Example 2.9 Figure 6 compares the behaviour of $\widehat{\varepsilon}$, ε^{**} and ε^* as functions of the time horizon, where $\hat{\varepsilon}$ is the optimal ϵ for the mean-variance problem, ε^{**} for the mean-$\tilde{\rho}_0$ problem and ε^* for the mean-$\tilde{\rho}_1$ problem. We have used the same data as in the foregoing example. To make the solutions of the three problems comparable we have chosen C in such a way that $\hat{\varepsilon}$, ε^{**} and ε^* coincide for T=5, i.e. for the variance $C = 107100$, for the CaR of the quantile $C = 384$ and for the CaR of the expected shortfall $C = 300$.

3 Capital-at-Risk portfolios and more general price processes

In this section we consider again the mean-CaR problem (2.24), but drop the assumption of log-normality of the stock price process. The self-financing condition, however, will still manifest itself in the form of the wealth equation

$$\frac{dX^\pi(t)}{X^\pi(t-)} = (1 - \pi'\underline{1})\frac{dP_0(t)}{P_0(t-)} + \sum_{i=1}^{d} \pi_i \frac{dP_i(t)}{P_i(t-)}\,, \quad t > 0\,, \qquad X^\pi(0) = x\,,$$

where the P_i model the dynamic of the stock price i. Of course, the explicit form of the P_i is crucial for the computability of the expected terminal wealth $X^\pi(T)$. To concentrate on these tasks we simplify the model in assuming $d = 1$ and assume for the bond $P_0(t) = e^{rt}$, $t \geq 0$, as before and for the dynamic of the risky asset

$$\frac{dP(t)}{P(t-)} = bdt + dY(t)\,, \quad t > 0\,, \qquad P(0) = p\,, \tag{3.1}$$

where $b \in \mathbb{R}$ and Y is a semimartingale with $Y(0) = 0$. Under these assumptions the choice of the portfolio π leads to the following explicit formula of the wealth process

$$\begin{aligned} X^\pi(t) &= x \exp\{(r + \pi(b-r))t\}\mathcal{E}(\pi Y(t)) \\ &= x \exp\{(r + \pi(b-r))t\} \exp\left\{\pi Y^c(t) - \tfrac{1}{2}\langle Y^c\rangle_t \prod_{0<s<t}(1 + \pi\Delta Y(s))\right\}\,, t \geq 0\,, \end{aligned} \tag{3.2}$$

where Y^c denotes the continuous part and ΔY the jump part of the process Y (more precisely, $\Delta Y(t)$ is the height of a (possible) jump at time t). This means that the wealth process is simply a multiple of the stochastic exponential $\mathcal{E}(\pi Y)$ of πY (see Protter (1990)). Analogously to Definitions 2.1 and 2.2 we define the CaR in this more general context.

Definition 3.1 *Consider the market given by a riskless bond with price $P_0(t) = e^{rt}$, $t \geq 0$, for $r \in \mathbb{R}$ and one stock with price process P satisfying (3.1) for $b \in \mathbb{R}$ and a semimartingale Y with $Y(0) = 0$. Assume that the dynamic of the wealth process is given by (3.2).*
Let x be the initial capital and T a given time horizon. For some portfolio $\pi \in \mathbb{R}$ and the corresponding terminal wealth $X^\pi(T)$ the α-quantile of $X^\pi(T)$ is given by

$$\rho_0(x, \pi, T) = x \exp\{(\pi(b-r) + r)T + \widetilde{z}_\alpha\}\,,$$

where $\widetilde{z}_\alpha$ is the α-quantile of $\widetilde{Y}(T) = \pi Y(T)$. Then we call

$$\mathrm{CaR}(x, \pi, T) = x \exp\{rT\} - E(X^\pi(T) | X^\pi(T) \leq \rho_0(x, \pi, T)) \tag{3.3}$$

the Capital-at-Risk *of the portfolio π (with initial capital x and time horizon T).* □

One of our aims of this section is to explore the behaviour of the solutions to the mean-CaR problem (2.24) if we model the returns of the price process by processes having heavier tails than the Brownian motion. We present some specific examples in the following subsections.

3.1 The Black-Scholes model with jumps

We consider a stock price process P, where the random fluctuations are generated by both a Brownian motion and a compound jump process, i.e. we consider the model (3.1) with

$$dY(t) = \sigma dW(t) + \sum_{i=1}^{n} (\beta_i dN_i(t) - \beta_i \lambda_i dt)\,, \quad t > 0\,, \quad Y(0) = 0\,, \tag{3.4}$$

where $n \in \mathbb{N}$, and for $i = 1, \ldots, n$ the process N_i is a homogeneous Poisson process with intensity λ_i. It counts the number of jumps of height β_i of Y. In order to avoid negative stock prices we assume

$$-1 < \beta_1 < \cdots < \beta_n < \infty .$$

An application of Itô's formula results in the explicit form

$$\begin{aligned} P(t) &= p \exp\left\{\left(b - \tfrac{1}{2}\sigma^2 - \sum_{i=1}^{n} \beta_i \lambda_i\right) t + \sigma W(t) + \sum_{i=1}^{n} \sum_{j=1}^{N_i(t)} \ln(1+\beta_i)\right\} \\ &= p \exp\left\{\left(b - \tfrac{1}{2}\sigma^2 - \sum_{i=1}^{n} \beta_i \lambda_i\right\} t + \sigma W(t) + \sum_{i=1}^{n} \left(N_i(t) \ln(1+\beta_i)\right)\right\} , \; t \geq 0 . \end{aligned} \tag{3.5}$$

In order to avoid the possibility of negative wealth after an "unpleasant" jump we restrict the portfolio π as follows

$$\pi \in \begin{cases} \left[-\dfrac{1}{\beta_n}, -\dfrac{1}{\beta_1}\right) & \text{if} \quad \beta_n > 0 > \beta_1 , \\ \left(-\infty, -\dfrac{1}{\beta_1}\right] & \text{if} \quad \beta_n < 0 , \\ \left[-\dfrac{1}{\beta_n}, \infty\right) & \text{if} \quad \beta_1 > 0 . \end{cases} \tag{3.6}$$

Under these preliminary conditions we obtain explicit representations of the expected terminal wealth and the CaR corresponding to a portfolio π similar to the equations (2.6) and (2.7).

Lemma 3.2 *With a stock price given by equation (3.5) let $(X^\pi(t))_{t\geq 0}$ be the wealth process corresponding to the portfolio π satisfying (3.6). Let $\rho_0(x,\pi,T)$ be the α-quantile of $X^\pi(T)$. Set*

$$B(x,\pi,T) = \exp\{(\pi(b-r) - \sum_{i=1}^{n} \pi\beta_i\lambda_i)T\}.$$

Then we have for some finite time horizon T:

$$E(X^\pi(T)) = \exp\{(r + \pi(b-r))T\} \tag{3.7}$$

and

$$\begin{aligned} \text{CaR } (x,\pi,T) &= xe^{rT} - E(X^\pi(T) | X^\pi(T) \leq \rho_0(x,\pi,T)) \\ &= xe^{rT} \left(1 - \frac{B(x,\pi,T)}{\alpha} \sum_{n_1,\ldots,n_n=0}^{\infty} \exp\left\{\sum_{i=1}^{n} \ln(1+\pi\beta_i)n_i - \lambda_i T\right\}\right. \\ &\quad \left. \times \prod_{i=1}^{n} \frac{(\lambda_i T)^{n_i}}{n_i!} \Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}(\tilde{z}_\alpha - \sum_{i=1}^{n} \ln(1+\pi\beta_i)n_i - |\pi\sigma|^2 T)\right)\right) . \end{aligned} \tag{3.8}$$

Here, $\tilde{z}_\alpha$ is the α-quantile of

$$\pi\sigma W(T) + \sum_{i=1}^{n} \ln(1+\pi\beta_i) N_i(T),$$

i.e. the real number $\tilde{z}_\alpha$ satisfying

$$\begin{aligned}
\alpha &= P\left(\pi\sigma W(T) + \sum_{i=1}^{n}(\ln(1+\pi\beta_i)N_i(T)) \le \tilde{z}_\alpha\right) \\
&= \sum_{n_1,\dots,n_n=0}^{\infty} \left(\Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}\left(\tilde{z}_\alpha - \sum_{i=1}^{n}(\ln(1+\pi\beta_i)n_i)\right)\right) e^{-T\sum_{i=1}^{n}\lambda_i} \prod_{i=1}^{n} \frac{(\lambda_i T)^{n_i}}{n_i!}\right) \quad (3.9)
\end{aligned}$$

Proof To obtain the expected value simply note that the two processes

$$\left(\exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W(t)\right\}\right)_{t\ge 0} \quad \text{and} \quad \left(\exp\left\{-\sum_{i=1}^{n}\beta_i\lambda_i t + \sum_{i=1}^{n}\sum_{j=1}^{N_i(t)} \ln(1+\beta_i)\right\}\right)_{t\ge 0}$$

are both martingales with unit expectation and they are independent. For the CaR recall (2.12). Hence for the shortfall we obtain

$$\begin{aligned}
&E(X^\pi(T) | X^\pi(T) \le \rho_0(x,\pi,T)) \\
&= \frac{E(X^\pi(T) I(X^\pi(T) \le \rho_0(x,\pi,T)))}{P(X^\pi(T) \le \rho_0(x,\pi,T))} \\
&= \frac{B(x,\pi,T)}{\alpha} \exp\left\{-\frac{1}{2}\pi^2\sigma^2 T + rT\right\} \times \\
&\quad E\left(\exp\left\{\pi\sigma W(T) + \sum_{i=1}^{n}(N_i(T)\ln(1+\pi\beta_i))\right\} I\left(\pi\sigma W(T) + \sum_{i=1}^{n}(N_i(T)\ln(1+\pi\beta_i)) \le \tilde{z}_\alpha\right)\right) \\
&= \frac{B(x,\pi,T)}{\alpha} \exp\left\{-\frac{1}{2}\pi^2\sigma^2 T + rT\right\} \sum_{n_1,\dots,n_n=0}^{\infty} \prod_{i=1}^{n} \frac{(\lambda_i T)^{n_i}}{n_i!} \exp\left\{\sum_{i=1}^{n} n_i \ln(1+\pi\beta_i) - \lambda_i T\right\} \times \\
&\quad \int_{-\infty}^{\frac{1}{|\pi\sigma|\sqrt{T}}(\tilde{z}_\alpha - \sum_{i=1}^{n} n_i \ln(1+\pi\beta_i))} \exp\{\sqrt{T}|\pi\sigma| x\}\varphi(x)dx \\
&= \frac{B(x,\pi,T)}{\alpha} \exp\{rT\} \sum_{n_1,\dots,n_n=0}^{\infty} \prod_{i=1}^{n} \frac{(\lambda_i T)^{n_i}}{n_i!} \exp\left\{\sum_{i=1}^{n} n_i \ln(1+\pi\beta_i) - \lambda_i T\right\} \times \\
&\quad \Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}(\tilde{z}_\alpha - \sum_{i=1}^{n} n_i \ln(1+\pi\beta_i)) - |\pi\sigma|\sqrt{T}\right) \\
&= \frac{B(x,\pi,T)}{\alpha} \exp\{rT\} \sum_{n_1,\dots,n_n=0}^{\infty} \exp\left\{\sum_{i=1}^{n} n_i \ln(1+\pi\beta_i) - \lambda_i T\right\} \times \\
&\quad \Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}\left(\tilde{z}_\alpha - \sum_{i=1}^{n}(n_i \ln(1+\pi\beta_i))\right) - |\pi\sigma|\sqrt{T}\right) \prod_{i=1}^{n} \frac{(\lambda_i T)^{n_i}}{n_i!}
\end{aligned}$$

□

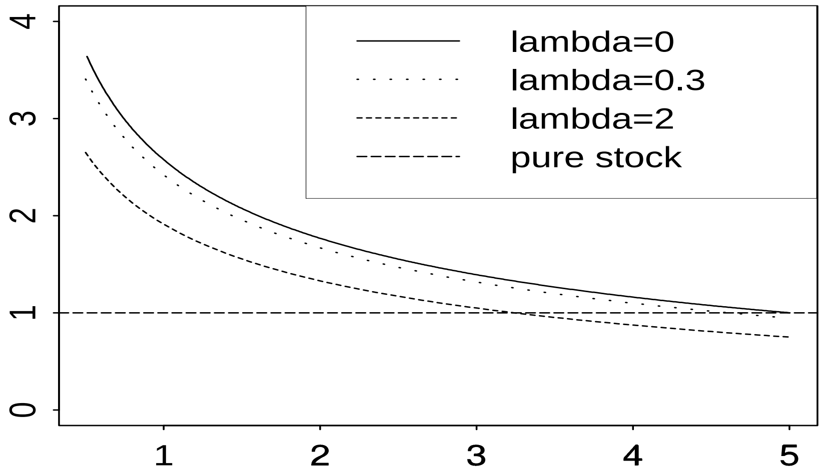

Figure 7: Optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 5$ for different jump parameters $\beta = -0.1$ and $\lambda = 0.3$ and $\lambda = 2$. The basic parameters are the same as in Figure 3.

Unfortunately, $\widetilde{z}_\alpha$ cannot be represented in such an explicit form as in the case without jumps. However, due to the explicit form of $E(X^\pi(T))$, it is obvious that the corresponding mean-CaR problem will be solved by the largest π that satisfies both the CaR constraint and requirement (3.6). Thus for an explicit example we obtain the optimal mean-CaR portfolio by a simple numerical iteration procedure. Comparisons of the solutions for the Brownian motion with and without jumps are given in Figure 7 and Figure 8.

Example 3.3 We have used the same parameters as in the examples of Section 2, but have included the possibility of a jump of height $\beta = -0.1$, occuring with intensity $\lambda = 0.3$, i.e. one would expect a jump approximately every three years, and with intensity $\lambda = 2$, i.e. one would expect a jump twice a year. An optimal portfolio for stock prices following a geometric Brownian motion with jumps is always below the optimal portfolio of the geometric Brownian motion (solid line) and the higher the intensity λ the lower is the portfolio. The reason for this is that the threat of a downwards jump of 10% leads an investor to a less risky behaviour.

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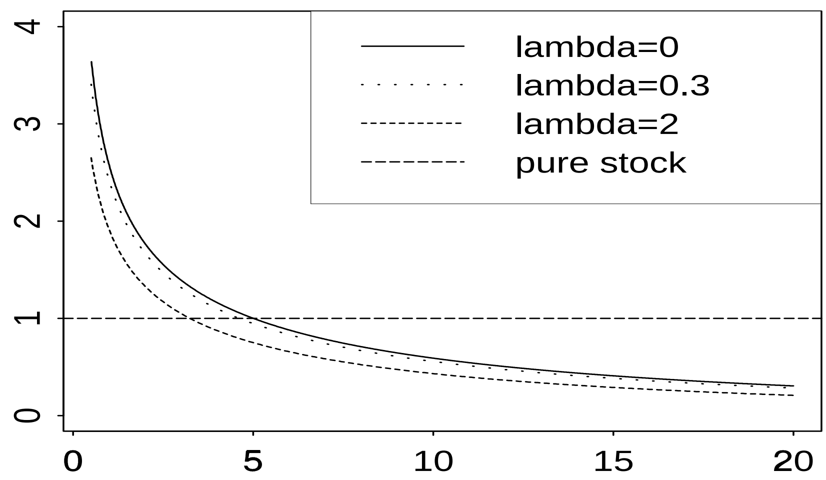

Figure 8: Optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 20$ for different jump parameters $\beta = -0.1$ and $\lambda = 0.3$ and $\lambda = 2$. The basic parameters are the same as in Figure 3.

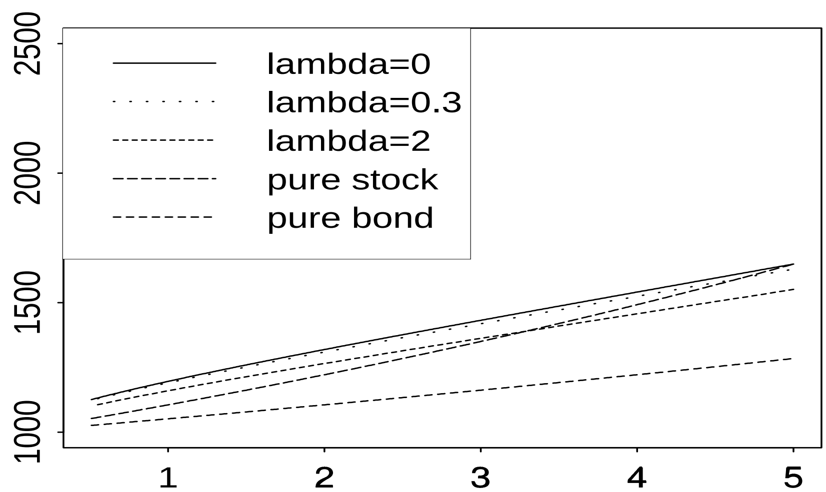

Figure 9: Expected terminal wealth corresponding to the optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 5$. The parameters are the same as in Figure 7.

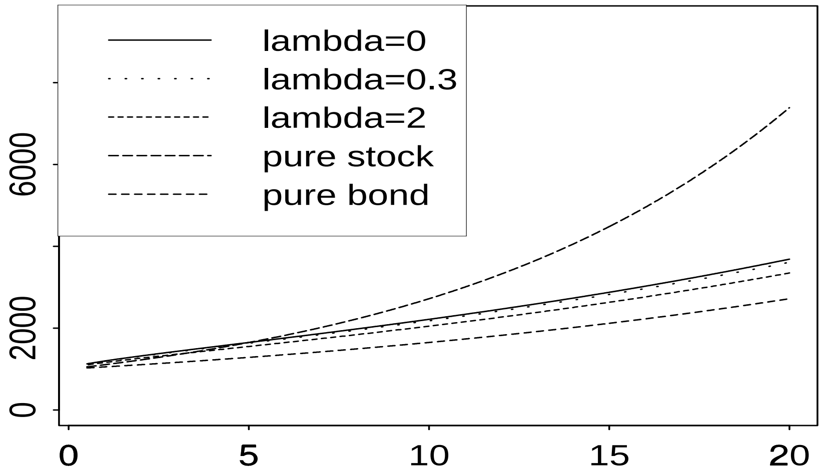

Figure 10: Expected terminal wealth corresponding to the optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 20$. The parameters are the same as in Figure 7.

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