High-frequency sampling and kernel estimation for continuous-time moving average processes

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Abstract

Interest in continuous-time processes has increased rapidly in recent years, largely because of the high-frequency data available in many applications, particularly in finance and turbulence. We develop a method for estimating the kernel function of a stationary Lévy-driven continuous-time moving average (CMA) process Y based on observations of the discrete-time process Y^Δ obtained by sampling Y on a discrete grid. We define a non-parametric estimator of g^Δ, based on the Wold representation of Y^Δ, and show that for every CARMA(p,q) process g^Δ converges pointwise to the kernel g of the process Y as $\Delta \downarrow 0$. Since the estimator is non-parametric, it is well-defined for all CMA processes with absolutely continuous spectral distribution function. We illustrate its performance by applying it to simulated high-frequency data from a CARMA process and a process with non-rational spectral density, and then to the Brookhaven turbulent wind speed data. In the course of the analysis we extend the results of Brockwell et al. (2012) to a wide class of CMA processes with strictly positive spectral densities, and we derive higher order asymptotic approximations, which are valid for CARMA processes of all orders.

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1 Introduction

We are concerned in this paper with causal continuous-time moving averages of the form

$$Y_t := \int_{-\infty}^{\infty} g(t-s)dL_s, \quad t \in \mathbb{R}, \tag{1.1}$$

where $\{L_t\}_{t\in\mathbb{R}}$ is a Lévy process with $EL_1 = 0$ and $EL_1^2 = \sigma^2 < \infty$. The kernel function g is assumed to be zero on $(-\infty, 0]$ (for causality) and square integrable, but it may be singular at the origin as in Example 4.8. The process Y defined by (1.1) is then a zero-mean strictly and weakly stationary process. For the estimation of g discussed in Sections 5 and 6 we make the additional assumption that L has been standardized so that $EL_1^2 = 1$, since otherwise g and σ^2 are confounded.

The integral in (1.1) is understood in the L^2-sense and, since we use only second-order properties in our analysis, the results apply more generally to processes defined by (1.1) with L a stationary orthogonal increment process with $EL_1 = 0$ and $EL_1^2 = \sigma^2$ as in Doob (1990) Ch. IX. It is important to note however that, when L is a given Lévy process, Y is completely characterized by g, while the spectral density of Y characterizes only the second-order properties. Throughout this paper, stationarity will always mean weak stationarity.

Examples of CMA processes are the Ornstein-Uhlenbeck process, with $g(t) = e^{\lambda t}\mathbf{1}_{(0,\infty)}$, where $\lambda < 0$, and the more general continuous-time autoregressive moving average (CARMA) processes studied by Doob (1944) for Gaussian L. State-space representations of these processes were exploited by Jones (1981) and Jones and Ackerson (1990) for dealing with missing values in time series, and by Brockwell (2001) for the study of Lévy-driven CARMA processes. Long-memory versions have been developed by Brockwell and Marquardt (2005) and Marquardt (2006). CMA processes constitute a very large class of continuous-time stationary processes (cf. Doob (1990), Theorem 5.3, Ch. XII and Yaglom (2005), Section 26.2). A gamma kernel function g corresponds to a process with Whittle-Matérn autocovariance function and spectral density of similar form to the von Kármán and Kaimal spectral densities, which have been widely employed in turbulence modelling. The extremal properties of CMA processes have been studied in Fasen (2009).

The work of the present paper was originally motivated by a study of the Brookhaven turbulence data (see Ferrazzano (2010) for a detailed description). The data consists of twenty million values, sampled at 5000Hz (i.e. 5000 values recorded per second over a time interval of 4000 seconds). In order to understand the relationship between such high-frequency data and the underlying continuous-time process Brockwell et al. (2012) studied the asymptotic behaviour of the sampled process $Y^\Delta = (Y_{n\Delta})_{n\in\mathbb{Z}}$ as $\Delta \downarrow 0$ when Y is a CARMA(p, q) process with $p - q \leq 3$.

In Section 2 of this paper we derive higher-order asymptotics for Y^Δ which apply (unlike those of Brockwell et al. (2012) to *all* CARMA(p, q) processes and we use the results of Section 2 in Section 3 to establish the pointwise convergence of a family of functions g^Δ, defined in terms of the Wold representation of Y^Δ, to g as $\Delta \downarrow 0$. In Sections 5 and 6, we use non-parametric estimators of g^Δ to estimate g based on observations of Y^Δ with Δ small and illustrate the performance of the estimators using both simulated data and the Brookhaven series described in the preceding paragraph. The outcome of a detailed statistical analysis for turbulence data is presented in Ferrazzano and Klüppelberg (2012).

In Section 4 we extend the asymptotic results of Brockwell et al. (2012) in a different direction by deriving analogous results for a broader class of CMA processes with strictly positive spectral density.

We use the following notation throughout: $\Re(z)$ denotes the real part of the complex number z; B denotes the backward shift operator, $BY_n^\Delta := Y_{n-1}^\Delta$ for $n \in \mathbb{Z}$; $a(\Delta) \sim b(\Delta)$ means $\lim_{\Delta \downarrow 0} a(\Delta)/b(\Delta) = 1$. As g can have a singularity in 0, the spectral densities of Y or Y^Δ may have a singularity in 0 as well, and we may have to restrict the range of frequencies for their spectral densities to $\Omega_c := \mathbb{R}\backslash\{0\}$ and $\Omega_d := [-\pi, \pi]\backslash\{0\}$.

2 Asymptotic behaviour of Y^Δ as $\Delta \downarrow 0$

The CMA process Y defined by (1.1) has autocovariance function

$$\gamma_Y(h) = \sigma^2 \int_{-\infty}^{\infty} g(x)g(x+h)dx, \quad h \in \mathbb{R},$$

and spectral density

$$f_Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega h}\gamma(h)dh = \frac{\sigma^2}{2\pi}|\tilde{g}(\omega)|^2, \ \omega \in \Omega_c, \tag{2.1}$$

where

$$\tilde{g}(\omega) := \int_{-\infty}^{\infty} e^{i\omega x} g(x)dx.$$

The spectral density of the sampled process, $Y^\Delta := (Y_{n\Delta})_{n\in\mathbb{Z}}$ is (Bloomfield (2000), p. 196, Eq. 9.17)

$$f_\Delta(\omega) = \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} f_Y\Big(\frac{\omega + 2k\pi}{\Delta}\Big), \quad \omega \in \Omega_d. \tag{2.2}$$

For the causal, finite variance Lévy-driven CARMA(p,q) process, autoregressive polynomial $a(z)$ and moving average polynomial $b(z)$, the spectral density is

$$f_Y(\omega) = \frac{\sigma^2}{2\pi} \left|\frac{b(i\omega)}{a(i\omega)}\right|^2, \quad -\pi \leq \omega \leq \pi, \tag{2.3}$$

where $a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$, $b(z) = b_0 + b_1 z + \cdots + b_q z^{p-1}$, $p > q$, $b_q = 1$, and the zeros of $a(z)$ all have strictly negative real parts. Without loss of generality we can also assume that $a(z)$ and $b(z)$ have no common zeros (see Brockwell and Lindner (2009), Theorem 4.1). The kernel is

$$g(t) = \frac{1}{2\pi i} \int_\rho \frac{b(z)}{a(z)} e^{tz} dz \ \mathbf{1}_{(0,\infty)}(t) = \sum_\lambda \mathrm{Res}_{z=\lambda} \left(e^z \frac{b(z)}{a(z)}\right) \mathbf{1}_{(0,\infty)}(t), \tag{2.4}$$

where the integration is anticlockwise around any simple closed curve ρ in the interior of the left half of the complex plane, encircling the distinct zeroes λ of $a(z)$, and $\mathrm{Res}_{z=\lambda}(f(z))$ denotes the residue of the function f at λ. For such processes it was shown in Brockwell et al. (2012), Section 2, that the spectral density f_Δ of the sampled process Y^Δ is

$$f_\Delta(\omega) = \frac{-\sigma^2}{4\pi^2 i} \int_\rho \frac{b(z)b(-z)}{a(z)a(-z)} \frac{\sinh(\Delta z)}{\cosh(\Delta z) - \cos(\omega)} dz, \ -\pi \leq \omega \leq \pi, \tag{2.5}$$

where the integral, as in (2.4), is anticlockwise around any simple closed contour ρ in the interior of the left half of the complex plane, enclosing the zeroes of $a(z)$. It is known that the sampled process Y^Δ satisfies the ARMA equations,

$$\phi_\Delta(B) Y_n^\Delta = \theta_\Delta(B) Z_n^\Delta, \quad n \in \mathbb{Z}, \quad \{Z_n^\Delta\}_{n\in\mathbb{Z}} \sim \mathrm{WN}(0, \sigma_\Delta^2) \tag{2.6}$$

where B is the backward shift operator, $\theta_\Delta(z)$ is a polynomial of degree less than p, $(Z_n^\Delta)_{n\in\mathbb{Z}}$ is an uncorrelated sequence of zero-mean random variables with variance, which we denote by σ_Δ^2,

$$\phi_\Delta(z) = \prod_{j=1}^{p}(1 - e^{\lambda_j\Delta}z), \quad z \in \mathbb{C},$$

and $\lambda_1, \dots, \lambda_p$ are the zeroes of the polynomial $a(z)$. Since the polynomial $\phi_\Delta(z)$ is known precisely for any given CARMA process, the second-order properties of the sampled process Y^Δ for small Δ can be determined by studying the properties of the moving average term, $X_n := \theta_\Delta(B)Z_n^\Delta$ in (2.6), as $\Delta \downarrow 0$. Denoting by f_{MA} the spectral density of X, we find from (2.6) that

$$f_{MA}(\omega) = 2^p e^{-a_1\Delta} f_\Delta(\omega) \prod_{j=1}^{p}(\cosh(\lambda_j\Delta) - \cos(\omega)), \quad -\pi \le \omega \le \pi. \tag{2.7}$$

Brockwell et al. (2012) determined the leading terms in the expansions of f_Δ and f_{MA} in powers of Δ. These terms determine the local second-order behaviour of the corresponding processes. In Section 4 we extend these results to a more general class of CMA processes.

In the following section we introduce a small-Δ approximation g^Δ to the kernel g of Y based on the Wold representation of the sampled process Y^Δ. In order to show the convergence of g^Δ to g as $\Delta \downarrow 0$ for CARMA(p, q) processes, we need to consider higher order expansions of the spectral densities f_Δ and f_{MA} than were considered in Brockwell et al. (2012). We conclude this section by deriving the required expansions.

From (2.5) it follows at once that the spectral density $f_\Delta(\omega)$ is $-\sigma^2/(2\pi)$ times the sum of the residues at the singularities of the integrand in the left half-plane, or more simply $\sigma^2/(4\pi)$ times the residue of the integrand at ∞, which is much simpler to calculate. Thus,

$$f_\Delta(\omega) = \frac{\sigma^2}{4\pi}\mathrm{Res}_{z=\infty}\left[\frac{b(z)b(-z)}{a(z)a(-z)}\frac{\sinh(\Delta z)}{\cosh(\Delta z) - \cos(\omega)}\right], \quad -\pi \le \omega \le \pi.$$

The spectral density can also be expressed as a power series,

$$f_\Delta(\omega) = \frac{\sigma^2}{4\pi}\sum_{j=0}^{\infty}\sigma^2\Delta^{2j+1}r_j c_j(\omega), \quad -\pi \le \omega \le \pi, \tag{2.8}$$

where $c_k(\omega)$ is the coefficient of z^{2k+1} in

$$\sum_{k=0}^{\infty} c_k(\omega)z^{2k+1} = \frac{\sinh z}{\cosh z - \cos\omega}, \quad -\pi \le \omega \le \pi,$$

and

$$r_j := \mathrm{Res}_{z=\infty}\left[z^{2j+1}\frac{b(z)b(-z)}{a(z)a(-z)}\right],$$

i.e. the coefficient of z^{2j} in the power series expansion,

$$\sum_{j=0}^{\infty} r_j z^{2j} = (-z^2)^{p-q-1}\frac{\prod_{i=1}^{q}(1-\mu_i^2 z^2)}{\prod_{i=1}^{p}(1-\lambda_i^2 z^2)}, \tag{2.9}$$

where $a(z) = \prod_{i=1}^{p}(z - \lambda_i)$ and $b(z) = \prod_{i=1}^{q}(z - \mu_i)$. The power series (2.8) is the required expansion for f_Δ. The expansion for f_{MA} is obtained from (2.7) and (2.8) as

$$f_{MA}(\omega) = \frac{2^p\sigma^2 e^{-a_1\Delta}}{4\pi}\prod_{i=1}^{p}\Big(1-\cos\omega+\sum_{j=1}^{\infty}\frac{(\lambda_i\Delta)^{2j}}{(2j)!}\Big)\sum_{k=0}^{\infty}r_k c_k(\omega)\Delta^{2k+1}, \quad -\pi\le\omega\le\pi.$$

This can be simplified by reexpressing it in terms of $x := 1-\cos\omega$. Thus

$$f_{MA}(\omega) = \frac{2^p\sigma^2 e^{-a_1\Delta}}{4\pi}\prod_{i=1}^{p}\Big(x+\sum_{j=1}^{\infty}\frac{(\lambda_i\Delta)^{2j}}{(2j)!}\Big)\sum_{k=0}^{\infty}r_k\alpha_k(x)\Delta^{2k+1}, \tag{2.10}$$

where $\alpha_k(x)$ is the coefficient of z^{2k+1} in the expansion,

$$\sum_{k=0}^{\infty}\alpha_k(x)z^{2k+1} = \frac{\sinh z}{\cosh z - 1 + x}.$$

In particular $\alpha_0(x) = 1/x, \alpha_1(x) = (x-3)/(3!x^2)$ and $\alpha_2(x) = (x^2 - 15x + 30)/(5!x^3)$. More generally, $\alpha_k(x)$ has the form.

$$\alpha_k(x) = \frac{1}{(2k+1)!x^{k+1}}\prod_{i=1}^{k}(x-\xi_{k,i}), \tag{2.11}$$

where

$$\prod_{i=1}^{k}\xi_{k,i} = (2k+1)!\,2^{-k}, \tag{2.12}$$

and the product, when $k=0$, is defined to be 1. Since $\alpha_{p-q-1}(x)$ plays a particularly important role in what follows, we shall denote its zeroes more simply as

$$\xi_i := \xi_{p-q-1,i}, \quad i = 1,\dots,p-q-1.$$

From (2.10), with the aid of (2.9) and (2.11), we can now derive the required higher-order approximation to $f_{MA}(\omega)$. Observe first that the expression on the right of (2.10), in spite of its forbidding appearance, is in fact a polynomial in x of degree less than p. We therefore collect together the coefficients of $x^{p-1}, x^{p-2}, \dots, x^0$. This gives (using the identity (2.12) and defining $y := \Delta^2$) the asymptotic expression as $\Delta \downarrow 0$,

$$f_{MA}(\omega) = \frac{2^p\sigma^2 e^{-a_1\Delta}\Delta^{2(p-q)-1}}{4\pi}\Big[x^p r_{p-q-1}\alpha_{p-q-1}(x) + o(1) + \sum_{j=1}^{q}\rho_j x^{q-j}y^j\Big], \tag{2.13}$$

with

$$\begin{aligned}\rho_j &= (-2)^{-(p-q-1+j)}\Big[r_{p-q-1+j} - r_{p-q-2+j}\sum_{i=1}^{p}\lambda_i^2\Big] + o(1)\\ &= 2^{-(p-q-1+j)}\sum\mu_{i_1}^2\dots\mu_{i_j}^2 + o(1),\end{aligned}$$

where the second line follows from (2.9) and the sum on the second line is over all subsets of size j of the q zeroes of the polynomal $b(z)$.

Finally, replacing r_{p-q-1} in (2.13) by $(-1)^{p-q-1}$, substituting for $\alpha_{p-q-1}(x)$ from (2.11) and using the continuity of the zeroes of a polynomial as functions of its coefficients, we can rewrite (2.13) (recalling that $x := 1 - \cos\omega$ and $\xi_i, i = 1, \ldots, p-q-1$ are the zeroes of $\alpha_{p-q-1}(x)$) as

$$f_{MA}(\omega) = \frac{\Delta(-\Delta^2)^{p-q-1}2^p\sigma^2 e^{-a_1\Delta}}{[2(p-q)-1]!4\pi} \prod_{i=1}^{p-1-q} [x - \xi_i(1+o(1))] \prod_{k=1}^{q} \left[x + \frac{\mu_k^2\Delta^2}{2}(1+o(1))\right]. \quad (2.14)$$

Observe now that we can write

$$x + \frac{\mu_k^2\Delta^2}{2}(1+o(1)) = \frac{1}{2\zeta_k}(1-\zeta_k e^{-i\omega})(1-\zeta_j e^{i\omega}), \quad -\pi \le \omega \le \pi,$$

where

$$\zeta_k = 1 \pm \mu_k\Delta + o(\Delta), \quad (2.15)$$

and the sign is chosen so that $\lim_{\Delta\downarrow 0}|\zeta_k| < 1$. Similarly we can write

$$x - \xi_i(1+o(1)) = -\frac{1}{2\eta(\xi_i)}(1+\eta(\xi_i)e^{-i\omega})(1+\eta(\xi_i)e^{i\omega}),$$

where

$$\eta(\xi_i) = \xi_i - 1 \pm \sqrt{(\xi_i-1)^2 - 1} + o(1), \quad (2.16)$$

and the sign is chosen so that $\lim_{\Delta\downarrow 0}|\eta(\xi_i)| \le 1$. If the zero ξ_i of $\alpha_{p-q-1}(x)$ is such that both choices of sign cause the limit to be 1, then either choice will do provided the same choice is made for $\eta(\bar{\xi}_i)$, where $\bar{\xi}_i$ denotes the complex conjugate of ξ_i .

These factorizations allow us to give the following asymptotic representation of the moving average process $X_n = \theta_\Delta(B)Z_n^\Delta$ appearing in (2.6).

Theorem 2.1. *The moving average process $\{X_n\}_{n\in\mathbb{Z}}$ with spectral density f_{MA} has the asymptotic representation, as $\Delta \downarrow 0$,*

$$X_n = \prod_{i=1}^{p-1-q}(1+\eta(\xi_i)B)\prod_{k=1}^{q}(1-\zeta_k B)Z_n^\Delta, \quad \{Z_n^\Delta\}_{n\in\mathbb{Z}} \sim \mathrm{WN}(0,\sigma_\Delta^2), \quad (2.17)$$

where

$$\sigma_\Delta^2 = \frac{\Delta^{2(p-q)-1}e^{-a_1\Delta}\sigma^2}{[2(p-q)-1]!\prod_{i=1}^{p-q-1}\eta(\xi_i)\prod_{k=1}^{q}\zeta_k}, \quad (2.18)$$

with ζ_k and $\eta(\xi_i)$ as in (2.15) and (2.16).

Proof. The result follows at once from (2.14), (2.15) and (2.16). □

Remark 2.2. (i) The parameters $\eta(\xi_i)$ and ζ_k may be complex but the moving average operator will have real coefficients because of the existence of corresponding complex conjugate parameters in the product.

(ii) The representation in Theorem 2.1 is a substantial generalization of the one in Corollary 2 of Brockwell et al. (2012), since it is not only of higher-order in Δ, but it applies to all CARMA(p,q) processes, not only to those with $p-q \le 3$. □

3 The Wold approximation to the CARMA(p,q) kernel

In this section we introduce an approximation g^Δ to the kernel g of the process Y defined in (1.1), which depends only on the Wold representation,

$$Y_n^\Delta = \sum_{j=0}^{\infty} \psi_j^\Delta Z_{n-j}^\Delta, \quad n \in \mathbb{Z}, \quad \{Z_n^\Delta\}_{n\in\mathbb{Z}} \sim \mathrm{WN}(0, \sigma_\Delta^2), \tag{3.1}$$

of the sampled process Y^Δ. The approximation is

$$g^\Delta(x) := \sum_{j=0}^{\infty} \frac{\sigma_\Delta}{\sqrt{\Delta}} \psi_j^\Delta \mathbf{1}_{[j\Delta,(j+1)\Delta)}(x). \tag{3.2}$$

Using Theorem 2.1, we shall show that, for all CARMA(p,q) processes, as $\Delta \downarrow 0$, g^Δ converges pointwise to σg, or to g if L is standardized so that $EL_1^2 = 1$. We first illustrate the convergence in the simple case when Y is a CARMA(1,0) or stationary Ornstein-Uhlenbeck process.

Example 3.1. [The CARMA(1,0) process] This a special case of (1.1) with kernel

$$g(x) = e^{\lambda x}\mathbf{1}_{(0,\infty)}(x) \text{ where } \lambda < 0.$$

The sampled process Y^Δ is the discrete-time AR(1) process satisfying

$$Y_n^\Delta = e^{\lambda\Delta} Y_{n-1}^\Delta + Z_n^\Delta, \quad n \in \mathbb{Z},$$

where $Z^\Delta = \left\{Z_n^\Delta\right\}_{n\in\mathbb{Z}}$ is the independent and identically distributed sequence defined by

$$Z_n^\Delta = \int_{(n-1)\Delta}^{n\Delta} e^{\lambda(n\Delta - u)} dL_u, \quad n \in \mathbb{Z}.$$

In this case it is easy to write down the coefficients ψ_j^Δ and the white noise variance σ_Δ^2 in the Wold representation of Y^Δ. From well-known properties of discrete-time AR(1) processes, they are $\psi_j^\Delta = e^{j\lambda\Delta}$, $j = 0, 1, 2, \ldots$, and $\sigma_\Delta^2 = \frac{\sigma^2}{2\lambda}(e^{2\lambda\Delta} - 1)$. Substituting these values in the definition (3.2) we find that

$$g^\Delta(x) = \sum_{j=0}^{\infty} \sigma \sqrt{\frac{e^{2\lambda\Delta} - 1}{2\lambda\Delta}} e^{j\lambda\Delta} \mathbf{1}_{[j\Delta,(j+1)\Delta)}(x),$$

which converges pointwise to σg as $\Delta \downarrow 0$. □

The approximation (3.2) is well defined for all processes (1.1) and there are standard methods for estimating the coefficients and white noise variance appearing in the definition from observations of Y^Δ. Example 3.1 shows that g^Δ converges pointwise to σg for CAR(1) processes. Our aim now is to establish this convergence for all CARMA(p,q) processes. We give the proof under the assumption that the zeroes $\lambda_1, \ldots, \lambda_p$ of the autoregressive polynomial $a(z)$ all have multiplicity one. Multiple roots can be handled by supposing them to be separated and letting the separation(s) converge to zero.

The kernel (2.4) of a causal CARMA(p,q) process Y whose autoregressive roots each have multiplicity one reduces (see e.g. Brockwell and Lindner (2009)) to

$$g(x) = \sum_{j=1}^{p} \frac{b(\lambda_i)}{a'(\lambda_i)} e^{\lambda_i x} \mathbf{1}_{(0,\infty)}(x), \tag{3.3}$$

where $a(z) = \prod_{i=1}^{p}(z - \lambda_i)$ and $b(z) = \prod_{i=1}^{q}(z - \mu_i)$ are the autoregressive and moving average polynomials respectively and a' denotes the derivative of the function a. We now establish the convergence, as $\Delta \downarrow 0$, of g^Δ as defined in (3.2) to σg. Theorem 2.1 is used to determine the parameters of the Wold representation appearing in the definition of g^Δ.

Theorem 3.2. *If Y is the CARMA(p,q) process with kernel (3.3),*

(i) the Wold coefficients and white noise variance of the sampled process Y^Δ are

$$\psi_j^\Delta = \sum_{r=1}^{p} \frac{\prod_{i=1}^{p-1-q}(1+\eta(\xi_i)e^{-\lambda_r\Delta})\prod_{k=1}^{q}(1-\zeta_k e^{-\lambda_r\Delta})}{\prod_{m\neq r}(1-e^{(\lambda_m-\lambda_r)\Delta})} e^{j\lambda_r\Delta}, \tag{3.4}$$

and

$$\sigma_\Delta^2 = \frac{\Delta^{2(p-q)-1}e^{-a_1\Delta}\sigma^2}{[2(p-q)-1]!\prod_{i=1}^{p-q-1}\eta(\xi_i)\prod_{k=1}^{q}\zeta_j}, \tag{3.5}$$

with ζ_k and $\eta(\xi_i)$ as in (2.15) and (2.16) and

(ii) the approximation g^Δ defined by (3.2) with ψ_j^Δ and σ_Δ^2 as in (3.4) and (3.5) converges pointwise to σg with g as in (3.3).

Proof. (i) The expression for σ_Δ^2 was found already as part of Theorem 2.1. The coefficient ψ_j^Δ is the coefficient of z^j in the power series expansion,

$$\sum_{j=0}^{\infty}\psi_j^\Delta z^j = \frac{\prod_{i=1}^{p-1-q}(1+\eta(\xi_i)z)\prod_{k=1}^{q}(1-\zeta_k z)}{\prod_{m=1}^{p}(1-e^{\lambda_m\Delta}z)},$$

which can be seen, by partial fraction expansion, to be equal to (3.4).

(ii) The convergence of g^Δ follows by substituting for ψ_j^Δ and σ_Δ^2 from (3.4) and (3.5) into (3.2), substituting for ζ_k from (2.15), letting $\Delta \downarrow 0$ and using the identities

$$a'(\lambda_r) = \prod_{m\neq r}(\lambda_r - \lambda_m)$$

and

$$\prod_{i=1}^{p-q-1}\frac{(1+\eta(\xi_i))^2}{\eta(\xi_i)} = \prod_{i=1}^{p-q-1}\frac{\xi_i}{2} = [2(p-q)-1]!,$$

the last equality following from (2.12).

□

Remark 3.3. Although we have established the convergence of g^Δ only for CARMA processes, the non-parametric nature of g^Δ strongly suggests that the result is true for all processes defined as in (1.1). In practice we have found that estimation of σg by estimation of g^Δ with Δ small works extremely well for simulated processes with non-rational spectral densities also. □

4 Asymptotics for a class of sampled CMA processes as $\Delta \downarrow 0$

Brockwell et al. (2012) derived first-order asymptotic expressions, as $\Delta \downarrow 0$, for the spectral density f_Δ when Y is a CARMA(p,q) process with $p - q \leq 3$. Although, as pointed out in Section 2, these asymptotic expressions are not sufficiently precise to establish the convergence of g^Δ to g, they do reveal the *local second-order behaviour* of the process Y. For example, if Y is a CARMA$(p, p-1)$ process driven by a Lévy process L with $\mathrm{Var}(L_1) = \sigma^2$ then equations (15) and (19) of Brockwell et al. (2012) give, as $\Delta \downarrow 0$,

$$f_\Delta(\omega) \sim \frac{\sigma^2 \Delta}{4\pi(1-\cos\omega)}, \quad -\pi \le \omega \le \pi,$$

showing that the spectral density of the *normalized differenced* sequence $\{(Y_{n\Delta} - Y_{(n-1)\Delta})/\sqrt{\Delta}\}_{n\in\mathbb{Z}}$ converges to that of white noise with variance σ^2 as $\Delta \downarrow 0$. In other words, for any fixed positive integer k, the sequence of observations $Y_{n\Delta}/\sqrt{\Delta},\ n = 1, \dots, k$, from a second-order point of view, behaves as $\Delta \downarrow 0$ like a sequence of observations of integrated white noise with white-noise variance σ^2.

In this section we derive analogous asymptotic approximations for the spectral densities of more general CMA processes and the implications for their local second-order behaviour. Since we allow in this section for spectral densities with a singularity at zero we recall the definition of the spectral domains,

$$\Omega_d := [-\pi, \pi]\backslash\{0\} \text{ and } \Omega_c := (-\infty, \infty)\backslash\{0\}.$$

We require the CMA processes to have spectral density satisfying a weak regularity condition at infinity. To formulate this condition we first need a definition.

Definition 4.1 (Regularly varying function (cf. Bingham et al. (1987))**.** *Let f be a positive, measurable function defined on $(0, \infty)$. If there exists $\rho \in \mathbb{R}$ such that*

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho, \quad \text{for all } \lambda > 0,$$

holds, f is called a regularly varying function of index ρ at ∞. *The convergence is then automatically locally uniform in λ. We shall denote this class of functions by $\mathcal{R}_\rho(\infty)$. Furthermore we shall say that $f(\cdot) \in \mathcal{R}_\rho(0+)$ if and only if $f(1/\cdot) \in \mathcal{R}_{-\rho}(\infty)$.*

The characterization theorem for regularly varying functions (Theorem 1.4.1. in Bingham et al. (1987)) tells us that $f \in \mathcal{R}_\rho(\infty)$ if and only if $f(x) = x^\rho L(x)$, where $L \in \mathcal{R}_0(\infty)$.

Theorem 4.2. *Let Y be the* CMA *process* (1.1) *with strictly positive spectral density f_Y such that $f_Y \in \mathcal{R}_{-\alpha}(\infty)$, where $\alpha > 1$, i.e., for $L \in \mathcal{R}_0(\infty)$,*

$$f_Y(\omega) = |\omega|^{-\alpha} L(|\omega|), \quad \omega \in \Omega_c. \tag{4.1}$$

Then the following assertions hold.

(a) The spectral density of the sampled process Y^Δ has for $\Delta \downarrow 0$ the asymptotic representation

$$f_\Delta(\omega) \sim L(\Delta^{-1})\Delta^{\alpha-1}\left[|\omega|^{-\alpha} + (2\pi)^{-\alpha}\zeta\left(\alpha, 1 - \frac{\omega}{2\pi}\right) + (2\pi)^{-\alpha}\zeta\left(\alpha, 1 + \frac{\omega}{2\pi}\right)\right], \quad \omega \in \Omega_d, \tag{4.2}$$

where $\zeta(s, r)$ is the Hurwitz zeta function, defined as

$$\zeta(s, r) := \sum_{k=0}^{\infty} \frac{1}{(r+k)^s}, \quad \Re(s) > 1, \quad r \ne 0, -1, -2, \dots.$$

(b) The right hand side of (4.2) is not integrable for any $\Delta > 0$. However, the corresponding asymptotic spectral density of the differenced sequence $(1-B)^{\alpha/2}Y^\Delta$ is integrable for each fixed $\Delta > 0$ and the spectral density of

$$\frac{(1-B)^{\alpha/2}}{L(\Delta^{-1})^{1/2}\Delta^{(\alpha-1)/2}}Y^\Delta \tag{4.3}$$

converges as $\Delta \downarrow 0$ to that of a short-memory stationary process, i.e. a stationary process with spectral density bounded in a neighbourhood of the origin.

(c) The variance of the innovations $\{Z_n^\Delta\}_{n\in\mathbb{Z}}$ in the Wold representation (3.1) of Y^Δ satisfies

$$\sigma_\Delta^2 \sim 2\pi C_\alpha L\left(\Delta^{-1}\right)\Delta^{\alpha-1}, \quad \Delta \downarrow 0,$$

where

$$C_\alpha = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[|\omega|^{-\alpha} + (2\pi)^{-\alpha}\zeta\left(\alpha, 1-\frac{\omega}{2\pi}\right) + (2\pi)^{-\alpha}\zeta\left(\alpha, 1+\frac{\omega}{2\pi}\right)\right]d\omega\right\}. \tag{4.4}$$

Proof. (a) Since f_Y is positive, Eq. (2.2) can be rewritten as

$$f_\Delta(\omega) = \Delta^{-1}f_Y(\Delta^{-1})\sum_{k=-\infty}^{\infty}\frac{f_Y(|\omega+2\pi k|\Delta^{-1})}{f_Y(\Delta^{-1})}, \quad \omega\in\Omega_d. \tag{4.5}$$

Each of the summands converges by regular variation to $|\omega+2\pi k|^{-\alpha}$. It remains to show that we can interchange the infinite sum with this limit. Invoking the Potter bounds (Theorem 1.5.6 (iii) of Bingham et al. (1987)), for every $\epsilon > 0$ there exists a Δ_ϵ, such that for all $\Delta \le \Delta_\epsilon$ and $|2\pi k+\omega| > 0$

$$(1-\epsilon)|2\pi k+\omega|^{-\alpha-\epsilon} < \frac{f_Y(|\omega+2\pi k|\Delta^{-1})}{f_Y(\Delta^{-1})} < (1+\epsilon)|2\pi k+\omega|^{-\alpha+\epsilon}. \tag{4.6}$$

We take $\epsilon > 0$ such that $\alpha - \epsilon > 1$. Then, using (4.6), we can bound (4.5) as follows:

$$(1-\epsilon)\frac{f_Y(\Delta^{-1})}{\Delta}\sum_{k=-\infty}^{\infty}|2\pi k+\omega|^{-\alpha-\epsilon} < f_\Delta(\omega) < (1+\epsilon)\frac{f_Y(\Delta^{-1})}{\Delta}\sum_{k=-\infty}^{\infty}|2\pi k+\omega|^{-\alpha+\epsilon}, \quad \omega\in\Omega_d. \tag{4.7}$$

Since ϵ can be chosen arbitrarily small, we conclude that as $\Delta \downarrow 0$

$$f_\Delta(\omega) \sim \frac{f_Y(\Delta^{-1})}{\Delta}\sum_{k=-\infty}^{\infty}|\omega+2k\pi|^{-\alpha}, \quad \omega\in\Omega_d.$$

We can rewrite the sum above as

$$\begin{aligned}\sum_{k=-\infty}^{\infty}|\omega+2k\pi|^{-\alpha} &= (2\pi)^{-\alpha}\sum_{k=-\infty}^{\infty}\left|\frac{\omega}{2\pi}+k\right|^{-\alpha} \\ &= |\omega|^{-\alpha} + (2\pi)^{-\alpha}\sum_{k=0}^{\infty}\left[\left(k+1-\frac{\omega}{2\pi}\right)^{-\alpha} + \left(k+1+\frac{\omega}{2\pi}\right)^{-\alpha}\right], \quad \omega\in\Omega_d.\end{aligned} \tag{4.8}$$

From this and the definition of ζ we obtain (4.2).

(b) We first note that the Hurwitz zeta function $\zeta(-\alpha, 1\pm\omega/2\pi)$ is bounded and strictly positive for all $\omega \in \Omega_d$, therefore, its integral over $[-\pi,\pi]$ is positive and finite. On the other hand, since $\alpha > 1$, the term $\omega^{-\alpha}$ is not integrable over $[-\pi,\pi]$. However, the differenced sequence $(1-B)^{\alpha/2}Y^\Delta$, has spectral density

$$h^\Delta(\omega) = 2^{\alpha/2}(1-\cos\omega)^{\alpha/2} f_\Delta(\omega), \quad \omega \in \Omega_d. \tag{4.9}$$

As $\Delta \downarrow 0$ we can write, for $\omega \in \Omega_d$, by (4.2)

$$\begin{aligned} h^\Delta(\omega) \quad \sim \quad & 2^{\alpha/2}(1-\cos\omega)^{\alpha/2} L(\Delta^{-1})\Delta^{\alpha-1} \times \\ & \left[|\omega|^{-\alpha} + (2\pi)^{-\alpha}\zeta\left(\alpha, 1-\frac{\omega}{2\pi}\right) + (2\pi)^{-\alpha}\zeta\left(\alpha, 1+\frac{\omega}{2\pi}\right)\right]. \end{aligned}$$

The right hand side is integrable over $[-\pi,\pi]$ and bounded in a neighbourhood of the origin, since $2^{\alpha/2}(1-\cos\omega)^{\alpha/2}\omega^{-\alpha} \to 1$ as $\omega \to 0$. Thus we conclude that the spectral density of the rescaled differenced sequence (4.3) converges to that of a short-memory stationary process.

(c) It is easy to check that the sampled CMA process has a Wold representation of the form (3.1) and that its one-step prediction mean-squared error based on the infinite past is σ^2_Δ. Kolmogorov's formula (see, e.g., Theorem 5.8.1 of Brockwell and Davis (1991)) states that the one-step prediction mean-squared error for a discrete-time stationary process with spectral density f is

$$\tau^2 = 2\pi \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\omega) d\omega\right\} \tag{4.10}$$

Applying it to the *differenced* process we find that its one-step prediction mean-squared error is

$$\begin{aligned} & 2\pi \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log h^\Delta(\omega) d\omega\right\} \\ = \quad & 2\pi \exp\left\{\frac{\alpha}{4\pi}\int_{-\pi}^{\pi} \log(2-2\cos\omega) d\omega\right\} \times \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f_\Delta(\omega) d\omega\right\} \; = \; \sigma^2_\Delta. \end{aligned}$$

Hence the differenced sequence has the same one-step prediction mean-squared error as Y^Δ itself. Since from (4.7), as $\Delta \downarrow 0$,

$$\log f_\Delta(\omega) - \log(L(\Delta^{-1})\Delta^{\alpha-1}) - \log\left[\sum_{-\infty}^{\infty} |2\pi k + \omega|^{-\alpha}\right] \to 0$$

pointwise on Ω_d, and since the left side is dominated by an integrable function on Ω_d, we conclude from the dominated convergence theorem that, as $\Delta \downarrow 0$,

$$\frac{1}{L(\Delta^{-1})\Delta^{\alpha-1}} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f_\Delta(\omega) d\omega\right\} \to \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\sum_{-\infty}^{\infty} |2\pi k + \omega|^{-\alpha} d\omega\right]\right\},$$

which, with (4.8) and (4.10), shows that as $\Delta \downarrow 0$,

$$\sigma^2_\Delta \sim 2\pi C_\alpha L\left(\Delta^{-1}\right)\Delta^{\alpha-1}. \tag{4.11}$$

□

Remark 4.3. (i) Theorem 4.2(b) means that, from a second-order point of view, a sample $\{Y_n^\Delta, n = 1, \ldots, k\}$ with k fixed and Δ small resembles a sample from an $(\alpha/2)$-times integrated short-memory stationary sequence. If in (b) we replace $(1-B)^{\alpha/2}$ by $(1-B)^\gamma$ where $\gamma > (\alpha-1)/2$, then the conclusion holds for the overdifferenced process. If, for example, we difference at order $\gamma = \lfloor(\alpha+1)/2\rfloor$ (the smallest integer greater than $(\alpha-1)/2$) we get a stationary process. In particular, if $1 < \alpha < 3$, then $\lfloor(\alpha+1)/2\rfloor = 1$ and, by (4.2) and (4.9), the differenced sequence $(1-B)Y^\Delta$ has the asymptotic spectral density, as $\Delta \downarrow 0$,

$$L(\Delta^{-1})\Delta^{\alpha-1}2(1-\cos\omega)\left[|\omega|^{-\alpha} + (2\pi)^{-\alpha}\zeta\left(\alpha, 1-\frac{\omega}{2\pi}\right) + (2\pi)^{-\alpha}\zeta\left(\alpha, 1+\frac{\omega}{2\pi}\right)\right], \quad \omega \in \Omega_d.$$

This is the spectral density of the increment process of a self-similar process with self-similarity parameter $H = (\alpha-1)/2$ (see Beran (1992), eq. (2)). Moreover, for a generic $\alpha > 1$ the asymptotic autocorrelation function of the filtered sequence has unbounded support. The only notable exception is when α is even, where the asymptotic autocorrelation sequence is the one of a moving-average process with order $\alpha/2$, as in Brockwell et al. (2012) or in Example 4.7.

(ii) The constant C_α of (4.4) is shown as a function of α in Figure 1. The values, when α is an even positive integer, can be derived from (3.5) since CARMA processes constitute a subclass of the processes covered by the theorem (see Example 4.7). It is clear from (4.4) that C_α is exponentially bounded as $\alpha \to \infty$. $\square$

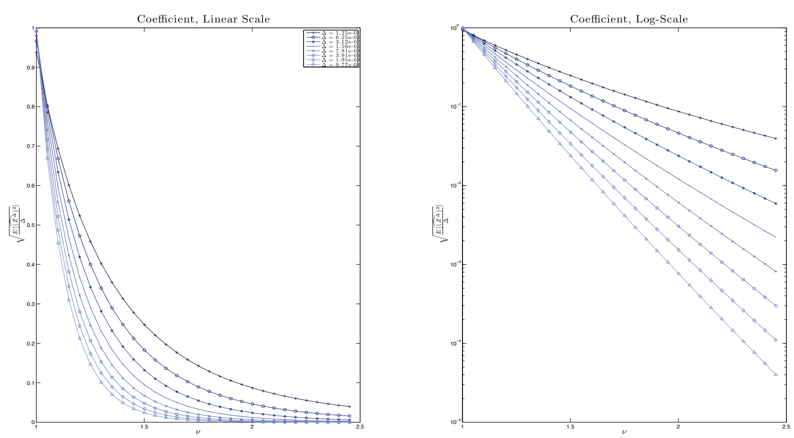

Figure 1: The constant C_α, as a function of the index of regular variation α, is shown on the left using a linear scale and on the right using a logarithmic scale. From Corollary 3.4 (a) of Brockwell et al. (2012) we know that $C_2 = 1$. The horizontal line indicates the value 1.

Corollary 4.4. *Let Y be a CMA process satisfying the assumptions of Theorem 4.2 with $1 < \alpha < 2p+1$. Then for $\Delta \downarrow 0$,*

$$\mathbb{E}[((1-B)^p Y_n^\Delta)^2] \sim 2^p S_{p,\alpha} L(\Delta^{-1})\Delta^{\alpha-1},$$

where

$$S_{p,\alpha} = \int_{-\pi}^{\pi}(1-\cos\omega)^p\left[|\omega|^{-\alpha} + (2\pi)^{-\alpha}\zeta\left(\alpha, 1-\frac{\omega}{2\pi}\right) + (2\pi)^{-\alpha}\zeta\left(\alpha, 1+\frac{\omega}{2\pi}\right)\right] d\omega.$$

Proof. By stationarity we have $\mathbb{E}[(1-B)^p Y_n^\Delta] = 0$ and, hence $\mathbb{E}[((1-B)^p Y_n^\Delta)^2]$ is the variance, of $((1-B)^p Y_n^\Delta)$ which can be calculated as the integral of its spectral density. Thus

$$\mathbb{E}[((1-B)^p Y_n^\Delta)^2] = 2^p \int_{-\pi}^{\pi} (1-\cos\omega)^p f_\Delta(\omega) d\omega.$$

Using the inequalities (4.7) and Lebesgue's dominated convergence theorem, we find that as $\Delta \downarrow 0$,

$$\frac{1}{L(\Delta^{-1})\Delta^{\alpha-1}} \int_{-\pi}^{\pi} (1-\cos\omega)^p f_\Delta(\omega) d\omega \to \int_{-\pi}^{\pi} (1-\cos\omega)^p \sum_{k=-\infty}^{\infty} |2k\pi + \omega|^{-\alpha} d\omega,$$

which, with the previous equation and (4.8), gives the result. □

The kernel of the CMA process (1.1) and its spectral density are linked by formula (2.1). Moreover, it has long been known that local properties of a function imply global properties of its Fourier transform (see e.g. Titchmarsh (1948), Theorems 85 and 86).

An Abelian theorem of Cline (1991) allows us to show, under the conditions of the following proposition, that CMA processes with regularly varying kernels at the origin have regularly varying spectral densities at infinity.

Proposition 4.5. *Let Y be a CMA process with kernel $g \in \mathcal{R}_{\nu-1}(0+)$ for $\nu > 1/2$. Assume that the derivatives in 0 satisfy the assumptions*

(A1) $g^{(\lfloor\nu\rfloor)}(0+) \neq 0$;
(A2) $g^{(\lfloor\nu-1\rfloor)} \in \mathcal{R}_\alpha(0+)$ *for* $\alpha \in [0,1)$ *(with* $g^{(-1)} := \int_0^t g(s)ds$*);*
(A3) *For some* $x_0 > 0$,

$$q(u) := \sup_{x \le x_0} \sup_{0\le w\le v\le 1} \left| \frac{g^{(\lfloor\nu-1\rfloor)}((u+v+w)x) - g^{(\lfloor\nu-1\rfloor)}((u+v)x) - g^{(\lfloor\nu-1\rfloor)}((u+w)x) + g^{(\lfloor\nu-1\rfloor)}(ux)}{g^{(\lfloor\nu-1\rfloor)}(x)} \right|,$$

is bounded and integrable on $[1,\infty)$.

Then

$$f_Y(|\cdot|) \in \mathcal{R}_{-2\nu}(\infty).$$

Proof. Under conditions ***(A1)-(A3)*** we can apply Theorem 2 of Cline (1991), which yields

$$F(g)(|\omega|) \sim \Gamma(\nu+1) e^{\pm i\nu\pi/2} \int_0^{1/|\omega|} g(s)ds, \quad \omega \to \pm\infty. \tag{4.12}$$

Moreover, Karamata's theorem (Theorem 1.5.11(ii) in Bingham et al. (1987)) gives

$$\int_0^{1/|\omega|} g(s)ds = \int_{|\omega|}^{\infty} s^{-2} g(1/s) ds \sim |\omega|^{-1} g(1/|\omega|)/\nu, \quad \omega \to \pm\infty,$$

where we used the fact $g(\cdot) \in \mathcal{R}_{\nu-1}(0+)$ means $g(1/\cdot) \in \mathcal{R}_{-\nu+1}(\infty)$.

Substituting (4.12) into (2.1) and recalling that $\Gamma(\nu+1) = \nu\Gamma(\nu)$, we obtain

$$f_Y(|\omega|) = \frac{1}{2\pi}|F(g)|^2(\omega) \sim \frac{\Gamma^2(\nu)}{2\pi}|\omega|^{-2} g^2(1/|\omega|), \quad \omega \to \pm\infty,$$

which gives the desired result. □

Remark 4.6. Condition (***A2***) can be replaced by a monotonicity condition on the derivative $g^{(\lfloor\nu\rfloor)}(\cdot)$ near the origin, so that the monotone density theorem (Bingham et al. (1987), Theorem 1.7.2.) can be applied. $\square$

Example 4.7. [CARMA(p,q) process]
The CARMA(p,q) process Y has spectral density (2.3), which clearly has the form

$$f_Y(\omega) = |\omega|^{-\alpha} L(|\omega|), \quad \omega \in \mathbb{R},$$

where $\alpha = 2(p-q)$ and $\lim_{\omega\to\infty} L(|\omega|) = \sigma^2/(2\pi)$. Hence, by Theorem 4.2(c), the white noise variance in the Wold representation of Y^Δ satisfies as $\Delta \downarrow 0$,

$$\sigma_\Delta^2 \sim \sigma^2 C_{2(p-q)} \Delta^{2(p-q)-1}, \tag{4.13}$$

where $C_{2(p-q)}$ can be calculated from (4.4). However $C_{2(p-q)}$ can also be calculated from (3.5) as $C_{2(p-q)} = [(2(p-q)-1)! \prod_{i=1}^{p-q-1} \lim_{\Delta\downarrow 0} \eta(\xi_i)]^{-1}$, where $\eta(\xi_i)$ was defined in (2.16). Theorem 4.2(b) implies that the spectral density of $\Delta^{q-p+1/2}(1-B)^{p-q} Y^\Delta$ converges to that of a short memory stationary process. From Theorem 2.1 we get the more precise result that the spectral density of $C_{2(p-q)}^{1/2} \Delta^{q-p+1/2}(1-B)^{p-q} \prod_{i=1}^{q}(1+\eta(\xi_i)B)^{-1} Y^\Delta$ converges to that of white noise with variance σ^2. $\square$

Example 4.8. [FICARMA(p,d,q) process, Brockwell and Marquardt (2005)]
The fractionally integrated causal CARMA(p,d,q) process has spectral density

$$f_Y(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{|\omega|^{2d}} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2, \quad \omega \in \Omega_c, \tag{4.14}$$

with $a(\cdot)$ and $b(\cdot)$ as in (2.3) and $0 < d < 0.5$. Hence

$$f_Y(\omega) = |\omega|^{-\alpha} L(|\omega|), \quad \omega \in \Omega_c,$$

where $\alpha = 2(p+d-q)$ and $\lim_{\omega\to\infty} L(|\omega|) = \sigma^2/(2\pi)$. The spectral density (4.14) has a singularity at frequency 0 which gives rise to the slowly decaying autocorrelation function associated with long memory. Applying Theorem 4.2(c) as in Example 4.7, the white noise variance in the Wold representation of Y^Δ satisfies as $\Delta \downarrow 0$

$$\sigma_\Delta^2 \sim \sigma^2 C_{2(p+d-q)} \Delta^{2(p+d-q)-1}, \tag{4.15}$$

where $C_{2(p+d-q)}$ can be calculated from (4.4). As $\Delta \downarrow 0$, the asymptotic spectral density f_Δ of Y^Δ is given by (4.2) with $\alpha = 2(p+d-q) > 1$ and is therefore not integrable for any $\Delta > 0$. However Theorem 4.2(b) implies that the spectral density of $\Delta^{q-p-d+1/2}(1-B)^{p+d-q} Y^\Delta$ converges to that of a short memory stationary process. $\square$

Our next two examples are widely used in the modelling of turbulence. Kolmogorov's famous 5/3 law (see Frisch (1996) Section 6.3.1, Pope (2000) Section 6.1.3) suggests a regularly varying spectral density model for turbulent flows.

Example 4.9. [Two turbulence models]
Denote by $\overline{U}$ the mean flow velocity, with ℓ the integral scale parameter and define $\overline{\ell} = \ell/\overline{U}$.
(i) The von Kármán (1948) spectrum models the isotropic energy spectrum. Its spectral density is, for C and c_ℓ positive, given by

$$f_Y(\omega) = C\overline{U}^{-2/3}|\omega|^{-5/3}\left(\frac{\omega^2}{\omega^2 + c_\ell/\overline{\ell}^2}\right)^{17/6}, \quad \omega \in \Omega_c.$$

Moreover, $f_Y \in \mathcal{R}_{-5/3}$, so it has a representation (4.1) and the conclusions of Theorem 4.2 hold with $\alpha = 5/3$.

(ii) The Kaimal spectrum for the longitudinal component of the energy spectrum is the current standard of the International Electrotechnical Commission; cf. IEC 61400-1 (1999). The spectral density is given by

$$f_Y(\omega) = v\frac{4\overline{\ell}}{(1+6\overline{\ell}\omega)^{5/3}}, \quad \omega \in \Omega_c, \tag{4.16}$$

where v is the variance of Y. Moreover, $f_Y \in \mathcal{R}_{-5/3}$, so it has a representation (4.1) and the conclusions of Theorem 4.2 hold with $\alpha = 5/3$. □

Example 4.10. [Gamma kernels and Whittle-Matérn autocorrelations]
The CMA process (1.1) with gamma kernel,

$$g(t) = t^{\nu-1}e^{-\lambda t}\mathbf{1}_{(0,\infty)}(t),\ \lambda > 0,\ \nu > 1/2, \tag{4.17}$$

has variance

$$\gamma_Y(0) = \sigma^2(2\lambda)^{1-2\nu}\Gamma(2\nu-1)$$

and autocorrelation function

$$\rho_Y(h) = \frac{2^{3/2-\nu}}{\Gamma(\nu-1/2)}|\lambda h|^{\nu-1/2}K_{\nu-1/2}(|\lambda h|), \tag{4.18}$$

which is the Whittle-Matérn autocorrelation function (see Guttorp and Gneiting (2005)) with parameter $\nu - 1/2$, evaluated at λh. The function $K_{\nu-1/2}$ in (4.18) is the modified Bessel function of the second kind with index $\nu - 1/2$ (Abramowitz and Stegun (1974), Section 9.6).

Note that $g \in \mathcal{R}_{\nu-1}(0+)$ and that it satisfies the assumptions of Proposition 4.5. From (2.1) with $\tilde{g}(\omega) = \Gamma(\nu)(\lambda - i\omega)^{-\nu}$, we obtain the spectral density

$$f_Y(\omega) = \frac{\sigma^2}{2\pi}|\tilde{g}(\omega)|^2 = \frac{\sigma^2}{2\pi}\frac{\Gamma^2(\nu)}{(\lambda^2+\omega^2)^\nu} = \omega^{-2\nu}\frac{\sigma^2\Gamma^2(\nu)}{2\pi\left((\lambda/\omega)^2+1\right)^\nu}, \quad \omega \in \Omega_c.$$

which belongs to $\mathcal{R}_{-2\nu}(\infty)$ and slowly varying function L such that $\lim_{\omega\to\infty} L(\omega) = \sigma^2\Gamma^2(\nu)/2\pi$.

Note that if $\nu = 5/6$, then f_Y, like the von Kármán spectral density of Example 4.9 (i), decays as $\omega^{-5/3}$ for $\omega \to \infty$, in accordance with Kolmogorov's 5/3 law.

Theorem 4.2 gives the asymptotic form of the spectral density of the sequence $\{(1-B)^\nu Y_n^\Delta\}_{n\in\mathbb{Z}}$ as $\Delta \downarrow 0$,

$$\begin{aligned} h^\Delta(\omega) \quad \sim \quad & \sigma^2\Gamma^2(\nu)(2\pi)^{-1}2^\nu\Delta^{2\nu-1}(1-\cos\omega)^\nu \ \times \\ & \left[|\omega|^{-2\nu} + (2\pi)^{-2\nu}\zeta\left(2\nu, 1-\frac{\omega}{2\pi}\right) + (2\pi)^{-2\nu}\zeta\left(2\nu, 1+\frac{\omega}{2\pi}\right)\right], \quad \omega \in \Omega_d. \end{aligned}$$

The second-order structure function, $S_2(\Delta) := \mathbb{E}[(Y_\Delta - Y_0)^2]$, plays an important role in the physics of turbulence. For the kernel (4.17) with $1/2 < \nu < 3/2$ its asymptotic behaviour as $\Delta \downarrow 0$ is given by

$$S_2(\Delta) = 2\gamma_Y(0)(1 - \rho_Y(\Delta)), \quad \Delta > 0,$$

which, by the asymptotic behaviour as $\Delta \downarrow 0$ of $K_{\nu-1/2}(\Delta)$ (see Abramowitz and Stegun (1974), Section 9.6), gives the asymptotic formulae,

$$\frac{S_2(\Delta)}{2\gamma_Y(0)} = \begin{cases} 2^{1-2\nu}\dfrac{\Gamma(3/2-\nu)}{\Gamma(\nu+1/2)}(\lambda\Delta)^{2\nu-1} + O(\Delta^2), & 1/2 < \nu < 3/2, \\[2ex] \frac{1}{2}(\lambda\Delta)^2|\log\Delta| + O(\Delta^3), & \nu = 3/2, \\[2ex] \dfrac{1}{4(\nu-3/2)}(\lambda\Delta)^2 + O(\Delta^{2\nu-1}), & \nu > 3/2, \end{cases}$$

which can be found in Pope (2000), Appendix G, and Barndorff-Nielsen et al. (2011). The first of these formulae can also be obtained as a special case of Corollary 4.4 with $p = 1$. □

5 Estimating the kernel function g

Given observations of Y^Δ with Δ small, we estimate the kernel g by estimating the approximation g^Δ defined in (3.2), which converges by Theorem 3.2 for all CARMA(p, q) processes pointwise, as $\Delta \downarrow 0$, to σg, or to g if the driving Lévy process is standardized so that Var(L_1)=1. In all of the examples which follow we make this assumption since without it σ and g are confounded. To estimate g^Δ it suffices to estimate the coefficients and white noise variance in the Wold representation (3.1) of Y^Δ, for which standard non-parametric methods are available. Being non-parametric they require no *a priori* knowledge of the order of the underlying CARMA process and moreover they can be applied to observations of any stationary causal CMA of the form (1.1). We now describe two useful algorithms for the estimation of g^Δ and apply them to simulated sampled data from a CARMA process, which has rational spectral density, and from a CMA process with non-rational spectral density. In Section 6 we estimate the kernel for a real data set without any parametric assumption.

5.1 The algorithms

The Wold coefficients ψ_j^Δ and white noise variance σ_Δ^2 appearing in the definition (3.2) of g^Δ are completely determined by the autocovariance function of the sequence Y^Δ and can be estimated using the sample autocovariances of the observations of Y^Δ. These will be based on high-frequency observations of Y_i^Δ, $i = 1, \ldots, n$, with n very large and $\Delta (> 0)$ very small. The estimator of g^Δ (and of g) obtained by substituting the estimated Wold coefficients and white noise variance into (3.2) will be denoted $\widehat{g}^\Delta$.

The estimation of the coefficients ψ_j^Δ for $j \in \mathbb{N}$, and the white noise variance σ_Δ^2 from the sample autocovariance function of Y^Δ is a time-domain version of the spectral factorization problem for which many algorithms have been developed (see Sayed and Kailath (2001) for a recent survey).

We shall apply and compare two algorithms for generating the required estimates. In both cases we fit high-order moving averages to Y^Δ. Note that g^Δ as defined by (3.2) is piecewise constant on the intervals $[j\Delta, (j+1)\Delta)$ for $j = 0, 1, 2, \ldots$. In Figures 2-5 and 6 c) the values of $\widehat{g}^\Delta$ are plotted at the mid-points $(n + 1/2)\Delta$, for $n = 0, 1, 2, \ldots$.

5.1.1 Durbin-Levinson algorithm

The first method determines the (high-order) causal AR(p) process whose autocovariances match the sample autocovariances of Y^Δ up to lag p by solving the Yule-Walker equations for the coefficients $\phi_1, \ldots, \phi_p$ of the autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$. This can be done efficiently using the Durbin-Levinson algorithm, which also yields an estimator of σ^2_Δ (Proposition 8.2.1, Brockwell and Davis (1991)).

The coefficients of the Wold representation (3.1) of the fitted AR(p) process are used as estimates of the coefficients ψ_j^Δ in (4.1). The estimate of ψ_j^Δ is the coefficient of z^j in the power-series expansion of $1/\phi(z)$. These coefficients are easily calculated recursively from the coefficients ϕ_j.

5.1.2 Innovations algorithm

A more direct approach to fitting a high-order moving average process based on the sample autocovariance function is to use the innovations algorithm (Definition 8.3.1 of Brockwell and Davis (1991)). Although this method is computationally slower than the Durbin-Levinson algorithm, under the conditions of Theorem 8.3.1 of Brockwell and Davis (1991), the estimates of the Wold coefficients are asymptotically jointly normal with simple covariance matrix, and the estimator of the white noise variance σ^2_Δ is consistent. If the driving process L in (1.1) is Brownian motion (as in the following simulations) and if Y is a CARMA process then Y and Y^Δ are Gaussian, the driving noise in the Wold representation of Y^Δ is iid Gaussian, and the conditions of Theorem 8.3.1 are satisfied.

In practice it has been found that the Durbin-Levinson algorithm gives better results except when the fitted autoregressive polynomial has zeroes very close to the unit circle.

5.2 Simulation examples

We shall illustrate our estimation method by a small simulation study based on the gamma kernel of Example 4.10 with standard Brownian motion as driving noise process, $\lambda = 1$, $\sigma^2 = 1$ and with the two values, $\nu = 1.05$ and $\nu = 2$, the latter corresponding to the CAR(2) model with $a(z) = (z-1)^2$ and $b(z) = 1$.

We applied the estimation method of Section 5.1, using both the Durbin-Levinson and the innovations algorithms. Since the models are known, we first applied both algorithms to the true, rather than the estimated autocovariances in order to eliminate the effect of sampling error. The estimated kernel functions are depicted in Figures 2-5 in the upper row.

Then we simulated a sample path for each regime $\nu = 1.05$ and $\nu = 2$ based on $n = 8 \cdot 10^6$ time steps and took this as a proxy for the continuous time true sample paths. From this we considered two different time resolutions for the sampled process Y^Δ: $\Delta = 2^{-2} = 0.25$ and $\Delta = 2^{-4} = 0.0625$. We fitted the kernel function $g(\cdot)$ based on observations up to time $T = 8$, that is, we estimate $\widehat{g}((j + \frac{1}{2})\Delta)$ for $j = 0, \ldots, N = 8/2^{-2} = 32$ and for $j = 0, \ldots, N = 8/2^{-4} = 128$, respectively.

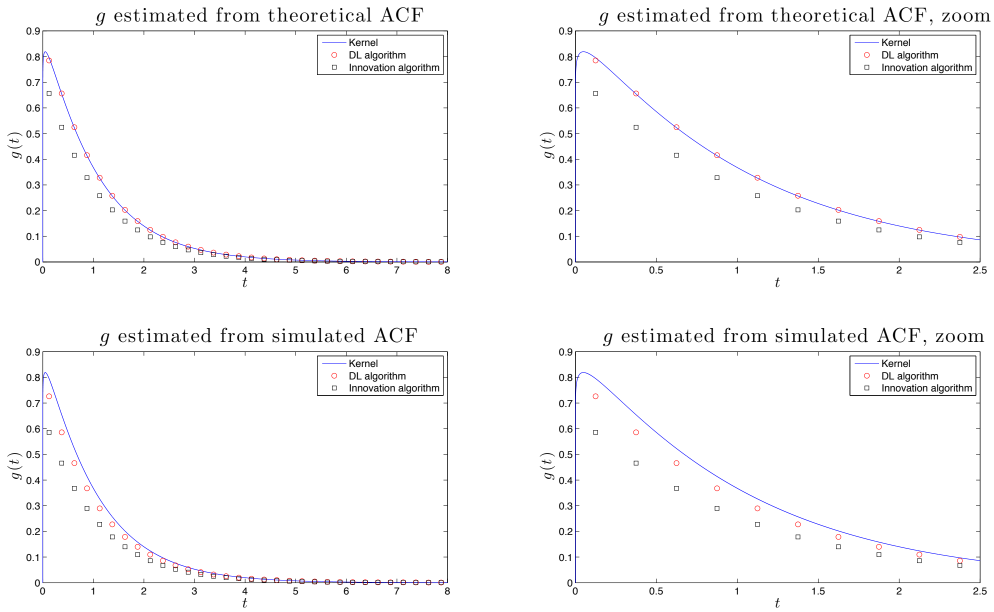

Figure 2: Estimation of the gamma kernel for $\nu = 1.05$ and $\Delta = 2^{-2}$.

In the case of the innovations algorithm, we used (for the known as well as for the estimated autocovariances) values of the discrete autocovariance functions up to $3N$.

The results are depicted in Figures 2-5 bottom row, where the squares denote the estimates from the innovations algorithm, and the circles the ones from the Durbin-Levinson algorithm. For reference the true kernel function is plotted with a solid line.

Comparing top and bottom rows of Figures 2-5 we find for the estimated autocovariance function an intrinsic finite-sample error, which influences the kernel estimation. We notice that in all cases considered, the Durbin-Levinson algorithm gives much better estimates. Furthermore, as expected, the estimates for both algorithms improve with decreasing grid spacing. The Durbin-Levinson algorithm, unlike the innovations algorithm, provides estimates with are in good agreement with the original kernel function even for a coarse grid with $\Delta = 0.25$.

6 An application to real data: mean flow turbulent velocities

We apply the algorithm of Section 5.1.1 to the Brookhaven turbulent wind-speed data, which consists of 20×10^6 measurements taken at 5000Hz; i.e. 5000 data points per second. The series thus covers a total time interval of approximately 67 minutes. This dataset displays a rather high Reynolds number (about 17000), thus it can be regarded as a good representative of turbulent phenomena. A more detailed presentation of turbulence phenomena and an application of the CMA model (1.1) in the context of turbulence modelling is given in Ferrazzano and Klüppelberg (2012); moreover we refer to Drhuva (2000), Ferrazzano (2010) for a precise description of the data, and to Pope (2000), Frisch (1996) for a comprehensive review of turbulence theory. The non-parametric kernel estimate for the CMA model (1.1) is then based on $\Delta = 1/5000 = 2\times10^{-4}$

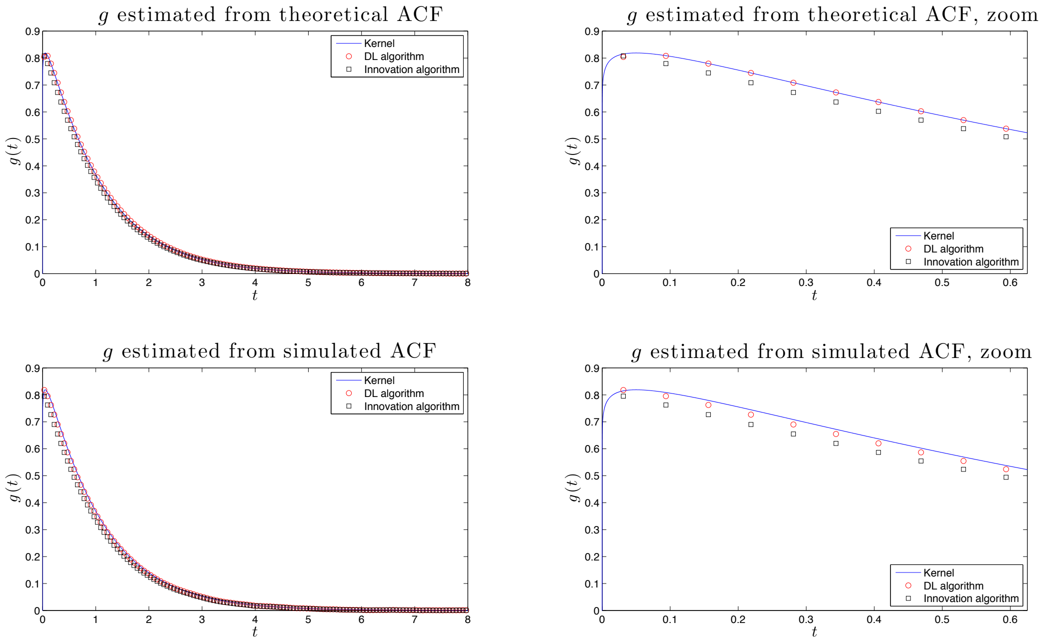

Figure 3: Estimation of the gamma kernel for $\nu = 1.05$ and $\Delta = 2^{-4}$.

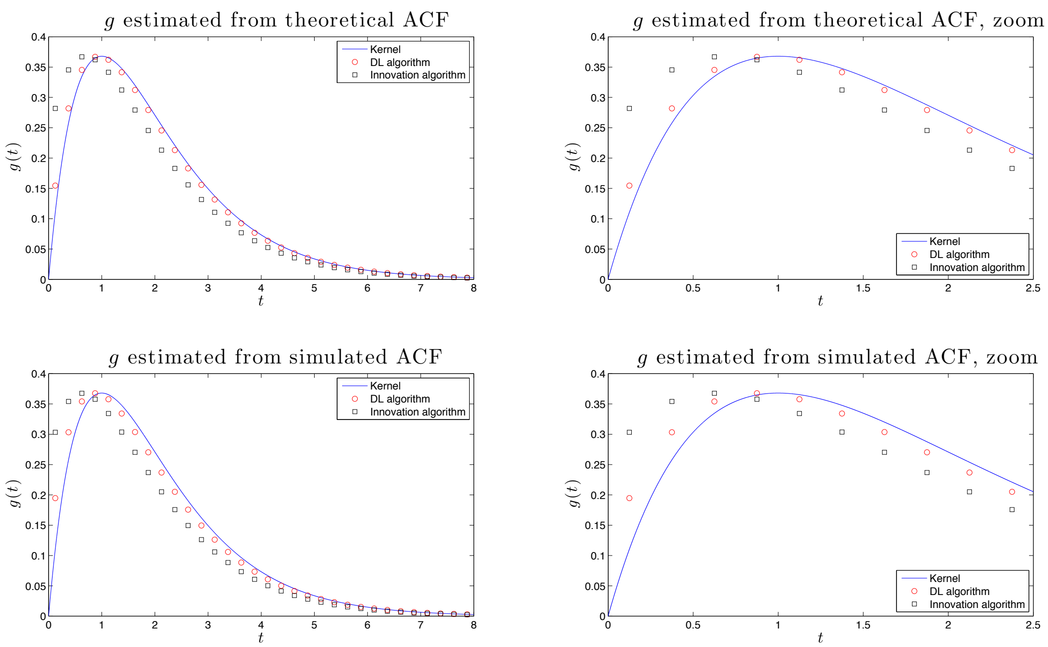

Figure 4: Estimation of the gamma kernel for $\nu = 2$ (CAR(2) process) and $\Delta = 2^{-2}$.

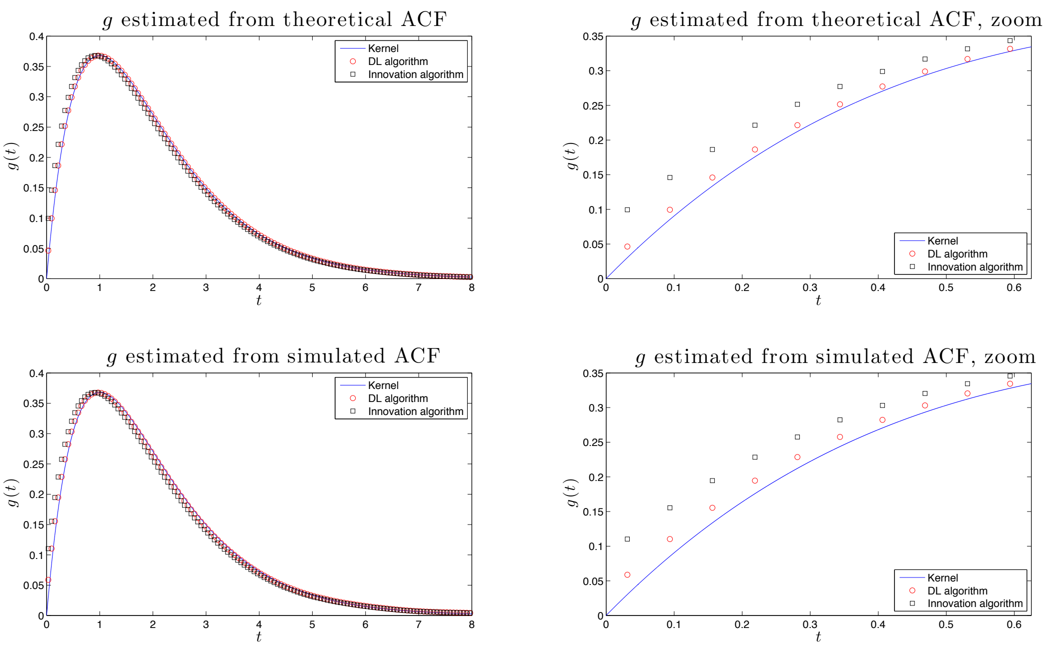

Figure 5: Estimation of the gamma kernel for $\nu = 2$ (CAR(2) process) and $\Delta = 2^{-4}$.

seconds. A CMA model (1.1) with a gamma kernel as in Example 4.10 has been suggested as a parametric model in Barndorff-Nielsen and Schmiegel (2009).

Figure 6 a) shows the sample autocorrelation function up to 120 seconds, which appears to be exponentially decreasing. In general, the data are not significantly correlated after a lag of 100 seconds.

The estimated spectral density of Y^{Δ} is shown in Figure 6 b), plotted against the frequency $\varphi = \omega/2\pi$, since ω in (2.1) have the units of an angular velocity. The estimates depicted with circles was estimated by Welch's method (Welch (1967)) with segments of 2^{22} data points (circa 14 minutes), windowed with a Hamming window and using an overlapping factor of 50%. This method allows a significant reduction of the variance of the estimate, sacrificing some resolution in frequency. In order to have a better resolution near to the frequency 0Hz, we estimated the spectral density for $\varphi \leq 10^{-3}$ Hz with the raw periodogram (Brockwell and Davis (1991), p. 322), which provides a better resolution in frequency at cost of a larger variance. The results are plotted in the leftmost part of Figure 6 b) with diamonds, and the two ranges of estimation are indicated by a vertical solid line. The spectral density is plotted on a log-log scale, so that a power-law relationship appears linear. The spectral density in a neighborhood of zero is essentially constant, and this is compatible with an exponentially decreasing autocorrelation function (as for instance for the gamma kernel function of Example 4.10).

Moreover, for a frequency φ between 10^{-2} and 200Hz, $\log f_Y$ decreases linearly with $\log \varphi$ with slope of approximately $-5/3$, in accordance with Kolmogorov's 5/3-law. As a reference, a solid line depicts the power-law $\varphi^{-5/3}$. For φ larger than 200Hz, the spectral density deviates from Kolmogorov's 5/3-law, decaying with a steeper slope. We stress the point that a spectral density decaying as prescribed by Kolmogorov's law in the neighborhood of ∞ would require a

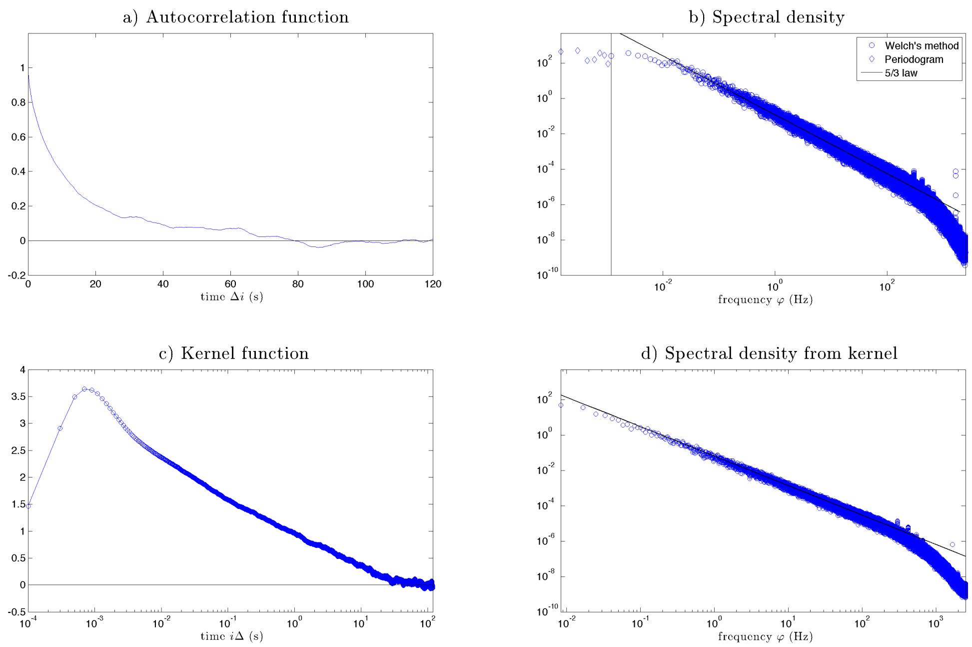

Figure 6: Estimates for the Brookhaven dataset: a) autocorrelation function b) spectral density (Welch estimator and periodogram) c) kernel function (linear-log scale) d) spectral density computed using the estimated kernel.

kernel behaving like $t^{-1/6}$ near to the origin, according to Proposition 4.5.

The estimated kernel function $\widehat{g}^{\Delta}(t)$ is plotted in Figure 6 c) in a log-linear scale, so that the behaviour of the kernel estimate for t near zero and for t to infinity is clearly visible. For large t the estimated $g(t)$ decays rapidly and it oscillates slightly around zero for $t > 100$ seconds. For small t the estimated $g(t)$ grows slowly up to $t \approx 10^{-3}$, corresponding to Kolmogorov's 5/3-law. For smaller t it drops off to zero corresponding to the steeper decay of the spectral density at high frequencies.

Figure 6 d) shows the spectral density computed directly from the estimated kernel function $\widehat{g}^{\Delta}$. Its close resemblance to the spectral density calculated by Welch's method provides justification for our estimator of g even when there is no underlying parametric model.

7 Conclusions

We studied the behaviour of the sequence of observations Y^{Δ} obtained when a CMA process of the form (1.1) is observed on a grid with spacing Δ as $\Delta \downarrow 0$.

In the particular case when Y is a CARMA process we obtained a more refined asymptotic representation of the sampled process than that found by Brockwell et al. (2012) and used it to show the pointwise convergence as $\Delta \downarrow 0$ of a sequence of functions defined in terms of the Wold representation of the sampled process to the kernel g. This suggested a non-parametric approach to the estimation of g based on estimation of the coefficients and white noise variance of the Wold representation of the sampled process.

For a larger class of CMA processes we found results analogous to those of Brockwell et al. (2012) and examined their implications for the local second-order properties of such processes, which include in particular fractionally integrated CARMA processes.

Finally we applied the non-parametric procedure for estimating g to simulated and real data with positive results.

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