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Multipliers for Hypergroups: Concrete Examples, Application to Time Series

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Preface

At the beginning of the last century, multipliers were first invented in the topic of harmonic analysis in the context of the summability of Fourier series. A sequence $\{d_n\}_{n\in\mathbb{N}_0}$ of real numbers was called "multiplier" or "factor" sequence whenever the mapping

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \mapsto \frac{1}{2}a_0 d_0 + \sum_{n=1}^{\infty} d_n(a_n\cos(nx) + b_n\sin(nx))$$

would map a certain class of Fourier series into itself or at least into another class of Fourier series. 1913 as the oldest reference we could find, Young [171, 172, 173] was concerned about the restrictions on functions such that their Fourier series becomes summable. He characterized those multiplier sequences, which map the conjugate Fourier series of an absolutely integrable function on $\mathbb{T}$ onto the Fourier series of a function in $L^1(\mathbb{T})$. He also found multiplier sequences, which carry functions of finite variation defined on $\mathbb{T}$ into absolutely continuous functions. Furthermore, Young proved already that the space of bounded measures on $\mathbb{T}$ is a subspace of all multipliers operating on $L^1(\mathbb{T})$ as well as of the space of those multipliers operating on the continuous functions on $\mathbb{T}$. The converse inclusion was shown in 1921 by Sidon [144]. Later in 1932, Sidon [145] defined an isometric isomorphism between $l^2(\mathbb{Z})$ and the space of bounded multipliers for the continuous functions on $\mathbb{T}$ into $l^1(\mathbb{Z})$.
1915 Mazurkiewicz [120] found multipliers for the space of absolutely integrable functions on $\mathbb{T}$ whose Fourier series are everywhere (C,1) summable.
Just as well in 1915 Steinhaus [151] observed that the algebra of bounded multipliers operating on $l^2(\mathbb{Z})$ is isometrically isomorphic to $l^\infty(\mathbb{Z})$. Further, he proved the inclusion of bounded multipliers mapping $l^2(\mathbb{Z})$ into $l^1(\mathbb{Z})$ in the set of all bounded multiplier operators on $l^1(\mathbb{Z})$.
Among the first mathematicians to invent multipliers was M. Riesz [134]. In his article of 1926 he proves in particular the norm decreasing inclusions of multipliers for $L^q(\mathbb{T})$ in the space of multipliers for $L^p(\mathbb{T})$ where $q \leq p \leq 2$ or $2 \leq p \leq q$.
In 1922 Fekete [38] was the first to generalize the concept of multipliers to several classes of Fourier series. He extended the results of Sidon and Young and proved further equalities of multiplier spaces. These equalities where generalized by Zygmund [174] in 1927.
While until 1929 only real multiplier sequences where considered, Bochner [6] was the first to transfer the multiplier theory into the complex form. He proved Fekete's inclusion results of multiplier spaces under this new point of view.
In 1929 Orlicz [127] was interested in multipliers for orthogonal series. He observed also the equality of the multiplier spaces of $l^p(\mathbb{Z})$ and those of $l^q(\mathbb{Z})$, where $1/p + 1/q = 1$. This result was transferred to functions on $\mathbb{T}$ by Hille and Tamarkin [80] in 1933. They proved further the equality of the multiplier space on absolutely integrable functions on $\mathbb{T}$ and the one on continuos functions on $\mathbb{T}$. Moreover, in 1930 Hille [79] published some necessary and sufficient condition for a sequence to be a factor sequence for bounded deviations.
Verblunsky [160] and Kaczmarz [87], took in 1932 /1933 a more general look on multiplier sequences by dividing Fourier series into different classes. Verblunsky characterized multipliers of six different subspaces of $L^1(\mathbb{T})$. 1935 Kaczmarz and Steinhaus [88] wrote a book about orthogonal series where they added also a chapter about multipliers. 1938/1939 Kaczmarz and Marcinkiewicz [89] characterized multipliers for orthogonal series and multipliers for general

Fourier series, see also [118]. In between Littlewood and Paley [113, 114, 115] proved some results about the convergence of Fourier series. They also gave explicit examples of multipliers for $L^p(\mathbb{T})$.

Later on starting around 1950 the concept of multipliers was generalized to many different areas of harmonic analysis such as to groups. Helson [74] and Wendel, [165, 166], were the first to prove pioneering results about characteristics of "multiplier operators" on a compact Abelian group G. They characterized those operators on the group algebra $L^1(G)$ which commute with translations, and called them "multiplier" or "centralizer". Helson proved that the space of all multipliers for $L^1(G)$ is isometrically isomorphic to the measure algebra of bounded measures on G. Using this result, Wendel found four equivalent definitions for such multipliers for $L^1(G)$. (This result was generalized to hypergroups by R. Lasser [105] in 1982, see also Theorem 3.1.5). 1957 these equivalent definitions where confirmed by Helgason [72], using a simpler proof. Furthermore, Wendel observed that the space of finite linear combinations of translation operators is strong operator dense in the space of all multipliers for $L^1(G)$. Also the fact that isometric multipliers for $L^1(G)$ are just translation operators multiplied with complex scalars of absolute value one, is due to Wendel.
Among the pioneers in characterizing multipliers was also Edwards, see [29, 30]. He calls a linear transformation multiplier, whenever it is the limit of finite linear combinations of translation operators in the ultra weak topology. This definition holds on all topological linear spaces of functions, measures or distributions on a locally compact group. On locally compact amenable groups this definition of a multiplier coincides with the one given by Helson and Wendel, see Derighetti, [22] chapter 4.
Furthermore, Edwards [29] characterized in 1955 each multiplier for $L^1(G)$ into $L^p(G)$ as the convolution with a Fourier transforms of a function in $L^p(G)$. The same holds for all multipliers for the set of bounded measures $M(G)$ into $L^p(G)$. Together with Brainerd, Edwards generalized these results in 1966 to all locally compact Abelian groups, see [7]. They also generalized Helson's result of the isometric isomorphism between the space of all multipliers for $L^1(G)$ and the measure algebra of bounded measures on G.
Moreover, multipliers for $L^p(G)$ for various groups G are quite well known, see for instance Gaudry [52, 53, 57]. Further interesting results on multipliers for Abelian groups where found 1952/1954 by Grothendieck [66, 67], 1955 by Sunouchi [153] and Tomič [157] and 1956-1958 by Helgason [71, 72, 73]. In 1956 S.G. Mihlin proved several results for multipliers of Fourier integrals, see[122] and [123].
Goes [63] reformulated known and new criteria for operators to be a multiplier by using complementary spaces in 1959. He defined new classes of multipliers by carrying forward the research on multipliers of Karamata [91, 92], Karamata and Tomič [93] and Katayama [94].
In 1960 Lars Hörmander expedited the theory of multipliers for $\mathbb{R}^n$ to a vast part, proving for example that every bounded translation invariant operator between $L^p(\mathbb{R})$ spaces is uniquely characterized by convolution with a distribution. He proved further basic properties of multipliers for $L^p(\mathbb{R})$ spaces and some inclusion results such as $M_p^p \subset M_2^2 = L^\infty$, see [84]. He focused also on the use of estimates for multipliers.
In 1965 Edwards [31] proved further results for multipliers for the character group of a compact Abelian group.
In the years following 1966 Gaudry [54] and Kitchen [97] published results concerning compact and weakly compact multipliers for $L^1(G)$. In this case the multiplier algebra is isometrically isomorphic to $L^1(G)$. Furthermore, Gaudry showed some results for multipliers for closely related but more general spaces, the weighted Lebesgue spaces and measure spaces, see [55]
Moreover, the theory of multipliers for translation-invariant Banach spaces on a non-compact locally compact group G is also quite investigated. Characterizations of multipliers for $L^p(G)$ for an abitrary locally compact group G can for instance be found in Eymard [36]. Along the pioneers on that area is Figà-Talamanca [39, 40, 41] who proved in 1965 that the dual of the Figà-Talamanca Herz algebras $A_p^p(G)$ is for every $1 < p < \infty$ isometrically isomorphic to the

multiplier space on $L^p(G)$ for every locally compact Abelian group G. Together with Gaudry [42] he extended this result to characterize multipliers in $L^p(G)$, which map into $L^q(G)$ for every locally compact Abelian group G. Furthermore, they proved strict inclusion results of the multiplier spaces on $L^p(G)$, $1 \leq p < \infty$, and mentioned several aspects of the spaces $A_p^p(G)$, see [42, 43, 44]. (Their inclusion results hold also for hypergroups, see Chapter 5). 1972 Bachelis and Gilbert [2] extended some of their results, proving for example that $A_p^q(G)$ is the dual space of compact multipliers in $M(L^p(G), L^q(G))$ for every compact group G. This leads to a double dual result, stating that the space of all multipliers $M(L^p(G), L^q(G))$ is the double dual of the space of compact multipliers in $M(L^p(G), L^q(G))$. Price [130] proved in 1970 the same strict inclusion results as Figà-Talamanca and Gaudry between Spaces of L^p-Multipliers, but used a different way to prove it. Some important inclusion results of multiplier spaces on function spaces over compact groups are also due to Akemann [1] and Iltis [85].
1967 Hahn [68] characterized a multiplier $f \in M_p(\hat{G})$ as a function f on G such that there exists a constant K and $\|\hat{f}\hat{\psi}\|_p \leq K\|\psi\|_p$ holds for every simple function ψ on G. (This result also holds for hypergroups and their duals, see Chapter 3 and 4).
Furthermore, many other remarkable results on multipliers for different groups are also found in De Leeuw [18], Hirschmann [81], Littman [116], Skvortsova [148], Stein and Zygmund [150], [175]. In 1969 Rieffel [133] transferred a lot of the known results in a very elegant way into the context of tensor products.
1974 G. I. Gaudry and I. R. Inglis gave some approximation theorems for multipliers for locally compact groups, extending the results of Edwards [32] and Ramirez [132].
Sato [141] characterized positive definite multipliers for locally compact groups in 1989.

In total, we see that a lot of investigations have been done on multipliers for spaces of functions and distributions on various groups. This theory on multipliers was extended by Hewitt and Ross [77] in 1970. One of the standard references for multipliers for groups is the book of Larsen [101], but there is also a lot to find in the books of Edwards [33, Chapter 16] and Gaudry [56, Chapter V, Vi and Vii]. Especially the history of multipliers is also explored quite detailed by Hewitt and Ross in [77, Notes of section 36] and by Larsen [101, section 0.3].
Multipliers defined on translation-invariant Banach spaces on non-commutative groups are studies intensively by Derighetti [22]. He recently published a lecture note volume, which presents all know and new found results on convolution operators on groups.
Multipliers for hypergroups have also been studied during the last decades.
1974/75 W. C. Connett and A. L. Schwartz [12, 13, 14] were interested in the topic of multipliers for ultraspherical series and Jacobi expansions. Their work is strongly connected to multipliers for polynomial hypergroups generated by the ultraspherical polynomials. Also the "multiplier criteria of Hörmander type for Jacobi expansions" published 1980 by G. Gasper and W. Trebels [51] have a strong correlation to the characterization of multipliers defined on the Jacobi hypergroup.
In 1982 R. Lasser [105] generalized Wendel's theorem to commutative hypergroups. 1986 K. Stempak [152] establishes a version of Hörmanders multiplier theorem on Bessel-Kingman hypergroups which are a special class of Chèbli-Trimèche hypergroups. Finally, W.R. Bloom and Z. Xu [5] generalized these results to the whole class of Chèbli-Trimèche hypergroups in 2000.
1985/86 F.Ghahramani and A.R.Medghalchifocused on compact multipliers for weighted hypergroup algebras, see [61, 62]. They defined an isometric isomorphism between the multipliers for a weighted hypergroup algebra $L_\omega(X)$ and the corresponding measure algebra $M_\omega(X)$.Their main theorem states, that every measure, which defines a compact multiplier for $L_\omega(X)$ is already an element in $L_\omega(X)$. Furthermore, they proved in the general setting of weighted hypergroup algebras that every weakly compact multiplier for $L_\omega(X)$ is already compact .Their work is based on the results of Vrem [161, 162, 163] about compact hypergroups.
1990 Obata proved in [126] that every surjective, isometric multiplier T for $L^1(K, m)$, is defined by an element $x \in G(K) := \{x \in K : \varepsilon_x * \varepsilon_{\tilde{x}} = \varepsilon_e = \varepsilon_{\tilde{x}} * \varepsilon_x\}$ and $\gamma \in \mathbb{C}$, $|\gamma| = 1$, by the equation $Tf = \gamma\varepsilon_x * f$.
Conversely, every measure $\mu = \gamma\varepsilon_x$ with $x \in G(K)$ and $\gamma \in \mathbb{C}$, $|\gamma| = 1$, defines a surjective,

isometric multiplier for $L^1(K,m)$. Hence, the surjective, isometric multipliers for $L^1(K,m)$ characterize the group part $G(K)$ of K, see [4, pp. 68].
In 2002 H.Emamirad and G.S.Heshmati [34] characterized multipliers for the dual hypergroup $K=[-1,1]$ of the ultraspherical polynomials $R_n^{(\alpha,\alpha)}(t)$.
Moreover, in 2007 Pavel [129], generalized the results of Brainerd and Edwards characterizing multipliers from $L^1(K,m)$ into $L^p(K,m)$, $1<p\leq\infty$, as convolutors by some $f\in L^p(K,m)$, see [129, Theorem 6]. Furthermore, compact multipliers are investigated in [129], too.
2007 V. Muruganandam [124], studied multipliers of Fourier spaces and gave necessary and sufficient conditions on a commutative hypergroup such that the Fourier space is a Banach algebra. He also introduced in 2008 a new class of hypergroups, spherical hypergroups and a subclass, the ultraspherical hypergroups, see [125]. The Fourier space of ultraspherical hypergroups form Banach algebras under pointwise product. This implies, that the set of absolutely integrable functions on the dual $\mathcal{S}$ admits convolution and is a Banach algebra with respect to a special norm.
In 2008 Teresa Martinez [119] characterized multipliers of Laplace transform type for ultraspherical expansions.
2009/10 N. Youmbi [169, 170] illustrated the relation of semigroups of operators to semigroups of multipliers and proved some results for multipliers for compact hypergroups, for instance an extension of Wendel's theorem for compact and not necessarily commutative hypergroups.

While there is a lot known about multipliers for groups, the literature about multipliers for hypergroups is rather thin. Our aim here is to generalize known results about multipliers for groups or for specific hypergroups to all commutative hypergroups. We will characterize multipliers in different settings and illustrate the consequences of these properties on various examples as for instance on polynomial hypergroups.
We will use all the common notations and elementary results of functional analysis without listing them all here.

This work is structured in the following way. It starts with a basic introduction into general properties of hypergroups and especially polynomial hypergroups.
The second part of this work includes theoretical results about multipliers for various Banach spaces over commutative hypergroups. We introduce an important tool in harmonic analysis in Chapter 2 by investigating the Hausdorff-Young transform for commutative hypergroups which is the extension of the Fourier transform and the Plancherel transform to all L^p-spaces where $1<p<2$. Further, we investigate the inverse Hausdorff-Young transform on $L^p(\hat{K},\pi)$, $1<p<2$. Our main theorem in Chapter 2 states that those extended transforms are inverse to each other. In contrast to the group case, this is not obvious, since the dual space $\hat{K}$ of an arbitrary hypergroup K admits in general no hypergroup structure. Moreover, we quote some consequences of this extension which will be very useful in the following chapters.
In Chapter 3 multipliers for the $L^p(K,m)$ spaces of a commutative hypergroup are characterized. Some applications of known results are quoted, for instance an application of Wendel's theorem (see [105]). Furthermore, new aspects are considered and some interesting examples of multipliers for polynomial hypergroups are added.
In Chapter 4 we investigate multipliers for the dual of a commutative hypergroup. Some results of Chapter 3 are transferred to those spaces. Since we miss in general a dual hypergroup structure, not all results are extendable. Nevertheless, using weak dual structures we can still create a good impression of multipliers for $L^p(\mathcal{S},\pi)$.
The relation of Figà-Talamanca Herz algebra $A_p(K)$ to multipliers for $L^p(K,m)$ is represented in Chapter 5. In contrast to the group case, we observe an isometric isomorphism of $A_p(K)$ into the set of bounded multipliers for $L^p(K,m)$ which is not necessarily surjective. We present some consequences of this relation such as some inclusion results. We prove strict inclusion results for the multiplier spaces as introduced in Chapter 3.
In Chapter 6 we take a look on those function spaces, for which all $\varphi\in C_0(\hat{K})$ define multipliers. We derive these spaces from the original $L^p(K,m)$ spaces. Thus, they are called derived

spaces. Figà-Talamanca [40] and Gaudry [43] introduced derived spaces for $L^p(G)$, where G is a locally compact Abelian group. Hörmander [84] proved similar results for Euclidean groups $G = \mathbb{R}^n$. We will transfer some of Figà-Talamanca's and Gaudry's results to hypergroups. In contrast to the group case not every dual $\mathcal{S}$ contains a set of uniqueness. Further, we only know that $|\alpha(x)| \leq 1$ for every $\alpha \in \hat{K}$ and $x \in K$, where as we obtain $|\alpha(x)| = 1$ for every x in a locally compact group G and $\alpha \in \hat{G}$. This leads to weaker results for the derived spaces over hypergroups. However, we conclude further inclusion results for different multiplier spaces by using characteristics of the derived spaces.
In Chapter 7 multipliers for homogeneous Banach spaces are investigated. We show that the multiplier spaces for different homogeneous Banach spaces coincide, even though these homogeneous Banach spaces differ in their structure and norm. This illustrates that the multiplier spaces predict little about the structure of the corresponding homogeneous Banach spaces. Introduced in Chapter 7, we will characterize multipliers for the p-Fourier spaces in Chapter 8. In contrast to Chapter 7, we will investigate results for arbitrary not necessarily strong hypergroups. The main result in this chapter states that the multiplier spaces for the p-Fourier spaces, $2 < p$, of an infinite, compact hypergroup are continuously linearly isomorphic to the dual space of a Banach space of continuous functions.
Some applications of the investigated theory are added in the last part of this work, which includes Chapter 9 and Chapter 10. In Chapter 9 multipliers for almost-convergent sequences with respect to polynomial hypergroups are studied. We prove that the multiplier space $M(AC)$ for AC, the space of almost-convergent sequences, coincides with AC_s, the set of all strongly almost-convergent sequences in the sense of Kerchy [96]. Finally, in Chapter 10 we discuss applications of multipliers in the theory of time series analysis.

I want to thank a number of people who encouraged me over the last years. Prof. Dr. R. Lasser gave me the inspiration to work on this topic in harmonic analysis. I thank him for a lot of useful discussions.
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Contents

Chapter 1

Commutative Hypergroups

The first to define hypergroups were Dunkl [27], Jewitt [86] and Spector [149]. Dunkl [27] and Spector [149] developed the theory of hypergroups in a very similar way. However, we will here quote hypergroups following the concept of Jewett, see [86]. For the theory of hypergroups and most of the basic properties we refer to [4].
Hypergroups generalize locally compact groups. Hence, many results of harmonic analysis can be shown for hypergroups, in particular for commutative hypergroups.
Hypergroups were also investigated by Vrem [162] who studied harmonic analysis on compact hypergroups and by Trimèche [159] who published a book entitled "Generalized Wavelets and Hypergoups" in 1997.

1.1 Definition of a Hypergroup and Basic Properties

For any locally compact Hausdorff space X let $C(X), C^b(X), C_0(X), C_c(X)$ be the spaces of all continuous functions on X, those that are bounded, those that vanish at infinity and those that have compact support, respectively. Furthermore, $M(X)$ denotes the space of all regular complex Radon measures on X. By Riesz's representation theorem we can identify $M(X)$ with $C_0(X)^*$, the dual space of $C_0(X)$. The subset of $M(X)$ which contains all probability measures on X is denoted by $M^1(X)$, while $M_+(X)$ denotes the subset of $M(X)$, which contains all positive measures on X. Further, we denote by ε_x the point measure of every $x \in X$.
Let $\mathcal{C}(X)$ be the space of all not empty compact subsets of X. We can define a topology on $\mathcal{C}(X)$, which is generated by the subbasis consisting of all sets of the form

$$\mathcal{C}_U(V) = \{A \in \mathcal{C}(X) : A \cap U \neq \emptyset \text{ and } A \subset V\},$$

where U and V are open subsets of X. This topology is called the **Michael topology** on $\mathcal{C}(X)$. The definition above is equivalent to Michaels definition of the finite topology [121, Definition 1.7] and the Hausdorff topology defined by Dellacherie [21]. For more information about the properties of the Michael topology, see e.g. [4].
Let K be a locally compact Hausdorff space and let

$$\omega : K \times K \to M^1(K)$$

be a continuous map with respect to the weak-*-topology on $M(K) = C_0(K)^*$. We can extend the mapping $(\varepsilon_x, \varepsilon_y) \mapsto \omega(x, y)$ (ε_x denotes the point measure at point x) to a bilinear mapping

$$(\mu, \nu) \mapsto \mu * \nu \qquad M(K) \times M(K) \to M(K)$$

defined by

$$\mu * \nu(f) := \int_K \int_K \omega(x, y)(f) d\mu(x) d\nu(y)$$

for each $f \in C_0(K)$. This mapping is called **canonical extension of** ω. It is commutative and fulfills

$$\|\mu * \nu\| \leq \|\mu\|\|\nu\|.$$

Furthermore, we can extend a homeomorphism on K denoted by $x \mapsto \tilde{x}$, $K \to K$, to an isometric isomorphism on $M(K)$ by

$$\mu \mapsto \tilde{\mu}, \qquad M(K) \to M(K),$$

where $\tilde{\mu}(E) := \mu(\tilde{E})$ for every Borel set $E \subset K$, see [4]. This mapping is called **canonical extension of** $x \mapsto \tilde{x}$

Definition 1.1.1. Let K be a locally compact Hausdorff space. The triple $(K, \omega, \tilde{\ })$ is called **hypergroup**, if it satisfies the following conditions.

(H1) $\omega : K \times K \to M^1(K)$ is a weak-*-topology continuous map, which fulfills

$$\varepsilon_x * \omega(y, z) = \omega(x, y) * \varepsilon_z$$

for all $x, y, z \in K$. We say that ω is associative with respect to the canonical extension. We call ω and its canonical extension to $M(K)$ **convolution** and denote also $x * y := \omega(x, y)$.

(H2) $\operatorname{supp}(\omega(x, y))$ is compact for every $x, y \in K$.

(H3) $\tilde{\ }$: $K \to K$ is a homeomorphism such that $\tilde{\tilde{x}} = x$ and $\widetilde{\omega(x, y)} = \omega(\tilde{y}, \tilde{x})$ for all $x, y \in K$. $\tilde{\ }$ and its canonical extension to $M(K)$ is called **involution**.

(H4) There exists a unique element $e \in K$ such that $\omega(x, e) = \varepsilon_x = \omega(e, x)$ for all $x \in K$. We call e **unit element**.

(H5) $e \in \operatorname{supp}(\omega(x, \tilde{y}))$ if and only if $x = y$.

(H6) $(x, y) \mapsto \operatorname{supp}(\omega(x, y))$, $K \times K \to \mathcal{C}(K)$ is continuous with respect to the Michael topology on $\mathcal{C}(K)$.

If $\omega(x, y) = \omega(y, x)$ for all $x, y \in K$, we call K a **commutative** hypergroup.

In the following we will write just K instead of $(K, \omega, \tilde{\ })$. Moreover, we assume throughout this thesis that K is a commutative hypergroup. For every function f on K we denote $\tilde{f}(y) := f(\tilde{y})$ and $f^*(y) := \overline{f(\tilde{y})}$.

Since it is often very difficult to ascertain if the Michael topology holds, we want to mention a result of T. H. Koornwinder and A. L. Schwartz [98]. They proved that the simpler Hausdorff topology for the compact subsets is equal to the Michael topology whenever K is a metric space and when both topologies are defined, see [98, Lemma 4.1]. The Hausdorff topology on $\mathcal{C}(X)$, where X is a metric space with metric d, is defined in the following way. First define for $A \in \mathcal{C}(X)$ and $r > 0$

$$V_r(A) = \{y \in X : \ d(x, y) < r \text{ for some } x \in A\},$$

and for $A, B \in \mathcal{C}(X)$ let $d(A, B) := \inf\{r : A \subset V_r(B) \text{ and } B \subset V_r(A)\}$. d is called the **Hausdorff metric** and the corresponding topology with basis consisting of the sets

$$N_r(A) = \{B \in \mathcal{C}(X) : \ d(A, B) < r\},$$

for $A \in \mathcal{C}(X)$ and $r > 0$, is called the **Hausdorff topology**.

Proposition 1.1.2. *Let X be a metric space. The Hausdorff topology and the Michael topology for $\mathcal{C}(X)$ coincide.*

Proof. (see [98, Lemma 4.1]) Let $A \in \mathcal{C}(X)$ and $r > 0$. Since A is compact, there is a finite sequence $a_1, ..., a_n \in A$ such that the sets $U_k = V_{r/2}(\{a_k\})$ form an open cover of A. We will show that $\bigcap_{k=1}^n \mathcal{C}_{U_k}(V_r(A)) \subset N_r(A)$. Suppose $B \in \bigcap_{k=1}^n \mathcal{C}_{U_k}(V_r(A))$, then on the one hand $B \in V_r(A)$. On the other hand, if $x \in A$, then $x \in U_k$ for some k, but $U_k \cap B \neq \emptyset$, therefore $x \in V_r(B)$ and $A \subset V_r(B)$. Hence, every Hausdorff-open subset of $\mathcal{C}(X)$ is also Michael-open. Now suppose that U and V are open subsets of X. Let $A \in \mathcal{C}_U(V)$. It will suffice to produce an $r > 0$ such that $N_r(A) \subset \mathcal{C}_U(V)$. Since $A \subset \mathcal{C}_U(V)$, $A \cap U$ must contain a point x, and since U is open, there is $r > 0$ such that $V_r(\{x\}) \subset U$ and $V_r(A) \subset V$. Now suppose $B \in \mathcal{C}(X)$ with $d(A, B) < r$. Then, $A \subset V_r(B)$ and there exists $y \in B$ such that $d(x, y) < r$. Hence, $y \in U$; thus $B \cap U \neq \emptyset$. Moreover, $B \subset V_r(A) \subset V$. Thus, $B \in \mathcal{C}_U(V)$, and the two topologies coincide. □

1.2 Harmonic Analysis on Hypergroups

Generally, we denote the space of bounded linear operators on a Banach space Y by $B(Y)$. $\| \ \|$ refers to the operator norm on $B(Y)$. Furthermore, $\simeq$ always terms an isometric isomorphism between Banach spaces. $\mathbb{C}$, $\mathbb{T}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}_0$ denote the complex numbers, the subset of $\mathbb{C}$ consisting of those numbers with absolute value equal to 1, the real numbers, all integers and all nonnegative integers, respectively.

The convolution in Definition 1.1.1 allows to define a translation operator on $C(K)$ by setting

$$L_x f(y) \ := \int_K f(z) \ d\omega(x, y)(z)$$

for $f \in C(K)$. For every $x \in K$ we have $L_x f \in C(K)$ for every $f \in C(K)$, $L_x f \in C_0(K)$ for every $f \in C_0(K)$, $L_x f \in C_c(K)$ for every $f \in C_c(K)$ and $L_x f \in C^b(K)$ such that $\|L_x f\|_\infty \leq \|f\|_\infty$ for every $f \in C^b(K)$, see [4, Proposition 1.2.16].
This translation can be extended to an operation of $M(K)$ on $C^b(K)$ by

$$L_\mu f(x) := \mu * f(x) := \int_K L_{\tilde{y}} f(x) d\mu(y)$$

for every $x \in K$. $L_\mu f$ is an element in $C^b(K)$ for every $f \in C^b(K)$ and an element in $C_0(K)$ for $f \in C_0(K)$. Moreover, we have $\|L_\mu f\|_\infty \leq \|\mu\| \|f\|_\infty$ for every $f \in C^b(K)$.
Furthermore, we note that $\varepsilon_{\tilde{y}} * f = L_y f$.

Spector [149] proved the existence of a Haar measure m for each commutative hypergroup. m is characterized by the left-invariance

$$\int_K L_x f(y) \ dm(y) \ = \int_K f(y) \ dm(y)$$

for all $x \in K$ and $f \in C_c(K)$. By this left-invariance the Haar measure m is uniquely determined up to a multiplicative positive constant. For a compact hypergroup K we will choose $m \in M_+(K)$ such that $m(K) = 1$.

The Banach spaces $L^p(K, m)$, $1 \leq p \leq \infty$, are defined in the ordinary way, i.e.

$$L^p(K, m) := \mathcal{L}^p(K, m)/\mathcal{N},$$

where

$$\mathcal{L}^p(K, m) = \{f : K \to \mathbb{C} \text{ Borel measurable} : \int_K |f(x)|^p dm(x) < \infty\}$$

for all $1 \leq p < \infty$ and

$$\mathcal{L}^\infty(K, m) := \{f : K \to \mathbb{C} \text{ Borel measurable} : \ f \text{ is } m\text{-almost everywhere bounded on } K\}$$

and

$$\mathcal{N} := \{f \in \mathcal{L}^p(K,m) : \ f = 0 \ m\text{-almost everywhere}\}.$$

$$\|f\|_p := \left(\int_K |f(x)|^p dm(x) \right)^{1/p}$$

defines a norm on $L^p(K,m)$ such that $L^p(K,m)$ becomes a Banach space for all $1 \le p < \infty$. The space $L^\infty(K,m)$ is also a Banach space with respect to the norm

$$\|f\|_\infty := \inf\{\alpha \ge 0 : \ \{x \in K : \ |f(x)| > \alpha\} \text{ is a locally } m-\text{zero set }\}.$$

On $\mathcal{L}^p(K,m)$, $1 \le p \le \infty$, we can also define a translation $L_x f(y) := \omega(x,y)(f)$ for all $x, y \in K$ and a convolution

$$L_\mu(f)(x) := \mu * f(x) \ = \ \int_K L_{\tilde{y}} f(x) \ d\mu(y),$$

for all $\mu \in M(K)$ and $f \in \mathcal{L}^p(K,m)$. Note that $\varepsilon_{\tilde{y}} * f = L_y f$. The spaces $L^p(K,m)$, $1 \le p \le \infty$, are invariant under the translation actions L_x, $x \in K$, and under the convolution operators L_μ, $\mu \in M(K)$, and we obtain

$$\|L_x f\|_p \le \|f\|_p \qquad \text{and} \qquad \|L_\mu f\|_p \le \|\mu\| \|f\|_p,$$

for all $1 \le p \le \infty$, see Proposition 1.3.5 and Lemma 1.4.6 in [4]. This is a difference to locally compact Abelian groups, where the translation L_x defines an isometry.
Furthermore, we conclude for all $x \in K$

$$\mu(L_x f) = \int_K L_x f(y) d\mu(y) = \int_K L_{\tilde{y}} f(x) d\mu(\tilde{y}) = \tilde{\mu} * f(x) = L_{\tilde{\mu}}(f)(x)$$

for all $f \in \mathcal{L}^p(K,m)$. Moreover, $L^1(K,m)$ is with respect to this convolution a Banach $*$-algebra, which is an ideal in $M(K)$, where we embed $L^1(K,m)$ into $M(K)$ by $f \mapsto fm$. In particular $L^1(K,m)$ acts on $L^p(K,m)$ via

$$f * g(x) \ = \ L_{fm}(g)(x) \ = \ \int_K f(y) \ L_{\tilde{y}} g(x) \ dm(y)$$

and we have $\|f * g\|_p \le \|f\|_1 \|g\|_p$ for $f \in L^1(K,m)$, $g \in L^p(K,m)$, $1 \le p \le \infty$.
The mapping $x \mapsto L_x f$, $K \to L^p(K,m)$ is continuous for all $1 \le p < \infty$, see [4, 1.2.1]. Hence, we can define a convolution on $L^p(K,m) \times L^q(K,m)$ to $C_0(K)$ by

$$f * g := \int_K f(y) L_{\tilde{y}} g(x) dm(y)$$

such that $\|f * g\|_\infty \le \|f\|_p \|g\|_q$ for $f \in L^p(K,m)$, $g \in L^q(K,m)$, $1 \le p < \infty$ and $1/p + 1/q = 1$, see [4, (1.4.10)].

Proposition 1.2.1. *For a commutative hypergroup K we have $f * g = g * f$ for $f \in L^q(K,m)$ and $g \in L^p(K,m)$, $1 \le p \le \infty$, $1/p + 1/q = 1$.*

Proof. By Theorem 1.3.21 in [4] holds

$$\int_K L_x f(y) g(y) dm(y) = \int_K f(y) L_{\tilde{x}} g(y) dm(y)$$

for all $x \in K$. We obtain further

$$\begin{aligned} f * g(x) &= \int_K f(y) \ L_{\tilde{y}} g(x) \ dm(y) = \int_K f(y) \ L_x g(\tilde{y}) \ dm(y) = \int_K \tilde{f}(y) \ L_x g(y) \ dm(y) \\ &= \int_K L_{\tilde{x}} \tilde{f}(y) g(y) \ dm(y) = \int_K \omega(\tilde{x}, y)(\tilde{f}) g(y) \ dm(y) \\ &= \int_K \omega(x, \tilde{y})(f) g(y) \ dm(y) = \int_K L_x f(\tilde{y}) g(y) dm(y) = g * f(x). \end{aligned}$$

□

Furthermore, let $1 \leq p \leq \infty$. For each continuous linear functional F on $L^p(K,m)$ there exists a function $g \in L^q(K,m)$, $1/p+1/q=1$, such that

$$F(f) = f * g^*(e) = \int_K f(x)\overline{g(x)}dm(x),$$

for every $f \in L^p(K,m)$. Conversely, every $g \in L^q(K,m)$ defines by this equation a continuous linear functional on $L^p(K,m)$.

The structure space of the commutative Banach $*$-algebra $L^1(K,m)$ can be identified with

$$\chi^b(K) \;=\; \{\alpha \in C^b(K) : \alpha(e)=1,\; L_x\alpha(y) \;=\alpha(x)\alpha(y) \text{ for all } x,y \in K\},$$

where $\chi^b(K)$ is equipped with the compact-open topology, which is equal to the Gelfand topology. The symmetric structure space of $L^1(K,m)$ can be identified with

$$\hat{K} \;=\; \{\alpha \in C^b(K) : \alpha(e)=1,\; L_x\alpha(y) \;=\alpha(x)\alpha(y) \text{ and } \alpha(\tilde{x}) = \overline{\alpha(x)} \text{ for all } x,y \in K\},$$

where $\hat{K}$ is equipped with the compact-open topology. We call the elements in $\chi^b(K)$ characters of the hypergroup K and those in $\hat{K}$ hermitian characters. Note that these two dual objects need not coincide, see [4, 2.2.49]. The Fourier transform of $f \in L^1(K,m)$ (the Fourier-Stieltjes transform of $\mu \in M(K)$) is defined by

$$\hat{f}(\alpha) \;=\; \int_K f(x)\,\overline{\alpha(x)}\,dm(x) \qquad (\hat{\mu}(\alpha) = \int_K \overline{\alpha(x)}\,d\mu(x))$$

for $\alpha \in \hat{K}$, respectively. $\hat{f}$ and $\hat{\mu}$ are bounded continuous complex-valued functions on $\hat{K}$, and $\hat{f}$ vanishes at infinity. We note that the space $\widehat{L^1(K,m)}$ is a dense subspace in $C_0(\hat{K})$. Furthermore, the algebra $L^1(K,m)$ admits a bounded approximate identity $(k_i)_{i\in I}$ satisfying $k_i \in C_c(K)$, $k_i \geq 0$, $\|k_i\|_1 = 1$, $\lim_i \text{supp} k_i = \{e\}$, $\hat{k_i} \in L^1_+(\hat{K},\pi)$ and $\lim_i \hat{k_i} = 1$ uniformly on compact subsets of $\hat{K}$, see [4, Theorem 2.2.28].

Considering the Hilbert space $L^2(K,m)$ there exists a positive Borel measure π on $\hat{K}$, called **Plancherel measure**, such that

$$\int_K |f(x)|^2\,dm(x) \;=\; \int_{\hat{K}} |\hat{f}(\alpha)|^2\,d\pi(\alpha)$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. We denote supp$\pi$ by $\mathcal{S}$. We emphasize that (in contrast to the group case) $\hat{K}$ does, in general, not bear a dual hypergroup structure. Moreover, whereas supp$m = K$, the support of π is in general a proper closed subset of $\hat{K}$. This leads to a great contrast between the harmonic analysis of commutative hypergroups and that of locally compact Abelian groups. A commutative hypergroup K which admits also a hypergroup structure on $\mathcal{S} = \hat{K}$ is called a **strong hypergroup**. Examples of strong hypergroups are induced by the Jacobi polynomials (see below) on $\mathbb{N}_0$ and on $[-1,1]$. Moreover, the Bessel hypergroups on $\mathbb{R}_+$ are strong hypergroups, see [4].

The Banach spaces $L^p(\mathcal{S},\pi)$, $1 \leq p \leq \infty$, with norm $\|\;\|_p$ are defined analogue to those on K, interchanging $\mathcal{S}$ and K and the measures π and m, respectively. The duality between $L^p(\mathcal{S},\pi)$ and $L^q(\mathcal{S},\pi)$, $1/p+1/q=1$ is also given by

$$\int_{\mathcal{S}} \varphi(\alpha)\overline{\psi(\alpha)}d\pi(\alpha)$$

The extension of the Fourier transform from $L^1(K,m) \cap L^2(K,m)$ to $L^2(K,m)$ is called the **Plancherel transform**. We denote the Plancherel transform of $f \in L^2(K,m)$ by $\wp(f)$. The

Plancherel transform is an isometric isomorphism from $L^2(K,m)$ onto $L^2(\mathcal{S},\pi)$, and for $f,g \in L^2(K,m)$ holds

$$\int_K f(x)\,\overline{g(x)}\,dm(x) \;=\; \int_{\mathcal{S}} \wp f(\alpha)\,\overline{\wp g(\alpha)}\,d\pi(\alpha),$$

and hence

$$\int_K f(x)\,g(x)\,dm(x) \;=\; \int_{\mathcal{S}} \wp(f)(\alpha)\,\wp(g)(\bar{\alpha})\,d\pi(\alpha)$$

(Parseval's formula).

The inverse Fourier transform of $\varphi \in L^1(\mathcal{S},\pi)$ (the inverse Fourier-Stieltjes transform of $\mu \in M(\hat{K})$) is defined by

$$\check{\varphi}(x) \;=\; \int_{\mathcal{S}} \varphi(\alpha)\,\alpha(x)\,d\pi(\alpha) \qquad \left(\check{\mu}(x) \;=\; \int_{\hat{K}} \alpha(x)\,d\mu(\alpha)\right)$$

for $x \in K$, respectively.

$\check{\varphi}$ and $\check{\mu}$ are bounded continuous complex-valued functions on K, and $\check{\varphi}$ vanishes at infinity. An inversion theorem holds. That means if $f \in L^1(K,m)$ and $\hat{f} \in L^1(\mathcal{S},\pi)$ then $f = (\hat{f})^\vee$ in $L^1(K,m)$. If f is also continuous, we have

$$f(x) = \int_{\mathcal{S}} \hat{f}(\alpha)\alpha(x)d\pi(\alpha) \text{ for all } x \in K.$$

Conversely, if $\varphi \in L^1(\mathcal{S},\pi)$ such that $\check{\varphi} \in L^1(K,m)$ then $\varphi = (\check{\varphi})^\wedge$ in $L^1(\mathcal{S},\pi)$. If φ is also continuous, we have

$$\varphi(\alpha) = \int_K \check{\varphi}(x)\overline{\alpha(x)}dm(x) \text{ for all } \alpha \in \mathcal{S}.$$

We will also use an inverse uniqueness theorem: If $\mu \in M(\hat{K})$ and $\check{\mu} = 0$, then $\mu = 0$ and if $\varphi \in L^1\mathcal{S},\pi)$ and $\check{\varphi} = 0$, then $\varphi = 0$.

Lasser proved in [108, Theorem 3.4, 3.5 and 3.6] the following relations between the topologies of a hypergroup K and its dual.

Theorem 1.2.2. *i) K is discrete if and only if $\mathcal{S}$ is compact.*

ii) $\mathcal{S}$ is compact if and only if $\hat{K}$ is compact.

iii) K is compact if and only if $\hat{K}$ is discrete and $\hat{K} = \mathcal{S}$. Moreover for K compact $\hat{K}$ is an orthogonal basis in $L^2(K,m)$.

Moreover, applying the Plancherel transform we can define a (rather weak) translation operator for $L^2(\mathcal{S},\pi)$. This translation is already introduced by Lasser in [108]. For every $f \in L^\infty(K,m)$ define $M_f \in B(L^2(\mathcal{S},\pi))$ by means of

$$M_f(\varphi) \;=\; \wp(\bar{f}\,\wp^{-1}(\varphi)) \qquad \text{for } \varphi \in L^2(\mathcal{S},\pi).$$

M_f is a bounded linear operator satisfying $\|M_f(\varphi)\|_2 \;\leq\; \|f\|_\infty\;\|\varphi\|_2$.

Proposition 1.2.3. *If $f,g \in L^\infty(K,m)$, then $M_{fg} = M_f \circ M_g$, $M_{\bar{f}} = (M_f)^*$ and $\|M_f\| \;=\; \|f\|_\infty$. Furthermore, $M_f = 0$ if and only if $f = 0$.*

The proof is straightforward, see [108].

Lemma 1.2.4. *Let $\varphi \in L^2(\mathcal{S},\pi)$. The mapping $\alpha \mapsto M_\alpha(\varphi)$, $\hat{K} \to L^2(\mathcal{S},\pi)$ is continuous.*

Proof. Let $\alpha_0 \in \hat{K},\ \epsilon > 0$. Since $\wp^{-1}(\varphi) \in L^2(K,m)$ there is a compact subset $C \subseteq K$ such that

$$\int_{K\setminus C} |\wp^{-1}(\varphi)(z)|^2 \, dm(z) \; < \; \frac{\epsilon}{8}.$$

Let $M = \int\limits_C |\wp^{-1}(\varphi)(z)|^2 \, dm(z)$ and

$$V(\alpha_0) \; = \; \left\{ \alpha \in \hat{K} : \; |\alpha(z) - \alpha_0(z)|^2 \; < \; \frac{\epsilon}{2M} \text{ for } z \in C \right\}.$$

For every $\alpha \in V(\alpha_0)$

$$\|M_\alpha(\varphi) - M_{\alpha_0}(\varphi)\|_2^2 \; = \; \|\bar{\alpha}\wp^{-1}(\varphi) \, - \bar{\alpha}_0\wp^{-1}(\varphi)\|_2^2$$

$$= \; \int_C |\alpha(z) - \alpha_0(z)|^2 \; |\wp^{-1}(\varphi)(z)|^2 \, dm(z) \; + \; \int_{K\setminus C} |\alpha(z) - \alpha_0(z)|^2 \; |\wp^{-1}(\varphi)(z)|^2 \, dm(z)$$

$$< \; \frac{\epsilon}{2} \; + \; \frac{\epsilon}{2} \; = \; \epsilon$$

□

We call $M_\alpha,\ \alpha \in \hat{K}$, translation operator on $L^2(\mathcal{S},\pi)$.

We can further introduce an action of $L^1(\mathcal{S},\pi)$ on $L^2(\mathcal{S},\pi)$. Given $\psi \in C_c(\mathcal{S})$ and $\varphi \in L^2(\mathcal{S},\pi)$ we use an $L^2(\mathcal{S},\pi)$-valued integral to define

$$\psi * \varphi := \int_{\mathcal{S}} \psi(\alpha) \; M_{\bar{\alpha}}(\varphi) \; d\pi(\alpha) \; \in \; L^2(\mathcal{S},\pi).$$

We have $\|\psi * \varphi\|_2 \; \leq \; \|\psi\|_1 \, \|\varphi\|_2$, and for any $\psi \in L^1(\mathcal{S},\pi)$ choose a sequence $(\psi_n)_{n\in\mathbb{N}}$, $\psi_n \in C_c(\mathcal{S})$, with $\|\psi - \psi_n\|_1 \to 0$ as n tends to infinity. It is easily shown that

$$\psi * \varphi := \lim_{n\in\mathbb{N}} \; \psi_n * \varphi \; \in \; L^2(\mathcal{S},\pi)$$

is a well-defined action of $L^1(\mathcal{S},\pi)$ on $L^2(\mathcal{S},\pi)$ with $\|\psi * \varphi\|_2 \; \leq \; \|\psi\|_1 \, \|\varphi\|_2$.

1.2.1 An Example for a Hypergroup derived from a Group

Hypergroups are obviously strongly related to groups and moreover, important examples for hypergroups are induced by groups. For instance, every locally compact Hausdorff group G defines in the canonical way a hypergroup. More examples can be found in [4].

We introduce one specific example of a hypergroup which is induced by a group. Let G be a locally compact (Hausdorff) group and let B denote a subgroup of the automorphism group $\mathrm{Aut}(G)$ that contains the group $\mathrm{I}(G)$ of inner automorphisms. G is called a $[FIA]_B$-group, if $\bar{B}$, the closure of B in $\mathrm{Aut}(G)$, is compact with respect to the Birkhoff topology. For each x in G denote by $[x] = \{\sigma(x) : \; \sigma \in \bar{B}\}$ the $\bar{B}$-orbit of x in G. The set G_B consisting of all $\bar{B}$-orbits is a commutative hypergroup with the operation

$$\varepsilon_{[x]} * \varepsilon_{[y]} := \int_{\bar{B}} \varepsilon_{[\sigma(x)y]} d\sigma,$$

where $d\sigma$ denotes the normalized Haar measure on $\bar{B}$, see [136] and [86, 8.3A].

The natural map $x \mapsto [x]$ is an orbital morphism from G onto G_B, see [86, 13.3].

Let $E(G,B)$ be the set of extreme points of B-invariant positive definite continuous functions f with $f(e) = 1$ endowed with the topology of compact convergence. Hartmann, Henrichs and Lasser [70] identified the spaces of hypergroup characters of G_B with $E(G,B)$. Moreover, they proved that $E(G,B)$ is a hypergroup, the dual hypergroup to G_B. Hence, G_B is a strong hypergroup.

As a special case, suppose $G = \mathbb{R}^n$ and $B = SO(n, \mathbb{R})$ the special orthogonal group, i.e. the set of orthogonal $n \times n$ matrices with determinant 1. Then the hypergroup G_B is the one-dimensional hypergroup on $\mathbb{R}_+$ associated to the Bessel function j_α of order α, see [4, 3.5.61]. Its dual is given by

$$\hat{G}_B = \{\phi_\lambda : \lambda \in \mathbb{R}_+\},$$

where

$$\phi_\lambda(x) := \begin{cases} 2^\alpha \Gamma(\alpha+1) \frac{j_\alpha(\lambda x)}{(\lambda x)^\alpha} & \text{if} \quad \lambda x \neq 0 \\ 1 & \text{if} \quad \lambda x = 0. \end{cases},$$

see [124].

1.3 Polynomial Hypergroups

As a special class of hypergroups we will deal with polynomial hypergroups or hypergroups of type [L] as they are called in [4]. Besides other applications, these hypergroups are important in the theory of time series analysis, see for instance [82, 110] and Chapter 10. In that case K is equal to $\mathbb{N}_0$ equipped with the discrete topology. Let $(R_n(x))_{n\in\mathbb{N}_0}$ be an orthogonal polynomial system on the real axis defined by a recurrence relation

$$R_0(x) = 1, \qquad R_1(x) = \frac{1}{a_0}(x - b_0)$$

$$R_1(x)R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x),$$

for $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. We want to point out, that with these assumptions we have $R_n(1) = 1$ for all $n \in \mathbb{N}_0$.
A convolution is generated by the orthogonal polynomial system $(R_n(x))_{n\in\mathbb{N}_0}$, see [106, 107], whenever $(R_n(x))_{n\in\mathbb{N}_0}$ has nonnegative linearization coefficients $g(m, n; k)$ of the product $R_m(x)R_n(x)$, i.e.

$$R_m(x)\ R_n(x) \ = \sum_{k=|n-m|}^{n+m} g(m,n;k)\ R_k(x).$$

Furthermore, assume $R_n(1) = 1$ for all $n \in \mathbb{N}_=$. Putting

$$\omega(n,m) = \varepsilon_m * \varepsilon_n \ = \sum_{k=|n-m|}^{n+m} g(m,n;k)\ \varepsilon_k,$$

a convex combination of the point measures ε_k, we get a convolution on $\mathbb{N}_0$. Together with $\tilde{n} = n$ as involution and $n = 0$ as unit. This convolution defines a hypergroup structure on $\mathbb{N}_0$. In this way, every orthogonal polynomial system $(R_n(x))_{n\in\mathbb{N}_0}$ such that the degree of R_n is n, $R_n(1) = 1$ for all $n \in \mathbb{N}_0$ and $g(m, n; k) \geq 0$ for all $m, n \in \mathbb{N}_0$ generates a hypergroup on $\mathbb{N}_0$. We call such a hypergroup **polynomial hypergroup on $\mathbb{N}_0$ induced by** $(R_n(x))_{n\in\mathbb{N}_0}$. Obviously all polynomial hypergroups are commutative.
Conversely, it is due to Favards' theorem that every commutative hypergroup on $\mathbb{N}_0$ with identity involution and zero as unit element and with

$$\{n-1, n+1\} \subset \operatorname{supp}(\omega(1,n)) \subset \{n-1, n, n+1\}$$

for every $n \in \mathbb{N}_0$, is a polynomial hypergroup induced by a certain orthogonal polynomial sequence $(R_n(x))_{n\in\mathbb{N}_0}$, see [106].

There are a lot of orthogonal polynomial systems with $g(m, n; k) \geq 0$ as for example, the generalized Chebyshev polynomials, associated ultraspherical polynomials, Pollaczek polynomials,

little q-Legendre polynomials, Jacobi polynomials and so on, see [4, 106, 107]. The Haar measure on polynomial hypergroups $\mathbb{N}_0$ is the counting measure with weights $h(n) = g(n,n;0)^{-1}$ at the points $n \in \mathbb{N}_0$.
The symmetric structure space of the Banach $*$-algebra $l^1(\mathbb{N}_0, h)$ can be identified with the set

$$D_s \;=\; \{t \in \mathbb{R} : \, |R_n(x)| \,\leq 1 \text{ for all } n \in \mathbb{N}_0\}$$

via the mapping $x \mapsto \alpha_x$, $\alpha_x(n) = R_n(x)$, see [4] or [106]. Hence, we consider $\widehat{\mathbb{N}_0}$ as a compact subset of $\mathbb{R}$ that contains $t = 1 \in \mathbb{R}$. The Plancherel measure π on D_s is exactly the orthogonalization measure of the orthogonal polynomial system $(R_n(x))_{n\in\mathbb{N}_0}$. Notice that $\mathcal{S} = \text{supp}\pi \subseteq D_s$, and the orthogonalization measure is (up to a multiplicative constant) uniquely determined. Hence, the Fourier transform of $d = (d(n))_{n\in\mathbb{N}_0} \in l^1(\mathbb{N}_0, h)$ is defined by

$$\hat{d}(n) = \sum_{k=0}^{\infty} d(k) R_n(k) h(k), \qquad x \in D_s.$$

Furthermore, the orthogonalization measure π on D_s of $(R_n(x))_{n\in\mathbb{N}_0}$ is the only measure such that the theorem of Plancherel-Levitan holds, i.e.

$$\sum_{k=0}^{\infty} |d(k)|^2 h(k) = \int_{D_s} |\hat{d}(x)|^2 d\pi(x)$$

for every $d \in l^1(\mathbb{N}_0, h)$.

1.3.1 The continuity Property (P)

The Jacobi polynomials are the only ones in the class of polynomial hypergroups, which possess a dual hypergroup structure (see below). For all other polynomial hypergroups generated by orthogonal polynomials $(R_n(x))_{n\in\mathbb{N}_0}$ we have a weaker condition such that a weak dual structure on D_s exists. We shall say that the polynomial hypergroup $K = \mathbb{N}_0$ fulfills the continuity property (P) if for all $s, t \in D_s$ there exists a probability measure $\mu_{s,t} \in M^1(D_s)$ such that

$$R_n(s)\, R_n(t) \;=\; \int_{D_s} R_n(u)\; d\mu_{s,t}(u).$$

If the continuity property (P) is satisfied we get a weak dual structure on D_s, see [106]. Obviously every strong polynomial hypergroup satisfies the continuity property (P). However, there exist also polynomial hypergroups which satisfy the continuity property (P), even though they are not strong. For example the hypergroup induced by the generalized Chebyshev polynomials, see [106], or the hypergroup induced by orthogonal polynomials related to homogeneous trees, see [9].

Lemma 1.3.1. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup satisfying the continuity property (P). Then $(s,t) \mapsto \mu_{s,t}$, $D_s \times D_s \to M^1(D_s)$ is continuous, where $M^1(D_s)$ bears the $\sigma(M^1(D_s), C(D_s))$-topology.*

Proof. Given any $\varphi \in C(D_s)$ and $\epsilon > 0$, choose $f \in l^1(\mathbb{N}_0, h)$ with finite support such that $\|\varphi - \hat{f}\|_\infty < \epsilon$. Given $s_0, t_0 \in D_s$ we conclude

$$\left|\int_{D_s} \varphi(u)\; d\mu_{s,t}(u) \;-\; \int_{D_s} \varphi(u)\; d\mu_{s_0,t_0}(u)\right| \leq\; 2\epsilon \;+\; \left|\int_{D_s} \hat{f}(u)\; d\mu_{s,t}(u) \;-\; \int_{D_s} \hat{f}(u)\; d\mu_{s_0,t_0}(u)\right|$$

$$\leq\; 2\epsilon \;+\; \left|\sum_{k\in\,\text{supp}\, f} f(k)\, h(k) \int_{D_s} R_k(u)\; d(\mu_{s,t} - \mu_{s_0,t_0})(u)\right|$$

$$\leq\ 2\epsilon\ +\ \sum_{k\in\,\mathrm{supp}\,f} |f(k)|\ |R_k(s)R_k(t) - R_k(s_0)R_k(t_0)|\ h(k)$$

Now it is obvious that we can find neighborhoods U_{s_0}, U_{t_0} of s_0 and t_0 such that

$$\left|\int_{D_s} \varphi(u)\ d\mu_{s,t}(u)\ -\ \int_{D_s} \varphi(u)\ d\mu_{s_0,t_0}(u)\right|\ \leq\ 3\epsilon$$

for all $s \in U_{s_0},\ t \in U_{t_0}$. □

We can now define the translation of any function $\varphi \in C(D_s)$ by $s \in D_s$. We define $L_s\varphi(t)\ = \mu_{s,t}(\varphi)$. By Lemma 1.3.1 we have $L_s\varphi \in C(D_s)$. The orthogonalization measure π behaves like a Haar measure on D_s.

Proposition 1.3.2. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup satisfying the continuity property (P). Then*

$$\int_{D_s} L_s\varphi(t)\ d\pi(t)\ =\ \int_{D_s} \varphi(t)\ d\pi(t)$$

holds for all $\varphi \in C(D_s)$ and $s \in D_s$.

Proof. For $s \in D_s$ let $\alpha_s(n) = R_n(s)$. For $f \in l^1(\mathbb{N}_0, h)$ and $s, t \in D_s$ we have

$$L_s\hat{f}(t)\ =\ \int_{D_s} \hat{f}(u)\ d\mu_{s,t}(u)\ =\ \int_{D_s} \sum_{n\in\mathbb{N}_0} f(n)\,R_n(u)\,h(n)\ d\mu_{s,t}(u)$$

$$=\ \sum_{n\in\mathbb{N}_0} f(n)\ R_n(s)\,R_n(t)\,h(n)\ =\ (\alpha_s\cdot f)^\wedge(t),$$

and then

$$\int_{D_s} L_s\hat{f}(t)\ d\pi(t)\ =\ \int_{D_s} \sum_{n\in\mathbb{N}_0} f(n)\,R_n(s)\,R_n(t)\,h(n)\ d\pi(t)\ =\ f(0) = \int_{D_s} \hat{f}(u)\ d\pi(u).$$

For any $\varphi \in C(D_s)$ and $\epsilon > 0$ there exist $f \in l^1(\mathbb{N}_0, h)$ (even with finite support) so that $\|\varphi - \hat{f}\|_\infty < \epsilon$. Then $\|L_s\varphi - L_s\hat{f}\|_\infty < \epsilon$ and it follows

$$\int_{D_s} L_s\varphi(t)\ d\pi(t)\ =\ \int_{D_s} \varphi(t)\ d\pi(t).$$

□

The next step is to consider Borel measurable functions on D_s. Based on Lemma 1.3.1 and Proposition 1.3.2 one can prove the following result, using the methods in [86].

Proposition 1.3.3. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup fulfilling the continuity property (P). Let $\psi : D_s \to [0,\infty]$ be a Borel measurable function. Then $(s,t) \mapsto \mu_{s,t}(\psi)$, $D_s \times D_s \to [0,\infty]$ is Borel measurable. For each complex-valued, Borel measurable function ψ with $\int_{D_s} |\psi(t)|\,d\pi(t) < \infty$ and $s \in D_s$, $L_s g(t) := \mu_{s,t}(g)$ satisfies*

$$\int_{D_s} L_s\psi(t)\ d\pi(t)\ =\ \int_{D_s} \psi(t)\ d\pi(t).$$

By Proposition 1.3.3 we know that $L_s\psi$ is a well-defined element of $L^1(D_s, \pi)$ for each $\psi \in L^1(D_s, \pi)$. Furthermore, we have $\|L_s\psi\|_1\ \leq\ \|\psi\|_1$. By Hölder's inequality we can transfer this result to $L^p(\mathcal{S}, \pi)$, $1 < p < \infty$.

Proposition 1.3.4. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup satisfying the continuity property (P). Let $\varphi \in L^p(\mathcal{S}, \pi)$, $1 \leq p < \infty$ and $t \in D_s$. Then $\|L_t\varphi\|_p \leq \|\varphi\|_p$.*

Proof.

$$\|L_t\varphi\|_p^p = \int_{D_s} |L_t\varphi(s)|^p d\pi(s) \le \int_{D_s} L_t(|\varphi|^p)(s) d\pi(s) = \int_{D_s} |\varphi|^p(s) d\pi(s) = \|\varphi\|_p^p.$$

□

Similarly one can prove that $L_s\psi$ is a well-defined element of $L^\infty(D_s,\pi)$ for each $\psi \in L^\infty(D_s,\pi)$ and we have $\|L_s\psi\|_\infty \le \|\psi\|_\infty$.
Hence, we can define an action of $M(\hat{K})$ on $L^p(\mathcal{S},\pi)$. For $\mu \in M(D_s)$ and $\psi \in L^p(D_s,\pi)$,

$$\mu * \psi(t) := \int_{D_s} L_s\psi(t)\, d\mu(s)$$

is a well-defined element of $L^p(D_s,\pi)$ with $\|\mu * \psi\|_p \le \|\mu\|\, \|\psi\|_p$. We note that $L^1(\mathcal{S},\pi)$ can isometrically be embedded in the set of measures $M(\hat{K})$ by $\varphi \mapsto \varphi\pi$. Hence, we can transfer the action of $M(\hat{K})$ on $L^p(\mathcal{S},\pi)$ to an action of $L^1(\mathcal{S},\pi)$ on $L^p(\mathcal{S},\pi)$, whenever the continuity property (P) is valid. We defined

$$\varphi * \psi(t) := \int_{D_s} L_s\varphi(t)\psi(s) d\pi(s)$$

for each $t \in D_s$. Furthermore, $\|\varphi * \psi\|_p \le \|\varphi\|_1\|\psi\|_p$ for $\varphi \in L^1(\mathcal{S},\pi)$ and $\psi \in L^p(\mathcal{S},\pi)$, $1 \le p \le \infty$.

Remark 1.3.5. If the continuity property (P) is fulfilled and $1 \in \mathcal{S}$, we obtain for every $s \in D_s$ and for every function $\varphi \in C_c(D_s)$ with $\varphi \ge 0$, $\varphi(s) > 0$ that $L_s\varphi(1) = \varphi(s) > 0$. Hence,

$$\int_{D_s} \varphi(t) d\pi(t) = \int_{D_s} L_s\varphi(t) d\pi(t) > 0,$$

since $1 \in \mathcal{S}$. This implies $s \in \mathcal{S}$ and we have $D_s = \mathcal{S}$.

1.4 The Jacobi Hypergroup

Important examples of polynomial hypergroups are generated by the **Jacobi polynomials** $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ which satisfy the recurrence formula

$$R_1^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x) = a_n R_{n+1}^{(\alpha,\beta)}(x) + b_n R_n^{(\alpha,\beta)}(x) + c_n R_{n-1}^{(\alpha,\beta)}(x)$$

with recurrence coefficients

$$\begin{aligned} a_n &= \frac{2(n+\alpha+\beta+1)(n+\alpha+1)(2+\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)2(\alpha+1)} \\ b_n &= \frac{\alpha-\beta}{2(\alpha+1)}\left[1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\right] \\ c_n &= \frac{2n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)2(\alpha+1)} \end{aligned}$$

for $\alpha, \beta \in \mathbb{R}$ and

$$a_0 = \frac{2(\alpha+1)}{\alpha+\beta+2}, \qquad b_0 = \frac{\beta-\alpha}{\alpha+\beta+2}.$$

The Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ are orthogonal with respect to

$$d\pi(x) = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta \chi_{[-1,1]}(x)\, dx\ ,$$

where $c_{\alpha,\beta}$ is a constant in $\mathbb{R}$ and $\text{supp}\pi^{(\alpha,\beta)} = [-1,1]$. We choose the normalization $R_n^{(\alpha,\beta)}(1) = 1$. If $\alpha \geq \beta > -1$ and $\alpha+\beta+1 \geq 0$, then $g(m,n;k) \geq 0$ and hence, in that case, $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ generates a polynomial hypergroup on $\mathbb{N}_0$. (The parameter region for the parameters (α,β) such that $g(m,n;k)$ are nonnegative is even a little bit larger). For

$$(\alpha,\beta) \in J = \left\{(\alpha,\beta) : \alpha \geq \beta > -1 \text{ and } (\beta \geq -\frac{1}{2} \text{ or } \alpha+\beta \geq 0)\right\},$$

there exists for any $x,y \in [-1,1]$ a probability Borel measure $\mu_{x,y}^{(\alpha,\beta)} \in M([-1,1])$ such that

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_{\mathcal{S}} R_n^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z)$$

for all $n \in \mathbb{N}_0$, see [46] and [50] and the symmetric structure space $D_s = [-1,1] = \mathcal{S}$ bears a dual hypergroup structure, see [4] or [106]. (For more information about the product formulas and the generated hypergroup see also [15].)
Furthermore, the Jacobi polynomials with $(\alpha,\beta) \in J$ are the only orthogonal polynomials which admit a dual hypergroup structure, see [4, Corollary 3.6.3]. The case $\alpha = \beta$ corresponds to the **ultraspherical (or Gegenbauer) polynomials**. For the sake of simplicity we fix the parameters $(\alpha,\beta) \in J$ and omit those from now on at all the notations of this chapter.
We define a Jacobi translation operator

$$L_y f(x) = \int_{\mathcal{S}} f(z)d\mu_{x,y}(z)$$

for all $f \in L^1(\mathcal{S},\pi)$ and $y \in \mathcal{S}$. Since $\mathcal{S}$ is a hypergroup corresponding to $\mu_{x,y}$, see [106], all the known results for translation operators on hypergroups are available. In particular, we have $L_y f \in L^p(\mathcal{S},\pi)$ for all $f \in L^p(\mathcal{S},\pi)$, $L_y f \in C(\mathcal{S})$ for all $f \in C(\mathcal{S})$ and $\|L_y f\|_p \leq \|f\|_p$ for $1 \leq p \leq \infty$.
Furthermore, we can define a Jacobi transform

$$\check{f}(n) := \int_{\mathcal{S}} f(x)R_n(x)d\pi(x)$$

for $f \in L^1(\mathcal{S},\pi)$ and $n \in \mathbb{N}_0$, and

$$\hat{d}(x) := \sum_{k=0}^{\infty} d(n)R_n(x)h(n)$$

for $d \in l^1(\mathbb{N}_0,h)$ and $x \in \mathcal{S}$. As the Fourier transform, the Jacobi transform also admits the uniqueness theorem stating that $\check{f} = 0$ implies $f = 0$ in $L^1(\mathcal{S},\pi)$, see [4]. Moreover, the Jacobi transform is also an isometric mapping from $L^2(\mathcal{S},\pi)$ into $l^2(\mathbb{N}_0,h)$ and, conversely, from $(l^1(\mathbb{N}_0,h),\|\ \|_2)$ into $L^2(\mathcal{S},\pi)$.
Using the Jacobi translation operator we define a commutative convolution on $L^1(\mathcal{S},\pi)$ by

$$f * g := \int_{\mathcal{S}} f(x)L_y g(x)d\pi(x)$$

for $f,g \in L^1(\mathcal{S},\pi)$, $y \in \mathcal{S}$. $f*g$ is again an element in $L^1(\mathcal{S},\pi)$ such that $\|f*g\|_1 \leq \|f\|_1\|g\|_1$. Similarly, $L^1(\mathcal{S},\pi)$ acts on $L^p(\mathcal{S},\pi)$, $1 \leq p < \infty$, see [46] or [4]. A simple consequence is

$$(L_y f)^\vee = R_n(y)\check{f}(n)$$

for all $f \in L^1(\mathcal{S},\pi)$, $y \in \mathcal{S}$.

Chapter 2

The Hausdorff-Young Theorem

A lot of results exist on the Hausdorff-Young transform on groups and their applications, see for instance [77]. Estimates for the norm of the L^p-Fourier transform on locally compact groups are established by Russo, see [137],[138],[139], and Fournier, see [48], [49]. For groups which are neither compact nor Abelian but which are unimodular the Hausdorff Young transform has been defined and a Hausdorff-Young theorem has been proven by Kunze, see [100].
We study the Hausdorff-Young transform for commutative hypergroups by extending the domain of the Fourier transform to encompass all functions in $L^p(K,m)$ and $L^p(\mathcal{S},\pi)$ respectively, where $1 \le p \le 2$. Our main theorem states that those extended transforms are inverse to each other. In contrast to the group case this is not obvious, since the dual space $\hat{K}$ of an arbitrary hypergroup K is in general not a hypergroup.

2.1 Main results

We extend the domain of the Fourier transform to all functions in $L^p(K,m)$ where $1 \le p \le 2$ using the Riesz-Thorin convexity theorem [26, VI.10.11].

The Fourier transform coincides on $L^1(K,m) \cap L^2(K,m)$ with the Plancherel transform. Therefore, the Riesz-Thorin convexity theorem yields the inequality

$$\|\hat{f}\|_q \le \|f\|_p$$

for $1 \le p \le 2$, $1/p + 1/q = 1$, for all simple functions f on K. Since we can approximate each function $f \in C_c(K)$ uniformly by simple functions, this inequality holds for all $f \in C_c(K)$ and the mapping

$$f \mapsto \hat{f}, \qquad C_c(K) \to L^q(\mathcal{S},\pi)$$

can be extended uniquely by continuity to the whole of $L^p(K,m)$. This extended map is called **Hausdorff-Young transform**. To sum up, we have the following important result:

Theorem 2.1.1 (Hausdorff-Young)**.** *Let $1 \le p \le 2$ and $1/p + 1/q = 1$. The Hausdorff-Young transform is a linear mapping from $L^p(K,m)$ into $L^q(\mathcal{S},\pi)$, $f \mapsto \hat{f}$ such that $\|\hat{f}\|_q \le \|f\|_p$.*

In the same way we can extend the inverse Fourier transform $f \mapsto \check{f}$ from $C_c(\mathcal{S})$ into $C_0(K) \subset L^\infty(K,m)$ by using the Riesz-Thorin convexity theorem once again. Its extension maps $L^p(\mathcal{S},\pi)$, $1 \le p \le 2$, into $L^q(K,m)$, $1/p + 1/q = 1$.

Theorem 2.1.2 (Inverse Hausdorff-Young)**.** *Let $1 \le p \le 2$ and $1/p + 1/q = 1$. The inverse Hausdorff-Young transform is a linear mapping from $L^p(\mathcal{S},\pi)$ into $L^q(K,m)$, $f \mapsto \check{f}$ such that $\|\check{f}\|_q \le \|f\|_p$.*

For later results it is important to know, whether the L^{p_1}-transform and the L^{p_2}-transform of a function f, which is contained in two different spaces $L^{p_1}(K,m)$ and $L^{p_2}(K,m)$, agree π-almost everywhere on $\hat{K}$. The same question holds for the dual versions.

Theorem 2.1.3. *Let $1 \le p_1, p_2 \le 2$.*
i) For $f \in L^{p_1}(K,m) \cap L^{p_2}(K,m)$ the L^{p_1}-transform of f and the L^{p_2}-transform of f agree π-almost everywhere on $\hat{K}$.
ii) For $f \in L^{p_1}(\mathcal{S},\pi) \cap L^{p_2}(\mathcal{S},\pi)$ the inverse L^{p_1}-transform of f and the inverse L^{p_2}-transform of f agree m-almost everywhere on K.

Proof. The proof follows the lines of [77, (31.26)]. □

Now it is natural to ask, whether the inverse Hausdorff-Young transform is indeed the inverse mapping of the Hausdorff-Young transform. We will prove this inverse relation, but we have to take into account that in general $\hat{K}$ is not a hypergroup. Thus, we need a few results in advance.

Lemma 2.1.4. *For $f, g \in L^2(K,m)$ and $h \in L^1(K,m)$ we have $f*g \in C_0(K)$, $g*h \in L^2(K,m)$ and*

$$\int_K f*g(y)h(\tilde{y})dm(y) = \int_K f(\tilde{y})g*h(y)dm(y).$$

Proof. It is well-known that $f*g \in C_0(K)$ for $f, g \in L^2(K,m)$ and $g*h \in L^2(K,m)$ for $g \in L^2(K,m)$, $h \in L^1(K,m)$, see [4]. Furthermore, $f*g(x) = g*f(x)$ by the commutativity of K. Finally, applying Fubini's theorem we conclude

$$\int_K f*g(y)h(\tilde{y})dm(y) = \int_K \int_K f(\tilde{x})L_y g(x)dm(x)h(\tilde{y})dm(y)$$

$$= \int_K f(\tilde{x}) \int_K L_y g(x)h(\tilde{y})dm(y)dm(x) = \int_K f(\tilde{x})g*h(x)dm(x).$$

□

Proposition 2.1.5. *Let K be a commutative hypergroup and $1 \le p \le 2$. Then the following holds:*

(i) For $f \in L^p(K,m)$ and $\varphi \in L^p(\mathcal{S},\pi)$ holds $(\hat{f}\varphi)^\vee = f*\check{\varphi}$.

(ii) For $\mu \in M(K)$ and $\varphi \in L^p(\mathcal{S},\pi)$ holds $(\hat{\mu}\varphi)^\vee = \mu*\check{\varphi}$ m-almost everywhere. Especially, for $f \in L^1(K,m)$ and $\varphi \in L^p(\mathcal{S},\pi)$ we have $(\hat{f}\varphi)^\vee = f*\check{\varphi}$ m-almost everywhere.

Proof. (i) $\hat{f}\varphi$ is by Theorem 2.1.1 and by an application of Hölder's inequality an element in $L^1(\mathcal{S},\pi)$. Hence, the inverse Fourier transform is well defined. Choosing $f \in C_c(K)$, $\varphi \in C_c(\mathcal{S})$, we obtain by Fubini's theorem

$$\begin{aligned}(\hat{f}\varphi)^\vee(x) &= \int_{\mathcal{S}} \hat{f}(\alpha)\varphi(\alpha)\alpha(x)d\pi(\alpha) = \int_{\mathcal{S}}\int_K f(y)\overline{\alpha(y)}dm(y)\varphi(\alpha)\alpha(x)d\pi(\alpha)\\ &= \int_{\mathcal{S}}\int_K \alpha(x)\overline{\alpha(y)}f(y)\varphi(\alpha)dm(y)d\pi(\alpha) = \int_K L_x\check{\varphi}(\tilde{y})f(y)dm(y) = f*\check{\varphi}(x).\end{aligned}$$

Using the continuity of the transformations and of the convolution, the statement follows from the denseness of $C_c(K)$ in $L^p(K,m)$ and the denseness of $C_c(\mathcal{S})$ in $L^p(\mathcal{S},\pi)$. Indeed, choosing a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$ such that $\lim_{n\to\infty} \|\varphi_n - \varphi\|_p = 0$, we conclude for each $f \in C_c(K)$ using Hölder's inequality

$$\left\|(\hat{f}\varphi)^\vee - f*\check{\varphi}\right\|_\infty \le \left\|\hat{f}(\varphi-\varphi_n)\right\|_1 + \|f*\check{\varphi}_n - f*\check{\varphi}\|_\infty \le 2\|f\|_p \|\varphi-\varphi_n\|_p \to 0$$

as n tends to infinity.
(ii) By [4, (2.2.15)] we know already that $(\hat{\mu}\varphi)^\vee = \mu * \check{\varphi}$ for all $\varphi \in C_c(\mathcal{S})$. The denseness of $C_c(\mathcal{S})$ in $L^p(\mathcal{S},\pi)$ yields for each $\varphi \in L^p(\mathcal{S},\pi)$ the existence of a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$, which converges to φ in $L^p(\mathcal{S},\pi)$. Hence,

$$\|(\hat{\mu}\varphi)^\vee - \mu * \check{\varphi}\|_q \leq \|\hat{\mu}(\varphi - \varphi_n)\|_p + \|\mu * (\varphi - \varphi_n)^\vee\|_q \leq 2\,\|\mu\|\,\|\varphi - \varphi_n\|_p \to 0$$

as n tends to infinity. The second statement follows by embedding $L^1(K,m)$ into $M(K)$ via the mapping $f \mapsto fm$, $L^1(K,m) \to M(K)$. □

We also use the following lemma proven in [45, Theorem 3.1].

Lemma 2.1.6. *Given a compact neighborhood C_α of $\alpha \in \mathcal{S}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c(K)$ such that $\|(f_n * f_n^*)^\wedge - \chi_{C_\alpha}\|_1 \to 0$ as n tends to infinity.*

The proof of the following proposition is essential for our main theorem of this chapter.

Proposition 2.1.7. *Let $f \in L^p(K,m)$, $1 \leq p \leq 2$, such that the Hausdorff-Young transform $\hat{f} \in L^2(\mathcal{S},\pi)$. Then $f \in L^2(K,m)$ and $f = \wp^{-1}(\hat{f})$ m-almost everywhere.*
The same holds true for the dual $\mathcal{S}$. Given $\varphi \in L^p(\mathcal{S},\pi)$, $1 \leq p \leq 2$, such that the inverse Hausdorff-Young transform $\check{\varphi} \in L^2(K,m)$. Then $\varphi \in L^2(\mathcal{S},\pi)$ and $\varphi = \wp(\check{\varphi})$ π-almost everywhere.

Proof. For $f \in L^p(K,m)$ exist functions $k_i \in C_c(K)$, $i \in I$, such that the net $(k_i * f)_{i\in I} \in L^p(K,m)$ converges in $L^p(K,m)$ to f, see [129]. Further $k_i * f \in L^p(K,m) \cap C_0(K) \subset L^p(K,m) \cap L^\infty(K,m) \subset L^2(K,m)$. Thus, $(k_i * f)^\wedge = \wp(k_i * f) \in L^2(\mathcal{S},\pi) \cap L^q(\mathcal{S},\pi)$, $1/p + 1/q = 1$. Furthermore, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c(K)$ such that $\|f_n - f\|_p \to 0$ as n tends to infinity. By

$$\left\|(k_i * f)^\wedge - \hat{k_i}\hat{f}\right\|_q \leq \|(k_i * f)^\wedge - (k_i * f_n)^\wedge\|_q + \left\|\hat{k_i}\hat{f_n} - \hat{k_i}\hat{f}\right\|_q \to 0$$

as n tends to infinity, we obtain $(k_i * f)^\wedge = \hat{k_i}\hat{f}$ π-almost everywhere. Hence, by Plancherel's theorem we have

$$\left\|k_i * f - \wp^{-1}(\hat{f})\right\|_2 = \left\|(k_i * f)^\wedge - \hat{f}\right\|_2 = \left\|\hat{k_i}\hat{f} - \hat{f}\right\|_2.$$

Since $(k_i)_{i\in I}$ can be chosen such that $(\hat{k_i})_{i\in I}$ converges uniformly to one on compact subsets of $\mathcal{S}$, see [4, Theorem 2.2.28], we can choose for each $\epsilon > 0$ a compact set $C \subset \mathcal{S}$ such that

$$\int_C |(\hat{k_i} - 1)\hat{f}(\alpha)|^2 d\pi(\alpha) + \int_{\mathcal{S}\setminus C} |(\hat{k_i} - 1)\hat{f}(\alpha)|^2 d\pi(\alpha) < \int_C |(\hat{k_i} - 1)\hat{f}(\alpha)|^2 d\pi(\alpha) + \epsilon/2 \to \epsilon/2.$$

Thus, $\left\|k_i * f - \wp^{-1}(\hat{f})\right\|_2 \to 0$ and we conclude $f = \wp^{-1}(\hat{f})$ m-almost everywhere.
To prove the second statement let $\alpha \in \mathcal{S}$, C a compact neighborhood of α and $(f_n)_{n\in\mathbb{N}}$ in $C_c(K)$ such that $\|(f_n * f_n^*)^\wedge - \chi_C\|_1 \to 0$ as n tends to infinity. Put $\psi = (f_n * f_n^*)^\wedge \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi) \cap C_0(\mathcal{S})$. For any $h \in L^1(K,m)$ we have, applying Parseval's formula and Proposition 2.1.5 ii),

$$\int_{\mathcal{S}} \wp(\check{\varphi})(\bar{\alpha})\hat{h}(\alpha)\psi(\alpha)d\pi(\alpha) = \int_K \check{\varphi}(x)\wp^{-1}(\hat{h}\psi)(x)dm(x) = \int_K \check{\varphi}(x)h * \check{\psi}(x)dm(x).$$

Denoting $\tilde{\psi}(\alpha) := \psi(\bar{\alpha})$ and $\tilde{h}(x) = h(\tilde{x})$ we easily obtain $\tilde{h} * (\tilde{\psi})^\vee(\tilde{x}) = h * \check{\psi}(x)$. Applying successively Fubini's theorem, Proposition 2.1.5 i) and Lemma 2.1.4, we conclude

$$\begin{aligned} \int_{\mathcal{S}} \varphi(\bar{\alpha})\hat{h}(\alpha)\psi(\alpha)d\pi(\alpha) &= \int_K h(\tilde{x})(\varphi\tilde{\psi})^\vee(x)dm(x) = \int_K h(\tilde{x})\check{\varphi} * (\tilde{\psi})^\vee(x)dm(x) \\ = \quad &\int_K \check{\varphi}(\tilde{x})\tilde{h} * (\tilde{\psi})^\vee(x)dm(x) = \int_K \check{\varphi}(x)h * (f_n * f_n^*)(x)dm(x). \end{aligned}$$

Hence, we have

$$\int_{\mathcal{S}} (\wp(\check{\varphi})(\bar{\alpha}) - \varphi(\bar{\alpha}))\hat{h}(\alpha)(f_n * f_n^*)^\wedge(\alpha) d\pi(\alpha) = 0$$

for all $h \in L^1(K,m)$. Since $\left\{\hat{h} : h \in L^1(K,m)\right\}$ is dense in $C_0(\hat{K})$ with respect to $\| \|_\infty$, see [4, Theorem 2.2.4 (ix)], we conclude for each $n \in \mathbb{N}$ that $(\wp(\check{\varphi}) - \varphi)(\bar{\alpha})(f_n * f_n^*)^\wedge(\alpha) = 0$ for almost all $\alpha \in \mathcal{S}$. Since the union of countable many $\pi-$zero sets is a $\pi-$zero set, we conclude $(\wp(\check{\varphi}) - \varphi)(\bar{\alpha})(f_n * f_n^*)^\wedge(\alpha) = 0$ for all $\alpha \in \mathcal{S}\backslash N$, where N is a $\pi-$zero set, for all $n \in \mathbb{N}$. Further, since $\|(f_n * f_n^*)^\wedge - \chi_C\|_1 \to 0$ we can find a subsequence $(f_{n_k} * f_{n_k}^*)_{k\in\mathbb{N}}$ of $(f_n * f_n^*)_{n\in\mathbb{N}}$ such that $(f_{n_k} * f_{n_k}^*)^\wedge(\alpha) - \chi_C(\alpha) \to 0$ for almost all $\alpha \in \mathcal{S}$. Thus, $\wp(\check{\varphi}) = \varphi$ π almost everywhere on C. Therefore, $\wp(\check{\varphi}) = \varphi$ $\pi-$almost everywhere, and in particular $\varphi \in L^2(\mathcal{S}, \pi)$. □

Now we are able to prove our main theorem:

Theorem 2.1.8 (Inversion Hausdorff-Young)**.** *Let $1 \le p \le 2$ and $1 \le r \le 2$. For $f \in L^p(K,m)$ with $\hat{f} \in L^r(\mathcal{S},\pi)$ holds $(\hat{f})^\vee = f$ in $L^p(K,m)$.*
Furthermore, for $g \in L^p(\mathcal{S},\pi)$ such that $\check{g} \in L^r(K,m)$ holds $(\check{g})^\wedge = g$ in $L^p(\mathcal{S},\pi)$.

Proof. First let $f \in L^p(K,m)$ such that $\hat{f} \in L^r(\mathcal{S},\pi)$. Then $\hat{f} \in L^q(\mathcal{S},\pi) \cap L^r(\mathcal{S},\pi) \subset L^2(\mathcal{S},\pi)$, $1/p + 1/q = 1$, and by Proposition 2.1.7 holds $f = \wp^{-1}(\hat{f}) = (\hat{f})^\vee$, since the inverse Hausdorff-Young transform and the inverse Plancherel transform coincide on $L^2(\mathcal{S},\pi) \cap L^r(\mathcal{S},\pi)$. The second statement follows similarly by Proposition 2.1.7. □

Remark 2.1.9. The special case $r = 1$ in Theorem 2.1.8 is of particular interest. If $f \in L^p(K,m)$ and $\hat{f} \in L^1(\mathcal{S},\pi)$, then the integral $\int_{\mathcal{S}} \hat{f}(\alpha)\alpha(x) d\pi(\alpha)$ is equal to $f(x)$ m-almost everywhere.

Corollary 2.1.10 (uniqueness theorem)**.** *Let $1 \le p \le 2$ and $f \in L^p(K,m)$ such that $\hat{f} = 0$ almost everywhere on $\mathcal{S}$. Then $f = 0$ almost everywhere.*
Let $g \in L^p(\mathcal{S},\pi)$ such that $\check{g} = 0$ almost everywhere, then $g = 0$ almost everywhere.

A further consequence of Proposition 2.1.5 is the following corollary.

Corollary 2.1.11. *Let $1 \le p \le 2$, $1/p + 1/q = 1$. Suppose that $f \in L^p(K,m)$, $g \in L^p(\mathcal{S},\pi)$ and $x \in K$. Further let $\varphi \in L^2(\mathcal{S},\pi)$ and $\beta \in \hat{K}$. Then*

i) $(L_x f)^\wedge(\alpha) = \alpha(x)\hat{f}(\alpha)$ for π-almost all $\alpha \in \hat{K}$ and
$(\hat{\epsilon}_x g)^\vee = L_{\tilde{x}}\check{g}$ m-almost everywhere.

ii) Denote by $f^*(x) = \overline{f(\tilde{x})}$.
Then $(f^*)^\wedge = \overline{\hat{f}}$ π-almost everywhere and $(\bar{g})^\vee = (\check{g})^*$ m-almost everywhere.

Proof. i) $(L_x f)^\wedge(\alpha) = (\varepsilon_{\tilde{x}} * f)^\wedge(\alpha) = \hat{\epsilon}_{\tilde{x}}(\alpha)\hat{f}(\alpha) = \alpha(x)\hat{f}(\alpha)$ for π-almost all $\alpha \in \hat{K}$. The second statement follows by Proposition 2.1.5.
ii) See [4, (2.2.32), (2.2.15)]. □

We can show similar results for bounded measures on K

Theorem 2.1.12. *Let $1 \le p \le 2$.*

i) Let $\mu \in M(K)$ such that $\hat{\mu} \in L^p(\mathcal{S},\pi)$ then $d\tilde{\mu} = (\hat{\mu})^\vee dm$.

ii) Let $\mu \in M(\hat{K})$ such that $\check{\mu} \in L^p(K,m)$ then $d\tilde{\mu} = (\check{\mu})^\wedge d\pi$.

Proof. Let $f, g \in L^1(K,m) \cap L^2(K,m)$ and put $h := f * g \in L^1(K,m) \cap L^2(K,m)$. The set $\{f * g : \ f, g \in L^1(K,m) \cap L^2(K,m)\}$ is sup-norm dense in $C_0(K)$. In fact, given some $g \in C_c(K)$ one can approximate g by $f_i * g$, $f_i \in C_c(K)$ with respect to $\| \|_\infty$. Similarly,

$\{(f * g)^\wedge : f, g \in L^1(K,m) \cap L^2(K,m)\}$ is sup-norm dense in $C_0(\hat{K})$.
To prove $i)$, we obtain by [4, Lemma 2.2.23] and by Plancherel's theorem [4, 2.2.34]

$$\int_K h d\tilde{\mu} = \int_{\hat{K}} \hat{h}\hat{\mu} d\pi = \int_K h(\hat{\mu})^\vee dm.$$

Since $\{f * g : f, g \in L^1(K,m) \cap L^2(K,m)\}$ is sup-norm dense in $C_0(K)$, we have $d\tilde{\mu} = (\hat{\mu})^\vee dm$.
To prove $ii)$, we have

$$\int_{\hat{K}} \hat{h}(\alpha) d\mu(\alpha) = \int_{\hat{K}} \int_K h(x)\overline{\alpha(x)} dm(x) d\mu(\alpha) = \int_K h(x)\check{\bar{\mu}}(x) dm(x).$$

Since, $\tilde{h} \in L^2(K,m)$ and $\check{\mu} \in L^p(K,m) \cap C^b(K) \subset L^2(K,m)$, we conclude by Plancherel's theorem [4, Theorem 2.2.34]

$$\int_K h(x)\check{\bar{\mu}}(x) dm(x) = \int_{\hat{K}} \hat{h}(\alpha)(\check{\bar{\mu}})^\wedge(\alpha) d\pi(\alpha).$$

Hence, we find $d\tilde{\mu} = (\check{\mu})^\wedge d\pi$. □

Remark 2.1.13. For $f \in L^1(K,m)$ we have $f dm \in M(K)$ and by Theorem 2.1.12 holds

$$(\hat{f})^\vee dm = \tilde{f} dm = f d\tilde{m}.$$

However, this is not a contradiction to Theorem 2.1.8, since the uniqueness of the Haar measure and the commutativity of K imply $m = \tilde{m}$, see [4, pp. 28].
The same holds for the dual case, since $\|\tilde{f}\|_2 = \|f\|_2$ for every function f on $\hat{K}$ and hence $d\tilde{\pi} = d\pi$ by definition.

Remark 2.1.14. Theorem 2.1.12 implies also, that every measure $\mu \in M(K)$ with $\hat{\mu} \in L^2(\hat{K},\pi)$ is absolutely continuous with respect to m and every measure $\mu \in M(\hat{K})$ with $\check{\mu} \in L^2(K,m)$ is absolutely continuous with respect to π.

2.2 Further Convolution Results

A similar result, but proven with standard arguments is the following.

Proposition 2.2.1. *Let $1 \le p \le 2$, $1/p + 1/q = 1$. For $f \in L^p(K,m)$ and each measure $\mu \in M(K)$ we have $(\mu * f)^\wedge = \hat{\mu}\hat{f}$ π-almost everywhere.*
Especially, for each function $g \in L^1(K,m)$ is $(g * f)^\wedge = \hat{g}\hat{f}$ π-almost everywhere.

Proof. By [4, Lemma 1.4.6], is $\mu * f \in L^p(K,m)$. Thus, $(\mu * f)^\wedge$ is defined by Theorem 2.1.1. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(K,m) \cap L^p(K,m)$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. We have $(\mu * f_n)^\wedge = \hat{\mu}\hat{f}_n$ for each $n \in \mathbb{N}$, see [4, (2.2.15)]. Since the Hausdorff-Young transform is norm decreasing, we obtain $(\mu * f)^\wedge = \hat{\mu}\hat{f}$ π-almost everywhere. The second statement follows by embedding $L^1(K,m)$ into $M(K)$ via the mapping $f \mapsto fm$, $L^1(K,m) \to M(K)$. □

Considering $p = 2$ we can conclude the following proposition immediately

Proposition 2.2.2. *Let $f, g \in L^2(\mathcal{S},\pi)$. Then $(fg)^\vee = \wp^{-1}(f) * \wp^{-1}(g)$ m-almost everywhere.*

Proof. Use [4, (2.2.15)] and Parseval's identity. □

Corollary 2.2.3. *The equality $L^2(K,m) * L^2(K,m) = L^1(\mathcal{S},\pi)^\vee$ holds.*
In particular, $L^2(K,m) * L^2(K,m)$ is a linear space.

In order to show a dual version of the last corollaries we need the following proposition. Note that the following results do only hold for a compact dual space.

Proposition 2.2.4. *Let $\mathcal{S}$ be compact, $\psi \in L^1(\mathcal{S},\pi)$ and $\varphi \in L^2(\mathcal{S},\pi)$. Then $(\psi * \varphi)^\vee = \check{\psi}\check{\varphi}$.*

Proof. Choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$ such that $\|\psi_n - \psi\|_1 \to 0$ as n tends to infinity. Since $\mathcal{S}$ is compact, we obtain $\psi_n * \varphi \in L^1(\mathcal{S},\pi)$ for all $n \in \mathbb{N}$ and by Fubini's theorem we conclude

$$(\psi_n * \varphi)^\vee(x) = \int_{\mathcal{S}}\int_{\mathcal{S}} \psi_n(\beta) M_{\bar{\beta}}\varphi(\alpha)\alpha(x)d\pi(\beta)d\pi(\alpha) = \check{\psi}_n(x)\check{\varphi}(x)$$

for all $x \in K$. The statement follows by

$$\left\|(\psi * \varphi)^\vee - \check{\psi}\check{\varphi}\right\|_\infty \le \|(\psi * \varphi) - (\psi_n * \varphi)\|_2 + \left\|\check{\psi}_n - \check{\psi}\right\|_\infty \|\check{\varphi}\|_\infty \to 0, \text{ as } n \to \infty.$$

□

Corollary 2.2.5. *Let $\mathcal{S}$ be compact. Then $(L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi))^\vee = L^1(K,m)$ as linear spaces.*

Proof. Let $\psi, \varphi \in L^2(\mathcal{S},\pi)$. By Proposition 2.2.4 holds $(\psi * \varphi)^\vee = \check{\psi}\check{\varphi} \in L^1(K,m)$.
Conversely, for each $h \in L^1(K,m)$ exist $\psi, \varphi \in L^2(\mathcal{S},\pi)$ such that $h = \check{\psi}\check{\varphi}$ in $L^1(K,m)$. Hence, $h \in (L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi))^\vee$. □

Corollary 2.2.6. *Let $\mathcal{S}$ be compact. Then $L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi) = L^1(K,m)^\wedge$ as linear spaces.*

Concerning the translation on the dual, the following result holds and can be proven with Proposition 2.2.4.

Corollary 2.2.7. *Let $\mathcal{S}$ be compact and $\varphi \in L^2(\mathcal{S},\pi)$.
Then $(M_\alpha\varphi)(\beta) = (M_\beta\varphi)(\alpha)$ for π-almost all $\alpha, \beta \in \hat{K}$.*

Proof. There exists a unique $g \in L^2(K,m)$ such that $\varphi = \wp(g)$ in $L^2(\mathcal{S},\pi)$. Further choose an arbitrary $f \in L^2(K,m)$. By Proposition 2.2.4 and Parseval's identity holds

$$\wp(f) * \wp(g)(\alpha) = (fg)^\wedge(\alpha) = \int_K f(x)g(x)\overline{\alpha(x)}dm(x) = \int_{\mathcal{S}} \wp(f)(\beta)\wp(\bar{\alpha}g)(\bar{\beta})d\pi(\beta).$$

$\wp(f) * \wp(g)(\alpha) = \int_{\mathcal{S}} \wp(f)(\beta)M_{\bar{\beta}}(\wp(g))(\alpha)d\pi(\beta)$ by definition. Hence, we conclude

$$\int_{\mathcal{S}} \wp(f)(\beta)[M_{\bar{\beta}}(\wp(g))(\alpha) - \wp(\bar{\alpha}g)(\bar{\beta})]d\pi(\beta) = 0.$$

Since f was chosen arbitrary, we obtain $M_{\bar{\beta}}(\varphi)(\alpha) = \wp(\bar{\alpha}g)(\bar{\beta}) = M_\alpha(\varphi)(\bar{\beta})$ for π-almost all $\alpha, \beta \in \hat{K}$. □

2.3 Further Consequences of the main Theorem

Theorem 2.3.1 (Generalization of Parseval's Identity)**.** *We have for $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

1. *For $f \in L^p(K,m)$, $g \in L^p(\mathcal{S},\pi)$ holds*
$$\int_K f(x)\overline{\check{g}(x)}dm(x) = \int_{\mathcal{S}} \hat{f}(\alpha)\overline{g(\alpha)}d\pi(\alpha).$$
2. *For K compact, $f \in L^p(K,m)$ and $g \in L^q(K,m)$ such that $\hat{g} \in L^p(\mathcal{S},\pi)$ holds*
$$\int_K f(x)\overline{g(x)}dm(x) = \int_{\mathcal{S}} \hat{f}(\alpha)\overline{\hat{g}(\alpha)}d\pi(\alpha).$$
3. *For $\mathcal{S}$ compact, $\varphi \in L^p(\mathcal{S},\pi)$ and $\psi \in L^q(\mathcal{S},\pi)$ such that $\check{\psi} \in L^p(K,m)$ is*
$$\int_{\mathcal{S}} \varphi(\alpha)\overline{\psi(\alpha)}d\pi(\alpha) = \int_K \check{\varphi}(x)\overline{\check{\psi}(x)}dm(x).$$

Proof. The proof follows the lines of [77, 31.48]. □

Theorem 2.3.2. *Let $1 < p < 2$ and $1/p+1/q = 1$. The mapping $f \mapsto \hat{f}$, $L^p(K,m) \to L^q(\mathcal{S},\pi)$ is onto if and only if K is finite.*

Proof. If K is finite the mapping $f \mapsto \hat{f}$, $L^p(K,m) \to L^q(\mathcal{S},\pi)$ is obviously onto. Conversely, let K be infinite and suppose that every function in $L^q(\mathcal{S},\pi)$ is the Hausdorff-Young transform of a function in $L^p(K,m)$. Thus, the mapping $f \mapsto \hat{f}$, $L^p(K,m) \to L^q(\mathcal{S},\pi)$ is linear, bijective and continuous. Hence, by a theorem of Banach the inverse mapping is also continuous and there exists a constant $C > 0$ such that $\left\|\hat{f}\right\|_q \le \|f\|_p \le C\left\|\hat{f}\right\|_q$.
Now consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(K,m)$, which converges weakly to zero in $L^p(K,m)$ and fulfills $\|f_{n_1} + f_{n_2} + ... + f_{n_m}\|_p = m^{1/p}$ for all subsets $\{f_{n_1}, f_{n_2}, ..., f_{n_m}\}$ of $(f_n)_{n\in\mathbb{N}}$, $m = 1,2,....$ Such a sequence exists by [75, Lemma A]. By Theorem 2.3.1 the sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ converges weakly to zero in $L^q(\mathcal{S},\pi)$. By Lemma B in [75], there exists a subsequence $(\hat{f}_{n_k})_{k\in\mathbb{N}}$ of $(\hat{f}_n)_{n\in\mathbb{N}}$ and a constant $A > 0$ such that $\left\|\sum_{k=1}^m \hat{f}_{n_k}\right\|_q \le Am^{1/2}$. We obtain

$$m^{1/p} = \|f_{n_1} + f_{n_2} + ... + f_{n_m}\|_p \le C\left\|\sum_{k=1}^m \hat{f}_{n_k}\right\|_q \le ACm^{1/2},$$

for all $m = 1,2,3....$ We see at once that $\frac{1}{p} \le \frac{1}{2}$, which contradicts our hypothesis. Hence, the mapping $f \mapsto \hat{f}$, $L^p(K,m) \to L^q(\mathcal{S},\pi)$ cannot be onto. □

The proof of Hewitt [75] is also extendable to the dual case and we conclude:

Theorem 2.3.3. *Let $1 < p < 2$ and $1/p+1/q = 1$. The mapping $f \mapsto \check{f}$, $L^p(\mathcal{S},\pi) \to L^q(K,m)$ is onto if and only if K is finite.*

If K is not necessarily finite, we can still prove that the range of the mapping $f \mapsto \check{f}$, $L^p(\mathcal{S},\pi) \to L^q(K,m)$, is dense in $L^q(K,m)$. For that, we need the following lemma.

Lemma 2.3.4. *Let A be a compact subset of K and H an open subset of K such that $A \subset H$. Then there is a function $\psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ such that $\check{\psi} \in C_c(K)$ and $\chi_A \le \check{\psi} \le \chi_H$.*

Proof. We may suppose that H has compact closure in K. Let P be a m-measurable symmetric neighborhood of $e \in K$ such that $P * P * A \subset H$. Let f be the function $f = \frac{1}{m(P)}\chi_{P*A} * \chi_P$ on K. By Corollary 2.2.3 there exists a function $\psi \in L^1(\mathcal{S},\pi)$ such that $f = \check{\psi}$. Further holds

$$\check{\psi}(x) = f(x) = \frac{1}{m(P)}(\int_P \omega(x,y)(P * A)dm(y))$$

and $(P * \{x\}) \cap (P * A) = \emptyset$ if and only if $\tilde{P} * P * A \cap \{x\} = \emptyset$. Hence, it is obvious that $\chi_A \le \check{\psi} \le \chi_{P*P*A} \le \chi_H$. Since $\check{\psi} \in L^2(K,m)$, we obtain $\psi \in L^2(\mathcal{S},\pi)$ with Proposition 2.1.7. □

Remark 2.3.5. M. Lashkarizadeh Bami, M. Pourgholamhossein and H. Samea proved Lemma 2.3.4, too, see [104, Proposition 2.4]. A similar proof for Abelian groups can be found in [77, (31.34)].

Remark 2.3.6. The dual spaces $\hat{K}$ or $\mathcal{S}$ do, in general, not bear a dual hypergroup structure. Therefore, we are not able to prove a dual version of Lemma 2.3.4 and to say something about the range of the mapping $f \mapsto \hat{f}$, $L^p(K,m) \to L^q(\mathcal{S},\pi)$.

The following proposition follows immediately.

Proposition 2.3.7. *The equality $L^1(\mathcal{S},\pi)^\vee C_c(K) = C_c(K)$ as linear spaces is valid.*

Proposition 2.3.8. *For $1 < p \le 2$ is $L^p(\mathcal{S},\pi)^\vee$ a $\| \|_q$-dense linear subspace in $L^q(K,m)$, $1/p + 1/q = 1$.*

Proof. For $p = 2$ the statement is obviously true. Therefore, suppose $1 < p < 2$. Consider a m-measurable subset B of K such that $m(B) < \infty$. Given $\epsilon > 0$, let A be a compact subset of B and H an open subset of K, such that $B \subset H$ and $m(H \setminus A) < \epsilon^q$. There exists a function $f \in (L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi))^\vee \subset L^p(\mathcal{S},\pi)^\vee$ such that $\chi_A \le f \le \chi_H$ by Lemma 2.3.4. Then $\|f - \chi_B\|_q < \|\chi_H - \chi_A\|_q < \epsilon$. Linear combinations of functions χ_B are dense in $L^q(K,m)$, and hence, $L^p(\mathcal{S},\pi)^\vee$ is dense in $L^q(K,m)$. □

Concluding, we give two further results which are interesting in the context of harmonic analysis.

Theorem 2.3.9. *Let $1 \le p \le 2$, μ be in $M(K)$, $f \in L^p(K,m)$ and suppose that $\hat{\mu} = \hat{f}$ π-almost everywhere on $\hat{K}$. Then f is in $L^1(K,m)$, μ is absolutely continuous and the Radon-Nikodym derivative of μ is f.*

Proof. The proof follows the lines of [77, 31.33]. □

Theorem 2.3.10. *Suppose that f is a function in $L^1(K,m) \cap L^\infty(K,m)$ and that $\hat{f}$ is a non-negative function. Then $\hat{f} \in L^1(\mathcal{S},\pi)$ and $\left\|\hat{f}\right\|_1 \le \|f\|_\infty$.*

Proof. Let $f \in L^1(K,m) \cap L^\infty(K,m)$. By Hölder's interpolation theorem holds $f \in L^2(K,m)$ and hence $\hat{f} = \wp(f) \in L^2(\mathcal{S},\pi)$. Let $(k_i)_{i\in I} \in C_c(K)$ be an approximate identity in $L^1(K,m)$. By Parseval's theorem follows for all $i \in I$ that

$$\int_{\mathcal{S}} \hat{f}(k_i * k_i^*)^\wedge d\pi = \int_K f\overline{(k_i * k_i^*)} dm \le \|f\|_\infty \|k_i * k_i^*\|_1 \le \|f\|_\infty .$$

$\hat{f}(k_i * k_i^*)^\wedge$ converges pointwise to $\hat{f}$ and $\hat{f}|\hat{k}_i|^2$ is by assumption not negativ. Applying Fatou's lemma we obtain

$$\int_{\mathcal{S}} \hat{f} d\pi = \int_{\mathcal{S}} \lim_i \hat{f}(k_i * k_i^*)^\wedge d\pi \le \limsup_i \int_{\mathcal{S}} \hat{f}(k_i * k_i^*)^\wedge d\pi \le \|f\|_\infty .$$

□

Remark 2.3.11. A result similar to Theorem 2.3.10 for Abelian groups can be found in [77, 31.42].

Remark 2.3.12. It is an open question whether Theorem 2.3.9 and Theorem 2.3.10 admit dual versions.

Remark 2.3.13. Even if it is not our intention to determine optimal estimates for the norm, we want to remark that Rodionov did establish expansions of functions in the L^p-space with respect to systems similar to orthogonal ones. His results are analogues of the Hausdorff-Young theorems in the theory of trigonometric series, see [135]. However, Rodionov's results apply to only a few polynomial hypergroups, since orthonormal polynomials, which are also bounded, are very rare.

Remark 2.3.14. M. S. Ramanujan and N. Tanović-Miller [131] generalized the Hausdorff-Young Theorem to mixed norm sequence spaces. Using their results they characterized multipliers for mixed norm sequence spaces.

Chapter 3

Multipliers for $L^p(K,m)$

As outlined in the introduction, multipliers for L^p spaces over various groups have been studied intensively in the past, see for instance [2], [22], [33],[42], [56], [84], [77] and [101]. Even multipliers for L^p spaces over hypergroups are investigated. Stempak [152] established a version of Hörmanders multiplier theorem on Bessel-Kingman hypergroups, which are a special class of Chèbli-Trimèche hypergroups. Hence, W.R. Bloom and Z. Xu [5] generalized these results to the whole class of Chèbli-Trimèche hypergroups. Moreover, H. Emamirad and G.S. Heshmati characterized multipliers for the dual hypergroup $K = [-1, 1]$ of the discrete hypergroup generated by the ultraspherical polynomials $R_n^{(\alpha,\alpha)}(t)$.
Our aim here is to generalize the characterizations of multipliers of specific hypergroups to all commutative hypergroups.
Some of the results in this chapter were already proven in the author's Master thesis [19]. This in particular concerns the theory of pseudomeasures and the basic characterizations of multipliers for $L^2(K,m)$ and $L^p(K,m)$, which are quoted in Theorem 3.2.8 and Theorem 3.3.4. These results are also quoted in [20]. However, there are also new results added as well as some example, e.g. 3.4.4.

3.1 Introduction and Multipliers for $L^1(K,m)$

Definition 3.1.1. A **multiplier** $T: L^p(K,m) \to L^p(K,m)$, $1 \le p < \infty$, is a bounded linear operator from $L^p(K,m)$ into $L^p(K,m)$, which commutes with all translation operators L_x, $x \in K$, i.e. $T \circ L_x = L_x \circ T$. We denote the space of multipliers for $L^p(K,m)$ by $M(L^p(K,m))$.

$M(L^p(K,m))$ is a closed subalgebra of $B(L^p(K,m))$, since the composition of operators in $B(L^p(K,m))$ is continuous.
In order to establish a complete characterization of the multipliers for $L^p(K,m)$, $1 \le p < \infty$, we basically use the arguments in the proof of Theorem 4.1.1 in [101] to prove that the space $M(L^p(K,m))$ coincides with the space of convolutors of $L^p(K,m)$, where $T \in B(L^p(K,m))$ is called **convolutor**, if $T(f * g) = f * Tg$ for all $f, g \in L^1(K,m) \cap L^p(K,m)$. Pavel proved this characterization of a multiplier in [129].

Proposition 3.1.2. *Let $T \in B(L^p(K,m))$ and $1 \le p < \infty$. Then T is an element of $M(L^p(K,m))$ if and only if*

$$Tf * g \;=\; T(f * g) \;=\; f * Tg \qquad \text{for all } f, g \in L^1(K,m) \cap L^p(K,m).$$

Proof. Let $T \in M(L^p(K,m))$. For each $f \in L^p(K,m)$ the mapping $y \mapsto L_y f$, $K \to L^p(K,m)$ is continuous. For $f, g \in C_c(K)$ the convolution $f * g$ is a $L^p(K,m)-$valued integral. Hence, we obtain by the multiplier characteristic of T

$$Tf * g = \int_K g(y) L_{\tilde{y}}(Tf)\,dm(y) = \int_K g(y) T(L_{\tilde{y}} f)\,dm(y).$$

Since $f, g \in C_c(K)$, we can interpret the integral as a limit of Riemann sums. By the additivity and the continuity of T we conclude

$$\int_K g(y) T\left(L_{\tilde{y}} f\right) dm(y) = T\left(\int_K g(y) L_{\tilde{y}} f \, dm(y)\right) = T(f * g).$$

For $f, g \in L^1(K, m) \cap L^p(K, m)$ we choose sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ in $C_c(K)$ such that

$$\|f_n - f\|_p \to 0, \qquad \|g_n - g\|_1 \to 0 \text{ for } n \to \infty.$$

Thus,

$$\begin{aligned}
& \|T(f * g) - Tf * g\|_p \\
\leq\ & \|T(f * g) - T(f_n * g_n)\|_p + \|T(f_n * g_n) - Tf_n * g\|_p + \|Tf_n * g - Tf * g\|_p \\
\leq\ & \|T\| \left(\|f * g - f_n * g\|_p + \|f_n * g - f_n * g_n\|_p\right) + \|Tf_n\|_p \|g_n - g\|_1 + \|T\| \|g\|_1 \|f_n - f\|_p \\
\leq\ & \|T\| \|g\|_1 \|f - f_n\|_p + \|T\| \|f_n\|_p \|g - g_n\|_1 + \|T\| \|f_n\|_p \|g_n - g\|_1 + \|T\| \|g\|_1 \|f_n - f\|_p \\
& \to 0
\end{aligned}$$

as n tends to infinity. Hence, we have $Tf * g = T(f * g)$ for all $f, g \in L^1(K, m) \cap L^p(K, m)$. Interchanging the roles of f and g leads to $Tf * g = T(f * g) = f * Tg$ for all $f, g \in L^1(K, m) \cap L^p(K, m)$.
Conversely, choose $f_i \in C_c(K),\ i \in I$ with $\lim_i \|f_i * g - g\|_p = 0$. Since $L_x f_i * g = L_x(f_i * g)$ we obtain for a convolutor T and any $g \in L^1(K, m) \cap L^p(K, m)$

$$T(L_x f_i * g) \ = \ L_x f_i * Tg \ = \ f_i * L_x Tg \ \longrightarrow \ L_x Tg$$

and by the continuity of T and L_x we have $T(L_x f_i * g) \ = \ T(L_x(f_i * g)) \ \longrightarrow \ T\, L_x g$ for all $g \in L^1(K, m) \cap L^p(K, m)$. Now it is obvious that $T \in M(L^p(K, m))$. □

Corollary 3.1.3. *Let $T, S \in M(L^p(K, m))$, $1 \leq p < \infty$. Then $T \circ S = S \circ T$.*

Corollary 3.1.4. *Let $1 \leq p \leq 2$ and $T \in M(L^p(K, m))$. T is bijective if and only if T^{-1} exists and $T^{-1} \in M(L^p(K, m))$.*

Proof. Let $T \in M(L^p(K, m))$ be bijective. Since $T \in M(L^p(K, m))$we obtain

$$T^{-1} f * g = T^{-1} T(T^{-1} f * g) = T^{-1}(f * g) = T^{-1}(f * TT^{-1} g) = f * T^{-1} g$$

for all $f, g \in L^1(K, m) \cap L^p(K, m)$. By Proposition 3.1.2 is $T^{-1} \in M(L^p(K, m))$. □

R. Lasser already generalized Wendel's classical result and Helson's result for multipliers for $L^1(K, m)$ in case of K being commutative, see [105, Corollary 2.2] or [4, Theorem 1.6.24]. For the sake of completeness, the characterizations of $T \in M(L^1(K, m))$ are formulated here again.

Theorem 3.1.5. *Let $T \in B(L^1(K, m))$. The following conditions are equivalent:*

i) $T \in M(L^1(K, m))$.

ii) $Tf * g \ = \ T(f * g) \ = \ f * Tg$ *for all* $f, g \in L^1(K, m)$.

iii) There exists a unique measure $\mu \in M(K)$ such that

$$Tf \ = \ \mu * f \qquad \text{for all } f \in L^1(K, m).$$

iv) There exists a unique measure $\mu \in M(K)$ such that

$$(Tf)^\wedge\, |\mathcal{S} \ = \ \hat{\mu}\, \hat{f} | \mathcal{S} \qquad \text{for all } f \in L^1(K, m).$$

v) There exists a unique function $\varphi \in C(\mathcal{S})$ such that

$$(Tf)^\wedge \,|\mathcal{S} \;=\; \varphi \hat{f}|\mathcal{S} \qquad \text{for all } f \in L^1(K,m).$$

Moreover, the correspondence between T and μ defines an isometric algebra isomorphism from $M(L^1(K,m))$ onto $M(K)$, such that $\|\varphi\|_\infty \;\le\; \|\mu\| \;=\; \|T\|$.

Proof. The proof of the equivalence of $i)$ to $v)$ is in [105, Corollary 2.2]. □

Remark 3.1.6. It should be noted that the implication $v) \Rightarrow iv)$ actually establishes that φ is a bounded function.

Remark 3.1.7. If a multiplier $T \in M(L^1(K,m))$ is submarkovian, i.e. $0 \le Tf \le 1$ m-almost everywhere for all $f \in L^1(K,m)$ with $0 \le f \le 1$ $m-$almost everywhere, then the corresponding measure $\mu \in M(K)$ is a positive contractive measure on K, see [4, Theorem 1.6.24].

Corollary 3.1.8. *The Banach algebra $M(L^1(K,m))$ is isometrically isomorphic to $M(K)$.*

There are two more results we can quote here. Similar results for locally compact Abelian group can be found in Larsen [101, Theorem 0.1.2].

Theorem 3.1.9. *The following hold:*

i) The set of all convolution operators $\{L_g :\; g \in L^1(K,m)\}$ is dense in $M(L^1(K,m))$ with respect to the strong operator topology.

ii) The set of all finite linear combinations of translation operators L_x, $x \in K$, is dense in $M(L^1(K,m))$ with respect to the strong operator topology.

Proof. Let $(k_i)_{i\in I}$ be an approximate identity in $L^1(K,m)$, i.e. $\lim_i \|k_i * f - f\|_1 = 0$ for all $f \in L^1(K,m)$. We have for all $\mu \in M(K)$

$$\lim_i \|\mu * f - \mu * k_i * f\|_1 \le \lim_i \|f - k_i * f\|_1 \|\mu\| = 0$$

for all $f \in L^1(K,m)$. Hence, we have for all $T \in M(L^1(K,m))$ with $Tf = \mu * f$,

$$\lim_i \|Tf - L_{\mu * k_i} f\|_1 = \lim_i \|\mu * f - \mu * k_i * f\|_1 = 0.$$

To show the second statement, it is sufficient to prove that if F is a strong operator continuous linear functional on the space of operators on $L^1(K,m)$, which vanishes on the space of finite linear combinations of translation operators, then it vanishes at every multiplier for $L^1(K,m)$. This is due to the Hahn-Banach Theorem. For an operator T on $L^1(K,m)$ a strong operator continuous linear functional has the form

$$F(T) = \sum_{i=1}^{n} h_i(Tf_i)$$

for $f_i \in L^1(K,m)$ and $h_i \in L^1(K,m)^*$, $i = 1,2,...,n$ (see [101, D. 8.1]). Since the dual of $L^1(K,m)$ is $L^\infty(K,m)$, there exist a $\varphi_i \in L^\infty(K,m)$ such that

$$h_i(Tf_i) = \int_K (Tf_i)(x)\overline{\varphi_i(x)}dm(x)$$

for all $i = 1,2,...,n$. Since F vanishes on the space of all translation operators, we obtain for every multiplier T, $Tf = \mu * f$, on $L^1(K,m)$ the following

$$F(T) \;=\; \sum_{i=1}^{n} \int_K Tf_i(x)\overline{\varphi_i(x)}dm(x) = \sum_{i=1}^{n} \int_K (\mu * f_i(x))\overline{\varphi_i(x)}dm(x)$$

$$= \sum_{i=1}^{n} \int_K \left[\int_K \omega(\tilde{y},x)(f_i) d\mu(y) \right] \overline{\varphi_i(x)} dm(x)$$
$$= \int_K \left[\sum_{i=1}^{n} \int_K L_{\tilde{y}} f_i(x) \overline{\varphi_i(x)} dm(x) \right] d\mu(y) = \int_K F(L_{\tilde{y}}) d\mu(y) = 0$$

□

Remark 3.1.10. In addition to Theorem 3.1.9 Obata [126, Corollary 3.8] proved that every surjective, isometric multiplier T for $L^1(K,m)$ is characterized by $Tf = \gamma\varepsilon_x * f$, where x is an element in the centre of K, i.e.

$$x \in G(K) := \{x \in K : \ \varepsilon_x * \varepsilon_{\tilde{x}} = \varepsilon_e = \varepsilon_{\tilde{x}} * \varepsilon_x\}$$

and $\gamma \in \mathbb{C}$, $|\gamma| = 1$.
Conversely, every measure $\mu = \gamma\varepsilon_x$ with $x \in G(K)$ and $\gamma \in \mathbb{C}$, $|\gamma| = 1$, defines a surjective, isometric multiplier for $L^1(K,m)$. Hence, the surjective, isometric multipliers for $L^1(K,m)$ characterize the group part $G(K)$ of K, see [4, pp. 68].

Before we continue to characterize multipliers for $L^p(K,m)$, $1 < p < \infty$, we want to mention that the set of multipliers for $L^1(K,m)$ is the same as the space of multipliers for $C_0(K)$ given the usual norm $\| \ \|_\infty$. Here, an operator T in $B(C_0(K))$ is called **multiplier** for $C_0(K)$, whenever it commutes with all translation operators, that is $L_x \circ T = T \circ L_x$ for all $x \in K$. The set of all multipliers for $C_0(K)$ is denoted by $M(C_0(K))$.

Theorem 3.1.11. *$M(C_0(K))$ is isometrically isomorphic to $M(K)$.*

Proof. Let $T \in M(C_0(K))$. Since T is continuous, $f \mapsto Tf(e)$ defines a continuous linear functional on $C_0(K)$. By Riesz's representation theorem there exists a unique $\mu \in M(K)$ such that $Tf(e) = \mu(f) = \tilde{\mu} * f(e)$. Applying this equation to $L_x f$ yields

$$Tf(x) = L_x Tf(e) = TL_x f(e) = \tilde{\mu} * L_x f(e) = \mu(L_x f) = \tilde{\mu} * f(x)$$

for all $x \in K$ and $f \in C_0(K)$. Moreover, $|\tilde{\mu}(f)| \leq \|T\| \|f\|_\infty$ for all $f \in C_0(K)$. Hence, we conclude $\|\tilde{\mu}\| \leq \|T\|$.
Conversely, the operator on $C_0(K)$ defined by $Tf := \tilde{\mu} * f$ is obviously a multiplier, since we have $\tilde{\mu} * L_x f(y) = \tilde{\mu} * \varepsilon_{\tilde{x}} * \varepsilon_{\tilde{y}} * f(e) = \varepsilon_{\tilde{x}} * \tilde{\mu} * f(y) = L_x(\tilde{\mu} * f(y))$, and $\|Tf\|_\infty = \|\tilde{\mu} * f\| \leq \|\tilde{\mu}\| \|f\|_\infty$. Hence, $M(K)$ is isometrically isomorphic to $M(C_0(K))$. □

Corollary 3.1.12. *$M(C_0(K))$ is isometrically isomorphic to $M(L^1(K,m))$.*

Remark 3.1.13. Following the lines of proof 3.1.5, we can transfer Theorem 3.1.5 to multipliers for $C_0(K)$.

An Application

We want to show a short application of Wendel's theorem. Figà-Talamanca and Gaudry [43] proved similar results in the context of locally compact groups. Sakai [140] who studied compact multipliers for $L^1(G)$, showed the same application for non-compact, non-Abelian groups. He proved also for a non-compact, locally compact group G, that zero is the only weakly compact multiplier of $L^1(G)$. Ghahramani and Medghalchi[61, 62] extended similar results to the hypergroup case.

Definition 3.1.14. An operator $T \in B(L^1(K,m))$ is called **spectrally continuous** if there exists a constant $r > 0$ satisfying

$$\|Th\|_1 \leq r \, \|L_h\|$$

for all $h \in L^1(K,m)$, $L_h \in B(L^2(K,m))$.

Theorem 3.1.15. *Let K be a commutative, non-compact hypergroup. Then each spectrally continuous multiplier $T \in M(L^1(K,m))$ is identically zero.*

Proof. Let $R(K,m)$ denote the closure of the set $\{L_f : f \in L^1(K,m)\}$ in the C^*-algebra $B(L^2(K,m))$, of all bounded linear operators on $L^2(K,m)$. Then $R(K,m)$ is a C^*-algebra. $T \in B(L^1(K,m))$ can also be interpreted as a well-defined operator on $\{L_f : f \in L^1(K,m)\}$ into $L^1(K,m)$. Since T is spectrally continuous, it can then be uniquely extended to a bounded operator $\tilde{T}$ of $R(K,m)$ into $L^1(K,m)$. Since $R(K,m)$ is a C^*-algebra, $\tilde{T}$ is weakly compact, see [140, Proposition 1]. Denote by S the unit sphere of $R(K,m)$. Since $\|L_f\| \le \|f\|_1$ for each $f \in L^1(K,m)$, we obtain that $S \cap L^1(K,m) := \{f \in L^1(K,m) : \|L_f\| \le 1\}$ contains the unit sphere of $L^1(K,m)$. Therefore, the set $\{Th : h \in L^1(K,m),\ \|h\|_1 \le 1\}$ is relatively weakly compact in $L^1(K,m)$, since $\tilde{T}$ is weakly compact and T and $\tilde{T}$ coincide on $\{h \in L^1(K,m) : \|h\|_1 \le 1\}$. This implies that T is weakly compact as an operator on $L^1(K,m)$. Since T is a multiplier for $L^1(K,m)$, by Wendel's theorem there exists a bounded measure $\mu \in M(K)$ such that $Th = \mu * h$ for all $h \in L^1(K,m)$. Let $f \in L^1(K,m)$, then the mapping

$$L^1(K,m) \to L^1(K,m), \qquad h \mapsto (f*\mu)^* * (f*\mu) * h$$

is weakly compact, where $f^*(x) := \overline{\tilde{f}(x)}$. Hence, by [3, Corollary 3.7] the mapping

$$L^1(K,m) \to L^1(K,m), \qquad h \mapsto (f*\mu)^* * (f*\mu) * (f*\mu)^* * (f*\mu) * h$$

is compact. Let $g = (f*\mu)^* * (f*\mu) * (f*\mu)^* * (f*\mu) \in L^1(K,m)$ and S_1 denote the unit sphere of $L^1(K,m)$. Then $g * S_1$ is a relatively compact subset in $L^1(K,m)$. Let $\{v_i\}_{i\in I}$ be a fundamental family of compact neighborhoods at a point s of K and let $\{f_i\}_{i\in I}$ be a family of continuous positive functions on K, such that the support of f_i is contained in v_i for all $i \in I$ and $\int_K f_i(x)dx = 1$. Then $\{f_i * g\}_{i\in I}$ converges to $L_s g$ in $L^1(K,m)$. L_s is a continuous operator on $L^1(K,m)$, therefore the set $\{L_s g : s \in K\}$ is relatively compact. Now suppose that $\|g\|_1 \neq 0$ and hence let $\|g\|_1 = 1$. Then there exists a finite set $\{L_{s_1}g, L_{s_2}g, ..., L_{s_n}g\}$, $s_i \in K$ for all $i \in I$, such that

$$\inf_{1\le i\le n} \|L_s g - L_{s_i} g\|_1 \le \frac{1}{2}$$

for all $s \in K$. On the other hand, let C be a compact subset of K such that

$$\int_{K\setminus C} |g(x)|dm(x) < \frac{1}{10} \quad \text{and} \quad \int_{K\setminus C} |L_{s_i} g(x)|dm(x) < \frac{1}{10},$$

for $i = 1, 2, ..., n$. Such a subset C exists, since $g, L_{s_i} g \in L^1(K,m)$ for $i = 1, 2, ..., n$. Further, let s be an element in K such that $s \notin C*C$. Then $\{\tilde{s}\}*C \cap C = \emptyset$, see [4, 1.2.11], and therefore

$$\begin{aligned}
\|L_s g - L_{s_i} g\|_1 &= \int_C |(L_s g - L_{s_i} g)(x)|dm(x) + \int_{K\setminus C} |(L_s g - L_{s_i} g)(x)|dm(x) \\
&\ge \int_C |L_{s_i} g(x)|dm(x) - \int_C |L_s g(x)|dm(x) \\
&\quad + \int_{K\setminus C} |L_s g(x)|dm(x) - \int_{K\setminus C} |L_{s_i} g(x)|dm(x) \\
&\ge (1 - \frac{1}{10}) - \int_C |L_s g(x)|dm(x) - \frac{1}{10}
\end{aligned}$$

Now we need to check the maximum value of $\int_C |L_s g| dm(x)$. We obtain

$$\begin{aligned}
&\int_C |L_s g(x)|dm(x) \le \int_C L_s(|g|)(x)dm(x) \\
= &\int_K \chi_C(x) L_s(|g|)(x)dm(x) = \int_K L_{\tilde{s}}\chi_C(x)|g|(x)dm(x).
\end{aligned}$$

Furthermore, we have

$$L_{\tilde{s}}\chi_C(x) = \int_{\text{supp}_{\omega(\tilde{s},x)}} \chi_C(z)d\omega(\tilde{s},x)(z) = \int_{\text{supp}_{\omega(\tilde{s},x)\cap C}} 1d\omega(\tilde{s},x)(z).$$

Since $\{\tilde{s}\} * C \cap C = \emptyset$ we have $L_{\tilde{s}}\chi_C(x) = 0$ for all $x \in C$. Therefore, we conclude

$$\begin{aligned} &\int_C |L_s g(x)|dm(x) \le \int_K L_{\tilde{s}}\chi_C(x)|g|(x)dm(x) \\ = &\int_{K\setminus C} L_{\tilde{s}}\chi_C(x)|g|(x)dm(x) \le \int_{K\setminus C} |g|(x)dm(x) \le \frac{1}{10}. \end{aligned}$$

All together, the inequalities above lead us to

$$\|L_s g - L_{s_i} g\|_1 \ge (1 - \frac{1}{10}) - \int_C |L_s g(x)|dm(x) - \frac{1}{10} \ge \frac{7}{10}$$

This is a contradiction to $\inf_{1\le i\le n} \|L_s g - L_{s_i} g\|_1 \le \frac{1}{2}$ for all $s \in K$. Hence, $g = 0$ and therefore $f * \mu = 0$. Since f is an arbitrary element of $L^1(K,m)$ we have $\mu = 0$. Thus, $T = 0$. □

Remark 3.1.16. The first part of proof 3.1.15 follows the lines of Sakai [140]. However, Sakai uses in the second part of his proof the isometric property of the left translation. This property is in general not given for hypergroups. Hence, we used different estimates to reach the contradiction.

Remark 3.1.17. Gaudry [53], Grothendieck [67] and Helgason [71, 72] proved the same result, but with various restrictions, for some special non-compact groups.

Remark 3.1.18. A similar result for compact groups G is available in [72], which states that every spectrally continuous multiplier $T \in M(L^1(G))$ equals a convolution operator L_f with $f \in L^2(G)$. However, the proof is based on the fact that $|\alpha(x)| = 1$ for all $x \in G$ and $\alpha \in \hat{G}$. Hence, this result cannot be proven for compact hypergroups in a similar way. In fact every commutative hypergroup with only unitary characters is already a locally compact Abelian group [4, Corollary 2.2.12].

3.2 Multipliers for $L^2(K,m)$

Now we consider $p = 2$. It is obvious, that every measure $\mu \in M(K)$ defines a multiplier for $L^2(K,m)$). For $p = 1$ this conversely characterizes every multiplier for $L^1(K,m)$. This is no longer true for $p > 1$. As a replacement for bounded measures on K we can define pseudomeasures. We will see, that every multiplier for $L^p(K,m)$, $p > 1$, can be characterized through a pseudomeasure on K. Hence, the space of pseudomeasures contains the set of all bounded measures on K. Pseudomeasures will be the main tool, to characterize multipliers operating on various Banach spaces. On the contrary, one needs quasimeasures, a generalization of pseudomeasures, to characterize multipliers which operate between different Banach spaces, see e.g. [52].
We continue with a few results concerning pseudomeasures on K.

Definition 3.2.1. We call the set $A(K) := \{\check{\varphi} : \varphi \in L^1(\mathcal{S},\pi)\}$ Fourier space of the hypergroup K. With the norm $\|\check{\varphi}\|_A := \|\varphi\|_1$, $A(K)$ becomes a Banach space.

Remark 3.2.2. By the uniqueness theorem for the inverse Fourier transform $\|\check{\varphi}\|_A := \|\varphi\|_1$ defines indeed a norm on $A(K)$, see [4, Theorem 2.2.35].

Remark 3.2.3. Muruganandam showed in [124] some properties of the Fourier space $A(K)$, for instance for every function $f \in A(K)$, we have also $\tilde{f}$, $\bar{f}$ and $L_x f$ in $A(K)$, for all $x \in K$.

Definition 3.2.4. The space of all continuous linear functionals on $A(K)$ is denoted by $P(K)$, the elements σ of $P(K)$ are called **pseudomeasures** on K and

$$\|\sigma\|_P = \sup\{\, |\sigma(\check{\varphi})| \; : \; \|\check{\varphi}\|_A \leq 1\}$$

is a norm on $P(K)$.

Since the dual space of $L^1(\mathcal{S}, \pi)$ is isometrically isomorphic to $L^\infty(\mathcal{S}, \pi)$, see [76, Theorem 12.18], $P(K)$ is isometrically isomorphic to $L^\infty(\mathcal{S}, \pi)$. The mapping $\Phi : P(K) \to L^\infty(\mathcal{S}, \pi)$, where for each $\sigma \in P(K)$ the element $\Phi(\sigma) \in L^\infty(\mathcal{S}, \pi)$ is uniquely determined by

$$\int\limits_{\mathcal{S}} \varphi(\alpha)\, \Phi(\sigma)(\bar{\alpha})\, d\pi(\alpha) \;=\; \sigma(\check{\varphi}) \qquad \text{for } \varphi \in L^1(\mathcal{S}, \pi),$$

defines an isometric isomorphism Φ from the Banach space $P(K)$ onto $L^\infty(\mathcal{S}, \pi)$. We can define a convolution of two pseudomeasures $\sigma_1, \sigma_2 \in P(K)$ by

$$\sigma_1 * \sigma_2 = \Phi^{-1}\left(\Phi(\sigma_1)\Phi(\sigma_2)\right),$$

such that Φ is also an algebra isomorphism from $P(K)$ onto $L^\infty(\mathcal{S}, \pi)$. The proof of these facts is exactly as in [101, Theorem 4.2.2]. We shall call $\Phi(\sigma)$ the Fourier transform of $\sigma \in P(K)$. If $\mu \in M(K)$ then

$$\int_K \check{\varphi}(x)\, d\mu(x) \;=\; \int_K \int_{\mathcal{S}} \alpha(x)\, \varphi(\alpha)\, d\pi(\alpha)\, d\mu(x) \;=\; \int_{\mathcal{S}} \varphi(\alpha)\, \hat{\mu}(\bar{\alpha})\, d\pi(\alpha)$$

for all $\varphi \in L^1(\mathcal{S}, \pi)$. Hence, each measure $\mu \in M(K)$ is a pseudomeasure and $\hat{\mu} = \Phi(\mu)$. Moreover, we have $\|\mu\|_P = \|\hat{\mu}\|_\infty \leq \|\mu\|$. We want to note that for K infinite, we have $M(K)^\wedge \subsetneq L^\infty(\mathcal{S}, \pi) = \Phi(P(K))$. Thus, $M(K) \subsetneq P(K)$. Furthermore,

$$\int_{\mathcal{S}} \varphi(\alpha)\, \Phi(\mu_1 * \mu_2)(\bar{\alpha})\, d\pi(\alpha) \;=\; \int_K \check{\varphi}(x)\, d\mu_1 * \mu_2(x) \qquad \text{for all } \varphi \in L^1(\mathcal{S}, \pi).$$

Hence, the convolution $\mu_1 * \mu_2$ of two measures $\mu_1, \mu_2 \in M(K)$ agrees with the convolution of μ_1 and μ_2 considered as pseudomeasures. Obviously, we have

$$\Phi(\sigma_1 * \sigma_2) \;=\; \Phi(\sigma_1)\, \Phi(\sigma_2) \qquad \text{for } \sigma_1, \sigma_2 \in P(K).$$

The above conclusions can be summarized as follows.

Proposition 3.2.5. *The Fourier transform $\Phi : P(K) \to L^\infty(\mathcal{S}, \pi)$ determined by*

$$\int_{\mathcal{S}} \varphi(\alpha)\, \Phi(\sigma)(\bar{\alpha})\, d\pi(\alpha) \;=\; \sigma(\check{\varphi}) \qquad \text{for all } \varphi \in L^1(\mathcal{S}, \pi)$$

is an isometric algebra isomorphism of $P(K)$ onto $L^\infty(\mathcal{S}, \pi)$.

Remark 3.2.6. We can consider $P(K)$ as the von Neumann algebra of K, see [124, Proposition 4.1].

We shall say that a pseudomeasure $\sigma \in P(K)$ belongs to $L^2(K, m)$ if there is a $g \in L^2(K, m)$ such that

$$\sigma(\check{\varphi}) \;=\; \int_K \check{\varphi}(x)\, g(x)\, dm(x) \qquad \text{for all } \varphi \in L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi).$$

Since the set $\{\check{\varphi} : \varphi \in L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)\}$ is dense in $L^2(K, m)$, g is uniquely determined. If $\sigma \in P(K)$ belongs to $L^2(K, m)$ and g is the corresponding element from $L^2(K, m)$ then Parseval's formula yields for $\varphi \in L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)$

$$\int_{\mathcal{S}} \varphi(\alpha)\, \Phi(\sigma)(\bar{\alpha})\, d\pi(\alpha) \;=\; \sigma(\check{\varphi}) \;=\; \int_K \check{\varphi}(x)\, g(x)\, dm(x)$$

$$= \int_{\mathcal{S}} \varphi(\alpha)\ \wp g(\bar{\alpha})\ d\pi(\alpha).$$

Hence, we conclude $\Phi(\sigma) = \wp g \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$, i.e. the Fourier transform of the pseudomeasure σ agrees with the Plancherel transform of g.

Conversely, let $\sigma \in P(K)$ such that $\Phi(\sigma) \in L^2(\mathcal{S},\pi)\cap L^\infty(\mathcal{S},\pi)$ then σ belongs to $L^2(K,m)$. Indeed, putting $g = \wp^{-1}(\Phi(\sigma)) \in L^2(K,m)$ and using Parseval's formula we obtain

$$\sigma(\check{\varphi}) = \int_{\mathcal{S}} \varphi(\alpha)\ \Phi(\sigma)(\bar{\alpha})\ d\pi(\alpha) = \int_K \check{\varphi}(x)\ g(x)\ dm(x)$$

for all $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$.

We may summarize the latter discussion in the following proposition.

Proposition 3.2.7. *A pseudomeasure $\sigma \in P(K)$ belongs to $L^2(K,m)$ if and only if $\Phi(\sigma) \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$. Moreover, the Fourier transform of σ as a pseudomeasure coincides with the Plancherel transform of the corresponding $g \in L^2(K,m)$.*

It should be noted that every $g \in L^1(K,m) \cap L^2(K,m)$ determines a pseudomeasure $\sigma \in P(K)$ such that $\sigma(\check{\varphi}) = \int_K \check{\varphi}(x)\, g(x)\, dm(x)$ is true for all $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Indeed, let $\sigma = \Phi^{-1}(\hat{g})$. Then

$$\Phi^{-1}(\hat{g})(\check{\varphi}) = \int_{\mathcal{S}} \varphi(\alpha)\ \Phi(\Phi^{-1}(\hat{g}))(\bar{\alpha})\ d\pi(\alpha) = \int_{\mathcal{S}} \varphi(\alpha)\ \wp g(\bar{\alpha})\ d\pi(\alpha) = \int_K \check{\varphi}(x)\ g(x)\ dm(x).$$

In particular, the convolution $\sigma * g = \Phi^{-1}(\Phi(\sigma)\hat{g})$ of $\sigma \in P(K)$ and $g \in L^1(K,m) \cap L^2(K,m)$ is well-defined as a convolution of pseudomeasures.

If $\sigma \in P(K)$ belongs to $L^2(K,m)$, we will further on denote the corresponding element of $L^2(K,m)$ by σ, as well.

Theorem 3.2.8. *Let $T \in B(L^2(K,m))$. The following conditions are equivalent:*

i) $T \in M(L^2(K,m))$.

ii) $Tf * g = T(f * g) = f * Tg$ *for all* $f, g \in L^1(K,m) \cap L^2(K,m)$.

iii) There exists a unique pseudomeasure $\sigma \in P(K)$ such that $\sigma * f$ belongs to $L^2(K,m)$ and

$$Tf = \sigma * f \qquad \text{for all } f \in L^1(K,m) \cap L^2(K,m).$$

iv) There exists a unique pseudomeasure $\sigma \in P(K)$ such that

$$\wp(Tf) = \Phi(\sigma)\ \wp(f) \qquad \text{for all } f \in L^2(K,m).$$

v) There exists a unique $\varphi \in L^\infty(\mathcal{S},\pi)$ such that

$$\wp(Tf) = \varphi\, \wp(f) \qquad \text{for all } f \in L^2(K,m)$$

Moreover, the correspondence between T, σ and φ defines isometric algebra isomorphisms between $M(L^2(K,m))$, $P(K)$ and $L^\infty(\mathcal{S},\pi)$ such that $\|\varphi\|_{\mathcal{S}} = \|\sigma\|_P = \|T\|$.

Proof. $i) \Leftrightarrow ii)$ is already proven in Proposition 3.1.2.

$v) \Rightarrow i)$: We have

$$(L_x f)^\wedge(\alpha) = \alpha(x)\ \hat{f}(\alpha) = \widehat{\varepsilon_{\tilde{x}}}(\alpha)\ \hat{f}(\alpha) \qquad \text{for all } f \in L^1(K,m),\ \alpha \in \hat{K},$$

and hence $\wp(L_x f) = \widehat{\varepsilon_{\tilde{x}}}\ \wp(f)$ for all $f \in L^2(K,m)$. Therefore, it follows by (v)

$$\wp(L_x(T(f))) = \widehat{\varepsilon_{\tilde{x}}}\ \wp(T(f)) = \widehat{\varepsilon_{\tilde{x}}}\ \varphi\ \wp(f) = \varphi\ \wp(L_x(f)) = \wp(T(L_x(f)))$$

for all $f \in L^2(K,m)$, and hence $T \in M(L^2(K,m))$.

i) $\Rightarrow$ *v)* : (The proof follows the lines of [101, Theorem 4.1.1]) *i)* is equivalent to (*ii*). Hence, we have

$$\wp(T(f))\,\hat{g} \;=\; \hat{f}\,\wp(T(g)) \qquad\qquad (*)$$

for all $f,g \in C_c(K)$. For $\alpha \in \mathcal{S}$, choose $f \in C_c(K)$ such that $\hat{f}(\alpha)$ is nonzero on a neighborhood of α and define on this neighborhood $\varphi = \wp(T(f))/\hat{f}$. By the identity $(*)$ φ is independent of the choice of f, and it follows that there is a unique locally measurable function φ on $\mathcal{S}$ such that $\wp(T(f)) = \varphi\,\hat{f}$ for all $f \in C_c(K)$. If $f \in L^2(K,m)$, there exists a sequence $(f_n)_{n\in\mathbb{N}_0}$, $f_n \in C_c(K)$ such that $\lim_{n\in\mathbb{N}_0} \|f - f_n\|_2 = 0$. Then $\lim\limits_{n\in\mathbb{N}_0} \|\wp(T(f)) - \wp(T(f_n))\|_2 = 0$, and replacing $(f_n)_{n\in\mathbb{N}_0}$ by a subsequence, we can suppose that $\hat{f}_n \to \wp(f)$ π-almost everywhere and $\wp(Tf_n) \to \wp(T(f))$ π-almost everywhere. It follows that $\wp(Tf) = \varphi\,\wp(f)$ π-almost everywhere.

It remains to prove that $\varphi \in L^\infty(\mathcal{S},\pi)$. (We will even show that $\|\varphi\|_{\mathcal{S}} \le \|T\|$.) Assume, in contrary, that there is a compact subset $C \subseteq \mathcal{S}$ such that $\pi(C) > 0$ and $|\varphi(\alpha)| > \|T\|$ for π-almost all $\alpha \in C$. Put $g \in L^2(K,m)$ such that $\wp(g) = \chi_C$. Then $\|\varphi\chi_C\|_2 > \|T\|\;(\pi(C))^{\frac{1}{2}}$, and on the other hand

$$\|\varphi\chi_C\|_2 \;=\; \|\varphi\,\wp(g)\|_2 \;=\; \|\wp(T(g))\|_2 \;=\; \|T(g)\|_2 \;\le\; \|T\|\;\|g\|_2 = \;\|T\|\;(\pi(C))^{\frac{1}{2}}.$$

Obviously, this is a contradiction. Hence, $\varphi \in L^\infty(\mathcal{S},\pi)$ and $\|\varphi\|_{\mathcal{S}} \le \|T\|$. In addition, we have $\|Tg\|_2 = \|\wp(Tg)\|_2 = \|\varphi\,\wp(g)\|_2 \le \|\varphi\|_{\mathcal{S}}\,\|g\|_2$, and we get $\|T\| \le \|\varphi\|_{\mathcal{S}}$.

iv) $\Leftrightarrow$ *v)* : The equivalence of *iv)* and *v)* is true by the assumptions on pseudomeasures in the Introduction.

iv) $\Rightarrow$ *iii)* : We have already shown that the convolution $\sigma * f$ of $\sigma \in P(K)$ and $f \in L^1(K,m) \cap L^2(K,m)$ yields a pseudomeasure. For pseudomeasures σ enjoying property (*iv*) we have

$$\Phi(\sigma * f) \;=\; \Phi(\sigma)\,\hat{f} \;=\; \wp(Tf)$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. Hence, $\Phi(\sigma * f) \in L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi)$ and (*iii*) follows by Proposition 3.2.7.

iii) $\Rightarrow$ *iv)* : Using Proposition 3.2.7 once again, we conclude

$$\wp(T(f)) \;=\; \wp(\sigma * f) \;=\; \Phi(\sigma * f) \;=\; \Phi(\sigma)\,\hat{f}$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. Since T is continuous and $L^1(K,m) \cap L^2(K,m)$ is dense in $L^2(K,m)$, the statement (*iv*) is shown. □

Remark 3.2.9. In [61] the equivalence of *i)* and *iii)* in Theorem 3.2.8 is shown for general hypergroups and weights on the measure algebra of K, see [61, Proposition 1].

We obtain a result concerning translation invariant subsets in $L^2(K,m)$. The proof follows closely the proof of a similar result for locally compact Abelian groups which can be found in Larsen [101, Theorem 4.1.1].

Corollary 3.2.10. *A subset X in $L^2(K,m)$ is a closed translation invariant linear subspace, if and only if there exists a Borel measurable subset E of $\mathcal{S}$ such that*

$$X = \left\{ f \in L^2(K,m) :\; \wp(f) = 0\;\pi - \textit{almost everywhere off } E \right\},$$

that is $\wp(X) = \chi_E \cdot L^2(\mathcal{S},\pi)$.

Proof. Let $X \in L^2(K,m)$ be a closed translation invariant linear subspace. Further let T be the Hilbert space projection of $L^2(K,m)$ onto X. T is obviously a linear operator with $\|T\| = 1$. Moreover, for every $f \in L^2(K,m)$ such that $f = f_1 + f_2$ with $f_1 \in X$ and $f_2 \in X^\perp$ we conclude $L_x f = L_x f_1 + L_x f_2$ for every $x \in K$. Further, $L_x f_1 \in X$ as X is translation invariant. By

$$\int_K L_x f_2(y) g(y) dm(y) = \int_K f_2(y) L_x g(y) dm(y) = 0$$

for all $g \in X$, we have $L_x f_2 \in X^{\perp}$. Hence,

$$TL_x f = T(L_x f_1 + L_x f_2) = L_x f_1 = L_x T f_1 = L_x(Tf_1 + Tf_2) = L_x Tf$$

for all $f \in L^2(K,m)$. Thus, T commutes with translations and we have $T \in M(L^2(K,m))$. Consequently, by Theorem 3.2.8 there exists a function $\varphi \in L^\infty(\mathcal{S},\pi)$ such that $\wp(Tf) = \varphi\wp(f)$ for all $f \in L^2(K,m)$. Since T is a projection, we conclude from

$$\wp^{-1}(\varphi\wp(f)) = Tf = T^2 f = \wp^{-1}(\varphi^2\wp(f)),$$

that $\varphi^2\wp(f) = \varphi\wp(f)$ for all $f \in L^2(K,m)$. Hence, $\varphi^2 = \varphi$ $\pi-$almost everywhere. Choosing a representative function φ_1 of φ and setting $E = \{x \in \mathcal{S} : \varphi_1(x) = 1\}$, we see that $\varphi = \chi_E$ $\pi-$almost everywhere and we obtain the existence of a Borel measurable subset E of $\mathcal{S}$ such that

$$X = \left\{f \in L^2(K,m) : \wp(f) = 0 \; \pi - \text{almost everywhere off } E\right\},$$

that is $\wp(X) = \chi_E \cdot L^2(\mathcal{S},\pi)$.
Conversely suppose there exists a Borel measurable subset E of $\mathcal{S}$ such that

$$X = \left\{f \in L^2(K,m) : \wp(f) = 0 \; \pi - \text{almost everywhere off } E\right\}.$$

Let $f \in X$. Then $\wp(L_x f) = \wp(\varepsilon_x)\wp(f) \in \chi_E \cdot L^2(\mathcal{S},\pi) = \wp(X)$ for all $x \in K$. Hence, $L_x f \in X$ and X is translation invariant. X is obviously linear. Furthermore, X is closed. Indeed, let a sequence $(f_n)_{n\in\mathbb{N}}$ in X converge to $f \in L^2(K,m)$, i.e. $\|f_n - f\|_2 \to 0$ as n tends to infinity. Then we have $\|\wp(f_n) - \wp(f)\|_2 = \|f_n - f\|_2 \to 0$ as n tends to infinity. Thus, $\wp(f) = 0$ $\pi-$almost everywhere off of E. □

3.3 Multipliers for $L^p(K,m)$, $p \neq 1$, $p \neq 2$

Now we investigate multipliers for $L^p(K,m)$, $p \neq 1$, $p \neq 2$. Basically with the same arguments used for Abelian groups we obtain inclusion results for $M(L^p(K,m))$. Let $\|T\|_p$ be the operator norm of $T \in M(L^p(K,m))$, $1 \leq p < \infty$. If it is clear which operator norm is meant in the context, we will omit p in this notation.

Following the lines of the proof [101, Theorem 4.1.2] we conclude:

Proposition 3.3.1. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists an isometric algebra isomorphism of $M(L^p(K,m))$ onto $M(L^q(K,m))$.*

Proof. Let $T \in M(L^p(K,m))$. By Proposition 3.1.2 the bounded linear functional

$$F_g(f) := f * Tg(e)$$

is well-defined on $C_c(K)$ for all $g \in C_c(K)$. F_g has a continuous extension on $L^p(K,m)$ such that $\|F_g\|_p \leq \|F_g\|$. With the duality between $L^p(K,m)$ and $L^q(K,m)$ it follows that $Tg \in L^q(K,m)$ and

$$\|Tg\|_q = \|F_g\| \leq \|T\|_p \|g\|_q .$$

Therefore, $T \in M(L^p(K,m))$ restricted to $C_c(K)$ defines a continuous transformation from $C_c(K)$ onto $L^q(K,m)$ which commutes with translations. Its unique extension from $L^q(K,m)$ to $L^q(K,m)$ will also be denoted by T. This extension T also commutes with translations and we have $\|T\|_q \leq \|T\|_p$. Interchanging the roles of p and q, we get an isometric algebra isomorphism from $M(L^p(K,m))$ onto $M(L^q(K,m))$. □

Using a form of the Riesz-Thorin convexity theorem we can derive the next result just as in Theorem 4.1.3 and Corollary 4.1.3 in [101].

Lemma 3.3.2. *Let $1 < p < \infty$. Then there exists a continuous algebra isomorphism Ψ of $M(L^p(K,m))$ into $M(L^2(K,m))$. Moreover, $\|\Psi(T)\|_2 \leq \|T\|_p$.*

Proof. Let $T \in M(L^p(K,m))$. By a form of Riesz-Thorin Convexity theorem [26, VI 10.8] the function $\log \|T\|_{\frac{1}{a}}$ is convex on $0 \leq a \leq 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$ and $\|T\|_p = \|T\|_q$ it follows

$$log \|T\|_2 \leq \frac{1}{p} log \|T\|_p + \frac{1}{q} log \|T\|_q = \left(\frac{1}{p} + \frac{1}{q}\right) log \|T\|_p = log \|T\|_p .$$

Thus, the restriction of $T \in M(L^p(K,m))$ to the integrable simple functions determines a unique element $T_2 \in M(L^2(K,m))$ with $\|T_2\|_2 \leq \|T\|_p$. □

Corollary 3.3.3. *Let $T \in M(L^p(K,m))$, $1 < p < \infty$. There exists a unique $\varphi \in L^\infty(\mathcal{S},\pi)$ such that*

$$Tf = \wp^{-1}(\varphi\wp(f))$$

for all $f \in L^2(K,m) \cap L^p(K,m)$. Moreover $\|\varphi\|_\infty = \|T\|_2 \leq \|T\|_p$.

Proof. Let $T \in M(L^p(K,m))$. T determines, when restricted to the integrable simple functions $\mathcal{S}(K,m)$, a unique element $T_2 \in M(L^2(K,m))$. By Theorem 3.2.8 there exists a unique $\varphi \in L^\infty(K,m)$ with $T_2 f = \wp^{-1}(\varphi\wp(f))$ for all $f \in \mathcal{S}(K,m)$. Let $f \in L^2(K,m) \cap L^p(K,m)$ be nonnegative almost everywhere and let $(g_n)_{n\in\mathbb{N}_0}$ be a monotone increasing sequence of non-negative simple functions which converges almost everywhere to f. By Lebesgues dominated convergence theorem and by the continuity of T and T_2 it follows with the usual argument that $Tf = T_2 f$ for all non-negative $f \in L^2(K,m) \cap L^p(K,m)$. By the Plancherel theorem $\lim_{n\to\infty} \|\wp(T_2 g_n) - \wp(Tf)\|_2 = 0$ and by Theorem 3.2.8,

$$\|\wp(T_2 g_n) - \varphi\wp(f)\|_2 = \|\varphi\wp(g_n) - \varphi\wp(f)\|_2 \leq \|\varphi\|_\infty \|g_n - f\|_2 \to 0$$

for $n \to \infty$. Therefore, $Tf = \wp^{-1}(\varphi\wp(f))$ for all nonnegative $f \in L^2(K,m) \cap L^p(K,m)$ and hence for all $f \in L^2(K,m) \cap L^p(K,m)$. □

We denote the set of all $\varphi \in L^\infty(\mathcal{S},\pi)$ such that $\wp(Tf) = \varphi\wp(f)$ for each $f \in L^2(K,m) \cap L^p(K,m)$ by $\mathcal{M}(L^p(K,m))$. Furthermore, we call each φ which corresponds to a multiplier $T \in M(L^p(K,m))$ the Fourier Transform of T and denote $\hat{T} := \varphi$.
We get the following characterizations for a multiplier for $L^p(K,m)$.

Theorem 3.3.4. *Let $T \in B(L^p(K,m))$, $1 < p < 2$. The following conditions are equivalent:*

i) $T \in M(L^p(K,m))$

ii) $Tf * g = T(f * g) = f * Tg$ *for all* $f, g \in L^1(K,m) \cap L^p(K,m)$

iii) There exists a unique pseudomeasure $\sigma \in P(K)$ such that

$$Tg = \sigma * g \text{ for all } g \in L^1(K,m) \cap L^2(K,m).$$

iv) There exists a unique pseudomeasure $\sigma \in P(K)$ such that $(Tg)^\wedge = \Phi(\sigma)\hat{g}$ for all $g \in L^p(K,m)$.

v) There exists a unique $\eta \in L^\infty(\mathcal{S},\pi)$ such that $(Tg)^\wedge = \eta\hat{g}$ for all $g \in L^p(K,m)$.

Moreover $\|\eta\|_\infty = \|\sigma\|_p = \|T\|_2 \leq \|T\|_p$.

Proof. The proof is completed by Lemma 3.1.2, Corollary 3.3.3 and Proposition 3.2.7. *iv)* and *v)* follow for all $g \in L^p(K,m)$ by extending the Fourier transform on $L^1(K,m)$ to the Hausdorff-Young transform on $L^p(K,m)$. □

Remark 3.3.5. By Proposition 3.3.1 Theorem 3.3.4 holds also for $2 < p < \infty$. However, since the Hausdorff-Young transform is only defined for $1 < p < 2$, we need to restrict statements *iv)* and *v)* to all functions in $L^1(K,m) \cap L^2(K,m)$.

Remark 3.3.6. If a multiplier $T \in M(L^p(K,m))$, $1 < p < \infty$, is submarkovian, i.e. $0 \le Tf \le 1$ m-almost everywhere for all $f \in L^p(K,m)$ with $0 \le f \le 1$ $m-$almost everywhere, then there exists even a positive, contractive measure $\mu \in M(K)$, such that $Tf = \mu * f$ for all $f \in L^p(K,m)$, see [4, Theorem 1.6.25].

Taking a look on our last results we can conclude for each commutative hypergroup K and $1 < p \le r \le 2$, that there exists a norm-decreasing algebra isomorphism J from $M(L^p(K,m))$ into $M(L^r(K,m))$. Indeed, $J(T)$, $T \in M(L^p(K,m))$, is the unique extension to $L^r(K,m)$ of the restriction $T|L^p(K,m) \cap L^r(K,m)$. Hence, J can also be interpreted as an inclusion. With a slight abuse of terminology we obtain the following inclusions for $1 < p < r < 2$:

$$M(K) \cong M(L^1(K,m)) \subseteq M(L^p(K,m)) \subseteq M(L^r(K,m)) \subseteq M(L^2(K,m)) \cong P(K),$$

and for $2 < r < p < \infty$

$$M(K) \cong M(C_0(K)) \subseteq M(L^p(K,m)) \subseteq M(L^r(K,m)) \subseteq M(L^2(K,m)) \cong P(K),$$

and the inclusion mappings are norm-decreasing. Later on we deal with the question whether those inclusions are strict, see Chapter 5.

Remark 3.3.7. Considering compact hypergroups we want to point out that Theorem 3.2.8 generalizes Theorem 1.1, Corollary 3.1 and Lemma 4.1 in [34], where the dual hypergroup $K = [-1,1]$ of the ultraspherical polynomials $R_n^{(\alpha,\alpha)}(t)$ is investigated.

We want to mention a short result, which follows from Theorems 3.1.5, 3.2.8, 3.3.4.
For a function $\varphi \in L^\infty(\mathcal{S},\pi)$ with $\varphi \neq 0$ π-almost everywhere, define $\frac{1}{\varphi}$ by choosing a representing function φ' on $\hat{K}$ such that $\varphi' = \varphi$ π-almost everywhere and set

$$\frac{1}{\varphi'}(\alpha) := \begin{cases} \frac{1}{\varphi'(\alpha)} & \text{if } \varphi'(\alpha) \neq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

$\frac{1}{\varphi}$ denotes the equivalence class in $L^\infty(\mathcal{S},\pi)$ of $\frac{1}{\varphi'}$. We obtain $\frac{1}{\varphi}\varphi = 1$ π-almost everywhere.

Proposition 3.3.8. *Let $1 \le p < \infty$ and $T \in M(L^p(K,m))$ such that $\hat{T} = \varphi$. T is invertible if and only if $\varphi \neq 0$ π-almost everywhere and $1/\varphi \in \mathcal{M}(L^p(K,m))$.*

Proof. Let without loss of generality $1 \le p \le 2$. Let $T_\varphi \in M(L^p(K,m))$ be invertible. There exists $T_\psi \in M(L^p(K,m))$ with corresponding $\psi \in \mathcal{M}(L^p(K,m))$ such that $T_\varphi \circ T_\psi = I = T_\psi \circ T_\varphi$ in $M(L^p(K,m))$. We obtain $\hat{f} = (T_\varphi \circ T_\psi f)^\wedge = \varphi(T_\psi f)^\wedge = \varphi\psi\hat{f}$ for all $f \in L^p(K,m)$. Since for all $\alpha \in \mathcal{S}$ there exists a function in $f \in C_c(K) \subset L^p(K,m)$ such that $\hat{f}(\alpha) \neq 0$ we conclude $\varphi \neq 0$ and $\psi = \frac{1}{\varphi}$ π-almost everywhere.
Conversely, $(T_\varphi \circ T_{\frac{1}{\varphi}} f)^\wedge = \hat{f}$. By the uniqueness theorem of the Fourier transform it follows that $T_\varphi \circ T_{\frac{1}{\varphi}} = I = T_{\frac{1}{\varphi}} \circ T_\varphi$ and T_φ is invertible. □

We want to give another criterion on $\varphi \in L^\infty(\mathcal{S},\pi)$, which indicates that φ is the Fourier transform of an operator $T \in M(L^p(K,m))$. The following criterion is analogous to the criterion given by Schoenberg, see [143], which characterizes the Fourier transform of measures. Also in the hypergroup case Schoenberg's theorem exists which states that $\varphi \in C^b(\hat{K})$ equals on $\mathcal{S}$ the Fourier transform of a bounded Borel measure μ on K if and only if there is a real number $B \ge 0$ such that $\left|\int_{\mathcal{S}} \varphi(\alpha)h(\alpha)d\pi(\alpha)\right| \le B \left\|\check{h}\right\|_\infty$ is satisfied for every $h \in C_c(\hat{K})$, see [78].
L.S. Hahn [68] found the following characterization for multipliers for locally compact groups. It can easily be transferred to hypergroups.

Proposition 3.3.9. *Let $1 < p < \infty$ and $\varphi \in L^\infty(\mathcal{S},\pi)$. Then the following assertions are equivalent* *i)* $\varphi \in \mathcal{M}(L^p(K,m))$

ii) There exists a constant B *such that*

$$|\int_{\mathcal{S}} \varphi \hat{f}\hat{g} d\pi| \leq B \|f\|_p \|g\|_q$$

for all $f, g \in C_c(K)$ *where* $1/p + 1/q = 1$.

Proof. Let $\varphi \in \mathcal{M}(L^p(K,m))$, then $\wp^{-1}(\varphi\wp(f)) \in L^p(K,m)$ for all $f \in L^p(K,m) \cap L^2(K,m)$ and we can define a continuous mapping

$$Tf := \wp^{-1}(\varphi\wp(f)), \qquad T : (C_c(K), \| \, \|_p) \to (L^p(K,m), \| \, \|_p).$$

Thus,

$$\|T\| = \sup \left\{ \left\| \wp^{-1}(\varphi\wp(f)) \right\|_p : f \in C_c(K), \ \|f\|_p \leq 1 \right\}.$$

Since $L^q(K,m)$ is the dual of $L^p(K,m)$ and $C_c(K)$ is norm dense in $L^q(K,m)$, we have

$$\|T\| = \sup \left\{ |\int_K (\wp^{-1}(\varphi\wp(f))g \, dm| : f, g \in C_c(K), \ \|f\|_p \leq 1 \ \|g\|_q \leq 1 \right\}.$$

By Plancherel's theorem, this in turn is the same as the supremum of

$$\left\{ |\int_K (\varphi\wp(f)\wp(g) d\pi| : f, g \in C_c(K), \ \|f\|_p \leq 1 \ \|g\|_q \leq 1 \right\}.$$

Conversely, consider ii) holds. Reversing the preceding arguments one sees that the operator $T : (C_c(K), \| \, \|_p) \to (L^p(K,m), \| \, \|_p)$ defined by $Tf := \wp^{-1}(\varphi\wp(f))$ has $L^p(K,m)-$norm less than or equal to B. Thus, T extends by continuity to all of $L^p(K,m)$ with the same norm, showing that $\varphi \in \mathcal{M}(L^p(K,m))$ and $\|\varphi\|_\infty \leq \|T\| = B$. □

We obtain two consequences of Proposition 3.3.9, which concern convergent nets of multipliers. Similar results for locally compact Abelian groups can be found in [47, Corollary 1.1, Corollary 1.2].

Corollary 3.3.10. *Let* $1 < p < \infty$ *and* $\{\varphi_i\}_{i\in I}$ *be a net of functions in* $\mathcal{M}(L^p(K,m))$ *such that the corresponding multipliers* $\{T_i\}_{i\in I}$ *fulfill* $\|T_i\|_p \leq B < \infty$ *for all* $i \in I$. *If* $\{\varphi_i\}_{i\in I}$ *converges in the weak*-topology of* $L^\infty(\mathcal{S},\pi)$ *to a function* φ, *then* φ *is also an element in* $\mathcal{M}(L^p(K,m))$ *and the corresponding multiplier* T_φ *fulfills* $\|T_\varphi\|_p \leq B$.

Proof. Let $1/p + 1/q = 1$. By Proposition 3.3.9 is

$$\left| \int_{\mathcal{S}} \varphi_i \hat{g} \hat{f} d\pi \right| \leq B \|f\|_p \|g\|_q,$$

for all $f, g \in C_c(\mathcal{S})$ and all $i \in I$. Since $\hat{f}, \hat{g} \in L^2(\mathcal{S},\pi)$, we have $\hat{f}\hat{g} \in L^1(\mathcal{S},\pi)$. The convergency assumption on $\{\varphi_i\}_{i\in I}$ implies that $\left| \int_{\mathcal{S}} \varphi_i \hat{f}\hat{g} d\pi \right|$ converges to $\left| \int_{\mathcal{S}} \varphi \hat{f}\hat{g} d\pi \right|$ for all $f, g \in C_c(\mathcal{S})$. Hence, we obtain

$$\left| \int_{\mathcal{S}} \varphi \hat{f}\hat{g} d\pi \right| \leq B \|f\|_p \|g\|_q,$$

for all $f, g \in C_c(\mathcal{S})$. By Proposition 3.3.9 is $\varphi \in \mathcal{M}(L^p(K,m))$ and $\|T\|_p \leq B$. □

Corollary 3.3.11. *Let* $1 < p < \infty$. *If* $\{T_i\}_{i\in I}$ *is a bounded net in* $M(L^p(K,m))$, $\|T_i\|_p \leq M < \infty$ *for all* $i \in I$, *and* $\{T_i\}_{i\in I}$ *converges to* T *in the weak operator topology over* $L^2(K,m)$, *then* $T \in M(L^p(K,m))$ *and* $\|T\|_p \leq M$.

Proof. Let $f, g \in L^2(K, m)$. By the convergency assumptions converges $\int_K (T_i f) g \; dm$ to $\int_K (Tf) g \; dm$. Hence, by Plancherel's theorem $\int_{\mathcal{S}} \varphi_i \wp(f) \wp(g) d\pi$ converges to $\int_{\mathcal{S}} \varphi \wp(f) \wp(g) d\pi$, where φ_i denote the corresponding functions of T_i in $L^\infty(\mathcal{S}, \pi)$ for all $i \in I$ and φ denotes the corresponding function to T. Since we can express every function in $L^1(\mathcal{S}, \pi)$ as a product of two functions in $L^2(\mathcal{S}, \pi)$, we conclude that $\{\varphi_i\}_{i \in I}$ converges to φ in the weak* topology of $L^\infty(\mathcal{S}, \pi)$. Corollary 3.3.10 completes the proof. □

Remark 3.3.12. Gaudry and Inglis [58] proved further approximation theorems for multipliers for a locally compact group G. These results are not extendable to hypergroups using similar proofs, since their proof is based on the fact that the dual of the Figà-Talamanca Herz algebras $A_p(G)$ is isometrically isomorphic to $M(L^p(G))$. We do not know, if this is also the case in the context of hypergroups (see Chapter 5).

3.4 Examples

We apply now Theorem 3.1.5 and Theorem 3.2.8, respectively, to polynomial hypergroups. In that case the measure space $M(\mathbb{N}_0)$ can be identified with $l^1(\mathbb{N}_0)$. Moreover, $\mathcal{S} = \text{supp}\pi$ is compact. Thus, $L^\infty(\mathcal{S}, \pi) \subseteq L^2(\mathcal{S}, \pi)$, and every pseudomeasure $\sigma \in P(\mathbb{N}_0)$ belongs to $l^2(\mathbb{N}_0, h)$.

Let $(\tau(m, n))_{m,n \in \mathbb{N}_0}$ be an infinite matrix of complex numbers. Consider the linear transformations T defined by

$$Tf(m) \;=\; g(m) \;=\; \sum_{n=0}^{\infty} \tau(m, n)\; f(n)\; h(n). \tag{3.4.1}$$

Then $g = (g(m))_{m \in \mathbb{N}_0}$ is defined at least whenever $f = (f(n))_{n \in \mathbb{N}_0} \in l_{fin}$, i.e. $f(n) \neq 0$ for at most finitely many $n \in \mathbb{N}_0$. We begin with a simple observation.

Lemma 3.4.1. *Define T by (3.4.1) for $f \in l_{fin}$. Then we have $T \circ L_n = L_n \circ T$ for all $n \in \mathbb{N}_0$ if and only if $\tau(m, n) = L_n \sigma(m)$ for all $m, n \in \mathbb{N}_0$, where $\sigma(k) = \tau(k, 0)$.*

Proof. Let $\varepsilon_l(k) = \delta_{k,l}$ for $k, l \in \mathbb{N}_0$. Then $T_n \varepsilon_0(k) = 0$ for $k \neq n$ and $T_n \varepsilon_0(n) = 1/h(n)$, and $T\varepsilon_0(k) = \tau(k, 0) = \sigma(k)$. Therefore, $T \circ L_n \varepsilon_0(m) = \tau(m, n)$ and $L_n \circ T\varepsilon_0(m) = L_n \sigma(m)$. It follows that $T \circ L_n = L_n \circ T$ implies $\tau(m, n) = L_n \sigma(m)$. Conversely, $\tau(m, n) = L_n \sigma(m)$ yields $T \circ L_n \varepsilon_0 = L_n \circ T\varepsilon_0$, and then

$$T \circ L_n \varepsilon_l \;=\; h(l)\; T \circ L_n \circ L_l \varepsilon_0 \;=\; h(l)\; L_l \circ L_n \circ T\varepsilon_0 \;=\; L_n \circ T\varepsilon_l,$$

for each $l \in \mathbb{N}_0$. Thus, the converse implication is also true. □

In view of Lemma 3.4.1 we focus our attention to the case $\tau(m, n) = T_n \sigma(m)$, where $\sigma = (\sigma(k))_{k \in \mathbb{N}_0}$. Notice that $\tau(m, n) = T_n \sigma(m) = T_m \sigma(n) = \tau(n, m)$. From Theorem 3.1.5 and Theorem 3.2.8 we can conclude the following characterizations.

Proposition 3.4.2. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup.*

(1) A necessary and sufficient condition that T is a bounded linear operator from $l^1(\mathbb{N}_0, h)$ into $l^1(\mathbb{N}_0, h)$ is that $\sigma = (\sigma(n))_{n \in \mathbb{N}_0} \in l^1(\mathbb{N}_0)$.

(2) A necessary and sufficient condition that T is a bounded linear operator from $l^2(\mathbb{N}_0, h)$ into $l^2(\mathbb{N}_0, h)$ is that $\sigma = (\sigma(n))_{n \in \mathbb{N}_0}$ is given by $\sigma(n) = \check{f}(n)$ for some $f \in L^\infty(\mathcal{S}, \pi)$.

Remark 3.4.3. One should compare Proposition 3.4.2(2) with Theorem 9.18 in [175, Ch.IV]. We want to mention that "the multipliers of $l^2(\mathbb{Z})$ correspond exactly with the so-called double infinite Toeplitz matrices", see [33, pp. 244]. For further information, we refer to Zygmund [176] and Widom [167].

If we consider polynomial hypergroups, which fulfill the continuity property (P), we can name a subset of $\mathcal{M}(l^p(\mathbb{N}_0,h))$. L.-S. Hahn formulated similar results for commutative groups, see [68, Theorem 5].

Theorem 3.4.4. *Let $K=\mathbb{N}_0$ be a polynomial hypergroup fulfilling the continuity property (P). Let $1\le p\le 2$, $1/p+1/q=1$. Then*

$$L^p(\mathcal{S},\pi)*L^q(\mathcal{S},\pi)\subset\mathcal{M}(l^r(\mathbb{N}_0,h)),$$

for every $\frac{2p}{3p-2}\le r\le\frac{2p}{2-p}$ or equivalently $\frac{2q}{q+2}\le r\le\frac{2q}{q-2}$.

Proof. For $p=1$ the theorem reduces to the already known result

$$L^1(\mathcal{S},\pi)*L^\infty(\mathcal{S},\pi)\subset L^\infty(\mathcal{S},\pi)\subset\mathcal{M}(l^2(\mathbb{N}_0,h)).$$

Let $1<p\le 2$ and $1/p+1/q=1$. Let $\psi,\varphi\in C_c(\mathcal{S})$ and $f,g\in C_c(\mathbb{N}_0)$. By Hölder's inequality and by Plancherel's theorem

$$\left|\int_{\mathcal{S}}\hat f\hat g(\varphi*\psi)d\pi\right|\le\|\varphi*\psi\|_\infty\int_{\mathcal{S}}|\hat f\hat g|d\pi\le\|\varphi\|_1\|\psi\|_\infty\|f\|_2\|g\|_2.$$

Furthermore, we have

$$\left|\int_{\mathcal{S}}\hat f\hat g(\varphi*\psi)d\pi\right|\le\|\varphi\|_2\|\psi\|_2\|f\|_1\|g\|_\infty.$$

Thus, the multilinear mapping

$$T: C_c(\mathbb{N}_0)\times C_c(\mathbb{N}_0)\times C_c(\mathcal{S})\times C_c(\mathcal{S})\to\mathbb{C},\qquad T(f,g,\varphi,\psi)=\int_{\mathcal{S}}\hat f\hat g(\varphi*\psi)d\pi,$$

satisfies

$$|T(f,g,\varphi,\psi)|\le\|f\|_{p'}\|g\|_{q'}\|\varphi\|_r\|\psi\|_s$$

for $p'=1$, $q'=\infty$, $r=s=2$ or $p'=q'=2$, $r=1$ and $s=\infty$. By the multilinear version of the Riesz-Thorin interpolation theorem, see [175, XII, Theorem 3.3], T satisfies this inequality for all $1/p'=(1-\lambda)\frac{1}{1}+\frac{\lambda}{2}$ and $1/r=(1-\lambda)\frac{1}{2}+\frac{\lambda}{1}$ where $0\le\lambda\le 1$. This leads to $1/p'+1/r=3/2$, see [68, pp. 325]. Hence, for $p'=p>1$ follows by Proposition 3.3.9 and the density of $C_c(\mathcal{S})$ in $L^p(\mathcal{S},\pi)$ and $L^q(\mathcal{S},\pi)$ that $L^p(\mathcal{S},\pi)*L^q(\mathcal{S},\pi)\subset\mathcal{M}(l^r(\mathbb{N}_0,h))$ for $1/r=3/2-1/p=(3p-2)/(2p)$. Proposition 3.3.1 and Lemma 3.3.2 complete the proof. □

Remark 3.4.5. For $p=2$ the Theorem above reduces to the already known result

$$L^2(\mathcal{S},\pi)*L^2(\mathcal{S},\pi)=L^1(\mathcal{S},\pi)^\vee\subset\mathcal{M}(l^1(\mathbb{N}_0,h)).$$

Remark 3.4.6. Combining Proposition 3.3.9 and Theorem 3.4.4, we conclude in particular that $\|T_{f*g}\|_r\le\|f\|_p\|g\|_q$ where T_{f*g} denotes the multiplier for $l^r(\mathbb{N}_0,h)$ corresponding to $f*g\in\mathcal{M}(l^r(\mathbb{N}_0,h))$.

Remark 3.4.7. We cannot strengthen the conclusions of Theorem 3.4.4. Even in the group case this is in general impossible, see [68, pp. 325].

3.5 On the Spectrum of a Multiplier

We define the **spectrum** of a bounded operator T on $L^p(K,m)$ as the set

$$\sigma^p(T):=\{\lambda\in\mathbb{C}:\ T-\lambda I\text{ is not bijective}\}.$$

For an operator T on $C_0(K)$ denote the spectrum $\sigma^0(T)$. As a direct consequence of Corollary 3.1.4 we obtain

Corollary 3.5.1. *Let $1 < p < 2$ and $1/p + 1/q = 1$. For $T \in M(L^p(K, m))$ we have $\sigma^p(T) = \sigma^q(T)$.*

Proof. Let $\lambda \notin \sigma^p(T)$ then $(T - \lambda I)$ is bijective and an element in $M(L^p(K, m))$. By Theorem 3.1.4 is $(T - \lambda I)^{-1} \in M(L^p(K, m))$. Hence, we conclude $(T - \lambda I)^{-1} \in M(L^q(K, m))$, as well. □

Corollary 3.5.2. *Let $T \in M(L^1(K, m))$. For the corresponding multiplier T^0 in $M(C_0(K))$ we have $\sigma^1(T) = \sigma^0(T^0)$.*

An application of Theorem 3.1.5, Theorem 3.2.8 and Theorem 3.3.4 is given in the spectral theory of multipliers for the Jacobi hypergroup $\mathcal{S} = [-1, 1]$. Note that the Jacobi polynomials define a hypergroup on $K = \hat{\hat{K}} = \mathbb{N}_0$ with a dual hypergroup structure on $\mathcal{S} = [-1, 1]$.
In Wong [168, Chapter 21] we can find similar results for $p = 2$ while studying filters on the unit circle.

Let $(R_n(x))_{n\in\mathbb{N}_0}$ denote the sequence of Jacobi polynomial as introduced in the first chapter.

Theorem 3.5.3. *Let $1 \le p < \infty$ and $d = (d(n))_{n\in\mathbb{N}_0} \in \mathcal{M}(L^p(\mathcal{S}, \pi)) \subseteq l^\infty(\mathbb{N}_0)$. Then $d(n)$ is an eigenvalue of $T_d \in M(L^p(\mathcal{S}, \pi))$ for all $n \in \mathbb{N}_0$. The corresponding eigenfunction is $R_n(x)$. Moreover, $d(n)$ are the only eigenvalues of T_d.*

Proof. Let $n \in \mathbb{N}_0$. We have $\check{R}_n(m) = \int_{-1}^{1} R_n(x)R_m(x)d\pi(x) = \delta_{m,n}h(n)^{-1}$, where $\delta_{m,n}$ denotes the Kroneker symbol. Hence, we obtain by Theorem 3.1.5, Theorem 3.2.8 and Theorem 3.3.4 that

$$T_d R_n = (d\check{R}_n)^\wedge = \sum_{k=0}^{\infty} d(n)h(n)^{-1}\delta_{k,n}R_k h(k) = d(n)R_n.$$

Thus, $d(n)$ is an eigenvalue of T_d with corresponding eigenfunction R_n.
Conversely, let λ be an eigenvalue of T_d, then there exists a corresponding eigenfunction $f \in L^p(\mathcal{S}, \pi)$ such that $T_d f = \lambda f$. We obtain $d(n)\check{f}(n) = (T_d f)^\vee(n) = \lambda \check{f}(n)$ for all $n \in \mathbb{N}$. Since $f \neq 0$ in $L^p(\mathcal{S}, \pi)$, by the uniqueness theorem of the Jacobi transform there exists an integer $n \in \mathbb{N}_0$ such that $d(n) = \lambda$. □

For $p = 2$ we can show even a bit more.

Theorem 3.5.4. *Let $d = (d(n))_{n\in\mathbb{N}_0} \in l^\infty(\mathbb{N}_0)$. Then $d(n)$ is an eigenvalue of $T_d \in M(L^2(\mathcal{S}, \pi))$ for all $n \in \mathbb{N}_0$. The corresponding eigenfunction is $R_n(x)$.*
Moreover, denote by $\{d(n) : n \in \mathbb{N}_0\}^c$ the closure in $\mathbb{C}$ of the set $\{d(n) : n \in \mathbb{N}_0\}$. The spectrum $\sigma^2(T_d)$ of T_d equals $\{d(n) : n \in \mathbb{N}_0\}^c$.

Proof. $d(n)$ is by Theorem 3.5.3 an eigenvalue of $T_d \in M(L^2(\mathcal{S}, \pi))$ for all $n \in \mathbb{N}_0$ with corresponding eigenfunction $R_n(x)$.
Since the spectrum of a bounded operator is always closed, the closure of the set $\{d(n) : n \in \mathbb{N}_0\}$ in $\mathbb{C}$ is contained in $\sigma(T_d)$.
Furthermore, let $\lambda \notin \{d(n) : n \in \mathbb{N}_0\}^c$. Then there exists a constant $C > 0$ such that $|\lambda - d(n)| > C$ for all $n \in \mathbb{N}_0$. By Plancherel's Theorem follows for every $\varphi \in L^2(\mathcal{S}, \pi)$

$$\|(T_d - \lambda I)\varphi\|_2 = \|((T_d - \lambda I)\varphi)^\vee\|_2 = \left(\sum_{k=0}^{\infty} |d(k) - \lambda|^2 |\check{\varphi}(k)|^2 h(k)\right)^{1/2} \ge C\|\varphi\|_2.$$

Hence, $(T_d - \lambda I)$ is injective.
We need to prove the surjectivity of $(T_d - \lambda I)$. Let $\psi \in L^2(\mathcal{S}, \pi)$. By Plancherel's theorem we find that $\check{\psi} \in l^2(\mathbb{N}_0, h)$ and, since $\frac{1}{\lambda - d} < \frac{1}{C}$, we have also $\frac{\check{\psi}}{\lambda - d} \in l^2(\mathbb{N}_0, h)$. Using Plancherel's theorem again, we conclude the existence of a function $\phi \in L^2(\mathcal{S}, \pi)$ such that $\check{\phi}(n) = \frac{\check{\psi}(n)}{\lambda - d(n)}$ for all $n \in \mathbb{N}_0$. Hence, $((T_d - \lambda I)\phi)^\vee = (\lambda - d(n))\check{\phi}(n) = \check{\psi}(n)$ for all $n \in \mathbb{N}_0$. By the uniqueness theorem of the Jacobi transform holds $(T_d - \lambda I)\phi = \psi$. □

Chapter 4

Multipliers for $L^p(\mathcal{S}, \pi)$

We investigate multipliers for the dual $\mathcal{S}$ of a commutative hypergroup K. Some results of Chapter 3 are transferred to $L^p(\mathcal{S}, \pi)$, $1 \leq p < \infty$. Since in general $\mathcal{S}$ does not admit a dual hypergroup structure, not all results are extendable. Nevertheless, using weak dual structures we can still gain a few insights into multipliers for $L^p(\mathcal{S}, \pi)$.

4.1 Multipliers for $L^1(\mathcal{S}, \pi)$

The dual spaces $\hat{K}$ or $\mathcal{S}$ do, in general, not bear a dual hypergroup structure, and hence there do not exist hypergroup-translation-operators on $\mathcal{S}$. Nevertheless, using the inverse Fourier transform we can characterize those functions f on K such that $f\,\check{\varphi} \in L^1(\mathcal{S}, \pi)^\vee$ for all $\varphi \in L^1(\mathcal{S}, \pi)$.

Definition 4.1.1. An operator T is called **multiplier** for $L^1(\mathcal{S}, \pi)$, if and only if there exists a corresponding continuous complex valued function f on K such that $(T\varphi)^\vee = f\,\check{\varphi} \in L^1(\mathcal{S}, \pi)^\vee$ for all $\varphi \in L^1(\mathcal{S}, \pi)$.
We denote by $M(L^1(\mathcal{S}, \pi))$ the set of all multipliers for $L^1(\mathcal{S}, \pi)$ and by $\mathcal{M}(L^1(\mathcal{S}, \pi))$ the set of corresponding continuous functions on K. Given a function $f \in \mathcal{M}(L^1(\mathcal{S}, \pi))$, we denote the corresponding multiplier by T_f.

By the uniqueness theorem for the inverse Fourier transform every multiplier operator T is well-defined through the equation $(T\varphi)^\vee = f\check{\varphi}$ and a linear operator from $L^1(\mathcal{S}, \pi)$ to $L^1(\mathcal{S}, \pi)$. Moreover, T is a closed mapping. In fact, if $\lim_{n\to\infty} \|\varphi_n - \varphi\|_1 = 0$ and $\lim_{n\to\infty} \|T\varphi_n - \kappa\|_1 = 0$ for $\varphi_n, \varphi, \kappa \in L^1(\mathcal{S}, \pi)$, then $\check{\kappa}(x) = \lim_{n\to\infty} (T\varphi_n)^\vee(x) = f(x) \lim_{n\to\infty} \check{\varphi}_n(x) = f(x)\check{\varphi}(x)$ for all $x \in K$. From the closed graph theorem we conclude that T is continuous.

Lemma 4.1.2. *Assume that $1 \in \mathcal{S} = supp\pi$. Then there exists a net $(k_i)_{i\in I}$ of functions $k_i \in C_c(\mathcal{S})$ such that $k_i \geq 0$, $\|k_i\|_1 = 1$ and $\check{k}_i(x) \to 1$ for all $x \in K$.*

Proof. We consider the set

$$V_{\epsilon,C} \; = \; \{\alpha \in \mathcal{S} : \; |\alpha(x) - 1| < \epsilon \text{ for all } x \in C\},$$

where $C \subseteq K$ is compact and $\epsilon > 0$, which is a member of the neighborhood basis of $1 \in \mathcal{S}$. We put $k_{\epsilon,C} = \chi_{V_{\epsilon,C}}/\pi(V_{\epsilon,C})$, introduce a corresponding index set $i \in I$ with the usual order and observe that $k^\vee_{\epsilon,C}(x) \to 1$, whenever $x \in C$ and $\epsilon \to 0$. Now the statement follows. ☐

Theorem 4.1.3. *Assume that $1 \in \mathcal{S}$. If $f \in \mathcal{M}(L^1(\mathcal{S}, \pi))$, i.e. $f\,\check{\varphi} \in L^1(\mathcal{S}, \pi)^\vee$ for all $\varphi \in L^1(\mathcal{S}, \pi)$, then there is a unique measure $\mu \in M(\hat{K})$ such that $\check{\mu} = f$.*

Proof. We know already that the linear operator T determined by $f\check{\varphi} = (T\varphi)^\vee$ is continuous, and hence, there exists some $M \geq 0$ with $\|T\varphi\|_1 \leq M\ \|\varphi\|_1$ for all $\varphi \in L^1(\mathcal{S},\pi)$. Consider a net $(k_i)_{i\in I}$ as in Lemma 4.1.2, and let $h \in C_c(K)$. Then

$$\int_K f(x)\ h(\tilde{x})\ dm(x) \ = \ \lim_i \int_K (f\,\check{k}_i)(x)\ h(\tilde{x})\ dm(x)$$

by Lebesgue's theorem of dominated convergence. We have with Fubini's theorem

$$\int_K (f\,\check{k}_i)(x)\ h(\tilde{x})\ dm(x) \ = \ \int_K (Tk_i)^\vee(x)\ h(\tilde{x})\ dm(x)$$

$$= \int_K \int_{\mathcal{S}} Tk_i(\alpha)\ \alpha(x)\ d\pi(\alpha)\ h(\tilde{x})\ dm(x) \ = \ \int_{\mathcal{S}} Tk_i(\alpha)\ \hat{h}(\alpha)\ d\pi(\alpha),$$

and then

$$\left|\int_K (f\check{k}_i)(x)\ h(\tilde{x})\ dm(x)\right| \ \leq \ \|\hat{h}\|_{\mathcal{S}}\ \|Tk_i\|_1 \ \leq \ M\ \|\hat{h}\|_{\mathcal{S}}.$$

Therefore, $\left|\int\limits_K f(x)h(\tilde{x})\,dm(x)\right| \ \leq \ M\ \|\hat{h}\|_{\mathcal{S}}$ for all $h \in C_c(K)$. The space $(C_c(K))^\wedge$ is dense in $C_0(\hat{K})$. Hence, the linear functional $F(\hat{h}) = \int\limits_K f(x)h(\tilde{x})\,dm(x)$ defined on $(C_c(K))^\wedge$ can be continuously extended to a continuous linear functional on $C_0(\hat{K})$. Riesz's representation theorem yields a unique measure $\mu \in M(\hat{K})$ such that

$$\int_K f(x)\ h(\tilde{x})\ dm(x) \ = \ F(\hat{h}) \ = \ \int_{\hat{K}} \hat{h}(\alpha)\ d\mu(\alpha)$$

for all $h \in C_c(K)$. By Fubini's theorem follows

$$\int_K \check{\mu}(x)\ h(\tilde{x})\ dm(x) \ = \ \int_{\hat{K}} \hat{h}(\alpha)\ d\mu(\alpha) \ = \ \int_K f(x)\ h(\tilde{x})\ dm(x)$$

for all $h \in C_c(K)$, and hence $\check{\mu} = f$. □

Remark 4.1.4. Theorem 4.1.3 implies that every $f \in \mathcal{M}(L^1(\mathcal{S},\pi))$ is a bounded function. The reader should note that $f = \check{\mu}$, $\mu \in M(\hat{K})$, is, in general, not an element in $\mathcal{M}(L^1(\mathcal{S},\pi))$. The reason is that a dual hypergroup structure is not always given on $\mathcal{S}$. In this case the space $\mathcal{M}(L^1(\mathcal{S},\pi))$ is, in general, smaller than $M(\hat{K})^\vee$. However, we assume some additional properties below, which imply that every measure $\mu \in M(\mathcal{S})$ defines a multiplier for $L^1(\mathcal{S},\pi)$. Furthermore, from the proof of Theorem 4.1.3 we know that the multiplier $T_\mu \in B(L^1(\mathcal{S},\pi))$ induced by $\check{\mu} \in \mathcal{M}(L^1(\mathcal{S},\pi))$ satisfies $\|\mu\| \leq \|T_\mu\|$.

We can say more about multipliers for $L^1(\mathcal{S},\pi)$. The following theorem has already been proven by Lasser in [105].

Theorem 4.1.5. *Let $f \in C(K)$. The following two conditions are equivalent:*

i) There is a measure $\mu \in M(\hat{K})$ such that $\check{\mu} = f$.

ii) For some constant $M \geq 0$ we have

$$\left|\int_K f(x)h(x)dm(x)\right| \leq M\left\|\hat{h}\right\|_\infty$$

for every $h \in C_c(K)$.

Moreover, these two conditions hold true, if $1 \in \mathcal{S}$ and $f \in \mathcal{M}(L^1(\mathcal{S},\pi))$.

Proof. We prove only the implication of *i)* to *ii)*. The rest follows by Theorem 4.1.3. Assume condition *i)* is valid. We obtain

$$\int_K \check{\mu}(x)h(x)dm(x) = \int_K \int_{\mathcal{S}} \alpha(x)d\mu(x)h(x)dm(x) = \int_{\mathcal{S}} \int_K \alpha(x)h(x)dm(x)d\mu(x) = \mu(\hat{h}).$$

Hence, we have

$$\left| \int_K f(x)h(x)dm(x) \right| \leq \|\mu\| \left\| \hat{h} \right\|_\infty .$$

□

V. Muruganandam characterized in [124] multipliers for the Fourier space $A(K) = L^1(\mathcal{S},\pi)^\vee$ with norm $\|\check{\varphi}\|_A = \|\varphi\|_1$, as those complex valued functions g on K with $g \cdot h \in A(K)$ for every $h \in A(K)$. He denoted the space of all multipliers for $A(K)$ by $MA(K)$.
The multiplier spaces $\mathcal{M}(L^1(\mathcal{S},\pi))$ and $MA(K)$ coincide by definition, since Muruganandam proved also that every $f \in MA(K)$ is always continuous, see [124, Proposition 3.2]. Hence, we note that the continuity assumption for every $f \in \mathcal{M}(L^1(\mathcal{S},\pi))$ is dispensable.
Furthermore, his characterizations for multipliers in $MA(K)$ are also true for functions in $\mathcal{M}(L^1(\mathcal{S},\pi))$. Muruganandam discovered, for instance, that the space $\mathcal{M}(L^1(\mathcal{S},\pi))$ forms a Banach algebra with respect to the operator norm $\|f\| := \|T_f\|$ on $L^1(\mathcal{S},\pi)$, see [124, Theorem 3.4]. He showed also that a bounded complex valued function g is an element in $MA(K)$ if and only if there exists a weakly continuous operator M_g on $M(L^2(K,m))$ satisfying

$$M_g(L_f) = L_{g\cdot f} \qquad \text{for all } f \in L^1(K,m),$$

and $\|g\|_{MA(K)} = \|M_g\|$, see [124, Theorem 3.5].
Muruganandam defined a subset $M_bA(K) = \{g \in MA(K) : \; g \text{ is bounded } \}$ in $MA(K)$, since it is unknown whether every multiplier for $A(K)$ is bounded. However, by Remark 4.1.4 we know for a commutative hypergroup K with $1 \in \mathcal{S}$, that $M_bA(K) = MA(K)$. Moreover, we proved that

$$\|f\|_\infty \leq \|T_f\|$$

for every $f \in \mathcal{M}(L^1(\mathcal{S},\pi))$. This coincides with the multiplier theory for groups, where $\|f\|_\infty \leq \|f\|_{MA(G)}$ for every multiplier f on $A(G)$, [124, pp. 11].
Now we investigate the additional properties which imply the converse statement of Theorem 4.1.3.

Definition 4.1.6. We say K **fulfills condition** (F) (or equivalently $\mathcal{S}$ **admits signed product formulas**), if there exists a constant $M > 0$ such that for every $\alpha, \beta \in \mathcal{S}$ there exists $\omega(\alpha,\beta) \in M(\mathcal{S})$ such that

(F1) $\|\omega(\alpha,\beta)\| < M$

(F2) $\alpha(x)\beta(x) = \int_{\mathcal{S}} \tau(x)d\omega(\alpha,\beta)(\tau) = \omega(\alpha,\beta)^\vee(x)$ for all $x \in K$.

Remark 4.1.7. Lasser gave Definition 4.1.6 and some consequences for hypergroups fulfilling (F) in [108]. He mentioned that $\omega(\alpha,\beta)(\mathcal{S}) = \alpha(e)\beta(e) = 1$ and hence $M \geq 1$.
Moreover, Lasser proved that $\mathcal{S}$ admits signed product formulas if and only if $\|(\bar{\alpha}f)^\wedge\|_\infty \leq M\|\hat{f}\|_\infty$ for all $f \in L^1(K,m)$ and $\alpha \in \mathcal{S}$, where M is the constant of Definition 4.1.6.

Remark 4.1.8. There are several examples of commutative hypergroups, which fulfill (F), listed in [124]. More examples can also be found in [4]. Every strong hypergroup obviously fulfills (F). Moreover, every polynomial hypergroup which satisfies the continuity property (P) fulfills (F), as well.

Theorem 4.1.9. *Let K be a commutative hypergroup fulfilling condition (F). Then*

$$M(\mathcal{S})^\vee \subset \mathcal{M}(L^1(\mathcal{S},\pi)).$$

Moreover, the multiplier $T_\mu \in B(L^1(\mathcal{S},\pi))$ induced by $\check{\mu} \in M(\mathcal{S})^\vee$ satisfies $\|T_\mu\| \leq M\|\mu\|$.

Proof. The proof follows by Theorem 4.12 in [124]. □

Corollary 4.1.10. *Let K be a commutative hypergroup fulfilling condition (F). Then $L^1(\mathcal{S},\pi)$ is an algebra with respect to the convolution*

$$(\varphi * \psi)^\vee := \check{\varphi} \cdot \check{\psi}$$

for all $\varphi, \psi \in L^1(\mathcal{S},\pi)$. Moreover, $\|\varphi * \psi\|_1 \leq M \|\varphi\|_1 \|\psi\|_1$ for all $\varphi, \psi \in L^1(\mathcal{S},\pi)$.

Proof. This convolution is well-defined by the uniqueness theorem of the inverse Fourier transform. The proof is completed by Theorem 4.1.9 and Corollary 4.13 in [124]. □

Remark 4.1.11. If the constant $M = 1$, $L^1(\mathcal{S},\pi)$ forms a Banach algebra.

Corollary 4.1.12. *Let K be a commutative hypergroup fulfilling condition (F). If $1 \in \mathcal{S}$, then $L^1(\mathcal{S},\pi)$ is a Banach algebra with respect to the convolution $(\varphi * \psi)^\vee := \check{\varphi} \cdot \check{\psi}$ for all $\varphi, \psi \in L^1(\mathcal{S},\pi)$ under the multiplier operator norm $\|\varphi\| := \|T_\varphi\|$.*
Moreover, the multiplier operator norm is equivalent to the usual norm $\| \ \|_1$.

Proof. The proof follows by Theorem 4.1.9, Corollary 4.1.10 and Corollary 4.14 in [124]. Concerning the equivalence of norms, we obtain by Remark 4.1.4 and Theorem 4.1.9 for every $\varphi \in L^1(\mathcal{S},\pi)$ and the corresponding multiplier $T_\varphi \in M(L^1(\mathcal{S},\pi))$ that $\|\varphi\|_1 \leq \|T_\varphi\| \leq M\|\varphi\|_1$. □

Finally, we can state the following characterizations for multipliers for $L^1(\mathcal{S},\pi)$:

Theorem 4.1.13. *Let K be a commutative hypergroup such that the property (F) is satisfied. Suppose that $\mathcal{S} = \hat{K}$. For an operator $T \in B(L^1(\mathcal{S},\pi))$ the following conditions are equivalent:*

i) $T \in M(L^1(\mathcal{S},\pi))$, *i.e.* $(T\varphi)^\vee = f\check{\varphi}$ *for some complex valued function f on K.*

ii) $\psi * T\varphi \;=\; T(\varphi * \psi) \;=\; \varphi * T\psi$ *for all* $\varphi, \psi \in L^1(\mathcal{S},\pi)$

iii) There exists a unique measure $\mu \in M(\mathcal{S})$ such that

$$(T\varphi)^\vee \;=\; \check{\mu}\,\check{\varphi} \qquad \text{for all } \varphi \in L^1(\mathcal{S},\pi).$$

Moreover, in this case there exists a constant M such that $\|\mu\| \leq \|T\| \leq M\|\mu\|$.

Proof. Equivalency *i*) and *iii*) follow by Theorem 4.1.3 and Theorem 4.1.9. Furthermore, we conclude by Remark 4.1.5 and Theorem 4.1.9 for every $\mu \in M(\mathcal{S})$ and the corresponding multiplier $T_\mu \in M(L^1(\mathcal{S},\pi))$ that $\|\mu\| \leq \|T_\mu\| \leq M\|\mu\|$, where the constant M is given as in Definition 4.1.6.
Let $T \in M(L^1(\mathcal{S},\pi))$ such that $(T\varphi)^\vee = f\check{\varphi}$ for every $\varphi \in L^1(\mathcal{S},\pi)$. We obtain

$$(T(\varphi * \psi))^\vee = f\check{\varphi}\check{\psi} = ((T\varphi) * \psi)^\vee.$$

ii) is proven by the uniqueness theorem of the inverse Fourier transform.
Conversely, let $T \in B(L^1(\mathcal{S},\pi))$ such that $T\varphi * \psi = T(\varphi * \psi)$. Hence, the function

$$f(n) := \frac{(T\varphi)^\vee(n)}{\check{\varphi}(n)} = \frac{(TR_n)^\vee(n)}{\check{R}_n(n)} = (TR_n)^\vee(n)h(n)$$

is well defined for all $n \in \mathbb{N}_0$. Furthermore, $(T\varphi)^\vee\check{\psi}(n) = (T\psi)^\vee\check{\varphi}(n) = 0$ for every $\varphi, \psi \in L^1(\mathcal{S},\pi)$ with $\check{\varphi}(n) = 0$ and $\check{\psi}(n) \neq 0$. Hence, the equation $(T\varphi)^\vee = f\check{\varphi}$ holds for all $\varphi \in L^1(\mathcal{S},\pi)$. □

Remark 4.1.14. Muruganandam invented in [125] a new class of hypergroups, the so-called spherical hypergroups. Those hypergroups are non-commutative, but their Fourier spaces also form Banach algebras under the point-wise product.

4.1.1 An Investigation on polynomial Hypergroups

We investigate the situation above in the case of polynomial hypergroups. This is indeed useful, since the Jacobi polynomials are the only ones in that class which possess a dual hypergroup structure. Let $K = \mathbb{N}_0$ be a polynomial hypergroup generated by $(R_n(x))_{n\in\mathbb{N}_0}$. Applying the inversion theorem we can easily show that each $f \in l^1(\mathbb{N}_0, h)$ defines a multiplier for $L^1(\mathcal{S},\pi)$. In fact, for every $\varphi \in L^1(\mathcal{S},\pi)$ we have $\psi := (f\check{\varphi})^\wedge|\mathcal{S} \in C(\mathcal{S}) \subseteq L^1(\mathcal{S},\pi)$ and $f\check{\varphi} = \check{\psi}$.

If the space $\mathcal{M}(L^1(\mathcal{S},\pi))$ is equal to $M(D_s)^\vee$ and if the correspondence between $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ is isometric, then dual product formulas for $R_n(x)$ are valid. We denote the multiplier operator corresponding to $\mu \in M(D_s)$ by T_μ. Recall that $\|\mu\| \leq \|T_\mu\|$.

Proposition 4.1.15. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup. Assume that $1 \in \mathcal{S}$. If the space of multipliers $M(L^1(\mathcal{S},\pi))$ is equal to $M(D_s)^\vee$, and if $\|T_\mu\| = \|\mu\|$, then $\mathcal{S} = D_s$ and for all $s,t \in D_s$ there exists a probability measure $\mu_{s,t} \in M^1(D_s)$ such that*

$$R_n(s)\,R_n(t) \;=\; \int_{D_s} R_n(u)\,d\mu_{s,t}(u) \qquad \text{for all } n \in \mathbb{N}_0.$$

Proof. Consider the points p_s and p_t of $s \in D_s$ and $t \in D_s$, respectively. There exist $L_{p_s}, L_{p_t} \in B(L^1(\mathcal{S},\pi))$ such that

$$(L_{p_s}(\varphi))^\vee(n) \;=\; \check{p}_s(n)\,\check{\varphi}(n) \;=\; R_n(s)\,\check{\varphi}(n)$$

and

$$(L_{p_t}(\varphi))^\vee(n) \;=\; R_n(t)\,\check{\varphi}(n)$$

for all $\varphi \in L^1(\mathcal{S},\pi)$ and $n \in \mathbb{N}_0$, and therefore

$$(L_{p_s} \circ L_{p_t}(\varphi))^\vee(n) \;=\; R_n(s)\,R_n(t)\,\check{\varphi}(n)$$

is valid for all $\varphi \in L^1(\mathcal{S},\pi)$, $n \in \mathbb{N}_0$. Now choose a net $(k_i)_{i\in I}$ with $k_i \in C(\mathcal{S}) \subseteq L^1(\mathcal{S},\pi)$, $\|k_i\|_1 = 1$ and $\check{k}_i(n) \to 1$ for all $n \in \mathbb{N}_0$, see Lemma 4.1.2.

Since $\|L_{p_s}\| = \|p_s\| = 1$ for all $s \in D_s$, $(L_{p_s} \circ L_{p_t}(k_i))_{i\in I}$ is a norm-bounded net in $L^1(D_s,\pi) \subseteq M(\mathcal{S})$. Thus, there is some subnet (we denote its index set again by I) such that $L_{p_s} \circ L_{p_t}(k_i)$ converges in the $\sigma(M(\mathcal{S}), C(\mathcal{S}))$-topology to some regular complex Borel measure $\mu_{s,t} \in M(\mathcal{S})$, where $\|\mu_{s,t}\| \leq 1$. Hence, we have

$$\int_{\mathcal{S}} R_n(u)\,d\mu_{s,t}(u) \;=\; \lim_i \int_{\mathcal{S}} R_n(u)\;L_{p_s} \circ L_{p_t}(k_i)(u)\,d\pi(u)$$

$$=\; \lim_i\,(L_{p_s} \circ L_{p_t}(k_i))^\vee(n) \;=\; R_n(s)R_n(t)\,\lim_i \check{k}_i(n) \;=\; R_n(s)\,R_n(t)$$

for all $n \in \mathbb{N}_0$. Putting $n = 0$ we have $\|\mu_{s,t}\| = \mu_{s,t}(\mathcal{S}) = 1$, and in view of the Jordan-Hahn decomposition of $\mu_{s,t}$ it follows that $\mu_{s,t}$ is a probability measure on $\mathcal{S}$. Finally, note that $\mu_{s,t} \in M(\mathcal{S})$ is uniquely determined by $R_n(s)R_n(t) = \check{\mu}_{s,t}(n)$ for all $n \in \mathbb{N}_0$. In particular, for $t = 1$ we have $\check{\mu}_{s,1}(n) = R_n(s) = \check{p}_s(n)$ and therefore each $s \in D_s$ is an element of $\mathcal{S} = \operatorname{supp}\pi$. □

Proposition 4.1.15 states that the continuity property (P) is necessarily satisfied if $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ are isometric and isomorphic (as Banach spaces), provided $1 \in \mathcal{S}$. We shall now prove the converse implication.

Theorem 4.1.16. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup, and suppose that $1 \in \mathcal{S}$. Then the following two conditions are equivalent.*

i) The space $\mathcal{M}(L^1(\mathcal{S},\pi))$ is equal to $M(D_s)^\vee$ and the correspondence between $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ is isometric, i.e. $\|T_\mu\| = \|\mu\|$.

ii) The continuity property (P) is fulfilled.

Proof. $i) \Rightarrow ii)$ is already shown in Proposition 4.1.15. Conversely, if the continuity property (P) is fulfilled, we obtain with Remark 1.3.5 that $D_s = \mathcal{S}$. Furthermore, we find for every $\nu \in M(D_s)$ and $\psi \in L^1(\mathcal{S},\pi)$ that $\nu * \psi \in L^1(\mathcal{S},\pi)$ and $\|\nu * \psi\|_1 \leq \|\nu\| \, \|\psi\|_1$. If $T \in M(L^1(\mathcal{S},\pi))$ and $\mu \in M(D_s)$, such that $(T\varphi)^\vee = \check{\mu}\check{\varphi}$ for all $\varphi \in L^1(\mathcal{S},\pi)$, then the inverse uniqueness theorem yields $T\varphi = \mu * \varphi$. Morover, we have $\|T\| \leq \|\mu\|$. □

Theorem 4.1.17. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup such that $1 \in \mathcal{S}$ and the continuity property (P) is satisfied. For an operator $T \in B(L^1(\mathcal{S},\pi))$ the following conditions are equivalent:*

i) $T \in M(L^1(\mathcal{S},\pi))$, i.e. $(T\varphi)^\vee = f\check{\varphi}$ for some complex valued function f on $\mathbb{N}_0$.

ii) $T \circ L_s = L_s \circ T$ for all $s \in D_s$

iii) $\psi * T\varphi = T(\varphi * \psi) = \varphi * T\psi$ for all $\varphi, \psi \in L^1(\mathcal{S},\pi)$

iv) There exists a unique measure $\mu \in M(D_s)$ such that

$$(T\varphi)^\vee = \check{\mu}\,\check{\varphi} \qquad \text{for all } \varphi \in L^1(\mathcal{S},\pi).$$

v) There exists a unique measure $\mu \in M(D_s)$ such that

$$T\varphi = \mu * \varphi \qquad \text{for all } \varphi \in L^1(\mathcal{S},\pi).$$

Moreover, the correspondence between $M(L^1(\mathcal{S},\pi))$ and $M(D_s)$ is isometric, i.e. $\|T\| = \|\mu\|$.

Proof. Let $\varphi \in L^1(\mathcal{S},\pi)$. By $L_s\varphi = \varepsilon_s * \varphi$ for all $s \in D_s$ follows $(L_s\varphi)^\vee(n) = R_n(s)\check{\varphi}(n)$ for all $n \in \mathbb{N}_0$. Hence $L_s \in M(L^1(\mathcal{S},\pi))$ for all $s \in D_s$ and we conclude for every $T \in M(L^1(\mathcal{S},\pi))$ that

$$(T \circ L_s\varphi)^\vee(n) = f(n)R_n(s)\check{\varphi}(n) = (L_s \circ T\varphi)^\vee(n)$$

for all $n \in \mathbb{N}_0$. Hence, $ii)$ holds be the uniqueness theorem of the inverse Fourier transform. Following the lines of Proposition 3.1.2 leads to the implication of $ii)$ to $iii)$.
The proof follows now by Chapter 1.3.1, Theorem 4.1.13 and Theorem 4.1.16. □

4.2 Multipliers for $L^2(\mathcal{S},\pi)$

Now we investigate multipliers for $L^2(\mathcal{S},\pi)$. Applying the Plancherel transform we can define a translation operator for $L^2(\mathcal{S},\pi)$, and derive five equivalent conditions for multipliers for $L^2(\mathcal{S},\pi)$.

We say that $T \in B(L^2(\mathcal{S},\pi))$ is a **multiplier** for $L^2(\mathcal{S},\pi)$ whenever

$$\wp^{-1}(T\varphi) = f\,\wp^{-1}(\varphi)$$

for all $\varphi \in L^2(\mathcal{S},\pi)$, where f is an element in $L^\infty(K,m)$. We denote the space of multipliers for $L^2(\mathcal{S},\pi)$ by $M(L^2(\mathcal{S},\pi))$ and the set of all corresponding functions $f \in L^\infty(K,m)$ by $\mathcal{M}(L^2(\mathcal{S},\pi))$.

To introduce pseudomeasures on $\hat{K}$ we proceed as before. Let

$$A(\hat{K}) := \{\hat{f} : f \in L^1(K,m)\}.$$

Take $\|\hat{f}\|_A := \|f\|_1$ as the norm on $A(\hat{K})$. Then $A(\hat{K})$ is a Banach space, and the dual space $A(\hat{K})^*$ is denoted by $P(\hat{K})$, the elements $s \in P(\hat{K})$ are called **pseudomeasures** on $\hat{K}$. The mapping $\Phi : P(\hat{K}) \to L^\infty(K,m)$, where for each $s \in P(\hat{K})$ the element $\Phi(s) \in L^\infty(K,m)$ is uniquely determined by

$$\int_K f(x)\, \Phi(s)(\tilde{x})\, dm(x) \;=\; s(\hat{f}) \qquad \text{for } f \in L^1(K,m),$$

is an isometric isomorphism from the Banach space $P(\hat{K})$ onto $L^\infty(K,m)$. A convolution of $s_1, s_2 \in P(\hat{K})$ is determined by $s_1 * s_2 \;=\Phi^{-1}(\Phi(s_1)\,\Phi(s_2))$, and Φ is then an algebra isomorphism. $\Phi(s)$ is called inverse Fourier transform of $s \in P(\hat{K})$. If $\mu \in M(\hat{K})$ we have

$$\int_{\hat{K}} \hat{f}(\alpha)\, d\mu(\alpha) \;=\; \int_{\hat{K}}\int_K f(x)\, \overline{\alpha(x)}\, dm(x)\, d\mu(\alpha) \;=\; \int_K f(x)\, \check{\mu}(\tilde{x})\, dm(x)$$

for all $f \in L^1(K,m)$. Hence, each measure $\mu \in M(\hat{K})$ is a pseudomeasure, $\check{\mu} = \Phi(\mu)$, and $\|\mu\|_P \;=\; \|\check{\mu}\|_\infty \;\le\; \|\mu\|$.

In conclusion we arrive at the following proposition

Proposition 4.2.1. *The inverse Fourier transform $\Phi : P(\hat{K}) \to L^\infty(K,m)$ determined by*

$$\int_K f(x)\, \Phi(s)(\tilde{x})\, dm(x) \;=\; s(\hat{f}) \qquad \textit{for all } f \in L^1(K,m)$$

is an isometric algebra isomorphism of $P(\hat{K})$ onto $L^\infty(K,m)$.

Remark 4.2.2. The convolution $\mu_1 * \mu_2$ of two measures $\mu_1, \mu_2 \in M(\hat{K})$ makes sense if we interpret μ_1 and μ_2 as pseudomeasures. Hence, $\mu_1 * \mu_2 \in P(\hat{K})$. However, in general $\mu_1 * \mu_2$ is not a member of $M(\hat{K})$.

Next, we will say that a pseudomeasure $s \in P(\hat{K})$ belongs to $L^2(\mathcal{S},\pi)$ if there is $\psi \in L^2(\mathcal{S},\pi)$ such that

$$s(\hat{f}) \;=\; \int_{\mathcal{S}} \hat{f}(\alpha)\, \psi(\alpha)\, d\pi(\alpha) \qquad \text{for all } f \in L^1(K,m) \,\cap\, L^2(K,m).$$

Similarly to Chapter 3 one can derive the following properties of the $\psi \in L^2(\mathcal{S},\pi)$ belonging to $s \in P(\hat{K})$: ψ is uniquely determined and

$$\int_K f(x)\, \Phi(s)(\tilde{x})\, dm(x) \;=\; s(\hat{f}) \;=\; \int_{\mathcal{S}} \hat{f}(\alpha)\, \psi(\alpha)\, d\pi(\alpha) = \int_K f(x)\, \wp^{-1}\psi(\tilde{x})\, dm(x)$$

for all $f \in L^1(K,m) \,\cap\, L^2(K,m)$.

In particular, we have $\Phi(s) \;= \wp^{-1}(\psi) \;\in L^2(K,m) \cap L^\infty(K,m)$. Conversely, every $s \in P(\hat{K})$ with $\Phi(s) \in L^2(K,m) \cap L^\infty(K,m)$ belongs to $L^2(\mathcal{S},\pi)$. For that, put $\psi = \wp(\Phi(s)) \;\in L^2(\mathcal{S},\pi)$. Hence, the following "dual" statement is valid.

Proposition 4.2.3. *A pseudomeasure $s \in P(\hat{K})$ belongs to $L^2(\mathcal{S},\pi)$ if and only if $\Phi(s) \in L^2(K,m) \,\cap\, L^\infty(K,m)$. Moreover, the inverse Fourier transform of s as a pseudomeasure coincides with the inverse Plancherel transform of the corresponding $\psi \in L^2(\mathcal{S},\pi)$.*

Furthermore, every $\psi \in L^1(\mathcal{S},\pi) \,\cap\, L^2(\mathcal{S},\pi)$ determines a pseudomeasure $s \in P(\hat{K})$ such that

$$s(\hat{f}) \;=\; \int_{\mathcal{S}} \hat{f}(\alpha)\, \psi(\alpha)\, d\pi(\alpha)$$

holds for all $f \in L^1(K,m) \cap L^2(K,m)$. For that, put $s = \Phi^{-1}(\check{\psi})$. In particular, the convolution $s * \psi = \Phi^{-1}(\Phi(s)\check{\psi})$ of $s \in P(\hat{K})$ and $\psi \in L^1(\mathcal{S},\pi) \,\cap\, L^2(\mathcal{S},\pi)$ is well-defined as a convolution of pseudomeasures.

Now, we have all the tools to give a complete characterization of multipliers for $L^2(\mathcal{S},\pi)$.

Theorem 4.2.4. *Let $T \in B(L^2(\mathcal{S},\pi))$. The following conditions are equivalent:*

i) $T \in M(L^2(\mathcal{S},\pi))$, *i.e.* $T\varphi = \wp(f\wp^{-1}(\varphi))$ *for some* $f \in L^\infty(K,m)$.

ii) $T \circ M_\alpha = M_\alpha \circ T$ *for all* $\alpha \in \hat{K}$

iii) $\psi * T\varphi = T(\varphi * \psi) = \varphi * T\psi$ *for all* $\varphi, \psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$

iv) There exists a unique pseudomeasure $s \in P(\hat{K})$ *such that* $s * \varphi$ *belongs to* $L^2(\mathcal{S},\pi)$ *and* $T\varphi = s * \varphi$ *for all* $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$.

v) There exists a unique pseudomeasure $s \in P(\hat{K})$ *such that*

$$\wp^{-1}(T\varphi) = \Phi(s)\, \wp^{-1}(\varphi) \qquad \text{for all } \varphi \in L^2(\mathcal{S},\pi).$$

Moreover, we have an isometric isomorphism between $M(L^2(\mathcal{S},\pi))$, $L^\infty(K,m)$ *and* $P(\hat{K})$.

Proof. The equivalence of *i)* and *v)* follows from Proposition 4.2.1.
Proposition 4.2.3 yields *iv)* $\Leftrightarrow$ *v)*.

i) $\Rightarrow$ *ii)* : We have $\wp^{-1}(M_\alpha\varphi) = \bar{\alpha}\wp^{-1}\varphi$ for all $\varphi \in L^2(\mathcal{S},\pi)$. Therefore, assumption *i)* implies

$$\wp^{-1}(M_\alpha T\varphi) = \bar{\alpha}\,\wp^{-1}(T\varphi) = \bar{\alpha}\, f\, \wp^{-1}(\varphi) = f\,\wp^{-1}(M_\alpha\varphi) = \wp^{-1}(TM_\alpha\varphi)$$

for all $\varphi \in L^2(\mathcal{S},\pi)$. Hence, statement *ii)* is valid.

ii) $\Rightarrow$ *iii)*: Let $\psi \in C_c(\mathcal{S})$, $\varphi \in L^2(\mathcal{S},\pi)$. Since $\psi * T\varphi$ and $\psi * \varphi$ are defined as $L^2(\mathcal{S},\pi)$-valued integrals, and since T is continuous we have

$$\begin{aligned}\psi * T\varphi &= \int_{\mathcal{S}} \psi(\alpha)\, M_{\bar{\alpha}}(T\varphi)\, d\pi(\alpha) = \int_{\mathcal{S}} \psi(\alpha)\, T(M_{\bar{\alpha}}\varphi)\, d\pi(\alpha)\\ &= T\left(\int_{\mathcal{S}} \psi(\alpha)\, M_{\bar{\alpha}}\varphi\, d\pi(\alpha)\right) = T(\psi * \varphi).\end{aligned}$$

If $\psi \in L^1(\mathcal{S},\pi)$ approximate ψ by a sequence $(\psi_n)_n \in \mathbb{N}$ in $C_c(\mathcal{S})$ such that $\|\psi - \psi_n\|_1 \to 0$. It follows that $\psi * T\varphi = T(\varphi * \psi)$ for $\psi \in L^1(\mathcal{S},\pi)$ and $\varphi \in L^2(\mathcal{S},\pi)$. Interchanging the roles of ψ and φ it follows that for all $\varphi, \psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$

$$\psi * T\varphi = T(\varphi * \psi) = \varphi * T\psi.$$

(One should note that $\varphi * \psi = \psi * \varphi$ for $\varphi, \psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$, see [86].)

To prove "*iii)* $\Rightarrow$ *i)*", one can proceed along the lines of the proof "*i)* $\to$ *v)*" of Theorem 4.1.15 to obtain $f \in L^\infty(K,m)$ such that $\wp^{-1}(T\varphi) = f\wp^{-1}(\varphi)$ for all $\varphi \in L^2(\mathcal{S},\pi)$.

It remains to prove that the spaces $M(L^2(\mathcal{S},\pi))$ and $L^\infty(K,m)$ are isometrically isomorphic. We have obviously $\|T\| \le \|f\|_\infty$. Assume $\|T\| < \|f\|_\infty$. Then there exists a compact subset $C \subset K$ such that $|f(x)| > \|T\|$ for m-almost all $x \in C$. Choose $\psi \in L^2(\mathcal{S},\pi)$ such that $\wp^{-1}(\psi) = \chi_C$. We obtain $\|f\chi_C\|_2 > \|T\| m(C)^{1/2}$. On the other hand, we find a contradiction by

$$\|f\chi_C\|_2 = \|f\wp^{-1}(\psi)\|_2 = \|\wp^{-1}(T\psi)\|_2 \le \|T\|\|\psi\|_2 = \|T\| m(C)^{1/2}.$$

Hence, $M(L^2(\mathcal{S},\pi))$ and $L^\infty(K,m)$ are isometrically isomorphic. □

Remark 4.2.5. We want to point out that Theorem 4.2.4 generalizes results of [175] to a vastly bigger class of orthogonal polynomials, viz. those that generate a polynomial hypergroup on $\mathbb{N}_0$.

Theorem 4.2.4 leads to another result concerning translation invariant subspaces of $L^2(\mathcal{S},\pi)$. Larsen quotes a similar result for commutative groups in [101, pp. 94].

Corollary 4.2.6. *A subset X in $L^2(\mathcal{S},\pi)$ is a closed translation invariant linear subspace, if and only if there exists a Borel measurable subset E of K such that*

$$X = \left\{\psi \in L^2(\mathcal{S},\pi) : \wp^{-1}(\psi) = 0 \; m - \textit{almost everywhere off } E\right\},$$

that is $\wp^{-1}(X) = \chi_E \cdot L^2(K,m)$.

Proof. We follow the lines of proof 3.2.10. □

4.3 Multipliers for $L^p(\mathcal{S},\pi)$, $p \neq 1$, $p \neq 2$

Let $1 < p < \infty$. We call a bounded operator T on $L^p(\mathcal{S},\pi)$ **multiplier** for $L^p(\mathcal{S},\pi)$, if there exists a function $f \in L^\infty(K,m)$ such that T is defined by $T\varphi = \wp(f\wp^{-1}(\varphi))$ for every $\varphi \in L^p(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. We denote the set of all multipliers T on $L^p(\mathcal{S},\pi)$ by $M(L^p(\mathcal{S},\pi))$ and the set of corresponding bounded functions f on K by $\mathcal{M}(L^p(\mathcal{S},\pi))$.
As for multipliers for $L^p(K,m)$ we can show the existence of an isometric isomorphism between $M(L^p(\mathcal{S},\pi))$ and $M(L^q(\mathcal{S},\pi))$, $1/p + 1/q = 1$.

Proposition 4.3.1. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then there exists an isometric isomorphism from $M(L^p(\mathcal{S},\pi))$ onto $M(L^q(\mathcal{S},\pi))$.*

Proof. Let $T \in M(L^p(\mathcal{S},\pi))$. Then there exists a bounded function f on K such that $T\varphi = \wp(f\wp^{-1}(\varphi))$ for all $\varphi \in L^p(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Let $\psi \in C_c(\mathcal{S})$ and define

$$F_\psi(\varphi) := \int_{\mathcal{S}} T\psi(\alpha)\varphi(\alpha)d\pi(\alpha) \qquad \text{for all } \varphi \in C_c(\mathcal{S}).$$

We conclude for all $\psi, \varphi \in C_c(\mathcal{S})$ by Plancherel' s theorem

$$\begin{aligned} F_\psi(\varphi) &= \int_{\mathcal{S}} T\psi(\alpha)\varphi(\alpha)d\pi(\alpha) = \int_{\mathcal{S}} \wp(f\wp^{-1}(\psi))(\alpha)\varphi(\alpha)d\pi(\alpha) \\ &= \int_K f\wp^{-1}(\psi)(x)\wp^{-1}(\varphi)(x)dm(x) = \int_{\mathcal{S}} \wp(f\wp^{-1}(\varphi))(\alpha)\psi(\alpha)d\pi(\alpha) \\ &= \int_{\mathcal{S}} T\varphi(\alpha)\psi(\alpha)d\pi(\alpha). \end{aligned}$$

Hence,

$$|F_\psi(\varphi)| = \left|\int_{\mathcal{S}} T\varphi(\alpha)\psi(\alpha)d\pi(\alpha)\right| \leq \|T\|_p \|\varphi\|_p \|\psi\|_q .$$

Thus, F_ψ defines a bounded linear functional on the norm dense subspace $C_c(\mathcal{S})$ of $L^p(\mathcal{S},\pi)$. Hence, F_ψ can be extended to such a functional on all of $L^p(\mathcal{S},\pi)$ without increasing the norm. By the duality of $L^p(\mathcal{S},\pi)$ and $L^q(\mathcal{S},\pi)$ and the definition of F_ψ it follows that $T\psi$ is an element in $L^q(\mathcal{S},\pi)$ and

$$\|T\psi\|_q = \|F_\psi\| \leq \|T\|_p \|\psi\|_q .$$

Thus, T restricted to $C_c(\mathcal{S})$ defines a continuous linear transformation of $C_c(\mathcal{S})$ into $L^q(\mathcal{S},\pi)$. Since $C_c(\mathcal{S})$ is dense in $L^q(\mathcal{S},\pi)$, T restricted to $C_c(\mathcal{S})$ can be uniquely extended to a continuous linear transformation of $L^q(\mathcal{S},\pi)$ without increasing the norm. We denote this extension again by T. Furthermore, by the above considerations $T\phi = \wp(f\wp^{-1}(\phi))$ holds for all $\phi \in L^q(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Indeed, let $\phi \in L^q(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ and $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $C_c(\mathcal{S})$ such that $\|\phi_n - \phi\|_q \to 0$ as n tends to infinity. Since T is continuous, this leads to $\|T\phi_n - T\phi\|_q \to 0$ as

n tends to infinity. We obtain by Plancherel's theorem for all $\psi \in C_c(\mathcal{S})$

$$\begin{aligned}\int_{\mathcal{S}} T\phi(\alpha)\psi(\alpha)d\pi(\alpha) &= \lim_n \int_{\mathcal{S}} T\phi_n(\alpha)\psi(\alpha)d\pi(\alpha) = \lim_n \int_{\mathcal{S}} \wp(f\wp^{-1}(\phi_n))(\alpha)\psi(\alpha)d\pi(\alpha) \\ &= \lim_n \int_K f(x)\wp^{-1}(\phi_n)(x)\wp^{-1}(\psi)(x)dm(x) \\ &= \lim_n \int_{\mathcal{S}} \phi_n(\alpha)\wp(f\wp^{-1}(\psi))(\alpha)d\pi(\alpha) = \int_{\mathcal{S}} \phi(\alpha)\wp(f\wp^{-1}(\psi))(\alpha)d\pi(\alpha) \\ &= \int_K f(x)\wp^{-1}(\phi)(x)\wp^{-1}(\psi)(x)dm(x) = \int_{\mathcal{S}} \wp(f\wp^{-1}(\phi))(\alpha)\psi(\alpha)d\pi(\alpha)\end{aligned}$$

Since $C_c(\mathcal{S})$ is norm dense in $L^p(\mathcal{S},\pi)$ we conclude $T\phi = \wp(f\wp^{-1}(\phi))$ for all $\phi \in L^q(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Hence, $T \in M(L^q(\mathcal{S},\pi))$ and $\|T\|_q \le \|T\|_p$.
Interchanging the roles of p and q proves that the isomorphism constructed above is onto and indeed isometric. □

Remark 4.3.2. The above proof shows especially, that given a bounded linear operator T on $L^p(\mathcal{S},\pi)$ such that $T\varphi = \wp(f\wp^{-1}(\varphi))$ for a function $f \in L^\infty(K,m)$ and all $\varphi \in C_c(\mathcal{S})$, then $T \in M(L^q(\mathcal{S},\pi))$, $1/p + 1/q = 1$. Furthermore, applying Proposition 4.3.1 we obtain also $T \in M(L^p(\mathcal{S},\pi))$. So, it is sufficient for an operator $T \in B(L^p(\mathcal{S},\pi))$ to be represented as $T\varphi = \wp(f\wp^{-1}(\varphi))$ for all $\varphi \in C_c(\mathcal{S})$, to be a multiplier for $L^p(\mathcal{S},\pi)$.

Proposition 4.3.3. *Let $1 \le p < \infty$. There exists a continuous algebra isomorphism from $M(L^p(\mathcal{S},\pi))$ into $M(L^2(\mathcal{S},\pi))$.*

Proof. Obviously every multiplier T for $L^p(\mathcal{S},\pi)$ with $T\varphi = \wp(f\wp^{-1}(\varphi))$ for $\varphi \in L^p(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ defines a multiplier for $L^2(\mathcal{S},\pi)$ by $T\varphi = \wp(f\wp^{-1}(\varphi))$ for every $\varphi \in L^2(\mathcal{S},\pi)$. Furthermore, we have $\|T\varphi\|_2 = \|f\wp^{-1}(\varphi)\|_2 \le \|f\|_\infty\|\varphi\|_2$. For $1 < p$, we have by Proposition 4.3.1 and the Riesz-Thorin Convexity Theorem $\|T\|_2 \le \|T\|_p$.
Especially, for $p = 1$ we find $\|T\varphi\|_2 \le \|\check{\mu}\|_\infty\|\varphi\|_2 \le \|\mu\|\|\varphi\|_2 \le \|T\|\|_1\varphi\|_2$. □

This leads to the next Theorem.

Theorem 4.3.4. *Let $T \in B(L^p(\mathcal{S},\pi))$, $1 < p < 2$. The following conditions are equivalent:*

i) $T \in M(L^p(\mathcal{S},\pi))$, i.e. $T\varphi = \wp(f\wp^{-1}(\varphi))$ for some $f \in L^\infty(K,m)$ and $\varphi \in L^2(\mathcal{S},\pi) \cap L^p(\mathcal{S},\pi)$.

ii) There exists a unique pseudomeasure $s \in P(\hat{K})$ such that $s * \varphi$ belongs to $L^2(\mathcal{S},\pi)$ and $T\varphi = s * \varphi$ for all $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$.

iii) There exists a unique pseudomeasure $s \in P(\hat{K})$ such that

$$(T\varphi)^\vee = \Phi(s)\, \check{\varphi} \text{ in } L^q(K,m) \qquad \text{for all } \varphi \in L^p(\mathcal{S},\pi).$$

Proof. The proof follows by Proposition 4.3.3 and Theorem 4.2.4. *iii*) follows by extending the plancherel transform on $L^2(\mathcal{S},\pi) \cap L^p(\mathcal{S},\pi)$ to the Hausdorff-Young Transform on $L^p(\mathcal{S},\pi)$. □

The next Proposition implies some inclusion results.

Proposition 4.3.5. *Let $1 \le p \le q \le 2$. Then there exists a continuous algebra isomorphism of $M(L^p(\mathcal{S},\pi))$ into $M(L^q(\mathcal{S},\pi))$.*

Proof. Let $T \in M(L^p(\mathcal{S},\pi))$. By Proposition 4.3.3 we know that $T \in M(L^2(\mathcal{S},\pi))$ and $\|T\|_2 \le \|T\|_p$. Using the Riesz-Thorin Convexity Theorem again, we obtain $T \in M(L^q(\mathcal{S},\pi))$ and $\|T\|_q \le \|T\|_p$. □

Remark 4.3.6. With a slight abuse of terminology we have for $1 \le p \le r \le 2$ or $2 \le r \le p < \infty$

$$M(L^1(\mathcal{S},\pi)) \subseteq M(L^p(\mathcal{S},\pi)) \subseteq M(L^r(\mathcal{S},\pi)) \subseteq M(L^2(\mathcal{S},\pi)).$$

Furthermore, we have $M(L^1(\mathcal{S},\pi)) \subset M(\hat{K}) \subset M(L^2(\mathcal{S},\pi))$. This leads to the question, whether there exists $1 < p < 2$ such that $M(L^p(\mathcal{S},\pi)) = M(\hat{K})$. Unfortunately, this is an open question.

In addition to Schoenberg's theorem 4.1.5 (see also [143]), we want to quote a similar theorem which characterizes multipliers for $L^p(\mathcal{S},\pi)$. L.S. Hahn [68] found the following characterization for multipliers for locally compact groups. It can easily be transferred to hypergroups.

Proposition 4.3.7. *Let $1 < p < \infty$ and f be a measurable, bounded function on K. Then the following assertions are equivalent*

i) $f \in \mathcal{M}(L^p(\mathcal{S},\pi))$

ii) There exists a constant B such that

$$\left|\int_K f\check{\psi}\check{\phi}dm\right| \le B\|\psi\|_p\|\phi\|_q$$

for all $\psi,\phi \in C_c(\mathcal{S})$ where $1/p + 1/q = 1$.

Proof. The proof follows the proof of 3.3.9 using Remark 4.3.2. □

Using Hahn's result for multipliers for locally compact groups, see [68, Theorem 5], Fisher proved the completeness of $\mathcal{M}(L^p(\mathcal{S},\pi))$ with respect to the weak-*-topology of $L^\infty(K,m)$, see [47, Corollary 1.1]. This result can easily be transferred to hypergroups.

Corollary 4.3.8. *If $1 < p < \infty$ and $\{f_i\}_{i\in I}$ be a net of functions in $\mathcal{M}(L^p(\mathcal{S},\pi))$ such that the corresponding multipliers $\{T_i\}_{i\in I}$ fulfill $\|T_i\|_p \le B < \infty$ for all $i \in I$. If $\{f_i\}_{i\in I}$ converges in the weak*-topology of $L^\infty(K,m)$ to a function f, then f is also an element in $\mathcal{M}(L^p(\mathcal{S},\pi))$ and the corresponding multiplier T_f fulfills $\|T_f\|_p \le B$.*

Proof. Following the proof of 3.3.10 □

Moreover, Fisher proved another similar result on bounded convergent nets of multipliers for $L^p(G)$ [47, Theorem 3] which can be transferred to hypergroups, too.

Corollary 4.3.9. *Let $1 < p < \infty$. If $\{T_i\}_{i\in I}$ is a bounded net in $M(L^p(\mathcal{S},\pi))$, $\|T_i\|_p \le M < \infty$ for all $i \in I$, and $\{T_i\}_{i\in I}$ converges to T in the weak operator topology over $L^2(\mathcal{S},\pi)$, then $T \in M(L^p(\mathcal{S},\pi))$ and $\|T\|_p \le M$.*

Proof. Following the lines of 3.3.11. □

We observed earlier that multipliers for polynomial hypergroups, which fulfill the continuity property (P), admit

$$l^2(\mathbb{N}_0,h) * l^2(\mathbb{N}_0,h) \subset \mathcal{M}(L^1(\mathcal{S},\pi)) \text{ and } l^1(\mathbb{N}_0,h) * l^\infty(\mathbb{N}_0,h) \subset \mathcal{M}(L^2(\mathcal{S},\pi)).$$

Now it is a natural question whether such a result can be generalized to $L^p(\mathcal{S},\pi)$. Indeed, we can state a similar but more generalized result.
Using the characterizations of Theorem 4.3.7 for functions in $\mathcal{M}(L^p(\mathcal{S},\pi))$, we are able to name a subset of $\mathcal{M}(L^p(\mathcal{S},\pi))$ for all $1 \le p < \infty$. L.-S. Hahn formulated similar results for commutative groups, see [68, Theorem 5].

Theorem 4.3.10. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup fulfilling the continuity property (P). Let $1 \leq p \leq 2$, $1/p + 1/q = 1$. Then*

$$l^p(\mathbb{N}_0, h) * l^q(\mathbb{N}_0, h) \subset \mathcal{M}(L^r(\mathcal{S},\pi)),$$

for every $\frac{2p}{3p-2} \leq r \leq \frac{2p}{2-p}$ or equivalently $\frac{2q}{q+2} \leq r \leq \frac{2q}{q-2}$.

Proof. Follows the proof of 3.4.4. □

Remark 4.3.11. Combining Proposition 4.3.7 and Theorem 4.3.10 we obtain especially that $\|T_{f*g}\|_r \leq \|f\|_p \|g\|_q$ where T_{f*g} denotes the multiplier for $L^r(\mathcal{S},\pi)$ corresponding to $f * g \in \mathcal{M}(L^r(\mathcal{S},\pi))$.

Chapter 5

The Figà-Talamanca Herz Algebras

The Figà-Talamanca Herz algebras $A_p(G)$ has first been investigated by A. Figà-Talamanca in 1965 for a group G which is either Abelian or compact. He also studied $A_2(G)$ for a non-compact, non-commutative unimodular group G [41]. The definition we are using below is analogous to the one for groups given by Eymard, see [36].
In this chapter we will focus on the relation of the Figà-Talamanca Herz algebras $A_p(K)$ to multipliers for $L^p(K,m)$.

5.1 The Figà-Talamanca Herz algebras

5.1.1 Definition

Let K be a commutative hypergroup, $1 < p < \infty$ and $1/p + 1/q = 1$. We define

$$A_p(K) := \{h \in C_0(K) \quad : \quad h = \sum_{i=1}^{\infty} f_i * \tilde{g}_i,\ f_i \in L^p(K,m),\ g_i \in L^q(K,m) \text{ for all } i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty\}.$$

By the inequality $\sum_{i=1}^{\infty} \|f_i * g_i\|_\infty \le \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q$, the sum $\sum_{i=1}^{\infty} |f_i * g_i|$ converges uniformly on K. For h in $A_p(K)$ there might be more than one possible way to write h as a sum of convolutions. Therefore, we define for $h \in A_p(K)$

$$\|h\|_{A_p} := \inf\{\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q : \ h = \sum_{i=1}^{\infty} f_i * \tilde{g}_i\}.$$

$\| \ \|_{A_p}$ is a norm on $A_p(K)$ such that $\|h\|_\infty \le \|h\|_{A_p}$. Furthermore, $A_p(K)$ is a Banach space with respect to $\| \ \|_{A_p}$, see [104]. We set $A_1(K) := C_0(K)$ with the usual sup-norm. These Banach spaces $A_p(K)$ are called **Figà-Talamanca Herz algebra**. The space $A_2(K)$ is equal to $A(K)$, the **Fourier space** of K, which we will discuss below.

Remark 5.1.1. Since we are considering a commutative hypergroup K, we have $A_p(K) = A_q(K)$ for $1/p + 1/q = 1$, $1 < p < \infty$.

Lemma 5.1.2. *Let $1 \le p < \infty$. For $u \in A_p(K)$, there exist sequences $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ in $C_c(K)$ such that $u = \sum_{i=1}^{\infty} f_i * \tilde{g}_i$. Hence, $A_{p,c} := A_p(K) \cap C_c(K)$ is dense in $A_p(K)$.*

Proof. See Lemma 2.2 and Corollary 2.5 in [104]. □

5.1.2 $A_2(K)$ and the Fourier Space of K

In this section we want to show that the space $A_2(K)$ is equal to the Fourier space of K, i.e. $A(K)$, which consists of all inverse Fourier transforms $\check{\varphi}$ with $\varphi \in L^1(\mathcal{S}, \pi)$. We introduced $A(K)$ already in Chapter 3. Recall that the set of pseudomeasures, $P(K)$, is the set of all linear functionals on $A(K)$.
A. Derighetti showed similar results for not necessarily commutative groups in [22, Chapter 3.2].

Lemma 5.1.3. *Let $h \in A_2(K)$ such that $h = \sum_{i=1}^{\infty} f_i * \tilde{g}_i$. Then there exists a unique $\varphi \in L^1(\mathcal{S}, \pi)$ such that*

$$\lim_{N\to\infty} \left\| \varphi - \sum_{i=1}^{N} \wp(f_i)\wp(\tilde{g}_i) \right\|_1 = 0.$$

Moreover, we have $\check{\varphi} = \sum_{i=1}^{\infty} f_i * \tilde{g}_i$ and $\|\varphi\|_1 \leq \sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2$.

Proof. Let $M, N \in \mathbb{N}$ with $M < N$. We have

$$\sum_{i=M}^{N} \|\wp(f_i)\wp(\tilde{g}_i)\|_1 \leq \sum_{i=M}^{N} \|\wp(f_i)\|_2 \|\wp(\tilde{g}_i)\|_2 \leq \sum_{i=1}^{\infty} \|f_i\|_2 \|\tilde{g}_i\|_2 < \infty.$$

Hence, $(\sum_{i=1}^{N} \wp(f_i)\wp(\tilde{g}_i))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\mathcal{S}, \pi)$ and there exists a $\varphi \in L^1(\mathcal{S}, \pi)$ such that $\lim_{N\to\infty} \left\| \varphi - \sum_{i=1}^{N} \wp(f_i)\wp(\tilde{g}_i) \right\|_1 = 0$. Moreover, $\|\varphi\|_1 \leq \sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2$.
Let $n \in \mathbb{N}$. Then

$$\left\| \sum_{i=1}^{\infty} f_i * \tilde{g}_i - \check{\varphi} \right\|_\infty \leq \sum_{i=n+1}^{\infty} \|f_i * \tilde{g}_i\|_\infty + \left\| \varphi - \sum_{i=1}^{n} \wp(f_i)\wp(\tilde{g}_i) \right\|_1 \to 0$$

as n tends to infinity. □

Theorem 5.1.4. *The inverse Fourier transform $^\vee$ is an isometric isomorphism of $L^1(\mathcal{S}, \pi)$ onto $A_2(K)$. Moreover, for every $h \in A_2(K)$ there exist $f, g \in L^2(K, m)$ such that $h = f * \tilde{g}$ and $\|h\|_{A_2} = \|f\|_2 \|g\|_2$.*

Proof. By Corollary 2.2.3 in Chapter 2, we have $L^1(\mathcal{S}, \pi)^\vee = L^2(K, m) * L^2(K, m) \subseteq A_2(K)$. Hence, for each $\varphi \in L^1(\mathcal{S}, \pi)$ exist $f, g \in L^2(K, m)$ such that $\check{\varphi} = f * \tilde{g}$. We choose in particular $\wp(f) = |\varphi|^{1/2}$ and for a representing function φ_1 of φ we set

$$\wp(\tilde{g})(\alpha) = \begin{cases} 0 \text{ if } \varphi_1(\alpha) = 0 \\ \frac{|\varphi_1(\alpha)|}{|\varphi_1(\alpha)|^{1/2}} \text{ if } \varphi_1(\alpha) \neq 0. \end{cases}$$

We obtain $\check{\varphi} = (\wp(f)\wp(\tilde{g}))^\vee = f * \tilde{g} \in A_2(K)$ and $\|\check{\varphi}\|_{A_2} \leq \|f\|_2 \|g\|_2 = \|\varphi\|_1$.
Conversely, by Lemma 5.1.3 holds $A_2(K) \subseteq L^1(\mathcal{S}, \pi)^\vee$ such that $\|\varphi\|_1 \leq \|\check{\varphi}\|_{A_2} \leq \|f\|_2 \|g\|_2$. Hence, we have $\|\varphi\|_1 = \|\check{\varphi}\|_{A_2} = \|f\|_2 \|g\|_2$.

□

Corollary 5.1.5. *The following equalities of linear spaces hold:*

$$A(K) = L^1(\mathcal{S}, \pi)^\vee = L^2(K, m) * L^2(K, m) = A_2(K).$$

Remark 5.1.6. V. Muruganandam defined $A(K)$ as the set $\{h \in C_0(K) : h = \sum_{k=0}^{\infty} f_i * \tilde{f}_i : f_i \in C_c(K) \; \forall i \in \mathbb{N}_0\}$. He also proved that $A(K) = L^1(\mathcal{S}, \pi)^\vee = L^2(K, m) * L^2(K, m)$ and some further equalities, see [124, Proposition 4.2].

Remark 5.1.7. Vrem [162] defines the Fourier space $A(K)$ for a compact hypergroup K as the space of all functions in $L^1(K,m)$ which have absolutely convergent Fourier series. By Corollary 5.1.5 we see that this definition coincides with the definition used here.

Proposition 5.1.8. *Let $\mu \in M(K)$. Then*

$$\|\hat{\mu}\|_\infty = \sup\{|\mu(h)| : \ h \in A_2(K) \text{ and } \|h\|_{A_2} \leq 1\}.$$

Proof. We obtain that $|\mu(h)| \leq \|\hat{\mu}\|_\infty \|h\|_{A_2}$. Hence,

$$\sup\{|\mu(h)| : \ h \in A_2(K) \text{ and } \|h\|_{A_2} \leq 1\} \leq \|\hat{\mu}\|_\infty .$$

Conversely, choose $\epsilon > 0$ and a compact subset $C \subset \mathcal{S}$ such that $|\hat{\mu}(c)| \geq \|\hat{\mu}\|_\infty - \epsilon$ and $\hat{\mu}(c) < 0$ for all $c \in C$ or $\hat{\mu}(c) > 0$ for all $c \in C$. This is possible by the continuity of $\hat{\mu}$. Hence, we have

$$|\int_{\mathcal{S}} \frac{\chi_C(\alpha)}{\pi(C)} \hat{\mu}(\alpha) d\pi(\alpha)| \geq \|\hat{\mu}\|_\infty - \epsilon.$$

Since ϵ was arbitrary, we conclude

$$\sup\{|\int_{\mathcal{S}} \varphi(\alpha)\hat{\mu}(\alpha) d\pi(\alpha)| : \ \varphi \in L^1(\mathcal{S},\pi) \text{ and } \|\varphi\|_1 \leq 1\} \geq \|\hat{\mu}\|_\infty .$$

Summing up, it follows from Theorem 5.1.4 that

$$\|\hat{\mu}\|_\infty = \sup\{|\mu(h)| : \ h \in A_2(K) \text{ and } \|h\|_{A_2} \leq 1\}.$$

□

5.2 The Role of $A_p(K)$ in the Theory of Multipliers

5.2.1 p-pseudomeasures

Let $1 \leq p < \infty$. We know that every bounded measure $\mu \in M(K)$ defines a multiplier L_μ on $L^p(K,m)$ by $L_\mu f := \mu * f$, see Chapter 3. If we denote the operator norm of L_μ in $B(L^p(K,m))$ by $\|L_\mu\|_p$, we obtain $\|L_\mu\|_p \leq \|\mu\|$. Hence, there exists a continuous algebra isomorphism λ_K^p from $M(K)$ into $M(L^p(K,m))$ defined by

$$\lambda_K^p : M(K) \to M(L^p(K,m)), \ \mu \mapsto \lambda_K^p(\mu) = L_\mu.$$

Theorem 5.2.1. *The map $\lambda_K^p : M(K) \to M(L^p(K,m)), \ \mu \mapsto L_\mu$ has the following properties:*

i) λ_K^p is an injective contraction of $M(K)$ into $M(L^p(K,m))$.

ii) For each $x \in K$ we denote by ε_x the point measure in x. For $f \in L^p(K,m)$ we have $\lambda_K^p(\varepsilon_x)(f) = L_{\tilde{x}} f$ and $\|\lambda_K^p(\varepsilon_x)\|_p \leq 1$.

iii) For each $x,y \in K$ is $\lambda_K^p(x * y) = \lambda_K^p(\varepsilon_x) \circ \lambda_K^p(\varepsilon_y)$.

iv) For $\mu_1, \mu_2 \in M(K)$ is $\lambda_K^p(\mu_1 * \mu_2) = \lambda_K^p(\mu_1) \circ \lambda_K^p(\mu_2)$.

Proof. Let $f \in L^p(K,m)$. *i)* and *iv)* are obviously true.
To prove ii) we conclude $\lambda_K^p(\varepsilon_x)(f) = \varepsilon_x * f = \int_K L_{\tilde{z}} f d\varepsilon_x(z) = L_{\tilde{x}} f$.
By $\lambda_K^p(x*y)(f) = \omega(x,y) * f = \int_K L_{\tilde{z}} f d\omega(x,y)(z) = L_{\tilde{x}} \circ L_{\tilde{y}}(f) = \lambda_K^p(\varepsilon_x)\lambda_K^p(\varepsilon_y)(f)$, iii) is proven. □

Remark 5.2.2. Let $1 < p < \infty$. The map $x \mapsto \lambda_K^p(\varepsilon_x)$ is a representation of the hypergroup K in the Banach space $B(L^p(K,m))$. For $p = 2$ the map is called the regular representation of K.

Proposition 5.2.3. *Let K be a finite hypergroup and $1 \leq p < \infty$. We denote by $\mathbb{C}^K$ the set of all complex valued functions on K. Then we have $M(L^p(K,m)) = \lambda_K^p(\mathbb{C}^K)$.*

Proof. Let $T \in M(L^p(K,m))$. For $f \in L^p(K,m) = \mathbb{C}^K$ we have $f = f * \varepsilon_e$. Hence, we find $Tf = T(f * \varepsilon_e) = f * T\varepsilon_e$ and therefore $T = \lambda_K^p(T\varepsilon_e)$. Conversely, it is obvious that $\lambda_K^p(\mathbb{C}^K) \subseteq M(L^p(K,m))$. □

Corollary 5.2.4. *Let $\mu \in M(K)$. Then*

$$\|L_\mu\|_2 = \|\hat{\mu}\|_\infty = \sup\{|\mu(h)| : \ h \in A_2(K) \text{ and } \|h\|_{A_2} \leq 1\}.$$

Proof. By Chapter 3 we know that $\|L_\mu\|_2 = \|\hat{\mu}\|_\infty$, see [20, Theorem 2]. The proof follows then by Proposition 5.1.8 □

Definition 5.2.5. Let $1 < p < \infty$. The topology on $B(L^p(K,m))$, associated to the family of semi-norms

$$T \mapsto |\sum_{i=1}^{\infty} \int_K Tf_i(x)\bar{g}_i(x)dm(x)|$$

with $\sum_{i=1}^{\infty} f_i * \tilde{g}_i \in A_p(K)$, is called the **ultraweak topology**.
This topology is locally convex and Hausdorff.

Remark 5.2.6. Let $T \in B(L^p(K,m))$ and $(S_j)_{j\in J}$ a net of $B(L^p(K,m))$ such that $\sup_j \|S_j\|_p < \infty$, D a dense subset of $L^p(K,m)$ and F a dense subset of $L^q(K,m)$, $1/p + 1/q = 1$. Suppose that $\lim_j \int_K S_jf(x)\bar{g}(x)dm(x) = \int_K Tf(x)\bar{g}(x)dm(x)$ for every $f \in D$ and $g \in F$. Then $\lim_j S_j = T$ for the ultraweak topology, see [22, pp. 48] or [146, pp. 20]

Definition 5.2.7. Let $1 < p < \infty$. The closure of $\lambda_K^p(M(K))$ in $B(L^p(K,m))$ with respect to the ultraweak topology is denoted $PM_p(K)$. Every element of $PM_p(K)$ is called **p-pseudomeasure**.

Remark 5.2.8. Clearly

$$PM_p(K) \subseteq M(L^p(K,m)),$$

but we do not know whether those sets are identical.
Even in the group case it is unknown whether $PM_p(G) = M(L^p(G))$ in general, see [22, pp. 48]. For a locally compact amenable group G and $1 < p < \infty$ those sets are equal, see [22, pp. 48, Corollary 3]. Unfortunately, we do not know if this result is also valid for hypergroups, since the proof in [22, pp. 48] uses the multiplicativity of translation operators on groups, i.e. $L_x(fg) = L_xfL_xg$ for every $x \in G$, $f, g \in L^p(G)$. This is in general not true for a commutative hypergroup. Moreover, concerning amenability hypergroups show very different behavior than groups, see for instance [109].

5.2.2 The Dual of $A_p(K)$

Figà-Talamanca [41] proved 1965 that the dual of the Figà-Talamanca Herz algebras $A_p(G)$ is for every $1 < p < \infty$ isometrically isomorphic to the multiplier space on $L^p(G)$ for every locally compact Abelian group G.
M. Lashkarizadeh Bami, M. Pourgholamhossein and H. Samea [104] transferred this result to hypergroups in the sense that the dual of $A_p(K)$ is isometrically isomorphic to the set $PM_p(K)$ of all p-pseudomeasures. Since we have only $PM_p(K) \subset M(L^p(K,m))$ we do not know whether the spaces $A_p(K)^*$ and $M(L^p(K,m))$ are isometrically isomorphic, too. Nevertheless, we will quote their result here.

Theorem 5.2.9. *Let $1 < p < \infty$ and $1/p+1/q = 1$. For any $F \in A_p(K)^*$ there exists a unique $F' \in PM_p(K)$ such that for all $f \in L^p(K,m)$ and $g \in L^q(K,m)$*

$$\int_K F'(f)(x)\bar{g}(x)dm(x) = F(f * \tilde{g}).$$

The mapping $\Psi_K^p : A_p(K)^* \to PM_p(K)$, $F \mapsto F'$ is a surjective isometry: it carries the weak-topology of $A_p(K)^*$ over to the ultraweak topology of $PM_p(K)$.*
If $h = \sum_{i=1}^\infty f_i * \tilde{g}_i \in A_p(K)$, then

$$F(h) = \sum_{i=1}^\infty \int_K F'(f_i)(x)\bar{g}_i(x)dm(x).$$

If $\mu \in M(K)$ and F_μ is the corresponding element in $A_p(K)^*$ defined by $F_\mu(h) := \int_K hd\mu$ for each $h \in A_p(K)$, then $\lambda_K^p(\mu) = F'_\mu$.

Proof. See Theorem 2.9 in [104]. □

Remark 5.2.10. V. Muruganandam proved similar results in [124] for $p = 2$.

Now we know for every $1 < p < \infty$ that

$$A_p(K)^* \simeq PM_p(K) \subset M(L^p(K,m)).$$

For $p = 2$ we can state even more. Note that in Chapter 3 we denoted the set of all continuous functionals on $A(K)$ by $P(K)$ and called its elements pseudomeasures. With this in mind we continue with

Corollary 5.2.11. *There exists an isometric isomorphism between $M(L^2(K,m))$ and $PM_2(K)$. In particular, every multiplier $T \in M(L^2(K,m))$ is the ultraweak limit of a net $\{\lambda_K^2(\mu_i)\}_{i\in I}$, $\mu_i \in M(K)$ for all $i \in I$.*

Proof. The proof follows from and Theorem 5.2.9, since $M(L^2(K,m))$ is isometrically isomorphic to $P(K) = A(K)^* = A_2(K)^*$ and $A_2(K)^*$ is isometrically isomorphic to $PM_2(K)$. □

Remark 5.2.12. For $p = 1$ define $A_1(K) := C_0(K)$. Note that $C_0(K)^* \simeq M(K) \simeq M(L^1(K,m))$.

Remark 5.2.13. Having Theorem 5.2.9 in mind, one could define for a multiplier $T \in M(L^p(K,m))$

$$\tilde{T}(h) := \sum_{k=0}^\infty Tf_k * \tilde{g}_k(e)$$

for $h = \sum_{k=0}^\infty f_k * \tilde{g}_k \in A_p(K)$. In the group case $\tilde{T}$ defines under some circumstances a linear functional on $A_p(K)$. However, to proof that $\tilde{T}$ is well-defined, the proof in [22, pp. 48] uses the multiplicativity of translation operators on groups, which is in general not valid on hypergroups.

This thought leads to the next proposition.

Proposition 5.2.14. *Let $1 < p < \infty$ and $T \in B(L^p(K,m))$. $T \in PM_p(K)$ if and only if*

$$\sum_{i=1}^\infty \int_K T(f_i)(x)\overline{g_i(x)}dm(x) = \sum_{i=1}^\infty \int_K T(f'_i)(x)\overline{g'_i(x)}dm(x)$$

for all $\sum_{i=1}^\infty f_i * \tilde{g}_i = \sum_{i=1}^\infty f'_i * \tilde{g'}_i \in A_p(K)$.

Proof. Let $T \in PM_p(K)$ and $h \in A_p(K)$ such that $\sum_{i=1}^{\infty} f_i * \tilde{g}_i = h = \sum_{i=1}^{\infty} f_i' * \tilde{g'}_i$. Then, by Theorem 5.2.9 holds

$$\sum_{i=1}^{\infty} \int_K T(f_i)(x)\overline{g_i(x)}dm(x) = (\Psi_K^p)^{-1}(T)(h) = \sum_{i=1}^{\infty} \int_K T(f_i')(x)\overline{g_i'(x)}dm(x).$$

Conversely, let $h = \sum_{i=1}^{\infty} f_i * \tilde{g}_i \in A_p(K)$. We define a linear functional on $A_p(K)$ by

$$F(h) := \sum_{i=1}^{\infty} \int_K T(f_i)(x)\overline{g_i(x)}dm(x).$$

Thus, $F \in A_p(K)^*$ and by Theorem 5.2.9 there exists an operator $S \in PM_p(K)$ such that

$$F(f * \tilde{g}) = \int_K Sf(x)\overline{g(x)}dm(x)$$

for all $f \in L^p(K, m)$ and $g \in L^q(K, m)$, $1/p + 1/q = 1$. Hence, by the continuity and the linearity of F it follows that $T = S \in PM_p(K)$. □

Even though we already know that $P(K)$ is isometrically isomorphic to $M(L^2(K, m))$, we want to show a second possibility to prove this result using Proposition 5.2.14. Derighetti proved similar results for locally compact groups in [22, Chapter 4.2].

Theorem 5.2.15. *$M(L^2(K, m)) = PM_2(K)$ and we have*

$$\sum_{i=1}^{\infty} \int_K T(f_i)(x)\overline{g_i(x)}dm(x) = (\Psi_K^2)^{-1}(T)(h) = \int_{\mathcal{S}} \varphi(\alpha)\hat{T}(\alpha)d\pi(\alpha)$$

for every $h = \sum_{i=1}^{\infty} f_i * \tilde{g}_i \in A_2(K)$, $\varphi \in L^1(\mathcal{S}, \pi)$ such that $\check{\varphi} = h$ and $T \in M(L^2(K, m))$ with corresponding $\hat{T} \in L^\infty(\mathcal{S}, \pi)$.

Proof. Let $T \in M(L^2(K, m))$. To prove that T is an element in $PM_2(K)$ it is by Proposition 5.2.14 sufficient to verify the equality

$$\sum_{i=1}^{\infty} \int_K T(f_i)(x)\overline{g_i(x)}dm(x) = \int_{\mathcal{S}} \varphi(\alpha)\hat{T}(\alpha)d\pi(\alpha)$$

for every $\check{\varphi} = h = \sum_{i=1}^{\infty} f_i * \tilde{g}_i \in A_2(K)$. Using Plancherel's theorem we find that

$$\begin{aligned} \sum_{i=1}^{N} \int_K T(f_i)(x)\overline{g_i(x)}dm(x) &= \sum_{i=1}^{N} \int_{\mathcal{S}} \wp(T(f_i))(\alpha)\wp(\tilde{g}_i)(\alpha)d\pi(\alpha) \\ &= \sum_{i=1}^{N} \int_{\mathcal{S}} \hat{T}(\alpha)\wp(f_i)(\alpha)\wp(\tilde{g}_i)(\alpha)d\pi(\alpha) \end{aligned}$$

for $N \in \mathbb{N}$. Furthermore, by Lemma 5.1.3 $\lim_{N\to\infty} \left\| \sum_{n=1}^{N} \wp(f_i)\wp(\tilde{g}_i) - \varphi \right\|_1 = 0$. Therefore, we obtain

$$\sum_{i=1}^{\infty} \int_K T(f_i)(x)\overline{g_i(x)}dm(x) = \int_{\mathcal{S}} \varphi(\alpha)\hat{T}(\alpha)d\pi(\alpha).$$

□

Remark 5.2.16. We define the linear bijective isometric map

$$\Omega : L^\infty(\mathcal{S}, \pi) \to L^1(\mathcal{S}, \pi)^*, \ \phi \mapsto \Omega(\phi), \ \text{where} \ \Omega(\phi)(\varphi) := \int_{\mathcal{S}} \phi(\alpha)\varphi(\alpha)d\pi(\alpha).$$

Further denote by Φ^T the transposed inverse Fourier transform on $L^1(\mathcal{S}, \pi)$ and by Λ the isometric isomorphism between $M(L^2(K,m))$ and $L^\infty(\mathcal{S}, \pi)$, i.e.

$$\Lambda: \; M(L^2(K,m)) \to L^\infty(\mathcal{S}, \pi), \; T \mapsto \hat{T}.$$

We obtain

$$\Lambda = \Omega^{-1} \circ \Phi^T \circ (\Psi_K^2)^{-1}.$$

Indeed, we have by Theorem 5.2.15 for $T \in M(L^2(K,m))$ and $\varphi \in L^1(\mathcal{S}, \pi)$ that

$$\begin{aligned} \Omega(\hat{T})(\varphi) &= \int_{\mathcal{S}} \hat{T}(\alpha)\varphi(\alpha)d\pi(\alpha) = (\Psi_K^2)^{-1}(T)(\check{\varphi}) \\ &= (\Psi_K^2)^{-1} \circ \Lambda^{-1}(\hat{T})(\check{\varphi}) = \Phi^T \circ (\Psi_K^2)^{-1} \circ \Lambda^{-1}(\hat{T})(\varphi). \end{aligned}$$

Hence, Λ is a homeomorphism of $M(L^2(K,m))$ with the ultraweak topology onto $L^\infty(\mathcal{S}, \pi)$ with the weak*- topology.

Remark 5.2.17. Following the lines of Derighetti in [22, pp 55] we conclude for $1 < p < \infty$ that

$$A_2(K) \subseteq A_p(K) \text{ and } \|h\|_{A_p} \leq \|h\|_{A_2}$$

for all $h \in A_2(K)$. The first statement follows also from the inclusion results in Chapter 3, i.e. with a slight abuse ob terminology

$$A_p(K)^* \subseteq M(L^p(K,m)) \subset M(L^2(K,m)) = A_2(K)^*.$$

Corollary 5.2.18. *Let $1 < p < \infty$. $A_2(K)$ is a dense subspace of $A_p(K)$.*

Proof. Since we have $M(L^p(K,m)) \subset M(L^2(K,m))$ and $A_p^* \subseteq M(L^p(K,m))$, we obtain $A_2(K) \subset A_p(K)$. To show that $A_2(K)$ is dense in $A_p(K)$ only requires noting that each element of $A_p(K)$ can be approximated by functions of the form $\sum_{i=1}^m f_i * g_i$ with $f_i, g_i \in C_c(K)$ for $i = 1, 2, ..., m$, see Lemma 5.1.2. □

5.2.3 An Application

We observe another consequence of Theorem 5.2.9. Using $A_p(K)$ we can reformulate Proposition 3.3.9. M. J. Fisher proved similar results for locally compact groups, see [47].

Proposition 5.2.19. *Let $\varphi \in L^\infty(\hat{K}, \pi)$ and $1 < p \leq 2$. The following conditions are equivalent:*

1. *$\varphi \in \mathcal{M}(L^p(K,m))$, i.e. φ is the Fourier transform of a multiplier $T \in M(L^p(K,m))$.*
2. *There exists a constant $B \geq 0$ such that*

$$\left| \int_{\mathcal{S}} \varphi(\alpha)\psi(\alpha)d\pi(\alpha) \right| \leq B \left\|\check{\psi}\right\|_{A_p}$$

for every $\psi \in L^1(\mathcal{S}, \pi)$, where $\left\|\check{\psi}\right\|_{A_p}$ denotes the norm of the inverse Fourier transform of ψ in $A_p(K)$.

Proof. Let $T \in M(L^p(K,m))$ and φ be the corresponding Fourier transform, i. e. $\varphi = \hat{T}$. Further, let $\psi \in L^1(\mathcal{S}, \pi)$. By Corollary 5.1.5 holds $\check{\psi} \in A_2(K)$ and by Lemma 5.1.2 exist sequences $(f_k)_{k \in \mathbb{N}_0}$, $(g_k)_{k \in \mathbb{N}_0}$ in $C_c(K)$ such that

$$\check{\psi}(x) = \sum_{k=0}^{\infty} f_k * g_k(x) \text{ and } \sum_{k=0}^{\infty} \|f_k\|_2 \|g_k\|_2 < \infty.$$

Furthermore, we have by Corollary 5.2.18 also $\check{\psi} \in A_p(K)$. We obtain by Plancherel's theorem

$$\sum_{k=0}^{\infty} \int_{\mathcal{S}} \varphi(\alpha) \hat{f}_k(\alpha) \hat{g_k}(\alpha) d\pi(\alpha) = \sum_{k=0}^{\infty} \int_K T f_k(x) g_k(x) dm(x) = \sum_{k=0}^{\infty} T f_k * \tilde{g}_k(e).$$

Since $T \in M(L^p(K,m))$ we have

$$\left| \sum_{k=0}^{\infty} T f_k * \tilde{g}_k(e) \right| \leq \sum_{k=0}^{\infty} \|T f_k\|_p \|g_k\|_q \leq \|T\|_p \|\check{\psi}\|_{A_p}.$$

Set $B = \|T\|_p$. By Lemma 5.1.3 $\psi = \sum_{k=0}^{\infty} \hat{f}_k \hat{g}_k$ and we obtain by Lebesque's theorem of dominated convergence

$$\left| \int_{\mathcal{S}} \varphi(\alpha) \psi(\alpha) d\pi(\alpha) \right| \leq B \|\check{\psi}\|_{A_p}.$$

Conversely, suppose that $\varphi \in L^\infty(\mathcal{S}, \pi)$ satisfies

$$\left| \int_{\mathcal{S}} \varphi(\alpha) \psi(\alpha) d\pi(\alpha) \right| \leq B \|\check{\psi}\|_{A_p}$$

for a given constant $B \geq 0$ and every $\psi \in L^1(\mathcal{S}, \pi)$. Define

$$\Phi(\check{\psi}) := \int_{\mathcal{S}} \varphi(\alpha) \psi(\alpha) d\pi(\alpha).$$

Then Φ admits by Corollary 5.2.18 a unique continuous linear extension to all of $A_p(K)$. Hence, we have $\Phi \in A_p(K)^*$. By Theorem 5.2.9 and by reversing the arguments above we conclude for the corresponding multiplier $T \in M(L^p(K,m))$ that $\hat{T} = \varphi$ π-almost everywhere on $\hat{K}$. $\square$

5.3 Inclusion Results

Another interesting application of Theorem 5.2.9 is the proof of strict inclusion results for multiplier spaces meant in the sense that there exists a continuous algebra isomorphism of one multiplier space into another one. We proved for $1 \leq p < r < 2$ or $2 < r < p < \infty$ the inclusions

$$M(L^p(K,m)) \subseteq M(L^r(K,m)) \subseteq M(L^2(K,m)).$$

It is a natural question whether those inclusions are strict. Obviously, it holds for every finite hypergroup K that $M(L^p(K,m)) = M(L^r(K,m)) = M(L^2(K,m))$. Indeed, if K is finite, it is compact and discrete. Hence, the dual space $\hat{K}$ is also compact and discrete and we obtain for every $\varphi \in L^\infty(\hat{K}, \pi)$ and every function $f \in L^p(K,m)$ that $(\varphi \hat{f})^\vee$ is well defined and an element in $L^p(K,m)$. Hence, $\mathcal{M}(L^p(K,m)) = L^\infty(\hat{K}, \pi) = \mathcal{M}(L^2(K,m))$.

The main result of this section is that for every infinite, compact hypergroup K these inclusions are strict. Moreover, we show that

$$\bigcup_{1 \leq q < p} M(L^q(K,m)) \subsetneq M(L^p(K,m)) \subsetneq \bigcap_{p < q \leq 2} M(L^q(K,m)),$$

for $1 < p < 2$, with the first inclusion remaining strict when $p = 2$ and the second inclusion remaining strict when $p = 1$. The same holds for non-compact hypergroups assuming some additional conditions.

In 1970 Figà-Talamanca and Gaudry [43] proved strict inclusion results for multipliers for locally compact groups. Price [130] extended their results and showed similar inclusion results for locally compact groups. The first part of this chapter is similar to section 4.5 in Larsen [101], but then we use these results to transfer Price's results to hypergroups.

5.3.1 Inclusion Results for compact Hypergroups

Lemma 5.3.1. *Let K be infinite and compact. Let $1 \le p < 2$. Then $L^2(K,m)$ is a proper subspace of $L^p(K,m)$.*

Proof. Suppose $L^2(K,m) = L^p(K,m)$. Since $\|f\|_p \le \|f\|_2$ for each $f \in L^p(K,m) \cap L^2(K,m)$, we deduce from the two norm theorem, see [101, D. 6.3] that there exists a constant $B > 0$ such that $\|f\|_2 \le B\|f\|_p$ for $f \in L^2(K,m) \cap L^p(K,m)$. We show now that this is impossible. Since K is not discrete, we have $m(\{e\}) = 0$. Indeed, suppose $m(\{e\}) = \epsilon > 0$, then

$$\epsilon = m(\{e\}) < m(\{e\} * \{x\}) = m(\{x\}).$$

Hence every point $x \in K$ has positive measure $m(\{x\}) \ge \epsilon$. Since K is not discrete, we can choose a compact subset $C \subset K$ which contains an infinite number points $x \in C$. Thus, $m(C) = \infty$. This is a contradiction.
Furthermore, by the regularity of the Haar measure m there exists a sequence of open neighborhoods U_n of the unit element e such that $0 < m(U_n) < \frac{1}{n}$. Define for all $n \in \mathbb{N}$ a function $\chi_{U_n}(x) = 1$ whenever $x \in U_n$ and $\chi_{U_n}(x) = 0$ whenever $x \notin U_n$. We conclude from $\|\chi_{U_n}\|_2 \le B\|\chi_{U_n}\|_p$ that

$$B \ge \frac{\|\chi_{U_n}\|_2}{\|\chi_{U_n}\|_p} = m(U_n)^{\frac{1}{2}-\frac{1}{p}} \ge n^{\frac{1}{p}-\frac{1}{2}}$$

for all $n \in \mathbb{N}$. This contradicts that B is constant. □

Lemma 5.3.2. *Let K be compact and ψ a function on $\hat{K}$. Then the following statements are equivalent:*

i) $\psi \in l^2(\hat{K},\pi)$ *ii)* $\varphi\psi \in L^1(K,m)^\wedge$ *for all* $\varphi \in C_0(\hat{K})$.

Proof. Let $\psi \in l^2(\hat{K},\pi)$. Then there exists a function $g \in L^2(K,m)$ such that $\hat{g} = \psi$. By identifying each $\varphi \in C_0(\hat{K})$ with a multiplier $T_\varphi \in M(L^2(K,m))$ we obtain immediately $\varphi\hat{g} \in l^2(\hat{K},\pi)$ for all $\varphi \in C_0(\hat{K})$. Since $L^2(K,m) \subset L^1(K,m)$, we have $\varphi\psi \in L^1(K,m)^\wedge$ for all $\varphi \in C_0(\hat{K})$.
Conversely, if $\varphi\psi \in L^1(K,m)^\wedge$ for all $\varphi \in C_0(\hat{K})$ then $\psi \in l^\infty(\hat{K},\pi)$. Indeed, assume $\psi \notin l^\infty(\hat{K})$, then there exists a sequence $\{r_n\} \subset \hat{K}$ such that $|\psi(r_n)| \ge n$. Let $\varphi(r_n) = \frac{1}{\sqrt{n}}$, $n \in \mathbb{N}$, and $\varphi(r) = 0$ if $r \ne r_n$ for all $n \in \mathbb{N}$. Since K is compact, $\hat{K}$ is discrete and we see that $\varphi \in C_0(\hat{K})$, but $|\varphi\psi(r_n)| \ge \sqrt{n}$. Thus, $\varphi\psi \notin L^1(K,m)^\wedge$ as $L^1(K,m)^\wedge \subset C_0(\hat{K})$.
Now we can define a bounded operator T_ψ on $(C(K), \|\ \|_1)$ into $L^2(K,m)$ which commutes with translation operators . Let $T_\psi f := \wp^{-1}(\psi\wp(f))$ for every $f \in C(K)$. To prove that T_ψ is continuous, let $f_n, f \in C(K)$ and $h \in L^2(K,m)$ such that $\lim_{n\to\infty}\|f_n - f\|_1 = 0$ and $\lim_{n\to\infty}\|T_\psi f_n - h\|_2 = 0$. By

$$\lim_{n\to\infty}\left\|\psi\hat{f_n} - \psi\hat{f}\right\|_\infty \le \lim_{n\to\infty}\|\psi\|_\infty\|f_n - f\|_1 = 0$$

$$\text{and } \lim_{n\to\infty}\|\wp(Tf_n) - \wp(h)\|_2 = \lim_{n\to\infty}\|\psi\wp(f_n) - \wp(h)\|_2 = 0,$$

we conclude $T_\psi f = h$. By the closed graph theorem T_ψ is continuous. Therefore, T_ψ can be extended to a bounded operator on $L^1(K,m)$ into $L^2(K,m)$ which commutes with translation operators, see [129]. By [129, Theorem 6] $\psi \in l^2(\hat{K},\pi)$, since for any given multiplier $T \in M(L^1(K,m), L^2(K,m))$ there exists only one unique $\psi \in l^2(\hat{K},\pi)$ with $Tf = \wp^{-1}(\psi\wp(f))$ for all $f \in L^1(K,m)$. □

Theorem 5.3.3. *Let K be infinite and compact. Then for each $p \ne 2$, $1 \le p < \infty$,*

$$\mathcal{M}(L^p(K,m)) \cap C_0(\hat{K}) \text{ is a proper subset of } C_0(\hat{K}).$$

Proof. Since $\mathcal{M}(L^p(K,m)) = \mathcal{M}(L^q(K,m))$ whenever $\frac{1}{p} + \frac{1}{q} = 1$, we may assume $1 \le p < 2$. Suppose that $C_0(\hat{K}) \subset \mathcal{M}(L^p(K,m))$. Then for all $f \in L^p(K,m)$ we have that $\varphi\hat{f} \in L^p(K,\pi)^\wedge \subset L^1(K,m)^\wedge$ for each $\varphi \in C_0(\hat{K})$. Note that since K is compact we have $L^p(K,m) \subseteq L^1(K,m)$. By Lemma 5.3.2 we obtain $\hat{f}$ is in $l^2(\hat{K},\pi)$ and therefore $f \in L^2(K,m)$. Thus, $L^p(K,m) \subseteq L^2(K,m)$ and since K is compact we obtain $L^p(K,m) = L^2(K,m)$. This is a contradiction to Lemma 5.3.1. □

Corollary 5.3.4. *Let K be infinite and compact. For each $p \neq 2$, $1 \le p < \infty$, is $\mathcal{M}(L^p(K,m))$ a proper subset of $\mathcal{M}(L^2(K,m))$.*

As a consequence of Corollary 5.3.4 we conclude the strictness of the inclusion $M(L^r(K,m)) \subsetneq M(L^2(K,m))$, $r \neq p$. We will use this proper inclusion to show the strictness of the other inclusions. Following exactly the lines of Price [130, Theoerem 4.1] we conclude the following theorem.

Theorem 5.3.5. *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a strictly positive sequence such that $\lambda_n / \sum_{m=1}^{n-1} \lambda_m$ is unbounded and increasing. Let $1 \le p < 2$ and suppose that there exists a sequence $\{T_n\}_{n\in\mathbb{N}}$ in $M(L^p(K,m))$ and numbers a,b such that*

i) $0 < a < \|T_n\|_p < b$ *for all* $n \in \mathbb{N}$.

ii) $T_n \to 0$ *in* $M(L^2(K,m))$ *as* $n \to \infty$.

Then there exists a sequence $n_1 < n_2 < \ldots$ of positive integers such that the series $\sum_{j=1}^{\infty} \lambda_j T_{n_j}$ converges in $M(L^q(K,m))$ for all $p < q \le 2$ to a unique operator T such that $T \notin M(L^p(K,m))$.

The topic of the next theorem, which proves the strict inclusions of multiplier spaces on a compact hypergroup, was also investigated by Price [130] for locally compact groups. However, for commutative hypergroups, we know only that $A_p(K)^*$ is isometrically isomorphic to a subset of the multiplier space $M(L^p(K,m))$, whereas in the group case $A_p(G)^*$ is isometrically isomorphic to the whole multiplier space $M(L^p(G))$. Hence, we need to adjust the proof given by Price to this situation.

Theorem 5.3.6. *Let K be infinite and compact. Then*

$$\bigcup_{1\le q<p} M(L^q(K,m)) \subsetneq M(L^p(K,m)) \subsetneq \bigcap_{p<q\le 2} M(L^q(K,m)),$$

if $1 < p < 2$. For $p = 1$ the second inclusion remains strict and for $p = 2$ the first inclusion remains strict.

Proof. For $1 \le p < 2$ the proof of the second strict inclusion will be based on the fact that $\| \, \|_p$ is a strictly stronger norm than $\| \, \|_2$ on $M(L^p(K,m))$.
Let $A_1(K) := C_0(K)$. We know by Theorem 2.9 in [104] that $A_p(K)^*$ equipped with the weak*-topology is isometrically isomorphic to the set of all p-pseudomeasures, $PM_p(K)$, which is a subset of $M(L^p(K,m))$. Hence, for each operator $T \in PM_p(K)$ the operator norm of T in $B(L^p(K,m))$ is equal to the operator norm of T in $A_p(K)^*$.
Moreover, since $A(K)$ is dense in $A_p(K)$, $1 \le p < 2$, we obtain also that the normed dual of $A(K)$ equipped with the $A_p(K)$-norm is a subset of $M(L^p(K,m))$. Now, assuming that $\| \, \|_p$ is equivalent to $\| \, \|_2$ on $M(L^p(K,m))$, leads to the equivalence of the $A_p(K)$-norm and the $A(K)$-norm on $A(K)$. Hence, we conclude with a slight abuse of terminology

$$(A(K)^*, \| \, \|_{A_2(K)^*}) = (A(K)^*, \| \, \|_{A_p(K)^*}) = (A_p(K)^*, \| \, \|_{A_p(K)*})$$

$$\subseteq (M(L^p(K,m)), \| \, \|_p) \subseteq (M(L^2(K,m)), \| \, \|_2) = (A(K)^*, \| \, \|_{A_2(K)^*}).$$

Therefore, if $\| \, \|_p$ is equivalent to $\| \, \|_2$ on $M(L^p(K,m))$, then the $A_p(K)$-norm is equivalent to the $A(K)$-norm on $A(K)$, and we obtain $M(L^p(K,m)) = M(L^2(K,m))$. This is a contradiction

to Theorem 5.3.4. Hence, $\| \|_p$ is strictly stronger than $\| \|_2$ on $M(L^p(K,m))$.
Hence, we may choose a sequence in $M(L^p(K,m))$ satisfying *i)* and *ii)* of Theorem 5.3.5. This leads to the existence of a multiplier $T \in \bigcap_{p<q\leq 2} M(L^q(K,m))$ such that $T \notin M(L^p(K,m))$.
To prove the strictness of the first inclusion when $1 < p \leq 2$, note first that from the above we have $M(L^q(K,m)) \subsetneq M(L^p(K,m))$ whenever $1 \leq q < p$. Assume $\bigcup_{1\leq q<p} M(L^q(K,m)) = M(L^p(K,m))$. Then let $\{q_m\}_{m\in\mathbb{N}}$ be an increasing sequence in $[1,p)$ such that $q_m \to p$ as $m \to \infty$. Hence, the injection maps $i_m : M(L^{q_m}(K,m)) \to M(L^p(K,m))$ are continuous and $\bigcup_{m\in\mathbb{N}} i_m(M(L^{q_m}(K,m))) = M(L^p(K,m))$. Thus, the hypotheses of 6.5.1 in Edward's Fourier series [33] are satisfied and we obtain $M(L^{q_m}(K,m)) = M(L^p(K,m))$ for one $m \in \mathbb{N}$ which is a contradiction. □

Remark 5.3.7. When p satisfies $2 \leq p < \infty$ we conclude the analogue by $M(L^p(K,m)) = M(L^{p'}(K,m))$ whenever $\frac{1}{p} + \frac{1}{p'} = 1$.

5.3.2 Inclusion Results for non-compact Hypergroups

To prove strict inclusion results for non-compact hypergroups in a similar way as we used above, requires the existence of a set of uniqueness for $L^p(K,m)$.

Definition 5.3.8. A **set of uniqueness for** $L^p(K,m)$, $1 \leq p \leq \infty$, is a π-measurable set E in $\hat{K}$ with the property that if $\varphi \in L^1(\mathcal{S},\pi)$ with $\varphi = 0$ π-almost everywhere off of E and $\check{\varphi} \in L^p(K,m)$, then $\varphi = 0$ in $L^1(\mathcal{S},\pi)$.

Example 5.3.9. Obviously, every measurable subset of $\widehat{K}$ with measure zero is a set of uniqueness for $L^p(K,m)$, $1 \leq p \leq \infty$.
For $p = 2$ and E a $\pi-$measurable subset of $\widehat{K}$ such that $\pi(E) < \infty$ we have $\varphi = \chi_E \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ and $\varphi = 0$ off of E. Further, $\check{\varphi} = \wp^{-1}(\varphi) \in L^2(K,m)$. Hence, $\varphi = 0$ in $L^2(\mathcal{S},\pi)$ if and only if $\pi(E) = 0$. Therefore, the only sets of uniqueness for $L^2(K,m)$ which have finite measure are those subsets E in $\widehat{K}$ with measure $\pi(E) = 0$. The same holds for all $2 \leq p \leq \infty$.

The prove of the existence of a set of uniqueness for non-compact Abelian groups is due to Figà-Talamanca and Gaudry [43]. Katznelson [95] established a bit earlier similar results for the circle group $\mathbb{T}$. Also interested in set of uniqueness and their connection to the spectra of multipliers for l^p were Devinatz and Hirschman [23].
To prove the existence of a set of uniqueness like Larsen [101], we need to construct a sequence of partitions $\{\Pi_n\}$ of a subset $F \subset \mathcal{S}$, $\pi(F) > 0$, in the following way: $\Pi_0 = F$. Suppose $\Pi_n = \{E_1, ..., E_{2^n}\}$ where the E_j are the mutually disjoint $\pi-$measurable subsets of F such that $\bigcup_{j=1}^{2^n} E_j = F$ and $\pi(E_j) = \pi(F)/2^n$, $j = 1,2,...,2^n$. Then for each $j = 1,2,...,2^n$ we let $E_{ij} \subset E_j$, $i = 1,2$, be a $\pi-$measurable subset of E_j such that $\pi(E_{ij}) = \pi(E_j)/2$, $i = 1,2$, and set $\Pi_{n+1} = \{E_{ij} : i = 1,2; j = 1,2,...,2^n\}$. Clearly Π_{n+1} partitions F into 2^{n+1} mutually disjoint $\pi-$measurable subsets each of measure $\pi(F)/2^{n+1}$.
To assure that this construction is possible, we need the existence of a suitable $F \subset \mathcal{S}$ such that $0 < \pi(F) < \infty$. Since the Plancherel measure is a regular positive Borel measure, we see by Halmos [69, pp. 174] that this construction is possible whenever $\mathcal{S}$ contains a compact subset $F \subset \mathcal{S}$, $\pi(F) > 0$, without an isolated point, i.e. for every point $\alpha \in F$ holds $\pi(\{\alpha\}) = 0$. This for instance is fulfilled for every non-compact strong hypergroup, see [4, Section 2.4].
In the case of a polynomial hypergroup $K = \mathbb{N}_0$ this requires that $\mathcal{S}$ contains an interval $[a,b] = F \in \mathcal{S}$ with $-\infty < a < b < \infty$. Unfortunately, not all dual spaces $\mathcal{S}$ contain such an interval. See for instance the hypergroup generated by the little q-Legendre polynomials, [4]. Here

$$\mathcal{S} = \{0\} \cup \{q^{2k} : \; k \in \mathbb{N}_0\}.$$

Therefore, it is unknown, whether the multiplier spaces $M(l^p(\mathbb{N}_0,h))$, $1 < p < 2$, on the little q-Legendre hypergroup are strictly included in one another.

However, there are also a lot of examples which fulfill this condition. Besides the Jacobi polynomials, where $\mathcal{S} = [-1, 1]$, we can also choose for instance the Karlin-McGregor polynomials which are defined by the recurrence coefficients $a_0 = 1$, $b_0 = 0$ and

$$a_n = \begin{cases} \frac{\alpha-1}{\alpha} & \text{for n odd} \\ \frac{\beta-1}{\beta} & \text{for n even,} \end{cases} \qquad b_n = 0,\ c_n = 1 - a_n$$

for $\alpha, \beta \geq 2$. Here the support of the Plancherel measure equals

$$\mathcal{S} = [-\gamma_1, -\gamma_2] \cup \{0\} \cup [\gamma_2, \gamma_1],$$

where $\gamma_1 = \frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} + \sqrt{\beta - 1})$ and $\gamma_2 = \frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} - \sqrt{\beta - 1})$.

Theorem 5.3.10. *Let K be a non-compact, commutative hypergroup and $\epsilon > 0$. If $\mathcal{S}$ contains a compact subset F such that $\pi(F) > 0$ and for all $\alpha \in F$ holds $\pi(\{\alpha\}) = 0$, then there exists a $\pi-$measurable set $E \subset F$ with*

1. *$\pi(E) > \pi(F) - \epsilon$*
2. *E is a set of uniqueness for $L^p(K, m)$, $1 \leq p < 2$.*

The proof follows the lines of Larsen [101, Theorem 4.4.1] with appropriate adjustments. We will omit it here, since it requires a lot of preliminaries especially concerning measure theory. We refer instead to Larsen [101, Section 4.4], where all preliminaries and the Theorem is proven for non-compact, commutative groups.
Whenever $\mathcal{S}$ contains a compact subset F such that $\pi(F) > 0$ and for all $\alpha \in F$ holds $\pi(\{\alpha\}) = 0$, we say $\mathcal{S}$ **contains a proper set of uniqueness**.
Theorem 5.3.10 enables us to examine the relationships between the spaces of multipliers $M(l^p(\mathbb{N}_0, h))$ for various values of p.

Theorem 5.3.11. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup such that $\mathcal{S}$ contains an interval $[a, b]$, $a < b$. For each $p \neq 2$, $1 < p < \infty$, $\mathcal{M}(l^p(\mathbb{N}_0, h))$ is a proper subset of $\mathcal{M}(l^2(\mathbb{N}_0, h))$.*

Proof. We choose a set of uniqueness $E \subset \mathcal{S}$ for $l^p(\mathbb{N}_0, h)$ with $\infty > \pi(E) > 0$. Define a function χ_E on $\widehat{\mathbb{N}}_0$ by $\chi_E(\alpha) = 1$ whenever $\alpha \in E$ and $\chi_E(\alpha) = 0$ elsewhere. Hence, χ_E defines a multiplier for $l^2(\mathbb{N}_0, h)$ as $L^\infty(\mathcal{S}, \pi) = \mathcal{M}(l^2(\mathbb{N}_0, h))$. However, χ_E is not in $\mathcal{M}(l^p(\mathbb{N}_0, h))$. Indeed, if we consider that χ_E defines a multiplier for $l^p(\mathbb{N}_0, h)$, then $\chi_E \hat{f} \in l^p(\mathbb{N}_0, h)^\wedge$ for all $f \in l^1(\mathbb{N}_0, h)$. In particular if $f = \check{\chi}_{\mathcal{S}}$. Indeed, by $R_0(x) = 1$ we have $\check{\chi}_{\mathcal{S}} \in l^1(\mathbb{N}_0, h)$ as

$$\|\check{\chi}_{\mathcal{S}}\|_1 = \sum_{n=0}^{\infty} |\int_{\mathcal{S}} R_n(x) d\pi(x)| h(n) = \sum_{n=0}^{\infty} |\int_{\mathcal{S}} R_n(x) R_0(x) d\pi(x)| h(n) = h(0) < \infty.$$

Hence, $\chi_E = \chi_E \hat{f} \in l^p(\mathbb{N}_0, h)^\wedge$. This means that there exists a unique function $g \in l^p(\mathbb{N}_0, h)$ such that $\hat{g} = \chi_E$. As $\chi_E \in L^1(\mathcal{S}, \pi)$ we get $\check{\chi_E} = g \in l^p(\mathbb{N}_0, h)$. This contradicts the fact that E is a set of uniqueness for $l^p(\mathbb{N}_0, h)$. □

Following the lines of Theorem 5.3.6 we obtain

Theorem 5.3.12. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup, such that $\mathcal{S}$ contains an interval $[a, b]$, $a < b$. Then*

$$\bigcup_{1 \leq q < p} M(l^q(\mathbb{N}_0, h)) \subsetneq M(l^p(\mathbb{N}_0, h)) \subsetneq \bigcap_{p < q \leq 2} M(L^q(\mathbb{N}_0, h)),$$

if $1 < p < 2$. For $p = 1$ the second inclusion remains strict and for $p = 2$ the first inclusion remains strict.

Remark 5.3.13. For $2 \leq p < \infty$ we conclude the analogue by $M(l^p(\mathbb{N}_0, h)) = M(l^{p'}(\mathbb{N}_0, h))$, $1/p + 1/p' = 1$.

Remark 5.3.14. There are other examples of hypergroups which are neither compact nor discrete and whose multiplier spaces are strictly included in one another. One example is the Bessel-Kingman hypergroup, see [4], which is a strong hypergroup.
In order to prove that $\mathcal{M}(L^p(K, m))$ is a proper subset of $\mathcal{M}(L^2(K, m))$ for a non-compact hypergroup in the way we used above, we need that $\mathcal{S}$ contains a proper set of uniqueness. Furthermore, we need the existence of a function $f \in L^1(K, m) \cap L^2(K, m)$ such that $\hat{f}|F = 1$. This is obviously guaranteed for every strong hypergroup, see Chapter 2. The existence of such a function is also guaranteed, whenever $L^1(K, m)$ is regular, that is if for every closed subset V of $\chi^b(K)$ and $\alpha \in \chi^b(K)\backslash V$ there exists a function $f \in L^1(K, m)$ with $\hat{f}|V = 0$ and $\hat{f}(\alpha) \neq 0$. Indeed, choose a compact subset C in $\mathcal{S}$ and for every $x \in C$ choose an open neighborhood U_x with compact closure. Then there exists a finite subcover such that $C \subset \bigcup_{j=1}^n U_{x_j}$. By Kaniuth [90, Corollary 4.2.9] there exists a function $f \in L^1(K, m)$ with $\hat{f}|C = 1$ and $\hat{f} = 0$ outside of $\bigcup_{j=1}^n U_{x_j}$. Since the closure of $\bigcup_{j=1}^n U_{x_j}$ is compact, we have $f \in L^2(K, m)$. However, the premise of regularity of $L^1(K, m)$ is much stronger than needed.
In the next chapter, we will prove that $\mathcal{M}(L^p(K, m))$ is a proper subset of $\mathcal{M}(L^2(K, m))$ for a non-compact hypergroup in a different way using derived spaces. We will see there that the assumption of the existence of a function $f \in L^1(K, m) \cap L^2(K, m)$ such that $\hat{f}|F = 1$ is unnecessary.

Remark 5.3.15. To prove similar strict inclusion results for multipliers for $L^p(\mathcal{S}, \pi)$ will be quite difficult, since in general we do not have a dual convolution and hence we cannot define the Figà-Talamanca Herz algebra on $\mathcal{S}$ like we did at the beginning of this chapter. Furthermore, given a relatively non-compact $\mathcal{S}$, we would need a translation operator on $\mathcal{S}$ to prove the existence of a proper set of uniqueness for $L^p(\mathcal{S}, \pi)$ in the way described in [101]. However, we will prove some strict inclusion results for the dual of a polynomial hypergroup as an application to derived spaces in the next chapter.

Chapter 6

Derived Spaces

In the previous chapters we characterized those functions $\varphi \in L^\infty(\mathcal{S}, \pi)$ which define a multiplier for the different L^p-spaces. Conversely, we now take a look on those function spaces, for which all $\varphi \in C_0(\hat{K})$ define multipliers. We derive these spaces from the original $L^p(K, m)$ spaces. Thus, they are called derived spaces.
Helgason [71, 72] discussed derived algebras of commutative Banach algebras. His results are also quoted in Larsen [101, Chapter 1.8] which are obviously also valid for $L^1(K, m)$ or for $L^p(K, m)$ whenever K is compact. (We note that for a compact, commutative hypergroup K, $L^p(K, m)$ with the usual convolution product is a semi-simple, self-adjoint commutative Banach algebra which contains an approximate identity. However, this approximate identity is not a minimal approximate identity.)
Figà-Talamanca [40] and Gaudry [43] invented derived spaces for $L^p(G)$, where G is a locally compact Abelian group. Their results are also quoted in Larsen [101, Chapter 4.6]. Hörmander [84] proved similar results for Euclidean groups $G = \mathbb{R}^n$. We will extend some of the results from Figà-Talamanca [40] and Gaudry [43] to hypergroups. In contrast to the group case not every dual $\mathcal{S}$ contains a proper set of uniqueness. Further we find $|\alpha(x)| \leq 1$ for every $\alpha \in \hat{K}$ and $x \in K$, whereas we obtain $|\alpha(x)| = 1$ for every x in a locally compact group G and $\alpha \in \hat{G}$. This leads to somehow weaker results for hypergroups.
In the last part of this chapter we prove applications of derived spaces for commutative hypergroups concerning strict inclusion results of multiplier spaces.

6.1 The Derived Space for $L^p(K, m)$

If K is non-compact, then $L^p(K, m)$, $p > 1$, is in general no longer a Banach algebra and the concept of the derived *algebra* is meaningless. Therefore, we define in a natural analogous manner the derived space of $L^p(K, m)$, $1 \leq p < \infty$. For each $f \in L^p(K, m)$, $1 \leq p < \infty$, denote

$$\|f\|_0 := \sup \left\{ \|h * f\|_p : h \in L^1(K, m), \left\|\hat{h}\right\|_\infty \leq 1 \right\}.$$

Let $L^p(K, m)^0$ be the linear subspace of $L^p(K, m)$ consisting of all those $f \in L^p(K, m)$ with $\|f\|_0 < \infty$. We call $L^p(K, m)^0$ the **derived space** of $L^p(K, m)$. We prove below that $L^p(K, m)^0$ is a Banach space with respect to the norm $\| \; \|_0$.
For $p = 1$ we see at once from Lemma 1.8.1 in [101], that $L^1(K, m)^0$ coincides with the derived algebra of $L^1(K, m)$, defined by

$$L^1(K, m)_0 = \left\{ f \in L^1(K, m) : \; \varphi \hat{f} \in L^1(K, m)^\wedge \quad \forall \varphi \in C_0(\hat{K}) \right\}.$$

Starting with this definition, we define for $1 \leq p \leq 2$

$$L^p(K,m)_0 := \left\{ f \in L^p(K,m) : \ \varphi \hat{f} \in L^p(K,m)^\wedge \quad \forall \varphi \in C_0(\hat{K}) \right\}.$$

Here, $\hat{f}$ denotes the Hausdorff-Young transform of an element in $L^p(K,m)$. Obviously $L^p(K,m)_0$ is a linear subspace of $L^p(K,m)$ and it equals $L^p(K,m)^0$ whenever $L^p(K,m)$ is a semi-simple self-adjoint Banach algebra, see [101, Chapter 1.8]. Moreover, since every bounded function on $\hat{K}$ defines a multiplier for $L^2(K,m)$, we see at once that $L^2(K,m)_0 = L^2(K,m)$. Further $L^2(K,m)^0 = L^2(K,m)$, because

$$\|h * f\|_2 = \|\wp(h*f)\|_2 = \left\| \hat{h} \wp(f) \right\|_2 \leq \left\| \hat{h} \right\|_\infty \|\wp(f)\|_2 = \left\| \hat{h} \right\|_\infty \|f\|_2$$

for all $h \in L^1(K,m)$ and $f \in L^2(K,m)$. Hence, $L^2(K,m)_0 = L^2(K,m)^0$ as linear spaces.

Our aim is now for an arbitrary hypergroup K and $1 < p < 2$ to prove that the two spaces $L^p(K,m)_0$ and $L^p(K,m)^0$ coincide. Furthermore, we will investigate some properties of $L^p(K,m)_0$ and $L^p(K,m)^0$.

Theorem 6.1.1. *Let* $1 \leq p \leq 2$. *Then* $L^p(K,m)_0 \subseteq L^p(K,m)^0$.

Proof. Let $f \in L^p(K,m)_0$. For $\varphi \in C_0(\hat{K})$ define a linear transformation A from $C_0(\hat{K})$ into $L^p(K,m)$ by $(A\varphi)^\wedge := \varphi \hat{f} \in L^p(K,m)^\wedge$. To prove that A is continuous, let $\lim_n \|\varphi_n - \varphi\|_\infty = 0$ and $\lim_n \|A\varphi_n - g\|_p = 0$. Then for $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left\| \varphi \hat{f} - \hat{g} \right\|_q \leq \left\| \varphi \hat{f} - (A\varphi_n)^\wedge \right\|_q + \|(A\varphi_n)^\wedge - \hat{g}\|_q \leq \|\varphi - \varphi_n\|_\infty \|f\|_p + \|A\varphi_n - g\|_p \to 0$$

as $n \to \infty$. Therefore, $(A\varphi)^\wedge = \varphi \hat{f} = \hat{g}$. By the uniqueness of the Hausdorff-Young transform A is a closed transformation and hence continuous by the closed graph theorem. Since $(h * f)^\wedge = \hat{h}\hat{f}$ for all $h \in L^1(K,m)$, we see that $\|h * f\|_p = \left\| A\hat{h} \right\|_p \leq \|A\| \left\| \hat{h} \right\|_\infty$. Thus, $\|f\|_0 \leq \|A\| < \infty$ and f is an element in $L^p(K,m)^0$. □

In the following, we characterize the space $L^p(K,m)^0$ in order to prove that it coincides with the linear space $L^p(K,m)_0$ for $1 \leq p \leq 2$.

Theorem 6.1.2. *Suppose* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the following are equivalent:*

i) $f \in L^p(K,m)^0$.

ii) For each $g \in L^q(K,m)$ *exists a unique* $\mu \in M(\hat{K})$ *such that* $f * g = \check{\mu}$.

Proof. Let $f \in L^p(K,m)^0$ and $g \in L^q(K,m)$. We define a continuous linear functional on $L^1(K,m)^\wedge$ with respect to the supremum norm by

$$F(\hat{h}) := h * f * g(e)$$

for each $h \in L^1(K,m)$. F is indeed continuous, since

$$|F(\hat{h})| \leq \|h * f * g\|_\infty \leq \|h * f\|_p \|g\|_q \leq \left\| \hat{h} \right\|_\infty \|f\|_0 \|g\|_q .$$

Consequently, F has a unique extension to a continuous linear functional on $C_0(\hat{K})$ and $\|F\| \leq \|f\|_0 \|g\|_q$. Let $\mu \in M(\hat{K})$ be the unique measure associated with this extension of F by the Riesz representation theorem. We obtain for each $h \in L^1(K,m)$

$$h * f * g(e) = \int_{\hat{K}} \hat{h}(\alpha) d\mu(\alpha).$$

Further, for each $h \in L^1(K,m)$

$$\begin{aligned} h * f * g(e) &= \int_{\hat{K}} \hat{h}(\alpha) d\mu(\alpha) = \int_{\hat{K}} \int_K h(x)\bar{\alpha}(x) dm(x) d\mu(\alpha) \\ &= \int_K h(x) \int_{\hat{K}} \alpha(\tilde{x}) d\mu(\alpha) dm(x) = \int_K h(x) L_{\tilde{x}} \check{\mu}(e) dm(x) = h * \check{\mu}(e). \end{aligned}$$

Let $x \in K$. Choosing $L_x h \in L^1(K,m)$ in the equations above, we conclude

$$h * f * g(x) = L_x h * f * g(e) = L_x h * \check{\mu}(e) = h * \check{\mu}(x)$$

for every $h \in L^1(K,m)$. Thus, $f * g = \check{\mu}$. μ is unique by the uniqueness theorem of the inverse Fourier-Stieltjes transform.
Conversely, suppose that $f \in L^p(K,m)$ such that for each $g \in L^q(K,m)$ there exists a unique measure $\mu \in M(\hat{K})$ with $f * g = \check{\mu}$. Clearly, this equation defines a linear transformation $S : L^q(K,m) \to M(\hat{K})$, $Sg := \mu$. Moreover, if $\lim_n \|g_n - g\|_q = 0$ and $\lim_n \|Sg_n - v\| = 0$ then

$$\begin{aligned} \|(Sg)^\vee - \check{v}\|_\infty &\leq \|(Sg)^\vee - (Sg_n)^\vee\|_\infty + \|(Sg_n)^\vee - \check{v}\|_\infty \\ &\leq \|f * g - f * g_n\|_\infty + \|Sg_n - v\| \leq \|f\|_p \|g - g_n\|_q + \|Sg_n - v\| \to 0. \end{aligned}$$

Hence, $Sg = v$ and by the Closed Graph Theorem S is continuous. We obtain for every $h \in L^1(K,m)$

$$\begin{aligned} |h * f * g(e)| &= |h * \check{\mu}(e)| = |\int_K h(x) L_{\tilde{x}} \check{\mu}(e) dm(x)| = |\int_K h(x) \int_{\hat{K}} \alpha(\tilde{x}) d\mu(\alpha) dm(x)| \\ &= |\int_{\hat{K}} \hat{h}(\alpha) d\mu(\alpha)| \leq \left\|\hat{h}\right\|_\infty \|\mu\| = \left\|\hat{h}\right\|_\infty \|Sg\| \leq \left\|\hat{h}\right\|_\infty \|S\| \|g\|_q . \end{aligned}$$

Consequently, we conclude with the Hahn-Banach Theorem

$$\|h * f\|_p = \sup\left\{ |h * f * g(e)| : \|g\|_q \leq 1 \right\} \leq \left\|\hat{h}\right\|_\infty \|S\| .$$

Hence, $\|f\|_0 \leq \|S\| < \infty$ and $f \in L^p(K,m)^0$. □

Theorem 6.1.2 shows that each $f \in L^p(K,m)^0$ defines a continuous linear transformation $S : L^q(K,m) \to M(\hat{K})$ by $(Sg)^\vee = \check{\mu} = f * g$ and $\|f\|_0 \leq \|S\|$. Moreover, we obtain

$$(SL_x g)^\vee = f * L_x g = L_x(f * g) = L_x(Sg)^\vee = \varepsilon_{\tilde{x}} * (Sg)^\vee$$

and further

$$\begin{aligned} (\hat{\epsilon}_{\tilde{x}} Sg)^\vee(y) &= \int_{\hat{K}} \alpha(y) \hat{\epsilon}_{\tilde{x}}(\alpha) d(Sg)(\alpha) = \int_{\hat{K}} \alpha(x)\alpha(y) d(Sg)(\alpha) \\ &= \int_{\hat{K}} L_y \alpha(x) d(Sg)(\alpha) = \int_{\hat{K}} \int_K \varepsilon_{\tilde{x}}(z) L_y \alpha(\tilde{z}) dm(z) d(Sg)(\alpha) \\ &= \int_{\hat{K}} \int_K L_{\tilde{y}} \tilde{\epsilon}_{\tilde{x}}(z) \alpha(z) dm(z) d(Sg)(\alpha) = \int_K L_y \varepsilon_{\tilde{x}}(\tilde{z}) \int_{\hat{K}} \alpha(z) d(Sg)(\alpha) dm(z) \\ &= \int_K L_y \varepsilon_{\tilde{x}}(z) (Sg)^\vee(\tilde{z}) dm(z) = \int_K \varepsilon_{\tilde{x}}(z) L_{\tilde{z}}(Sg)^\vee(y) dm(z) = \varepsilon_{\tilde{x}} * (Sg)^\vee(y) \end{aligned}$$

for all $x, y \in K$. Hence, S commutes with translations in the sense that $S(L_x g) = \hat{\epsilon}_{\tilde{x}} Sg$ for all $x \in K$. The next theorem shows that the converse is also true. We prove that each continuous linear transformation $S : L^q(K,m) \to M(\hat{K})$ with $S(L_x g) = \hat{\epsilon}_{\tilde{x}} Sg$ for all $x \in K$ defines a unique function $f \in L^p(K,m)^0$.

Theorem 6.1.3. *Suppose* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $S : L^q(K,m) \to M(\hat{K})$ *be a linear transformation. The following assertions are equivalent* *i) There exists a unique $f \in L^p(K,m)^0$ such that $(Sg)^\vee = f * g$ for all $g \in L^q(K,m)$.*

ii) S is continuous and $S(L_x g) = \hat{\epsilon}_{\tilde{x}}(Sg)$ for all $x \in K$.

Moreover, the correspondence between f and S defines a linear isometry from $L^p(K,m)^0$ with norm $\| \, \|_0$ onto the Banach space of all continuous linear transformations $S : L^q(K,m) \to M(\hat{K})$ such that $SL_x g = \hat{\epsilon}_{\tilde{x}} Sg$ for all $x \in K$.

Proof. It remains to prove that *ii)* implies *i)*. For $g \in L^q(K,m)$ denote $Sg = \mu_g \in M(\hat{K})$. We define a functional F on $L^q(K,m)$ by

$$F(g) := \int_{\hat{K}} d\mu_g.$$

Since S is linear, F is also linear and we obtain $|F(g)| = \|\tilde{\mu}_g\| = \|Sg\| \leq \|S\| \, \|g\|_q$ for all $g \in L^q(K,m)$. Thus, F is continuous and there exists a unique function $f \in L^p(K,m)$ such that $F(g) = f * g(e)$ for all $g \in L^q(K,m)$. We conclude for all $x \in K$

$$\begin{aligned} f * g(x) &= L_x(f * g)(e) = f * L_x g(e) = F(L_x g) \\ &= \int_{\hat{K}} d(SL_x g) = \int_{\hat{K}} d(\hat{\epsilon}_{\tilde{x}} Sg) = \int_{\hat{K}} \alpha(x) d\mu_g(\alpha) = \check{\mu}_g(x). \end{aligned}$$

Hence, for all $g \in L^q(K,m)$ we have $f * g = (Sg)^\vee$ and $f \in L^p(K,m)^0$ by Theorem 6.1.2. Furthermore, the equivalence of *i)* and *ii)* defines obviously a mapping from $L^p(K,m)^0$ onto the Banach space of all continuous linear transformations $S : L^q(K,m) \to M(\hat{K})$ which fulfill $S(L_x g) = \hat{\epsilon}_{\tilde{x}}(Sg)$ for all $x \in K$. By the proof of Theorem 6.1.2 we have $\|f\|_0 \leq \|S\|$ whenever $f * g = (Sg)^\vee$ for all $g \in L^q(K,m)$. On the other hand we find for each $g \in L^q(K,m)$ by the proof of Theorem 6.1.2 that

$$|h * f * g(e)| = |\int_{\hat{K}} \hat{h}(\alpha) d(Sg)(\alpha)|$$

for all $h \in L^1(K,m)$. Since $L^1(K,m)^\wedge$ is norm dense in $C_0(\hat{K})$ we conclude that

$$\begin{aligned} \|Sg\| &= \sup \left\{ |\int_{\hat{K}} \hat{h}(\alpha) d\mu(\alpha)| : \hat{h} \in L^1(K,m)^\wedge, \left\|\hat{h}\right\|_\infty \leq 1 \right\} \\ &= \sup \left\{ |h * f * g(e)| : h \in L^1(K,m), \left\|\hat{h}\right\|_\infty \leq 1 \right\} \\ &\leq \sup \left\{ \|h * f\|_p \|g\|_q : h \in L^1(K,m), \left\|\hat{h}\right\|_\infty \leq 1 \right\} = \|f\|_0 \|g\|_q . \end{aligned}$$

Thus, we obtain in total that $\|S\| = \|f\|_0$ and the mapping is an isometry. ☐

Corollary 6.1.4. *Let $1 < p < \infty$. With the norm $\| \, \|_0$, $L^p(K,m)^0$ is a Banach space.*

The next theorem leads to the inclusion $L^p(K,m)^0 \subset L^p(K,m)_0$ for $1 < p \leq 2$.

Theorem 6.1.5. *Let $1 < p \leq 2$ and $\varphi \in C^b(\hat{K})$. Then there exists a bounded linear operator $T : L^p(K,m)^0 \to L^p(K,m)^0$ such that T commutes with translations and $\varphi \hat{f} = (Tf)^\wedge$ for all $f \in L^p(K,m)^0$.*

Proof. Let $f \in L^p(K,m)^0$. By Theorem 6.1.2 there exists for every $g \in L^q(K,m)$ a unique measure $\mu \in M(\hat{K})$ such that $f * g = \check{\mu}_g$. Define a linear functional F on $L^q(K,m)$ by

$$F(g) := \int_{\hat{K}} \tilde{\varphi}(\alpha) d\mu_g(\alpha).$$

Since $|F(g)| \leq \|\varphi\|_\infty \|\mu_g\| \leq \|\varphi\|_\infty \|f\|_0 \|g\|_q$, F is bounded. Thus, there exists a unique $Tf \in L^p(K,m)$ such that $F(g) = Tf * g(e)$. Further, we obtain for all $y \in K$

$$Tf * g(y) = \int_K Tf(x) L_{\tilde{x}} g(y) dm(x) = F(L_y g).$$

By Theorem 6.1.3

$$F(L_y g) = \int_{\hat{K}} \tilde{\varphi}(\alpha) d(SL_y g)(\alpha) = \int_{\hat{K}} \tilde{\varphi}(\alpha) d(\hat{\epsilon}_{\tilde{y}} Sg)(\alpha) = \int_{\hat{K}} \tilde{\varphi}(\alpha)\alpha(y) d\mu_g(\alpha) = \check{v}(y)$$

where $dv = \tilde{\varphi} d\mu_g$. By Theorem 6.1.2 $Tf \in L^p(K,m)^0$ and we conclude by Theorem 6.1.3 $\|Tf\|_0 \leq \|f\|_0 \|\varphi\|_\infty$.
Moreover, by Chapter 2 we have $(\check{v})^\wedge d\pi = d\tilde{v} = \varphi d\tilde{\mu_g} = \varphi(\check{\mu_g})^\wedge d\pi$. Hence, we obtain $(Tf)^\wedge \hat{g} = (Tf * g)^\wedge = (\check{v})^\wedge = \varphi(\check{\mu}_g)^\wedge = \varphi(f * g)^\wedge = \varphi \hat{f} \hat{g}$ for all $g \in L^1(K,m) \cap L^q(K,m)$. For each $\alpha \in \hat{K}$ choose a function $g \in L^1(K,m) \cap L^q(K,m)$ such that $\hat{g}(\alpha) \neq 0$. Hence, we conclude $(Tf)^\wedge(\alpha) = \varphi \hat{f}(\alpha)$ for all $\alpha \in \hat{K}$ and T obviously commutes with translations. □

Corollary 6.1.6. *Let $1 \leq p \leq 2$. Then $L^p(K,m)^0 = L^p(K,m)_0$*

Remark 6.1.7. We want to remark that Theorem 6.1.5 holds for all $\varphi \in C^b(\hat{K})$. This leads to the equality of the spaces

$$\left\{ f \in L^p(K,m) : \ \varphi \hat{f} \in L^p(K,m)^\wedge \quad \forall \varphi \in C^b(\hat{K}) \right\}$$

$$\text{and} \ \left\{ f \in L^p(K,m) : \ \varphi \hat{f} \in L^p(K,m)^\wedge \quad \forall \varphi \in C_0(\hat{K}) \right\}$$

for $1 \leq p \leq 2$.

We characterized the derived spaces for $1 \leq p \leq 2$ very precisely. For $2 < p < \infty$ we can at least determine a dense linear subset of $L^p(K,m)^0$.

Proposition 6.1.8. *Let $2 \leq p < \infty$ and $1/p + 1/q = 1$. Then $L^q(\mathcal{S},\pi)^\vee \subset L^p(K,m)^0$.*

Proof. Let $f = \check{\varphi}$, $\check{\varphi} \in L^q(\mathcal{S},\pi)^\vee \subset L^p(K,m)$. Then for $h \in L^1(K,m)$ is $\hat{h}\varphi \in L^q(\mathcal{S},\pi)$ with $(\hat{h}\varphi)^\vee = h * f$. Thus, $\|h * f\|_p \leq \left\|\hat{h}\varphi\right\|_q \leq \left\|\hat{h}\right\|_\infty \|\varphi\|_q$. Therefore, $f \in L^p(K,m)^0$ and $\|f\|_0 \leq \|\varphi\|_q$. □

We mentioned earlier that $L^2(\hat{K},\pi)^\vee = L^2(K,m) = L^2(K,m)^0$. For $2 < p$ we do not know whether the spaces $L^q(\mathcal{S},\pi)^\vee$ and $L^p(K,m)^0$ are equal. However, we can prove that $L^q(\mathcal{S},\pi)^\vee$ is a $\| \ \|_0$ −dense subset in $L^p(K,m)^0$, $1/p + 1/q = 1$.

Proposition 6.1.9. *Let $2 \leq p < \infty$. $L^q(\mathcal{S},\pi)^\vee$ is a $\| \ \|_0$-dense linear subspace in $L^p(K,m)^0$, $1/p + 1/q = 1$.*

Proof. For each $f \in L^p(K,m)^0$ there exists $f_n \in L^q(\mathcal{S},\pi)^\vee$, $n \in \mathbb{N}$ such that $\|f - f_n\|_p \to 0$ as n tends to infinity and $f_n \in L^p(K,m)^0$ for all $n \in \mathbb{N}$, see Chapter 2. Hence, $\frac{\|(f-f_n)*h\|_p}{\|\hat{h}\|_\infty} \to 0$ for all $h \in L^1(K,m)$ as n tends to infinity. Thus, $\|f - f_n\|_0 \to 0$ as n tends to infinity. □

6.2 The Derived Spaces for compact Hypergroups

Now we want to get a better idea of how the spaces $L^p(K,m)_0 = L^p(K,m)^0$ appear for different $1 \leq p \leq 2$. We already know that $L^2(K,m) = L^2(K,m)_0 = L^2(K,m)^0$. For a compact hypergroup K we can also show that the two spaces $L^2(K,m)$ and $L^p(K,m)^0$ coincide for $1 \leq p \leq 2$. The corresponding result for a commutative, compact group G states that $L^p(G)_0$ is algebraically and topologically isomorphic to $L^2(G)$ and $2^{-1/2}\|f\|_2 \leq \|f\|_0 \leq \|f\|_2$, see [101, Theorem 1.9.1]. For a locally compact group G we know for all characters $\alpha \in \hat{G}$ that $|\alpha(x)| = 1$ for all $x \in G$. In contrast to the group case, we have for a commutative hypergroup K only $|\alpha(x)| \leq 1$ for $\alpha \in \hat{K}$ and $x \in K$. This is why the next proposition is weaker than the corresponding result for a commutative and compact group G.

Proposition 6.2.1. *Let K be compact and $1 \le p \le 2$. Then $L^2(K,m) = L^p(K,m)^0$ as linear spaces.*

Proof. Since K is compact, we have $L^2(K,m) \subset L^p(K,m) \subset L^1(K,m)$. Given a function $f \in L^2(K,m)$ we obtain for each $h \in L^1(K,m)$

$$\|f * h\|_p \le \|f * h\|_2 = \|\wp(f * h)\|_2 \le \left\|\hat{f}\right\|_2 \left\|\hat{h}\right\|_\infty = \|f\|_2 \left\|\hat{h}\right\|_\infty .$$

Thus, $\|f\|_0 \le \|f\|_2$. Conversely, let $f \in L^p(K,m)^0$. For $1 \le q \le p \le 2$ we have

$$\|f * h\|_q / \left\|\hat{h}\right\|_\infty \le \|f * h\|_p / \left\|\hat{h}\right\|_\infty$$

for all $h \in L^1(K,m)$. Thus, $L^p(K,m)^0 \subseteq L^q(K,m)^0 \subseteq L^1(K,m)^0$. Hence, f is also an element in $L^1(K,m)^0 = L^1(K,m)_0$ and we obtain for each $\varphi \in C_0(\hat{K})$ that $\varphi\hat{f} \in L^1(K,m)^\wedge$. Furthermore, by Lemma 5.3.2 and Proposition 2.1.7 we have $f \in L^2(K,m)$. This leads to $L^2(K,m) = L^p(K,m)^0 = L^p(K,m)_0$. □

Remark 6.2.2. In addition to Chapter 5 we can reformulate Theorem 5.3.3 and Corollary 5.3.4. Lemma 5.3.1 implies that for an infinite and compact hypergroup K, $1 \le p < 2$, the inclusion $L^p(K,m)_0 \subsetneq L^p(K,m)$ is proper. Moreover, there exists $\varphi \in C_0(K)$ such that $\varphi \notin \mathcal{M}(L^p(K,m))$.

We can say even a bit more about the derived spaces of $L^p(K,m)$, $1 < p < \infty$, for a compact hypergroup K. A similar result for compact Abelian groups can be found in [101, Theorem 1.9.1 (ii)].

Theorem 6.2.3. *Let K be compact and $1 < p < \infty$. If $\varphi \in (L^p(K,m)^0)^\wedge$ and ψ is a function on $\hat{K}$ such that $|\psi(\alpha)| / \|\alpha\|_2^2 \le |\varphi(\alpha)|$ for all $\alpha \in \hat{K}$, then $\psi \in (L^p(K,m)^0)^\wedge$.*

Proof. If $f \in L^p(K,m)_0$ and $h \in L^q(K,m)$, $1/p + 1/q = 1$, then $f * h$ has an absolutely convergent Fourier series. To see this, define

$$F(g) := \int_K f * g(x)\tilde{h}(x)dm(x)$$

for $g \in L^p(K,m)$. Clearly, F is linear and since $f \in L^p(K,m)_0$ we have by Hölder's inequality

$$|F(g)| \le \|f * g\|_p \|h\|_q \le \|f\|_0 \|\hat{g}\|_\infty \|h\|_q .$$

Thus, F defines a bounded linear functional on $L^p(K,m)^\wedge$ and hence on all of $C_0(\hat{K})$, since $L^p(K,m)^\wedge$ is dense in $C_0(\hat{K})$, see [4, 2.2.4]. Consequently, there exists a unique bounded regular Borel measure $\mu \in M(\hat{K})$, such that

$$\int_K f * g(x)h(\tilde{x})dm(x) = \sum_{\alpha \in \hat{K}} \mu(\{\alpha\})\hat{g}(\alpha),$$

for all $g \in L^p(K,m)$. Moreover, since K is compact, $\hat{K}$ defines an orthogonal basis on $L^2(\hat{K},\pi)$ and we obtain $\hat{\alpha}(\beta) = \varepsilon_\alpha(\beta)\|\alpha\|_2^2$ for all $\alpha, \beta \in \hat{K}$. In particular, for all $\alpha \in \hat{K} \subset L^p(K,m) \cap L^2(K,m)$

$$\begin{aligned} \|\alpha\|_2^2 \mu(\{\alpha\}) &= \hat{\alpha}(\alpha)\mu(\{\alpha\}) = \int_K f * \alpha(x)h(\tilde{x})dm(x) = \int_K \int_K L_{\tilde{y}}\alpha(x)f(y)dm(y)h(\tilde{x})dm(x) \\ &= \int_K \int_K \overline{\alpha(y)} f(y)dm(y)\alpha(x)h(\tilde{x})dm(x) = \hat{f}\hat{h}(\alpha) = (f * h)^\wedge(\alpha). \end{aligned}$$

Since μ is bounded and $\|\alpha\|_2^2 \le \|\alpha\|_\infty^2 = 1$ for all $\alpha \in \hat{K}$, we have $\|(f * h)^\wedge\|_1 \le \|\mu\| < \infty$ and therefore, $f * h$ has an absolutely convergent Fourier series.

Now suppose $f \in L^p(K,m)^0$, $\hat{f} = \varphi$ and ψ is a function on $\hat{K}$ such that $|\psi(\alpha)| / \|\alpha\|_2^2 \leq |\varphi(\alpha)|$ for all $\alpha \in \hat{K}$. In light of the previous remarks it is apparent that for each $h \in L^q(K,m)$ we have

$$\sum_{\alpha \in \hat{K}} |\varphi(\alpha)\hat{h}(\alpha)| < \infty$$

and hence

$$\sum_{\alpha \in \hat{K}} |\psi(\alpha)\hat{h}(\alpha)| / \|\alpha\|_2^2 < \infty.$$

The convergence of the sum forces that only countable many summands differ from 0. Hence, $\psi(\alpha)$ vanishes except for countable many $\alpha \in \hat{K}$, say except for $\alpha_1, \alpha_2, \ldots$. For each positive integer $n \in \mathbb{N}$ define

$$g_n(x) := \sum_{i=1}^{n} \psi(\alpha_i)\alpha_i(x) / \|\alpha_i\|_2^2 .$$

Then we have for each $h \in L^q(K,m)$

$$\int_K g_n(x)h(\tilde{x})dm(x) = \sum_{i=1}^{n} \psi(\alpha_i) / \|\alpha_i\|_2^2 \int_K \overline{\alpha_i(x)}h(x)dm(x) = \sum_{i=1}^{n} \psi(\alpha_i)\hat{h}(\alpha_i) / \|\alpha_i\|_2^2 < \infty.$$

It follows at once that $(g_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $L^p(K,m)$. Since $L^p(K,m)$ is weakly complete, see [101, D.10] there exists a function $g \in L^p(K,m)$ such that

$$\lim_{n\to\infty} \int_K g_n(x)h(\tilde{x})dm(x) = \int_K g(x)h(\tilde{x})dm(x),$$

for all $h \in L^q(K,m)$. Further,

$$\hat{g}(\alpha) = \lim_{n\to\infty} \int_K g_n(x)\overline{\alpha(x)}dm(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \psi(\alpha_i)\hat{\alpha}(\alpha_i) / \|\alpha_i\|_2^2 = \psi(\alpha)$$

for all $\alpha \in \hat{K}$. Finally, since $f \in L^p(K,m)^0$, we have for every $d \in L^p(K,m) \subset L^1(K,m)$

$$\|f * d\|_p \leq \|f\|_0 \left\|\hat{d}\right\|_\infty .$$

Thus, if $d(x) = \sum_{i=1}^n \hat{d}(\alpha_i)\alpha_i(x)$ is a polynomial in $L^p(K,m)$ we obtain

$$\begin{aligned} \|g * d\|_p &= \left\|\sum_{i=1}^{n} \hat{d}(\alpha_i)\hat{g}(\alpha_i)\alpha_i\right\|_p = \left\|\sum_{i=1}^{n} \hat{d}(\alpha_i)\psi(\alpha_i)\alpha_i\right\|_p = \left\|f * \sum_{i=1}^{n} \hat{d}(\alpha_i)\psi(\alpha_i)[\varphi(\alpha_i)]^{-1}\alpha_i\right\|_p \\ &\leq \|f\|_0 \left\|(\sum_{i=1}^{n} \hat{d}(\alpha_i)\psi(\alpha_i)[\varphi(\alpha_i)]^{-1}\alpha_i)^\wedge\right\|_\infty \leq \|f\|_0 \left\|\hat{d}\psi\varphi^{-1}\right\|_\infty \leq \|f\|_0 \left\|\hat{d}\right\|_\infty \end{aligned}$$

since $|\psi(\alpha)| \leq |\varphi(\alpha)| \|\alpha\|_2^2 \leq |\varphi(\alpha)|$. Since the trigonometric polynomials are norm dense in $L^p(K,m)$ we can conclude that $\|g * d\|_p \leq \|f\|_0 \left\|\hat{d}\right\|_\infty$ for all $d \in L^p(K,m)$. Thus, $\|g\|_0 < \infty$ and therefore is $g \in L^p(K,m)^0$. Hence, $\psi = \hat{g} \in (L^p(K,m)^0)^\wedge$. □

6.3 The Derived Spaces for non-compact Hypergroups

Now we want to characterize the derived spaces for a non-compact hypergroup K. Therefore, we need again the existence of a proper set of uniqueness, see Chapter 5. Hence, in contrast to the group case, this restricts our results for non-compact hypergroups. For K non-compact we will show that under some circumstances $L^p(K,m)^0$ equals $\{0\}$.

Theorem 6.3.1. *Let K be non-compact and $1 \leq p < 2$. If γ is a function on $\hat{K}$ such that $\varphi\gamma \in L^p(K,m)^\wedge$ for all $\varphi \in C_0(\hat{K})$ then $\gamma = 0$ almost everywhere on each compact set without an isolated point.*

Proof. If $\mathcal{S}$ does not contain a set of uniqueness, then $\mathcal{S}$ does not contain a compact set F without an isolated point such that $\pi(F) > 0$. Hence, the theorem is obvious.
Thus, we suppose that $\mathcal{S}$ contains a proper set of uniqueness. Further suppose γ does not vanish almost everywhere on each compact set without an isolated point. Then there exists a compact set $C \subset \hat{K}$ without an isolated point, such that $\pi(C) > 0$ and γ does not vanish almost everywhere on C. For $\psi \in C_c(\hat{K})$ such that $\psi = 1$ on C, $\psi\gamma$ does not vanish almost everywhere on C. Moreover, since $\varphi\psi \in C_0(\hat{K})$ for all $\varphi \in C_0(\hat{K})$, it follows by assumption that $\varphi\psi\gamma \in L^p(K,m)^\wedge$ for all $\varphi \in C_0(\hat{K})$.
Therefore, we may assume without loss of generality that γ vanishes outside of some compact set F without an isolated point and with $\pi(F) > 0$.
For $p = 1$ we have $\varphi\gamma \in L^1(K,m)^\wedge$ for all $\varphi \in C_0(\hat{K})$. Thus, for all $\varphi \in C_0(\hat{K})$ there exists a unique $g_\varphi \in L^1(K,m)$ such that $\hat{g}_\varphi = \varphi\gamma$. Since γ vanishes outside of F, we conclude that $\varphi\gamma \in C_c(\hat{K}) \subset L^2(\mathcal{S},\pi)$. We obtain $g_\varphi \in L^1(K,m) \cap L^2(K,m)$ and hence $g_\varphi \in L^p(K,m)$, $1 \leq p \leq 2$. Consequently $\varphi\gamma \in L^p(K,m)^\wedge$, $1 \leq p \leq 2$, for all $\varphi \in C_0(\hat{K})$. Thus, we may further assume without loss of generality, that $\varphi\gamma \in L^p(K,m)^\wedge$ for all $\varphi \in C_0(\hat{K})$ for some fixed p, $1 < p < 2$.
Choosing a $\varphi \in C_0(\hat{K})$ which is identically one on F, leads to $\gamma \in L^p(K,m)^\wedge \subset L^q(\mathcal{S},\pi)$, $\frac{1}{p} + \frac{1}{q} = 1$. Thus, the estimate

$$\begin{aligned}\int_{\hat{K}} |\gamma(\alpha)|^2 \, d\pi(\alpha) &= \int_{\hat{K}} \chi_F(\alpha) \, |\gamma(\alpha)|^2 \, d\pi(\alpha) = \|\chi_F\gamma\|_2^2 \\ &\leq \left(\int_{\hat{K}} |\gamma(\alpha)|^q \, d\pi(\alpha)\right)^{2/q} \left(\int_{\hat{K}} \chi_F(\alpha) d\pi(\alpha)\right)^{1-2/q} < \infty\end{aligned}$$

shows that $\gamma \in L^2(\mathcal{S},\pi)$. Now for each $\varphi \in C_0(\hat{K})$ let $A\varphi$ denote the unique element of $L^p(K,m)$ such that $(A\varphi)^\wedge = \varphi\gamma$. Since γ is an element in $L^2(\mathcal{S},\pi)$ with compact support, we have $(A\varphi)^\wedge \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ and it is apparent that $A\varphi = \wp^{-1}(\varphi\gamma) \in L^2(K,m)$. It needs to be noted that by the construction of the Hausdorff-Young transform, it is apparent that the Hausdorff-Young transform coincides with the Fourier transform or the Plancherel Isomorphism on $L^p(K,m) \cap L^1(K,m)$ and $L^p(K,m) \cap L^2(K,m)$, respectively.
In this way we define a linear mapping A from $C_0(\hat{K})$ to $L^p(K,m)$. Moreover, by the Closed Graph Theorem using standard argumentation, we deduce that A is continuous. Thus, there exists a constant $B > 0$ such that $\|A\varphi\|_p \leq B \|\varphi\|_\infty$ for all $\varphi \in C_0(\hat{K})$.
Now let $E \subset \hat{K}$ be any compact set. We wish to show that $(\chi_E\gamma)^\vee \in L^p(K,m)$. Therefore, let the sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(\hat{K})$ be such that $0 \leq \varphi_n(\alpha) \leq 1$ for all $\alpha \in \hat{K}$ and $\varphi_n(\alpha) = 1$ for $\alpha \in E$. Further, let E_n be the support of φ_n, $E_{n+1} \subset E_n$ and $\bigcap_{n=1}^\infty E_n = E$. Clearly $(\varphi_n)_{n\in N}$ converges pointwise to χ_E. Furthermore, $\|A\varphi_n\|_p \leq B \|\varphi_n\|_\infty \leq B$ shows that $(A\varphi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(K,m)$. Hence, the sequence $(A\varphi_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(A\varphi_k)_{k\in\mathbb{N}_0}$. Let $g \in L^p(K,m)$ be such that

$$\lim_k \int_K A\varphi_k(x) h(\tilde{x}) dm(x) = \int_K g(x) h(\tilde{x}) dm(x)$$

for every $h \in L^q(K,m)$, $\frac{1}{p} + \frac{1}{q} = 1$. Further, the sequence $(\varphi_k\gamma)_{k\in\mathbb{N}_0}$ clearly converges pointwise to $\chi_E\gamma$ and $|\varphi_k f(\alpha)| \leq |f(\alpha)|$ for all $\alpha \in \hat{K}$.
Since $(A\varphi_k)_{k\in\mathbb{N}_0}$ is in $L^p(K,m) \cap L^2(K,m)$, we have by Parseval's formula and Lebesque's

theorem of dominated convergence for all $h \in C_c(K) \subset L^1(K,m) \cap L^2(K,m)$ that

$$\int_K g(x)h(\tilde{x})dm(x) = \lim_k \int_K A\varphi_k(x)h(\tilde{x})dm(x) = \lim_k \int_{\hat{K}} (A\varphi_k)^\wedge(\alpha)\hat{h}(\alpha)d\pi(\alpha)$$
$$= \lim_k \int_{\hat{K}} \varphi_k\gamma(\alpha)\hat{h}(\alpha)d\pi(\alpha) = \int_{\hat{K}} \chi_E\gamma(\alpha)\hat{h}(\alpha)d\pi(\alpha) = \int_K (\chi_E\gamma)^\vee(x)h(\tilde{x})dm(x).$$

The applications of Parseval's formula are justified since $\chi_E\gamma \in L^2(\mathcal{S},\pi) \cap L^1(\mathcal{S},\pi)$, see [4, Lemma 2.2.20], and by the Hausdorff-Young Theorem. Since $C_c(K)$ is norm dense in $L^q(K,m)$ we conclude that $(\chi_E\gamma)^\vee = g$ almost everywhere. Therefore, $(\chi_E\gamma)^\vee \in L^p(K,m)$.
There exists a measurable subset $E \subset F \subset \hat{K}$, $\pi(E) > 0$, such that γ does not vanish almost everywhere on E and E is a set of uniqueness for $L^p(K,m)$, see Chapter 5. Since the Plancherel measure is regular, see [86], and a measurable subset of a set of uniqueness is again a set of uniqueness, it is apparent that we may assume E is compact. Then $(\chi_E\gamma)^\vee \in L^p(K,m)$ while $\chi_E\gamma$ does not vanish almost everywhere. This is a contradiction to the fact that E is a set of uniqueness for $L^p(K,m)$. □

Corollary 6.3.2. *Let K be non-compact and $1 \le p < 2$. If $\mathcal{S}$ does not have an isolated point, then $L^p(K,m)^0 = L^p(K,m)_0 = \{0\}$.*

Corollary 6.3.3. *Let K be non-compact and $1 < p < \infty$, $p \neq 2$. Suppose $\varphi \in \mathcal{M}(L^p(K,m))$ has the property that if $\psi \in C_0(\hat{K})$ and $|\psi(\alpha)| \le |\varphi(\alpha)|$ for almost all $\alpha \in \hat{K}$ then ψ is also in $\mathcal{M}(L^p(K,m))$. Then $\varphi = 0$ almost everywhere on each compact set without an isolated point.*

Proof. Assume $1 < p < 2$, since $\mathcal{M}(L^p(K,m)) \cong \mathcal{M}(L^q(K,m))$, if $\frac{1}{p} + \frac{1}{q} = 1$.
By Chapter 3 we see that $\varphi\hat{f} \in L^p(K,m)^\wedge$ for each $f \in L^p(K,m)$. If $\psi \in C_0(\hat{K})$, for $\psi \neq 0$ holds $|\psi\varphi(\alpha)| / \|\psi\|_\infty \le |\varphi(\alpha)|$ for almost all $\alpha \in \hat{K}$ and by assumption $\psi\varphi/\|\psi\|_\infty \in \mathcal{M}(L^p(K,m))$. Therefore, $\psi\varphi \in \mathcal{M}(L^p(K,m))$ for all $\psi \in C_0(\hat{K})$. Then we have in particular $\psi\varphi\hat{f} \in L^p(K,m)^\wedge$ for all $\psi \in C_0(\hat{K})$ and $f \in L^p(K,m)$. By Theorem 6.3.1 it follows $\varphi\hat{f} = 0$ almost everywhere on each compact set without an isolated point for all $f \in L^p(K,m)$. For every compact set F without an isolated point we can choose $f \in L^p(K,m)$ such that $\hat{f}|F \neq 0$. Indeed, we know for an approximate identity $(k_i)_{i\in I}$ that $(k_i * k_i)^\wedge = \hat{k}_i\hat{k}_i$ converges to 1 uniformly on each compact subset. Choosing the function $f = k_i * k_i \in L^p(K,m)$ for an appropriate $i \in I$, we obtain $\hat{f}\chi_F \neq 0$.
Thus, $\varphi = 0$ almost everywhere on each compact set without an isolated point. □

Remark 6.3.4. By Theorem 6.3.1 it follows in particular for each non-compact hypergroup K such that $\mathcal{S}$ contains a proper set of uniqueness, that the inclusion $L^p(K,m)_0 \subsetneq L^p(K,m)$ is proper. Indeed, $\mathcal{S}$ contains a compact subset F, $\pi(F) > 0$, without an isolated point. Hence, for every $f \in L^p(K,m)_0$ it follows by Theorem 6.3.1 that $\hat{f}|F = 0$ almost everywhere. However, there exists a function $f \in L^p(K,m)$ such that $\hat{f}\chi_F \neq 0$.

6.4 Applications to inclusion Results

Using Theorem 6.3.1 we derive some inclusion results concerning different multiplier spaces.

Corollary 6.4.1. *Let K be non-compact and $1 \le p < 2$. If $\mathcal{S}$ contains a proper set of uniqueness, then $C_0(\hat{K}) \cap \mathcal{M}(L^p(K,m))$ is a proper subset of $C_0(\hat{K})$.*
Moreover, $\mathcal{M}(L^p(K,m))$ is a proper subset of $\mathcal{M}(L^2(K,m))$.

Following the lines of Theorem 5.3.6, we obtain

Theorem 6.4.2. *Let K be non-compact. If $\mathcal{S}$ contains a proper set of uniqueness, then*

$$\bigcup_{1\le q<p} M(L^q(K,m)) \subsetneq M(L^p(K,m)) \subsetneq \bigcap_{p<q\le 2} M(L^q(K,m)),$$

if $1 < p < 2$. For $p = 1$ the second inclusion remains strict and for $p = 2$ the first inclusion remains strict.

Using Theorem 6.3.1 we obtain also a result concerning the inclusion results of multipliers for the dual space of a polynomial hypergroup $K = \mathbb{N}_0$.

Theorem 6.4.3. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup which fulfills the continuity property (P). If $\mathcal{S}$ contains a proper set of uniqueness, then $M(\widehat{\mathbb{N}}_0)^\vee$ is a proper subset of $\mathcal{M}(L^p(\mathcal{S},\pi))$ for all $1 < p < \infty$.*

Proof. Since $K = \mathbb{N}_0$ satisfies the continuity property (P), we have proven in Chapter 3 that $M(\widehat{\mathbb{N}}_0)^\vee \subset \mathcal{M}(L^p(\mathcal{S},\pi))$ for all $1 < p < \infty$. For $p = 2$ we have further $\mathcal{M}(L^2(\mathcal{S},\pi)) = L^\infty(\mathbb{N}_0)$ and the inclusion is obviously proper. Furthermore, for $p \neq 2$ we can assume $1 < p < 2$ by Proposition 4.3.5. Let $r = \frac{2p}{3p-2}$ then $1 < r < 2$ and $p = \frac{2r}{3r-2}$ and by Theorem 4.3.10 we obtain

$$l^r(\mathbb{N}_0,h) * l^s(\mathbb{N}_0,h) \subset \mathcal{M}(L^r(\mathcal{S},\pi)),$$

$1/r + 1/s = 1$. To prove that the inclusion $M(\widehat{\mathbb{N}}_0)^\vee \subset \mathcal{M}(L^p(\mathcal{S},\pi))$ is proper, it is sufficient to show that $l^r(\mathbb{N}_0,h) * l^s(\mathbb{N}_0,h) \not\subset M(\widehat{\mathbb{N}}_0)^\vee$.
Now assume $l^r(\mathbb{N}_0,h) * l^s(\mathbb{N}_0,h) \subset M(\widehat{\mathbb{N}}_0)^\vee$. We define a complete norm on $M(\widehat{\mathbb{N}}_0)^\vee$ by $\|\check{\mu}\| = \|\mu\|$. This is indeed a norm by the uniqueness theorem of the inverse Fourier-Stieltjes transform. For each $g \in l^r(\mathbb{N}_0,h)$ define a linear transformation

$$T_g : \ l^s(\mathbb{N}_0,h) \to M(\widehat{\mathbb{N}}_0) \text{ by } T_g(f) := g * f.$$

To prove that T_g is continuous, choose a sequence $(f_n)_{n\in\mathbb{N}_0}$ in $l^s(\mathbb{N}_0,h)$ and $f \in l^s(\mathbb{N}_0,h)$ such that $\|f_n - f\|_s \to 0$ and $\|T_g f_n - \check{\mu}\| = 0$ as n tends to infinity. Then

$$\begin{aligned} \|T_g f - \check{\mu}\|_\infty &\le \|T_g f - T_g f_n\|_\infty + \|T_g f_n - \check{\mu}\|_\infty \\ &\le \|g * f - g * f_n\|_\infty + \|T_g f_n - \check{\mu}\| \le \|g\|_r \|f - f_n\|_s + \|T_g f_n - \check{\mu}\| \to 0 \end{aligned}$$

as n tends to infinity. Thus, $T_g f = \check{\mu}$ and by the Closed Graph Theorem T_g is continuous. Hence, T_g is bounded and for each $g \in L^r(\mathbb{N}_0,h)$ there exists a constant $M(g) \ge 0$ such that

$$\|T_g f\| = \|f * g\| \le M(g)\|f\|_s.$$

If we choose $f, g \in C_c(\mathbb{N}_0)$, we conclude $\hat{g}\hat{f} \in L^1(\mathcal{S},\pi)$ and $g * f = (\hat{g}\hat{f})^\vee$. Since, we can embed $L^1(\mathcal{S},\pi)$ into $M(\widehat{\mathbb{N}}_0)$ we obtain by the assumptions above

$$\|\hat{g}\hat{f}\|_1 = \|(\hat{g}\hat{f})^\vee\| = \|g * f\| \le M(g)\|f\|_s.$$

For $\varphi \in C_0(\widehat{\mathbb{N}}_0)$, $\|\varphi\|_\infty < 1$, we conclude therefore

$$\|\varphi\hat{g}\hat{f}\|_1 \le \|\varphi\|_\infty \|(\hat{g}\hat{f})^\vee\| \le M(g)\|f\|_s.$$

By Chapter 2 it follows $(\varphi\hat{g}\hat{f})^\vee = (\varphi\hat{g})^\vee * f \in C_0(\mathbb{N}_0)$ for all $f \in C_c(\mathbb{N}_0)$. Hence, we have

$$|\sum_{k=0}^{\infty} (\varphi\hat{g})^\vee(k) f(k) h(k)| \ = \ |(\varphi\hat{g})^\vee * f(0)| \le \|(\varphi\hat{g})^\vee * f\|_\infty \le \|\varphi\hat{g}\hat{f}\|_1 \le M(g)\|f\|_s$$

for all $f \in C_c(\mathbb{N}_0)$. Since $C_c(\mathbb{N}_0)$ is dense in $l^s(\mathbb{N}_0,h)$, $(\varphi\hat{g})^\vee$ defines a continuous linear functional on $l^s(\mathbb{N}_0,h)$ and we can conclude $(\varphi\hat{g})^\vee \in l^r(\mathbb{N}_0,h)$. This leads to $\varphi\hat{g} \in l^r(\mathbb{N}_0,h)^\wedge$ for all $\varphi \in C_0(\widehat{\mathbb{N}}_0)$. By Theorem 6.3.1 it follows $\hat{g} = 0$ π-almost everywhere on each compact set without an isolated point. Since $\mathcal{S}$ contains a proper set of uniqueness, there exists a compact set $F \subset \mathcal{S}$ without an isolated point and $\pi(F) > 0$. Hence, $\hat{g}|F = 0$. This contradicts the fact that g was chosen arbitrarily in $C_c(\mathbb{N}_0)$. Therefore, $l^r(\mathbb{N}_0,h) * l^s(\mathbb{N}_0,h)$ is not contained in $M(\widehat{\mathbb{N}}_0)^\vee$ and $M(\widehat{\mathbb{N}}_0)^\vee$ is properly contained in $\mathcal{M}(l^p(\mathbb{N}_0,h))$. □

Corollary 6.4.4. *Let $K = \mathbb{N}_0$ be a polynomial hypergroup which fulfills the continuity property (P). If $\mathcal{S}$ contains a proper set of uniqueness, then $\mathcal{M}(L^1(\mathcal{S},\pi))$ is a proper subset of $\mathcal{M}(L^p(\mathcal{S},\pi))$ for all $1 < p < \infty$.*

Chapter 7

Multipliers for homogeneous Banach Spaces

In the context of Fourier analysis homogeneous Banach spaces on the unit circle are of great interest, see [95], [105] and [164]. For instance, in homogeneous Banach spaces we can apply all the classical approximation procedures on functions on the unit circle and their Fourier expansions.
Dales and Pandey [16] have studied the class S_p of Segal algebras and proved weakly amenability. Using this results, Ghahramani and Lau [59, 60] characterized further for various classes of Segal algebras derivations and multipliers from a Segal algebra into itself and into its dual module. Furthermore, multipliers of Segal algebras on locally compact groups are also investigated in [8],[28], [64], [65], [99], [155, 156] and [158]. Even multipliers into general homogeneous Banach spaces on groups are investigated by Feichtinger [37].
Segal algebras are a specific type of homogeneous Banach spaces. They are often even defined equivalent to homogeneous Banach spaces.
Homogeneous Banach spaces determined by the Jacobi translation operator are introduces by G. Fischer and R. Lasser in [46]. They give a lot of examples of homogeneous Banach spaces and Banach algebras consisting of functions on $\mathcal{S} = [-1, 1]$. These spaces are determined by the Jacobi translation operator, which is generated by the Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$.
Our aim here is to characterize multipliers for homogeneous Banach spaces in the hypergroup setting. Moreover, we will give several examples of homogeneous Banach spaces, e.g. the Wiener algebra, the Beurling space or the Sobolev space and study their multiplier properties individually. The investigation of multipliers for the Wiener algebra was started in the author's Master thesis [19], but is here continued to a vast part.

7.1 Homogeneous Banach spaces determined by the Jacobi translation operator

The Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}}$, $\alpha, \beta > -1$, are orthogonal with respect to $\pi^{(\alpha,\beta)}$, where supp $\pi^{(\alpha,\beta)} = [-1, 1] = \mathcal{S}$. For the sake of simplicity we fix the parameters $(\alpha, \beta) \in J$ and omit those from now on at all the notations in this chapter.

Definition 7.1.1. We call a linear subspace B of $L^1(\mathcal{S}, \pi)$ a **homogeneous Banach space on $\mathcal{S}$ with respect to** (α, β), if it is endowed with a norm $\| \, \|_B$ such that

(B1) $R_n \in B$ for all $n \in \mathbb{N}_0$

(B2) B is complete with respect to $\| \, \|_B$ and $\| \, \|_1 \le \| \, \|_B$

(B3) For every $f \in B$, $x \in \mathcal{S}$ we have $L_x f \in B$ and $\|L_x f\|_B \leq \|f\|_B$

(B4) For every $f \in B$ the map $x \mapsto L_x f$, $S \to B$ is continuous.

A homogeneous Banach space is called character-invariant, if

(B5) For every $f \in B$, $n \in \mathbb{N}_0$ we have $R_n \cdot f \in B$ and $\|R_n \cdot f\|_B \leq \|f\|_B$.

Every homogeneous Banach space B on $\mathcal{S}$ with respect to (α, β) is in fact a $L^1(\mathcal{S}, \pi)$-module, since for each $g \in B$ and $f \in L^1(\mathcal{S}, \pi)$ we have $f * g \in B$ and $\|f * g\|_B \leq \|f\|_1 \|g\|_B$. Furthermore, B is a Banach algebra with convolution as multiplication, [46].

Some obvious examples for a homogeneous Banach space B on $\mathcal{S}$ with respect to (α, β) are $B = L^p(\mathcal{S}, \pi)$, $1 \leq p < \infty$, with norm $\| \;\|_p$ and $B = C(\mathcal{S})$ with norm $\| \;\|_\infty$.
We are now in the position to examine multipliers for certain homogeneous Banach spaces on $\mathcal{S}$ with respect to (α, β).

7.2 Multipliers for homogeneous Banach spaces

Let B be a homogeneous Banach space on $\mathcal{S}$ with respect to (α, β).

Definition 7.2.1. We call a bounded linear operator T on B **multiplier**, if and only if T commutes with the Jacobi translation operator L_y for all $y \in \mathcal{S}$, e.g. $T \circ L_y = L_y \circ T$. We denote by $M(B)$ the set of all multipliers for B.

Before we take a look on multipliers for some specific homogeneous Banach spaces on $\mathcal{S}$, we will first characterize multipliers for a general homogeneous Banach space B on $\mathcal{S}$. Later on, we will examine some examples for homogeneous Banach spaces on $\mathcal{S}$ and their corresponding multiplier spaces.

Theorem 7.2.2. *A bounded linear operator T on B is a multiplier for B, $T \in M(B)$, if and only if*

$$T(f * g) = f * Tg$$

for all $f \in L^1(\mathcal{S}, \pi)$ and $g \in B$. Moreover, we have $Tf * g = T(f * g) = f * Tg$ for all $f, g \in B$.

Proof. Following the lines of proof 3.1.2. □

The next Theorem shows that each multiplier T for B is uniquely defined by its values on R_n for $n \in \mathbb{N}_0$.

Theorem 7.2.3. *A bounded linear operator T on B is a multiplier for B, $T \in M(B)$, if and only if there exists a unique function $\varphi \in l^\infty(\mathbb{N}_0)$ such that*

$$(Tf)^\vee = \varphi \check{f}$$

for all $f \in B$. Moreover, we have $\varphi(n) = (TR_n)^\vee(n)h(n)$ for all $n \in \mathbb{N}_0$ and $\|\varphi\|_\infty \leq \|T\|$.

Proof. By $(B1)$ is $R_n \in B$ for all $n \in \mathbb{N}_0$ and we have $\check{R}_n(m) = \delta_{m,n} h(n)^{-1}$, where $\delta_{m,n}$ denotes the Kroneker symbol. Thus, for every $n \in \mathbb{N}_0$ there exists $R_n \in B$ with $\check{R}_n(n) = h(n)^{-1} \neq 0$. Further, for $n \in \mathbb{N}_0$ and $f, g \in B$ such that $\check{f}(n) \neq 0$ and $\check{g}(n) \neq 0$ we have by Theorem 7.2.2

$$(Tf)^\vee(n)/\check{f}(n) = (Tg)^\vee(n)/\check{g}(n).$$

This equation shows that the definition

$$\varphi(n) := (Tf)^\vee(n)/\check{f}(n) = (TR_n)^\vee(n)h(n)$$

is independent on the choice of $f \in B$. Hence, $\varphi(n)$ is well-defined on $\mathbb{N}_0$. If $\check{f}(n) \neq 0$ and $\check{g} = 0$ then $(Tg)^\vee(n)\check{f}(n) = (Tf)^\vee(n)\check{g}(n) = 0$. Hence, the equation $(Tg)^\vee(n) = \varphi(n)\check{g}(n)$ is valid for all $n \in \mathbb{N}_0$ and $g \in B$.
If $\psi \in l^\infty(\mathbb{N}_0)$ is a second function with $(Tg)^\vee(n) = \varphi(n)\check{g}(n)$ for all $n \in \mathbb{N}_0$ and $g \in B$, then we obtain $(\psi(n) - \varphi(n))\check{g}(n) = 0$ for all $n \in \mathbb{N}_0$ and all $g \in B$. This implies $\varphi = \psi$.
To prove that φ is bounded, we define

$$K_n := \sup_{f\in B} \left\{ \left|\check{f}(n)\right| : \|f\|_B = 1 \right\}.$$

By $\left\|\check{f}\right\|_\infty \leq \|f\|_1 \leq \|f\|_B = 1$ for all $f \in B \subset L^1(\mathcal{S}, \pi)$, we have $0 < K_n \leq 1$. Further holds $|\check{g}(n)| \leq K_n \|g\|_B$. Moreover,

$$|\varphi(n)\check{g}(n)| = |(Tg)^\vee(n)| \leq K_n \|Tg\|_B \leq K_n \|T\| \|g\|_B$$

for all $g \in B$. By choosing only those $g \in B$ with $\|g\|_B = 1$ and $\check{g}(n) \neq 0$ we have

$$|\varphi(n)| \leq K_n \|T\| \inf \left\{ \frac{1}{|\check{g}(n)|} : \|g\|_B = 1 \text{ and } \check{g}(n) \neq 0 \right\} = \|T\|,$$

where the second equality holds, since the $g \in B$ with $\|g\|_B = 1$ and $\check{g}(n) = 0$ do not contribute to the value of K_n. Hence, φ is bounded by $\|\varphi\|_\infty \leq \|T\|$.
Conversely, let T be a bounded linear operator on B such that T is defined by $(Tf)^\vee = \varphi\check{f}$ for a given $\varphi \in l^\infty(\mathbb{N}_0)$ and all $f \in B$. Then

$$\begin{aligned} (T \circ L_x f)^\vee(n) &= \varphi(n)(L_x f)^\vee(n) = \varphi(n)R_n(x)\check{f}(n) \\ &= R_n(x)\varphi(n)\check{f}(n) = (L_x \circ Tf)^\vee(n) \end{aligned}$$

for all $n \in \mathbb{N}_0$ and $f \in B$. By the uniqueness of the Jacobi transform it follows that $T \circ L_x = L_x \circ T$. □

The next Theorem depends on whether the homogeneous Banach space B is a subset in $L^2(\mathcal{S}, \pi)$ or not. If $B \subset L^2(\mathcal{S}, \pi)$, we have $\check{f} \in l^2(\mathbb{N}_0, h)$ for all $f \in B$ and we can show the existence of a pseudomeasure $\sigma \in P(\mathcal{S})$, such that $Tf = \sigma * f$ belongs to $L^2(\mathcal{S}, \pi)$ for all $f \in B$. For the theory of pseudomeasures we refer to Chapter 3.

Theorem 7.2.4. *Let $B \subset L^2(\mathcal{S}, \pi)$. For a bounded linear operator T on B the following conditions are equivalent:*

i) $T \in M(B)$, i.e. $T \circ L_x = L_x \circ T$ for all $x \in \mathcal{S}$.

ii) There exists a unique pseudomeasure $\sigma \in P(\mathcal{S})$, such that $Tf = \sigma * f$ for all $f \in B$.

Moreover, there exists a continuous algebra isomorphism on $M(B)$ into $P(\mathcal{S})$ with $\|\sigma\|_P \leq \|T\|$.

Proof. Let $T \in M(B)$. By Theorem 7.2.3 there exists a unique $\varphi \in l^\infty(\mathbb{N}_0)$ such that $Tf = \wp(\varphi\check{f})$ for all $f \in B$. Furthermore, by the assumptions in Chapter 3 (see also [20]) there exists an isometric isomorphism $\Phi : P(\mathcal{S}) \to l^\infty(\mathbb{N}_0)$. Set $\sigma := \Phi^{-1}(\varphi)$. Moreover, we know that for $f \in B \subset L^1(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)$, $\sigma * f$ exists as a pseudomeasure and we obtain by definition

$$\Phi(\sigma * f) = \Phi(\sigma)\check{f} = \varphi\check{f} \in l^2(\mathbb{N}_0, h) \cap l^\infty(\mathbb{N}_0).$$

Hence, $\Phi^{-1}(\varphi\check{f}) = \sigma * f$ belongs to $L^2(\mathcal{S}, \pi)$ and

$$\sigma * f = \Phi^{-1}(\varphi\check{f}) = \wp(\varphi\check{f}) = Tf.$$

Since Φ is isometric, we have by Theorem 7.2.3 $\|\sigma\|_P = \|\varphi\|_\infty \leq \|T\|$.
Conversely is every bounded linear operator T on B, which is defined by $Tf = \sigma * f$ for $\sigma \in P(\mathcal{S})$, a multiplier for B. Indeed, $Tf = \sigma * f = \Phi^{-1}(\Phi(\sigma)\check{f}) = \wp(\varphi\check{f})$ for $\varphi = \Phi(\sigma) \in l^\infty(\mathbb{N}_0)$. The rest follows by Theorem 7.2.3. □

If $B \not\subset L^2(\mathcal{S},\pi)$ we have to take compromises. We are only able to prove a much weaker result for $T \in M(B)$.

Theorem 7.2.5. *Let $B \not\subset L^2(\mathcal{S},\pi)$. If a bounded linear operator T on B is a multiplier for B, i.e. $T \circ L_x = L_x \circ T$ for all $x \in \mathcal{S}$, then there exists a unique pseudomeasure $\sigma \in P(\mathcal{S})$, such that $Tf = \sigma * f$ for all $f \in B \cap L^2(\mathcal{S},\pi)$.*
Moreover, there exists a continuous algebra isomorphism on $M(B)$ into $P(\mathcal{S})$ with $\|\sigma\|_P \leq \|T\|$.

Proof. The proof follows the lines of the first part of proof 7.2.4. □

7.2.1 Multipliers for the Wiener algebra $W(\mathcal{S})$

An interesting example of homogeneous Banach spaces determined by the Jacobi translation operator is the Wiener algebra. Denote by

$$W(\mathcal{S}) := \left\{ f \in C(\mathcal{S}) : \check{f} \in l^1(\mathbb{N}_0, h) \right\}.$$

The **Wiener algebra**, $W(\mathcal{S})$, is with the norm $\|f\|_W := \|\check{f}\|_1$, $f \in W(\mathcal{S})$, a homogeneous Banach space on $\mathcal{S}$, see [46] (We introduced $W(\mathcal{S})$ already in Chapter 3). Furthermore, $W(\mathcal{S})$ is a Banach algebra with respect to the convolution and with respect to the pointwise multiplication of functions.

Theorem 7.2.6. *For an operator $T \in B(W(\mathcal{S}))$, the following conditions are equivalent:*

i) $T \in M(W(\mathcal{S}))$, i.e. $T \circ L_x = L_x \circ T$ for all $x \in \mathcal{S}$.

ii) For all $f, g \in W(\mathcal{S})$ we have $Tf * g = T(f * g) = f * Tg$.

iii) There exists a unique bounded function φ on $\mathbb{N}_0$, such that $(Tf)^\vee = \varphi \check{f}$ for all $f \in W(\mathcal{S})$.

Moreover $\|\varphi\|_\infty = \|T\|$.

We will introduce a different possibility of a proof and show the implication *i)* to *ii)* again, even though we have already proven this equivalency in the section above, see Theorem 7.2.2.

Proof. Let $T \in M(W(\mathcal{S}))$. Since $W(\mathcal{S})$ is a homogeneous Banach space, we have $L_x f \in W(\mathcal{S})$ for all $x \in \mathcal{S}$ and $f \in W(\mathcal{S})$.
By

$$\|f\|_W = \|\check{f}\|_1 \leq \|f\|_1 + \|\check{f}\|_1 \leq \|f\|_\infty + \|\check{f}\|_1 \leq 2\|\check{f}\|_1 = 2\|f\|_W$$

the two norms $\|\ \|_W$ and $\|.\|^1 := \|.\|_1 + \|\check{.}\|_1$ are equivalent on $W(S)$. Thus, each continuous functional F on $W(S)$ w.r.t. $\|\ \|_W$ is also continuous w.r.t. $\|\ \|^1$.
Further, it is evident that the mapping $\Psi : W(S) \to L^1(\mathcal{S},\pi) \times l^1(\mathbb{N}_0, h)$ defined by $\Psi(f) := (f, \check{f})$ for each $f \in W(\mathcal{S})$ is a linear isometry of $(W(\mathcal{S}), \|\ \|^1)$ into the Banach space $L^1(\mathcal{S},\pi) \times l^1(\mathbb{N}_0, h)$ equipped with the sum of the ordinary norms as productnorm, i.e. $\|\ \|_1 + \|\ \|_1$. Thus, we may consider $W(\mathcal{S})$ as a closed linear subspace of $L^1(\mathcal{S},\pi) \times l^1(\mathbb{N}_0, h)$. Since the dual space of $L^1(\mathcal{S},\pi) \times l^1(\mathbb{N}_0, h)$ is isomorphic to $L^\infty(\mathcal{S},\pi) \times l^\infty(\mathbb{N}_0)$, by the theorem of Hahn-Banach we can consider every continuous linear functional F on $W(\mathcal{S})$ w.r.t. $\|\ \|^1$ to be of the following form:

$$F(f) = \int_{\mathcal{S}} f(x)\overline{a(x)}d\pi(x) + \sum_{k=0}^{\infty} \check{f}(k)\overline{b(k)}h(k),$$

for $(a, b) \in L^\infty(\mathcal{S},\pi) \times l^\infty(\mathbb{N}_0)$. (The pair (a, b) corresponding to a given functional may not be unique.)
Now let F be such a continuous functional on $W(\mathcal{S})$ w.r.t. $\|\ \|^1$. Then $F \circ T$ is also a continuous

linear functional on $W(\mathcal{S})$ w.r.t. $\| \|^1$. Hence, there exist (a,b) and (α,β) in $L^\infty(\mathcal{S},\pi)\times l^\infty(\mathbb{N}_0)$ such that for each $f\in W(\mathcal{S})$ we have

$$F(f) = \int_{\mathcal{S}} f(x)\overline{a(x)}d\pi(x) + \sum_{k=0}^{\infty} \check{f}(k)\overline{b(k)}h(k)$$

$$F\circ T(f) = \int_{\mathcal{S}} f(x)\overline{\alpha(x)}d\pi(x) + \sum_{k=0}^{\infty} \check{f}(k)\overline{\beta(k)}h(k).$$

Consequently, for $f,g\in W(\mathcal{S})$ is

$$\begin{aligned}
F(Tf*g) &= \int_{\mathcal{S}} (Tf*g)(x)\overline{a(x)}d\pi(x) + \sum_{k=0}^{\infty}(Tf*g)^\vee(k)\overline{b(k)}h(k)\\
&= \int_{\mathcal{S}}\int_{\mathcal{S}} L_yTf(x)g(y)d\pi(y)\overline{a(x)}d\pi(x) + \sum_{k=0}^{\infty}(Tf)^\vee\int_{\mathcal{S}} g(y)R_k(y)d\pi(y)\overline{b(k)}h(k)\\
&= \int_{\mathcal{S}} g(y)\int_{\mathcal{S}} L_yTf(x)\overline{a(x)}d\pi(x)d\pi(y) + \int_{\mathcal{S}} g(y)\sum_{k=0}^{\infty}(Tf)^\vee R_k(y)\overline{b(k)}h(k)d\pi(y)\\
&= \int_{\mathcal{S}} g(y)\left[\int_{\mathcal{S}} TL_yf(x)\overline{a(x)}d\pi(x) + \sum_{k=0}^{\infty}(TL_yf)^\vee\overline{b(k)}h(k)\right]d\pi(y)\\
&= \int_{\mathcal{S}} g(y)F\circ T(L_yf)d\pi(y)\\
&= \int_{\mathcal{S}} g(y)\left[\int_{\mathcal{S}} L_yf(x)\overline{\alpha(x)}d\pi(x) + \sum_{k=0}^{\infty}(L_yf)^\vee\overline{\beta(k)}h(k)\right]d\pi(y)\\
&= \int_{\mathcal{S}}\int_{\mathcal{S}} L_yf(x)g(y)d\pi(y)\overline{\alpha(x)}d\pi(x) + \sum_{k=0}^{\infty}\check{f}\int_{\mathcal{S}} g(y)R_k(y)d\pi(y)\overline{\beta(k)}h(k)\\
&= \int_{\mathcal{S}} (f*g)(x)\overline{\alpha(x)}d\pi(x) + \sum_{k=0}^{\infty}(f*g)^\vee(k)\overline{\beta(k)}h(k)\\
&= F\circ T(f*g) = F(T(f*g))
\end{aligned}$$

Moreover, we know by Theorem 7.2.3 the existence of a unique $\varphi\in l^\infty(\mathbb{N}_0)$ such that $(Tf)^\vee = \varphi\check{f}$ and $\|T\|\geq\|\varphi\|_\infty$. We obtain further

$$\|Tf\|_W = \|(Tf)^\vee\|_1 = \left\|\varphi\check{f}\right\|_1 \leq \|\varphi\|_\infty \|f\|_W.$$

Hence, $\|T\| = \|\varphi\|_\infty$. □

Theorem 7.2.7. *For $T\in B(W(\mathcal{S}))$ the following conditions are equivalent:*

i) $T\in M(W(\mathcal{S}))$, i.e. $T\circ L_x = L_x\circ T$ for all $x\in\mathcal{S}$.

ii) There exists a unique pseudomeasure $\sigma\in P(\mathcal{S})$, such that $Tf=\sigma*f$ for all $f\in W(\mathcal{S})$.

Moreover, there exists an isometric algebra isomorphism on $M(W(\mathcal{S}))$ onto $P(\mathcal{S})$.

Proof. The assertion follows by Theorem 7.2.4, since $W(\mathcal{S})\subset L^2(\mathcal{S},\pi)$. Further, Φ is isometric and we have by Theorem 7.2.6 $\|\sigma\|_P = \|\varphi\|_\infty = \|T\|$. Moreover, each $\varphi\in l^\infty(\mathbb{N}_0)$ defines a multiplier for $W(\mathcal{S})$ by $Tf := (\varphi\check{f})^\wedge$. Hence,

$$M(W(\mathcal{S}))\simeq l^\infty(\mathbb{N}_0)\simeq P(\mathcal{S}).$$

□

7.2.2 Multipliers for $A^p(\mathcal{S},\pi)$

Motivated by the Wiener algebra, $W(\mathcal{S})$, we introduce further homogeneous Banach spaces

$$A^p(\mathcal{S},\pi) := \left\{ f \in L^1(\mathcal{S},\pi) : \check{f} \in l^p(\mathbb{N}_0,h) \right\},$$

$1 \le p < \infty$, with norm $\|f\|^p := \|f\|_1 + \left\|\check{f}\right\|_p$. Notice that $(A^1(\mathcal{S},\pi), \|.\|^1)$ is the Wiener algebra $W(\mathcal{S})$ with a different but equivalent norm. The equivalence of the two norms $\|.\|^1$ and $\|.\|_W$ was shown in the proof of Theorem 7.2.6.
Larsen introduced in [101, Chapter 6] these spaces on groups. He presents various characterizations for multipliers for functions with Fourier transform in $L_p(\hat{G})$. Many of these results are extendable to commutative hypergroups. Here we focus on $A^p(\mathcal{S},\pi)$.
We want to remark, that the following relations between those spaces holds obviously for $1 \le p \le q \le 2 \le r \le s < \infty$,

$$W(\mathcal{S}) \subset A^p(\mathcal{S},\pi) \subset A^q(\mathcal{S},\pi) \subset A^2(\mathcal{S},\pi) = L^2(\mathcal{S},\pi) \subset A^r(\mathcal{S},\pi) \subset A^s(\mathcal{S},\pi) \subset L^1(\mathcal{S},\pi)$$

Before we can characterize the multipliers for $A^p(\mathcal{S},\pi)$, we need to check, if those spaces are indeed homogeneous Banach spaces on $\mathcal{S}$. The proof equals partial the proof of Proposition 3.6 in [46].

Proposition 7.2.8. *$A^p(\mathcal{S},\pi) := \left\{ f \in L^1(\mathcal{S},\pi) : \check{f} \in l^p(\mathbb{N}_0,h) \right\}$, $1 \le p < \infty$, is with norm $\|f\|^p := \|f\|_1 + \left\|\check{f}\right\|_p$ a character-invariant homogeneous Banach space on $\mathcal{S}$.*

Proof. $A^p(\mathcal{S},\pi)$ is obviously a linear subspace of $L^1(\mathcal{S},\pi)$ and $\|\cdot\|^p$ is a norm by the uniqueness theorem of the Jacobi transform. Since $\check{R}_m(n) = h(n)^{-1}\delta_{m,n} \in l^p(\mathbb{N}_0,h)$, where $\delta_{m,n}$ denotes the Kronecker symbol, we have $R_n \in A^p(\mathcal{S},\pi)$ for all $n \in \mathbb{N}_0$. Hence, $(B1)$ holds.
For $f \in A^p(\mathcal{S},\pi)$ we have $\|f\|_1 \le \|f\|_1 + \left\|\check{f}\right\|_p$. Further, $A^p(\mathcal{S},\pi)$ is complete with respect to $\|\cdot\|^p$, because each Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $A^p(\mathcal{S},\pi)$ is a Cauchy sequence in $L^1(\mathcal{S},\pi)$ and $(\check{f}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $l^p(\mathbb{N}_0,h)$. Since $L^1(\mathcal{S},\pi)$ and $l^p(\mathbb{N}_0,h)$ are complete with respect to $\|.\|_1$ and $\|.\|_p$ respectively, there exists $f \in L^1(\mathcal{S},\pi)$ with $\lim_{n\to\infty} \|f_n - f\|_1 = 0$ and $d \in l^p(\mathbb{N}_0,h)$ such that $\lim_{n\to\infty} \left\|\check{f}_n - d\right\|_p$. Furthermore, $\check{f} = d$, since

$$\left\|\check{f} - d\right\|_\infty \le \left\|\check{f} - \check{f}_n\right\|_\infty + \left\|\check{f}_n - d\right\|_p \le \|f - f_n\|_1 + \left\|\check{f}_n - d\right\|_p \to 0$$

as n tends to infinity. Thus, we have proven $(B2)$.
To show $(B3)$ note that $\sup_{x\in\mathcal{S}} |R_n(x)| = 1$ for all $n \in \mathbb{N}_0$. Hence, for each $f \in A^p(\mathcal{S},\pi)$ and $x \in \mathcal{S}$ we have $L_x f \in L^1(\mathcal{S},\pi)$ and $(L_x f)^\vee(n) = R_n(x)\check{f}(n)$ is an element in $l^p(\mathbb{N}_0,h)$. Further follows

$$\|L_x f\|^p = \|L_x f\|_1 + \left(\sum_{k=0}^{\infty} \left|R_k(x)\check{f}(k)\right|^p h(k) \right)^{1/p} \le \|f\|_1 + \sup_{k\in\mathbb{N}_0} |R_k(x)| \left\|\check{f}\right\|_p \le \|f\|^p .$$

Now we want to show the continuity of $x \mapsto L_x f$, $\mathcal{S} \to A^p(\mathcal{S},\pi)$ for all $f \in A^p(\mathcal{S},\pi)$. Fix $f \in A^p(\mathcal{S},\pi)$ and let $x_0 \in \mathcal{S}$, $\epsilon > 0$. There exists $N \in \mathbb{N}$ and $g \in A^p(\mathcal{S},\pi)$ such that $\left\|\check{g} - \check{f}\right\|_p < \frac{\epsilon}{4}$, $\check{g}(n) = \check{f}(n)$ for all $n \le N$ and $\check{g}(n) = 0$ for all $n > N$. Indeed, since $\check{f} \in l^p(\mathbb{N}_0,h)$ there exists a $N \in \mathbb{N}_0$ such that $\sum_{k=N+1}^{\infty} \|\check{f}(k)|^p h(k) < \frac{\epsilon}{4}$. We choose $g = \sum_{k=0}^{N} \check{f}(k) R_k h(k)$. Furthermore, since $\check{g}(n) = 0$ for all $n > N$ there exists $\delta > 0$ with $\|(L_x g)^\vee - (L_{x_0} g)^\vee\|_p < \frac{\epsilon}{2}$ for all $x \in \mathcal{S}$ such that $|x - x_0| < \delta$. Thus, we obtain $\|(L_x f)^\vee - (L_{x_0} f)^\vee\|_p < \epsilon$ for all $x \in \mathcal{S}$ with $|x - x_0| < \delta$ and finally $\|L_x f - L_{x_0} f\|^p < 2\epsilon$ for all $x \in \mathcal{S}$ with $|x - x_0| < \delta'$, since $L^1(\mathcal{S},\pi)$ is a homogeneous Banach space. Thus, the map $x \mapsto L_x f$, $\mathcal{S} \to B$ is continuous for all $f \in A^p(\mathcal{S},\pi)$ and $(B4)$ is proven.
We conclude $R_n \cdot f \in A^p(\mathcal{S},\pi)$, since $\check{R}_n = \frac{1}{h(n)}\varepsilon_n$ and hence,

$$(R_n \cdot f)^\vee = \check{R}_n * \check{f} \in l^1(\mathbb{N}_0,h) * l^p(\mathbb{N}_0,h) \in l^p(\mathbb{N}_0,h).$$

Moreover,

$$\begin{aligned}\|R_n \cdot f\|^p &= \|R_n \cdot f\|_1 + \|(R_n \cdot f)^\vee\|_p \\ &\leq \|R_n\|_\infty \|f\|_1 + \|\check{R}_n * \check{f}\|_p \\ &\leq \|f\|_1 + \|\check{R}_n\|_1 \|\check{f}\|_p = \|f\|_1 + \left\|\frac{1}{h(n)}\varepsilon_n\right\|_1 \|\check{f}\|_p = \|f\|^p .\end{aligned}$$

for all $n \in \mathbb{N}_0$. Hence, $A^p(\mathcal{S},\pi)$ is a character-invariant homogeneous Banach space on $\mathcal{S}$. □

Remark 7.2.9. $A^2(\mathcal{S},\pi)$ is with norm $\|f\| := \|\check{f}\|_2$ also a character-invariant homogeneous Banach space.

Now we continue characterizing multipliers for $A^p(\mathcal{S},\pi)$.

Theorem 7.2.10. *For a bounded linear operator T on $A^p(\mathcal{S},\pi)$, $1 \leq p < \infty$, the following conditions are equivalent:*

i) $T \in M(A^p(\mathcal{S},\pi))$, i.e. $T \circ L_x = L_x \circ T$ for all $x \in \mathcal{S}$.

ii) For all $f, g \in A^p(\mathcal{S},\pi)$ we have $Tf * g = T(f * g) = f * Tg$.

iii) There exists a unique bounded function φ on $\mathbb{N}_0$, such that $(Tf)^\vee = \varphi\check{f}$ for all $f \in A^p(\mathcal{S},\pi)$.

Moreover, $\|T\| \geq \|\varphi\|_\infty$. For $1 \leq p \leq 2$ is further $\|T\| \leq 2\|\varphi\|_\infty$.

Proof. The proof follows directly by Theorem 7.2.2 and Theorem 7.2.3. (One could also follow the lines of the proof of Theorem 7.2.6.)
For the last statement we have for $1 \leq p \leq 2$

$$\|Tf\|^p \leq \|Tf\|_2 + \|\varphi\|_\infty \|\check{f}\|_p = \|(Tf)^\vee\|_2 + \|\varphi\|_\infty \|\check{f}\|_p$$

$$\leq \|\varphi\|_\infty \|\check{f}\|_2 + \|\varphi\|_\infty \|\check{f}\|_p \leq 2\|\varphi\|_\infty \|\check{f}\|_p \leq 2\|\varphi\|_\infty \|f\|^p .$$

□

The next characterization of multipliers in $A^p(\mathcal{S},\pi)$ depends on whether $p \leq 2$ or $p > 2$. This is do to the fact that for $1 \leq p \leq 2$, we have $A^p(\mathcal{S},\pi) \subset L^2(\mathcal{S},\pi)$.
Conversely is $L^2(\mathcal{S},\pi) \subsetneq A^p(\mathcal{S},\pi)$ whenever $2 < p$. Indeed, for $p > 2$ and $1/p + 1/q = 1$ suppose $L^2(\mathcal{S},\pi) = A^2(\mathcal{S},\pi) = A^p(\mathcal{S},\pi)$. Given a continuous function g on $\mathcal{S}$ the Riesz-Thorin convexity theorem yields the inequality $\|\check{g}\|_p \leq \|g\|_p$. Now let $f \in L^q(\mathcal{S},\pi) \subset L^1(\mathcal{S},\pi)$. By approximating f through continuous functions on $\mathcal{S}$, we obtain $\check{f} \in l^p(\mathbb{N}_0, h)$. Our assumption implies $f \in L^2(\mathcal{S},\pi)$. Hence, $L^q(\mathcal{S},\pi) = L^2(\mathcal{S},\pi)$. This is a contradiction, since $\mathcal{S}$ is infinite.
In case $2 < p$ we can only quote Theorem 7.2.5, but in case $p \leq 2$ we can say more:

Theorem 7.2.11. *For a bounded linear operator T on $A^p(\mathcal{S},\pi)$ and $1 \leq p \leq 2$, the following conditions are equivalent:*

i) $T \in M(A^p(\mathcal{S},\pi))$, i.e. $T \circ L_x = L_x \circ T$ for all $x \in \mathcal{S}$.

ii) There exists a unique pseudomeasure $\sigma \in P(\mathcal{S})$, such that $Tf = \sigma * f$ for all $f \in A^p(\mathcal{S},\pi)$.

Moreover, there exists a continuous algebra isomorphism from $M(A^p(\mathcal{S},\pi))$ onto $P(\mathcal{S})$ such that $\|\sigma\|_P \leq \|T\| \leq 2\|\sigma\|_P$.

Proof. The proof follows by Theorem 7.2.4 and Theorem 7.2.10, since $P(\mathcal{S})$ and $l^\infty(\mathbb{N}_0)$ are isometrically isomorphic. Moreover, each $\varphi \in l^\infty(\mathbb{N}_0)$ defines by $Tf := \wp(\varphi\check{f})$ for all $f \in A^p(\mathcal{S},\pi)$ a multiplier for $A^p(\mathcal{S},\pi)$. Hence, the algebra isomorphism from $M(A^p(\mathcal{S},\pi))$ into $P(\mathcal{S})$ is surjective. □

Proposition 7.2.12. *Let $1 \le p \le 2$. Then there exists a norm-increasing algebra isomorphism from $M(W(\mathcal{S}))$ into $M(A^p(\mathcal{S},\pi))$.*

Proof. We have $W(\mathcal{S}) \subset A^p(\mathcal{S},\pi)$. Hence, $M(A^p(\mathcal{S},\pi)) \subset M(W(\mathcal{S}))$, since for $T \in M(A^p(\mathcal{S},\pi))$ and $f \in W(\mathcal{S})$ we obtain $(Tf)^\vee = \varphi \check{f} \in l^1(\mathbb{N}_0, h)$, for $\varphi \in l^\infty(\mathbb{N}_0)$ as in Theorem 7.2.10, and $Tf = (\varphi \check{f})^\wedge \in C(\mathcal{S})$. Hence, $Tf \in W(\mathcal{S})$. Furthermore,

$$\|Tf\|_W = \|(Tf)^\vee\|_1 \le \|\varphi\|_\infty \left\|\check{f}\right\|_1 = \|\varphi\|_\infty \|f\|_W .$$

Thus, T is a bounded linear operator on $W(\mathcal{S})$, which commutes with translation, that is $T \in M(W(\mathcal{S}))$. Further, by Theorem 7.2.7 and Theorem 7.2.11 there exists an isomorphism between $M(W(\mathcal{S}))$ and $P(\mathcal{S})$ and between $P(\mathcal{S})$ and $M(A^p(\mathcal{S},\pi))$. Hence, $M(W(\mathcal{S}))$ and $M(A^p(\mathcal{S},\pi))$ are algebraic isomorphic. Moreover, we have

$$\|T\|_W = \|\varphi\|_\infty \le \|T\|^p ,$$

where $\|T\|_W$ denotes the operator norm of T defined on $W(\mathcal{S})$ and $\|T\|^p$ denotes the operator norm of T defined on $A^p(\mathcal{S},\pi)$. Hence, the algebra isomorphism is norm-increasing. □

Remark 7.2.13. We want to point out, that by Proposition 7.2.12 the multiplier spaces of $W(\mathcal{S})$ and $A^p(\mathcal{S},\pi)$, $1 < p \le 2$, coincide, despite that fact that $W(\mathcal{S})$ is a proper linear subset in $A^p(\mathcal{S},\pi)$ and the norms of those spaces are not equivalent. This leads to the observation that the multipliers for a homogeneous Banach space contribute little information about the homogeneous Banach space itself.
Moreover, we remark the existence of a norm-increasing algebra isomorphism from $M(A^p(\mathcal{S},\pi))$ into $M(A^q(\mathcal{S},\pi))$ for $1 \le p < q \le 2$.

Remark 7.2.14. In contrast to $1 \le p < q \le 2$ where $M(A^p(\mathcal{S},\pi)) = M(A^q(\mathcal{S},\pi))$, for $2 \le p < q < \infty$ we can only show that

$$M(A^q(\mathcal{S},\pi)) \subset M(A^p(\mathcal{S},\pi)).$$

Since for $T \in M(A^q(\mathcal{S},\pi))$ and $f \in A^p(\mathcal{S},\pi) \subseteq A^q(\mathcal{S},\pi)$ we have $(Tf)^\vee = \varphi\check{f} \in l^p(\mathbb{N}_0)$ as $f \in A^p(\mathcal{S},\pi)$ and φ is bounded. Further holds

$$\|Tf\|^p \le \|Tf\|_1 + \|\varphi\|_\infty \left\|\check{f}\right\|_p \le \|Tf\|^q + \|\varphi\|_\infty \left\|\check{f}\right\|_p$$

$$\le \|T\|^q \|f\|^q + \|\varphi\|_\infty \left\|\check{f}\right\|_p \le (\|T\|^q + \|\varphi\|_\infty) \|f\|^p ,$$

where $\|T\|^q$ denotes the operator norm of T defined on $A^q(\mathcal{S},\pi)$. Hence, T is a bounded linear operator on $A^p(\mathcal{S},\pi)$ and T commutes with all Jacobi translation operators.

Remark 7.2.15. Using Theorem 4 in [20] we have for all $1 \le p < \infty$

$$l^1(\mathbb{N}_0,h) = M(D_s) \simeq M(L^1(\mathcal{S},\pi)) \subset M(A^p(\mathcal{S},\pi)) \subset M(W(\mathcal{S})) \simeq P(\mathcal{S}).$$

7.2.3 Multipliers for the Beurling space $W_*(\mathcal{S})$

The space

$$W_*(\mathcal{S}) := \left\{ f \in W(\mathcal{S}) : \sum_{k=0}^\infty \sup_{l \ge k} |\check{f}(l)| h(k) < \infty \right\}$$

is called **Beurling space**. The Beurling space $W_*(\mathcal{S})$ is with norm

$$\|f\|_{W_*} := \sum_{k=0}^\infty \sup_{l \ge k} |\check{f}(l)| h(k)$$

a homogeneous Banach space, see [46]. Furthermore, $W_*(\mathcal{S})$ is a Banach algebra with respect to the convolution and with respect to the pointwise multiplication of functions.
G. Fischer and R. Lasser proved in [46] that $W(\mathcal{S})\backslash W_*(\mathcal{S}) \neq \emptyset$ by giving an example of a function in $W(\mathcal{S})\backslash W_*(\mathcal{S})$. Hence, it makes indeed sense to examine the multipliers for $W_*(\mathcal{S})$, even though we already know the multipliers for $W(\mathcal{S})$. Despite the fact that the two homogeneous Banach spaces are different and their norms are not equivalent, we will prove that their multiplier spaces coincide.
It is easy to see, that the multiplier space $M(W(\mathcal{S}))$ is included in $M(W_*(\mathcal{S}))$. Indeed, choose a multiplier $T \in M(W(\mathcal{S}))$ with the corresponding function $\varphi \in l^\infty(\mathbb{N}_0)$ such that $(Tg)^\vee = \varphi\check{g}$ for all $g \in W(\mathcal{S})$ and let $f \in W_*(\mathcal{S}) \subset W(\mathcal{S})$. We obtain

$$\|Tf\|_{W_*} = \sum_{k=0}^{\infty} \sup_{l\geq k} |(Tf)^\vee(l)| h(k) = \sum_{k=0}^{\infty} \sup_{l\geq k} |\varphi(l)||\check{f}(l)| h(k) \leq \|\varphi\|_\infty \|f\|_{W_*}.$$

Thus, T is a bounded linear operator on $W_*(\mathcal{S})$, which commutes with Jacobi translation operators. Hence, $T \in M(W_*(\mathcal{S}))$. In particular each $\varphi \in l^\infty(\mathbb{N}_0)$ defines a multiplier for $W_*(\mathcal{S})$, see Theorem 7.2.7
There is even more we can say about the multipliers for $W_*(\mathcal{S})$:

Theorem 7.2.16. *For a bounded linear operator T on $W_*(\mathcal{S})$ the following conditions are equivalent:*

i) $T \in M(W_*(\mathcal{S}))$

ii) For all $f, g \in W_*(\mathcal{S})$ we have $Tf * g = T(f * g) = f * Tg$.

iii) There exists a unique function $\varphi \in l^\infty(\mathbb{N}_0)$ such that $(Tf)^\vee = \varphi\check{f}$ for all $f \in W_*(\mathcal{S})$.

Moreover is $\|T\| = \|\varphi\|_\infty$.

Proof. The equivalencies of *i)*, *ii)* and *iii)* follow by Theorem 7.2.2 and 7.2.3.
Furthermore, we have $\|T\| \geq \|\varphi\|_\infty$ by Theorem 7.2.3. By

$$\|Tf\|_{W_*} = \sum_{k=0}^{\infty} \sup_{l\geq k} |(Tf)^\vee(l)| h(k) = \sum_{k=0}^{\infty} \sup_{l\geq k} |\varphi(l)||\check{f}(l)| h(k) \leq \|\varphi\|_\infty \|f\|_{W_*}.$$

we obtain $\|\varphi\|_\infty = \|T\|$.

□

As a consequence of Theorem 7.2.16 we conclude

Theorem 7.2.17. *For a bounded linear operator T on $W_*(\mathcal{S})$ the following conditions are equivalent:*

i) $T \in M(W_*(\mathcal{S}))$

ii) There exists a unique pseudomeasure $\sigma \in P(\mathcal{S})$ such that $Tf = \sigma * f$, for all $f \in W_*(\mathcal{S})$.

Moreover, there exists an isometric algebra isomorphism from $M(W_*(\mathcal{S}))$ onto $P(\mathcal{S})$.

Proof. The proof follows by Theorem 7.2.4 and by Theorem 7.2.16.
Moreover, the isometric algebra isomorphism from $M(W_*(\mathcal{S}))$ into $P(\mathcal{S})$ is surjective, since each $\varphi \in l^\infty(\mathbb{N}_0)$ defines a multiplier for $W_*(\mathcal{S})$ by $Tf := (\varphi\check{f})^\wedge$. Hence, we have an isometric algebra isomorphism from $M(W_*(\mathcal{S}))$ onto $l^\infty(\mathbb{N}_0) \simeq P(\mathcal{S})$. □

Corollary 7.2.18. *There exists an isometric algebra isomorphism between $M(W(\mathcal{S}))$ and $M(W_*(\mathcal{S}))$.*

7.2.4 Multipliers for the Sobolev space $H_2^{(1)}(\mathcal{S})$

We denote by $s_n := R_n'(1)$ for $n \in \mathbb{N}$.
Define a subspace

$$D(B) := \left\{ f \in L^2(\mathcal{S},\pi) : \sum_{k=0}^{\infty} s_k^2 |\check{f}(k)|^2 h(k) < \infty \right\}$$

and an operator

$$B : D(B) \to L^2(\mathcal{S},\pi), \;\; Bf := \sum_{k=0}^{\infty} s_k \check{f}(k) R_k h(k)$$

for all $f \in D(B)$. Furthermore, we put

$$H_2^{(1)}(\mathcal{S}) := \left\{ f \in L^2(\mathcal{S},\pi) : \lim_{x\to 1^-} \frac{f - L_x f}{1-x} \text{ exists in } L^2(\mathcal{S},\pi) \right\}$$

and call

$$D : H_2^{(1)}(\mathcal{S}) \to L^2(\mathcal{S},\pi), \;\; Df := \lim_{x\to 1^-} \frac{f - L_x f}{1-x}$$

the **Jacobi differential operator** with respect to $(\alpha,\beta) \in J$.
G.Fischer and R.Lasser showed in [46] that the Jacobi differential operator D fullfils

i) $H_2^{(1)}(\mathcal{S}) = D(B)$

ii) $Df = \lim_{x\to 1^-} \frac{f-L_x f}{1-x} = \sum_{k=0}^{\infty} s_k \check{f}(k) R_k h(k) = Bf$ for all $f \in H_2^{(1)}(\mathcal{S})$

We call the space $H_2^{(1)}(\mathcal{S})$ **Sobolev space induced by** D and choose

$$\|f\|_{2,1} := \|f\|_2 + \|Df\|_2$$

as norm on $H_2^{(1)}(\mathcal{S})$. With this norm $H_2^{(1)}(\mathcal{S})$ becomes a homogeneous Banach space on $\mathcal{S}$ with respect to $(\alpha,\beta) \in J$, see [46].
Sobolev spaces are very important in the theory of partial differential equations. Sobolev spaces defined on the torus $\mathbb{T}$ are investigated in [142].

Theorem 7.2.19. *For a bounded linear operator T on $H_2^{(1)}(\mathcal{S})$ the following conditions are equivalent:*

i) T is a multiplier for $H_2^{(1)}(\mathcal{S})$, i.e. $T \in M(H_2^{(1)}(\mathcal{S}))$.

ii) For all $f,g \in H_2^{(1)}(\mathcal{S})$ we have $Tf * g = T(f*g) = f * Tg$.

iii) There exists a unique function $\varphi \in l^\infty(\mathbb{N}_0)$ such that $(Tf)^\vee = \varphi\check{f}$ for all $f \in H_2^{(1)}(\mathcal{S})$.

Moreover, $\|\varphi\|_\infty = \|T\|$.

Proof. The equivalencies $i)$, $ii)$ and $iii)$ follow by Theorem 7.2.2 and 7.2.3. Theorem 7.2.3 yields$\|\varphi\|_\infty \leq \|T\|$. Further, we obtain $\|T\| \leq \|\varphi\|_\infty$ by

$$\|Tf\|_{2,1} = \|Tf\|_2 + \|D(Tf)\|_2 = \|(Tf)^\vee\|_2 + \left\| \sum_{k=0}^{\infty} s_k (Tf)^\vee(k) R_k h(k) \right\|_2$$

$$= \;\; \|\varphi\check{f}\|_2 + \left\| \sum_{k=0}^{\infty} s_k \varphi(k) \check{f}(k) R_k h(k) \right\|_2 \leq \|\varphi\|_\infty \|\check{f}\|_2 + \left(\sum_{k=0}^{\infty} s_k^2 (\varphi(k)\check{f}(k))^2 h(k) \right)^{1/2}$$

$$\leq \;\; \|\varphi\|_\infty \|f\|_2 + \|\varphi\|_\infty \left(\sum_{k=0}^{\infty} s_k^2 (\check{f}(k))^2 h(k) \right)^{1/2} = \|\varphi\|_\infty (\|f\|_2 + \|Df\|_2) = \|\varphi\|_\infty \|f\|_{2,1}.$$

□

Theorem 7.2.20. *Let T be a bounded linear operator on $H_2^{(1)}(\mathcal{S})$. The following conditions are equivalent:*

i) $T \in M(H_2^{(1)}(\mathcal{S}))$.

ii) There exists a unique pseudomeasure $\sigma \in P(\mathcal{S})$ such that $Tf = \sigma * f$ for all $f \in H_2^{(1)}(\mathcal{S})$.

Moreover, there exists an isometric algebra isomorphism from $M(H_2^{(1)}(\mathcal{S}))$ onto $P(\mathcal{S})$.

Proof. The proof follows by Theorem 7.2.4 and Theorem 7.2.19.
Furthermore, each $\varphi \in l^\infty(\mathbb{N}_0)$ defines by $Tf := \wp(\varphi\check{f})$ a multiplier for $H_2^{(1)}(\mathcal{S})$. Indeed, we have

$$\|Tf\|_{2,1} = \|Tf\|_2 + \left(\sum_{k=0}^{\infty} s_k^2 (Tf)^\vee(k)^2 h(k)\right)^{1/2} = \left\|\varphi\check{f}\right\|_2 + \left(\sum_{k=0}^{\infty} s_k^2 \varphi(k)^2 \check{f}(k)^2 h(k)\right)^{1/2}$$

$$\leq \|\varphi\|_\infty \left\|\check{f}\right\|_2 + \|\varphi\|_\infty \left(\sum_{k=0}^{\infty} s_k^2 \check{f}(k)^2 h(k)\right)^{1/2} = \|\varphi\|_\infty \|f\|_{2,1} < \infty$$

for all $f \in H_2^{(1)}(\mathcal{S})$. Hence, Tf is a bounded linear operator on $H_2^{(1)}(\mathcal{S})$, which commutes with Jacobi translation operators. Thus, $T \in M(H_2^{(1)}(\mathcal{S}))$ and we obtain an isometric algebra isomorphism from $M(H_2^{(1)}(\mathcal{S}))$ onto $P(\mathcal{S})$. □

Corollary 7.2.21. *There exists an isometric algebra isomorphism between the spaces $M(W(\mathcal{S}))$, $M(W_*(\mathcal{S}))$ and $M(H_2^{(1)}(\mathcal{S}))$.*

Remark 7.2.22. We want to point out, that the homogeneous Banach spaces $W(\mathcal{S})$, $W_*(\mathcal{S})$ and $H_2^{(1)}(\mathcal{S})$ are all very different in their structure and the spaces of bounded operators $B(W(\mathcal{S}))$, $B(W_*(\mathcal{S}))$ and $B(H_2^{(1)}(\mathcal{S}))$ on $W(\mathcal{S})$, $W_*(\mathcal{S})$ and $H_2^{(1)}(\mathcal{S})$, respectively, differ. However, their multiplier spaces coincide. The basic tool to prove this quite remarkable fact is the theory of pseudomeasures.

Chapter 8

Multipliers for $p-$Fourier Spaces

Inspired by the last chapter where we introduced the homogeneous Banach spaces $A^p(\mathcal{S})$ defined on the Jacobi hypergroup $\mathcal{S} = [-1, 1]$, we generalize the definition of $A^p(\mathcal{S})$ to a general commutative hypergroup K. We will discuss the Banach algebras $A^p(K, m)$ consisting of functions in $L^1(K, m)$ with Fourier transform in $L^p(\mathcal{S}, \pi)$ and give an overview of the structure of the multiplier spaces $M(A^p(K, m))$.
1964 Larsen, Liu and Wang [103] introduced the spaces $A^p(G)$ for a locally compact Abelian group G. Later on the spaces $A^p(G)$ and their multipliers were intensively studied by Larsen [102], Figà-Talamanca and Gaudry [43]. Larsen presents the main results about multipliers for $A^p(G)$ in [101, Chapter 6]. Our aim is to generalize the known results about multipliers for $A^p(G)$ to commutative hypergroups.

8.1 The Banach algebra $A^p(K, m)$

We denote by $A^p(K, m)$ the set of all functions in $L^1(K, m)$ with Fourier transform in $L^p(\mathcal{S}, \pi)$, $1 \le p \le \infty$,

$$A^p(K, m) := \left\{ f \in L^1(K, m) : \hat{f} \in L^p(\mathcal{S}, \pi) \right\}.$$

We note that $A^p(K, m)$ for $p = \infty$ is the space $L^1(K, m)$. Moreover, by Hölder's interpolation theorem holds $A^1(K, m) \subseteq A^p(K, m) \subseteq A^q(K, m)$ for $1 \le p \le q \le \infty$.
Further, by Proposition 2.1.7 in Chapter 2 follows $L^1(K, m) \cap L^2(K, m) = A^2(K, m)$.
A hypergroup K is discrete, if and only if $\hat{K}$ is compact, see [108]. Hence, $A^p(K, m) = L^1(K, m)$ for each $1 \le p < \infty$. Therefore, we will omit this case in the whole chapter.
In this section we take a closer look on the structure of the spaces $A^p(K, m)$. We note first that the space $A^p(K, m)$, $1 \le p \le \infty$, is translation invariant, since for all $x \in K$ holds $(L_x f)^\wedge = (\varepsilon_{\tilde{x}} * f)^\wedge = \hat{\varepsilon}_{\tilde{x}} \hat{f} \in L^p(\mathcal{S}, \pi)$ for all $f \in A^p(K, m)$. Furthermore, by $(f * g)^\wedge = \hat{f}\hat{g}$ for all $f \in A^p(K, m)$ and $g \in L^1(K, m)$ is $A^p(K, m)$ an ideal in $L^1(K, m)$.

Theorem 8.1.1. *$A^p(K, m)$, $1 \le p \le \infty$, is with the norm*

$$\|f\|^p := \|f\|_1 + \left\|\hat{f}\right\|_p$$

and the usual convolution in $L^1(K, m)$ a commutative Banach algebra.

Proof. By the uniqueness theorem for Fourier transforms is $\| \ \|^p$ obviously a norm on $A^p(K, m)$. Since K is a commutative hypergroup, the convolution in $A^p(K, m)$ is commutative, too. Furthermore, $A^p(K, m)$ is an ideal in $L^1(K, m)$ which is a commutative Banach algebra itself.
In order to prove the completeness of $A^p(K, m)$ let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $A^p(K, m)$. Clearly $(f_n)_{n\in\mathbb{N}}$ and $(\hat{f}_n)_{n\in\mathbb{N}}$ are Cauchy sequences in the Banach spaces $L^1(K, m)$ and $L^p(\mathcal{S}, \pi)$,

respectively. Let $f \in L^1(K,m)$ and $g \in L^p(\mathcal{S},\pi)$ be such that $\lim_n \|f_n - f\|_1 = 0$ and $\lim_n \left\|\hat{f}_n - g\right\|_p = 0$. Since the Fourier transform is norm decreasing, the first assertion implies that $\lim_n \left\|\hat{f}_n - \hat{f}\right\|_\infty = 0$. The second assertion implies the existence of a subsequence of $(\hat{f}_n)_{n\in\mathbb{N}}$ which converges point-wise almost everywhere to g. Thus, $\hat{f} = g$ almost everywhere. Hence, $f \in A^p(K,m)$ and $A^p(K,m)$ is with norm $\| \|^p$ complete.
To show the submultiplicativity of $\| \|^p$ we observe that for each $f,g \in A^p(K,m)$

$$\|f*g\|^p = \|f*g\|_1 + \left\|\hat{f}\hat{g}\right\|_p \leq \|f\|_1\|g\|_1 + \left\|\hat{f}\right\|_\infty \|\hat{g}\|_p$$

$$\leq \|f\|_1 (\|g\|_1 + \|\hat{g}\|_p) \leq \|f\|^p \|g\|^p .$$

Thus, $A^p(K,m)$ is a commutative Banach algebra. □

Remark 8.1.2. Larsen, Liu and Wang [103] proved Theorem 8.1.1 for locally compact Abelian groups.

Lemma 8.1.3. *$A^p(K,m)$ is a $\| \|_1$-norm dense subspace in $L^1(K,m)$ for all $1 \leq p \leq \infty$.*

Proof. Let $f,g \in L^1(K,m)\cap L^2(K,m)$. Then $f*g \in L^1(K,m)\cap L^2(K,m)\cap C_0(K)$ and we know by the existence of an approximative identity for $L^1(K,m)$ in $C_c(K)$ that the set $\{f*g : f,g \in L^1(K,m)\cap L^2(K,m)\}$ is norm dense in $L^1(K,m)$. Furthermore, for each element $h = f*g$, $f,g \in L^1(K,m)\cap L^2(K,m)$, holds

$$\hat{h} = (f*g)^\wedge = \hat{f}\hat{g} \in L^1(\mathcal{S},\pi)\cap L^2(\mathcal{S},\pi)\cap C_0(\hat{K}).$$

Thus, the set $A^1(K,m) = \left\{f \in L^1(K,m) : \hat{f} \in L^1(\mathcal{S},\pi)\right\}$ is norm dense in $L^1(K,m)$. By Hölder's interpolation theorem follows that $A^p(K,m) = \left\{f \in L^1(K,m) : \hat{f} \in L^p(\mathcal{S},\pi)\right\}$ is norm dense in $L^1(K,m)$ for all $1 \leq p \leq \infty$. □

Theorem 8.1.4. *$A^p(K,m)$, $1 \leq p \leq \infty$, contains an approximate identity $(k_i)_{i\in I}$ in $A^1(K,m)$.*

Proof. Consider $W(e,C,\epsilon) := \{y \in K : |\alpha(y) - \alpha(e)| < \epsilon \ \forall \alpha \in C\}$, where $C \subset \mathcal{S}$ is compact and $\epsilon > 0$. $W(e,C,\epsilon)$ is a member of the neighborhood basis of $e \in K$. Put

$$f := \chi_{W(e,C,\epsilon)}/m(W(e,C,\epsilon))$$

and introduce an index set $j \in J$ corresponding to ϵ with the usual order. We observe that $\hat{f}_j(\alpha) \to 1$ whenever $\alpha \in C$ and $\epsilon \to 0$.
Let $(k_i)_{i\in I}$ be an approximate identity for $L^1(K,m)$ such that $k_i \geq 0$ and $\mathrm{supp} k_i \to \{e\}$ for each $i \in I$. Since $A^1(K,m)$ is a norm dense subset of $L^1(K,m)$, we can choose $(k_i)_{i\in I}$ in $A^1(K,m)$. Indeed, choose for each $i \in I$ a sequence $(k_{ij})_{j\in\mathbb{N}}$ in $A^1(K,m)$ such that k_{ij} converges to k_i in $L^1(K,m)$. It obviously holds for every $g \in L^1(K,m)$ that

$$\|k_{ij}*g - g\|_1 \leq \|k_{ij}*g - k_i*g\|_1 + \|k_i*g - g\|_1 \to 0$$

as i and j tend to infinity. Furthermore, we can also choose $k_{ij} > 0$ for each $j \in \mathbb{N}$ and $\|k_{ij}\|_1 = \|k_{ij} - k_i + k_i\|_1 \leq \|k_{ij} - k_i\|_1 + \|k_i\|_1 \to 1$ as j tends to infinity for all $i \in I$. We denote $(k_{ij})_{j\in\mathbb{N},i\in I}$ again by $(k_i)_{i\in I}$.
We prove that $(\hat{k_i})_{i\in I}$ converges uniformly to one on compact subsets of $\mathcal{S}$. Let $C \subset \mathcal{S}$ compact and $(f_j)_{j\in J}$ in $L^1(K,m)$ as indicated above, such that $\hat{f}_j(\alpha) \to 1$ for each $\alpha \in C$. Since $(k_i)_{i\in I}$ is an approximate identity for $L^1(K,m)$, the desired conclusion is evident from the inequality

$$|\hat{k_i}(\alpha) - 1| = \lim_{j\to\infty} |\hat{k_i}\hat{f}_j(\alpha) - \hat{f}_j(\alpha)| \leq \lim_{j\to\infty} \left\|\hat{k_i}\hat{f}_j - \hat{f}_j\right\|_\infty$$

$$\leq \lim_{j\to\infty} \|k_i * f_j - f_j\|_1 \to 0,$$

for all $\alpha \in C$.
Now suppose $g \in A^p(K,m)$ and $\epsilon > 0$. Let $C \subseteq \mathcal{S}$ be a compact subset such that

$$\int_{\mathcal{S}\setminus C} |\hat{g}(\alpha)|^p d\pi(\alpha) < \epsilon^p/(2\cdot 3^p).$$

Moreover, since $(\hat{k}_i)_{i\in I}$ converges uniformly to one on C it is easily seen that there exists $i_0 \in I$ such that for $i > i_0$

$$\int_C |\hat{g}(\alpha)\hat{k}_i(\alpha) - \hat{g}(\alpha)|^p d\pi(\alpha) < \epsilon^p/2.$$

Further, let $(k_i)_{i\in I}$ fulfill $\|k_i\|_1 < 2$ for each $i \in I$. By the assumptions above this is not a loss of generality Thus, for $i > i_0$

$$\left\|\hat{g}\hat{k_i} - \hat{g}\right\|_p = \left[\int_C |\hat{g}\hat{k_i}(\alpha) - \hat{g}(\alpha)|^p d\pi(\alpha) + \int_{\mathcal{S}\setminus C} |\hat{g}\hat{k_i}(\alpha) - \hat{g}(\alpha)|^p d\pi(\alpha)\right]^{1/p}$$

$$\leq \left[\epsilon^p/2 + \left\|\hat{k_i} - 1\right\|_\infty^p \int_{\mathcal{S}\setminus C} |\hat{g}(\alpha)|^p \, d\pi(\alpha)\right]^{1/p} \leq \left[\epsilon^p/2 + (\left\|\hat{k_i}\right\|_\infty + 1)^p \epsilon^p/(2\cdot 3^p)\right]^{1/p} \leq \epsilon.$$

Hence, $\lim_i \left\|\hat{g}\hat{k_i} - \hat{g}\right\|_p = 0$ and $\lim_i \|g * k_i - g\|_1 = 0$ as i tends to infinity. Therefore, $\lim_i \|g * k_i - g\|^p = 0$ for each $g \in A^p(K,m)$ and $(k_i)_{i\in I}$ is an approximate identity for $A^p(K,m)$. □

Remark 8.1.5. We want to remark that the approximate identity $(k_i)_{i\in I} \subset A^1(K,m)$ is not necessarily in $C_c(K)$ and it might be that $\|k_i\|_1 \neq 1$. However, as shown in the proof above we have $\|k_i\|_1 \to 1$ as i tends to infinity.

The existence of an approximate identity for $A^p(K,m)$ in $A^1(K,m)$ reveals that each $A^p(K,m)$ is norm dense in $A^r(K,m)$, $r > p$, since the spaces $A^p(K,m)$, $1 \leq p < \infty$, are ideals in $L^1(K,m)$.
We can say even more about the relation of different $A^p(K,m)-$spaces.

Proposition 8.1.6. *Let K be an infinite and compact hypergroup. Then*

$$A^r(K,m) \subsetneq A^p(K,m), \qquad 1 \leq r < p \leq 2$$

$$L^2(K,m) = A^2(K,m) \subsetneq A^p(K,m), \qquad 2 < p.$$

Proof. We note that $\mathcal{S}$ is discrete, since K is compact. For $1 \leq r < p \leq 2$ choose a function $f \in l^p(\mathcal{S},\pi)\setminus l^r(\mathcal{S},\pi)$. By the Hausdorff-Young theorem holds that $\check{f} \in L^q(K,m) \subset L^1(K,m)$, $1/p + 1/q = 1$, and further $(\check{f})^\wedge = f \in l^p(\mathcal{S},\pi)\setminus l^r(\mathcal{S},\pi)$. Hence, $\check{f} \in A^p(K,m)\setminus A^r(K,m)$.
From the Hausdorff-Young theorem we find $A^2(K,m) = L^2(K,m)$. For $2 < p$ suppose $A^p(K,m) = L^2(K,m)$, that is for each $f \in L^1(K,m)$ with $\hat{f} \in l^p(\mathcal{S},\pi)$ holds $f \in L^2(K,m)$. Let $f \in L^q(K,m) \subset L^1(K,m)$, $1/p + 1/q = 1$. By the Hausdorff-Young theorem follows $\hat{f} \in l^p(\mathcal{S},\pi)$ thus $f \in L^2(K,m)$. Hence, $L^q(K,m) \subseteq L^2(K,m)$. Since K is compact, this implies $L^q(K,m) = L^2(K,m)$. This is impossible by Lemma 4.5.1 in [101]. □

Proposition 8.1.7. $\left\{f \in L^1(K,m) \cap L^\infty(K,m) : \ \hat{f} \geq 0\right\} \subseteq A^1(K,m)$.

Proof. Following the lines of Hewitt and Ross [77, 31.42], we obtained in Chapter 2, Theorem 1.3.8, for every function in $L^1(K,m) \cap L^\infty(K,m)$ such that $\hat{f}$ is a nonnegative function that $\hat{f} \in L^1(\mathcal{S},\pi)$ and $\left\|\hat{f}\right\|_1 \leq \|f\|_\infty$. □

8.2 Multipliers for $A^p(K,m)$

A continuous linear operator T on $A^p(K,m)$, i.e. $T \in B(A^p(K,m))$, is called **multiplier** for $A^p(K,m)$, if and only if T commutes with translations, i.e.

$$T \circ L_x = L_x \circ T$$

for all $x \in K$. We denote the set of all multipliers for $A^p(K,m)$ by $M(A^p(K,m))$.
Before we start with the first characterization we need to consider a short additional result. The mapping

$$\Psi_p : A^p(K,m) \to L^1(K,m) \times L^p(\mathcal{S},\pi), \quad \Psi_p(f) := (f,\hat{f}),$$

for each $f \in A^p(K,m)$ is obviously a linear isometry from $A^p(K,m)$ into the Banach space $L^1(K,m) \times L^p(\mathcal{S},\pi)$ with the sum norm $\|(f,g)\| := \|f\|_1 + \|g\|_p$. Thus, we may consider $A^p(K,m)$ as a closed subspace of $L^1(K,m) \times L^p(\mathcal{S},\pi)$. The dual space of $L^1(K,m) \times L^p(\mathcal{S},\pi)$ is isomorphic with $L^\infty(K,m) \times L^q(\mathcal{S},\pi)$, $1/p + 1/q = 1$. Thus, by an application of the Hahn-Banach theorem follows that every continuous linear functional F on $A^p(K,m)$ must be of the form

$$F(f) = \int_K f(x)\overline{g(x)}dm(x) + \int_{\hat{K}} \hat{f}(\alpha)\overline{h(\alpha)}d\pi(\alpha),$$

where $(g,h) \in L^\infty(K,m) \times L^q(\mathcal{S},\pi)$. However, the pair (g,h) corresponding to a given functional F may not be unique.

Theorem 8.2.1. *Let $1 \le p \le \infty$ and $T \in B(A^p(K,m))$. The following assertions are equivalent:*

i) $T \in M(A^p(K,m))$, i.e. $T \circ L_x = L_x \circ T$ for all $x \in K$.

ii) $T(f * g) = Tf * g = f * Tg$ for all $f,g \in A^p(K,m)$.

iii) There exists a unique $\varphi \in C^b(\mathcal{S})$ such that $(Tf)^\wedge|\mathcal{S} = \varphi\hat{f}|\mathcal{S}$ for all $f \in A^p(K,m)$ and $\|\varphi\|_\infty \le \|T\|$.

Proof. The prove of the equivalence of $i)$ to $ii)$ follows the lines of prove 3.1.2. However, we want to present a different way to prove the implication $i)$ to $ii)$ here as well. Suppose T is a continuous linear operator on $A^p(K,m)$ which commutes with translations. Let F be a continuous linear functional on $A^p(K,m)$. By the previous comment exist (g,h), $(g',h') \in L^\infty(K,m) \times L^q(\mathcal{S},\pi)$, $1/p + 1/q = 1$, such that for $f \in A^p(K,m)$ holds

$$F(f) = \int_K f(t)\overline{g(t)}dm(t) + \int_{\hat{K}} \hat{f}(\alpha)\overline{h(\alpha)}d\pi(\alpha)$$

$$F \circ T(f) = \int_K f(t)\overline{g'(t)}dm(t) + \int_{\hat{K}} \hat{f}(\alpha)\overline{h'(\alpha)}d\pi(\alpha).$$

Consequently for $f_1, f_2 \in A^p(K,m)$ we have

$$\begin{aligned}
F(Tf_1 * f_2) &= \int_K Tf_1 * f_2(t)\overline{g(t)}dm(t) + \int_{\hat{K}} (Tf_1 * f_2)^\wedge(\alpha)\overline{h(\alpha)}d\pi(\alpha) \\
&= \int_K \int_K L_{\tilde{x}} Tf_1(t) f_2(x) dm(x)\overline{g(t)}dm(t) + \int_{\hat{K}} (Tf_1)^\wedge(\alpha)\hat{f}_2(\alpha)\overline{h(\alpha)}d\pi(\alpha) \\
&= \int_K f_2(x) \int_K L_{\tilde{x}} Tf_1(t)\overline{g(t)}dm(t)dm(x) \\
&\quad + \int_K f_2(x) \int_{\hat{K}} (Tf_1)^\wedge(\alpha)\overline{\alpha(x)h(\alpha)}d\pi(\alpha)dm(x) \\
&= \int_K f_2(x) \left[\int_K T(L_{\tilde{x}} f_1)(t)\overline{g(t)}dm(t) + \int_{\hat{K}} (TL_{\tilde{x}} f_1)^\wedge(\alpha)\overline{h(\alpha)}d\pi(\alpha)\right] dm(x) \\
&= \int_K f_2(x) F(TL_{\tilde{x}} f_1) dm(x) \\
&= \int_K f_2(x) \left[\int_K L_{\tilde{x}} f_1(t)\overline{g'(t)}dm(t) + \int_{\hat{K}} (L_{\tilde{x}} f_1)^\wedge(\alpha)\overline{h'(\alpha)}d\pi(\alpha)\right] dm(x) \\
&= \int_K \int_K L_{\tilde{x}} f_1(t) f_2(x) dm(x)\overline{g'(t)}dm(t) + \int_{\hat{K}} \hat{f}_1(\alpha)\hat{f}_2(\alpha)\overline{h'(\alpha)}d\pi(\alpha) \\
&= \int_K (f_1 * f_2)(t)\overline{g'(t)}dm(t) + \int_{\hat{K}} (f_1 * f_2)^\wedge(\alpha)\overline{h'(\alpha)}d\pi(\alpha) \\
&= F \circ T(f_1 * f_2) = F(T(f_1 * f_2)).
\end{aligned}$$

Since this holds for every continuous linear functional F on $A^p(K,m)$, we obtain $T(f * g) = Tf * g$ for all $f, g \in A^p(K,m)$. Changing the roles of f and g implies assertion *ii*).
Since $A^p(K,m)$ is dense in $L^1(K,m)$, there exists for each $\alpha \in \mathcal{S}$ a function $f \in A^p(K,m)$ such that $\hat{f}(\alpha) \neq 0$. The rest of the proof follows the lines of the proof of 7.2.3. □

Proposition 8.2.2. *Let $1 \leq p < r \leq \infty$. Then $M(A^r(K,m)) \subseteq M(A^p(K,m))$.*

Proof. As mentioned above holds $A^p(K,m) \subseteq A^r(K,m)$. Thus, for each multiplier T in $M(A^r(K,m))$ and each function $f \in A^p(K,m)$ is $Tf \in A^r(K,m) \subset L^1(K,m)$ and further $(Tf)^\wedge = \varphi\hat{f}$ in $L^p(\mathcal{S},\pi)$. Hence, $Tf \in A^p(K,m)$. Furthermore, let $f \in A^p(K,m)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $A^p(K,m)$ such that $\lim_{n\to\infty} \|f - f_n\|^p = 0$ and $\lim_{n\to\infty} \|Tf_n - g\|^p = 0$ for some $g \in A^p(K,m)$. We obtain

$$(Tf)^\wedge(\alpha) = \varphi\hat{f}(\alpha) = \lim_{n\to\infty} \varphi\hat{f}_n(\alpha) = \lim_{n\to\infty} (Tf_n)^\wedge(\alpha) = \hat{g}(\alpha)$$

for all $\alpha \in \mathcal{S}$. By the uniqueness theorem of the Fourier transform follows $Tf = g$ in $A^p(K,m)$ and hence by the Closed Graph Theorem is $T|_{A^p(K,m)}$ continuous with respect to $\|.\|^p$. Thus,

$$M(A^r(K,m)) \subseteq M(A^p(K,m)).$$

□

Every bounded measure $\mu \in M(K)$ defines obviously a multiplier for $A^p(K,m)$, $1 \leq p \leq \infty$, by $Tf := \mu * f$. Conversely, it is an open question whether there exists a measure $\mu \in M(K)$ for each multiplier T for $A^p(K,m)$ such that $Tf = \mu * f$. However, we will prove the existence of a unique pseudomeasure $\sigma \in P(K)$ for each $T \in M(A^p(K,m))$, $1 \leq p < \infty$, such that $Tf = \sigma * f$ for $f \in A^2(K,m) \cap A^p(K,m)$. For an introduction into the theory of pseudomeasures see Chapter 3. Below we will discuss if the embedding of $M(A^p(K,m))$ into the set of pseudomeasures $P(K)$ is onto.

Theorem 8.2.3. *Let $1 \le p < \infty$. For $T \in M(A^p(K,m))$ there exists a unique pseudomeasure $\sigma \in P(K)$ such that*

$$Tf = \sigma * f,$$

for each $f \in A^2(K,m) \cap A^p(K,m)$. Moreover, the correspondence between T and σ defines a continuous algebra isomorphism from $M(A^p(K,m))$ into $P(K)$.

Proof. Let $f \in A^2(K,m) \cap A^p(K,m)$ and $T \in M(A^p(K,m))$. For $p < 2$ is $A^p(K,m) \subset A^2(K,m)$ and therefore $Tf \in A^p(K,m) \cap A^2(K,m)$. For $p > 2$ holds $A^2(K,m) \cap A^p(K,m) = A^2(K,m)$ and by $M(A^p(K,m)) \subset M(A^2(K,m))$ we have $Tf \in A^2(K,m)$. Thus, $Tf \in A^2(K,m) \cap A^p(K,m)$ for all $1 \le p < \infty$.
By Theorem 8.2.1 exists a unique $\varphi \in C^b(\mathcal{S})$ such that $(Tf)^\wedge|\mathcal{S} = \varphi \hat{f}|\mathcal{S}$ and $\|\varphi\|_\infty \le \|T\|$. We choose $\sigma \in P(K)$ sucht that $\Phi(\sigma) = \varphi$.
The convolution $\sigma * f$ of a pseudomeasure $\sigma \in P(K)$ and a function $f \in A^2(K,m) \cap A^p(K,m) \subset L^1(K,m) \cap L^2(K,m)$ is well defined and yields a pseudomeasure in $P(K)$. Thus, we obtain

$$\Phi(\sigma * f) = \Phi(\sigma)\wp(f) = \wp(Tf) \text{ in } L^p(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi) \cap L^\infty(\mathcal{S},\pi),$$

for all $f \in A^2(K,m) \cap A^p(K,m)$. Hence, we have $\sigma * f = Tf$ in $A^p(K,m)$ for all $f \in A^2(K,m) \cap A^p(K,m)$. Since Φ is isometric, we obtain $\|T\| \ge \|\varphi\|_\infty = \|\sigma\|$ and the correspondence between T and σ defines a continuous algebra isomorphism from $M(A^p(K,m))$ into $P(K)$. □

Remark 8.2.4. For $1 \le p \le 2$ we have $A^p(K,m) \subset L^1(K,m) \cap L^2(K,m)$ and we can omit the limitation $f \in A^p(K,m) \cap L^2(K,m)$ in Theorem 8.2.3.

Denote by $\mathcal{M}(A^p(K,m))$ the set of all $\varphi \in C^b(\mathcal{S})$ such that there exists a multiplier $T \in M(A^p(K,m))$ with $(Tf)^\wedge|\mathcal{S} = \varphi \hat{f}|\mathcal{S}$ for all $f \in A^p(K,m)$.

Corollary 8.2.5. *Let $1 \le p < \infty$. Then there exists an isometric algebra isomorphism from $\mathcal{M}(A^p(K,m))$ into $P(K)$.*

With a slight abuse of terminology we conclude

$$M(K) \subseteq M(A^p(K,m)) \subseteq P(K).$$

In the following, we will check if those inclusions are proper. We will prove for a compact hypergroup K and $1 \le p \le 2$ that the second inclusion is onto. In contrary, we will show the strictness of the second inclusion for a compact, infinite hypergroup K and $p > 2$. For a non-compact hypergroup K holds $C^b(\mathcal{S}) \subsetneq L^\infty(\mathcal{S},\pi)$. Hence, the second inclusion is proper.

Remark 8.2.6. For a locally compact, non-compact Abelian group G the sets $M(A^p(G))$ and $M(G)$ coincide. This was first proven in [103]. The prove quoted in [101, Chapter 6.3] is based on the fact that the translation on $L^1(G)$ is a linear isometry. However, in the case of commutative hypergoups the translation is in general only norm decreasing. Therefore, it is unclear whether $M(A^p(K,m))$ and $M(K)$ are isomorphic.

8.2.1 The Multipliers for $A^p(K,m)$ for a compact Hypergroup K

In this section we take a closer look on multipliers for $A^p(K,m)$ on a compact, infinite hypergroup K. For $p > 2$ we show that $M(A^p(K,m))$ is a proper subset of $P(K)$. However, we name a normed linear space whose dual is isomorphic to $M(A^p(K,m))$. Larsen [102] proved similar results for locally compact Abelian groups.

Proposition 8.2.7. *Let K be infinite and compact and 1$\le p \le 2$. There exists a homeomorphism between the spaces $M(A^p(K,m))$ and $P(K)$.*

Proof. We know from Chapter 3 that $l^\infty(\mathcal{S},\pi)$ is isometrically isomorphic to $P(K)$. Therefore, we choose a $\varphi \in l^\infty(\mathcal{S},\pi)$ and $f \in A^p(K,m)$. Then $\varphi\hat{f} \in l^p(\mathcal{S},\pi) \cap l^\infty(\mathcal{S},\pi)$ and by Hölder's interpolation theorem follows $\varphi\hat{f} \in l^2(\mathcal{S},\pi)$. By Plancherel's theorem there exists a function $Tf \in L^2(K,m)$ such that $(Tf)^\wedge = \varphi\hat{f}$. Thus, $Tf \in A^p(K,m)$. Moreover, by Hölder's interpolation theorem follows

$$\|Tf\|^p = \|Tf\|_1 + \|(Tf)^\wedge\|_p \le \|Tf\|_2 + \|\varphi\|_\infty \|\hat{f}\|_p \le \|\varphi\|_\infty (\|\hat{f}\|_2 + \|\hat{f}\|_p) \le 2\|\varphi\|_\infty \|f\|^p.$$

Hence, the equation $(Tf)^\wedge = \varphi\hat{f}$ defines a continuous operator on $A^p(K,m)$ and we obtain $\|T\| \le 2\|\varphi\|_\infty$. Thus, we have a continuous mapping from $l^\infty(\mathcal{S},\pi)$ into $M(A^p(K,m))$.
Conversely, let $T \in M(A^p(K,m))$, there exists by Theorem 8.2.1 a function $\varphi \in l^\infty(\mathcal{S},\pi)$ such that $(Tf)^\wedge = \varphi\hat{f}$ for each $f \in A^p(K,m)$ and $\|\varphi\|_\infty \le \|T\|$. □

Remark 8.2.8. For $1 \le p \le 2$ and K compact holds with a slight abuse of terminology

$$M(A^1(K,m)) = M(A^p(K,m)) = M(A^2(K,m)) \simeq l^\infty(\mathcal{S},\pi) \simeq P(K).$$

We want to point out that $M(A^p(K,m))$ is in general not isometric to $l^\infty(\mathcal{S},\pi)$. However, as shown in the prove above we obtain

$$\|\varphi\|_\infty \le \|T\| \le 2\,\|\varphi\|_\infty\,.$$

We will now take a closer look on the situation for $2 < p$. We obtain for $2 < p < \infty$, $1/p + 1/q = 1$, the following inclusions (mostly by applying the Hausdorff-Young Theorem)

$$A^1(K,m) \subseteq A^q(K,m) \subset L^p(K,m) \subset L^2(K,m) = A^2(K,m) \subseteq L^q(K,m) \subset A^p(K,m).$$

To prove the next proposition we will use the Derived space $A^p(K,m)_0$ of $A^p(K,m)$, denoted by

$$A^p(K,m)_0 := \left\{ f \in A^p(K,m) : \varphi\hat{f} \in A^p(K,m)^\wedge \;\forall\, \varphi \in C_0(\mathcal{S}) \right\}.$$

Note that for a compact hypergroup K holds for $1 \le p \le 2$,

$$A^p(K,m) = A^p(K,m)_0 \subset L^2(K,m)$$

and for $2 \le p$

$$L^2(K,m) = A^p(K,m)_0.$$

Indeed, for $1 \le p \le 2$ we have $A^p(K,m) \subset A^2(K,m) = L^2(K,m)$. Let $f \in A^p(K,m)$ and $\varphi \in C_0(\mathcal{S})$. We obtain $\varphi\hat{f} \in L^p(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Consequently, by the Plancherel theorem there exists $g \in L^2(K,m)$ such that $\wp(g) = \varphi\hat{f}$. Since $g \in L^2(K,m) \subset L^1(K,m)$ and $\wp(g) \in L^p(\mathcal{S},\pi)$, we have $g \in A^p(K,m)$. Thus, $f \in A^p(K,m)_0$ and $A^p(K,m) = A^p(K,m)_0$ for $1 \le p \le 2$.
For $p > 2$ holds $A^p(K,m)_0 \subseteq L^1(K,m)_0 = L^2(K,m)$ by Chapter 6. Further, $L^2(K,m) = A^2(K,m)_0 \subset A^p(K,m)_0$. Thus, $L^2(K,m) = A^p(K,m)_0$.

Proposition 8.2.9. *Let K be infinite and compact. For $2 < p$ exists $\varphi \in C_0(\mathcal{S})$ such that $\varphi \notin \mathcal{M}(A^p(K,m))$.*

Proof. Suppose $\varphi A^p(K,m)^\wedge \subseteq A^p(K,m)^\wedge$ for each $\varphi \in C_0(\mathcal{S})$, then we have by the remarks above $A^p(K,m) = A^p(K,m)_0 = L^2(K,m) = A^2(K,m)$. This contradicts Proposition 8.1.6. □

Corollary 8.2.10. *Let K be compact and infinite. For $2 < p$ holds $M(A^p(K,m)) \subsetneq P(K)$.*

Corollary 8.2.11. *Let K be compact and infinite. For $1 \le r \le 2 < p$ holds*

$$M(A^p(K,m)) \subsetneq M(A^r(K,m)).$$

Now we construct a certain Banach space of continuous functions on K whose dual is continuously isomorphic to $M(A^p(K,m))$. However, the isomorphism involved is not an isometry. Fix $p > 2$. For $T \in M(A^p(K,m))$ and $\varphi \in l^\infty(\mathcal{S},\pi)$ such that $(Tf)^\wedge = \varphi\hat{f}$ in $l^p(\mathcal{S},\pi)$ for each $f \in A^p(K,m)$, we set

$$\beta(T)(f) := \int_{\mathcal{S}} (Tf)^\wedge(\alpha) d\pi(\alpha) = \int_{\mathcal{S}} \varphi(\alpha)\hat{f}(\alpha) d\pi(\alpha)$$

and define

$$\|f\|_B := \sup\left\{|\beta(T)(f)| : T \in M(A^p(K,m)), \|T\| \leq 1\right\},$$

for all $f \in A^1(K,m)$. It is evident that these definitions make sense as each $T \in M(A^p(K,m))$ is also an element in $M(A^1(K,m))$. $\| \, \|_B$ is obviously a norm on the linear space $A^1(K,m)$ and we shall denote $A^1(K,m)$ with this norm by $B_p(K,m)$. The preceding definitions also show for each $T \in M(A^p(K,m))$ that $\beta(T)$ defines a continuous linear functional on the normed linear space $B_p(K,m)$. Thus, we obtain a mapping $\beta : M(A^p(K,m)) \to B_p(K,m)^*$. Following the lines of Larsen [101, Chapter 6.4] we obtain the next theorem. The theory in [101] is for Abelian groups, but without further issues it can be extended to hypergroups.

Theorem 8.2.12. *Let K be infinite and compact. For each $2 < p < \infty$ is the mapping $\beta : M(A^p(K,m)) \to B_p(K,m)^*$ defined by*

$$\beta(T)(f) = \int_{\mathcal{S}} (Tf)^\wedge(\alpha) d\pi(\alpha),$$

for $f \in B_p(K,m)$, a continuous linear isomorphism of $M(A^p(K,m))$ onto $B_p(K,m)^*$. Moreover, $\|\beta(T)\|_{B^*} \leq \|T\|^p \leq 2\|\beta(T)\|_{B^*}$.

We denote by $\bar{B}_p(K,m)$ the completion of the normed linear space $B_p(K,m)$. Then the dual of $\bar{B}_p(K,m)$ is obviously the same as the dual of $B_p(K,m)$. In particular, the preceding theorem establishes the existence of a continuous linear isomorphism between $M(A^p(K,m))$ and $\bar{B}_p(K,m)^*$. Moreover, for a compact hypergroup K follows from the Fourier inversions theorem that $A^1(K,m) = l^1(\mathcal{S},\pi)^\vee$ as linear spaces. In particular we can assume $A^1(K,m)$ as a norm dense subset in $C_0(K)$. Hence, we have

$$B_p(K,m) = A^1(K,m) = l^1(\mathcal{S},\pi)^\vee \subset C(K).$$

$\bar{B}_p(K,m)$ may also be consider as a linear subspace of $C(K)$.

Theorem 8.2.13. *Let K be infinite and compact. For all $p > 2$ exists a continuous, linear, injective mapping τ of $\bar{B}_p(K,m)$ onto a subspace of $C(K)$.*

Proof. For $f \in B_p(K,m)$ is by the Fourier inversions theorem $(\hat{f})^\vee = f$ in $L^1(K,m)$. Thus, for each $x \in K$ follows

$$\begin{aligned} |f(x)| = |(\hat{f})^\vee(x)| &= \left|\int_{\mathcal{S}} \alpha(x)\hat{f}(\alpha) d\pi(\alpha)\right| \\ &= \left|\int_{\mathcal{S}} (L_x f)^\wedge(\alpha) d\pi(\alpha)\right| = |\beta(L_x)(f)| \\ &\leq \sup\left\{|\beta(T)(f)| : T \in M(A^p(K,m)), \|T\| \leq 1\right\} = \|f\|_B . \end{aligned}$$

Hence, $\|f\|_\infty = \left\|(\hat{f})^\vee\right\|_\infty \leq \|f\|_B$ for all $f \in B_p(K,m)$.

We consider the elements of $\bar{B}_p(K,m)$ as Cauchy sequences of elements of $B_p(K,m)$. Hence, by the preceding inequality follows for each Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $B_p(K,m)$, that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(K)$. This leads to the existence of a unique function $f \in C(K)$ such

that $\lim_n \|f_n - f\|_\infty \to 0$ as n tends to infinity.
We define a linear mapping from $\bar{B}_p(K,m)$ onto a subspace of $C(K)$ by

$$\tau((f_n)_{n\in\mathbb{N}}) := f.$$

It is apparent by the previous inequality that τ is a continuous linear mapping.
We want to prove that τ is also injective. Here, it is sufficient to show for a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $B_p(K,m)$ such that $\lim_n \|f_n\|_\infty = 0$ follows $\lim_n \|f_n\|_B = 0$. Now if $T \in M(A^p(K,m))$, we have by Theorem 8.2.12 that $\beta(T) \in B_p(K,m)^*$. Moreover, the inequality

$$|\beta(T)(f_n) - \beta(T)(f_m)| \le \|\beta(T)\|_{B^*} \|f_n - f_m\|_B$$

shows that $(\beta(T)(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence of numbers. Define

$$G(T) := \lim_n \beta(T)(f_n).$$

We claim that $G(T) = 0$ for each $T \in M(A^p(K,m))$ and hence $(f_n)_{n\in\mathbb{N}}$ converges weakly to zero. Indeed, given $g \in L^1(K,m)$ we denote by T_g the multiplier for $A^p(K,m)$ defined by $T_g f := g * f$ for all $f \in A^p(K,m)$. Since $(f_n)_{n\in\mathbb{N}}$ is a subset in $A^1(K,m)$, we may apply the Fourier inversions theorem to deduce that for all $n \in \mathbb{N}$

$$|\beta(T_g)(f_n)| = |\int_{\mathcal{S}} (T_g f_n)^\wedge(\alpha) d\pi(\alpha)| = |\int_{\mathcal{S}} (g * f_n)^\wedge(\alpha) d\pi(\alpha)| = |((g * f_n)^\wedge)^\vee(e)| \le \|g\|_1 \|f_n\|_\infty .$$

Hence, $G(T_g) = 0$ for all $g \in L^1(K,m)$.
Furthermore, suppose $(k_i)_{i\in I} \subset A^1(K,m)$ is an approximate identity for $A^1(K,m)$ such that $\lim_i \|k_i\|^1 \to 1$. If $T \in M(A^p(K,m)) \subset M(A^1(K,m))$, we see that L_{Tk_i} converges to T in the strong operator topology on $M(A^1(K,m))$. We obtain for all $f \in B_p(K,m)$ and each $i \in I$

$$\begin{aligned} |\beta(T)(f) - \beta(L_{Tk_i})(f)| &= |\int_{\mathcal{S}} (Tf)^\wedge(\alpha) d\pi(\alpha) - \int_{\mathcal{S}} (L_{Tk_i} f)^\wedge(\alpha) d\pi(\alpha)| \\ &\le \|[(T - L_{Tk_i})(f)]^\wedge\|_1 \le \|(T - L_{Tk_i})(f)\|^1 . \end{aligned}$$

Consequently holds $\lim_i \beta(L_{Tk_i})(f) = \beta(T)(f)$ for all $f \in B_p(K,m)$. Suppose $T \in M(A^p(K,m))$ and $\epsilon > 0$. Then, since $G(L_{Tk_i}) = 0$ for all $i \in I$ we have

$$\begin{aligned} |G(T)| &= |G(T) - G(L_{Tk_i})| \\ &\le |G(T) - \beta(T)(f_n)| + |\beta(T)(f_n) - \beta(L_{Tk_i})(f_n)| + |\beta(L_{Tk_i})(f_n) - G(L_{tk_i})| \\ &\le |G(T) - \beta(T)(f_n)| + \|(T - L_{Tk_i}) f_n\|^1 + \|f_n\|_\infty \|Tk_i\|_1 . \end{aligned}$$

Since $\lim_n \beta(T)(f_n) = G(T)$ and $\lim_n \|f_n\|_\infty = 0$, exists an integer $N \in \mathbb{N}$ such that

$$|\beta(T)(f_N) - G(T)| \le \epsilon/3 \text{ and } \|f_N\|_\infty \le \epsilon/3.$$

For this N choose $j \in I$ such that

$$\left\|(T - L_{Tk_j})(f_N)\right\|^1 \le \epsilon/3.$$

We conclude $|G(T)| < \epsilon$. Since $\epsilon > 0$ was chosen arbitrary, we obtain $G(T) = 0$ for all $T \in M(A^p(K,m))$ and $(f_n)_{n\in\mathbb{N}}$ converges weakly to zero.
Let $\epsilon > 0$ and for all $n \in N$ choose $T_n \in M(A^p(K,m))$ such that $\|T_n\| \le 1$ and $\|f_n\|_B < |\beta(T_n)(f_n)| + \epsilon/3$. This is possible by the definition of $\| \ \|_B$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B_p(K,m)$, there exists an integer $N \in \mathbb{N}$ such that for all $m, n \ge N$ holds $\|f_n - f_m\|_B < \epsilon/3$. In particular, since $\|\beta(T_m)\|_{B^*} \le \|T_m\|$ for all $m > N$, we have

$$\begin{aligned} \|f_N\|_B &< |\beta(T_N)(f_N)| + \epsilon/3 \le |\beta(T_N)(f_N - f_m)| + |\beta(T_N)(f_m)| + \epsilon/3 \\ &\le \|T_N\| \|f_N - f_m\|_B + |\beta(T_N)(f_m)| + \epsilon/3 \\ &\le 2\epsilon/3 + |\beta(T_N)(f_m)|. \end{aligned}$$

However, since $(f_m)_{m>N}$ converges weakly to zero, we see that $\lim_m |\beta(T_N)(f_m)| = 0$. Hence, $\|f_N\|_B \leq 2\epsilon/3$. Thus, $\lim_n \|f_n\|_B = 0$. Hence, τ is injective. □

We can summarize the previous theorems in the next result.

Theorem 8.2.14. *Let K be infinite and compact. For all $2 < p$ exists a continuous linear isomorphism of $M(A^p(K,m))$ onto the dual space of a Banach space of continuous functions.*

Remark 8.2.15. We want to remark, that the norm in the Banach space of continuous functions in Theorem 8.2.14 is not the sup-norm.

Theorem 8.2.16. *Let K be infinite and compact and $p > 2$. The space of finite linear combinations of functions in $\{\beta(L_x) : x \in K\}$ is weak* dense in $B_p(K,m)^*$.*

Proof. Suppose $f \in B_p(K,m)$ and $\beta(L_x)(f) = 0$ for all $x \in K$. Then for each $x \in K$ is

$$0 = \beta(L_x)(f) = \int_{\mathcal{S}} (L_x f)^{\wedge}(\alpha) d\pi(\alpha) = \int_{\mathcal{S}} \hat{f}(\alpha)\bar{\alpha}(x) d\pi(\alpha) = (\hat{f})^{\vee}.$$

By the Fourier Inversions theorem is $f = 0$ and $\beta(T)(f) = 0$ for all $T \in M(A^p(K,m))$. Consequently, every weak* continuous linear functional which vanishes on $\{\beta(L_x) : x \in K\}$, vanishes on all of $B_p(K,m)^*$. Thus, we conclude that the space of finite linear combinations of functions in $\{\beta(L_x) : x \in K\}$ is weak* dense in $B_p(K,m)^*$. □

Remark 8.2.17. Theorem 8.2.16 is an analogue for $M(A^p(K,m))$ to Theorem 3.1.9 in Chapter 3.

8.3 Multipliers for $A^p(\mathcal{S},\pi)$

We investigate briefly multipliers for

$$A^p(\mathcal{S},\pi) := \left\{\varphi \in L^1(\mathcal{S},\pi) : \check{\varphi} \in L^p(K,m)\right\}, \qquad 1 \leq p \leq \infty.$$

In the last chapter we characterized multipliers for $A^p(\mathcal{S},\pi)$ on the Jacobi hypergroup $\mathcal{S} = [-1,1]$, which is a strong hypergroup. Characterizing the multipliers for $A^p(\mathcal{S},\pi)$ for an arbitrary or discrete hypergroup K is more difficult, since we cannot use argumentations relaying on convolutions or translations. However, we will give a short impression.

Theorem 8.3.1. *$A^p(\mathcal{S},\pi)$ is with norm $\|\varphi\|^p := \|\varphi\|_1 + \|\check{\varphi}\|_p$ a Banach space.*

Proof. Follows the lines of the first part of proof 8.1.1. □

For a compact hypergroup K holds $A^p(\mathcal{S},\pi) = L^1(\mathcal{S},\pi)$. This case was investigated in Chapter 4. Thus, we will omit that case here.
Moreover, by Hölder's interpolation theorem and an application of the Hausdorff-Young theorem holds

$$\begin{aligned} A^p(\mathcal{S},\pi) &\subseteq A^q(\mathcal{S},\pi), \quad \text{for} \quad 1 \leq p \leq q \leq \infty, \text{ and} \\ A^p(\mathcal{S},\pi) &\subseteq L^2(\mathcal{S},\pi), \quad \text{for} \quad p \leq 2. \end{aligned}$$

Whenever $\mathcal{S}$ is infinite and compact, we show that the first inclusion is proper.

Proposition 8.3.2. *Let $\mathcal{S}$ be infinite and compact. Then*

$$\begin{aligned} A^p(\mathcal{S},\pi) &\subsetneq A^q(\mathcal{S},\pi), \quad \textit{for} \quad 1 \leq p < q \leq 2, \textit{ and} \\ L^2(\mathcal{S},\pi) = A^2(\mathcal{S},\pi) &\subsetneq A^p(\mathcal{S},\pi), \quad \textit{for} \quad 2 < p. \end{aligned}$$

Proof. Follows the lines of proof 8.1.6. □

We obtain the analogue to Lemma 8.1.3. Gaudry [53, Theorem 7.1] proved similar results for locally compact groups.

Lemma 8.3.3. *The set* $\{\psi \in L^1(\mathcal{S},\pi) : \check{\psi} \in C_c(K)\}$ *is dense in* $L^1(\mathcal{S},\pi)$ *with respect to* $\|\ \|_1$.

Proof. First note that the set of functions in $L^2(\mathcal{S},\pi)$ with inverse Plancherel transform in $C_c(K)$ is dense in $L^2(\mathcal{S},\pi)$. This is obvious from Plancherel's theorem. Therefore, the set of functions in $L^1(\mathcal{S},\pi)$ with inverse transform in $C_c(K)$ is dense in $L^1(\mathcal{S},\pi)$. Indeed, given $\varphi \in L^1(\mathcal{S},\pi)$, write $\varphi = \psi \cdot \gamma$ with $\psi,\gamma \in L^2(\mathcal{S},\pi)$. Now approximate ψ and γ in $L^2(\mathcal{S},\pi)$ by functions ψ_i,γ_i with $\check{\psi}_i,\check{\gamma}_i \in C_c(K)$. Then φ is approximated in $L^1(\mathcal{S},\pi)$ by $\psi_i \cdot \gamma_i$ and $(\psi_i \cdot \gamma_i)^\vee = \check{\psi}_i * \check{\gamma}_i \in C_c(K)$. Hence, $\{\psi \in L^1(\mathcal{S},\pi) : \check{\psi} \in C_c(K)\}$ is dense in $L^1(\mathcal{S},\pi)$. □

Corollary 8.3.4. *Let* $1 \le p \le \infty$. $A^p(\mathcal{S},\pi)$ *is* $\|.\|_1$*-norm dense in* $L^1(\mathcal{S},\pi)$.

Remark 8.3.5. By the uniqueness theorem of inverse Fourier transforms and by the Hausdorff-Young Theorem holds for $\mathcal{S}$ being compact and infinite and for each $2 < p < \infty$, $1/p + 1/q = 1$, that

$$A^1(\mathcal{S},\pi) \subseteq A^q(\mathcal{S},\pi) \subseteq L^p(\mathcal{S},\pi) \subset L^2(\mathcal{S},\pi) = A^2(\mathcal{S},\pi) \subseteq L^q(\mathcal{S},\pi) \subseteq A^p(\mathcal{S},\pi) \subseteq L^1(\mathcal{S},\pi).$$

Furthermore, we have $A^1(\mathcal{S},\pi) = L^1(K,m)^\wedge$ as linear spaces.

Multipliers for $A^p(\mathcal{S},\pi)$

We call a bounded function $f \in C^b(K)$ **multiplier** for $A^p(\mathcal{S},\pi)$, if and only if

$$f\check{\varphi} \in A^p(\mathcal{S},\pi)^\vee = L^1(\mathcal{S},\pi)^\vee \cap L^p(K,m)$$

for each $\varphi \in A^p(\mathcal{S},\pi)$, $1 \le p < \infty$.
We can define a continuous linear mapping $T : A^p(\mathcal{S},\pi) \to A^p(\mathcal{S},\pi)$ by $(T\varphi)^\vee := f\check{\varphi}$. The set of all multiplier operators T on $A^p(\mathcal{S},\pi)$ is denoted by $M(A^p(\mathcal{S},\pi))$.
For $1 \le p \le q \le \infty$ holds $M(L^1(\mathcal{S},\pi)) \subseteq M(A^q(\mathcal{S},\pi)) \subseteq M(A^p(\mathcal{S},\pi))$ using straight forward argumentation.
Moreover, for K discrete and hence $\mathcal{S}$ compact each function in $A^1(\mathcal{S},\pi)^\vee$ defines a multipler for $A^p(\mathcal{S},\pi)$, $1 \le p < \infty$. Indeed, let $f \in A^1(\mathcal{S},\pi)$. Then $\check{f}\check{\varphi} \in L^1(K,m) \cap L^p(K,m)$ and therefore $(\check{f}\check{\varphi})^\wedge \in C_0(\mathcal{S}) \subset L^1(\mathcal{S},\pi)$. Hence, $\check{f}\check{\varphi} \in A^p(\mathcal{S},\pi)^\vee$.

Theorem 8.3.6. *Let* $1 \le p < \infty$. *For each* $T \in M(A^p(\mathcal{S},\pi))$ *exists a unique pseudomeasure* $s \in P(\hat{K})$ *such that*

$$T\varphi = s * f$$

for each $\varphi \in A^1(\mathcal{S},\pi)$. *Moreover, the correspondence between* T *and* s *defines a continuous algebra isomorphism from* $M(A^p(\mathcal{S},\pi))$ *into* $P(\hat{K})$.

Proof. Let $T \in M(A^p(\mathcal{S},\pi))$ and $f \in C^b(K)$ such that $(T\varphi)^\vee = f\check{\varphi} \in A^p(\mathcal{S},\pi)^\vee$ for all $\varphi \in A^p(\mathcal{S},\pi)$. By Chapter 4 exists an isometric isomorphism $\Phi : P(\hat{K}) \to L^\infty(K,m)$ and hence a unique pseudomeasure $s \in P(\hat{K})$ such that $\Phi(s) = f$. The convolution $s * \varphi = \Phi^{-1}(\Phi(s)\check{\varphi})$ of a pseudomeasure $s \in P(\hat{K})$ and an element $\varphi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ is well-defined as a convolution of pseudomeasures. Further, by an application of the Hausdorff-Young Theorem is $A^1(\mathcal{S},\pi) \subset L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$. Thus,

$$\Phi(s * \varphi) = \Phi(s)\check{\varphi} = (T\varphi)^\vee,$$

for all $\varphi \in A^1(\mathcal{S},\pi)$. Hence, $\Phi(s * \varphi) \in L^1(K,m) \cap C_0(K)$. The rest of the proof follows by Hölder's interpolation Theorem and the arguments used in Chapter 4. □

Even though the space $L^1(\mathcal{S},\pi)$ admits in general no convolution, we can still investigate a characterization for multipliers in $M(A^p(\mathcal{S},\pi))$ analogue to Theorem 8.2.1.

Corollary 8.3.7. *Let $T \in M(A^p(\mathcal{S},\pi))$ and $\varphi,\psi \in A^1(\mathcal{S},\pi)$. Then $\varphi * T\psi = T\varphi * \psi$ as a convolution of pseudomeasures.*

Proof. By $(T\varphi)^\vee = f\check{\varphi} \in L^1(K,m) \cap C_0(K)$ holds $T\varphi \in A^1(\mathcal{S},\pi)$ and the convolution $T\varphi * \psi$ is well-defined. Further,

$$T\varphi * \psi = \Phi^{-1}(\Phi(T\varphi)\Phi(\psi)) = \Phi^{-1}(f\check{\varphi}\check{\psi}) = \varphi * T\psi.$$

□

In the following let $\mathcal{S}$ be compact and infinite. Obviously holds $A^2(\mathcal{S},\pi) = L^2(\mathcal{S},\pi)$. Furthermore, by Proposition 8.3.2 is $A^p(\mathcal{S},\pi)$ a $\| \ \|^q$-dense subset in $A^q(\mathcal{S},\pi)$.

Theorem 8.3.8. *Let $\mathcal{S}$ be compact and infinite and $1 \le p \le 2$. There exists a homeomorphism between the spaces $M(A^p(\mathcal{S},\pi))$ and $P(\hat{K})$. Moreover, $\|s\|_P \le \|T\| \le 2\|s\|_P$.*

Proof. The proof follows the lines of proof 8.2.7. □

Remark 8.3.9. If $K = \hat{\hat{K}}$ we call the hypergroup pontryagin. In general this is not the case. Hence, K is in general not an orthogonal basis for $L^2(\mathcal{S},\pi)$ and the prove of Theorem 8.2.12 cannot be transferred to the dual situation.

Chapter 9

Multipliers for almost-convergent Sequences

Lorentz [117] formulated the theory of almost convergence for bounded, complex sequences. His concept of almost convergence was then studied in the context of amenable semigroups, see for instance [11, 17, 25]. Lasser extended this theory to polynomial hypergroups, see [111].
In this chapter we want to introduce multipliers for the set of all almost convergent sequences, i. e. AC, in the context of polynomial hypergroups. We will give six equivalent characterizations of multipliers for AC.

9.1 Almost-convergent Sequences

In the following let $K = \mathbb{N}_0$ denote a polynomial hypergroup which is generated by the orthogonal polynomials $(R_n(x))_{n\in\mathbb{N}_0}$. We denote the set of all invariant means (with respect to $(R_n(x))_{n\in\mathbb{N}_0}$) by

$$\mathbb{M} := \{\mu \in l^\infty(\mathbb{N}_0)^* : \ \mu(1) = 1, \ \mu \geq 0, \text{ and } \mu(L_m f) = \mu(f) \text{ for all } m \in \mathbb{N}_0\}.$$

$\mathbb{M}$ is nonempty, see [147]. Further denote

$$P_1(h) := \left\{\varphi \in l^1(\mathbb{N}_0, h) : \varphi \geq 0, \ \|\varphi\|_1 = 1\right\}.$$

$P_1(h)$ is weak-*-dense in the set of all means, see [128].

Definition 9.1.1. A sequence $f \in l^\infty(\mathbb{N}_0)$ is called **almost convergent** to a constant $d(f)$, if

$$\mu(f) = d(f) \text{ for all } \mu \in \mathbb{M}.$$

We will denote the set of all almost convergent sequences in $l^\infty(\mathbb{N}_0)$ by AC. The subset of all almost convergent sequences in $l^\infty(\mathbb{N}_0)$ such that $\mu(f) = 0$ for all $\mu \in \mathbb{M}$ is denoted by AC_0.

Lasser [111] proved that AC_0 equals the closed linear span of $\{a - L_n a : a \in l^\infty(\mathbb{N}_0), \ n \in \mathbb{N}\}$. Furthermore, he verified that

$$AC = \mathbb{C}1 \oplus AC_0.$$

L. Kerchy [96] defined a stronger form of almost convergence for bounded complex sequences. We extend his notations to the context of polynomial hypergroups.

Definition 9.1.2. A sequence $(f(n))_{n\in\mathbb{N}_0} \in l^\infty(\mathbb{N}_0)$ is **strongly almost convergent** to $d(f) \in \mathbb{C}$, if

$$\mu(|f - d(f)1|) = 0 \text{ for all } \mu \in \mathbb{M}.$$

Here 1 stands for the constant sequence $(1, 1, 1, ...)$. We denote the set of all strongly almost convergent sequences in $l^\infty(\mathbb{N}_0)$ by AC_s.

Definition 9.1.3. A sequence $(f(n))_{n\in\mathbb{N}_0} \in l^\infty(\mathbb{N}_0)$ is $\mathbb{N}-$**convergent** to a constant $d(f) \in \mathbb{C}$, if for each $\epsilon > 0$ there exists a set $A \subseteq \mathbb{N}$ such that $\mu(\chi_A) = 0$ for all $\mu \in \mathbb{M}$ and $|f(n)-d(f)| < \epsilon$ for all $n \in \mathbb{N}\backslash A$.

Remark 9.1.4. Kerchy's definition of strongly almost convergence obviously implies $\mathbb{N}$-convergence. Conversely, let $f \in l^\infty(\mathbb{N}_0)$ be $\mathbb{N}-$convergent to $d(f) \in \mathbb{C}$. For each $\epsilon > 0$ we obtain $\mu(|f - d(f)1|) \leq \mu(\chi_{\mathbb{N}\backslash A})\epsilon = \epsilon$. Hence, $\mu(|f - d(f)1|) = 0$ and $f \in AC_s$.

Remark 9.1.5. Each convergent sequence $(f_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}_0)$ is obviously strongly almost convergent, since we conclude by Example 2 in [111] for each finite set $A \in \mathbb{N}$ that $\chi_A \in AC_0$. The converse implication is in general false.

9.2 Multipliers for almost-convergent Sequences

Definition 9.2.1. A function $f \in l^\infty(\mathbb{N}_0)$ such that $f \cdot AC \subset AC$ is called **multiplier** for AC. We denote the set of all multipliers for AC by $M(AC)$.

Chou and Duran, [10] showed results for multipliers for AC in the context of semigroups. Inspired by Chou and Duran, we will study the space $M(AC)$. Our main result in this chapter is the equivalence

$$M(AC) = AC^0 \oplus \mathbb{C}1,$$

where we denote $AC^0 := \{f \in l^\infty(\mathbb{N}_0) : |f| \in AC_0\}$.

Theorem 9.2.2. *Let K be a polynomial hypergroup and $f \in l^\infty(\mathbb{N}_0)$. f is a multiplier for AC, i.e. $f \cdot AC \subset AC$, if and only if $f \in AC$ and*

$$\mu(f \cdot g) = \mu(f)\mu(g) = d(f)d(g)$$

for all $g \in AC$ and $\mu \in \mathbb{M}$.

Proof. Each function $f \in AC$, which fullfils $\mu(f \cdot g) = \mu(f)\mu(g) = d(f)d(g)$ for all $g \in AC$ and $\mu \in \mathbb{M}$, is obviously a multiplier for AC.
Conversely, let $f \in M(AC)$. Since $1 \in AC$ we have $f = f \cdot 1 \in AC$. To prove for every $g \in AC$ and $\mu \in \mathbb{M}$ that $\mu(f \cdot g) = \mu(f)\mu(g)$, we consider $l^1(\mathbb{N}_0, h)^* = l^\infty(\mathbb{N}_0)$ and $l^1(\mathbb{N}_0, h)^{**} = l^\infty(\mathbb{N}_0)^*$. If $\psi \in l^1(\mathbb{N}_0, h)$, $a \in l^\infty(\mathbb{N}_0)$, then

$$a(\psi) = \psi(a) = \sum_{k=0}^{\infty} \psi(k)a(k)h(k).$$

Since $P_1(h)$ is weak-*-dense in the set of all means, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $P_1(h)$ such that $\lim_n(\varphi_n(a) - \varphi_n(L_m a)) = 0$ for each $a \in l^\infty(\mathbb{N}_0)$ and $m \in \mathbb{N}$. Obviously, it holds for each $g \in AC$ that

$$\lim_n \varphi_n(g) = d(g). \tag{1}$$

Now let $k \in \mathbb{N}$ be fixed. Set $\psi_n = \varphi_n f - L_k(\varphi_n f)$ for each $n \in \mathbb{N}_0$. Then $(\psi_n)_{n\in\mathbb{N}_0}$ is a sequence in $l^1(\mathbb{N}_0, h) \subset l^\infty(\mathbb{N}_0)^*$. We claim that ψ_n is a weak Cauchy sequence in $l^1(\mathbb{N}_0, h)$. Indeed, for $a \in l^\infty(\mathbb{N}_0)$ we obtain

$$\begin{aligned}
\psi_n(a) &= (\varphi_n f - L_k(\varphi_n f))(a) \\
&= \sum_{j=0}^{\infty}[\varphi_n(j)f(j)a(j)h(j) - L_k(\varphi_n f)(j)a(j)h(j)] \\
&= \sum_{j=0}^{\infty}[\varphi_n(j)f(j)a(j)h(j) - \varphi_n(j)f(j)L_k a(j)h(j)] \\
&= \sum_{j=0}^{\infty} \varphi_n(j)f(j)(a - L_k a)(j)h(j) = \varphi_n(f(a - L_k a)).
\end{aligned}$$

Since f is a multiplier for AC and $(a - L_k a) \in AC$, we have $f(a - L_k a) \in AC$. Hence, by equation (1) we find that

$$\lim_n \psi_n(a) = \lim_n \varphi_n(f(a - L_k a)) = d(f(a - L_k a)) \tag{2}$$

for all $a \in l^\infty(\mathbb{N}_0)$. Therefore, $\lim_n \psi_n(a)$ exists for all $a \in l^\infty(\mathbb{N}_0)$ and ψ_n is a weak Cauchy sequence as we claimed.
Since $l^1(\mathbb{N}_0, h)$ is weakly sequentially complete, see [26, Theorem IV 8.6] there exists $\psi \in l^1(\mathbb{N}_0, h)$ such that $\psi = \lim_n \psi_n$ in the weak topology. Certainly, for every point measure $\varepsilon_t \in l^\infty(\mathbb{N}_0)$, $t \in \mathbb{N}_0$, holds

$$\psi(t)h(t) = \psi(\varepsilon_t) = \lim_n \psi_n(\varepsilon_t) = \lim_n \psi_n(t)h(t).$$

Since $h(t) \neq 0$ for all $t \in \mathbb{N}_0$, we have $\psi(t) = \lim_n \psi_n(t)$ for every $t \in \mathbb{N}_0$. On the other hand by [111, Example 2] we have

$$\lim_n \varphi_n(t)h(t) = \lim_n \varphi_n(\varepsilon_t) = 0.$$

Hence, $\lim_n \varphi_n(t) = 0$ and

$$|\psi(t)| = |\lim_n (\varphi_n(t)f(t) - L_k(\varphi_n f)(t))| \leq |\lim_n L_k(\varphi_n f)(t)| = |\lim_n \sum_{j=|k-t|}^{k+t} g(k,t;j)\varphi_n(j)f(j)| = 0.$$

We obtain $\psi \equiv 0$. By (2) follows for all $\mu \in \mathbb{M}$

$$\mu(f(a - L_k a)) = d(f(a - L_k a)) = \lim_n \psi_n(a) = 0 = \mu(f)\mu(a - L_k a).$$

Since AC_0 equals the closed linear span of $\{a - L_k a : a \in l^\infty(\mathbb{N}_0),\ k \in \mathbb{N}\}$, the statement follows by $AC = \mathbb{C}1 \oplus AC_0$, see [111, Theorem 2]. □

Theorem 9.2.3. *Let K be a polynomial hypergroup and $f \in l^\infty(\mathbb{N}_0)$. f is a multiplier for AC, i.e. $f \cdot AC \subset AC$, if and only if $f \in AC^0 \oplus \mathbb{C}1$.*

Proof. Let $f \in AC^0 \oplus \mathbb{C}1$. Then there exists a constant $d(f) \in \mathbb{C}1$ and a function $f_0 \in AC^0$ such that $f = f_0 + d(f)$. Given an arbitrary function $g \in AC$ and a mean $\mu \in \mathbb{M}$ we have

$$|\mu(f_0 g)| \leq \mu(|f_0 g|) \leq \|g\|_\infty \mu(|f_0|) = 0.$$

Hence, we obtain $\mu(f_0 g) = 0$ and

$$\mu(fg) = \mu((f_0 + d(f))g) = d(f)\mu(g) = d(f)d(g) \in \mathbb{C}.$$

By Theorem 9.2.2 holds $fg \in AC$.
Conversely, let $f \in l^\infty(\mathbb{N}_0)$ such that $f \cdot AC \subset AC$. Since $1 \in AC$ we have $f = f \cdot 1 \in AC$. Identify each mean in the canonical way with a measure on $\mathbb{N}_0$, see [24]. To show that f is an element in $AC^0 \oplus \mathbb{C}1$ it is sufficient to show that

$$f \equiv d(f) \text{ on } \operatorname{supp}\mu \text{ for each } \mu \in \mathbb{M}.$$

Indeed, if $f \equiv d(f)$ on $\operatorname{supp}\mu$, then there exists a function $g \in l^\infty(\mathbb{N}_0)$ such that $f = d(f) + g$ and $g \equiv 0$ on $\operatorname{supp}\mu$ for each $\mu \in \mathbb{M}$. Hence,

$$\mu(|g|) = \mu(|g\chi_{K\setminus \operatorname{supp}\mu}|) \leq \|g\|_\infty \mu(\chi_{K\setminus \operatorname{supp}\mu}) = 0$$

for each $\mu \in \mathbb{M}$ and we obtain $g \in AC^0$.
Now let $\mu \in \mathbb{M}$ be fixed. By Theorem 9.2.2 holds

$$\mu(fg) = \mu(f)\mu(g)$$

for all $g \in AC$. Hence,

$$\mu((f - d(f))^2) = \mu(f - d(f))^2 = (\mu(f) - d(f))^2 = 0$$

and $f \equiv d(f)$ on $\operatorname{supp}\mu$ as we wanted. □

We conclude the following characterization of multipliers for AC.

Theorem 9.2.4. *Let K be a polynomial hypergroup and $f \in l^\infty(\mathbb{N}_0)$. The following assertions are equivalent:*

i) f *is a multiplier for* AC, *i.e.* $f \cdot AC \subset AC$.

ii) $f \in AC$ *and* $\mu(f \cdot g) = \mu(f)\mu(g) = d(f)\mu(g)$ *for all* $g \in l^\infty(\mathbb{N}_0)$ *and* $\mu \in \mathbb{M}$.

iii) $f \equiv d(f)$ *on* $supp\mu$ *for all* $\mu \in \mathbb{M}$.

iv) f *is* $\mathbb{N}-$*convergent to a constant* $d(f)$.

v) $f \in AC_s$.

vi) $f \in AC^0 \oplus \mathbb{C}1$.

Proof. Suppose f is strongly almost convergent to $d(f) \in \mathbb{C}$ and let $g \in l^\infty(\mathbb{N}_0)$, $\mu \in \mathbb{M}$. Then, we have $|(f(n) - d(f))g(n)| \leq |f(n) - d(f)|\|g\|_\infty$. Thus,

$$0 \leq |\mu((f - d(f)1)g)| \leq \mu(|f - d(f)1|)\|g\|_\infty = 0.$$

Hence, $\mu(f - d(f)1)g) = 0$ and we obtain $\mu(fg) = d(f)\mu(g)$ for all $g \in l^\infty(\mathbb{N}_0)$. Hence, *v)* implies *ii)*.
The rest of the proof follows by Theorem 9.2.2 and Theorem 9.2.3. □

Corollary 9.2.5. *$M(AC)$ defines with the sup-norm a commutative Banach algebra with respect to the point-wise multiplication.*

Proof. $M(AC)$ is a closed subspace of $l^\infty(\mathbb{N}_0)$, since AC is a closed subspace of $l^\infty(\mathbb{N}_0)$ and every $\mu \in \mathbb{M}$ is continuous. Let $f, g \in M(AC)$. Obviously holds $\|f \cdot g\|_\infty \leq \|f\|_\infty\|g\|_\infty$. Furthermore, for each $a \in AC$ we have $f \cdot g \cdot a \in AC$. Thus, $f \cdot g \in M(AC)$. □

The following example shows that the set of all multipliers for AC is in general a proper subset of AC.

Example 9.2.6. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of ultraspherical polynomials, which is normed such that $R_1(x) = x$. Since, the characters of the polynomial hypergroup are exactly given by $\alpha_x(n) := R_n(x)$, $x \in [-1, 1]$, the sequences α_x, $x \in [-1, 1]$, fulfill $L_m\alpha_x(n) = \alpha_x(n)\alpha_x(m)$. Hence, they are almost convergent, see [111]. For any $\mu \in \mathbb{M}$ we have

$$\mu(\alpha_x) = \mu(T_1\alpha_x) = R_1(x)\mu(\alpha_x).$$

This is for $x \in [-1, 1]$, $x \neq 1$, only possible if $\mu(\alpha_x) = 0$. Therefore, $d(\alpha_x) = 0$ for $x \in [-1, 1]$, $x \neq 1$ and $d(\alpha_1) = 1$, see [111, Example 1].
In particular holds $\alpha_{-1} \in AC_0$. Furthermore, (note that $R_n(-1) = (-1)^n$)

$$\alpha_{-1} \cdot \alpha_{-1}(n) = R_n(-1)R_n(-1) = (-1)^{2n} = 1.$$

This leads to

$$\mu(\alpha_{-1}\alpha_{-1}) = \mu(1) = 1 \neq 0 = \mu(\alpha_{-1})\mu(\alpha_{-1})$$

for each $\mu \in \mathbb{M}$, which contradicts *ii)* in Theorem 9.2.4. Therefore, $\alpha_{-1} \in AC$ cannot be a multiplier for AC.

Chapter 10

Multipliers and invariant linear Systems in Time Series Analysis

Time series which obey stationarity induced by polynomial hypergroups are investigated in a series of papers, see [35], [82, 83], [110], [112]. This stationarity takes into account effects caused by delays, see [35]. Besides modeling delays, there are various other fields of applications where stationarity induced by polynomial hypergroups provides an appropriate access, see for instance [83]. In a recent paper [110] Lasser formulated the basic ideas and tools in the context of sequences $(X(n))_{n\in\mathbb{N}_0}$ in Hilbert spaces.

10.1 General Theory

We recall the relevant facts from [110]. Note that for the application in time series the Hilbert space H is always a L^2-space on a certain probability space. Throughout this chapter $(R_n(x))_{n\in\mathbb{N}_0}$ denotes an orthogonal polynomial sequence that generates a polynomial hypergroup on $\mathbb{N}_0$.
We define the translation of a sequence $(X(n))_{n\in\mathbb{N}_0}$ in a Hilbert spaces H by

$$L_nX(m) := \sum_{k=|n-m|}^{m+n} g(m,n;k)X(k)$$

for all $n, m \in \mathbb{N}_0$, where $g(m,n;k)$ denote the linearization coefficients of the product $R_n(x)R_m(x)$ as introduced in Chapter 1.3.

Definition 10.1.1. A sequence $(X(n))_{n\in\mathbb{N}_0}$ in a Hilbert spaces H with corresponding scalar-product $< . , . >$ is called R_n-**stationary**, if

$$< X(m), X(n) > = < L_nX(m), X(0) >$$

for all $n, m \in \mathbb{N}_0$.

Given a sequence $(X(n))_{n\in\mathbb{N}_0}$ in a Hilbert spaces H we consider $\varphi(n) := < X(n), X(0) >$. It is easily shown that $(\varphi(n))_{n\in\mathbb{N}_0}$ is a sequence of complex numbers satisfying

$$\sum_{j,k=0}^{n} \lambda_j \overline{\lambda_k} L_j\varphi(k) \geq 0$$

for all $n \in \mathbb{N}_0$ and $\lambda_0, ..., \lambda_n \in \mathbb{C}$. This is exactly the property of positive definiteness with respect to $(R_n(x))_{n\in\mathbb{N}_0}$, see [110]. Therefore, we can apply Bochner's theorem (or often called Cramer's Theorem) for commutative hypergroups, see [86], in order to obtain the spectral measure μ for the sequence $(X(n))_{n\in\mathbb{N}_0}$.

Theorem 10.1.2 (Cramer)**.** *Let $(X(n))_{n\in\mathbb{N}_0}$ be an R_n-stationary sequence such that the corresponding sequence $(\varphi(n))_{n\in\mathbb{N}_0}$, $\varphi(n) \;=\; < X(n), X(0) >$, is bounded. Then there exists a unique positive bounded measure μ on $\widehat{\mathbb{N}}_0 = D_s$ such that*

$$< X(m), X(n) > = \; L_n\varphi(m) \; = \; \int_{\widehat{\mathbb{N}}_0} R_m(x)R_n(x)d\mu(x).$$

We call μ the spectral measure of $(X(n))_{n\in\mathbb{N}_0}$.

Remark 10.1.3. $(\varphi(n))_{n\in\mathbb{N}_0}$ is bounded if and only if $(X(n))_{n\in\mathbb{N}_0}$ is bounded in H.

Consider the Hilbert space $L^2(\widehat{\mathbb{N}}_0, \mu)$. $\{R_n :\; n \in \mathbb{N}_0\}$ is a linear independent subset of $L^2(\widehat{\mathbb{N}}_0, \mu)$ and the linear span of $\{R_n :\; n \in \mathbb{N}_0\}$ is dense in $L^2(\widehat{\mathbb{N}}_0, \mu)$. We denote a sub Hilbert space of H by $H_0 \;:=\; \overline{\text{span}\{X(n) :\; n \in \mathbb{N}_0\}}$. Furthermore, we define a linear map

$$\Phi :\; \text{span}\{R_n :\; n \in \mathbb{N}_0\} \to H_0, \qquad \Phi\left(\sum_{k=1}^{N} c_k R_{n_k}\right) \;:=\; \sum_{k=1}^{N} c_k X(n_k).$$

Φ is well-defined. Moreover, Φ can be uniquely extended to an isometric isomorphism from $L^2(\widehat{\mathbb{N}}_0, \mu)$ onto H_0. We may assume that $H = H_0$.

The classical theory of time-invariant systems in time series analysis can be generalized to R_n-stationary sequences in an obvious manner. In the classical theory of time-invariant systems the index-set is $\mathbb{Z}$. A stationary sequence $(Y(n))_{n\in\mathbb{N}_0}$ in a L^2-Hilbert space satisfies $< Y(m), Y(n) > = < Y(m-n), Y(0) >$.
A time-invariant system is nothing else than a bounded linear operator which transforms the given stationary sequence $(Y(n))_{n\in\mathbb{N}_0}$ into a sequence $(Y'(n))_{n\in\mathbb{N}_0}$ which is stationary again. Such operators are called **multipliers**, i.e. operators which commute with translation operators. For a bounded, R_n-stationary sequence $(X(n))_{n\in\mathbb{N}_0}$ in H the corresponding multipliers are exactly those bounded linear operators $T \in B(H)$ which obey

$$T \circ L_n = L_n \circ T \qquad \text{for all } \; n \in \mathbb{N}_0.$$

Such an operator T is called **multiplier of** $(X(n))_{n\in\mathbb{N}_0}$. The following proposition holds, see [110, Proposition 6.1].

Proposition 10.1.4. *Let $(X(n))_{n\in\mathbb{N}_0}$ be a bounded, R_n-stationary sequence in H and $T \in B(H)$ be a multiplier of $(X(n))_{n\in\mathbb{N}_0}$. Put $Y(n) = TX(n)$ for $n \in \mathbb{N}_0$. Then $(Y(n))_{n\in\mathbb{N}_0}$ is a bounded, R_n-stationary sequence in H.*

We can say even more about multipliers of $(X(n))_{n\in\mathbb{N}_0}$. Let $\phi \in L^\infty(\widehat{\mathbb{N}}_0, \mu)$. Define a bounded linear operator $T_\phi \in B(H)$ by

$$T_\phi(Z) = \Phi(\phi\Phi^{-1}(Z)), \qquad Z \in H.$$

We obtain $\Phi(R_n\psi) = L_n\Phi(\psi)$ for all $\psi \in L^2(\widehat{\mathbb{N}}_0, \mu)$ and $n \in \mathbb{N}_0$ by

$$\begin{aligned} < \Phi(R_n\psi), X(m) > \;&=\; < R_n\psi, R_m > = < \psi, R_nR_m > \\ &=\; < \Phi(\psi), \Phi(R_nR_m) > = < \Phi(\psi), L_nX(m) > = < L_n\Phi(\psi), X(m) > \end{aligned}$$

for all $m \in \mathbb{N}_0$. Thus,

$$T_\phi \circ L_nZ = \Phi(\phi\Phi^{-1}(L_nZ)) = \Phi(\phi R_n\Phi^{-1}(Z)) = L_n \circ T_\phi Z$$

for all $Z \in H$. Hence, T_ϕ is a multiplier of $(X(n))_{n\in\mathbb{N}_0}$.
In [110] it is proven that for every multiplier T of $(X(n))_{n\in\mathbb{N}_0}$ there exists a corresponding $\phi \in L^\infty(\widehat{\mathbb{N}}_0, \mu)$ such that $T = T_\phi$. Therefore, the operators T_ϕ, $\phi \in L^\infty(\widehat{\mathbb{N}}_0, \mu)$, are exactly the multipliers of $(X(n))_{n\in\mathbb{N}_0}$. The following characterization for multipliers of $(X(n))_{n\in\mathbb{N}_0}$ is valid, see [110, Theorem 6.3]

Theorem 10.1.5. *Let $(X(n))_{n\in\mathbb{N}_0}$ be a bounded, R_n−stationary sequence in a Hilbert space H. Assume that the linear span of the $X(n)$ is dense in H. Let $T \in B(H)$. Then the following are equivalent:*

(i) T is a multiplier of $(X(n))_{n\in\mathbb{N}_0}$.

(ii) There exists a unique $\phi \in L^\infty(\widehat{\mathbb{N}}_0, \mu)$ such that $T = T_\phi$.

10.2 Examples of Multipliers for R_n−stationary sequences

We consider now some examples of multipliers $T = T_\phi$ for bounded, R_n−stationary sequences.

Multipliers for R_n−white noise

$(Z(n))_{n\in\mathbb{N}_0}$ is called R_n−**white noise** in a Hilbert space H if

$$< Z(n), Z(m) > = \delta_{n,m}\frac{1}{h(n)}.$$

If $(Z(n))_{n\in\mathbb{N}_0}$ is R_n−white noise, $(Z(n))_{n\in\mathbb{N}_0}$ is obviously a bounded, R_n−stationary sequence and the corresponding spectral measure μ of $(Z(n))_{n\in\mathbb{N}_0}$ is exactly the orthogonalization measure π. We assume again $H = H_0 := \overline{\text{span}\{Z(n):\ n \in \mathbb{N}_0\}}$.
Thus, we can illustrate the connection to the multiplier results in Chapter 3.
The map

$$\text{span}\{Z(n):\ n \in \mathbb{N}_0\} \to l^2(\mathbb{N}_0, h), \qquad Z(n) \mapsto \frac{\delta_n}{h(n)} =: \epsilon_n$$

can be uniquely extended to an isometric isomorphism between $H = \overline{\text{span}\{Z(n):\ n \in \mathbb{N}_0\}}$ and $l^2(\mathbb{N}_0, h)$. This leads to the following observation.
On one hand, Theorem 3.2.8 characterizes $T \in M(l^2(\mathbb{N}_0, h))$ by the existence of a function $\phi \in L^\infty(\mathcal{S}, \pi)$ such that $\wp(Td) = \phi\wp(d)$ for all $d \in l^2(\mathbb{N}_0, h)$, where $\wp$ denotes the usual Plancherel transform.
On the other hand, Theorem 10.1.5 describes the multipliers of $(Z(n))_{n\in\mathbb{N}_0}$ as operators of the form $T = T_\phi$ as constructed above, where $\phi \in L^\infty(\mathcal{S}, \pi)$. Identifying $l^2(\mathbb{N}_0, h)$ with $L^2(\mathcal{S}, \pi)$ via the Plancherel isomorphism $\wp$ and H with $L^2(\widehat{\mathbb{N}}_0, \pi) = L^2(\mathcal{S}, \pi)$ via Φ^{-1} (as described above), we see that both characterizations yield the same $\phi \in L^\infty(\mathcal{S}, \pi)$ as multiplier, only the "representation" of the Hilbert spaces are different.

Furthermore, let $(Z(n))_{n\in\mathbb{N}_0}$ be R_n−white noise and choose any $\phi \in L^\infty(\mathcal{S}, \pi)$. By Proposition 10.1.4 the sequence $(Y(n))_{n\in\mathbb{N}_0}$ with $Y(n) := T_\phi Z(n)$ is a bounded, R_n-stationary sequence in $H = \overline{\text{span}\{Z(n):\ n \in \mathbb{N}_0\}}$. The corresponding spectral measure μ for $(Y(n))_{n\in\mathbb{N}_0}$ fulfills $d\mu(x) = |\phi(x)|^2 d\pi(x)$. In fact, we can prove even a more generalized result.

Proposition 10.2.1. *Let $(X(n))_{n\in\mathbb{N}_0}$ be a bounded, R_n−stationary sequence in a Hilbert space H with corresponding spectral measure μ. For $\phi \in L^\infty(\widehat{\mathbb{N}}_0, \mu)$ the sequence $Y(n) := T_\phi X(n)$ fulfills*

$$< Y(n), Y(0) > = \int_{\widehat{\mathbb{N}}_0} R_n(x)|\phi(x)|^2 d\mu(x), \qquad n \in \mathbb{N}_0,$$

i.e. $|\phi(x)|^2 d\mu(x)$ defines the spectral measure corresponding to $(Y(n))_{n\in\mathbb{N}_0}$.

Proof. Sinec Φ is an isometry, we obtain

$$\begin{aligned} < Y(n), Y(0) > &= < T_\phi X(n), T_\phi X(0) > = < \Phi(\phi\Phi^{-1}(X(n))), \Phi(\phi\Phi^{-1}(X(0))) > \\ &= < \phi\Phi^{-1}(X(n)), \phi\Phi^{-1}(X(0)) > = \int_{\widehat{\mathbb{N}}_0} R_n(x)|\phi(x)|^2 d\mu(x). \end{aligned}$$

□

Example of a Multiplier for an autoregressive Process

An important class of examples of multipliers lead to the so called autoregressive processes. We give a short introduction of this class.
Let $(Z(n))_{n\in\mathbb{N}_0}$ be R_n-white noise in a Hilbert space H. A R_n-stationary sequence $(Y(n))_{n\in\mathbb{N}_0}$ is called **autoregressive of order** q **(with respect to** R_n**)**, if there exist $b_1, ..., b_q \in \mathbb{C}$, $b_q \neq 0$, such that

$$Y(n) + b_1 L_1 Y(n) + b_2 L_2 Y(n) + ... + b_q L_q Y(n) = Z(n)$$

holds for all $n \in \mathbb{N}_0$.

If $1 \notin \mathcal{S} = \mathrm{supp}\pi$, then the function

$$\phi(x) := \frac{1}{1 - R_1(x)}, \quad x \in \widehat{\mathbb{N}}_0$$

is an element in $L^\infty(\mathcal{S}, \pi)$. Hence, the elements $Y(n) := T_\phi Z(n)$ form a bounded, R_n-stationary sequence with corresponding spectral measure

$$d\mu(x) = \frac{1}{|1 - R_1(x)|^2} d\pi(x).$$

$(Y(n))_{n\in\mathbb{N}_0}$ is an autoregressive process. Indeed, it obviously holds that

$$R_n\phi - R_1 R_n \phi = R_n \qquad \text{for all } n \in \mathbb{N}_0.$$

We apply $\Phi : \overline{\mathrm{span}\{Z(n) : \ n \in \mathbb{N}_0\}} \to L^2(\mathcal{S}, \pi)$ on both sides of the equation. By

$$R_1 \Phi^{-1}(Z(n)) = \Phi^{-1}(L_1 Z(n))$$

we obtain that

$$\Phi(\phi\Phi^{-1}(Z(n))) - \Phi(\phi\Phi^{-1}(L_1 Z(n))) = Z(n) \qquad \text{for all } n \in \mathbb{N}_0.$$

By the definition of $Y(n)$ and the fact that T_ϕ commutes with translations, we have the following autoregressive equation of order 1

$$Y(n) - L_1 Y(n) = Z(n) \qquad \text{for all } n \in \mathbb{N}_0.$$

Orthogonal polynomial sequences $(R_n(x))_{n\in\mathbb{N}_0}$ with $1 \notin \mathcal{S} = \mathrm{supp}\pi$ are for example the Karlin-McGregor polynomials (see Chapter 5) or the polynomials related to homogeneous trees, see [9]. The reader should note that in the classical theory of time series the function

$$g(e^{it}) = \frac{1}{|1 - e^{it}|}$$

can never define a multiplier.

In case of $1 \in \mathcal{S}$ one can construct bounded, R_n-stationary sequences $(X(n))_{n\in\mathbb{N}_0}$ satisfying the same autoregressive equation, see [110, Example 5], provided $\phi(x) := \frac{1}{1-R_1(x)} \in L^1(\mathcal{S}, \pi)$. However, the sequence $(X(n))_{n\in\mathbb{N}_0}$ is constructed from an R_n-white noise $(Z(n))_{n\in\mathbb{N}_0}$ in a different way, see [110, Theorem 7.4].

Bibliography

[1] C. A. Akemann, *Some mapping properties of the group algebras of a compact group*, Pacific. J. Math. 22 (1967), 1-8

[2] G. F. Bachelis and J. E. Gilbert, *Banach spaces of compact multipliers and their dual spaces*, Math. Z. 125 (1972), 285-297

[3] R. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Can. J. Math. 7 (1955), 289-305.

[4] W. R. Bloom and H.Heyer, *Harmonic analysis of probability measures on hypergroups,* de Gruyter, Berlin (1995)

[5] W. R. Bloom and Z. Xu, *Fourier multipliers for L^p on Chèbli-Trimèche hypergroups*, Proc. London Math. Soc. (3) 80 (2000), 643-664

[6] T. K. Bochner, *Über Faktorfolgen für Fouriersche Reihen*, Acta Sci. Math. (Szeged) 4 (1928-1929), 125-129

[7] B. Brainerd and R. E. Edwards, *Linear operators which commute with translations I, II*, J. Austral. Math: Soc. 6 (1966), 289-327 and 328-350

[8] J. T. Burnham and R. R. Goldberg, *Multipliers from $L^1(G)$ to a Segal algebra*, Bull Inst. Math. Acd. Sinica 2 (1974), 153-164

[9] D. I. Cartwright, G. Kuhn and P.M. Soardi, *A Product Formular for spherical representations of a group of automorphisms of a homogeneous tree I*, Tran. Amer. Math. Soc., Vol. 353, no. 1 (2000), 349-364

[10] C. Chou and J. P. Duran, *Multipliers for the space of almost-convergent functions on a semigroup*, PAMS, Vol. 39, no. 1, June (1973)

[11] C. Chou, *Weakly almost periodic functions and almost convergent functions on a group*, Trans. Amer. Math. Soc. 206 (1975), 175-200

[12] W. C. Connett and A. L. Schwartz, *A multiplier theorem for ultraspherical series*, Studia Math. 51 (1974), 51-70

[13] W. C. Connett and A. L. Schwartz, *A multiplier theorem for Jacobi expansions*, Studia Math. 52 (1975), 243-261

[14] W. C. Connett and A. L. Schwartz, *The theory of ultraspherical multipliers*, Mem. Amer. Math. Soc. Vol. 9 (2), no. 183 (1977)

[15] W. C. Connett and A. L. Schwartz, *Product formulas, hypergroups and the Jacobi polynomials*, Bull. Amer. Math. Soc. Vol. 22, no.1 (1990), 91- 96

[16] H. G. Dales and S. S. Pandey, *Weak amenability of Segal algebras*, Proc. Amer. Math. Soc. Vol. 128, no. 5 (1999), 1419-1425

[17] M. M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509-544

[18] K. De Leeuw, *On L_p multipliers*, Ann. of Math (2) (1965), 364-379

[19] S. Degenfeld-Schonburg, *Multiplikatoren von Algebren über Hypergruppen*, Master thesis (2009)

[20] S. Degenfeld-Schonburg and R. Lasser, *Multipliers on L^p-spaces for hypergroups*, to appear in Rocky Mountain Journal of Mathematics

[21] C. Dellacherie, *Ensembles analytiques capacites mesures de Hausdorff*, Lecture Notes in Mathematics, Vol. 295, Springer, Berlin (1972)

[22] A. Derighetti, *Convolution operators on groups*, Springer, Heidelberg, London New York (2011)

[23] A. Devinatz and I.I. Hirschman, Jr., *The spectra of multiplier transforms on l^p*, Amer. J. Math. 80 (1958), 829-842

[24] R. G. Douglas, *On the measure-theoretic character of an invariant mean*, PAMS, Vol. 16, no. 1 (1965), 30-36

[25] S. A. Douglas, *On a concept of summability in amenable semigroups*, Math. Scand 23 (1968), 96-102

[26] N. Dunford und J. T. Schwartz, *Linear operators, part 1: General theory*, New York: Interscience Publishers (1958)

[27] C. F. Dunkl. *The measure algebra of a locally Compact Hypergroup*, Trans. Amer. Math. Soc. 179 (1973), 331-348

[28] M. Dutta and U. B. Tewari, *On Multipliers of Segal algebras*, Proceedings of the American Mathematical Society, Vol. 72, no. 1, October (1978), 121-124

[29] R. E. Edwards, *On factor functions*, Pacific J. Math. 5 (1955), 367-378

[30] R. E. Edwards, *Endomorphisms of function spaces which leave stable all translation invariant manifolds*, Pacific J. Math. 14 (1964), 31-47

[31] R. E. Edwards, *Changing signs of Fourier coefficients*, Pacific J. Math. 15 (1965), 463-475

[32] R. E. Edwards, *Uniform approximation on noncompact spaces* Trans. Amer. Math. Soc. 122 (1966), 249-276

[33] R. E. Edwards, *Fourier series I, II: A modern introduction*, Hold, Rinehart and Winston, Inc., New York (1967)

[34] H. Emamirad and G. S. Heshmati, *Pseudomeasure character of the ultraspherical semigroups*, Semigroup Forum 65 (2002), 336-347.

[35] K. Ey and R. Lasser, *Facing linear difference equations through hypergroup methods*, J. Difference Equ. Appl. 13 (2007), 953-965

[36] P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, C. R. Acad. Sci. Paris Sèr A 256 (1963), 1429-1431

[37] H. G. Feichtinger, *Multipliers from $L^1(G)$ to a homogeneous Banach space*, J. Math. Analysis and Applications Vol. 61, no. 2 (1977), 341-356

[38] M. Fekete, *Über die Faktorenfolgen welche die "Klasse" einer Fourierschen Reihe unverändert lassen*, Acta Sci. Math. (Szeged) 1 (1923), 148-166

[39] A. Figà-Talamanca, *Multipliers of p-integrable functions*, Bull. Amer. Math. Soc. 70 (1964), 666-669

[40] A. Figà-Talamanca, *On the subspace of L^p invariant under multiplication of transforms by bounded continuous functions*, Rend. Sem. Mat. Univ. Padova 35 (1965), 176-189

[41] A. Figà-Talamanca, *Translation invariant operators in L^p*, Duke Math. J. 32 (1965), 495-501

[42] A. Figà-Talamanca and G. I. Gaudry, *Density and representation theorems for multipliers of type (p, q)*, J. Austral. Math Soc. 7 (1967), 1-6

[43] A. Figà-Talamanca and G. I. Gaudry, *Multipliers and set of uniqueness of L^p*, Michigan Math. J. 17 (1970), 179-191

[44] A. Figà-Talamanca and G. I. Gaudry, *Multipliers of L^p which vanish at infinity*, J. Functional Analysis 7 (1971), 475-486

[45] F. Filbir and R. Lasser, *Reiter's condition P_2 and the Plancherel measure for hypergroups*, Illinois J. Math. 44 (2000), 20-32

[46] G. Fischer and R. Lasser, *Homogeneous Banach spaces with respect to Jacobi polynomials*, Rendiconti del circolo matematico di palermo, Ser. II, Suppl. 76 (2005), 331-353

[47] M. J. Fisher, *Recognition and limit theorems for L_p-multipliers*, Stud. Math. T. L. (1974), 31-41

[48] J. F. Fournier, *Local complements to the Hausdorff -Young theorem*, Michigan Math. J. 20 (1973), 263-276.

[49] J. J. Fournier, *On the Hausdorff-Young theorem for amalgams*, Monatsh. Math. 95 (1983), 117-135

[50] G. Gasper, *Banach algebras for Jacobi series and positivity of a kernel*, Ann. of Math. 95 (1972), 261-280

[51] G. Gasper and W. Trebels, *Multiplier criteria of Hörmander type for Jacobi expansions*, Studia Math. T. LXVIII (1980), 187-197

[52] G. I. Gaudry, *Quasimeasures and operators commuting with convolutions*, Pacific J. Math. 18 (1966), 461-476

[53] G. I. Gaudry, *Multipliers of type (p,q)*, Pacific J. Math. 18 (1966), 477-488

[54] G. I. Gaudry, *Quasimeasures and multiplier problems*, Doctoral dissertation, Australian National University, Canberra, Australia (1966)

[55] G. I. Gaudry, *Multipliers of weighted Lebesgue and measure spaces*, Proc. London Math. Soc. 19 (1969), 327-340

[56] G. I. Gaudry, *Topics in harmonic analysis*, Lecture notes, Department of Mathematics, Yale University, New Haven, Ct. (1969)

[57] G. I. Gaudry, *Bad behavior and inclusion results for multipliers of type (p,q)*, Pacific J. Math. 35 (1970), 83-94

[58] G. I. Gaudry and I. R. Inglis, *Approximation of multipliers*, Proc. oAmer. Math. Soc. Vol 44, no. 2, (1974), 381-384

[59] F. Ghahramani and A. T. M. Lau, *Weak amenability of certain classes of Banach algebras without bounded approximate identities*, Math. Proc. Camb. Phil. Soc. 133 (2002), 357-371

[60] F. Ghahramani and A. T. M. Lau, *Approximate weak amenability, derivations and Arens regularity of Segal algebras*, Studia Math. 169 (2005), 189-205

[61] F. Ghahramani and A. R. Medgalchi, *Compact multipliers on weighted hypergroup algebras*, Math. Proc. Cambridge Philos. Soc. 98 (1985), 493-500.

[62] F. Ghahramani and A. R. Medgalchi, *Compact multipliers on weighted hypergroup algebras II*, Math. Proc. Cambridge Philos. Soc. 100 (1986), 145-149.

[63] G. Goes, *Komplementäre Fourierkoeffizientenräume und Multiplikatoren*, Math. Ann. 137 (1959), 371-384

[64] R. R. Goldberg, *Multipliers from $L^1(G)$ to a Segal algebra*, Notices Amer. Math. Soc. 22 (1975), A-12

[65] R. R. Goldberg and S. E. Seltzer, *Uniformly concentrated sequences and multipliers of Segal algebras*, J. Math. Anal. Appl. 59 (1977), 488-497

[66] A. Grothendieck, *Résumé des résultàts essentiels dans la thèorie des produits tensoriels topologiques et des espaces nuclèaires*, Ann. Inst. Fourier 4 (1952), 73-112

[67] A. Grothendieck, *Résultàts nouveaux dans la thèorie des opèrations linèaires I, II*, C. R. Acad. Sci. Paris 239 (1954), 577-579 and 607-609

[68] L. S. Hahn,*On Multipliers of p-integrable functions*, TAMS 128 (1967), 321-335

[69] P. Halmos, *Measure theory*, Princeton, N.J.:D. van Nostrand Company, Inc. (1950)

[70] K. Hartmann, R. W. Henrichs and R. Lasser, *Duals of orbit spaces in groups with relatively compact Inner automorphism groups are hypergroups*, Mh. Math. 88 (1979), 229-238

[71] S. Helgason, *Multipliers of Banach algebras*, Ann. of Math (2) 64 (1956), 240-254

[72] S. Helgason, *Topologies of group algebras and a theorem of Littlewood*, Trans. Amer. Math. Soc. 86 (1957), 269-283

[73] S. Helgason, *Lacunary fourier series on noncommutative groups*, Proc. Amer. Math. Soc. 9 (1958), 782-790

[74] H. Helson,*Isomorphisms of abelian group algebras*, Ark. Mat. 2 (1953), 475-487

[75] E. Hewitt, *Fourier transforms of the class L_p*, Ark. Mat. 2 (1954), 571-574

[76] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer, Berlin (1963)

[77] E. Hewitt and K. A. Ross, *Abstract harmonic analysis II*, Springer, Berlin (1963)

[78] H. Heyer, *Probability theory on hypergroups: A survey*, Lecture Notes in Mathematics, Volume 1064 (1984), 481-550

[79] E. Hille, *On functions of bounded derivations*, Proc. London Math. Soc. (2) 31 (1930), 165-173

[80] E. Hille and J. D. Tamarkin, *On the summability of Fourier series II* , Ann. of Math. (2) 34 (1933), 329-348

[81] I. Hirschman, Jr., *On multiplier transformations*, Duke Math. J. 26 (1959), 221-242

[82] V. Hösel and R. Lasser, *One-Step Prediction for P_n-weakly stationary processes*, Mh. Math. 113 (1992), 199-212

[83] V. Hösel and R. Lasser, *Prediction of weakly stationary sequences on polynomial hypergroups*, Ann. Probab. 31 (2003), 93-114

[84] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93-140

[85] R. Iltis, *Harmonic analysis on compact topological groups*, Doctoral dissertation, University of Oregon (1966)

[86] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math. 18 (1975), 1-101.

[87] S. Kaczmarz, *On some classes of Fourier series*, J. London Math. Soc 8 (1933), 39-46

[88] S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Monografje Matematyczne, tom VI. Warszawa (1935)

[89] S. Kaczmarz and J. Marcinkiewicz, *Sur les multiplicateurs des séries orthogonales*, Studia Math. 7 (1938), 73-81

[90] E. Kaniuth:, *A course in commutative Banach algebras*, Springer, Heidelberg (2008)

[91] J. Karamata, *Suite de fonctionelles linèaires et facteurs de convergence de sèries de Fourier*, J. Math. Pures Appl. 35 (1956), 87-95

[92] J. Katamata, *Sur les facteurs de convergence uniforme des sèries de Fourier*, Rev. Fac. Sci. Univ. Istanbul Ser. A 22 (1957), 35-43

[93] J. Karamata and M. Tomiè, *Sur la sommation des sèries de Fourier des fonctions continues*, Acad. Serbe. Sci. Publ. Inst. Math. 8 (1955), 123-138

[94] M. Katayama, *Fourier series. VII. Uniform convergence factors of Fourier series*, J. Fac. Sci. Hokkaido Univ. Ser. I 13 (1957), 121-129

[95] Y. Katznelson, *Set of uniqueness for some classes of trigonometric series*, Bull. Amer. Math. Soc. 70 (1964), 722-723

[96] L. Kerchy, *Operators with regular norm-sequences*, Acta. Sci. Math. (Szeged) 63 (1997), 571-605

[97] J. W. Kitchen, Jr. , *The almost periodic measures on a compact abelian group*, Monatsh. Math. 72 (1968), 217-219

[98] T. H. Koornwinder and A. L. Schwartz, *Product formulas and associated hypergroups for orthogonal polynomials on the simplex and on a parabolic biangle*, Constr. Approx. 13 (1997), 537-567

[99] H. E. Krogstad, *Multipliers of Segal algebras*, Math. Scand 38 (1976), 285-303

[100] R. A. Kunze, *L_p-Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc. 89 (1958), 519-540

[101] R. Larsen, *An introduction to the theory of multipliers*, Die Grundlehren der mathematischen Wissenschaft. Band 175, Springer, Heidelberg (1971)

[102] R. Larsen, *The multipliers for functions with Fourier transform in L_p*, Math. Scand,. 28 (1971), 1-11

[103] R. Larsen, T. S. Liu and J. K. Wang, *On functions with Fourier transform in L_p*, Michigan Math. J. 11 (1964), 369-378

[104] M. Lashkarizadeh Bami, M. Pourgholamhossein and H. Samea, *Fourier algebras on locally compact hypergroups*, Math. Nachr. 282, no. 1 (2009), 16-25

[105] R. Lasser, *Fourier-Stieltjes transforms on hypergroups*, Analysis 2 (1982), 281-303.

[106] R. Lasser, *Orthogonal polynomials and hypergroups*, Rend. Math. 3 (1983), 185-209.

[107] R. Lasser, *Orthogonal polynomials and hypergroups II – The symmetric case*, Trans. Amer. Math. Soc. 341 (1994), 749-770.

[108] R. Lasser, *On the character space of commutative hypergroups*, DMV Jahresbericht 104 (2002), 3-16.

[109] R. Lasser, *Amenability and weak amenability of L^1-algebras of polynomial hypergroups*, Studia Math., 182(2) (2007), 183-196

[110] R. Lasser, *On positive definite and stationary sequences with respect to polynomial hypergroups*, J. App. Ana. 17 (2011), 207-230

[111] R. Lasser, *Almost-convergent sequences with respect to polynomial hypergroups*, to appear in Acta Math. Hungar.

[112] R. Lasser and M. Leitner, *Stochastic processes indexed by hypergroups I*, J. Theoret. Probab. 2 (1989), 301-311

[113] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series I* , J. London Math. Soc. 6 (1931), 230-233

[114] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series II* , Proc. London Math. Soc. (2) 42 (1937), 52-89

[115] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series III* Proc. London Math. Soc. (2) 43 (1937), 105-126

[116] W. Littman, *Multipliers in L^p and interpolation*, Bull. Amer. Math. Soc. 71 (1965), 764-766

[117] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 (1948), 167-190

[118] J. Marcinkiewicz, *Sur les multiplicateurs des séries de Fourier*, Studia Math. 8 (1939), 78-91

[119] T. Martinez, *Multipliers of Laplace transform type for ultraspherical expansions*, Math. Nachr. 281, no. 7 (2008), 978-988

[120] S. Mazurkiewicz, *O sumowalności szeregów kształtu* $\sum_{n=1}^{\infty} a_n u_n$. C. R. Soc. Sci. Varsovie 8 (1915), 649-655

[121] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., 71 (1951), 152182.

[122] S. G. Mihlin, *On the multipliers of Fourier integrals*. Dokl. Akad. Naulc SSSR (N. S.), 109 (1956), 701-703 (Russian).

[123] S. G. Mihlin, *Fourier integrals and multiple singular integrals*. Vestnik Leningrad. Univ. Set. Mat. Mech. Astr., 12 no. 7 (1957), 143-155. (Russian.)

[124] V. Muruganandam, *Fourier algebra of a hypergroup. I*, J. Aust. Math. Soc. 82 (2007), 59-83

[125] V. Muruganandam, *Fourier algebra of a hypergroup- II. Spherical hypergroups*, Math. Nachr. 281, no. 11 (2008), 1590-1603

[126] N. Obata, *Isometric operators on L^1-algebras of hypergroups*, Probability measures on groups, X (Oberwolfach 1990), 315-328. Plenum, New York-London (1991)

[127] W. Orlicz, *Beiträge zur Theorie der Orthogonalentwicklungen*, Studia Math. 1 (1929), 1-39

[128] A. L. T. Paterson, *Amenability*, Amer. Math. Soc., Providence, (1988)

[129] L. Pavel, *Multipliers for the L_p-spaces of a hypergroup*, Rocky Mountain Journal of Mathematics 37 (2007), 987-1000.

[130] J. F. Price, *Some strict inclusions between spaces of L^p-Multipliers*, Trans. Amer Math Soc., Vol. 152 (1970)

[131] M. S. Ramanujan and N. Tanović-Miller, *A generalization of the Hausdorff-Young theorem*, Acta Math. Hungar. 81 (4) (1998), 279-303

[132] D. E. Ramirez, *Uniform approximation by Fourier- Stieltjes transforms*, Proc. Cambridge Philos. Soc. 64 (1968), 323-333

[133] M. Rieffel, *Multipliers and tensor products of L^p-spaces of locally compact groups*, Studia. Math. 33 (1969), 71-82

[134] M. Riesz, *Sur les maxima des formes bilinaires et sur les fonctionnelles linaires*, Acta Math. 49 (1926), 465-497

[135] T. V. Rodionov, *Analogues of the Hausdorff-Young and Hardy-Littlewood theorems*, Izv. Math. 65 (3) (2001), 589-606

[136] K. A. Ross, *Centres of hypergroups*, Trans. Amer. Math. Soc. Vol. 243 (1978), 251-269

[137] B. Russo, *The norm of the L^p-fourier transform on unimodular groups*, Trans. Amer. Math. Soc. 192 (1974), 293-305

[138] B. Russo, *The norm of the L^p-fourier transform,II*, Canad. J. Math., Vol. XXVIII, no. 6 (1976), 1121-1131

[139] B. Russo, *Recent advances in the Hausdorff Young theorem*, Symposia Mathematica XXII (1976), 173-181

[140] S. Sakai, *Weakly compact operators on operator algebras*, Pacific J. Math. Volume 14, no. 2 (1964), 659-664.

[141] E. Sato, *Certain measures on locally compact abelian groups*, Contemp. Math. Vol. 91 (1989), 273-280

[142] H.-J. Schmeisser and H. Triebel, *Topics in Fourier analysis and function spaces*, Geest and Portig, Leipzig / Wiley, Chichester (1987)

[143] I. J. Schoenberg, *A remark on the preceding note by Bochner*, Bul. Amer. Math. Soc. 40 (1934), 277-278

[144] S. Sidon, *Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen*, Math. Z. 10 (1921), 121-127

[145] S. Sidon, *Ein Satz über die Fourierschen Reihen stetiger Funktionen*, Math. Z. 34 (1932), 485-486

[146] A. I. Singh, *Completely positive hypergroup actions*, Mem. Amer. Math. Soc. Vol. 124, no. 593 (1996)

[147] M. Skantharajah, *Amenable hypergroups*, Illinois J. Math. 36 (1992), 15-46

[148] M. G. Skvortsova, *Fourier series multipliers*, Siberian Math. J. 10 (1969), 97-135

[149] R. Spector, *Mesures invariantes sur les hypergroupes*, Trans. Amer. Math. Soc. 239 (1978), 147-165.

[150] E. M. Stein and A. Zygmund, *Boundedness of translation invariant operators on Hölder spaces and L^p spaces*, Ann. of Math. 85 (1967), 337-349

[151] H. Steinhaus, *Sur quelques propriétés de séries trigonomtriques et celles de puissances*, Rospravy Akademji Umiejetności, Cracow (1915), 175-225

[152] K. Stempak, *La thèorie de Littlewood-Paley pour la transformation de Fourier-Bessel*, C.R. Acad. Sci. Paris Ser. I Math. 303 (1986), 15-18

[153] G. Sunouchi, *Theorems on power series of the class H^p*, Tôhoku Math. J. 7 (1955), 96-109

[154] G. Szegö, *Orthogonal polynomials*, American Mathematical Society, Providence, RI (1975)

[155] U. B. Tewari, *Multipliers of Segal algebras*, Proc. Amer. Math. Soc. Vol. 54 (1976), 157-161

[156] U. B. Tewari, *Compact multipliers of Segal algebras*, Indian J. pure appl. Math. 14 (2) (1983), 194-201

[157] M. Tomiċ, *Sur les facteurs de convergence des sèries de Fourier des fonctions continues*, Acad. Serbe Sci. Publ. Inst. Math. 8 (1955), 23-32

[158] B. J. Tomiuk, *On some properties of Segal algebras and their multipliers*, Manuscripta Math. 27 (1979), 1-18

[159] K. Trimèche, *Generalized wavelets and hypergroups*, Gordon and Breach Science Publishers (1997)

[160] S. Verblundsky, *On some classes of Fourier series*, Proc. London Math. Soc. (2) 33 (1932), 287-327

[161] R. C. Vrem, *Lacunarity on compact hypergroups*, Math. Z. 164 (1978), 93-104

[162] R. C. Vrem, *Harmonic analysis on compact hypergroups*, PacificJ. Math. Vol. 85(1) (1979), 239-251

[163] R. C. Vrem, *Continuous measures and lacunarity on hypergroups*, Trans. Amer. Math. Soc. (2) 269 (1982), 549-557

[164] H.-C. Wang, *homogeneous Banach algebras*, Marcel Dekker, New York (1977)

[165] J.G. Wendel, *On isometric isomorphisms of group algebras* Pacific. J. Math. 1 (1951), 305-311

[166] J.G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), 251-261

[167] H. Widom, *Toeplitz operators on H_p*, Pacific J. Math. 19 (1966), 573-582.

[168] M. W. Wong, *Discrete Fourier analysis*, Birkhäuser, Basel (2011)

[169] N. Youmbi, *Semigroup of operators commuting with translations on compact commutative hypergroups*, Int. J. Math. Analysis, Vol 3, no. 17 (2009), 801-813

[170] N. Youmbi, *Some multipliers results on compact hypergroups*, Int. Math. Forum 5, no. 52 (2010), 2569-2580

[171] W. H. Young, *On a certain series of Fourier*, Proc. London Math. Soc. (1) 11 (1913), 357-366.

[172] W. H. Young, *On the Fourier series of bounded functions*, Proc. London Math. Soc.(2) 12 (1913), 41-70

[173] W. H. Young, *On Fourier series and functions of bounded variations*, Proc. Roy. Soc. Ser. A 88 (1913), 561-568

[174] A. Zygmund, *Sur un théorem de M. Fekete*, Bull. Int. Acad. Polon Sci. Lett. Ser. A Math Cracowie, no. 6A (1927), 343-347

[175] A. Zygmund, *Trigonometric series I and II*, Cambridge University Press, New York (1959)

[176] A. Zygmund *On the preservation of classes of functions*, J. Math. Mach. 8 (1959), 889-895