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Transfer Operators in the Context of Orthogonal Polynomials

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Abstract

We use transfer operators, a standard tool in dynamical systems theory, together with the theory of orthogonal polynomials, polynomial hypergroups and harmonic analysis to define a new transfer operator. We define this transfer operator via the preimages of certain orthogonal polynomials, namely the Chebyshev polynomials of the first kind. This transfer operator acts on various function spaces which are determined by orthogonal polynomials. We find that the defined transfer operator is bounded on the function spaces that we study and that the transformation operator coincides with the adjoint operator and the right inverse, respectively. Using a quadratic transformation which is given by the second order Chebyshev polynomial of the first kind, we construct an orthogonal polynomial sequence which generates a polynomial hypergroup the transfer operator acts on. Similarly, we construct an orthogonal polynomial sequence using a cubic transformation. However, in the cubic case, the orthogonal polynomial sequence does not generate a hypergroup. Furthermore, a brief investigation of the inverse branches of the Chebyshev polynomials shows that these are infinite. The concepts we established can serve as a starting point for an orthogonal polynomial point of view in transfer operator theory and can be transferred to the classical transfer operator theory in dynamical systems and wavelet theory.

Zusammenfassung

Wir verwenden Transfer Operatoren, ein Standradwerkzeug aus dem Gebiet der dynamischen Systeme, in Kombination mit der Theorie von orthogonalen Polynomen, polynomialen Hypergruppen und harmonischer Analysis, um einen neuen Transfer Operator zu definieren. Wir definieren diesen Transfer Operator durch die Urbilder bestimmter orthogonaler Polynome, der Tschebyscheff Polynome erster Art. Dieser Transfer Operator operiert auf verschieden Funktionenräumen, die durch orthogonale Polynome bestimmt sind. Wir erhalten, dass der Transfer Operator auf den untersuchten Funktionenräumen beschränkt ist, und dass der Einsetzoperator mit dem adjungierten Operator beziehungsweise mit der Rechtsinversen übereinstimmt. Mit Hilfe einer quadratischen Transformation, die durch das zweite Tschebyscheff Polynom erster Art gegeben ist, konstruieren wir eine Folge orthogonaler Polynome, die eine polynomiale Hypergruppe erzeugt, auf der der Transfer Operator operiert. In gleicher Weise verwenden wir eine kubische Transformation, um eine Folge orthogonaler Polynome zu konstruieren. Im kubischen Fall erzeugt die Folge orthogonaler Polynome jedoch keine Hypergruppe. Des weiteren zeigt eine kurze Untersuchung der Inversenzweige der Tschebyscheff Polynome erster Art, dass diese unendlich sind. Die Konzepte, die wir eingeführt haben, können als Ausgangspunkt für eine Sichtweise vom Standpunkt der orthogonalen Polynome in der Theorie der Transfer Operatoren dienen und übertragen werden in die klassische Theorie von Transfer Operatoren im Kontext von dynamischen Systemen und Wavelet-Theorie.

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Contents

Introduction

Transfer operators, also called Ruelle transfer operators, Ruelle operators or Ruelle-Perron-Frobenius operators,

$$R\phi(y) = \sum_{x\in f^{-1}(y)} \psi(x)\,\phi(x) \tag{0.1}$$

have first been introduced by David Ruelle in the 1960s in the context of thermodynamical formalism (see [**55**]). He proved in [**47**], [**49**] that the Ruelle-Perron-Frobenius operator acting on a Hölder continuous function space has a unique maximal positive eigenvalue with a positive eigenfunction if the given dynamical system f is the one-side shift on a symbolic space of finite type and when the given function is positive and Hölder continuous (see also [**8**]). D. Ruelle used this result to complete a mathematical understanding of the existence and the uniqueness of the equilibrium measure, the so-called Gibbs measure, for a Hölder continuous positive function on a symbolic space of finite type in thermodynamical formalism. His theorem is an important result in modern thermodynamical formalism. Since then, transfer operators have become a standard tool in dynamical systems, ergodic theory and other branches of mathematics as well as mathematical physics. P. Walters proved Ruelle's theorem in a more general setting, that is, the dynamical system can be a positive summable variational function (see [**60**]). There are many textbooks and articles about Ruelle's theorem, see for example [**5**], [**8**], [**18**], [**19**], [**20**], [**40**]. The spectral properties of transfer operators have been used to obtain results on various types of dynamical systems and to study their ergodic properties. D. Ruelle himself used transfer operators to investigate various types of dynamical systems, e.g. piecewise monotone maps (see [**53**]), Axiom A flows (see [**48**]), expanding maps and Anosov flows (see [**50**]) and to established correspondences of transfer operators to the so-called dynamical zeta function (see [**52**], [**54**]) and to Fredholm determinants (see [**51**]). V. Baladi, G. Keller and F. Hofbauer also employed transfer operators to study dynamical zeta functions (see [**4**],[**3**], [**25**]) as well as C. Liverani who gained new results on certain dynamical systems using the spectral properties of transfer operators (see [**35**], [**36**]).
In the 1990s P.E.T. Jorgensen started using a transfer operator in harmonic analysis and wavelet theory in order to find orthogonal wavelet bases. In [**29**], he found an interconnection between the cascade refinement operator, which is well-known in wavelet theory, and a transfer operator, as well as a one-to-one correspondence between the eigenfunctions of a transfer operator to the eigenvalue 1 and representations of certain C^*-algebras. In [**9**], O. Bratteli and P.E.T. Jorgensen obtained a relation between the spectral properties of a transfer operator and the question of convergence of the cascade algorithm for the approximation of the corresponding scaling function.
More recently, transfer operators have also been utilized in applied mathematics, e.g. in the analysis of biomolecular systems (see [**56**]).
In this thesis, we achieve synergies of the theory of orthogonal polynomials, polynomial

hypergroups, harmonic analysis and transfer operators. We define a new transfer operator which is based on certain orthogonal polynomials rather than using the dynamical systems context. The underlying preimage for the transfer operator which is usually given by a dynamical system f, will in our case be given by an orthogonal polynomial. We investigate the action of this transfer operator on function spaces determined by various orthogonal polynomial sequences, e.g. on homogeneous Banach spaces like the Wiener algebra and its p-versions. The orthogonal polynomials which we use to define our transfer operator will be the Chebyshev polynomials of the first kind as up to similarity they and the powers are the only chains, that is, they are a sequence of polynomials, each of positive degree which contains at least one of each degree and such that every two polynomials in it are permutable (see Theorem 1.1.42). Furthermore, they exhibit a symmetric distribution of the preimages in the even case which will be crucial for our invstigation. The Chebyshev polynomials of the first kind also form a polynomial hypergroup (see Example 1.2.27 (1)) which our transfer operator will act on. This case can be compared to P.E.T. Jorgensen's invstigations on the torus as for the Chebyshev polynomials of the first kind, the transfer operator and the corresponding transformation operator map the spaces they operate on into themselves. Moreover, the transformation operator coincides with the adjoint operator for the case of the Chebyshev polynomials of the first kind. We do not stop our investigation at this point, but consider the ultraspherical case which carries our studies over to the polynomial hypergroup induced and function spaces determined by the generalized Chebyshev polynomials. In this case, the spaces the transfer operator and the corresponding transformation operator as well as their iteratives operate on and the spaces they map into are not the same. We will consider an unweighted version of the transfer operator, that is, the function which corresponds to ψ in (0.1) is identical one, and a weighted version.
Furthermore, we even consider the case of arbitrary orthogonal polynomials which are symmetric. We construct an orthogonal polynomial sequence based on a quadratic transformation which is given by the second order Chebyshev polynomial of the first kind, $T_2(x) = 2x^2 - 1$. We can show that the constructed orthogonal polynomial sequence induces a hypergroup structure on $\mathbb{N}_0$, and we study the transfer operator on this polynomial hypergroup as well as the Wiener algebra induced by these polynomials.
In the same way, we construct an orthogonal polynomial sequence based on a cubic transformation which is given by the third order Chebyshev polynomial of the first kind, $T_3(x) = 4x^3 - 3x$.
Finally, we briefly analyze the inverse branches of the Chebyshev polynomials of the first kind and give an outlook on future work.
The first chapter of this thesis will provide the necessary background on orthogonal polynomials and hypergroups which will be needed in the subsequent chapters. We will give an introduction to orthogonal polynomials following the general setting of T.S. Chihara (see **[11]**) who uses a linear functional, the moment functional, to prove some main results on orthogonal polynomials. In this context, we will prove the fundamental recurrence formula (see Theorem 1.1.14) which will in Chapter 3 be crucial for the construction of the orthogonal polynomial sequences based on quadratic and cubic transformations, respectively, as well as Favard's theorem (see Theorem 1.1.21), which states that there is a unique moment functional for a polynomial sequence that satisfies the fundamental recurrence formula, and the Perron-Favard theorem, which states that there exists an unique measure which the polynomial sequence is orthogonal for if the fundamental recurrence

formula is satisfied (see Theorem 1.1.23). Based on the Riesz representation theorem, we will give a proof of the Perron-Favard theorem. We also deal with the question of symmetry for an orthogonal polynomial sequence. Since the Chebyshev polynomials of the first kind are a symmetric orthogonal polynomial sequence, symmetry will play a crucial role throughout Chapter 3 when we study a transfer operator based on orthogonal polynomials. Referring to T.J. Rivlin (see **[43]**), we provide that the Chebyshev polynomials of the first kind and the powers are up to similarity the only orthogonal polynomials which pairwise commute, which will be essential for our study of the transfer operator.

In the second part of the first chapter, we give a brief introduction to hypergroups and homogeneous Banach spaces following an unpublished book manuscript of R. Lasser as well as **[32]**, **[33]**, **[34]** and G. Fischer and R. Lasser (see **[22]**). We first give some general facts on hypergroups and discrete hypergroups and then turn to the study of polynomial hypergroups, which our transfer operator in Chapter 3 will be defined on. We will also provide some material on the linearization coefficients of polynomial hypergroups which we need in Section 3.7 for the orthogonal polynomial sequences defined through cubic transformations. The remainder of the first chapter is dedicated to the introduction of homogeneous Banach spaces which our transfer operator will act on.

The second chapter of this thesis will give a short glimpse of the theory of transfer operators both of the dynamical systems and the harmonic analysis point of view. This chapter will be the inspiration and the starting point for the framework that we develop in Chapter 3 concerning a transfer operator defined via orthogonal polynomials. However, we do not mean to extend the deep results on transfer operators provided here. The first section of Chapter 2 introduces transfer operators in the context of dynamical systems following Y.P. Jiang (see **[27]**). We will discuss Ruelle's theorem which consists of two parts. The first part states the existence and the simplicity of a unique maximal eigenvalue for a Ruelle-Perron-Frobenius operator acting on a Hölder continuous function space. The second part concerns the existence and uniqueness of the Gibbs measure. Y.P. Jiang found an elementary but elegant proof to the first part of Ruelle's theorem (see **[26]**) and the proof of the second part is a combination of the ideas in **[26]** and in **[14]** (see **[19]**).

We begin the second part of Chapter 2 with a short introduction to wavelet theory following I. Daubechies **[13]**. We give the definitions for wavelets and multiresolution analysis and conditions for the existence of an orthonormal wavelet basis both for infinitely supported wavelets and compactly supported wavelets. After having introduced the basic notions of wavelet theory, we discuss the results of P.E.T. Jorgensen **[29]** and O. Bratteli and P.E.T. Jorgensen **[9]** on the harmonic analysis of transfer operators. The spectral properties of the transfer operator which arise from polynomial wavelet filters are studied. In **[29]** P.E.T. Jorgensen defined a transfer operator of the form

$$Rf(z) = \sum_{\omega^N = z} |m_0(\omega)|^2 f(\omega), \qquad f \in L^1(\mathbb{T}), z \in \mathbb{T}, \tag{0.2}$$

where $m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ is a polynomial wavelet filter. There is a one-to-one correspondence between representations of a C^*-algebra and functions which are harmonic for the transfer operator (0.2), that is, functions $h \in L^1(\mathbb{T})$, $h \geqslant 0$ that satisfy $R(h) = h$. In **[9]** the transfer operator (0.2) is studied for the case that $N = 2$. The spectral properties of the transfer operator are related to the convergence question for the cascade algorithm for approximation of the corresponding wavelet scaling function. In Chapter 3, we define a transfer operator which is based on the Chebyshev polynomials of the first kind,

$\{T_n(x)\}_{n=0}^{\infty}$, by

$$R_{(m_N,T_N)}f(y) = \sum_{T_N(x)=y} m_N(x) f(x), \tag{0.3}$$

with the weight function $m_N(x) = \sum_{k=0}^{\infty} b_k T_k(x) h(k)$. We get for the unweighted Ruelle operator

$$R_{(1,T_N)}(T_n) = \begin{cases} T_{\frac{n}{N}}, & n = Nl, l \in \mathbb{N}_0 \\ 0, & else \end{cases}.$$

We find that the Ruelle operator acting on $C([-1,1])$ is a bounded linear operator and that $\|R_{(1,T_N)}\| = 1$. For the Ruelle operator acting on the Wiener algebra $A([-1,1])$ we have that $R_{(1,T_N)} \in B(A([-1,1]))$, $\|R_{(1,T_N)}\| = 1$ and $R_{(1,T_N)}f(y) = \sum_{n=0}^{\infty} \check{f}(Nn)$ $T_n(y) h(n)$ for each $f \in A([-1,1])$. We also prove similar results for the p-versions $A^p([-1,1])$ of the Wiener algebra and for $L^p([-1,1],\pi)$. Corresponding to the action of the Ruelle operator $R_{(1,T_N)}$ on $L^2([-1,1],\pi)$ we study an adjoint operator E of the Ruelle operator and obtain for $f, g \in C(S)$,

$$\int_{-1}^{1} R_{(1,T_N)}f(y)\overline{g(y)}d\pi(y) = \int_{-1}^{1} f(x)\overline{Eg(x)}d\pi(x). \tag{0.4}$$

We find that $E \circ R_{(1,T_N)}$ is a Morkovian projection as well as that the spectral radius of $R_{(1,T_N)}$ equals 1.
We obtain for the weighted Ruelle operator based on the Chebyshev polynomials of the first kind

$$\begin{aligned} &R_{(m_N,T_N)}(T_n) \\ &= \begin{cases} T_{\frac{n}{N}}, & n = Nl, l \in \mathbb{N}_0 \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\left(2+\sum_{k=1}^{\infty} 2b_{2k}T_{2k}(x_i)\right)h(2k)\right]T_n(x_i)\right), & n, N \text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\sum_{k=1}^{\infty} 2b_{2k-1}T_{2k-1}(x_i)h(2k-1)\right]T_n(x_i)\right), & n \text{ odd}, N \text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N}\left[\left(1+\sum_{j=1}^{N-1}\sum_{k=1}^{\infty} b_{k+j}T_{k+j}(x_i)\right)h(k+j)\right]T_n(x_i)\right), & N \text{ odd}, n \in \mathbb{N} \end{cases}, \end{aligned}$$

as well as that $R_{(m_N,T_N)}$ is a bounded operator on $C([-1,1])$, $A([-1,1])$, $A^p([-1,1])$ with the operator norm equal to 1 and on $L^p([-1,1],\pi)$ with the p-norm equal to $\|m_N\|$. Furthermore, we investigate the unweighted and the weighted Ruelle operator on function spaces determined by the generalized Chebyshev polynomials $\{T_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ and get for even N for the unweighted Ruelle operator that

$$R_{(1,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2} 2P_k^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & k = 2n \\ 0, & k = 2n+1 \end{cases}.$$

For N=2, we can prove that $R_{(1,T_2)} \in B\left(A_T^{(\alpha,\beta)}([-1,1]), A_P^{(\alpha,\beta)}([-1,1])\right)$ and $\|R_{(1,T_2)}\| = 1$ as well as $R_{(1,T_2)} \in B\left(L^p\left([-1,1],\pi_{\alpha,\beta}^T\right), L^p\left([-1,1],\pi_{\alpha,\beta}^P\right)\right)$ and $\|R_{(1,T_2)}\| = 1$, $1 \leqslant p < \infty$. We also investigate a transformation operator E for the unweighted Ruelle operator and get that E is the right inverse of $R_{(1,T_2)}$. In the weighted case, we deal with two different kinds of weight functions $m_N(x)$ defined as above, and $\tilde{m_N}(x)$ given by $\tilde{m_N}(x) = \sum_{k=0}^{\infty} \tilde{b}_k T_k^{(\alpha,\beta)}(x) h(k)$. We obtain that for $\tilde{m_N}(x)$, the Ruelle operator

$R_{(\tilde{m}_N,T_N)}$ acting on $\{T_n^{(\alpha,\beta)(x)}\}_{n=0}^{\infty}$ coincides with the unweighted case as well as $R_{(m_2,T_2)}$. Moreover, we construct an orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ from a given orthogonal polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ which satisfies

$$Q_{2n}(x) = P_n(T_2(x))$$

using the fundamental recurrence formula. We establish conditions for the recurrence coefficients so that $\{Q_n(x)\}_{n=0}^{\infty}$ generates a polynomial hypergroup and find that the Ruelle operator satisfies

$$R_{(1,T_2)}(Q_n) = \begin{cases} P_{\frac{n}{2}}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

We can also show that the Wiener algebra generated by the polynomials Q_n is a Banach space and that $R_{(1,T_2)} \in B(A_Q([-1,1]), A_P([-1,1]))$. For its right inverse E, we see that $E \in B(A_P([-1,1]), A_Q([-1,1]))$. For the operator norms, we have $\|R_{(1,T_2)}\| = 1$ and $\|E\| = 1$. Moreover,

$$\int_S R_{(1,T_2)} f(y) g(y) \, d\pi_P(y) = \int_S f(x) g(T_2(x)) \, d\pi_Q(x) = \int_S f(x) E(g)(x) \, d\pi_Q(x).$$

For the weighted case, we consider three different kinds of weight functions, $m_2(x)$, $\tilde{m}_2(x)$, defined as above, and $\tilde{\tilde{m}}_2(x)$ given by $\tilde{\tilde{m}}_2(x) = \sum_{k=0}^{\infty} \tilde{\tilde{b}}_k Q_k(x) h_Q(x)$. For $\tilde{m}_2(x)$ we find again that $R_{\tilde{m}_2,T_2}$ acting on $\{Q_n(x)\}_{n=0}^{\infty}$ coincides with the unweighted case. Considering the other weight functions, we get that

$$R_{(m_2,T_2)}(Q_n)(y) = \begin{cases} \frac{1}{2}\left[2b_0 T_0\left(x_1^{(2)}\right) h(0)\right] P_l(y) = P_l(y), & n = 2l \\ \frac{1}{2}\left[\sum_{k=0}^{\infty} 2b_{2k+1} T_{2k+1}\left(x_1^{(2)}\right) h(2k+1)\right] \frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}, & n = 2l+1 \end{cases},$$

and

$$R_{\left(\tilde{\tilde{m}}_2,T_2\right)}(Q_n)(y) = \begin{cases} P_l(y), & n = 2l \\ \frac{1}{2}\left[\sum_{k=0}^{\infty} 2\tilde{\tilde{b}}_{2k+1} \frac{P_{k+1}(y)+P_k(y)}{x_1^{(2)}} h_Q(2k+1)\right] \frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}, & n = 2l+1 \end{cases}.$$

We obtain for both weights that $R_{(m_2,T_2)} \in B(A_Q(S), A_P(S))$ and for its right inverse E, $E \in B(A_P(S), A_Q(S))$. For the operator norms, we have $\|R_{(m_2,T_2)}\| = 1$ and $\|E\| = 1$.
Then we construct another orthogonal polynomial sequence $\{C_n(x)\}_{n=0}^{\infty}$ from a given orthogonal polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ in the same way as in the quadratic case, but this time we use a cubic transformation

$$C_{3n}(x) = P_n(T_3(x)).$$

We find a representation for the polynomials $\{C_{3n}(x)\}_{n=0}^{\infty}$, and for the products, we calculate the corresponding linearization coefficients. For the connection coefficients we obtain

$$C_{3n}(x) = P_n(T_3(x)) = \sum_{k=0}^{n} \kappa_{n,k} T_k(T_3(x)) = \sum_{k=0}^{n} \kappa_{n,k} T_{3k}(x)$$

for $P_n(x) = \sum_{k=0}^{n} \kappa_{n,k} T_k(x)$.
For the cubic Ruelle operator $R_{(1,T_3)}$ acting on $\{C_n(x)\}_{n=0}^{\infty}$ we get

$$R_{(1,T_3)}(C_{3n}) = P_n, \qquad R_{(1,T_3)}(C_{3n+1}) = 0 = R_{(1,T_3)}(C_{3n+2}).$$

Moreover, for the cubic Ruelle operator $R_{(1,T_3)}$ and the cubic transformation operator E, $Eg = g \circ T_3$, we have:

$$\int_S R_{(1,T_3)} f(y)\, g(y)\, d\pi_P(y) = \int_S f(x)\, Eg(x)\, d\pi_C(x),$$

and $R_{(1,T_3)} \in B(A_C(S), A_P(S))$, $E \in B(A_P(S), A_C(S))$. For the operator norms, we find $\|R_{(1,T_3)}\| = 1$ and $\|E\| = 1$.
In Chapter 4 we conclude with the result that the inverse branches of the Chebyshev polynomials of the first kind are infinite and give a short outlook.
The concepts we established in Chapter 3 and 4 serve as a starting point for an orthogonal polynomial point of view in transfer operator theory and can be transferred to the classical transfer operator theory in dynamical systems and wavelet theory introduced in Chapter 2 which will be work that goes beyond the amount of a dissertation.

1 Orthogonal polynomials and hypergroups

1.1 Orthogonal polynomials

In this section, we will introduce orthogonal polynomials following Chihara [**11**] and provide the background that we will need for the study of polynomial hypergroups in Section 1.2.3 and throughout Chapter 3 for the Ruelle operator that we define as well as for the function spaces our Ruelle operator will act on and for the construction of certain orthogonal polynomial sequences. For some additional properties of the Chebychev polynomials of the first kind and for the generalized Chebychev polynomials we will consult [**43**], [**23**], [**31**], [**33**] and [**2**].

Chihara's approach to the theory of orthogonal polynomials using a so-called moment functional is motivated by a high level of generality which will be illustrated on the following two pages.

Let w be a nonnegative and integrable function on an interval (a, b) which satisfies $w(x) > 0$ on a sufficiently large subset of (a, b) and

$$\int_a^b w(x)\, dx > 0.$$

If (a, b) is unbounded, then the *moments* μ_n are additionally required to satisfy

$$\mu_n = \int_a^b x^n w(x)\, dx < \infty, \qquad n \in \mathbb{N}_0. \tag{1.1}$$

If there is a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, P_n of degree n, such that

$$\int_a^b P_m(x)\, P_n(x)\, w(x)\, dx = 0, \qquad m \neq n, \tag{1.2}$$

then $\{P_n(x)\}_{n=0}^{\infty}$ is called an *orthogonal polynomial sequence* with respect to the *weight function* w on (a, b). Now, we can write for any function f,

$$\mathcal{M}[f] = \int_a^b f(x)\, w(x)\, dx, \tag{1.3}$$

and thus we get for (1.1) and (1.2)

$$\begin{aligned} \mathcal{M}[x^n] &= \mu_n, \qquad n \in \mathbb{N}_0, & (1.4)\\ \mathcal{M}[P_m(x)\, P_n(x)] &= 0, \qquad m \neq n, \qquad m, n \in \mathbb{N}_0. & (1.5) \end{aligned}$$

$\mathcal{M}$ is a linear functional

$$\mathcal{M}\left[af\left(x\right)+bg\left(x\right)\right]=a\mathcal{M}\left[f\left(x\right)\right]+b\mathcal{M}\left[g\left(x\right)\right] \tag{1.6}$$

for arbitrary constants a, b and integrable functions f, g, and (1.4) and (1.6) suffice to define $\mathcal{M}\left[Q\left(x\right)\right]$ for any polynomial $Q\left(x\right)$, i.e.

$$\mathcal{M}\left[\sum_{k=0}^{n}c_k x^k\right]=\sum_{k=0}^{n}c_k\mu_k.$$

Thus, more generally, we consider an arbitrary sequence of real or complex numbers $\{\mu_n\}_{n=0}^{\infty}$. Hence a linear functional $\mathcal{M}$ on the vector space of all polynomials in one real variable can be defined by (1.4) and (1.6), and if there is a sequence $\{P_n\left(x\right)\}_{n=0}^{\infty}$ of polynomials satisfying (1.5) and

$$\mathcal{M}\left[P_n^2\left(x\right)\right]\neq 0,$$

then it is called an orthogonal polynomial sequence with respect to $\mathcal{M}$.
In order to illustrate the generality of the use of the functional $\mathcal{M}$ to define orthogonal polynomial sequences, we will introduce an orthogonality relation which seems to be of a different type than (1.2):
Let

$$G\left(x,w\right)=e^{-aw}\left(1+w\right)^x=\sum_{m=0}^{\infty}\frac{\left(-a\right)^m w^m}{m!}\sum_{n=0}^{\infty}\binom{x}{n}w^n \tag{1.7}$$

with parameter $a\neq 0$. Then when forming the Cauchy product of the two series in (1.7), we have

$$G\left(x,w\right)=\sum_{n=0}^{\infty}P_n\left(x\right)w^n, \tag{1.8}$$

where

$$P_n\left(x\right)=\sum_{k=0}^{n}\binom{x}{k}\frac{\left(-a\right)^{n-k}}{\left(n-k\right)!}. \tag{1.9}$$

Since $\binom{x}{k}=\frac{1}{k!}x\left(x-1\right)\cdots\left(x-k+1\right)$ for $k=1,2,\ldots,n$, we get that $P_n\left(x\right)$ is a polynomial of degree n. $P_n\left(x\right)$ or some constant multiple of $P_n\left(x\right)$ is called a *Charlier polynomial* and $G\left(x,w\right)$ is said to be the *generating function* for $\{P_n\left(x\right)\}_{n=0}^{\infty}$.
With (1.7) we have that

$$a^xG\left(x,v\right)G\left(x,w\right)=e^{-a\left(v+w\right)}\left[a\left(1+v\right)\left(1+w\right)\right]^x,$$

thus

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{a^kG\left(k,v\right)G\left(k,w\right)}{k!} &= e^{-a\left(v+w\right)}e^{a\left(1+v\right)\left(1+w\right)}\\ &= e^ae^{avw}=\sum_{n=0}^{\infty}\frac{e^aa^n\left(vw\right)^n}{n!},\end{aligned}$$

and with (1.8) we have

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{a^k G(k,v)\,G(k,w)}{k!} &= \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{m,n=0}^{\infty} P_m(k)\,P_n(k)\,v^m w^n \\ &= \sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} P_m(k)\,P_n(k)\,\frac{a^k}{k!} v^m w^n.\end{aligned}$$

Comparison of the coefficients of $v^m w^n$ yields

$$\sum_{k=0}^{\infty} P_m(k)\,P_n(k)\,\frac{a^k}{k!} = \begin{cases} 0, & m \neq n \\ \frac{e^a a^n}{n!}, & m = n \end{cases}. \tag{1.10}$$

Then we call $\{P_n(x)\}_{n=0}^{\infty}$ an orthogonal polynomial sequence with respect to the discrete mass distribution which has mass $a^k/k!$ at the point k, $k \in \mathbb{N}_0$. Now, we write

$$\mathcal{M}[x^n] = \sum_{k=0}^{\infty} k^n \frac{a^k}{k!}, \qquad n \in \mathbb{N}_0 \tag{1.11}$$

and define $\mathcal{M}$ for all polynomials by linearity. Then, we see that (1.10) can be written as

$$\mathcal{M}[P_m(x)\,P_n(x)] = \frac{e^a a^n}{n!}\delta_{mn}, \qquad m, n \in \mathbb{N}_0.$$

Hence both (1.2) and (1.10) can be described by the linear functional $\mathcal{M}$.

1.1.1 Orthogonal polynomials sequences

Definition 1.1.1. (1) Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, called the *moment sequence*, and let $\mathcal{M}$ be a complex valued function, the *moment functional*, defined on the vector space of polynomials with complex coefficients in one variable $\mathcal{P}$ by

$$\begin{aligned}\mathcal{M}[x^n] &= \mu_n, \qquad n \in \mathbb{N}_0, \\ \mathcal{M}[\alpha Q(x) + \beta R(x)] &= \alpha \mathcal{M}[Q(x)] + \beta \mathcal{M}[R(x)] \qquad \forall R(x), Q(x) \in \mathcal{P}, \alpha, \beta \in \mathbb{C}.\end{aligned}$$

The number μ_n denotes the moment of order n.

(2) A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is called an *orthogonal polynomial sequence* (OPS) with respect to the moment functional $\mathcal{M}$ if for all nonnegative integers m and n,

(i) $P_n(x)$ is a polynomial of degree n,

(ii) $\mathcal{M}[P_m(x)\,P_n(x)] = 0$ for $m \neq n$,

(iii) $\mathcal{M}[P_n^2(x)] \neq 0$.

(3) If $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS for $\mathcal{M}$ and $\mathcal{M}[P_n^2(x)] = 1$, then $\{P_n(x)\}_{n=0}^{\infty}$ is an *orthonormal polynomial sequence*.

Examples 1.1.2. (1) The *Chebyshev polynomials of the first kind*, $\{T_n(x)\}_{n=0}^{\infty}$, defined by

$$\begin{aligned} T_n(x) = \cos(n\theta) &= \cos\left(n\cos^{-1}(x)\right) \\ &= \cos(n\arccos(x)), \qquad \theta \in [0,\pi], x = \cos\theta \in [-1,1], \end{aligned}$$

form an OPS with respect to the weight function $w(x) = (1-x^2)^{-1/2}$.
We have

$$\begin{aligned} &T_0(x) = 1, && T_1(x) = x, \\ &T_2(x) = 2x^2 - 1, && T_3(x) = 4x^3 - 3x, \text{ etc.} \end{aligned}$$

(2) The *Chebyshev polynomials of the second kind*, $\{U_n(x)\}_{n=0}^{\infty}$, defined by

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)} = \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))}, \quad \theta \in [0,\pi], x = \cos\theta \in [-1,1],$$

are an OPS with respect to the weight function $w(x) = (1-x^2)^{1/2}$.
We have

$$\begin{aligned} &U_0(x) = 1, && U_1(x) = 2x, \\ &U_2(x) = 4x^2 - 1, && U_3(x) = 8x^3 - 4x, \text{ etc.} \end{aligned}$$

Moreover, we have

$$U_{n-1}(x) = \frac{1}{n}T_n'(x) = \frac{\sin(n\theta)}{\sin(\theta)}, \qquad x = \cos(\theta), \tag{1.12}$$

and by an easy calculation

$$T_n^2(x) - \left(x^2-1\right)U_{n-1}^2(x) = 1. \tag{1.13}$$

(3) The *Legendre polynomials*, $\{L_n(x)\}_{n=0}^{\infty}$, are given by

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right], \qquad x \in [-1,1].$$

They form an OPS with respect to the weight function $w(x) = \chi_{[-1,1]}$. We have

$$\begin{aligned} &L_0(x) = 1, && L_1(x) = x, \\ &L_2(x) = \frac{1}{2}\left(3x^2-1\right), && L_3(x) = \frac{1}{2}\left(5x^3-3x\right), \text{ etc.} \end{aligned}$$

(4) The *Jacobi polynomials*, $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, can be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2),$$

where

$${}_qF_p(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n\,x^n}{(\beta_1)_n\cdots(\beta_q)_n\,n!}$$

is the hypergeometric function and $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the Pochhammer symbol.
When $\alpha, \beta > -1$, the Jacobi polynomials form an OPS on $[-1,1]$ with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$.
For $\alpha = \beta = 0$, the Jacobi polynomials equal the Legendre polynomials, for $\alpha = \beta = -\frac{1}{2}$, they equal the Chebyshev polynomials of the first kind and for $\alpha = \beta = \frac{1}{2}$, they equal the Chebyshev polynomials of the second kind.

(5) For $\alpha = \beta$ the Jacobi polynomials are called *ultraspherical polynomials* or *Gegenbauer polynomials*, $\{P_n^{(\alpha)}(x)\}_{n=0}^\infty$, and are defined by

$$P_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{n!} {}_2F_1\left(-n, 2\alpha+n; \alpha+\frac{1}{2}; (1-x)/2\right).$$

For fixed α the ultraspherical polynomials form an OPS on $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^\alpha$. They are related to the Jacobi polynomials by the following equation:

$$\begin{aligned} P_n^{(\alpha)}(x) &= \frac{(2\alpha)_n}{\left(\alpha+\frac{1}{2}\right)_n} P_n^{(\alpha,\alpha)}(x) \\ &= \frac{(2\lambda+1)_n}{(\lambda\ +1)_n} P_n^{(\lambda,\lambda)}(x), \qquad \lambda = \alpha - \frac{1}{2}. \end{aligned}$$

(6) The *generalized Chebyshev polynomials*, $\{T_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $\alpha, \beta > -1$, are given by

$$T_n^{(\alpha,\beta)}(x) = \begin{cases} R_k^{(\alpha,\beta)}(2x^2-1), & n = 2k \\ xR_k^{(\alpha,\beta+1)}(2x^2-1), & n = 2k+1 \end{cases},$$

where

$$R_k^{(\alpha,\beta)}(x) = \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} = {}_2F_1\left(-k, k+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)\right).$$

They are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^\alpha$ $|x|^{2\beta+1}$ and are normalized by $T_n^{\alpha,\beta}(1) = 1$.
For $\beta = -\frac{1}{2}$ the generalized Chebyshev polynomials are related to the ultraspherical polynomials by

$$T_n^{\left(\alpha,-\frac{1}{2}\right)}(x) = \frac{P_n^{(\alpha)}(x)}{P_n^{(\alpha)}(1)} = \frac{n!}{(2\alpha+1)_n} P_n^{(\alpha)}(x).$$

For $\alpha = \beta = -\frac{1}{2}$ the generalized Chebyshev polynomials equal the Chebyshev polynomials of the first kind.

Theorem 1.1.3. *Let $\mathcal{M}$ be a moment functional and let $\{P_n(x)\}_{n=0}^\infty$ be a sequence of polynomials. Then the following properties are equivalent:*

(i) $\{P_n(x)\}_{n=0}^\infty$ is an OPS with respect to $\mathcal{M}$,

(ii) $\mathcal{M}[Q(x)P_n(x)] = 0$ *for every polynomial* $Q(x)$ *of degree* $m < n$, *and* $\mathcal{M}[Q(x)P_n(x)] \neq 0$ *if* $m = n$,

(iii) $\mathcal{M}[x^m P_n(x)] = K_n \delta_{mn}$, *where* $K_n \neq 0$, $m = 0, 1, \ldots, n$.

Proof. Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS for $\mathcal{M}$. Each $P_k(x)$ is of degree k, thus $\{P_0(x), P_1(x), \ldots, P_m(x)\}$ is a basis for the vector subspace of polynomials of degree $\leqslant m$. Hence if $Q(x)$ is a polynomial of degree m, then there exist constants c_k such that

$$Q(x) = \sum_{k=0}^{m} c_k P_k(x), \qquad c_m \neq 0.$$

The linearity of $\mathcal{M}$ implies,

$$\mathcal{M}[Q(x)P_n(x)] = \begin{cases} \sum_{k=0}^{m} c_k \mathcal{M}[P_k(x)P_n(x)] = 0, & m < n \\ c_k \mathcal{M}[P_n^2(x)], & m = n \end{cases}.$$

This proves $(i) \Rightarrow (ii)$. Clearly, $(ii) \Rightarrow (iii) \Rightarrow (i)$ follows from the definitions of the Kronecker delta δ_{mn}, $\mathcal{M}$ and the OPS. □

Theorem 1.1.4. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be an OPS with respect to* $\mathcal{M}$. *Then for every polynomial* $Q(x)$ *of degree* n,

$$Q(x) = \sum_{k=0}^{n} c_k P_k(x)$$

with

$$c_k = \frac{\mathcal{M}[Q(x)P_k(x)]}{\mathcal{M}[P_k^2(x)]}, \qquad k = 0, 1, \ldots, n. \tag{1.14}$$

Proof. If $Q(x)$ is a polynomial of degree n, then there are constants c_k such that

$$Q(x) = \sum_{k=0}^{n} c_k P_k(x).$$

Multiplying both sides of this equation with $P_m(x)$ and applying $\mathcal{M}$ gives

$$\mathcal{M}[Q(x)P_m(x)] = \sum_{k=0}^{n} c_k \mathcal{M}[P_k(x)P_m(x)] = c_m \mathcal{M}\left[P_m^2(x)\right].$$

By Definition 1.1.1 (2)(iii), $\mathcal{M}[P_m^2(x)] \neq 0$, hence (1.14) follows. □

Corollary 1.1.5. *If* $\{P_n(x)\}_{n=0}^{\infty}$ *is an OPS for* $\mathcal{M}$, *then each* $P_n(x)$ *is determined uniquely up to multiplication with a nonzero constant. That is, if* $\{Q_n(x)\}_{n=0}^{\infty}$ *is another OPS for* $\mathcal{M}$, *then there are constants* $c_n \neq 0$ *such that*

$$Q_n(x) = c_n P_n(x), \qquad n \in \mathbb{N}_0. \tag{1.15}$$

Proof. If $\{Q_n(x)\}_{n=0}^{\infty}$ is an OPS for $\mathcal{M}$, then Theorem 1.1.3 (ii) implies that

$$\mathcal{M}[P_k(x)Q_n(x)] = 0, \qquad k < n.$$

By setting $Q(x) = Q_n(x)$ in Theorem 1.1.4, (1.15) is obtained. □

Definition 1.1.6. (1) A polynomial $P_n(x)$ is called a *monic* polynomial if its leading coefficient equals 1.

(2) Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS and let each $P_n(x)$ be monic, then $\{P_n(x)\}_{n=0}^{\infty}$ is called a *monic OPS*.

Now, we turn to the question of the existence of OPS. For this purpose, we introduce the determinants

$$\triangle_n = \det(\mu_{i+j})_{i,j=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}. \tag{1.16}$$

Theorem 1.1.7. *Let $\mathcal{M}$ be a moment functional and $\{\mu_n\}_{n=0}^{\infty}$ the corresponding moment sequence. There exists an OPS for $\mathcal{M}$ if and only if*

$$\triangle_n \neq 0, \qquad n \in \mathbb{N}_0.$$

Proof. We set

$$P_n(x) = \sum_{k=0}^{n} c_{nk} x^k.$$

Then by Theorem 1.1.3,

$$\mathcal{M}[x^m P_n(x)] = \sum_{k=0}^{n} c_{nk}\mu_{k+m} = K_n \delta_{mn}, \qquad K_n \neq 0,\ m \leqslant n, \tag{1.17}$$

is equivalent to

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix} \begin{bmatrix} c_{n0} \\ c_{n1} \\ \cdot \\ \cdot \\ c_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ K_n \end{bmatrix}. \tag{1.18}$$

Thus if an OPS for $\mathcal{M}$ exists, then it is uniquely determined by the constants K_n in (1.17) has a unique solution so that $\triangle_n \neq 0$, $n \geqslant 0$.
Conversely, if $\triangle_n \neq 0$, then for $K_n \neq 0$, (1.18) has a unique solution and $P_n(x)$ satisfying (1.17) exists. Furthermore,

$$c_{nn} = \frac{K_n \triangle_{n-1}}{\triangle_n} \neq 0, \qquad n \geqslant 0, \tag{1.19}$$

where $\triangle_{-1} = 1$. Hence $P_n(x)$ is of degree n and thus, $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS for $\mathcal{M}$. □

As we will use (1.19) later on, we will prove the following theorem:

Theorem 1.1.8. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS for $\mathcal{M}$. Then for any polynomial $Q_n(x)$ of degree n,*

$$\mathcal{M}\left[Q_n(x) P_n(x)\right] = a_n \mathcal{M}\left[x^n P_n(x)\right] = \frac{a_n k_n \triangle_n}{\triangle_{n-1}}, \qquad \triangle_{-1} = 1, \tag{1.20}$$

with a_n the leading coefficient of Q_n and k_n the leading coefficient of $P_n(x)$.

Proof. If we set

$$Q_n(x) = a_n x^n + Q_{n-1}(x),$$

where $Q_{n-1}(x)$ denotes a polynomial of degree $n-1$, then

$$\begin{aligned} \mathcal{M}\left[Q_n(x) P_n(x)\right] &= a_n \mathcal{M}\left[x^n P_n(x)\right] + \mathcal{M}\left[Q_{n-1}(x) P_n(x)\right] \\ &= a_n \mathcal{M}\left[x^n P_n(x)\right]. \end{aligned}$$

Thus for $k_n = c_{nn}$, (1.19) implies (1.20). □

Definition 1.1.9. (1) A moment functional $\mathcal{M}$ is called *positive-definite*, if $\mathcal{M}\left[Q(x)\right] > 0$ for every polynomial that is not identical zero and nonnegative for all $x \in \mathbb{R}$.

(2) $\mathcal{M}$ is called *quasi-definite* if and only if $\triangle_n \neq 0$, $n \geqslant 0$.

Theorem 1.1.10. *If the moment functional $\mathcal{M}$ is positive-definite, then the corresponding moment sequence $\{\mu_n\}_{n=0}^{\infty}$ is real and there exists a corresponding OPS which consists of real polynomials.*

Proof. Let $\mathcal{M}$ be a positive-definite moment functional, then

$$\mu_{2k} = \mathcal{M}\left[x^{2k}\right] > 0,$$

and since

$$0 < \mathcal{M}\left[(x+1)^{2n}\right] = \sum_{k=0}^{2n} \binom{2n}{k} \mu_{2n-k},$$

we get by induction that μ_{2k+1} is real.
Then by the Gram-Schmidt process (see **[11]**, pp. 13, 14) an OPS can be constructed. □

Lemma 1.1.11. *Let $Q(x)$ be a polynomial which is nonnegative for real x. Then there are real polynomials $P(x)$ and $R(x)$ satisfying*

$$Q(x) = P^2(x) + R^2(x).$$

Proof. If $Q(x) \geqslant 0$ for real x, then $Q(x)$ is a real polynomial, its real zeros have even multiplicity and its non-real zeros occur in conjugate pairs, that is

$$Q(x) = S^2(x) \prod_{k=1}^{m} (x - \alpha_k - \beta_k i)(x - \alpha_k + \beta_k i),$$

with $S(x)$ a real polynomial and $\alpha_k, \beta_k \in \mathbb{R}$. By setting

$$\prod_{k=1}^{m} (x - \alpha_k - \beta_k i) = A(x) + iB(x),$$

where $A(x)$ and $B(x)$ are real polynomials, we have

$$Q(x) = S^2(x)\left[A^2(x) + B^2(x)\right].$$

□

Theorem 1.1.12. *$\mathcal{M}$ is positive-definite if and only if its moment sequence $\{\mu_n\}_{n=0}^{\infty}$ is real and $\triangle_n > 0$, $n \geqslant 0$.*

Proof. Let μ_n be real and $\triangle_n > 0$, $n \geqslant 0$. By Theorem 1.1.7 an OPS $\{P_n(x)\}_{n=0}^{\infty}$ exists for $\mathcal{M}$. Without loss of generality $P_n(x)$ is monic. Then by Theorem 1.1.8 we have

$$\mathcal{M}\left[P_n^2(x)\right] = \frac{\triangle_n}{\triangle_{n-1}} > 0.$$

By (1.18) $P_n(x)$ is real and thus if $P(x)$ is a real polynomial of degree m, then

$$P(x) = \sum_{k=0}^{m} a_k P_k(x),$$

where $a_k \in \mathbb{R}$ for all k and $a_m \neq 0$. Hence

$$\mathcal{M}\left[P^2(x)\right] = \sum_{j,k=0}^{m} a_j a_k \mathcal{M}\left[P_j(x) P_k(x)\right] = \sum_{k=0}^{m} a_k^2 \mathcal{M}\left[P_k^2(x)\right] > 0.$$

Finally, Lemma 1.1.11 implies that $\mathcal{M}$ is positive-definite.
Conversely, if $\mathcal{M}$ is positive-definite, then by Theorem 1.1.10 its moment sequence $\{\mu_n\}_{n=0}^{\infty}$ is real and there exists an OPS, $\{P_n(x)\}_{n=0}^{\infty}$, for $\mathcal{M}$. Again $\{P_n(x)\}_{n=0}^{\infty}$ is supposed to be monic, and again we have

$$0 < \mathcal{M}\left[P_n^2(x)\right] = \frac{\triangle_n}{\triangle_{n-1}}, \qquad n \geqslant 0.$$

Since $\triangle_{-1} = 1$, it follows that $\triangle_n > 0$, $n \geqslant 0$. □

An immediate consequence of the previous theorem is the following corollary:

Corollary 1.1.13. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS for $\mathcal{M}$. If $P_n(x)$ is real and $\mathcal{M}\left[P_n^2(x)\right] > 0$, then $\mathcal{M}$ is positive-definite.*

1.1.2 The fundamental recurrence formula

The following theorem provides the three-term recurrence formula for orthogonal polynomials, i.e. any three consecutive polynomials can be related by an easy relation. We will use this recurrence formula throughout Section 3.5 and Section 3.7 in order to construct orthogonal polynomials via a quadratic and a cubic transformation, respectively, which induce function spaces the Ruelle operator will act on.

Theorem 1.1.14. *Let $\mathcal{M}$ be a quasi-definite moment functional and let $\{P_n(x)\}_{n=0}^{\infty}$ be the corresponding monic OPS. Then there are constants c_n and $\lambda_n \neq 0$ satisfying*

$$P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n \in \mathbb{N}, \tag{1.21}$$

with $P_{-1}(x) = 0$.
Moreover, if $\mathcal{M}$ is positive-definite, then c_n is real and $\lambda_{n+1} > 0$ for $n \geqslant 1$.

Proof. Since $xP_n(x)$ is a polynomial of degree $n+1$,

$$xP_n(x) = \sum_{k=0}^{n+1} a_{nk}P_k(x), \qquad a_{nk} = \frac{\mathcal{M}[xP_n(x)P_k(x)]}{\mathcal{M}[P_k^2(x)]}.$$

But $xP_k(x)$ is a polynomial of degree $k+1$ so that $a_{nk}=0$ for $0 \leqslant k \leqslant n-1$ and as $xP_n(x)$ is monic, $a_{n,n+1}=1$. Thus we have

$$xP_n(x) = P_{n+1}(x) + a_{nn}P_n(x) + a_{n,n-1}P_{n-1}(x), \qquad n \geqslant 1. \tag{1.22}$$

If we replace n by $n-1$, then (1.22) rewrites as

$$xP_{n-1}(x) = P_n(x) + c_nP_{n-1}(x) + \lambda_nP_{n-2}(x), \qquad n \geqslant 2.$$

which is equivalent to (1.21) for $n \geqslant 2$. If we set $P_{-1}(x)=0$ and $c_1 = -P_1(0)$ (λ_1 is arbitrary), then (1.21) is also valid for $n=1$. We get from (1.21)

$$\mathcal{M}\left[x^{n-2}P_n(x)\right] = \mathcal{M}\left[x^{n-1}P_{n-1}(x)\right] - c_n\mathcal{M}\left[x^{n-2}P_{n-1}(x)\right] - \lambda_n\mathcal{M}\left[x^{n-2}P_{n-2}(x)\right],$$

$$0 = \mathcal{M}\left[x^{n-1}P_{n-1}(x)\right] - \lambda_n\mathcal{M}\left[x^{n-2}P_{n-2}(x)\right].$$

Now, by Theorem 1.1.8, we get for $n \geqslant 1$,

$$\lambda_{n+1} = \frac{\mathcal{M}[x^nP_n(x)]}{\mathcal{M}[x^{n-1}P_{n-1}(x)]} = \frac{\triangle_{n-2}\triangle_n}{\triangle_{n-1}^2}, \qquad \triangle_{-1}=1.$$

This implies that $\lambda_n \neq 0$ if $\mathcal{M}$ is quasi-definite and $\lambda_n > 0$, $n \geqslant 2$, if $\mathcal{M}$ is positive-definite. Since the $P_k(x)$ are real, c_n is real. □

The proof of Theorem 1.1.14 provides additional facts that will be stated in the next theorem:

Theorem 1.1.15. *Let $n \geqslant 1$, then:*

(i)
$$\lambda_{n+1} = \frac{\mathcal{M}[P_n^2(x)]}{\mathcal{M}\left[P_{n-1}^2(x)\right]} = \frac{\triangle_{n-2}\triangle_n}{\triangle_{n-1}^2},$$

(ii) $\mathcal{M}[P_n^2(x)] = \lambda_1\lambda_2\cdots\lambda_{n+1}$, *where* $\lambda_1 := \mu_0 = \triangle_0$,

(iii)
$$c_n = \frac{\mathcal{M}\left[xP_{n-1}^2(x)\right]}{\mathcal{M}\left[P_{n-1}^2(x)\right]},$$

(iv) The coefficient of x^{n-1} in $P_n(x)$ is $-(c_1+c_2+\cdots+c_n)$.

Proof. The proof of Theorem 1.1.14 provides the formula in (i), and (ii) follows form (i).
If both sides of (1.21) are multiplied by $P_n(x)$ and $\mathcal{M}$ is applied, then (iii) follows.
Let d_n be the coefficient of x^{n-1} in $P_n(x)$, then comparison of the coefficients of x^{n-1} on both sides of (1.21) provides that $d_n = d_{n-1} - c_n$, which implies (iv). □

Remark 1.1.16. In the case that $\{P_n(x)\}_{n=0}^{\infty}$ is not monic, then its recurrence formula is of the form

$$P_{n+1}(x) = (A_n x + B_n) P_n(x) - C_n P_{n-1}(x), \qquad n \geqslant 0. \tag{1.23}$$

Writing $P_n(x) = k_n \hat{P}_n(x)$, where $\hat{P}_n(x)$ is monic, gives

$$A_n = k_n^{-1} k_{n+1}, \quad B_n = -c_{n+1} k_n^{-1} k_{n+1}, \quad C_n = \lambda_{n+1} k_{n-1}^{-1} k_{n+1}, \qquad n \geqslant 0, \tag{1.24}$$

with $k_{-1} = 1$, and c_n, λ_n given by Theorem 1.1.15 in terms of $\{\hat{P}_n(x)\}_{n=0}^{\infty}$.
Particularly, $A_n \neq 0$, $C_n \neq 0$, and the positive-definiteness condition for $\mathcal{M}$ can be rewritten as

$$C_n A_n A_{n-1} > 0, \qquad n \geqslant 1. \tag{1.25}$$

Example 1.1.17. For the Chebyshev polynomials of the first kind we use the trigonometric identity

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(n\theta)\cos(\theta), \qquad n \geqslant 1,$$

to obtain

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \qquad n \geqslant 1. \tag{1.26}$$

Since $T_0(x) = 1$, $T_1(x) = x$, we obtain from (1.26) by induction that the leading coefficient of $T_n(x)$ is 2^{n-1}. This implies that the corresponding monic polynomials are

$$\hat{T}_0(x) = T_0(x), \quad \hat{T}_n(x) = 2^{1-n} T_n(x), \qquad n \geqslant 1.$$

Thus the recurrence formula (1.21) takes the form

$$\begin{aligned} \hat{T}_n(x) &= x\hat{T}_{n-1}(x) - \frac{1}{4}\hat{T}_{n-2}(x), \qquad n \geqslant 3, \\ \hat{T}_2(x) &= x\hat{T}_1(x) - \frac{1}{2}\hat{T}_0(x). \end{aligned}$$

Definition 1.1.18. A moment functional $\mathcal{M}$ is called *symmetric* if all its moments of odd order, $\{\mu_n\}$, $n = 2l+1$ with $l \in \mathbb{N}_0$, are 0.

In the case that $\mathcal{M}$ is given in terms of a weight function, then it will be symmetric if $a = -b$ and w is an even function on $[-b, b]$.
In the following, we will also call the corresponding OPS symmetric.

Examples 1.1.19. (1) The Chebyshev polynomials of the first kind are a symmetric OPS. $T_n(x)$ is an odd or an even function depending on whether n is odd or even; all moments of odd order of the corresponding moment functional equal 0. This is also related to the fact that in its recurrence formula the coefficient corresponding to c_n in (1.21) is 0.

(2) The Legendre polynomials are a symmetric OPS.

Theorem 1.1.20. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic OPS with respect to a quasi-definite moment functional, then the following assertions are equivalent:*

(i) $\mathcal{M}$ is symmetric,

(ii) $P_n(-x) = (-1)^n P_n(x)$, $n \geqslant 0$,

(iii) The coefficient c_n, $n \geqslant 1$, *in the corresponding recurrence formula equals* 0.

Proof. We will prove $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.
If $\mathcal{M}$ is symmetric, then $\mathcal{M}[Q(-x)] = \mathcal{M}[Q(x)]$ for any polynomial $Q(x)$. This implies

$$\mathcal{M}[P_m(-x) P_n(-x)] = \mathcal{M}[P_m(x) P_n(x)].$$

Thus by Corollary 1.1.5, $P_n(-x) = a_n P_n(x)$ where a_n is a constant. Then by comparison of the leading coefficients, $a_n = (-1)^n$.
Conversely, if $P_n(-x) = (-1)^n P_n(x)$, then $P_n(x)$ contains only the odd powers of x when n is odd. Hence

$$\mathcal{M}[P_1(x)] = \mu_1 = 0, \quad \mathcal{M}[P_3(x)] = \mu_3 = 0, \quad \text{etc.},$$

and we get inductively $\mu_{2n+1} = 0$, $n \geqslant 0$, which completes the proof of $(i) \Leftrightarrow (ii)$.
Now, we prove the second equivalence relation, $(ii) \Leftrightarrow (iii)$. With (1.21) we get for $(-1)^n P_n(x) = Q_n(x)$,

$$Q_n(x) = (x + c_n) Q_{n-1}(x) - \lambda_n Q_{n-2}(x), \qquad n \geqslant 1.$$

It follows that for $Q_n(x) = P_n(x)$ subtraction of the above equation from (1.21) yields that $2c_n P_{n-1}(x) = 0$ and thus $c_n = 0$ for $n \geqslant 1$.
Conversely, if $c_n = 0$ in (1.21) for $n \geqslant 1$, then $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies the same recurrence formula as $\{P_n(x)\}_{n=0}^{\infty}$. The fact that $Q_{-1}(x) = P_{-1}(x)$ and $Q_0(x) = P_0(x)$, implies that $Q_n(x) = P_n(x)$ for all n. □

The following theorem will provide the converse to Theorem 1.1.14, that is, that any polynomial which satisfies a recurrence relation of the form (1.21) is an OPS.

Theorem 1.1.21 (Favard)**.** *Let* $\{c_n\}_{n=0}^{\infty}$ *and* $\{\lambda_n\}_{n=0}^{\infty}$ *be sequences of complex numbers and let* $\{P_n\}_{n=0}^{\infty}$ *be defined by the recurrence formula*

$$\begin{aligned} &P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n \in \mathbb{N}, \\ &P_{-1}(x) = 0, \quad P_0(x) = 1. \end{aligned} \tag{1.27}$$

Then there is a unique moment functional $\mathcal{M}$ *such that*

$$\mathcal{M}[1] = \lambda_1, \quad \mathcal{M}[P_m(x) P_n(x)] = 0, \qquad \text{for } m \neq n,\ m, n \in \mathbb{N}_0.$$

Furthermore, $\mathcal{M}$ *is quasi-definite and* $\{P_n(x)\}_{n=0}^{\infty}$ *is the corresponding monic OPS if and only if* $\lambda_n \neq 0$ *while* $\mathcal{M}$ *is positive-definite if and only if* c_n *is real and* $\lambda_n > 0$, $n \geqslant 1$.

Proof. We define the moment functional $\mathcal{M}$ inductively by

$$\mathcal{M}[1] = \mu_0 = \lambda_1, \quad \mathcal{M}[P_n(x)] = 0, \qquad n \in \mathbb{N}. \tag{1.28}$$

Thus we have defined μ_1 by the condition, $\mathcal{M}[P_1(x)] = \mu_1 - c_1\mu_0 = 0$, μ_2 by $\mathcal{M}[P_2(x)] = \mu_2 - (c_1 + c_2)\mu_1 + (\lambda_2 - c_1 c_2)\mu_0 = 0$, etc. Rewriting (1.27) in the form

$$x P_n(x) = P_{n+1}(x) + c_{n+1} P_n(x) + \lambda_{n+1} P_{n-1}(x), \qquad n \geqslant 1, \tag{1.29}$$

(1.28) implies

$$\mathcal{M}\left[xP_n(x)\right] = 0, \qquad n \geqslant 2. \tag{1.30}$$

Multiplication of both sides of (1.29) by x implies together with (1.30) that

$$\mathcal{M}\left[x^2 P_n(x)\right] = 0, \qquad n \geqslant 3.$$

By induction,

$$\begin{aligned} \mathcal{M}\left[x^k P_n(x)\right] &= 0, \qquad 0 \leqslant k < n, \\ \mathcal{M}\left[x^n P_n(x)\right] &= \lambda_{n+1} \mathcal{M}\left[x^{n-1} P_{n-1}(x)\right], \qquad n \geqslant 1. \end{aligned}$$

Thus $\mathcal{M}\left[P_m(x) P_n(x)\right] = 0$, for $m \neq n$, while as in the proof of Theorem 1.1.8, we have

$$\mathcal{M}\left[P_n^2(x)\right] = \mathcal{M}\left[x^n P_n(x)\right] = \lambda_1 \lambda_2 \cdots \lambda_{n+1}, \qquad n \geqslant 0.$$

Hence $\mathcal{M}$ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is the corresponding OPS if and only if $\lambda_n \neq 0$ for $n \geqslant 1$.
The moments $\{\mu_n(x)\}_{n=0}^{\infty}$ are real if c_n and λ_n are all real, thus by Theorem 1.1.12 $\mathcal{M}$ is positive-definite if and only if $\lambda_n > 0$ for $n \geqslant 1$. □

Next, we will proof the Theorem of Perron-Favard. We first state a lemma that we will need for the proof. The proof which we will give for the Perron-Favard Theorem, to our knowledge, is not available in the literature so far.

Lemma 1.1.22. *Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\mathcal{M}$ be given as in Theorem 1.1.21. Then*

(i) *$P_n(x)$ has n distinct real zeros $x_{n1}, \ldots, x_{nn}$,*

(ii)

$$|x_{nk}| \leqslant \max_{0 \leqslant j \leqslant n-1} (1 + \lambda_{j+1}) + \max_{0 \leqslant j \leqslant n-1} |c_{j+1}| =: b, \tag{1.31}$$

(iii) *For $n \in \mathbb{N}$ there are numbers $A_{nk} > 0$ with $A_{n1} + \cdots + A_{nn} = 1$ such that*

$$\mathcal{M}\left[Q(x)\right] = \sum_{k=1}^{n} A_{nk} Q(x_{nk}) \qquad \forall Q(x) \in \mathcal{P} \text{ with } \deg(Q) \leqslant 2n-1.$$

Proof. (i) and (iii), respectively, will be proven later on in Theorem 1.1.31 and in Theorem 1.1.37 respectively.
In order to prove (ii), we use that

$$xP_k^2(x) = P_k(x) P_{k+1}(x) + c_{k+1} P_k^2(x) + \lambda_{k+1} P_k(x) P_{k-1}(x), \qquad k \in \mathbb{N}_0,$$

hence

$$x \sum_{j=0}^{n-1} P_j^2(x) = \sum_{j=0}^{n-2} (1 + \lambda_{j+1}) P_j(x) P_{j-1}(x) + P_{n-1}(x) P_n(x) + \sum_{j=0}^{n-1} c_{j+1} P_j^2(x).$$

Thus for $k = 1, \dots, n$

$$|x_{nk}| \sum_{j=0}^{n-1} P_j^2(x_{nk}) \leqslant \max_{0 \leqslant j \leqslant n-1} (1+\lambda_{j+1}) \sum_{j=0}^{n-1} |P_j(x_{nk})| |P_{j+1}(x_{nk})| + \max_{0 \leqslant j \leqslant n-1} |c_{j+1}| \sum_{j=0}^{n-1} |P_j^2(x_{nk})|.$$

Using the Cauchy-Schwarz inequality, we get for $k = 1, \dots, n$

$$|x_{nk}| \sum_{j=0}^{n-1} P_j^2(x_{nk}) \leqslant \left(\max_{0 \leqslant j \leqslant n-1} (1+\lambda_{j+1}) + \max_{0 \leqslant j \leqslant n-1} |c_{j+1}| \right) \sum_{j=0}^{n-1} P_j^2(x_{nk}),$$

where we used that $P_n(x_{nk}) = 0$. Finally, we obtain that

$$|x_{nk}| \leqslant \max_{0 \leqslant j \leqslant n-1} (1+\lambda_{j+1}) + \max_{0 \leqslant j \leqslant n-1} |c_{j+1}|,$$

and thus (ii) holds true. □

Theorem 1.1.23 (Perron-Favard)**.** *Let $\{P_n(x)\}_{n=0}^{\infty}$ be defined as in (1.27), let b be the number defined in Lemma 1.1.22 (ii), set $a := -b$ and let K_n be the number in Theorem 1.1.3 (iii). Then there is a unique measure π on $[a, b]$ such that*

$$\int_a^b P_n(x) P_m(x) \, d\pi(x) = K_n \delta_{n,m}. \tag{1.32}$$

Proof. The moment functional $\mathcal{M}$ is continuous, because by Lemma 1.1.22 for $Q(x) \in \mathcal{P}$, $\deg(Q) \leqslant 2n-1$, we have

$$|\mathcal{M}[Q(x)]| \leqslant \sum_{k=1}^{\infty} A_{nk} |Q(x_{nk})| \leqslant \sup_{x \in [a,b]} |Q(x)|.$$

By the Theorem of Weierstrass (see **[61]** p. 8) $\mathcal{M} : \mathcal{P} \to \mathbb{C}$ has a unique continuous extension to a continuous positive linear functional on $C([a, b])$, which will also be denoted by $\mathcal{M}$. Then the Riesz representation theorem (see **[44]** p.130 Theorem 6.19) yields the existence of a uniquely determined regular complex-valued measure $d\pi(x)$ on $[a, b]$ with

$$\int_a^b P_m(x) P_n(x) \, d\pi(x) = \mathcal{M}[P_m(x) P_n(x)] = K_n \delta_{m,n}.$$

Since the functional $\mathcal{M}$ is positive, the measure π is positive. □

The next theorem states the "Christoffel-Darboux-Identity".

Theorem 1.1.24. *Let $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the recurrence formula (1.27) with $\lambda_n \neq 0$, $n \geqslant 1$. Then*

$$\sum_{k=0}^{n} \frac{P_k(x) P_k(u)}{\lambda_1 \lambda_2 \cdots \lambda_{k+1}} = (\lambda_1 \lambda_2 \cdots \lambda_{n+1})^{-1} \frac{P_{n+1}(x) P_n(u) - P_n(x) P_{n+1}(u)}{x - u}. \tag{1.33}$$

Proof. Since $\{P_n(x)\}_{n=0}^{\infty}$ satisfies (1.27), we have

$$\begin{aligned} xP_n(x)P_n(u) &= P_{n+1}(x)P_n(u) + c_{n+1}P_n(x)P_n(u) + \lambda_{n+1}P_{n-1}(x)P_n(u), \\ uP_n(u)P_n(x) &= P_{n+1}(u)P_n(x) + c_{n+1}P_n(u)P_n(x) + \lambda_{n+1}P_{n-1}(u)P_n(x). \end{aligned}$$

Subtracting the second equation from the first, it follows that

$$\begin{aligned} (x-u)P_n(x)P_n(u) &= P_{n+1}(x)P_n(u) - P_{n+1}(u)P_n(x) \\ &\quad -\lambda_{n+1}\left[P_n(x)P_{n-1}(u) - P_n(u)P_{n-1}(x)\right]. \end{aligned}$$

By setting $F_n(x,u) = (\lambda_1\lambda_2\cdots\lambda_{n+1})^{-1}\frac{P_{n+1}(x)P_n(u)-P_n(x)P_{n+1}(u)}{x-u}$, the above equation can be rewritten as

$$\frac{P_m(x)P_m(u)}{\lambda_1\lambda_2\cdots\lambda_{m+1}} = F_m(x,u) - F_{m-1}(x,u), \qquad m \geqslant 0.$$

Now, (1.33) follows from the summation of the last equation from 0 to n and by setting $F_{-1}(x,u) = 0$. ☐

Remark 1.1.25. For the corresponding orthonormal polynomials $p_n(x)$ given by

$$p_n(x) = k_nP_n(x), \qquad k_n = (\lambda_1\lambda_2\cdots\lambda_{n+1})^{-1/2}, \tag{1.34}$$

(1.33) rewrites as

$$\sum_{k=0}^{n} p_k(x)p_k(u) = \frac{k_n}{k_{n+1}}\frac{p_{n+1}(x)p_n(u) - p_n(x)p_{n+1}(u)}{x-u}.$$

Theorem 1.1.26. *The following "confluent form" of (1.33) is also valid.*

$$\sum_{k=0}^{n}\frac{P_k^2(x)}{\lambda_1\lambda_2\cdots\lambda_{k+1}} = \frac{P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)}{\lambda_1\lambda_2\cdots\lambda_{n+1}}. \tag{1.35}$$

Proof. We can write the right-hand side of (1.33) as

$$\begin{aligned} P_{n+1}(x)P_n(u) - P_n(x)P_{n+1}(u) &= \left[P_{n+1}(x)P_{n+1}(u)\right]P_n(x) \\ &\quad -\left[P_n(x)P_n(u)\right]P_{n+1}(x). \end{aligned}$$

Thus for $u \to x$ (1.35) follows from (1.33). ☐

Corollary 1.1.27. *Let $\mathcal{M}$ be a positive-definite moment functional, then*

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0, \qquad x \in \mathbb{R}.$$

1.1.3 Zeros of orthogonal polynomials

We will now explore the zeros of orthogonal polynomials and observe that for a positive moment functional these zeros exhibit a certain regularity in their behavior. For this purpose we refine our concept of positive-definiteness.

Definition 1.1.28. Let $E \subset (-\infty, \infty)$. A moment functional $\mathcal{M}$ is called *positive-definite on* E if and only if $\mathcal{M}[Q(x)] > 0$ for every real polynomial $Q(x)$ which is nonnegative on E and is not identical zero on E. The set E is the *supporting set* for $\mathcal{M}$.

Example 1.1.29. The moment functionals for the Chebyshev polynomials of both the first and second kind are positive-definite on $[-1, 1]$.

The next theorem shows that positive-definiteness on any infinite set implies positive-definiteness.

Theorem 1.1.30. *Let $\mathcal{M}$ be positive-definite on E, where E is an infinite set. Then*

(i) $\mathcal{M}$ is positive-definite on every set that contains E,

(ii) $\mathcal{M}$ is positive-definite on every dense subset of E.

Proof. Let $Q(x)$ be a real polynomial which is nonnegative and not identical zero on a set S.
If $S \supset E$, then trivially $Q(x) \geqslant 0$ on E. If $S \subset E$ and S dense in E, then $Q(x) \geqslant 0$ by continuity. Since $Q(x)$ does not vanish everywhere on an infinite set, it follows in either case that $\mathcal{M}[Q(x)] > 0$. □

Theorem 1.1.31. *Let $\mathcal{M}$ be a positive-definite moment functional and $\{P_n(x)\}_{n=0}^{\infty}$ the corresponding monic OPS. Suppose I is an interval which is a supporting set for $\mathcal{M}$. Then the zeros of $P_n(x)$ are all real, simple and located in the interior of I.*

Proof. $\mathcal{M}[P_n(x)] = 0$ implies that $P_n(x)$ must change sign at least once in the interior of I because of Theorem 1.1.30. That is, $P_n(x)$ has at least one zero of odd multiplicity located in the interior of I. Let $x_1, x_2, \ldots, x_k$ denote the distinct zeros of odd multiplicity which are located in the interior of I. Set

$$p(x) = (x - x_1) \cdots (x - x_k).$$

Then $p(x) P_n(x)$ has no zeros of odd multiplicity in the interior of I. Thus $p(x) P_n(x) \geqslant 0$ for $x \in I$ and $\mathcal{M}[p(x) P_n(x)] > 0$, which is a contradiction to Theorem 1.1.3 (ii) unless $k \geqslant n$. That is, $k = n$ and $P_n(x)$ has n distinct zeros in the interior of I. □

Let us denote the zeros of $P_n(x)$ by x_{ni}. We order the zeros x_{ni} by increasing size:

$$x_{n1} < x_{n2} < \cdots < x_{nn}, \qquad n \geqslant 1.$$

$P_n(x)$ has a positive leading coefficient, thus

$$P_n(x) > 0 \qquad \text{for } x > x_{nn}, \quad sgn\,(P_n(x)) = (-1)^n \qquad \text{for } x < x_{n1}.$$

Now, $P_n'(x)$ has exactly one zero on each of the intervals $(x_{n,k-1}, x_{nk})$ and the sign of $P_n'(x_{nk})$ alternates as k varies from 1 to n. The leading coefficient of $P_n'(x)$ is also positive, hence

$$sgn\,(P_n'(x_{nk})) = (-1)^{n-k}, \qquad k = 1, 2, \ldots, n. \tag{1.36}$$

Examples 1.1.32. (1) The zeros of the Chebyshev polynomials of the first kind $\{T_n(x)\}_{n=0}^{\infty}$ are given by

$$x_{ni} = \cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right), \qquad i = 1, \ldots, n.$$

The extrema of $\{T_n(x)\}_{n=0}^{\infty}$ are given by

$$y_{nk} = \cos\left(\frac{k\pi}{n}\right), \qquad k = 0, 1, \ldots, n.$$

The y_{nk} are all distinct, lie in $[-1, 1]$ and satisfy

$$T_n(y_{nk}) = (-1)^k, \qquad k = 0, \ldots, n.$$

(2) The zeros of the Chebyshev polynomials of the second kind $\{U_n(x)\}_{n=0}^{\infty}$ are given by the extrema of $\{T_n(x)\}_{n=0}^{\infty}$, y_{nk}, for $i = 1, \ldots, n$.

Theorem 1.1.33 (Separation theorem for the zeros)**.** *Let $\{P_n(x)\}_{n=0}^{\infty}$ be the corresponding monic OPS to a positive-definite moment functional $\mathcal{M}$. The zeros of $P_n(x)$ and $P_{n+1}(x)$ mutually separate each other, that is*

$$x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \qquad i = 1, 2, \ldots, n. \tag{1.37}$$

Proof. Corollary 1.1.27 provides that

$$P'_{n+1}(x) P_n(x) - P'_n(x) P_{n+1}(x) > 0,$$

which implies

$$P'_{n+1}(x_{n+1,k}) P_n(x_{n+1,k}) > 0, \qquad k = 1, 2, \ldots, n+1.$$

By (1.36), $P_n(x_{n+1,k}) = (-1)^{n+1-k}$. Thus $P_n(x)$ has exactly one zero on each of the n intervals, $(x_{n+1,k}, x_{n+1,k+1})$, $k = 1, 2, \ldots, n$. □

Corollary 1.1.34. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be the corresponding monic OPS to a positive-definite moment functional $\mathcal{M}$. Then for each $k \geqslant 1$, $\{x_{nk}\}_{n=k}^{\infty}$ is a decreasing sequence and $\{x_{n,n-k+1}\}_{n=k}^{\infty}$ is an increasing sequence. In particular, the limits*

$$\xi_i = \lim_{n\to\infty} x_{ni}, \quad \eta_j = \lim_{n\to\infty} x_{n,n-j+1}, \qquad i, j \in \mathbb{N},$$

exist, at least in $[-\infty, \infty]$.

Definition 1.1.35. Let $\mathcal{M}$ be positive-definite moment functional. The closed interval $[\xi_1, \eta_1]$ is called the *true interval of orthogonality* of the OPS.

Remark 1.1.36. (1) In the case of Theorem 1.1.23, we have $\operatorname{supp}\pi \subseteq [\xi_1, \eta_1]$.

(2) The true interval of orthogonality is the smallest closed interval that contains all of the zeros of all $P_n(x)$.

1.1.4 Gauss quadrature

We will now turn to the Gauss quadrature formula which will serve as a tool to explore the moment functional $\mathcal{M}$ in the positive-definite case and which we used earlier to prove the Perron-Favard Theorem.

Let $\{t_1, t_2, \ldots, t_n\}$ be any set of $n \geqslant 1$ distinct numbers and set

$$F(x) = \prod_{i=1}^{n} (x - t_i).$$

Now, $F(x)/(x - t_i)$ is a polynomial of degree $n-1$ and

$$\lim_{x \to t_k} \frac{F(x)}{x - t_k} = F'(t_k) \neq 0.$$

Then

$$l_k(x) = \frac{F(x)}{(x - t_k) F'(t_k)}$$

is a polynomial of degree $n-1$ satisfying

$$l_k(t_j) = \delta_{jk}.$$

For any set $\{y_1, y_2, \ldots, y_n\}$ of numbers, the polynomial

$$L_n(x) = \sum_{k=1}^{n} y_k l_k(x)$$

is of degree, at most, $n-1$, and satisfies

$$L_n(t_j) = \sum_{k=1}^{n} y_k \delta_{jk} = y_j, \qquad j = 1, 2, \ldots, n.$$

$L_n(x)$ is called the *Lagrange interpolation polynomial* corresponding to the *nodes* t_i and the *ordinates* y_i, and provides a unique solution to the problem of constructing a polynomial of degree at most $n-1$ passing through the points (t_i, y_i), $i = 1, 2, \ldots, n$.

Theorem 1.1.37 (Gauss quadrature formula)**.** *Let $\mathcal{M}$ be positive-definite. Then there are numbers $A_{n1}, A_{n2}, \ldots, A_{nn}$ such that for every polynomial $Q(x)$ of degree at most $2n-1$,*

$$\mathcal{M}[Q(x)] = \sum_{k=1}^{n} A_{nk} Q(x_{nk}). \tag{1.38}$$

The numbers A_{nk} are all positive and have the property,

$$A_{n1} + A_{n2} + \cdots + A_{nn} = \mu_0. \tag{1.39}$$

Proof. Let $Q(x)$ be an arbitrary polynomial of degree at most $2n-1$ and construct the Lagrange interpolation polynomial corresponding to the nodes x_{nk} and the ordinates $Q(x_{nk})$, $1 \leqslant k \leqslant n$:

$$L_n(x) = \sum_{k=1}^{n} Q(x_{nk}) l_k(x),$$

where

$$l_k(x) = \frac{P_n(x)}{(x - x_{nk}) P_n'(x_{nk})}.$$

Set $S(x) = Q(x) - L_n(x)$, then $S(x)$ is a polynomial of degree at most $2n-1$ which vanishes at x_{nk}, $k = 1, \ldots, n$, that is

$$S(x) = R(x) P_n(x),$$

where $R(x)$ is a polynomial of degree at most $n-1$. By Theorem 1.1.3,

$$\begin{aligned} \mathcal{M}[Q(x)] &= \mathcal{M}[L_n(x)] + \mathcal{M}[R(x) P_n(x)] = \mathcal{M}[L_n(x)] \\ &= \sum_{k=1}^{n} Q(x_{nk}) \mathcal{M}[l_k(x)]. \end{aligned}$$

Thus we have (1.38) with $A_{nk} = \mathcal{M}[l_k(x)]$. If we set $Q(x) = l_m^2(x)$ in (1.38), then

$$0 < \mathcal{M}\left[l_m^2(x)\right] = \sum_{k=1}^{n} A_{nk} l_m^2(x_{nk}) = A_{nm},$$

hence the A_{nk} are all positive. By choosing $Q(x) = 1$ in (1.38), we obtain (1.39). □

Remark 1.1.38. (1) The weights A_{nk} in the Gauss quadrature formula do not depend on the fact that $P_n(x)$ is monic.

(2) If $\mathcal{M}$ is defined as in (1.3),

$$\mathcal{M}[f] = \int_a^b f(x) w(x) dx$$

provided the integral converges, then (1.38) suggests the approximation

$$\mathcal{M}[f] = \int_a^b f(x) w(x) dx \approx \sum_{k=1}^{n} A_{nk} f(x_{nk}) \equiv \mathcal{M}_n[f].$$

Formulas of this general form in which A_{nk} and x_{nk} are numbers independent from f are called *approximate quadrature formulas.*

Theorem 1.1.37 can be used to gain additional information about the separation properties of the zeros of orthogonal polynomials:

Theorem 1.1.39. *Between any zeros of $P_N(x)$ there is at least one zero of $P_n(x)$ for every $n > N \geqslant 2$.*

***Proof*.** Assume that for some $n > N$, $P_n(x)$ has no zero between x_{Np} and $x_{N,p+1}$, $1 \leqslant p < N$. We have that

$$p(x) = \frac{P_N(x)}{(x - x_{Np})(x - x_{N,p+1})}$$

is a polynomial of degree $N-2$ and

$$p(x) P_N(x) \geqslant 0 \qquad \text{for } x \notin (x_{Np}, x_{N,p+1}). \tag{1.40}$$

With (1.38) we have

$$\mathcal{M}[p(x) P_N(x)] = \sum_{k=1}^{n} A_{nk} p(x_{nk}) P_N(x_{nk}).$$

Now, (1.40) implies that $\mathcal{M}[p(x) P_N(x)] > 0$ as $p(x) P_N(x)$ cannot vanish at every x_{nk}, but this contradicts the orthogonality properties. □

1.1.5 Iterative properties of the Chebyshev polynomials of the first kind

In the rest of this section, we will deal with a special property of the Chebyshev polynomials of the first kind, $\{T_n(x)\}_{n=0}^{\infty}$, following [**43**], and if we say Chebyshev polynomials, we will automatically mean the Chebyshev polynomials of the first kind. The fact that the Chebyshev polynomials besides the powers are the only orthogonal polynomials up to similarity that are permutable, will be crucial in Chapter 3 when we study the Ruelle operator defined by the preimages of the Chebyshev polynomials.

Lemma 1.1.40. *For $m, n \geqslant 0$ the Chebyshev polynomials, $\{T_n(x)\}_{n=0}^{\infty}$, satisfy the following semi-group property:*

$$T_m(T_n(x)) = T_{mn}(x). \tag{1.41}$$

Proof. We have

$$\begin{aligned} T_m(T_n(x)) &= \cos(m \arccos(\cos(n \arccos(x)))) \\ &= \cos(m(n \arccos(x))) = T_{mn}(x), \end{aligned}$$

which proves the lemma. □

Definition 1.1.41. (1) Let $P(x)$, $Q(x)$ be two arbitrary polynomials. $P(x)$, $Q(x)$ are called *permutable* if $P(Q(x)) = Q(P(x))$ for all x. We will denote the composition $P(Q(x))$ by $P \circ Q(x)$, the n-fold composition $P \circ P \circ \ldots \circ P(x)$ by $P^{\{n\}}(x)$ and say that $P(x)$ commutes with $Q(x)$ and vice versa if $P(x)$ and $Q(x)$ are permutable.

(2) A sequence of polynomials, each of positive degree which contains at least one of each degree and such that every two polynomials in it are permutable is called a *chain*.

Since

$$T_m(T_n(x)) = T_n(T_m(x)) = T_{mn}(x),$$

the Chebyshev polynomials, $\{T_n(x)\}_{n=0}^{\infty}$, are permutable and they form a chain. So do the powers x^j, $j \in \mathbb{N}$.

We will now see that no polynomials other than Chebyshev polynomials commute with a given $T_n(x)$ if $n \geqslant 2$.

Theorem 1.1.42 (Bertram)**.** *Let $P(x)$ be a polynomial of degree $k \geqslant 1$ and let $n \geqslant 2$. If $P(x)$ commutes with $T_n(x)$, then $P(x) = T_k(x)$ for n even and $P(x) = \pm T_k(x)$ for n odd.*

Proof. By Lemma 1.1.43 (see below) $\pm T_m(x)$ are the only polynomial solutions of

$$\left(1 - x^2\right)(y')^2 = m^2\left(1 - y^2\right) \tag{1.42}$$

for $m > 0$.

We will show that, if $P(x)$ commutes with $T_n(x)$, then $y = P(x)$ satisfies (1.42) with $m = k$.

The polynomial

$$Q(x) = \left(1 - x^2\right)\left(P'(x)\right)^2 - k^2\left(1 - P^2(x)\right)$$

is in $\mathcal{P}_{2k-1}$, the polynomials of degree at most $2k-1$, since the coefficient of x^{2k} is zero, but with the permutability of $P(x)$ and $T_n(x)$ and the fact that $T_n(x)$ satisfies (1.42) with $m = n$, we have

$$\begin{aligned} n^2 Q \circ T_n(x) &= n^2\left(1 - T_n^2(x)\right)\left(P' \circ T_n(x)\right)^2 - n^2k^2\left(1 - \left(P \circ T_n(x)\right)^2\right) \\ &= \left(1 - x^2\right)\left(T_n'(x)\right)^2\left(P' \circ T_n(x)\right)^2 - k^2\left(1 - P^2(x)\right)\left(T_n' \circ P(x)\right)^2. \end{aligned}$$

Now,

$$\left(P' \circ T_n(x)\right) T_n'(x) = \left(P \circ T_n(x)\right)' = \left(T_n \circ P(x)\right)' = \left(T_n' \circ P(x)\right) P'(x),$$

thus

$$\begin{aligned} n^2 Q \circ T_n(x) &= \left(1 - x^2\right)\left(P'(x)\right)^2\left(T_n' \circ P(x)\right)^2 - k^2\left(1 - P^2(x)\right)\left(T_n' \circ P(x)\right)^2 \\ &= \left(T_n' \circ P(x)\right)^2\left(\left(1 - x^2\right)\left(P'(x)\right)^2 - k^2\left(1 - P^2(x)\right)\right) \\ &= \left(T_n' \circ P(x)\right)^2 Q(x) \end{aligned} \tag{1.43}$$

Suppose $Q(x) \neq 0$ is of degree $s \leqslant 2k-1$, then by (1.43) $sn = 2(n-1)k + s$ so that $s = 2k > 2k-1$ which is a contradiction. Hence $Q(x)$ is identically zero and $P(x) = \pm T_k(x)$. For n even, $T_n \circ (-T_k(x)) = T_n \circ T_k(x) = T_k \circ T_n(x) \neq -T_k \circ T_n(x)$, thus $P(x) = T_k(x)$. For n odd, $T_n \circ (-T_k(x)) = -T_n \circ T_k(x) = -T_k \circ T_n(x)$, thus $P(x) = \pm T_k(x)$. □

Lemma 1.1.43. *For $-1 \leqslant x \leqslant 1$ the differential equation*

$$\left(1 - x^2\right)\left(y'\right)^2 = m^2\left(1 - y^2\right)$$

is solved by $y = \pm T_m(x)$ and has no other polynomial solution for $m > 0$.

Proof. See [**43**] pp.87-89, p.39. □

For

$$\lambda(x) = ax + b, \qquad a \neq 0, \tag{1.44}$$

we have

$$\lambda^{-1}(x) = \frac{x - b}{a}.$$

If $P(x)$ and $Q(x)$ commute, clearly $\lambda^{-1} \circ P \circ \lambda(x)$ and $\lambda^{-1} \circ Q \circ \lambda(x)$ also commute.

Definition 1.1.44. We use the same notation as above.

(1) $P(x)$ and $\lambda^{-1} \circ P \circ \lambda(x)$ are called *similar*.

(2) Two chains are called *similar*, if there exists a $\lambda(x)$ satisfying (1.44) such that each polynomial in one chain is similar to the polynomial in the other chain of the same degree via $\lambda(x)$ (see also Remark 1.1.47).

In the following, we will see that $\{T_n(x)\}_{n=0}^{\infty}$ and $\{x^n\}_{n=0}^{\infty}$ are the only chains up to similarities.

Theorem 1.1.45. *Let $n \geqslant 2$. If $P(x)$ is a polynomial of degree $k \geqslant 1$ which commutes with x^n, then $P(x) = x^k$ for n even and $P(x) = \pm x^k$ for n odd.*

Proof. $y = x^n$ satisfies

$$xy' = ny. \tag{1.45}$$

The polynomial $Q(x) = xP'(x) - kP(x)$ is in $\mathcal{P}_{k-1}$ as the coefficient of x^k is zero. Analogously to the proof of Theorem 1.1.42, we get $nQ \circ x^n = (nx^{n-1} \circ P(x))\, Q(x)$, and if $Q(x)$ is of degree $s \geqslant 0$, then $sn = k(n-1) + s$ yields that $s = k$ which is a contradiction. Thus $Q(x)$ is identically zero. Hence $y = P(x)$ satisfies (1.45) with n replaced by k, that is, $P(x) = cx^k$ with $c \neq 0$. The commutativity of $P(x)$ and x^n implies that $cx^{kn} = c^n x^{kn}$ and $c^{n-1} = 1$. Since c must be real, $c = 1$ if n is even and $c = \pm 1$ if n is odd. □

Theorem 1.1.46. *There is at most one polynomial of degree $k \geqslant 1$ permutable with a given quadratic, $S(x) = a_0 + a_1 x + a_2 x^2$, $a_2 \neq 0$.*

Proof. If we put

$$\lambda(x) = \frac{x}{a_2} - \frac{a_1}{2a_2}, \tag{1.46}$$

then we get

$$\left(\lambda^{-1} \circ S \circ \lambda(x)\right) = x^2 + c,$$

where $c = a_0 a_2 + \frac{a_1}{2} - \frac{a_1^2}{4}$. Thus in order to prove the theorem it suffices to show that there are no two distinct polynomials of degree k which commute with $x^2 + c$. If $U(x)$ and $V(x)$ are distinct polynomials of degree k which commute with $S(x)$, then there are distinct polynomials of degree k similar to $U(x)$ and $V(x)$ via (1.46) commuting with $x^2 + c$.

Suppose that $P(x)$ and $Q(x)$ are distinct polynomials satisfying

$$\begin{aligned} P\left(x^2 + c\right) &= P^2(x) + c, \\ Q\left(x^2 + c\right) &= Q^2(x) + c. \end{aligned} \tag{1.47}$$

Then by comparison of the leading coefficients on both sides of each equality, we obtain that $P(x)$ and $Q(x)$ both have leading coefficient 1. Thus $R(x) = P(x) - Q(x) \in \mathcal{P}_{k-1}$ and

$$R\left(x^2 + c\right) = P^2(x) - Q^2(x) = R(x)\left(P(x) + Q(x)\right). \tag{1.48}$$

If $R(x)$ is of degree $t \geqslant 0$, then by (1.48), $2t = t + k$ or $t = k$ which is a contradiction. Therefore, $R(x)$ is identically zero and $P(x) = Q(x)$. This contradiction proves the theorem. □

Remark 1.1.47. An immediate consequence of Theorem 1.1.46 is that each chain contains exactly one polynomial of each positive degree.

Theorem 1.1.48. *Every chain is either similar to $\{x^j\}_{j=1}^{\infty}$ or to $\{T_j(x)\}_{j=1}^{\infty}$.*

Proof. Let $\{P_j(x)\}_{j=1}^{\infty}$ be a chain with $P_2(x) = a_0 + a_1 x + a_2 x^2$ and $\{Q_j(x)\}_{j=1}^{\infty}$ a chain similar to $\{P_j(x)\}_{j=1}^{\infty}$ with $\lambda(x)$ as in (1.46). Then $Q_2(x) = x^2 + c$, and $Q_3(x)$ commutes with $Q_2(x)$ thus

$$Q_3\left(x^2 + c\right) = Q_3^2(x) + c. \tag{1.49}$$

Hence $Q_3^2(-x) = Q_3^2(x)$, and since $Q_3(x)$ is of degree 3, we have that $Q_3(-x) = -Q_3(x)$, that is, $Q_3(x)$ is an odd polynomial. We set

$$Q_3(x) = b_1 x + b_3 x^3. \tag{1.50}$$

By substitution of (1.50) into (1.49) we obtain $b_3 = 1$, $b_1 = \frac{3}{2}c$,

$$c(c+2) = 0 \qquad \text{and} \qquad c(2+c)(2c-1) = 0.$$

Thus c either equals -2 or 0. For the case that $c = 0$, $Q_2(x) = x^2$ and by Theorem 1.1.45 $Q_j(x) = x^j$, $j \in \mathbb{N}$. Hence $\{P_j(x)\}_{j=1}^{\infty}$ is similar to $\{x^j\}_{j=1}^{\infty}$.
For the case that $c = -2$, we consider the chain $\{\mu^{-1} \circ Q_j \circ \mu(x)\}_{j=1}^{\infty}$ with $\mu(x) = 2x$. Since

$$\left(\mu^{-1} \circ Q_2 \circ \mu(x)\right) = T_2(x),$$

we get with Theorem 1.1.42 that

$$\mu^{-1} \circ Q_j \circ \mu(x) = T_j(x), \qquad j \in \mathbb{N}.$$

Hence $\{P_j(x)\}_{j=1}^{\infty}$ is similar to $\{T_j(x)\}_{j=1}^{\infty}$ via the linear transformation $\lambda \circ \mu(x)$. □

1.2 Hypergroups and homogeneous Banach spaces

In this section, we will introduce hypergroups and some of their basic properties with a special focus on polynomial hypergroups. First, we will provide a definition and basic facts on hypergroups in general following an unpublished book manuscript of R. Lasser and [**7**]. Then, we will turn to discrete hypergoups and polynomial hypergroups on $\mathbb{N}_0$ following R. Lasser's manuscript as well as [**32**], [**33**] and [**34**]. Polynomial hypergroups will be most important for us, as in the next chapter, we will define a transfer operator which acts on polynomial hypergroup structures induced by certain orthogonal polynomials as well as homogeneous Banach spaces which we will give a short overview in the last part of this section following [**22**].

1.2.1 Definition and basic properties of hypergroups

Let K be a locally compact Hausdorff space, $C_0(K)$ the set of continuous functions on K which vanish at infinity, $M(K)$ the space of complex measures and $M^1(K)$ the subset of probability measures. By the Riesz representation theorem (see [**44**], p.130, Theorem 6.19) $M(K)$ is the dual of $C_0(K)$ which will be used throughout this section.

Lemma 1.2.1. *Let K be a locally compact Hausdorff space and*

$$\omega : K \times K \to M^1(K)$$

a continuous map, where $M^1(K)$ possesses the weak-$*$-topology with respect to the duality $M^1(K) = C_0(K)^*$. For μ, $\nu \in M(K)$ we define

$$\mu * \nu(f) := \int_{K \times K} \omega(x,y)(f)\, d(\mu \times \nu)(x,y) \qquad \forall f \in C_0(K). \tag{1.51}$$

Then

$$(\mu, \nu) \mapsto \mu * \nu, \qquad M(K) \times M(K) \to M(K)$$

is a bilinear extension of the mapping $(\epsilon_x, \epsilon_y) \mapsto \omega(x, y)$*, where* ϵ_x *is the point measure of* $x \in K$*. It is called canonical extension of* ω*. Moreover,*

$$\|\mu * \nu\| \leqslant \|\mu\| \|\nu\|$$

and

$$\mu * \nu(f) = \int_K \int_K \omega(x, y)(f)\, d\mu(x)\, d\nu(y) = \int_K \int_K \omega(x, y)(f)\, d\nu(y)\, d\mu(x) \quad \forall f \in C_0(K).$$

Proof. For $f \in C_0(K)$, $(x, y) \mapsto \omega(x, y)(f)$ is continuous and $|\omega(x, y)(f)| \leqslant \|f\|_\infty$ for all $x, y \in K$, thus

$$|\mu * \nu(f)| \leqslant \int_{K \times K} |\omega(x, y)(f)|\, d|\mu \times \nu|(x, y) \leqslant \|f\|_\infty \|\mu \times \nu\|.$$

That is, $\mu * \nu \in C_0(K)^* = M(K)$ and by Fubini's theorem (see **[44]** p. 164, Theorem 8.8)

$$\mu * \nu(f) = \int_K \int_K \omega(x, y)(f)\, d\mu(x)\, d\nu(y) = \int_K \int_K \omega(x, y)(f)\, d\nu(y)\, d\mu(x).$$

Hence $|\mu * \nu(f)| \leqslant \|f\|_\infty \|\mu\| \|\nu\|$ for all $f \in C_0(K)$ and thus $\|\mu * \nu\| \leqslant \|\mu\| \|\nu\|$. The rest of the lemma is evident. □

Lemma 1.2.2. *Let* K *be a locally compact Hausdorff space and* $x \mapsto \tilde{x}$*,* $K \to K$*, a homeomorphism from* K *onto* K*. For* $\mu \in M(K)$ *and* E *a Borel set, we define*

$$\tilde{\mu}(E) = \mu\left(\tilde{E}\right) \qquad \forall E \subset K. \tag{1.52}$$

Then $\mu \mapsto \tilde{\mu}$ *is an isometric isomorphism of* $M(K)$ *onto* $M(K)$ *and called the canonical extension of* $x \mapsto \tilde{x}$*.*

Proof. We have that $|\tilde{\mu}| = |\mu|$, the rest of the proof is clear. □

Definition 1.2.3. Let K be a locally compact Hausdorff space. The triple $(K, \omega, {}^\sim)$ is called a *hypergroup* if it satisfies the following properties:

(H1) $\omega : K \times K \to M^1(K)$ is a weak-$*$-topology continuous map such that associativity holds with respect to the canonical extension, $\epsilon_x * \omega(y, z) = \omega(x, y) * \epsilon_z$ for all $x, y, z \in K$.

(H2) supp $(\omega(x, y))$ is compact for every $x, y \in K$.

(H3) ${}^\sim : K \to K$ is a homeomorphism such that $\tilde{\tilde{x}} = x$ and $(\omega(x, y))^\sim = \omega(\tilde{y}, \tilde{x})$ for all $x, y \in K$.

(H4) There exists a (necessarily unique) element $e \in K$ such that $\omega(e, x) = \epsilon_x = \omega(x, e)$ for all $x \in K$.

(H5) We have $e \in$ supp $(\omega(x, \tilde{y}))$ if and only if $x = y$.

(H6) Let $\mathcal{C}(K)$ denote the space of nonempty compact subsets of K given the Michael topology (see below). The mapping $(x,y) \mapsto \operatorname{supp}(\omega(x,y))$, $K \times K \to \mathcal{C}(K)$, is continuous.

We call the mapping ω and its extension to $M(K)$ a *convolution*, $\tilde{}$ and its extension to $M(K)$ an *involution* and e the *unit element*. The convolution $\omega(x,y)$ can also be written as $\epsilon_x * \epsilon_y$ or $\rho_x * \rho_y$. If $\omega(x,y) = \omega(y,x)$ for all $x, y \in K$, then K is called a *commutative hypergroup*.
We will just use K to denote the hypergroup $(K, \omega, \tilde{})$.

Definition 1.2.4. Let $\mathcal{C}(K)$ denote the space of nonempty compact subsets of K and write $\mathcal{C}_A(B) := \{C \in \mathcal{C}(K) : C \cap A \neq \varnothing \text{ and } C \subset B\}$. Then $\mathcal{C}(K)$ can be given the *Michael topology* which is generated by the subbasis of all $\mathcal{C}_U(V)$ for which U and V are open subsets of K.

Remark 1.2.5. The Michael topology has the following properties:

(i) If K is compact, then $\mathcal{C}(K)$ is compact.

(ii) $\mathcal{C}(K)$ is a locally compact Hausdorff space.

(iii) The mapping $x \mapsto \{x\}$ is a homeomorphism of K onto a closed subset of $\mathcal{C}(K)$.

(iv) The collection of nonempty finite subsets of K is dense in $\mathcal{C}(K)$.

(v) If Ω is a compact subset of $\mathcal{C}(K)$, then $B := \cup\{A : A \in \Omega\}$ is a compact subset of K.

Example 1.2.6. Every locally compact group together with its usual convolution structure is a hypergroup.

Using the convolution ω, we can define generalized translation operators, the left-translation L_x and the right-translation R_x,

$$L_x f(y) := \omega(x,y)(f) \qquad \text{and} \qquad R_x f(y) := \omega(y,x)(f), \qquad x, y \in K, \tag{1.53}$$

where f is a continuous complex-valued function on K. By (H2) the integrals in (1.53) are well-defined.

Lemma 1.2.7. *Let $f : K \to \mathbb{C}$ be a continuous function. Then $(x,y) \mapsto \omega(x,y)(f)$, $K \times K \to \mathbb{C}$, is a continuous function, i.e. $L_x f$ and $R_x f$ are continuous functions on K.*

Proof. Let $x_0, y_0 \in K$, $\epsilon > 0$ and U an open set with compact closure $\overline{U}$ such that $\operatorname{supp}(\omega(x_0,y_0)) \subseteq U$. By (H6) there are neighborhoods V_{x_0}, V_{y_0} of x_0 and y_0 such that $\operatorname{supp}(\omega(x,y))) \subseteq U$ for all $x \in V_{x_0}$ and $y \in V_{y_0}$. Applying Urysohn's lemma (see **[61]**, p.7), we have that there is a function $g \in C_c(K)$ satisfying $g|\overline{U} = f|\overline{U}$. By (H1) there are neighborhoods U_{x_0} of x_0 and U_{y_0} of y_0 with $U_{x_0} \subseteq V_{x_0}$ and $U_{y_0} \subseteq V_{y_0}$ such that

$$|\omega(x,y)(g) - \omega(x_0,y_0)(g)| < \epsilon, \qquad x \in U_{x_0}, y \in U_{y_0}.$$

For each $x \in U_{x_0}$, $y \in U_{y_0}$ we also have $\operatorname{supp}\omega(x,y) \subseteq U$, thus $\omega(x,y)(g) = \omega(x,y)(f)$, which implies the statements of the lemma. □

In order to deal with convolutions of sets, we define

$$A * B := \bigcup_{x\in A, y\in B} \operatorname{supp} \omega(x,y) \qquad \text{and} \qquad \tilde{A} := \{\tilde{x} : x \in A\}.$$

If $A, B \subseteq K$ are compact, then $A * B$ is also compact which follows from Remark 1.2.5 (v). By (H6) the collection $\{\operatorname{supp} \omega(x,y) : x \in A, y \in B\}$ is a compact subset in $\mathcal{C}(K)$, and $(A * B) * C = A * (B * C)$.

Lemma 1.2.8. *For $A, B, C \subseteq K$ we have that $(A * B) \cap C \neq \emptyset$ if and only if $\left(\tilde{A} * C\right) \cap B \neq \emptyset$.*

Proof. $(A * B) \cap C \neq \emptyset$ if and only if $e \in (A * B)^{\sim} * C = \tilde{B} * \left(\tilde{A} * C\right)$ if and only if $B \cap \left(\tilde{A} * C\right) \neq \emptyset$. □

Proposition 1.2.9. *Let $\mu, \nu \in M(K)$, $\mu, \nu \geqslant 0$. Then*

$$\textit{supp}\ (\mu * \nu) = ((\textit{supp}\ \mu) * (\textit{supp}\ \nu))^{c}.$$

If, moreover, μ and ν have compact support, then $\mu * \nu$ has compact support, too, and supp $(\mu * \nu) = (\text{supp}\ \mu) * (\text{supp}\ \nu)$.

Proof. If $z \notin ((\operatorname{supp} \mu) * (\operatorname{supp} \nu))^{c}$, then we choose a neighborhood U of z such that $U \cap ((\operatorname{supp} \mu) * (\operatorname{supp} \nu))^{c} = \emptyset$. Hence for every continuous function f with $\operatorname{supp} f \subseteq U$, we have $\omega(x,y)(f) = 0$ for all $x \in \operatorname{supp} \mu$, $y \in \operatorname{supp} \nu$. Thus $\mu * \nu(f) = 0$, i.e. $z \notin \operatorname{supp}\ (\mu * \nu)$.
In order to show the other inclusion, suppose $z \in (\operatorname{supp} \mu) * (\operatorname{supp} \nu)$, i.e. $z \in \operatorname{supp} \omega(x,y)$ for some $x \in \operatorname{supp} \mu$, $y \in \operatorname{supp} \nu$. Given a neighborhood U of z there exists a continuous function $f \geqslant 0$ with $\operatorname{supp} f \subseteq U$ and $\omega(x,y)(f) > 0$. By Lemma 1.2.7 $\mu * \nu(f) > 0$, that is, $z \in \operatorname{supp}\ (\mu * \nu)$. If $\operatorname{supp} \mu$ and $\operatorname{supp} \nu$ are compact subsets of K, then $(\operatorname{supp} \mu) * (\operatorname{supp} \nu)$ is compact and thus it is closed. □

Theorem 1.2.10. *Let $x, y \in K$, L_x and R_x the translation operators defined in (1.53). If $f \in C_c(K)$, then $L_x f \in C_c(K)$ and $R_x f \in C_c(K)$. If $f \in C_0(K)$, then $L_x f \in C_0(K)$ and $R_x f \in C_0(K)$, and if $f \in C^b(K)$, then $L_x f \in C^b(K)$ and $R_x f \in C^b(K)$. Furthermore, if $f \in C^b(K)$, then $\|L_x f\|_\infty \leqslant \|f\|_\infty$ and $\|R_x f\|_\infty \leqslant \|f\|_\infty$.*

Proof. By Lemma 1.2.7 $L_x f$ and $R_x f$ are continuous functions. Evidently, for $f \in C^b(K)$, $\|L_x f\|_\infty \leqslant \|f\|_\infty$ holds. Suppose that $f \in C_c(K)$. If $L_x f \neq 0$, then $\operatorname{supp} f \cap \operatorname{supp}\ (\omega(x,y)) \neq \emptyset$, that is, $\operatorname{supp} f \cap (\{x\} * \{y\}) \neq \emptyset$. By Lemma 1.2.8 this is equivalent to $y \in \{\tilde{x}\} * \operatorname{supp} f$, and thus

$$\{y \in K : L_x f(y) \neq 0\} \subseteq \{\tilde{x}\} * \operatorname{supp} f.$$

The compactness of $\operatorname{supp} f$ implies that $\operatorname{supp} L_x f$ is also compact, thus $L_x f \in C_c(K)$. Since $C_c(K)$ is dense in $C_0(K)$, we have that $L_x f \in C_0(K)$ for every $f \in C_0(K)$. The proof for the right translation can be done analogously. □

Theorem 1.2.11. *The measure space $M(K)$ together with the convolution and the involution defined in Definition 1.2.3 is a Banach-$*$-algebra with unit.*

Proof. We will only check the associativity law. (H1) is equivalent to

$$\omega(y,z)(L_x f) = \omega(x,y)(R_z f) \tag{1.54}$$

for all $x, y, z \in K$ and $f \in C_0(K)$. Hence for $\lambda, \mu, \nu \in M(K)$ and $f \in C_0(K)$, we have

$$\begin{aligned}
\lambda * (\mu * \nu)(f) &= \int_K \int_K L_u f(v)\, d(\mu * \nu)(v)\, d\lambda(u) \\
&= \int_K \int_K \int_K \omega(y,z)(L_u f)\, d\mu(y)\, d\nu(z)\, d\lambda(u) \\
&= \int_K \int_K \int_K \omega(u,y)(R_z f)\, d\mu(y)\, d\nu(z)\, d\lambda(u) \\
&= \int_K \int_K R_z f(v)\, d(\lambda * \mu)(v)\, d\nu(z) \\
&= (\lambda * \mu) * \nu(f).
\end{aligned}$$

□

Proposition 1.2.12. *Let $x, y \in K$ and $f \in C^b(K)$, then*

$$L_x \circ L_y f(z) = \int_K L_u f(z)\, d\omega(y,x)(u) \tag{1.55}$$

and

$$R_x \circ R_y f(z) = \int_K R_u f(z)\, d\omega(x,y)(u) \tag{1.56}$$

for all $z \in K$.

Proof. Using (1.54) we get

$$\begin{aligned}
L_x \circ L_y f(z) &= \omega(x,z)(L_y f) = \omega(y,x)(R_z f) \\
&= \int_K R_z f(u)\, d\omega(y,x)(u) = \int_K L_u f(z)\, d\omega(y,x)(u).
\end{aligned}$$

Analogously (1.56) can be shown. □

The translation by elements of the hypergroup K can be extended to a module operation of $M(K)$ on $C^b(K)$ and $C_0(K)$. For $\mu \in M(K)$ and $f \in C^b(K)$, we define

$$L_\mu f(x) := \mu * f(x) := \int_K \omega(\tilde{y}, x)(f)\, d\mu(y) = \tilde{\mu}(R_x f) \tag{1.57}$$

and

$$R_\mu f(x) := f * \mu(x) := \int_K \omega(x, \tilde{y})(f)\, d\mu(y) = \tilde{\mu}(L_x f) \tag{1.58}$$

for $x \in K$. We have $\epsilon_{\tilde{y}} * f = L_y f$ and $f * \epsilon_{\tilde{y}} = R_y f$.

Theorem 1.2.13. *For $\mu \in M(K)$ and $f \in C^b(K)$, L_μ and R_μ, defined in (1.57) and (1.58), are bounded linear operators from $C^b(K)$ into $C^b(K)$ and from $C_0(K)$ into $C_0(K)$.*
For $f \in C^b(K)$, we have additionally $\|L_\mu f\|_\infty \leqslant \|\mu\| \|f\|_\infty$ and $\|R_\mu f\|_\infty \leqslant \|\mu\| \|f\|_\infty$.
If $\mu \in M(K)$ has compact support and $f \in C_c(K)$, then $L_\mu f \in C_c(K)$ and $R_\mu f \in C_c(K)$.

Proof. We first prove $L_\mu f \in C^b(K)$ for $f \in C^b(K)$ and $\mu \in M(K)$ with compact support. Let $x_0 \in K$ and $\epsilon > 0$. Since supp μ is compact by Lemma 1.2.7, we get that there exists a neighborhood U of x_0 such that

$$|\omega(\tilde{y}, x)(f) - \omega(\tilde{y}, x_0)(f)| < \epsilon, \qquad x \in U, y \in \text{supp } \mu.$$

Thus

$$|\mu * f(x) - \mu * f(x_0)| < \epsilon\|\mu\|, \qquad x \in U,$$

which shows $L_\mu f \in C^b(K)$. Moreover, the measures $\mu \in M(K)$ with compact support are dense in $M(K)$, thus $L_\mu f$ is also continuous. Evidently, $\|L_\mu f\|_\infty \leqslant \|\mu\|f\|_\infty$ holds.
Now, let $f \in C_c(K)$ and $\mu \in M(K)$ with compact support. If $L_\mu f(x) \neq 0$, then we have supp $\tilde{\mu} \cap$ supp $R_x f \neq \varnothing$. Since supp $R_x f \subseteq$ supp $f * \{\tilde{x}\}$, we have that supp $\tilde{\mu} \cap$ (supp $f * \{\tilde{x}\}) \neq \varnothing$. Using Lemma 1.2.8, we get supp $f *$ supp $\tilde{\mu} \cap \{\tilde{x}\} \neq \varnothing$, hence $x \in$ supp $\mu *$ supp f by means of (H3). Thus $L_\mu f \in C_c(K)$.
Finally, we let $f \in C_0(K)$, $f \neq 0$ and $\mu \in M(K)$ arbitrary. For given $\epsilon > 0$, we choose a measure $\nu \in M(K)$ with compact support, $\nu \neq 0$, such that $\|\mu - \nu\| < \epsilon/\|f\|_\infty$ and then $g \in C_c(K)$ with $\|f - g\|_\infty < \epsilon/\|\nu\|$. Thus $\|L_\mu f - L_\mu g\|_\infty < 2\epsilon$, and $L_\mu f \in C_0(K)$ follows. The assertions for R_μ can be proven analogously. □

1.2.2 Discrete hypergroups

We will now turn to discrete hypergroups, following **[34]**, which will enable us to completely avoid notions of measure theory. What we have proven for general hypergroups above, of course, also holds for discrete hypergroups.

Let K be a set and for $x \in K$, let ϵ_x be the Dirac function on K, that is, $\epsilon_x(x) = 1$ and $\epsilon_x(y) = 0$ for $y \in K$ with $y \neq x$. The Banach space of all functions $f : K \to \mathbb{C}$, $f = \sum_{n=1}^{\infty} a_n \epsilon_{x_n}$ with $a_n \in \mathbb{C}$, $\sum_{n=1}^{\infty} |a_n| < \infty$ and x_n distinct points in K with the norm $\|f\| = \sum_{n=1}^{\infty} |a_n|$ will be denoted by ℓ^1. If $f \in \ell^1$ is a finite convex combination of Dirac functions, we write $f \in \ell^1_{co}$, $f = \sum_{n=1}^{N} \alpha_n \epsilon_{x_n}$ with $\alpha_n \geqslant 0$ and $\sum_{n=1}^{N} \alpha_n = 1$. Now, we define the discrete versions of the convolution, the involution and their extensions defined in definition 1.2.3. In the discrete case, the convolution $\omega : K \times K \to \ell^1_{co}$ is defined by

$$\omega(f, g) = \sum_{n,m=1}^{\infty} a_n b_m \omega(x_n, y_m), \tag{1.59}$$

where $f = \sum_{n=1}^{\infty} a_n \epsilon_{x_n}$ and $g = \sum_{m=1}^{\infty} b_m \epsilon_{y_m}$. Any rearrangement of the series can be applied, thus we have

$$\|\omega(f, g)\| \leqslant \sum_{n,m=1}^{\infty} |a_n||b_m|\|\omega(x_n, y_m)\| = \|f\|\|g\|,$$

and the bilinear extension $\omega : \ell^1 \times \ell^1 \to \ell^1$ of ω defined in (1.59) is well-defined. We define the involution $^\sim : K \to K$, $x \mapsto \tilde{x}$ and extend it to ℓ^1 by setting

$$\tilde{f} = \sum_{n=1}^{\infty} a_n \epsilon_{\tilde{x_n}},$$

where $f = \sum_{n=1}^{\infty} a_n \epsilon_{x_n}$.

We will now provide the definition of a discrete hypergroup which is much simpler than the definition of hypergroups in general.

Definition 1.2.14. Let K be a set. The triplet $(K, \omega, \tilde{\ })$ is called a *discrete hypergroup* if it satisfies the following properties:

(DH1) $\omega : K \times K \to \ell^1_{co}$ is a mapping such that associativity holds, $\omega(\epsilon_x, \omega(y, z)) = \omega(\omega(x, y), \epsilon_z)$ for all $x, y, z \in K$.

(DH2) $\tilde{\ } : K \to K$ is a bijective mapping such that $\tilde{\tilde{x}} = x$ and $\omega(x, y)^{\tilde{\ }} = \omega(\tilde{y}, \tilde{x})$ for all $x, y \in K$.

(DH3) There exists a (necessarily unique) element $e \in K$, the unit element, such that $\omega(e, x) = \epsilon_x = \omega(x, e)$ for all $x \in K$.

(DH4) We have $e \in \mathit{supp}(\omega(x, \tilde{y}))$ if and only if $x = y$.

We will also define the left-translation and the right-translation operators for the discrete case. Let $f : K \to \mathbb{C}$ be a function and $x \in K$. Then the left-translation is defined by

$$L_x f : K \to \mathbb{C}, \qquad L_x f(y) = \sum_{n=1}^{N} a_n f(u_n),$$

where $\omega(x, y) = \sum_{n=1}^{N} a_n \epsilon_{u_n}$. Since $L_x \epsilon_u(y) = \omega(x, y)(u)$ for each $u \in K$, we can write

$$L_x f(y) = \omega(x, y)(f).$$

The right-translation will be defined by

$$R_x f : K \to \mathbb{C}, \qquad R_x f(y) = \sum_{n=1}^{M} b_n f(v_n),$$

where $\omega(y, x) = \sum_{n=1}^{M} b_n \epsilon_{v_n}$.

Definition 1.2.15. Let K be a set. A positive function $h : K \to [0, \infty)$ is called *left-invariant* if for each $f : K \to \mathbb{C}$ with $|\text{supp } f| < \infty$ and $y \in K$

$$\sum_{x \in K} L_y f(x) h(x) = \sum_{x \in K} f(x) h(x).$$

A left-invariant positive function $h : K \to [0, \infty)$, $h \neq 0$, is called *Haar function*.
Right-invariance can be defined in the same way.

Theorem 1.2.16. *Let K be a set and $(K, \omega, \tilde{\ })$ a discrete hypergroup, then there exists a Haar function $h : K \to [0, \infty)$. If $h(e) = 1$, then*

$$h(x) = (\omega(\tilde{x}, x)(e))^{-1}, \qquad x \in K.$$

The Haar function is unique and positive up to multiplication by a positive constant.

Proof. Applying the associativity, we obtain for $x, y, z \in K$

$$\begin{aligned}\sum_{t\in K}\omega\left(x,t\right)\left(e\right)\omega\left(y,z\right)\left(t\right) &= \omega\left(\epsilon_x,\omega\left(y,z\right)\right)\left(e\right)\\ &= \omega\left(\omega\left(x,y\right),\epsilon_z\right)\left(e\right)\\ &= \sum_{t\in K}\omega\left(x,y\right)\left(t\right)\omega\left(t,z\right)\left(e\right).\end{aligned} \tag{1.60}$$

By (DH4) we have $\omega\left(x,y\right)\left(e\right) > 0$ if and only if $x = \tilde{y}$ and thus (1.60) reduces to

$$\omega\left(x,\tilde{x}\right)\left(e\right)\omega\left(y,z\right)\left(\tilde{x}\right) = \omega\left(x,y\right)\left(\tilde{z}\right)\omega\left(\tilde{z},z\right)\left(e\right). \tag{1.61}$$

By setting

$$h\left(x\right) := \left(\omega\left(\tilde{x},x\right)\left(e\right)\right)^{-1}$$

and with (DH2) (1.61) can be rewritten as

$$h\left(z\right)\omega\left(y,z\right)\left(\tilde{x}\right) = h\left(\tilde{x}\right)\omega\left(x,y\right)\left(\tilde{z}\right). \tag{1.62}$$

Summation over all $z \in K$ (1.62) gives

$$\sum_{z\in K}\omega\left(y,z\right)\left(\tilde{x}\right)h\left(z\right) = \sum_{z\in K}\omega\left(x,y\right)\left(\tilde{z}\right)h\left(\tilde{x}\right) = h\left(\tilde{x}\right), \tag{1.63}$$

where the last equation in (1.63) follows by the fact that $\omega\left(x,y\right) \in \ell^1_{co}$, particularly $\sum_{z\in K}\omega\left(x,y\right)\left(\tilde{z}\right) = 1$.
Since $L_y\epsilon_x\left(z\right) = \omega\left(y,z\right)\left(x\right)$, we have for each $y \in K$

$$\sum_{z\in K}L_y\epsilon_x\left(z\right)h\left(z\right) = \sum_{z\in K}\epsilon_x\left(z\right)h\left(z\right).$$

The uniqueness follows immediately, as we have for any Haar function h'

$$\omega\left(x,\tilde{x}\right)\left(e\right)h'\left(\tilde{x}\right) = \sum_{z\in K}L_x\epsilon_e\left(z\right)h'\left(z\right) = \sum_{z\in K}\epsilon_e\left(z\right)h'\left(z\right) = h'\left(e\right),$$

that is, $h'\left(x\right) = h'\left(e\right)h\left(x\right)$. □

We will usually use a Haar function which is normed by $h\left(e\right) = 1$, i.e. $h\left(x\right) \geqslant 0$ for all $x \in K$.
By (DH2) (1.62) can be rewritten as

$$h\left(z\right)\omega\left(y,z\right)\left(x\right) = h\left(\tilde{x}\right)\omega\left(\tilde{y},x\right)\left(z\right), \qquad x,y,z \in K, \tag{1.64}$$

which we will use in the rest of this section. In the next theorem, we will use (1.64) to prove a stronger property of the Haar function.

Theorem 1.2.17. *Let $f : K \to \mathbb{C}$, $g : K \to \mathbb{C}$ be functions with finite support. Then*

$$\sum_{z\in K}L_yf\left(z\right)g\left(z\right)h\left(z\right) = \sum_{z\in K}f\left(z\right)L_{\tilde{y}}g\left(z\right)h\left(z\right), \qquad y \in K.$$

Proof. It is sufficient to consider $f = \epsilon_x$ and $g = \epsilon_u$. By the application of (1.64), we obtain

$$\begin{aligned}\sum_{z\in K} L_y\epsilon_x(z)\,\epsilon_u(z)\,h(z) &= L_y\epsilon_x(u)\,h(u) = \omega(y,u)(x)\,h(u)\\ &= h(x)\,\omega(\tilde{y},x)(u) = L_{\tilde{y}}\epsilon_u(x)\,h(x)\\ &= \sum_{z\in K}\epsilon_x(z)\,L_{\tilde{y}}\epsilon_u(z)\,h(z).\end{aligned}$$

□

By Theorem 1.2.11 ℓ^1 is a Banach-$*$-algebra with unit ϵ_e where the algebra operations are given by the convolution $f * g := \omega(f,g)$ for $f, g \in \ell^1$ and the $*$-operation by $f^* = \overline{\left(\tilde{f}\right)}$ for $f \in \ell^1$.
The corresponding Banach space to the L^1-algebra is the Banach space

$$\ell^1(h) := \{f : K \to \mathbb{C} : \sum_{z\in K}|f(z)|h(z) < \infty\} \text{ with the norm } \|f\|_1 = \sum_{z\in K}|f(z)|h(z).$$

The Banach space $\ell^1(h)$ together with the operations

$$f * g(x) = \sum_{z\in K} f(z)\,L_{\tilde{z}}g(x)\,h(z) \tag{1.65}$$

and $f^* = \overline{\tilde{f}}$ becomes a Banach-$*$-algebra. We will show that $f * g \in \ell^1(h)$ and $\|f * g\|_1 \leqslant \|f\|_1\|g\|_1$: The functions with finite support are dense in $\ell^1(h)$, thus

$$\sum_{z\in K} L_y f(z)\,h(z) = \sum_{z\in K} f(z)\,h(z) \qquad \text{for every } f \in \ell^1(h).$$

Hence

$$\begin{aligned}\|L_y f\|_1 &= \sum_{z\in K}|L_y f(z)|h(z) \leqslant \sum_{z\in K} L_y|f|(z)\,h(z)\\ &= \sum_{z\in K}|f|(z)\,h(z) = \|f\|_1,\end{aligned}$$

therefore, L_y is a norm-decreasing operator in $\ell^1(h)$. By changing the summation, we get

$$\begin{aligned}\|f * g\|_1 &= \sum_{x\in K}|f * g(x)|h(x)\\ &\leqslant \sum_{x\in K}\sum_{z\in K}|f(z)||L_{\tilde{z}}g(x)|h(z)\,h(x) \leqslant \|f\|_1\|g\|_1.\end{aligned}$$

The following theorem summarizes the above results:

Theorem 1.2.18. *(i) The Banach space $\ell^1(h)$ together with the convolution $f * g$ (see (1.65)) and the $*$-operation $f^* = \overline{\left(\tilde{f}\right)}$ is a Banach-$*$-algebra with unit ϵ_e.*

(ii) The mapping $f \mapsto fh$, $\ell^1(h) \to \ell^1$ is an isometric isomorphism from the Banach space $\ell^1(h)$ onto the Banach space ℓ^1. It is also an algebra homomorphism. If $h(x) = h(\tilde{x})$ for all $x \in K$, then it is also a $*$-homomorphism.

Proof. Only (ii) is left to prove. We have

$$(\epsilon_x h) * (\epsilon_y h) = h(x)\, h(y)\, \omega(x, y)$$

and by (1.64)

$$\begin{aligned}(\epsilon_x * \epsilon_y)(u)\, h(u) &= (L_{\tilde{x}}\epsilon_y(u)\, h(x))\, h(u) = \omega(\tilde{x}, u)(y)\, h(x)\, h(u) \\ &= \omega(x, y)(u)\, h(x)\, h(y), \qquad u \in K.\end{aligned}$$

Evidently, $f^* h = (fh)^*$, if $h(x) = h(\tilde{x})$. □

The Banach-$*$-algebras ℓ^1 and $\ell^1(h)$ act on the Banach spaces $\ell^p(h)$, $1 \leqslant p \leqslant \infty$, where

$$\ell^p(h) = \{f : K \to \mathbb{C} : \sum_{z \in K} |f(z)|^p h(z) < \infty\}, \qquad 1 \leqslant p < \infty$$

and

$$\ell^\infty(h) = \ell^\infty = \{f : K \to \mathbb{C} : f \text{ bounded}\}$$

with the norms $\|f\|_p = \left(\sum_{z \in K} |f(z)|^p h(z)\right)^{1/p}$ and $\|f\|_\infty = \sup_{x \in K} |f(x)|$.

Proposition 1.2.19. *Let $1 \leqslant p \leqslant \infty$, K be a set and $y \in K$. For $f \in \ell^p(h)$ we have $L_y f \in \ell^p(h)$ and*

$$\|L_y f\|_p \leqslant \|f\|_p.$$

Proof. Let $1 \leqslant p < \infty$ and $\omega(y, z) = \sum_{n=1}^N a_n \epsilon_{x_n}$. Since $\sum_{n=1}^N a_n = 1$, the Hölder inequality implies

$$|L_y f(z)|^p = \left|\sum_{n=1}^N a_n f(x_n)\right|^p \leqslant \sum_{n=1}^N a_n |f|^p(x_n) = L_y |f|^p(z).$$

Thus

$$\begin{aligned}\|L_y f\|_p^p &= \sum_{z \in K}^N |L_y f(z)|^p h(z) \leqslant \sum_{z \in K} L_y |f|^p(z)\, h(z) \\ &= \sum_{z \in K} |f|^p(z)\, h(z) = \|f\|_p^p\end{aligned}$$

for $f : K \to \mathbb{C}$ with $|\text{supp } f| < \infty$. Since the functions with finite support are dense in $\ell^p(h)$, this proves the proposition for $1 \leqslant p < \infty$.
For $p = \infty$

$$|L_y f(z)| \leqslant \sum_{n=1}^N a_n |f(x_n)| \leqslant \|f\|_\infty, \qquad z \in K.$$

□

Lemma 1.2.20. *Let $\mu \in \ell^1$, $\mu \geqslant 0$, and $f \in \ell^p(h)$, $f \geqslant 0$, with $1 \leqslant p \leqslant \infty$. If we set*

$$\mu * f(x) := \sum_{z \in K} L_{\tilde{z}} f(x)\, \mu(z),$$

then $\mu * f(x)$ is finite for all $x \in K$, $\mu * f \in \ell^p(h)$ and $\|\mu * f\|_p \leqslant \|\mu\| \|f\|_p$.

Proof. Suppose $p = \infty$, then by Proposition 1.2.19 $\|L_{\tilde{z}} f\|_\infty \leqslant \|f\|_\infty$, and thus

$$|\mu * f(x)| \leqslant \|\mu\| \|f\|_\infty, \qquad x \in K.$$

For $1 \leqslant p < \infty$, we know that $\mu * f(x)$ is finite since $\ell^p(h) \subseteq \ell^\infty$. In order to prove that $\mu * f \in \ell^p(h)$, we can assume that $\sum_{z \in K} \mu(z) = 1$. Then the Hölder inequality yields

$$\begin{aligned}(\mu * f)^p(x) &\leqslant \sum_{z\in K} (L_{\tilde{z}} f)^p(x)\,\mu(z) \leqslant \sum_{z \in K} L_{\tilde{z}}(f^p)(x)\,\mu(z) \\ &= \mu * (f^p)(x).\end{aligned}$$

Hence by the left-invariance of h

$$\begin{aligned}\sum_{x\in K} (\mu * f)^p(x)\,h(x) &\leqslant \sum_{x \in K}\sum_{z\in K} L_{\tilde{z}}(f^p)(x)\,\mu(z)\,h(x) \\ &= \sum_{x\in K} f^p(x)\,h(x),\end{aligned}$$

that is, $\|\mu * f\|_p \leqslant \|f\|_p$. □

Now, $\mu * f(x)$ can be defined for arbitrary $\mu \in \ell^1$ and $f \in \ell^p(h)$, $1 \leqslant p \leqslant \infty$, as above by

$$\mu * f(x) = \sum_{z\in K} L_{\tilde{z}} f(x)\,\mu(z). \tag{1.66}$$

Theorem 1.2.21. *Let $\mu \in \ell^1$, $f \in \ell^p(h)$, $1 \leqslant p \leqslant \infty$. Then $\mu * f \in \ell^p(h)$ and $\|\mu * f\|_p \leqslant \|\mu\|\|f\|_p$.*

Proof. Since $|\mu * f(x)| \leqslant |\mu| * |f|(x)$, $\mu * f(x)$ is finite.
We assume that $\sum_{z\in K} |\mu(z)| = 1$. Then we get as in the proof of Lemma 1.2.20

$$|\mu * f|^p(x) \leqslant (|\mu| * |f|(x))^p \leqslant |\mu| * |f|^p(x)$$

and thus

$$\begin{aligned}\sum_{x\in K} |\mu * f|^p(x)\,h(x) &\leqslant \sum_{x\in K} |\mu| * |f|^p(x)\,h(x) \\ &= \sum_{x\in K} |f|^p(x)\,h(x).\end{aligned}$$

□

Proposition 1.2.22. *Let $1 < p < \infty$, $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \ell^p(h)$ and $g \in \ell^q(h)$. For $x \in K$ we define*

$$f * g(x) = \sum_{z\in K} f(z)\,L_{\tilde{x}} g(z)\,h(z).$$

Then $f * g$ is a bounded function and $\|f * g\|_\infty \leqslant \|f\|_p \|g\|_q$. Moreover, $f * g$ can be uniformly approximated by functions with finite support.

Proof. The Hölder inequality and Proposition 1.2.19 imply

$$\left|\sum_{z\in K} f(z)\, L_{\tilde{x}}g(z)\, h(z)\right| \leqslant \|f\|_p \|L_{\tilde{x}}g\|_q \leqslant \|f\|_p\|g\|_q.$$

Therefore, $f * g$ is defined for every $x \in K$ and $\|f * g\|_\infty \leqslant \|f\|_p\|g\|_q$. If f and g have finite support, then so does $f * g$ and by Theorem 1.2.10 supp $L_{\tilde{x}}g \subseteq \omega(\{x\}, \text{supp } g)$. Hence Lemma 1.2.8 implies

$$\begin{aligned} \text{supp } f * g &\subseteq \{x \in K : \text{supp } f \cap \omega(\{x\}, \text{supp } g)\} \\ &\subseteq \omega(\text{supp } f, \text{supp } g). \end{aligned}$$

Let $f \in \ell^p(h)$, $g \in \ell^q$ be arbitrary and choose f_n, g_n with finite support and $\|f - f_n\|_p \to 0$, $\|g - g_n\|_q \to 0$ for $n \to \infty$. Then

$$\begin{aligned} \|f * g - f_n * g_n\|_\infty &\leqslant \|f * (g - g_n)\|_\infty + \|(f - f_n) * g_n\|_\infty \\ &\leqslant \|f\|_p\|g - g_n\|_q + \|f - f_n\|_p\|g_n\|_q \to 0, \end{aligned}$$

for $n \to \infty$. □

1.2.3 Polynomial hypergroups

We will now introduce polynomial hypergroups referring to an unpublished book manuscript of R. Lasser, [**32**] and [**33**].
Let $\{P_n(x)\}_{n=0}^\infty$ be an OPS defined on $\mathbb{R}$ with respect to a probability measure $\pi \in M^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} P_n(x)\, P_m(x)\, d\pi(x) = \delta_{nm}\mu_m, \qquad \mu_m > 0.$$

We assume that $P_n(1) \neq 0$ and that (after renorming)

$$P_n(1) = 1, \qquad n \in \mathbb{N}_0.$$

Since the degree of $P_n(x)$ is n for each $n \in \mathbb{N}_0$ and $\{P_n(x)\}_{n=0}^\infty$ is an OPS, we have the following recurrence relation (see (1.22))

$$P_1(x)\, P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) \tag{1.67}$$

for $n \in \mathbb{N}$ and

$$P_0(x) = 1, \qquad P_1(x) = \frac{1}{a_0}(x - b_0) \tag{1.68}$$

with $a_n > 0$ for all $n \in \mathbb{N}_0$, $c_n > 0$ for all $n \in \mathbb{N}$ and $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$.
Conversely, the Theorem of Perron-Favard (see Theorem 1.1.23) states that a polynomial sequence $\{P_n(x)\}_{n=0}^\infty$ which is defined recursively by (1.67) and (1.68) is an OPS with respect to a certain measure $\pi \in M^1(\mathbb{R})$. The condition $P_n(1) = 1$ implies that $a_n + b_n + c_n = 1$ for each $n \in \mathbb{N}$ and $a_0 + b_0 = 1$. The following lemma extends the recurrence relation in (1.67).

Lemma 1.2.23. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS with respect to a measure $\pi \in M^1(\mathbb{R})$ satisfying $P_n(1) = 1$. Then the products $P_m(x)P_n(x)$ can be linearized by*

$$P_m(x)P_n(x) = \sum_{k=|n-m|}^{n+m} g(m,n;k)P_k(x) \tag{1.69}$$

with $g(m,n;k) \in \mathbb{R}$ for $k = |n-m|, \dots, n+m$. Furthermore, $g(m,n;|n-m|) \neq 0$ and $g(m,n;n+m) \neq 0$.

***Proof*.** The OPS $\{P_n(x)\}_{n=0}^{\infty}$ forms a basis in the real vector space of all polynomials. As the degree of $P_n(x)$ is n for each $n \in \mathbb{N}_0$, we have a unique representation

$$P_m(x)P_n(x) = \sum_{k=0}^{n+m} g(m,n;k)P_k(x)$$

where $g(m,n;k) \in \mathbb{R}$ for $k = 0, \dots, n+m$ and $g(m,n;n+m) \neq 0$. It remains to show that $g(m,n;k) = 0$ for $k = 0, \dots, |n-m|-1$ and $g(m,n;|n-m|) \neq 0$. In order to show that $g(m,n;k) = 0$ for $k = 0, \dots, |n-m|-1$, we assume that $m < n$. If $k < n-m$, then the degree of $(P_mP_k)(x)$ is strictly smaller than n, and thus

$$\begin{aligned} 0 = \int_{\mathbb{R}} P_m(x)P_k(x)P_n(x)\,d\pi(x) &= \sum_{j=0}^{n+m} g(m,n;j)\int_{\mathbb{R}} P_j(x)P_k(x)\,d\pi(x) \\ &= g(m,n;k)\,\mu_k. \end{aligned} \tag{1.70}$$

Since $\mu_k > 0$, we have $g(m,n;k) = 0$ for $k = 0, \dots, |n-m|-1$. Now, we want to show that $g(m,n;|n-m|) \neq 0$ and assume $g(m,n;n-m) = 0$. Then we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} P_{n-m}(x)P_m(x)P_n(x)\,d\pi(x) = \sum_{k=|n-2m|}^{n} g(m,n-m;k)\int_{\mathbb{R}} P_k(x)P_n(x)\,d\pi(x) \\ &= g(m,n-m;n)\,\mu_n, \end{aligned}$$

which is a contradiction to $\mu_n > 0$, hence $g(m,n-m;n) \neq 0$. □

Proposition 1.2.24. *The linearization coefficents $g(m,n;k))$ in (1.69) satisfy the following properties for $k = |n-m|, \dots, n+m$:*

(i) $g(m,n;k) = g(n,m;k)$ *for all* $n,m \in \mathbb{N}_0$,

(ii) $g(0,n;n) = g(n,0;n) = 1$ *for all* $n \in \mathbb{N}_0$,

(iii) $g(1,n;n+1) = a_n$, $g(1,n;n) = b_n$ *and* $g(1,n;n-1) = c_n$ *for all* $n \in \mathbb{N}_0$,

(iv) $\sum_{k=|n-m|}^{n+m} g(m,n;k) = 1$ *for all* $n,m \in \mathbb{N}_0$,

(v) $g(n,n;0) = \mu_n$ *for all* $n \in \mathbb{N}_0$,

(vi) $g(m,n;k)\,\mu_k = g(m,k;n)\,\mu_n$ *for all* $n,m \in \mathbb{N}_0$,

(vii) $g(m,n;n+m) = g(m-1,n;n+m-1)\,\frac{a_{n+m-1}}{a_{m-1}}$,

$$g(m,n;n-m) = g(m-1,n;n-m+1)\,\tfrac{c_{n-m+1}}{a_{m-1}}, \qquad 2 \leqslant m \leqslant n,$$

(viii) $g(m,n;n+m) = \frac{a_n a_{n+1} a_{n+2} \cdot \ldots \cdot a_{n+m-1}}{a_1 a_2 \cdot \ldots \cdot a_{m-1}}$,

$$g(m,n;n-m) = \frac{c_n c_{n-1} c_{n-2} \cdot \ldots \cdot c_{n-m+1}}{a_1 a_2 \cdot \ldots \cdot a_{m-1}}, \qquad 2 \leqslant m \leqslant n,$$

(ix) *For all* $2 \leqslant m \leqslant n$

$$\begin{aligned} g(m,n;n+m-1) &= \frac{g(m-1,n;n+m-2)\,a_{n+m-2}}{a_{m-1}} \\ &\quad + \frac{g(m-1,n;n+m-1)\,(b_{n+m-1}-b_{m-1})}{a_{m-1}}, \end{aligned}$$

and for $k = 2, \ldots, 2m-2$

$$\begin{aligned} g(m,n;n+m-k) &= \frac{g(m-1,n;n+m-k-1)\,a_{n+m-k-1}}{a_{m-1}} \\ &\quad + \frac{g(m-1,n;n+m-k+1)\,c_{n+m-k+1}}{a_{m-1}} \\ &\quad + \frac{g(m-1,n;n+m-k)\,(b_{n+m-k}-b_{m-1})}{a_{m-1}} \\ &\quad - \frac{g(m-2,n;n+m-k)\,c_{m-1}}{a_{m-1}}, \\ g(m,n;n+m+1) &= \frac{g(m-1,n;n-m+1)\,(b_{n-m+1}-b_{m-1})}{a_{m-1}} \\ &\quad + \frac{g(m-1,n;n-m+2)\,c_{n-m+2}}{a_{m-1}}. \end{aligned}$$

Proof. Assertions (i)-(iv) obviously hold.

(v) We have

$$\mu_n = \int_{\mathbb{R}} P_n^2(x)\,d\pi(x) = \sum_{k=0}^{2n} g(n,n;k) \int_{\mathbb{R}} P_k(x)\,d\pi(x) = g(n,n;0).$$

(vi) We get

$$\begin{aligned} \int_{\mathbb{R}} P_m(x)\,P_n(x)\,P_k(x)\,d\pi(x) &= \sum_{j=|m-n|}^{m+n} g(m,n;j) \int_{\mathbb{R}} P_j(x)\,P_k(x)\,d\pi(x) \\ &= g(m,n;k)\,\mu_k. \end{aligned}$$

and linearizing $P_m(x)\,P_k(x)$ gives

$$\begin{aligned} \int_{\mathbb{R}} P_m(x)\,P_n(x)\,P_k(x)\,d\pi(x) &= \sum_{j=|m-k|}^{m+k} g(m,k;j) \int_{\mathbb{R}} P_j(x)\,P_n(x)\,d\pi(x) \\ &= g(m,k;n)\,\mu_n. \end{aligned}$$

(vii) We write

$$\begin{aligned} P_1(x)\left(P_{m-1}(x)\,P_n(x)\right) &= \sum_{k=n-m+1}^{n+m-1} g(m-1,n;k)\,P_1(x)\,P_k(x) \\ &= g(m-1,n;n+m-1)\,a_{n+m-1}P_{n+m}(x)+\ldots, \end{aligned}$$

where the remaining summands have degree smaller that $n+m$. We also have that

$$\begin{aligned} \left(P_1(x)\,P_{m-1}(x)\right)P_n(x) &= \left(a_{m-1}P_m(x)+b_{m-1}P_{m-1}(x)+c_{m-1}P_{m-2}(x)\right)P_n(x) \\ &= a_{m-1}g(m,n;n+m)\,P_{n+m}(x)+\ldots. \end{aligned}$$

Thus $a_{m-1}g(m,n;n+m) = g(m-1,n;n+m-1)\,a_{n+m-1}$. Similarly, the second equation in (vii) can be shown.

(viii) follows from (vii).

(ix) See **[33]**. □

Theorem 1.2.25. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS with $P_n(1)=1$ and assume that the coefficients $g(m,n;k)$ (see (1.69)) satisfy*

$$g(m,n;k) \geqslant 0 \qquad \text{for } k=|m-n|,\ldots,m+n.$$

Then the triplet $(\mathbb{N}_0,\omega,\tilde{}\,)$ is a hypergroup with unit element 0, where the convolution ω on $\mathbb{N}_0$ (with discrete topology) is defined by

$$\omega(m,n) = \sum_{k=|n-m|}^{n+m} g(n,m;k)\,\epsilon_k, \qquad n,m\in\mathbb{N}_0,$$

and the involution is given by the identity-mapping $\tilde{n} := n$.

Proof. Since

$$\sum_{k=|n-m|}^{n+m} g(m,n;k) = 1, \qquad m,n\in\mathbb{N}_0,$$

$\omega(m,n)$ are probability measures on $\mathbb{N}_0$ with compact support. By Proposition 1.2.24 (v), we have that $g(m,n;0)=0$ if $m\neq n$, hence $0\notin \operatorname{supp}\omega(m,n)$ if and only if $m=n$. Finally, we check the associativity (H1) and set $e_k(j) := \delta_{k,j}$. Then $\omega(m,n)(e_k) = g(m,n;k)$ and hence

$$\begin{aligned} \epsilon_l*\omega(m,n)(e_k) &= \sum_{j=0}^{\infty}\omega(l,j)(e_k)\,d\omega(m,n)(j) = \sum_{j=|n-m|}^{n+m} g(m,n;j)\,g(j,l;k) \\ &= \frac{1}{\mu_k}\sum_{j=|n-m|}^{n+m} g(m,n;j)\int_{\mathbb{R}} P_l(x)\,P_j(x)\,P_k(x)\,d\pi(x) \\ &= \frac{1}{\mu_k}\int_{\mathbb{R}} P_l(x)\left(P_m(x)\,P_n(x)\right)P_k(x)\,d\pi(x) \\ &= \frac{1}{\mu_k}\int_{\mathbb{R}} \left(P_l(x)\,P_m(x)\right)P_n(x)\,P_k(x)\,d\pi(x) \\ &= \sum_{j=|l-m|}^{l+m} g(l,m;j)\,g(j,n;k) = \omega(l,m)*\epsilon_n(e_k)\,. \end{aligned}$$

Then by considering linear combinations of e_k, we get

$$\epsilon_l * \omega(m,n) = \omega(l,m) * \epsilon_n.$$

□

Such hypergroups are called *polynomial hypergroups* on $\mathbb{N}_0$ induced by $\{P_n(x)\}_{n=0}^{\infty}$.

Remark 1.2.26. (1) Favard's theorem (see Theorem 1.1.21) implies that every commutative hypergroup on $\mathbb{N}_0$ with identity involution and 0 as a unit element which satisfies

$$\{n-1, n+1\} \subseteq \operatorname{supp}\ (\omega(1,n)) \subseteq \{n-1, n, n+1\}$$

for all $n \in \mathbb{N}$ is a polynomial hypergroup induced by the orthogonal sequence $\{P_n(x)\}_{n=0}^{\infty}$.

(2) The corresponding orthogonal polynomials depend on the choice of $P_1(x)$, that is, on the choice of $a_0, b_0 \in \mathbb{R}$.

(3) For a polynomial hypergroup we have the following translation operator

$$L_n f(m) = \sum_{k=|n-m|}^{n+m} g(n,m;k)\, f(k).$$

(4) By Theorem 1.2.16 we have for the Haar function

$$h(n) = (\omega(n,n)(0))^{-1} = g(n,n;0)^{-1} = \mu_n^{-1}.$$

We will now give a few examples for polynomial hypergroups on $\mathbb{N}_0$ that are induced by polynomials introduced in Example 1.1.2.

Examples 1.2.27. (1) The Chebyshev polynomials of the first kind $\{T_n(x)\}_{n=0}^{\infty}$ induce the *Chebyshev hypergroup* on $\mathbb{N}_0$. The addition theorem for the cosine-function gives

$$T_m(x)\, T_n(x) = \frac{1}{2} T_{|n-m|}(x) + T_{n+m}(x), \qquad n, m \in \mathbb{N}_0,$$

particularly $g(m,n;|n-m|) = \frac{1}{2} = g(m,n;n+m)$ and $g(m,n;k) = 0$ otherwise. The convolution is given by

$$\omega(m,n) = \frac{1}{2}\epsilon_{|n-m|} + \frac{1}{2}\epsilon_{n+m},$$

and the Haar weights are $h(0) = 1$ and $h(n) = 2$ for $n \geqslant 2$. The orthogonalization measure is given by

$$d\pi(x) = \frac{1}{\pi}\chi_{[-1,1]}\left(1-x^2\right)^{-1/2} dx.$$

(2) For the *Jacobi polynomials* and $\alpha, \beta \in \mathbb{R}$ such that $\alpha \geqslant \beta > -1$ and $\alpha + \beta + 1 \geqslant 0$, we have for $n \in \mathbb{N}$,

$$\begin{aligned} a_0 &= \frac{2(\alpha+1)}{\alpha+\beta+2}, \qquad b_0 = \frac{\beta-\alpha}{\alpha+\beta+2}, \\ a_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(\alpha+1)}, \\ b_n &= \frac{\alpha-\beta}{2(\alpha+1)}\left[1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\right], \\ c_n &= \frac{n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)(\alpha+1)}. \end{aligned}$$

The Haar weights are

$$h(0) = 1, \qquad h(n) = \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n(\alpha+1)_n}{(\alpha+\beta+1)\,n!\,(\beta+1)_n}, \qquad n \geqslant 1,$$

and the orthogonalization measure is given by

$$d\pi(x) = \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \chi_{[-1,1]}(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

(3) If we set $\alpha = \beta$, $\alpha \geqslant -\frac{1}{2}$, for the Jacobi polynomials, we get the *ultraspherical* case with

$$\begin{aligned} a_0 &= 1, \qquad b_0 = 0, \\ a_n &= \frac{n+2\alpha+1}{2n+2\alpha+1}, \qquad b_n = 0, \qquad c_n = \frac{n}{2n+2\alpha+1}, \qquad n \in \mathbb{N} \end{aligned}$$

and for $m \leqslant n$,

$$\begin{aligned} g(n,m;k) = {} & \frac{n!m!\left(\alpha+\frac{1}{2}\right)_k\left(\alpha+\frac{1}{2}\right)_{n-k}\left(\alpha+\frac{1}{2}\right)_{m-k}(2\alpha+1)_{n+m-k}}{k!(n-k)!(m-k)!\left(\alpha+\frac{1}{2}\right)_{n+m-k}(2\alpha+1)_n(2\alpha+1)_m} \\ & \cdot \frac{\left(n+m+\alpha+\frac{1}{2}-2k\right)}{\left(n+m+\alpha+\frac{1}{2}-k\right)}, \end{aligned}$$

for $k \in \{n-m, n-m+2, n-m+4, \ldots, n+m\}$, and

$$g(n,m;k) = 0,$$

for $k \in \{n-m+1, n-m+3, n-m+5, \ldots, n+m-1\}$.
The Haar weights are given by

$$h(0) = 1, \qquad h(n) = \frac{(2n+2\alpha+1)(2\alpha+1)_n}{(2\alpha+1)\,n!}, \qquad n \geqslant 1,$$

and the orthogonalization measure by

$$d\pi(x) = \frac{1}{2^{2\alpha+1}} \frac{\Gamma(2\alpha+2)}{(\Gamma(\alpha+1))^2} \chi_{[-1,1]}\left(1-x^2\right)^{\alpha} dx.$$

(4) The *generalized Chebyshev polynomials* induce a polynomial hypergroup with

$$a_0 = 1, \qquad b_0 = 0,$$

$$a_n = \begin{cases} \frac{k+\alpha+\beta+1}{2k+\alpha+\beta+1}, & n = 2k, k \in \mathbb{N} \\ \frac{k+\alpha+1}{2k+\alpha+\beta+2}, & n = 2k+1, k \in \mathbb{N}_0 \end{cases}, \qquad c_n = \begin{cases} \frac{k}{2k+\alpha+\beta+1}, & n = 2k, k \in \mathbb{N} \\ \frac{k+\beta+1}{2k+\alpha+\beta+2}, & n = 2k+1, k \in \mathbb{N}_0 \end{cases}$$

and $b_n = 0$, $n \in \mathbb{N}$. The Haar weights are

$$h(0) = 1, \qquad h(n) = \begin{cases} \frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k(\alpha+1)_k}{k!(\alpha+\beta+1)(\beta+1)_k}, & n = 2k, k \in \mathbb{N}, \\ \frac{(2k+\alpha+\beta+2)(\alpha+\beta+2)_k(\alpha+1)_k}{k!(\beta+1)_{k+1}}, & n = 2k+1, k \in \mathbb{N}_0, \end{cases} \qquad n \geqslant 1,$$

and the orthogonalization measure is given by

$$d\pi(\alpha) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \chi_{[-1,1]} \left(1-x^2\right)^{\alpha} |x|^{2\beta+1}.$$

1.2.4 Homogeneous Banach spaces

Referring to **[22]**, we will give a short introduction on homogeneous Banach spaces together with some examples. In Section 3 we will define a transfer operator on the spaces introduced in this paragraph.

Let $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ be the Jacobi polynomials defined in Example 1.1.2 (4) (see also Example 1.2.27 (2) for the orthogonalization measure and the Haar weights) with $(\alpha,\beta) \in J := \{(\alpha,\beta) : \alpha \geqslant \beta > -1 \text{ and } \left(\beta \geqslant -\frac{1}{2} \text{ or } \alpha+\beta \geqslant 0\right)\}$. Then for any $x, y \in S := [-1,1]$ there exists a probability Borel measure $\mu_{x,y}^{(\alpha,\beta)} \in M(S)$ such that

$$P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) = \int_{-1}^{1} P_n^{(\alpha,\beta)}(z)\, d\mu_{x,y}^{(\alpha,\beta)}(z), \qquad n \in \mathbb{N}_0.$$

Let $L^p(S,\pi)$, $1 \leqslant p < \infty$, be the Banach space with the norm

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p\, d\pi(x)\right)^{1/p},$$

and $C(S)$ the Banach space with norm $\|f\|_\infty = \sup_{x\in S} |f(x)|$. The measures $\mu_{x,y}^{(\alpha,\beta)}$ induce a generalized translation operator which can be defined for $f \in C(S)$ or $f \in L^1(S,\pi)$ and $y \in S$ by

$$T_y f(x) = \int_{-1}^{1} f(z)\, d\mu_{x,y}^{(\alpha,\beta)}(z). \tag{1.71}$$

For the Jacobi polynomials this hypergroup structure is dual to the one on $\mathbb{N}_0$. For this translation we have that $T_y f \in L^p(S,\pi)$ if $f \in L^p(S,\pi)$, and $T_y f \in C(S)$ if $f \in C(S)$. Furthermore, $\|T_y f\|_p \leqslant \|f\|_p$ and $\lim_{y\to 1-} \|T_y f - f\|_p = 0$ (see **[7]**, p. 42 Lemma 1.4.6 (ii)). Using the generalized translation in (1.71), we can define a convolution by

$$f * g(y) = \int_{-1}^{1} f(x)\, T_y g(x)\, d\pi(x), \qquad f, g \in L^1(S,\pi), y \in S, \tag{1.72}$$

which is an element of $L^1(S,\pi)$, such that $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$. So we have that $L^1(S,\pi)$ with the convolution as multiplication is a Banach algebra. Similarly, using (1.72) with $f \in L^1(S,\pi)$ and $g \in L^p(S,\pi)$, $1 \leqslant p < \infty$, $L^1(S,\pi)$ acts on $L^p(S,\pi)$.

Definition 1.2.28. A linear subspace B of $L^1(S,\pi)$ is called a *homogeneous Banach space* on S with respect to some fixed pair $(\alpha,\beta) \in J$ if it is endowed with a norm $\|\cdot\|_B$ such that

(B1) $P_n^{(\alpha,\beta)} \in B$ for all $n \in \mathbb{N}_0$.

(B2) B is complete with respect to $\|\cdot\|_B$ and $\|\cdot\|_1 \leqslant \|\cdot\|_B$.

(B3) For $f \in B$, $x \in S$, $T_x f \in B$ and $\|T_x f\|_B \leqslant \|f\|_B$.

(B4) For $f \in B$, the map $x \mapsto T_x f$, $S \to B$ is continuous.

Furthermore, a homogeneous Banach space B is called *character-invariant* if

(B5) For every $f \in B$, $n \in \mathbb{N}_0$, $P_n^{(\alpha,\beta)} \cdot f \in B$ and $\|P_n^{(\alpha,\beta)} \cdot f\|_B \leqslant \|f\|_B$.

Proposition 1.2.29. *Let B be a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$. Then for $g \in B$ and $f \in L^1(S,\pi)$, $f * g \in B$ and $\|f * g\|_B \leqslant \|f\|_1 \|g\|_B$.*

Proof. For the B-valued integral with $g \in B$, we have

$$f * g = \int_{-1}^{1} f(x)\, T_x g d\pi(x) \quad \in B,$$

and $\|f * g\|_B \leqslant \|f\|_1 \|g\|_B$, where f is a continuous function on S. If $f \in L^1(S,\pi)$, then choose a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in C(S)$ with $\|f - f_n\|_1 \to 0$ as $n \to \infty$. Thus it follows that $f * g \in B$ and $\|f * g\|_B \leqslant \|f\|_1 \|g\|_B$. ☐

Corollary 1.2.30. *Every homogeneous Banach space B on S with respect to $(\alpha,\beta) \in J$ is a Banach algebra with convolution as multiplication.*

Proof. The corollary follows immediately as $B \subseteq L^1(S,\pi)$ and $\|\cdot\|_1 \leqslant \|\cdot\|_B$. ☐

Examples 1.2.31. (1) Let denote

$$\check{f}(n) = \int_{-1}^{1} f(x)\, P_n^{(\alpha,\beta)}(x)\, d\pi(x)$$

if $f \in L^1(S,\pi)$, and

$$\hat{d}(x) = \sum_{n=0}^{\infty} d(n)\, P_n^{(\alpha,\beta)}(x)\, h(n)$$

if $d \in l^1(\mathbb{N}_0, h)$. For $f \in L^1(S,\pi)$ the *Wiener algebra* $A(S) := \{f \in C(S) : \sum_{n=0}^{\infty} |\check{f}(n)|\, h(n) < \infty\}$ with the norm $\|f\|_{A(S)} = \sum_{n=0}^{\infty} |\check{f}(n)|\, h(n)$ is a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$. It is also character-invariant.

Proof. $A(S)$ is a linear space and by the uniqueness theorem (see **[7]**) $\|\cdot\|_{A(S)}$ is a norm. (B1) is satisfied since $\check{P}_m^{(\alpha,\beta)}(n) = h(n)^{-1}\, \delta_{m,n}$. $\left(A(S), \|\cdot\|_{A(S)}\right)$ is complete because $f \mapsto \check{f}$, $A(S) \to l^1(\mathbb{N}_0, h)$ is an isometric isomorphism from $A(S)$ onto the Banach space $l^1(\mathbb{N}_0, h)$. (B2) holds as $\|f\|_{A(S)} = \|\check{f}\|_1 \geqslant \|\check{f}\|_2 = \|f\|_2 \geqslant \|f\|_1$. For $f, g \in L^1(S,\pi)$ we have

$$\int_{-1}^{1} f(x)\, T_y g(x)\, d\pi(x) = \int_{-1}^{1} T_y f(x)\, g(x)\, d\pi(x). \tag{1.73}$$

Then it is easy to show that

$$(T_y f)\check{}\,(n) = P_n^{(\alpha,\beta)}(y)\,\check{f}(n)$$

for every $f \in L^1(S,\pi)$, $y \in S$. Hence $T_x f \in A(S)$ and $\|T_x f\|_{A(S)} \leqslant \|f\|_{A(S)}$. Now, let $x_0 \in S$ and $\epsilon > 0$. There exists $N \in \mathbb{N}$ and $g \in A(S)$ such that $\|g - f\|_{A(S)} < \frac{\epsilon}{4}$ with $\check{g}(n) = \check{f}(n)$ for all $n \in \{0,\dots,N\}$ and $\check{g}(n) = 0$ for $n \geqslant N+1$. Moreover, there is some $\delta > 0$ satisfying $\|T_x g - T_{x_0} g\|_{A(S)} < \frac{\epsilon}{2}$ for all $x \in S$ such that $|x - x_0| < \delta$. Thus $\|T_x f - T_{x_0} f\|_{A(S)} < \epsilon$ for all $x \in S$ with $|x - x_0| < \delta$ which shows (B4). (B5) holds since we have for $f \in A(S)$ and $n \in \mathbb{N}_0$

$$\left(P_n^{(\alpha,\beta)} \cdot f\right)\check{}\,(m) = L_n \check{f}(m), \qquad m \in \mathbb{N}_0,$$

where L_n denotes the translation operator on $l^1(\mathbb{N}_0, h)$. Thus $P_n^{(\alpha,\beta)} \cdot f \in A(S)$ and $\|P_n^{(\alpha,\beta)} \cdot f\|_{A(S)} \leqslant \|f\|_{A(S)}$. $\square$

(2) We use the same notation as in (1), then for $f \in L^1(S,\pi)$ the p-versions, $1 \leqslant p < \infty$, of the Wiener algebra $A^p(S) := \{f \in L^1(S,\pi) : \check{f} \in \ell^p(h)\}$ with the norm $\|f\|^p = \|f\|_1 + \|\check{f}\|_p$ is a homogeneous Banach space on S with respect to $(\alpha,\beta) \in J$. It is also character-invariant.

Proof. Again $A^p(S)$ is a linear space, by the uniqueness theorem $\|\cdot\|^p$ is a norm and as in (1), (B1) is satisfied since $\check{P}_m^{(\alpha,\beta)}(n) = h(n)^{-1}\,\delta_{m,n}$. The completeness of $(A^p(S), \|\cdot\|^p)$ is obvious and $\|f\|^p = \|f\|_1 + \|\check{f}\|_p \geqslant \|f\|_1$. For $f \in L^1(S,\pi)$, $y \in S$ we have with (1.73)

$$(T_y f)\check{}\,(n) = P_n^{(\alpha,\beta)}(y)\,\check{f}(n).$$

Hence $\|T_x f\|^p = \|T_x f\|_1 + \|(T_x f)\check{}\,\|_p \leqslant \|f\|_1 + \|\check{f}\|_p \leqslant \|f\|^p$ and $T_x f \in A^p(S)$. Now, let $x_0 \in S$ and $\epsilon > 0$. Since $C(S)$ is dense in $L^1(S)$, and for $f \in L^1(S)$ $\check{f}$ is continuous, (B4) follows similarly as in (1) by $\|T_x f - T_{x_0} f\|^p = \|T_x f - T_{x_0} f\|_1 + \|T_x f - T_{x_0} f\|_p < \epsilon$ for all $x \in S$ with $|x - x_0| < \delta$. (B5) holds since we have for $f \in A^p(S)$ and $n \in \mathbb{N}_0$

$$\left(P_n^{(\alpha,\beta)} \cdot f\right)\check{}\,(m) = L_n \check{f}(m), \qquad m \in \mathbb{N}_0,$$

where L_n denotes the translation operator on $l^p(\mathbb{N}_0, h)$. Thus $P_n^{(\alpha,\beta)} \cdot f \in A^p(S)$ and $\|P_n^{(\alpha,\beta)} \cdot f\|^p = \|P_n^{(\alpha,\beta)} \cdot f\|_1 + \|\left(P_n^{(\alpha,\beta)} \cdot f\right)\check{}\,\|_p \leqslant \|f\|_1 + \|\check{f}\|_p = \|f\|^p$. $\square$

Remark 1.2.32. (1) For $p = 1$, $A^1(S) \cong A(S)$. This holds because $f \in A^1(S) \subseteq C(S)$ since $f(x) = \sum_{n=0}^{\infty} \check{f}(n)\,P_n^{(\alpha,\beta)} h(n)$. Moreover, $\|f\|_1 \leqslant \|f\|_\infty \leqslant \|\check{f}\|_1$ and thus $\|f\|^1 = \|f\|_1 + \|\check{f}\|_1 \leqslant 2\|\check{f}\|_1$. Together we have that the norms are equivalent $\|\check{f}\|_1 \leqslant \|f\|^1 \leqslant 2\|\check{f}\|_1$. Hence $A^1(S)$ and $A(S)$ are isometrically isomorphic.

(2) For $\alpha = \beta = -\frac{1}{2}$ we have the Wiener algebra with respect to the Chebychev polynomials of the first kind and for $\alpha = \beta$, $\alpha \geqslant \frac{1}{2}$ we have the ultraspherical case.

2 Transfer operators

2.1 Transfer operators and spectral properties

Transfer operators with positive weight associated with certain dynamical systems are called Ruelle-Perron-Frobenius (RPF) operators and often occur in thermodynamical formalism.
The main result presented in this chapter will be Ruelle's theorem, the existence and the simplicity of a unique maximal eigenvalue for the Ruelle-Perron-Frobenius operator on a Hölder continuous function space and the existence and uniqueness of the Gibbs measure. Its proof will follow work done by Y.P. Jiang (see [**27**]). The original proof done by D. Ruelle (see [**47**], [**49**]) uses the Hilbert projective metric on cones in Banach spaces. He showed that the Ruelle-Perron-Frobenius operator acting on the Hölder continuous function space contracts the Hilbert projective metric of positive functions on the convex cones in the Hölder continuous function space. Thus the contracting fixed point theorem implies the existence and the uniqueness of a maximal eigenvalue. Y.P. Jiang found a shorter proof, which we will present. We won't consider further work done on transfer operator concerning dynamical zeta functions or Fredholm determinants (see for example [**52**], [**51**], [**4**],[**3**], [**25**]).

2.1.1 Geometry of expanding and mixing dynamical systems

We will now give the preliminaries that are needed to state Ruelle's theorem.

Definition 2.1.1. Let (X, d) be a compact metric space, and let $B(x, r)$ denote the open ball centered at x with radius $r > 0$.

(1) For each $n \geqslant 0$

$$d_n(x, y) := \max_{0 \leqslant i \leqslant n} \left\{ d\left(f^i(x), f^i(y)\right) \right\}$$

is called the *n-Bowen metric.*

(2) $B_n(x, r) = \{y \in X : d_n(x, y) < r\}$ is called the *n-Bowen ball* centered at x with radius $r > 0$.

Remark 2.1.2. The 0-Bowen metric and a 0-Bowen ball are just the original metric d and a ball for d.

Definition 2.1.3. Let (X, d) be a compact metric space and $f : X \to X$ a continuous map. Let $\{f^n\}_{n=0}^{\infty}$ denote a dynamical system on X; for simplicity we call f itself a dynamical system.

(1) The dynamical system f is called *locally expanding* if there are constants $\lambda > 1$ and $b > 0$ such that

$$d\left(f\left(x\right), f\left(x'\right)\right) \geqslant \lambda d\left(x, x'\right), \qquad x, x' \in X \text{ with } d\left(x, x'\right) \leqslant b.$$

Then (λ, b) denotes the *primary expanding parameter*.

(2) The dynamical system f is called *mixing* if for any open set $U \subset X$, there exists an integer $n > 0$ with $f^n\left(U\right) = X$.

Remark 2.1.4. In general, locally expanding can be defined as follows: There are three constants $C, b > 0$ and $\lambda > 1$ such that

$$d\left(f^n\left(x\right), f^n\left(x'\right)\right) \geqslant C\lambda^n d\left(x, x'\right), \qquad x, x' \in X \text{ with } d_n\left(x, x'\right) \leqslant b.$$

Proposition 2.1.5. *Let (X, d) be a compact metric space and $f : X \to X$ locally expanding. Then $f|B\left(x, b\right)$ is homeomorphic for any $x \in X$.*

Proof. It is clear that $f|\overline{B\left(x, b\right)}$ is injective. Since f is continuous on $\overline{B\left(x, b\right)}$ and $\overline{B\left(x, b\right)}$ is compact, the inverse of $f|\overline{B\left(x, b\right)}$ is also continuous. But $f : B\left(x, b\right) \to f\left(B\left(x, b\right)\right)$ is bijective, so $f|B\left(x, b\right)$ is homeomorphic. $\square$

Proposition 2.1.6. *Let (X, d) be a compact metric space and $f : X \to X$ locally expanding. Then there is a constant $0 < a < b$ such that for any $y \in X$ with $f^{-1}\left(y\right) = \{x_1, \ldots, x_n\}$, there are local inverses $g_1, \ldots, g_n$ of f defined on $\overline{B\left(y, a\right)}$ satisfying $g_i\left(y_i\right) = x_i$ and $\left\{g_i\left(\overline{B\left(y, a\right)}\right)\right\}_{i=1}^{n}$ are pairwise disjoint. Moreover, there is a constant integer $n_0 \geqslant 0$ such that $\#\left(f^{-1}\left(y\right)\right) \leqslant n_0$ for all $y \in X$.*

Proof. $\#\left(f^{-1}\left(y\right)\right)$ is finite for each $y \in X$ because otherwise f would not be bijective about a limit point of $f^{-1}\left(y\right)$. Define

$$d\left(y\right) := \inf_{1 \leqslant k \neq j \leqslant n} d\left(x_k, x_j\right)$$

to be the shortest distance between the preimages of y, so clearly $d\left(y\right) < b$. There exists $0 < r \leqslant \frac{b}{2}$ such that $f : B\left(x_i, r\right) \to f\left(B\left(x_i, r\right)\right)$ is homeomorphic for each $1 \leqslant i \leqslant n$. Since y is contained in the open set $\bigcap_{i=1}^{n} f\left(B\left(x_i, r\right)\right)$, $\overline{B\left(y, r_y\right)} \subset \bigcap_{i=1}^{n} f\left(B\left(x_i, r\right)\right)$ must hold for $r_y > 0$ sufficiently small, such that the inverse g_{i_y} which maps y to x_i satisfies

$$g_{i_y} : \overline{B\left(y, r_y\right)} \to g_{i_y}\left(\overline{B\left(y, r_y\right)}\right) \subset B\left(x_i, r\right).$$

Thus $g_{i_y}\left(\overline{B\left(y, r_y\right)}\right)$ are disjoint, because $B\left(x_i, r\right)$ are disjoint. Let now be $\left\{B\left(y_j, r_{y_j}\right)\right\}$ a finite number of balls such that $\left\{B\left(y_j, \frac{r_{y_j}}{2}\right)\right\}$ form a cover of X and set

$$a = \frac{1}{2}\min_j r_{y_j}$$

so that it satisfies the proposition.

For any $y \in X$, $y \in B\left(y_j, \frac{r_{y_j}}{2}\right)$ for some j, then $\overline{B\left(y, a\right)} \subset \overline{B\left(y_j, r_{y_j}\right)}$. Let

$$g_i = g_{i_{y_j}}|\overline{B\left(y_j, a\right)}, \qquad 1 \leqslant i \leqslant n,$$

then $g_1, \ldots, g_n$ are local inverses of $f|\overline{B(y,a)}$ with $g_i(y) = x_i$ and $\left\{g_i\left(\overline{B(y,a)}\right)\right\}_{i=1}^{n}$ are pairwise disjoint.
$\#(f^{-1}(y))$ is a locally constant function of y, thus it is bounded, since X is compact, that is, there exists an integer $n_0 \geqslant 1$ with $\#(f^{-1}(y)) \leqslant n_0$ for all $y \in X$. □

Remark 2.1.7. We call (λ, a) an expanding parameter for f, where a is the number in Proposition 2.1.6; for any $0 < a' < a$ and $1 < \lambda' < \lambda$, (λ', a') is also an expanding parameter for f.
Let g be an inverse branch of f on $B(y,r)$ for any $0 < r \leqslant a$. Furthermore, let $y \in X$ and $x = g(y)$. For any $z, z' \in B(y,r)$

$$d(g(z), g(z')) \leqslant \frac{1}{\lambda} d(z, z'),$$

hence g is contracting on $B(y,r)$. This implies

$$g(B(y,r)) \subseteq B\left(x, \frac{r}{\lambda}\right).$$

Moreover, $B_1(x,r) = g(B(y,r))$ and $f : B_1(x,r) \to B(y,r)$ is homeomorphic.

Proposition 2.1.8. *Let (X,d) be a compact metric space and $f : X \to X$ locally expanding and mixing. For any $0 < r \leqslant a$ and $x \in X$, $f^n : B_n(x,r) \to B(f^n(x), r)$ is homeomorphic.*

Proof. Let $x, f(x), \ldots, f^n(x)$ be a finite orbit of f, then there are n local inverses, $h_1, \ldots, h_n$, of f which satisfy

$$x \overset{h_1}{\leftarrow} f(x) \overset{h_2}{\leftarrow} f^2(x) \overset{h_3}{\leftarrow} \cdots \overset{h_{n-1}}{\leftarrow} f^{n-1}(x) \overset{h_n}{\leftarrow} f^n(x).$$

By Remark 2.1.7,

$$h_n : B(f^n(x), r) \to B_1\left(f^{n-1}(x), r\right)$$

is homeomorphic.
Thus

$$h_n(B(f^n(x), r)) = B_1\left(f^{n-1}(x), r\right).$$

Now, assume that

$$h_{n-k+1} \circ \cdots \circ h_n(B(f^n(x), r)) = B_k\left(f^{n-k}(x), r\right), \qquad 1 \leqslant k < n-1$$

is already proven, then

$$h_{n-k} \circ \cdots \circ h_n(B(f^n(x), r)) = h_{n-k}\left(B_k\left(f^{n-k}(x), r\right)\right).$$

Thus

$$\begin{aligned} z \in h_{n-k}\left(B_k\left(f^{n-k}(x), r\right)\right) &\Leftrightarrow f(z) \in B_k\left(f^{n-k}(x), r\right) \\ &\Leftrightarrow d\left(f^i(f(z)), f^{i(n-k)}(x)\right) < r, \qquad 0 \leqslant i \leqslant k. \end{aligned}$$

Hence

$$d\left(f(z), f^{n-k}(x)\right) < r, \ldots, d\left(f^{k+1}(z), f^n(x)\right) < r,$$

which together with

$$d\left(z, f^{n-k-1}(x)\right) \leqslant \frac{1}{\lambda} d\left(f(z), f^{n-k}(x)\right) < \frac{r}{\lambda}$$

implies that

$$h_{n-k} \circ \cdots \circ h_n \left(B\left(f^n(x), r\right)\right) = B_{k+1}\left(f^{n-k-1}(x), r\right).$$

Then, by induction

$$h_1 \circ \cdots \circ h_n \left(B\left(f^n(x), r\right)\right) = B_n(x, r),$$

hence $f^n\left(B_n(x,r)\right) = B\left(f^n(x), r\right)$. Since f^n is injective on $B_n(x,r)$, $f^n : B_n(x,r) \to B\left(f^n(x), r\right)$ is homeomorphic. □

Proposition 2.1.9. *Let (X,d) be a compact metric space and $f : X \to X$ locally expanding and mixing. For any $0 < r \leqslant a$, there exists an integer $p = p(r) \geqslant 1$ with $f^p\left(B(x,r)\right) = X$ for any $x \in X$.*

Proof. Let $\left\{B\left(y_i, \frac{r}{2}\right)\right\}_{i \in J}$, where J is an index set, be a finite ball cover of X, then for all $i \in J$ there is an integer $p_i = p(y_i) > 0$ such that $f^{p_i}\left(B\left(y_i, \frac{r}{2}\right)\right) = X$.
Set $p = \max_i \{p_i\}$. For any $y \in X$, there exists an i with $y \in B\left(y_i, \frac{r}{2}\right)$. Thus

$$B(y,r) \supset B\left(y_i, \frac{r}{2}\right) \quad \text{and} \quad f^p\left(B(y,r)\right) \supseteq f^{p-p_i}\left(f^{p_i}\left(B\left(y_i, \frac{r}{2}\right)\right)\right) \supseteq f^{p-p_i}(X) = X.$$

□

Proposition 2.1.10. *Let (X,d) be a compact metric space and $f : X \to X$ locally expanding and mixing. For any $0 < r \leqslant a$, let $p = p(r)$ be the integer in Proposition 2.1.9 and n_0 the integer in Proposition 2.1.6, then*

$$1 \leqslant \#\left(f^{-(n+p)}(y) \cap B_n(x,r)\right) \leqslant n_0^p, \qquad x, y \in X, n \geqslant 1.$$

Proof. We have that $f^n : B_n(x,r) \to B\left(f^n(x), r\right)$ is a homeomorphism, this implies $f^{n+p}\left(B_n(x,r)\right) = f^p\left(B\left(f^n(x), r\right)\right) = X$. Thus $f^{-(n+p)}(y) \cap B_n(x,r) \neq \varnothing$. On the other hand, $\#\left(f^{-p}(y)\right) \leqslant n_0^p$ and every $z \in f^{-p}(y) \cap B\left(f^n(x), r\right)$ has exactly one preimage in $B_n(x,r)$ under f^n. Hence

$$1 \leqslant \#\left(f^{-(n+p)}(y) \cap B_n(x,r)\right) \leqslant n_0^p.$$

□

2.1.2 Maximal eigenvalues for Ruelle-Perron-Frobenius operators

Definition 2.1.11. Let (X,d) be a compact metric space and $f : X \to X$ locally expanding and mixing. Let $\mathbb{R}$ denote the real line and $C(X) := C(X, \mathbb{R})$ the space of all continuous functions $\phi : X \to \mathbb{R}$ with the supremum norm

$$\|\phi\| = \max_{x \in X} \{|\phi(x)|\}.$$

(1) Let $0 < \alpha \leqslant 1$. A function $\phi \in C(X)$ is said to be *α-Hölder continuous* if

$$[\phi]_\alpha = \sup_{x,y:0<d(x,y)\leqslant a} \frac{|\phi(x) - \phi(y)|}{d(x,y)^\alpha} < \infty,$$

where $[\phi]_\alpha$ denotes the *local Hölder constant* for ϕ. $C^\alpha(X) := C^\alpha(X,\mathbb{R})$ denotes the space of all α-Hölder continuous functions.

(2) Two functions ϕ_1 and ϕ_2 are said to be $\phi_1 \geqslant \phi_2$ if $\phi_1(x) \geqslant \phi_2(x)$ for all $x \in X$. A function is said to be *positive* if $\phi > 0$.

(3) A positive function in $C^\alpha(X)$ is called a *potential.*

(4) We define $C^\alpha_{K,s}(X) := C^\alpha_{K,s}(X,\mathbb{R}) = \{\phi \in C^\alpha(X) : \phi \geqslant s, [\log\phi]_\alpha \leqslant K\}$ for constants $K, s > 0$.

The following lemma is a consequence of Arzela-Ascoli's theorem (see **[44]** p. 245):

Lemma 2.1.12. *Any bounded sequence in $C^\alpha_{K,s}(X)$ has a convergent subsequence in $C(X)$ whose limit is in $C^\alpha_{K,s}(X)$.*

Definition 2.1.13. Let (X,d) be a compact metric space and ψ a potential. The *Ruelle-Perron-Frobenius (RPF) operator* with weight ψ is defined as

$$R\phi(y) = \sum_{x\in f^{-1}(y)} \psi(x)\phi(x) = \sum_{i=1}^{k} \psi(x_i)\phi(x_i),$$

where $\{x_1,\ldots,x_k\} = f^{-1}(y)$.

Proposition 2.1.14. *Since $R(C(X)) \subset C(X)$, $R : C(X) \to C(X)$ is a linear operator. Moreover, for any α, $R : C^\alpha(X) \to C^\alpha(X)$ is a linear operator.*

Proof. Consider $y, y' \in X$ with $d(y,y') \leqslant a$ for any $\phi \in C^\alpha(X)$. Let $\{x_1,\ldots,x_n\} = f^{-1}(y)$ and $\{x'_1,\ldots,x'_n\} = f^{-1}(y')$ be the corresponding inverse images of y and y' with $d(x_i,x'_i) \leqslant \frac{1}{\lambda}d(y,y')$ for all $1 \leqslant i \leqslant k$. Then

$$\begin{aligned}\left|R\phi(y) - R\phi\left(y'\right)\right| &= \left|\sum_{i=1}^{k}\psi(x_i)\phi(x_i) - \sum_{i=1}^{k}\psi\left(x'_i\right)\phi\left(x'_i\right)\right| \\ &= \left|\sum_{i=1}^{k}\left(\psi(x_i)\left(\phi(x_i) - \phi\left(x'_i\right)\right) + \phi\left(x'_i\right)\left(\psi\left(x'_i\right) - \psi(x_i)\right)\right)\right|.\end{aligned}$$

Hence

$$[R\phi]_\alpha \leqslant \frac{n_0}{\lambda^\alpha}\left(\|\psi\|[\phi]_\alpha + \|\phi\|[\psi]_\alpha\right) < \infty,$$

and thus $R(C^\alpha(X)) \subset C^\alpha(X)$, so $R : C^\alpha(X) \to C^\alpha(X)$ is a linear operator. □

Remark 2.1.15. The weight ψ can be normalized such that $\min_{x\in X}\psi(x) = 1$ for the purpose of the study of the eigenvalues of R. In the rest of this section, it is always assumed that ψ is a normalized element in $C^\alpha_{K_0,1}(X)$ for some constant $K_0 > 0$.

Lemma 2.1.16. *Let (X, d) be a compact metric space and $0 < s < 1$, $K > \frac{K_0}{\lambda^\alpha - 1}$ two fixed constants. Then there exists an integer $N > 0$ such that*

$$R^n \phi \in C^\alpha_{K,s}(X) \qquad \forall 0 \leqslant \phi \in C^\alpha(X) \ \text{ with } \ \|\phi\| = 1, n \geqslant N.$$

Proof. We have $\|\phi\| = 1$ which implies that there exists $y \in X$ with $\phi(y) = 1$. Thus there exists a neighborhood U of y with $\phi(y') > s$ for all $y' \in U$. Since f is mixing, there is an integer $n_1 > 0$ with $f^n(U) = X$ for all $n \geqslant n_1$. This implies for any $z \in X$, $f^{-n}(z) \cap U \neq \varnothing$, $n \geqslant n_1$, and hence $R^n\phi(z) \geqslant s$.
Let for any $y, y' \in X$ with $d(y, y') \leqslant a$, $\{x_1, \ldots, x_n\} = f^{-1}(y)$ and $\{x'_1, \ldots, x'_n\} = f^{-1}(y')$ be the corresponding inverse images of y and y' with $d(x_i, x'_i) \leqslant \frac{1}{\lambda} d(y, y')$ for all $1 \leqslant i \leqslant k$. Now, set $K' = [\log R^{n_1}\phi]_\alpha$, then

$$\begin{aligned} R(R^{n_1}\phi)(y') &= \sum_{i=1}^{k} \psi(x'_i) R^{n_1}\phi(x'_i) \\ &\leqslant \sum_{i=1}^{k} \psi(x_i) \exp\left(K_0 d(x_i, x'_i)^\alpha\right) R^{n_1}\phi(x_i) \exp\left(K' d(x_i, x'_i)^\alpha\right) \\ &\leqslant \exp\left((K_0 + K') \frac{1}{\lambda^\alpha} d(y, y')^\alpha\right) R(R^{n_1}\phi)(y), \quad y, y' \in X, d(y, y') \leqslant a. \end{aligned}$$

Thus inductively for $K_n = K_0 \left(\sum_{i=1}^n \lambda^{-\alpha i}\right) + K' \lambda^{-\alpha n}$,

$$R^n(R^{n_1}\phi)(y') \leqslant \exp\left(K_n d(y, y')^\alpha\right) R^n(R^{n_1}\phi)(y), \qquad y, y' \in X, d(y, y') \leqslant a.$$

Then $K_n \overset{n\to\infty}{\to} \frac{K_0}{\lambda^\alpha - 1}$ and there exists an integer $n_2 > 0$ such that

$$R^n(R^{n_1}\phi)(y') \leqslant \exp\left(K d(y, y')^\alpha\right) R^n(R^{n_1}\phi)(y), \qquad y, y' \in X, d(y, y') \leqslant a, n \geqslant n_2.$$

Hence $N = n_1 + n_2$ satisfies the lemma. □

Remark 2.1.17. By Lemma 2.1.16, R also has an eigenfunction in $C^\alpha_{K,s}(X)$ with respect to μ if $\mu > 0$ is an eigenvalue of $R : C^\alpha(X) \to C^\alpha(X)$ with a nonzero eigenfunction $\phi \geqslant 0$. Thus $C^\alpha_{K,s}(X)$ can be used to find positive eigenvalues of $R : C^\alpha(X) \to C^\alpha(X)$ with nonnegative eigenfunctions, where the calculation in the proof of Lemma 2.1.16 provides that

$$R\left(C^\alpha_{K_0,s}(X)\right) \subset C^\alpha_{K,s}(X),$$

because $(K_0 + K)\lambda^{-\alpha} < K$ for $K > \frac{K_0}{\lambda^\alpha - 1}$.

Lemma 2.1.18. *Define $S := \left\{\mu \in \mathbb{R} : \mu > 0, \exists \phi \in C^\alpha_{K,s}(X) \ \text{ s.t. } R\phi \geqslant \mu\phi\right\}$. Then S is a nonempty bounded subset in the real line $\mathbb{R}$.*

Proof. Consider $\phi \in C^\alpha_{K,s}(X)$, then

$$R\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x) = \left(\sum_{x \in f^{-1}(y)} \frac{\phi(x)}{\phi(y)} \psi(x)\right) \phi(y) \geqslant \frac{s}{\|\phi\|} \phi(y), \qquad x, y \in X.$$

This implies $\mu = \frac{s}{\|\phi\|} \in S$.
Set

$$m = \sup_{y \in X} \sum_{x \in f^{-1}(y)} \psi(x),$$

and let for any $\phi \in C^{\alpha}_{K,s}(X)$, $\phi(y) = \|\phi\|$. Then

$$R\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x) \leqslant \phi(y) \sum_{x \in f^{-1}(y)} \psi(x) \leqslant m\phi(y),$$

and therefore any $\mu > m$ is not in S. Hence $S \subset \mathbb{R}$ is bounded. $\square$

Theorem 2.1.19 (Ruelle)**.** *The linear operator $R : C^{\alpha}(X) \to C^{\alpha}(X)$ has an unique maximal positive eigenvalue whose corresponding eigenspace is one-dimensional.*

Proof. Set $\delta = \sup S > 0$, then there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ in S which converges to δ. Let $\phi_n \in C^{\alpha}_{K,s}(X)$ be the corresponding functions with $R\phi_n \geqslant \mu_n\phi_n$ and normalize ϕ_n with $\min_{x \in X}\{\phi_n(x)\} = s$. Hence $\{\phi_n\}_{n=1}^{\infty}$ is bounded in $C^{\alpha}_{K,s}(X)$ and by Lemma 2.1.12, it has a convergent subsequence in $C(X)$ whose limit is in $C^{\alpha}_{K,s}(X)$.
Assume that $\{\phi_n\}_{n=1}^{\infty}$ itself converges to ϕ_0, thus $R\phi_0 \geqslant \delta\phi_0$. Now, suppose that there exists $y \in X$ such that

$$R\phi_0(y) > \delta\phi_0(y).$$

This implies that there is a neighborhood U of y such that

$$R\phi_0(y') - \delta\phi_0(y') > 0, \qquad y' \in U.$$

Since f is mixing, there is an integer $n > 0$ with

$$f^n(U) = X,$$

and thus

$$R^n(R\phi_0 - \delta\phi_0) > 0,$$

which implies $R(R^n\phi_0) > \delta R^n\phi_0$. Thus for $\phi = R^n\phi_0$, there exists a $\mu > \delta$ with $R\phi \geqslant \mu\phi$, which is a contradiction to the maximality of δ. This proves

$$R\phi_0 = \delta\phi_0.$$

Now, it will be shown that δ is simple, that is, the eigenspace

$$E_\delta = \{\phi \in C^{\alpha}(X) : R\phi = \delta\phi\}$$

has dimension one. Suppose $\phi \in E_\delta$ and set $a = \min_{x \in X}\left\{\frac{\phi(x)}{\phi_0(x)}\right\}$ and $\phi_1 = \phi - a\phi_0$. Then $\phi_1 \in E_\delta$ and $\phi_1 \geqslant 0$, moreover, there exists $y \in X$ such that $\phi_1(y) = 0$, this implies $\phi_1(x) = 0$ for all $x \in f^{-1}(y)$. Hence inductively $\phi_1 = 0$ on $X_y = \bigcup_{n=0}^{\infty} f^{-n}(y)$, and since f is mixing, X_y is dense in X. This implies $\phi_1 \equiv 0$ on X, and so $\phi \equiv a\phi_0$.
It remains to prove that δ is the biggest eigenvalue of R. Assume that $\mu \neq \delta$ is an eigenvalue of $R : C^{\alpha}(X) \to C^{\alpha}(X)$, then there is $0 \neq \phi \in C^{\alpha}(X)$ with $\|\phi\| = 1$ such that $R\phi = \mu\phi$. Thus as above

$$R|\phi| \geqslant |\mu||\phi|.$$

Now, there exists an integer $N > 0$ such that $R^N |\phi| \in C^{\alpha}_{K,s}(X)$ and

$$R\left(R^N |\phi|\right) \geqslant |\mu| R^N |\phi| .$$

It follows $|\mu| \in S$, hence $|\mu| \leqslant \delta$. If $|\mu| < \delta$ there is nothing to prove, and if $|\mu| = \delta$, the mixing property implies (as above) $|\phi| = a\phi_0$ for some $a > 0$. Hence $\phi = \pm a\phi_0$ and $\mu = \delta$. □

2.1.3 The Gibbs property

We will now introduce chains of Markovian projections which we will need for the proof of the second part of Ruelle's theorem, the existence and the uniqueness of the Gibbs measure. We will also use Gibbs distributions to prove a theorem on the spectrum of the Ruelle-Perron-Frobenius operator.

Let X be a compact Hausdorff space and $\mathcal{F}_X$ the standard σ-algebra generated by all open sets in X. Let $M(X) = (C(X))^*$ be the dual space of $C(X)$. By the Riesz representation theorem (see **[44]** p. 40) this is the space of all measures on X with respect to $\mathcal{F}_X$. Let $M^1(X) \subset M(X)$ denote the space of all probability measures and

$$\langle \mu, \phi \rangle = \int_X \phi d\mu$$

the integral of a function with respect to a measure $\mu \in M(X)$.

Definition 2.1.20. A linear map $P : C(X) \to C(X)$ is called a *projection* if $P^2 = P$, P is called *Markovian* if $P\mathbf{1} = 1$ and $P\phi \geqslant 0$ whenever $\phi \geqslant 0$.

We denote the kernel and the image of P by $Ker(P) = \{\phi \in C(X) : P\phi = 0\}$ and $Im(P) = P(C(X))$, respectively.

Proposition 2.1.21. *Let P and Q be projections. Then*

(i) $C(X) = Ker(P) \bigoplus Im(P)$.

(ii) $\phi \in Im(P)$ *if an only if* $P\phi = P$.

(iii) $PQ = Q$ *if and only if* $Im(Q) \subseteq Im(P)$.

(iv) $QP = Q$ *if and only if* $Ker(P) \subseteq Ker(Q)$.

Proof. The proof of the assertions is evident. □

For an operator $P : C(X) \to C(X)$, $P^* : M(X) \to M(X)$ denotes its adjoint operator.

Proposition 2.1.22. *Let P and Q be Markovian projections on $C(X)$, then we have*

(i) $\|P\| = \|P^*\| = 1$.

(ii) $P^{*2} = P^*$.

(iii) $P^*(M^1(X)) \subseteq M^1(X)$.

(iv) $PQ = Q$ if and only if $Q^*P^* = Q^*$.

Proof. The proof of the assertions is evident. □

Definition 2.1.23. (1) A sequence of Markovian projections $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ defined on $C(X)$ is called a *chain of Markovian projections* (CMP) if

$$P_m P_n = P_m, \qquad 1 \leqslant n \leqslant m.$$

(2) For $\mathcal{P}$ a chain of Markovian projections, define

$$\mathcal{G}_n := \{\mu \in M^1(X) : P_n^* \mu = \mu\}, \qquad 1 \leqslant n < \infty.$$

Since $M^1(X) \subseteq M(X)$ is weakly compact and convex, the Schauder-Tychonoff theorem (see [**16**]) implies that $\mathcal{G}_n \neq \varnothing$. Set $\mathcal{G}_\infty = \cap_{n=1}^{\infty} \mathcal{G}_n$. A measure $\mu \in \mathcal{G}_\infty$ is called a *G-measure* with respect to $\mathcal{P}$. The given CMP is called *uniquely ergodic* if $\mathcal{G}_\infty$ is a singleton.

Theorem 2.1.24. *Let $\mathcal{P}$ be a CMP, then $\mathcal{G}_\infty \neq \varnothing$, i.e. if $\{\mu_n\}_{n=1}^{\infty}$ is a sequence in $M^1(X)$, then any weak limit of $\{P_n^*\mu_n\}_{n=1}^{\infty}$ is in $\mathcal{G}_\infty$.*

Proof. $\|P_n^*\mu_n\| = 1$ implies that there is a weak limit of $\{P_n^*\mu_n\}_{n=1}^{\infty}$. Suppose ν is such a weak limit, then there is a subsequence $P_{n_i}^*\mu_{n_i}$ which weakly converges to ν as i goes to infinity. Thus for any $\phi \in C(X)$,

$$\lim_{i\to\infty}\langle P_n^* P_{n_i}^* \mu_{n_i}, \phi\rangle = \lim_{i\to\infty}\langle P_{n_i}^*\mu_{n_i}, P_n\phi\rangle = \langle \nu, P_n\phi\rangle = \langle P_n^*\nu, \phi\rangle,$$

and by Proposition 2.1.21

$$\begin{aligned}\langle P_n^*\nu, \phi\rangle &= \lim_{i\to\infty}\langle P_{n_i}^*\mu_{n_i}, P_n\phi\rangle = \lim_{i\to\infty}\langle \mu_{n_i}, P_{n_i}P_n\phi\rangle \\ &= \lim_{i\to\infty}\langle \mu_{n_i}, P_{n_i}\phi\rangle = \lim_{i\to\infty}\langle P_{n_i}^*\mu_{n_i}, \phi\rangle = \langle \nu, \phi\rangle.\end{aligned}$$

Hence $P_n^*\nu = \nu$ for all $n \geqslant 1$, that is, $\nu \in \mathcal{G}_\infty$. □

Theorem 2.1.24 states that $\mathcal{G}_\infty$ is weakly compact, clearly it is also convex, i.e., $t\mu_1 + (1-t)\mu_2 \in \mathcal{G}_\infty$ if $\mu_1, \mu_2 \in \mathcal{G}_\infty$ and $0 \leqslant t \leqslant 1$.

Theorem 2.1.25. *If $\mathcal{P}$ is a CMP defined on $C(X)$, then the following statements are equivalent:*

(i) The CMP is uniquely ergodic.

(ii) For $\phi \in C(X)$, $P_n\phi$ converges uniformly on X to a constant.

(iii) For $\phi \in C(X)$, $P_n\phi$ converges pointwise on X to a constant.

Proof. Clearly (iii) follows from (ii). We suppose that (iii) holds true, then for any $\mu \in \mathcal{G}_\infty$ the constant is $\langle \mu, \phi\rangle$, since by the Lebesgue theorem (see [**1**])

$$\langle \mu, \phi\rangle = \langle P_n^*\mu, \phi\rangle = \lim_{n\to\infty}\langle \mu, P_n\phi\rangle = \langle \mu, \lim_{n\to\infty} P_n\phi\rangle = \lim_{n\to\infty} P_n\phi.$$

Hence for any $\mu, \nu \in \mathcal{G}_\infty$ and $\phi \in C(X)$,

$$\langle \mu, \phi \rangle = \langle \nu, \phi \rangle = \lim_{n \to \infty} P_n \phi.$$

This implies that $\mu = \nu$, and therefore, (i) holds.
We assume that (i) holds and that μ is the unique element in $\mathcal{G}_\infty$. Suppose that (ii) is false, then there is a $\phi \in C(X)$ such that $P_n \phi$ does not converge uniformly to $\langle \mu, \phi \rangle$. That is, there is a constant $\epsilon > 0$, a subsequence of integers $\{n_i\}_{i=1}^\infty$ and a sequence of points $\{x_i\}_{i=1}^\infty$ with

$$|P_{n_i} \phi(x_i) - \langle \mu, \phi \rangle| \geqslant \epsilon, \qquad i \geqslant 1.$$

For δ_{x_i} the Dirac measure concentrated at x_i,

$$P_{n_i} \phi(x_i) = \langle P_{n_i}^* \delta_{x_i}, \phi \rangle,$$

and by Theorem 2.1.24 any weak limit ν of $\{P_{n_i}^* \delta_{x_i}\}_{i=0}^\infty$ is in $\mathcal{G}_\infty$. But $|\langle \nu, \phi \rangle - \langle \mu, \phi \rangle| \geqslant \epsilon$, which is a contradiction to (i). $\square$

Definition 2.1.26. (1) A CMP $\mathcal{P} = \{P_n\}_{n=1}^\infty$ is called *compatible* if

- $P_n(\phi\chi) = \chi P_n \phi$ if $\chi \in Im P_n$,
- $P_n P_m = P_m = P_m P_n$ if $m \geqslant n$.

(2) If $\mathcal{P} = \{P_n\}_{n=1}^\infty$ is a compatible CMP, $\mathcal{F}_0$ the standard σ-algebra on $\mathbb{R}$ generated by all open sets and $\phi : X \to \mathbb{R}$ a function, then we define $\mathcal{F}_\phi = \phi^{-1}(\mathcal{F}_0)$ to be the *pull-back σ-algebra* on X. For a family of functions Γ, $\mathcal{F}_\Gamma$ denotes the minimal σ-algebra containing all σ-algebras $\mathcal{F}_\phi$ for $\phi \in \Gamma$ and we set $\mathcal{F}_n = \mathcal{F}_{Im(\mathcal{P}_n)}$ for all $n \geqslant 1$. Then $\{\mathcal{F}_n\}_{n=1}^\infty$ is a decreasing sequence of sub-σ-algebras in $\mathcal{F}_X$, i.e., $\cdots \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_X$, since by Proposition 2.1.21 $\cdots \subseteq \mathrm{Im}P_{n+1} \subseteq \mathrm{Im}P_n \subseteq \cdots \subseteq \mathrm{Im}P_1 \subseteq C(X)$.
Finally, we define $\mathcal{F}_\infty$ to be the σ-algebra generated by the limit of $\{\mathcal{F}_n\}_{n=1}^\infty$, that is,

$$\mathcal{F}_\infty = \cup_{n=1}^\infty \cap_{m \geqslant n} \mathcal{F}_m.$$

(3) A G-measure is called *$\mathcal{P}$-ergodic* if $\mu|\mathcal{F}_\infty$ is trivial, that is, $\mu(A) = 0$ or 1 for any $A \in \mathcal{F}_\infty$.

(4) For $\mu \in M^1(X)$, $\phi \in C(X)$ and $n \geqslant 1$, we have a measure defined on the sub-σ-algebra $\mathcal{F}_n$ by

$$\mu_n(A) = \int_A \phi d\mu, \qquad A \in \mathcal{F}_n.$$

Clearly, μ_n is absolutely continuous with respect to $\mu|\mathcal{F}_n$ and by the Radon-Nikodym theorem (see **[12]** p.193) there exists a unique (modulo zero sets) $L^1(X, \mathcal{F}_n, \mu)$-function $E(\phi|\mathcal{F}_n)$ called the *conditional expectation* of ϕ given $\mathcal{F}_n$ satisfying

$$\mu_n(A) = \int_A E(\phi|\mathcal{F}_n), \qquad A \in \mathcal{F}_n.$$

The function $E(\phi|\mathcal{F}_n)$ is defined uniquely a.e. by

- $\int_A E(\phi|\mathcal{F}_n) d\mu = \int_A \phi d\mu$, $A \in \mathcal{F}_n$,

- $E(\phi|\mathcal{F}_n) \in L^1(X, \mathcal{F}_n, \mu)$,

and the operator $E(\cdot|\mathcal{F}_n)$ satisfies

- $E(\phi\phi'|\mathcal{F}_n) = \phi' E(\phi|\mathcal{F}_n)$ for all $\phi \in L^1(X, \mathcal{F}_n, \mu), \phi' \in L^\infty(X, \mathcal{F}_n, \mu)$.

Theorem 2.1.27 (Decreasing martingale theorem). *Let $\cdots \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subseteq \mathcal{F}_X$ be a decreasing sequence of sub-σ-algebras satisfying $\cap_{n=1}^{\infty}\mathcal{F}_n = \mathcal{F}_\infty$. Then $E(\phi|\mathcal{F}_n) \to E(\phi|\mathcal{F}_\infty)$ a.e. and in $L^1(X, \mathcal{F}_X, \mu)$ if $\phi \in L^1(X, \mathcal{F}_X, \mu)$.*

Proof. See [**39**] pp. 30. □

If $\mu \in \mathcal{G}_\infty$ is a G-measure for $\mathcal{P}$, then for any $\phi' \in ImP_n$

$$\langle \mu, \phi' P_n \phi \rangle = \langle \mu, P_n(\phi\phi') \rangle = \langle P_n^* \mu, \phi\phi' \rangle = \langle \mu, \phi\phi' \rangle.$$

Hence $P_n\phi = E(\phi|\mathcal{F}_n)$, μ-a.e., and then with the decreasing martingale theorem the limit of $P_n\phi$ exists μ-a.e., furthermore, a $\mathcal{P}$-invariant measure μ is $\mathcal{P}$-ergodic if and only if $\lim_{n\to\infty} P_n\phi = \langle \mu, \phi \rangle$ μ-a.e. for $\phi \in C(X)$. So, we have the classical ergodicity theorem:

Theorem 2.1.28. *Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ be a compatible CMP. Then*

(i) If $\mu_1, \mu_2 \in \mathcal{G}_\infty$ are $\mathcal{P}$-ergodic, then either $\mu_1 = \mu_2$ or $\mu_1 \perp \mu_2$.

(ii) $\mu \in \mathcal{G}_\infty$ is $\mathcal{P}$-ergodic if and only if μ is an extremal point in $\mathcal{G}_\infty$.

Proof. (i) Suppose $\mu_1 \neq \mu_2$, then there is a $\phi \in C(X)$ with

$$\langle \mu_1, \phi \rangle \neq \langle \mu_2, \phi \rangle.$$

We set $A_1 = \{x \in X : \lim_{n\to\infty} P_n\phi = \langle \mu_1, \phi \rangle\}$ and $A_2 = \{x \in X : \lim_{n\to\infty} P_n\phi = \langle \mu_2, \phi \rangle\}$. Hence $\mu_1(A_1) = 1$ and $\mu_2(A_1) = 0$, and $\mu_1(A_2) = 0$ and $\mu_2(A_2) = 1$. Thus $\mu_1 \perp \mu_2$ follows.

(ii) Suppose that $\mu \in \mathcal{G}_\infty$ is $\mathcal{P}$-ergodic and $\mu = t\mu_1 + (1-t)\mu_2$ with $\mu_1, \mu_2 \in \mathcal{G}_\infty$ and $0 < t < 1$. Then by the ergodicity of μ, $\mu(A) = 0$ or 1 for any $A \in \mathcal{F}_\infty$ and $\mu_1(A) = \mu_2(A) = 0$ or 1 because $0 < t < 1$. Thus μ_1 and μ_2 are also $\mathcal{P}$-ergodic. Now, (i) implies that if $\mu_1 \neq \mu_2$, then there is a $A \in \mathcal{F}_\infty$ with $\mu_1(A) = 1$ and $\mu_2(A) = 0$. Hence $\mu(A) = t$ which is a contradiction to the ergodicity of μ. So $\mu_1 = \mu_2$ and μ is extremal.
Conversely, suppose that $\mu \in \mathcal{G}_\infty$ is not $\mathcal{P}$-ergodic. Let $A \in \mathcal{F}_\infty$ with $0 < t = \mu(A) < 1$ and set

$$\mu_1 = \frac{1}{t}\mu\chi_A, \qquad \mu_2 = \frac{1}{1-t}\mu\chi_{X\backslash A}.$$

We will show that $\mu_1, \mu_2 \in \mathcal{G}_\infty$. For any $n \geqslant 1$

$$\langle P_n^*\mu_1, \phi \rangle = \langle \mu, \frac{1}{t}\chi_A P_n\phi \rangle, \qquad \phi \in C(X).$$

Since $A \in \mathcal{F}_\infty \subseteq \mathcal{F}_n$,

$$\begin{aligned} \langle \mu, \frac{1}{t}\chi_A P_n\phi \rangle &= \frac{1}{t}\sup\langle \mu, \phi' P_n\phi \rangle = \frac{1}{t}\sup\langle P_n^*\mu, \phi\phi' \rangle \\ &= \frac{1}{t}\sup\langle \mu, \phi'\phi \rangle = \langle \frac{1}{t}\chi_B\mu, \phi \rangle, \end{aligned}$$

where the supremum is taken over $\{\phi' \in ImP_n : \phi' \leqslant \chi_A\}$. Thus $P_n^* \mu_1 = \mu_1$ for all $n \geqslant 1$, that is $\mu \in \mathcal{G}_\infty$. Similarly, $\mu_2 \in \mathcal{G}_\infty$. But

$$\mu = t\mu_1 + (1-t)\,\mu_2, \qquad 0 < t < 1,$$

implies that μ is not an extremal point.

□

$M(X)$ is a locally convex topological space which is metrizable and $\mathcal{G}_\infty$ is a compact metrizable convex subset in $M(X)$. So, Theorem 2.1.28 states that the set of $\mathcal{P}$-ergodic measures $\mu \in \mathcal{G}_\infty$ is comprised of all extremal points of $\mathcal{G}_\infty$. The following theorem relates $\mathcal{G}_\infty$ and the set of its extremal points.

Theorem 2.1.29 (Choquet representation theorem). *Let $\mu \in \mathcal{G}_\infty$, then there exists a Borel probability measure m on $\mathcal{G}_\infty$, supported on the set of extremal points of $\mathcal{G}_\infty$, such that*

$$\mu = \int_{\mathcal{G}_\infty} \nu dm(\nu)\,, \qquad \nu \in \mathcal{G}_\infty.$$

Proof. See [**38**] pp. 1-32. □

Originally, Gibbs distributions are motivated by physics, more precisely statistical mechanics. It is a physical fact that for a system of n states with the corresponding energies $E_1, \ldots, E_n$ which is put into contact with a much larger heat source being at temperature T, where T is constant, the probability p_j that the state j occurs is given by the Gibbs distribution

$$p_j = \frac{e^{-\beta E_j}}{\sum_{i=1}^n e^{-\beta E_i}},$$

with $\beta = \frac{1}{kT}$ and k is a physical constant. This is the starting point for the thermodynamical formalism which, however, studies Gibbs measures for more general systems.

We suppose that f is a locally expanding and mixing dynamical system with an expanding parameter (λ, a) and $0 < \psi \in C^\alpha(X)$, $0 < \alpha \leqslant 1$ is a potential. We set

$$G_n(x) = \prod_{i=0}^{n-1} \psi\left(f^i(x)\right), \qquad x \in X, n \geqslant 1.$$

Let

$$R\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\,\phi(x)$$

be the RPF operator with weight ψ. We assume that $\delta > 0$ is the maximal eigenvalue, $0 < h \in C^\alpha(X)$ a corresponding eigenvector of R and $R^* : M(X) \to M(X)$ the adjoint operator of R.

Theorem 2.1.30. *There exists a unique probability measure $\nu = \nu_\psi \in M^1(X)$ such that $R^*\nu = \delta\nu$ and for any $0 < r < \frac{a}{2}$, there is a constant $C = C(r) > 0$ with*

$$C^{-1} \leqslant \frac{\nu(B_n(x,r))}{\delta^{-n} G_n(x)} \leqslant C, \qquad x \in X, n \geqslant 1. \tag{2.1}$$

Moreover, for h satisfying $\int_X h d\nu = 1$ and for any $\phi \in C(X)$, $\lim_{n\to\infty} \delta^{-n} R^n \phi = \langle \nu, \phi \rangle h$ uniformly.

The inequality in (2.1) is called *Gibbs Property* and the probability measure $\mu = h\nu$ is called *Gibbs measure* for (f, ψ). Theorem 2.1.30 is proven by normalizing the RPF operator R and using Theorem 2.1.31 below. The transfer operator can be normalized as follows: Define

$$\widetilde{\psi}(x) := \frac{h(x)}{\delta h(f(x))}\psi(x)$$

which is still a positive function in $C^{\alpha}(X)$. Now, let

$$\widetilde{R}\phi(x) = R_{\widetilde{\psi}}\phi(x) = \sum_{y \in f^{-1}(x)} \widetilde{\psi}(y)\phi(y)$$

be the normalized transfer operator having the property that $\widetilde{R}1 = 1$. Let $\widetilde{R}^*$ be the adjoint operator of $\widetilde{R}$ acting on $M(X)$ and

$$\widetilde{G_n}(x) = \prod_{i=0}^{n-1} \widetilde{\psi}\left(f^i(x)\right), \qquad x \in X, n \geqslant 1.$$

Then, we have

$$\widetilde{G_n} = \frac{h}{\delta^n h \circ f^n} G_n$$

and thus the following relations between R and $\widetilde{R}$ and R^* and $\widetilde{R}^*$

$$R^n\phi = \delta^n h \widetilde{R}^n\left(\phi h^{-1}\right) \qquad \text{and} \qquad R^{*n}\nu = \delta^n h^{-1}\widetilde{R}^{*n}(h\nu). \tag{2.2}$$

Theorem 2.1.31. *Let R be the normalized RPF operator. Then there exists a unique probability measure $\mu \in M^1(X)$ such that $R^*\mu = \mu$ and for any $0 < r < \frac{a}{2}$, there is a constant $C = C(r) > 0$ with*

$$C^{-1} \leqslant \frac{\mu(B_n(x,r))}{G_n(x)} \leqslant C, \qquad x \in X, n \geqslant 1. \tag{2.3}$$

Moreover, for any $\phi \in C(X)$, $\lim_{n\to\infty} R^n\phi = \langle \mu, \phi \rangle$.

For the remainder of this section we assume that the RPF R is normalized. We set

$$P_n\phi(x) = R^n\phi(f^n(x)) = \sum_{y \in f^{-n}(f^n(x))} G_n(y)\phi(y)$$

which defines a linear operator from $C(X)$ into itself with $P_n\mathbf{1} = 1$ and $P_n\phi > 0$ for $\phi > 0$. Moreover, P_n satisfies the assertions in the following three lemmas.

Lemma 2.1.32. *For $m \geqslant n \geqslant 1$, $P_m P_n = P_n P_m = P_m$.*

Proof. We will show that $P_m P_n = P_m$. Note that $\sum_{y\in f^{-n}(w)} G_n(y) = 1$.

$$\begin{aligned}
P_m P_n \phi(x) &= \sum_{y\in f^{-m}(f^m(x))} G_m(y)\, P_n\phi(y) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))} \sum_{y\in f^{-n}(w)} G_{m-n}(w)\, G_n(y)\, P_n\phi(y) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))} \sum_{y\in f^{-n}(w)} G_{m-n}(w)\, G_n(y) \sum_{z\in f^{-n}(f^n(y))} G_n(z)\,\phi(z) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))} \sum_{y\in f^{-n}(w)} G_n(y) \sum_{z\in f^{-n}(w)} G_{m-n}(w)\, G_n(z)\,\phi(z) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))} \left(\sum_{y\in f^{-n}(w)} G_n(y)\right)\left(\sum_{z\in f^{-n}(w)} G_m(z)\,\phi(z)\right) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))} \sum_{z\in f^{-n}(w)} G_m(z)\,\phi(z) \\
&= \sum_{z\in f^{-m}(f^m(x))} G_m(z)\,\phi(z) \\
&= P_m\phi(x).
\end{aligned}$$

This also implies that P_n is a projection, i.e. $P_n^2 = P_n$. Analogously, we get $P_n P_m = P_m$. □

Lemma 2.1.33. *For $\phi \in C(X)$ and $\chi \in ImP_n$, $P_n(\phi\chi) = \chi P_n\phi$.*

Proof. We assume that $\chi(x) = \sum_{y\in f^{-n}(f^n(x))} G_n(y)\,\beta(y)$. Then with Lemma 2.1.32

$$\begin{aligned}
P_n(\phi\chi)(x) &= \sum_{z\in f^{-n}(f^n(x))} G_n(z)\,\phi(z) \sum_{y\in f^{-n}(f^n(z))} G_n(y)\,\beta(y) \\
&= \sum_{y\in f^{-n}(f^n(x))} \sum_{z\in f^{-n}(f^n(x))} G_n(y)\, G_n(z)\,\phi\beta \\
&= \sum_{y\in f^{-n}(f^n(x))} G_n(y)\,\beta(y) \sum_{z\in f^{-n}(f^n(x))} G_n(z)\,\phi(z) \\
&= \chi(x)\, P_n\phi(x).
\end{aligned}$$

□

The Lemmas 2.1.32 and 2.1.33 provide that $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a compatible CMP.

Lemma 2.1.34. *For $\phi \in C(X)$, $P_n\phi$ converges to a constant if and only if $R^n\phi$ converges to the same constant. Furthermore, the constant equals $\langle \mu, \phi\rangle$ for any G-measure μ.*

Proof.

$$P_n\phi(x) = \sum_{y\in f^{-n}(f^n(x))} \psi(y)\,\phi(y) = (R^n\phi)(f^n(x))$$

and $f : X \to X$ is surjective, thus

$$\|P_n\phi(x) - c\| = \|R^n\phi(x) - c\|.$$

Hence $P_n\phi$ converges to c if and only if $R^n\phi$ converges to c.
If $P_n\phi$ converges to c, then

$$c = \lim_{n\to\infty} P_n\phi = \lim_{n\to\infty}\langle \mu, P_n\phi\rangle = \lim_{n\to\infty}\langle P_n^*\mu, \phi\rangle = \langle \mu, \phi\rangle.$$

□

Lemma 2.1.35 (Naive distortion lemma)**.** *There is a constant $C > 0$ such that for any $n \geqslant 0$ and any $x, y \in X$ with $d_n(x, y) \leqslant a$,*

$$C^{-1} \leqslant \frac{G_n(x)}{G_n(y)} \leqslant C.$$

Proof. Let $x_i = f^i(x)$ and $y_i = f^i(y)$ for $0 \leqslant i \leqslant n$, then $d(x_i, y_i) \leqslant \lambda^{n-i} d(x_n, y_n)$. We get

$$\begin{aligned} |\log G_n(x) - \log G_n(y)| &\leqslant \sum_{i=0}^{n-1} |\log\psi(x_i) - \psi(y_i)| \\ &\leqslant \frac{[\psi]_\alpha}{A}\sum_{i=0}^{n-1} d(x_i, y_i)^\alpha \\ &\leqslant \frac{[\psi]_\alpha}{A}\sum_{i=0}^{n-1} \lambda^{-\alpha(n-i)} d(x_n, y_n)^\alpha \\ &\leqslant C_0 \end{aligned}$$

with $A = \min_{x\in X}\psi(x)$, $[\psi]_\alpha$ the Hölder constant for ψ, and $C_0 = \frac{[\psi]_\alpha a^\alpha\lambda^\alpha}{A(\lambda^\alpha - 1)}$. Hence for $C = e^{C_0}$, we have

$$C^{-1} \leqslant \frac{G_n(x)}{G_n(y)} \leqslant C.$$

□

Proof of Theorem 2.1.31. First, we prove the Gibbs property. Let μ be a G-measure and r a real number with $0 < 2r \leqslant a$. Let further ϕ be a function such that $\chi_{B_n(x,r)} \leqslant \phi \leqslant \chi_{B_n(x,2r)}$. Then

$$\mu(B_n(x, r)) \leqslant \int \phi d\mu = \int \phi dP_n^*\mu = \int P_n\phi d\mu$$

with

$$P_n\phi(y) = \sum_{z\in f^{-n}(f^n(y))} G_n(z)\,\phi(z) \leqslant \sum_{z\in f^{-n}(f^n(y))} G_n(z)\,\chi_{B_n(x,2r)}(z).$$

Using Lemma 2.1.35 and Proposition 2.1.10, we have that there is a constant $C > 0$ such that

$$\#\left(f^{-n}(f^n(y)) \cap B_n(x, 2r)\right) \leqslant C$$

and

$$G_n(z) \leqslant C G_n(x), \qquad z \in B_n(x, 2r).$$

This implies $\mu(B_n(x,2r)) \leqslant C^2 G_n(x)$.
On the other hand, we have

$$\mu(B_n(x,r)) \geqslant \int \phi d\mu = \int \phi dP^*_{n+p}\mu \geqslant \int P_{n+p}\phi d\mu,$$

where p is the integer in Proposition 2.1.9 and

$$\begin{aligned} P_{n+p}\phi(y) &= \sum_{z\in f^{-n-p}(f^{n+p}(y))} G_{n+p}(z)\,\phi(z) \\ &\geqslant \sum_{z\in f^{-n-p}(f^{n+p}(y))} G_{n+p}(z)\,\chi_{B_n(x,r)}(z). \end{aligned}$$

By Proposition 2.1.10 at least one term in the sum is non-zero. This implies together with Lemma 2.1.35 that there is a positive constant C so that

$$\mu(B_n(x,2r)) \geqslant CG_{n+p}(x) \geqslant CA^p G_n(x) \tag{2.4}$$

for $A = \min_{x\in X}\psi(x)$. Let s be the least integer sucht that $\lambda^s \geqslant 2$, then $B_n(x,r) \supset B_{n+s}(x,\lambda^s r) \supset B_{n+s}(x,2r)$. Using (2.4), we get

$$\mu(B_n(x,r)) \geqslant CA^{p+s}G_n(x).$$

Thus we have a positive constant which depends on r only, denoted by C, satisfying

$$C^{-1} \leqslant \frac{\mu(B_n(x,r))}{G_n(x)} \leqslant C.$$

Following our previous investigation on CMPs together with th Gibbs Property, it remains to prove that a G-measure is unique. Using the Choquet representation theorem (Theorem 2.1.29) it suffices to show that a $\mathcal{P}$-ergodic G-measure is unique. Theorem 2.1.28 provides that any two $\mathcal{P}$-ergodic G-measures are either equal or totally singular. We will use the Gibbs Property to show that any two $\mathcal{P}$-ergodic G-measures μ and ν are mutually absolutely continuous, that is, that there is a constant $C > 0$ such that

$$C^{-1}\nu(U) \leqslant \mu(U) \leqslant C\nu(U)$$

for all open subsets $U \subset X$. For this purpose we fix a real number r, $0 < 2r \leqslant a$ and let $\{x_1,\dots,x_m\}$ be a $2r$-net in (X,d), that is, the balls $\{B(x_i,r)\}_{1\leqslant i\leqslant m}$ are disjoint and the balls $\{B(x_i,2r)\}_{1\leqslant i\leqslant m}$ form a cover of X. Then we define

$$\begin{aligned} A_1 &= B(x_1,2r)\setminus(B(x_2,r)\cup\dots\cup B(x_m,r)), \\ A_i &= B(x_i,2r)\setminus(A_1\cup\dots\cup A_{i-1}), \qquad 2\leqslant i\leqslant m. \end{aligned}$$

So we get a partition $Q_0 = \{A_i\}_{i=1}^m$ of X which satisfies

$$B(x_i,r) \subseteq A_i \subseteq B(x_i,2r), \qquad 1\leqslant i\leqslant m.$$

We denote $f^{-n}(x_i) = \{z_j\}_{j=1}^{k_{ni}}$ for $n \geqslant 1$, $1\leqslant i\leqslant m$ and let g_{jn} be the inverse branches of $f^n : B_n(z_j,2r) \to B(x_i,2r)$. $A_{nij} := g_{jn}(A_i)$ is called a n-component of $f^{-n}|Q_0$ and Q_n is the set of all n components of $f^{-n}|Q_0$ which is again a partition of X satisfying

$$B_n(c_A,r) \subseteq A \subseteq B_n(c_A,2r) \qquad \forall A\in Q_n,$$

where $c_A \in A$ (called the *center of* A) such that $f^n(c_A) = x_j$.
Let U be an arbitrary open set in X. Let further, for $n \geqslant 1$, $Q_n(U)$ be the family of all elements A of the partition Q_n such that the n-Bowen ball $B_n(c_A, r)$ is entirely contained in U. Set $V_n = \cup_{A \in Q_n(U)} A$ which is a Borel subset of U being a countable union of disjoint sets. Now, with the Gibbs Property we get

$$\begin{aligned}
\mu(V_n) &= \sum_{A \in Q_n(U)} \mu(A) \leqslant \sum_{A \in Q_n(U)} \mu(B_n(c_A, 2r)) \\
&\leqslant C \sum_{A \in Q_n(U)} G_n(c_A) \leqslant C^2 \sum_{A \in Q_n(U)} \nu(B_n(c_A, r)) \\
&\leqslant C^2 \sum_{A \in Q_n(U)} \nu(A) = C^2 \nu\left(\cup_{A \in Q_n(U)} A\right) = C^2 \nu(V_n).
\end{aligned}$$

Then by Fatou's lemma (see **[46]** p. 376) together with $U = \liminf_{n\to\infty} V_n$, $\mu(U) \leqslant C^2 \nu(U)$ and similarly $\nu(U) \leqslant C^2 \mu(U)$, thus a G-measure is unique.
If μ is a unique G-measure, then by Theorem 2.1.25, $P_n \phi \to \langle \mu, \phi \rangle$ as $n \to \infty$ for any $\phi \in C(X)$. Hence by Lemma 2.1.34, $R^n \phi \to \langle \mu, \phi \rangle$ as $n \to \infty$, which completes the proof. □

2.1.4 Spectra of Ruelle-Perron-Frobenius operators

Now, let $C_{\mathbb{C}}(X) = C(X, \mathbb{C})$ be the space of all continuous complex-valued functions $\phi : X \to \mathbb{C}$ with the supremum norm

$$\|\phi\| = \max_{x \in X} \{|\phi(x)|\}.$$

For $0 < \alpha \leqslant 1$, let $C^\alpha_{\mathbb{C}}(X) = C^\alpha(X, \mathbb{C})$ be the space of all α-Hölder complex-valued continuous functions ϕ in $C_{\mathbb{C}}(X)$. A function $\phi \in C_{\mathbb{C}}(X)$ is said to be α-Hölder continuous if

$$[\phi]_\alpha = \sup_{x,y: 0 < d(x,y) \leqslant a} \frac{|\phi(x) - \phi(y)|}{d(x,y)^\alpha} < \infty .$$

$\phi \in C_{\mathbb{C}}(X)$ can be written as

$$\phi = \phi_1 + \imath \phi_2, \qquad \phi_1, \phi_2 \in C(X),$$

then $\phi \in C^\alpha_{\mathbb{C}}(X)$ if and only if $\phi_1 \in C^\alpha(X)$ and $\phi_2 \in C^\alpha(X)$.
We have $R\phi = R\phi_1 + \imath R\phi_2$ because ψ is a real-valued function, and thus $R : C_{\mathbb{C}}(X) \to C_{\mathbb{C}}(X)$ is a bounded linear operator. $C^\alpha_{\mathbb{C}}(X)$ equipped with the norm

$$\|\phi\|_\alpha = \|\phi\| + [\phi]_\alpha$$

is a Banach space, and $R_\alpha = R : C^\alpha_{\mathbb{C}}(X) \to C^\alpha_{\mathbb{C}}(X)$ is a bounded linear operator (see Proposition 2.1.14).

Corollary 2.1.36. *The maximal eigenvalue δ is the spectral radius of $R_0 = R : C^0_{\mathbb{C}}(X) \to C^0_{\mathbb{C}}(X)$.*

Proof. By **[42]**, p. 407, the spectral radius can be calculated as

$$\rho(R_0) = \lim_{n\to\infty} \|R_0^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\sup_{\phi\in C^0_{\mathbb{C}}(X),\ \|\phi\|\leqslant 1} \|R_0^n \phi\| \right)^{\frac{1}{n}}.$$

By Theorem 2.1.30

$$\left\|\delta^{-n} R_0^n \phi\right\| \leqslant |\langle \nu, \phi\rangle| \, \|h\| + 1 \leqslant \|h\| + 1 \qquad \forall \phi \in C^0_{\mathbb{C}}(X) \text{ with } \|\phi\| \leqslant 1, \text{ n large.}$$

Thus $\delta^{-n}\|R_0^n\| \leqslant \|h\| + 1$ and $\|R_0^n\|^{\frac{1}{n}} \leqslant (\|h\|+1)^{\frac{1}{n}}\delta$ for large n. Hence $\rho(R_0) \leqslant \delta$. As δ is a spectral point, $\rho(R_0) = \delta$. □

Furthermore, we have for the normalized transfer operator the following corollary which is a direct consequence of the relation between R and its normalization (see (2.2)).

Corollary 2.1.37. *The maximal eigenvalue δ is the spectral radius of $R_\alpha = R : C^\alpha_{\mathbb{C}}(X) \to C^\alpha_{\mathbb{C}}(X)$. The rest of the spectrum is in a disk of center 0 with radius strictly less than δ.*

Corollary 2.1.38. *For the normalized RPF operator, the maximal eigenvalue 1 is the spectral radius of $R_\alpha = R : C^\alpha_{\mathbb{C}}(X) \to C^\alpha_{\mathbb{C}}(X)$.*
The rest of the spectrum is in a disk of center 0 with radius strictly less than 1.

Proof. The spectral radius can be calculated as

$$\rho(R_\alpha) = \lim_{n\to\infty} \|R_\alpha^n\|_\alpha^{\frac{1}{n}} = \lim_{n\to\infty} \left(\sup_{\phi\in C^\alpha_{\mathbb{C}}(X),\ \|\phi\|_\alpha\leqslant 1} \|R_\alpha^n \phi\|_\alpha \right)^{\frac{1}{n}}.$$

Consider for any $\phi \in C^\alpha_{\mathbb{C}}(X)$ with $\|\phi\|_\alpha \leqslant 1$ and $x, y \in X$ with $d(x,y) \leqslant a$ the corresponding inverse images of x and y, $f^{-1}(x) = \{x_1, \ldots, x_n\}$ and $f^{-1}(y) = \{y_1, \ldots, y_n\}$, such that $d(x_i, y_i) \leqslant \frac{1}{\lambda} d(x,y)$ for all i. Then

$$\begin{aligned} |R_\alpha\phi(x) - R_\alpha\phi(y)| &\leqslant \sum_{i=1}^n |\psi(x_i)\phi(x_i) - \psi(y_i)\phi(y_i)| \\ &\leqslant \sum_{i=1}^n |\psi(x_i) - \psi(y_i)|\,|\phi(y_i)| + \sum_{i=1}^n \psi(x_i)\,|\phi(x_i) - \phi(y_i)|. \end{aligned}$$

As $d(x_i, y_i) \leqslant \frac{1}{\lambda} d(x,y)$ and $\sum_{i=1}^n \psi(x_i) = 1$, we have

$$[R_\alpha\phi]_\alpha \leqslant \frac{[\psi]_\alpha n_0}{\lambda^\alpha}\|\phi\| + \frac{1}{\lambda^\alpha}[\phi]_\alpha,$$

and with $C_1 := 1 - \frac{1}{\lambda^\alpha} + [\psi]_\alpha \frac{n_0}{\lambda^\alpha}$

$$\|R_\alpha\phi\|_\alpha \leqslant C_1\|\phi\| + \frac{1}{\lambda^\alpha}\|\phi\|_\alpha.$$

Then by induction,

$$\left\|R_\alpha^{n-1}\phi\right\|_\alpha \leqslant C_{n-1}\|\phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n-1}\|\phi\|_\alpha.$$

Thus

$$\begin{aligned}\|R_\alpha^n \phi\|_\alpha &\leqslant C_{n-1}\|R_\alpha \phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n-1}\|R_\alpha\phi\|_\alpha \\ &\leqslant C_{n-1}\|\phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n-1} C_1\|\phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n}\|\phi\|_\alpha \\ &= C_n\|\phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n}\|\phi\|_\alpha,\end{aligned}$$

where $C_n = C_{n-1} + \left(\frac{1}{\lambda^\alpha}\right)^{n-1} C_1 \leqslant C = C_1\frac{\lambda^\alpha}{\lambda^\alpha - 1}$. Hence we get that

$$\|R_\alpha^n \phi\|_\alpha \leqslant C\|\phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n}\|\phi\|_\alpha, \qquad n \geqslant 1.$$

Set $C_{\mathbb{C}}^{\alpha\perp}(X) = \{\phi \in C_{\mathbb{C}}^{\alpha}(X) : \langle \mu, \phi\rangle = 0\}$, then $C_{\mathbb{C}}^{\alpha}(X) = C_{\mathbb{C}}^{\alpha\perp}(X) \oplus \mathbb{C}$ because $\phi = (\phi - \langle\mu,\phi\rangle) + \langle\mu,\phi\rangle$. It suffices to prove that the spectral radius of $R_\alpha | C_{\mathbb{C}}^{\alpha\perp}(X) : C_{\mathbb{C}}^{\alpha\perp}(X) \to C_{\mathbb{C}}^{\alpha\perp}(X)$ is strictly less than 1, in order to prove that the rest of the spectrum of R_α is in a disk of center 0 with radius less than 1. To show this, suppose $n, k > 0$. Then

$$\begin{aligned}\left\|R_\alpha^{n+k}\phi\right\|_\alpha &\leqslant C\left\|R_\alpha^k\phi\right\| + \left(\frac{1}{\lambda^\alpha}\right)^{n}\left\|R_\alpha^k\phi\right\|_\alpha \\ &\leqslant C\left\|R_\alpha^k\phi\right\| + C\left(\frac{1}{\lambda^\alpha}\right)^{n}\|\phi\| + \left(\frac{1}{\lambda^\alpha}\right)^{n+k}\|\phi\|_\alpha.\end{aligned}$$

We have that $\left\{\phi \in C_{\mathbb{C}}^{\alpha\perp}(X) : \|\phi\|_\alpha \leqslant 1\right\}$ is a uniformly bounded and equicontinuous family, thus it is a compact set in $C_{\mathbb{C}}(X)$. Hence by Theorem 2.1.31, for any $0 < \tau < 1$ there are $m, k > 0$ such that

$$\left\|R_\alpha^{m+k}\phi\right\|_\alpha \leqslant \tau \qquad \forall \phi \in C_{\mathbb{C}}^{\alpha\perp}(X) \text{ with } \|\phi\|_\alpha \leqslant 1.$$

So, $\left\|R_\alpha^{m+k}\right\|_\alpha \leqslant \tau$, and it follows that

$$\lim_{n\to\infty}\left\|R^n | C_{\mathbb{C}}^{\alpha\perp}(X)\right\|_\alpha^{\frac{1}{n}} \leqslant \tau^{\frac{1}{m+k}} < 1.$$

□

2.2 Harmonic analysis for the transfer operator on $\mathbb{R}$ and $\mathbb{T}$

2.2.1 Wavelets

In this section, we will state all the basic facts on wavelets and their multiresolution analysis, following **[13]**, which we will need for the harmonic analysis of the transfer operator.

Definition 2.2.1. A *multiresolution analysis* consists of a sequence of closed subspaces V_j which satisfy

(a)
$$\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \tag{2.5}$$

with
$$\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}), \tag{2.6}$$

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}. \tag{2.7}$$

(b) All the spaces V_j are scaled versions of the space V_0,
$$f \in V_j \Leftrightarrow f(2^j \cdot) \in V_0. \tag{2.8}$$

(c) Invariance of V_0 under integer translations,
$$f \in V_0 \Rightarrow f(\cdot - n) \in V_0, \qquad n \in \mathbb{Z}. \tag{2.9}$$

(d) There exists $\phi \in V_0$ such that
$$\{\phi_{0,n} : n \in \mathbb{Z}\} \text{ is an orthonormal basis in } V_0, \tag{2.10}$$

with $\phi_{j,n}(x) = 2^{-\frac{j}{2}} \phi(2^{-j}x - n)$. The function ϕ is called the *scaling function* of the multiresolution analysis.

Remark 2.2.2. (1) For P_j, the orthogonal projection operator onto V_j, condition (2.6) provides that $\lim_{j \to -\infty} P_j f = f$ for all $f \in L^2(\mathbb{R})$.

(2) Condition (2.9) together with condition (2.8) implies that if $f \in V_j$, then $f(\cdot - 2^j n) \in V_j$ for all $n \in \mathbb{Z}$.

(3) The fact that $\{\phi_{j,n} : n \in \mathbb{Z}\}$ is an orthonormal basis for V_j for all $j \in \mathbb{Z}$ is provided by the combination of condition (2.8) and (2.10).

(4) Condition (2.10) can be relaxed considerably, e.g. to Riesz bases. (A basis $\{e_k\}_{k=1}^{\infty}$ is called a *Riesz basis* if it is equivalent to an orthonormal basis $\{v_k\}_{k=1}^{\infty}$, that is, $\sum_{k=1}^{\infty} c_k e_k$ converges if and only if $\sum_{k=1}^{\infty} c_k v_k$ converges).

Theorem 2.2.3. *If a sequence of closed subspaces $(V_j)_{j \in \mathbb{Z}}$ in $L^2(\mathbb{R})$ meets the conditions (2.5) - (2.10), then there is an associated orthonormal wavelet basis $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$ such that*
$$P_{j-1} = P_j + \sum_{k \in \mathbb{Z}} \langle \cdot, \psi_{j,k} \rangle \psi_{j,k} \ . \tag{2.11}$$

The wavelet ψ in Theorem 2.2.3 can be constructed explicitly (see also [**13**], pp. 130-135). One possibility for the construction involves a function m_0, whose properties are crucial for the next section.
Since $\phi \in V_0 \subset V_{-1}$, and the $\phi_{-1,n}$ are an orthonormal basis in V_{-1}, we have
$$\phi = \sum_{n \in \mathbb{Z}} h_n \phi_{-1,n} \ , \tag{2.12}$$

with

$$h_n = \langle \phi, \phi_{-1,n} \rangle \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |h_n|^2 = 1. \tag{2.13}$$

(2.12) can either be rewritten as

$$\phi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \phi(2x - n) \tag{2.14}$$

or

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n e^{-in\xi/2} \hat{\phi}(\xi/2), \tag{2.15}$$

where convergence in both sums hold in L^2-sense. Let us denote

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n e^{-in\xi}, \tag{2.16}$$

then (2.12) can be written as

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \tag{2.17}$$

where equality holds pointwise almost everywhere. Furthermore, (2.13) indicates that m_0 is a 2π-periodic function in $L^2([0, 2\pi])$.
The orthonormality of $\phi(\cdot - k)$ provides some special properties for m_0:

$$\begin{aligned} \delta_{k,0} &= \int \phi(x)\overline{\phi(x-k)}dx = \int |\hat{\phi}(\xi)|^2 e^{ik\xi} d\xi \\ &= \int_0^{2\pi} e^{ik\xi} \sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi l)|^2 d\xi. \end{aligned}$$

Thus we have

$$\sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi l)|^2 = \frac{1}{2\pi} \text{ a.e.} \tag{2.18}$$

By substitution in (2.17), ($\zeta = \xi/2$), we get

$$\sum_{l\in\mathbb{Z}} |m_0(\zeta + \pi l)|^2 |\hat{\phi}(\zeta + \pi l)|^2 = \frac{1}{2\pi}; \tag{2.19}$$

splitting the sum into even and odd l, using the periodicity of m_0 and applying (2.18) finally gives

$$|m_0(\zeta)|^2 + |m_0(\zeta + \pi)|^2 = 1 \text{ a.e.} \tag{2.20}$$

The scaling function ϕ can also be used as a starting point for the construction of a multiresolution analysis. First, V_0 is constructed from the $\phi(\cdot - k)$, and then all other V_j can be obtained. For this construction we choose ϕ such that

$$\phi(x) = \sum_{n\in\mathbb{Z}} a_n \phi(2x - n), \tag{2.21}$$

with $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$, and

$$0 < \alpha \leqslant \sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi l)|^2 \leqslant \beta < \infty. \tag{2.22}$$

Then V_j can be defined to be the closed subspace spanned by the $\phi_{j,k}$, $k \in \mathbb{Z}$, where $\phi(x) = 2^{-j/2}\phi(2^{-j}x - k)$. The conditions (2.21) and (2.22) give that $\{\phi_{j,k} : k \in \mathbb{Z}\}$ is a Riesz basis in each V_j, and that (2.5) is satisfied. It can be shown that the V_j also satisfy the conditions (2.6)-(2.10). (2.21) can be rewritten as

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \tag{2.23}$$

with $m_0(\xi) = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{in\xi}$. If $\phi \in L^\infty(\mathbb{R})$, $\hat{\phi}(0) \neq 0$ and $\hat{\phi}$ is continuous in 0, we have $\hat{\phi}(0) = m_0(0)\hat{\phi}(0)$, and thus $m_0(0) = 1$ or

$$\sum_{n\in\mathbb{Z}} a_n = 2. \tag{2.24}$$

Furthermore, m_0 is continuous by (2.23), except possibly near the zeros of $\hat{\phi}$.

Before proceeding with the next section, we finally introduce compactly supported wavelets, including some properties of m_0 and the scaling function which result from the compact support. The easiest way to obtain compact support for the wavelet ψ is to choose the scaling function ϕ with compact support.
For compactly supported ϕ the 2π-periodic function m_0,

$$m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} h_n e^{-in\xi}, \tag{2.25}$$

becomes a trigonometric polynomial, and as in (2.20), it follows from the orthonormality of the $\phi_{0,n}$ that,

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1. \tag{2.26}$$

We do not need to suppose "almost everywhere" anymore as m_0 is continuous, and thus (2.26) is satisfied for all ξ if it is satisfied almost everywhere. Since $\hat{\phi}(0) \neq 0$ and $m_0(0) = 1$, with (2.26) we have that $m_0(\pi) = 0$. Consequently, for all $k \in \mathbb{Z}$, $k \neq 0$,

$$\begin{aligned}
\hat{\phi}(2k\pi) &= \hat{\phi}\left(2\cdot 2^l(2m+1)\pi\right) \qquad\qquad \text{(for some } l \geqslant 0, m \in \mathbb{Z}) \\
&= \left[\prod_{j=1}^{l} m_0\left(2^{l+1-j}(2m+1)\pi\right)\right] m_0((2m+1)\pi)\,\hat{\phi}((2m+1)\pi) \\
&= m_0(\pi)\,\hat{\phi}((2m+1)\pi) = 0.
\end{aligned}$$

Equation (2.18) provides a normalization of ϕ by $|\hat{\phi}(0)| = \frac{1}{\sqrt{2\pi}}$, or $|\int \phi(x)\,dx| = 1$. Together with (2.23) this implies

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty} m_0\left(2^{-j}\xi\right). \tag{2.27}$$

This product makes sense because $\sum_{n\in\mathbb{Z}} |h_n||n| < \infty$, $m_0(0) = 1$, and $m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} h_n e^{-in\xi}$ satisfies for a constant C

$$|m_0(\xi)| \leqslant 1 + |m_0(\xi) - 1| \leqslant 1 + \sqrt{2}\sum_{n\in\mathbb{Z}} |h_n||\sin\frac{n\xi}{2}| \leqslant 1 + C|\xi| \leqslant e^{C|\xi|},$$

thus

$$\prod_{j=1}^{\infty} |m_0\left(2^{-j}\xi\right)| \leqslant \exp\left(\sum_{j=1}^{\infty} C|2^{-j}\xi|\right) \leqslant e^{C|\xi|}.$$

Hence the right hand side of (2.27) converges absolutely and uniformly on compact sets.

Lemma 2.2.4 (Mallat)**.** *If m_0 is a 2π-periodic function which satisfies (2.26), and if $\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty} m_0\left(2^{-j}\xi\right)$ converges pointwise a.e., then its limit $\hat{\phi}(\xi)$ is in $L^2(\mathbb{R})$, and $\|\phi\|_2 \leqslant 1$.*

Proof. If we set $f_k(\xi) = \frac{1}{\sqrt{2\pi}}\left[\prod_{j=1}^{\infty} m_0\left(2^{-j}\xi\right)\right]\chi_{[-\pi,\pi]}\left(2^{-k}\xi\right)$ with
$\chi_{[-\pi,\pi]}(\zeta) = \begin{cases} 1 & , \quad \text{if } |\zeta| \leqslant \pi \\ 0 & , \quad \text{otherwise} \end{cases}$, then $f_k \to \hat{\phi}$ pointwise a.e.
With the 2π-periodicity of m_0, we get

$$\begin{aligned}
\int |f_k(\xi)|^2 d\xi &= \frac{1}{2\pi}\int_{-2^k\pi}^{2^k\pi} \prod_{j=1}^{k} |m_0\left(2^{-j}\xi\right)|^2 d\xi \\
&= \frac{1}{2\pi}\int_{0}^{2^{k+1}\pi} \prod_{j=1}^{k} |m_0\left(2^{-j}\xi\right)|^2 d\xi \\
&= \frac{1}{2\pi}\int_{0}^{2^k\pi} \left[\prod_{j=1}^{k-1} m_0\left(2^{-j}\xi\right)^2\right]\left[|m_0\left(2^{-k}\xi\right)|^2 + |m_0\left(2^{-k}\xi+\pi\right)|^2\right] d\xi \\
&\overset{(2.26)}{=} \frac{1}{2\pi}\int_{0}^{2^k\pi} \prod_{j=1}^{k-1} |m_0\left(2^{-j}\xi\right)|^2 d\xi \\
&= \|f_{k-1}\|^2 .
\end{aligned}$$

Consequently, for all k,

$$\|f_k\|^2 = \|f_{k-1}\|^2 = \cdots = \|f_0\|^2 = 1,$$

and by Fatou's lemma (see **[46]** p.376),

$$\int |\hat{\phi}(\xi)|^2 d\xi = \limsup_{k\to\infty} \int |f_k(\xi)|^2 d\xi \leqslant 1.$$

□

2.2.2 The trigonometric case

In the previous section, we encountered wavelets and their construction in the general case. Now, we will turn to the trigonometric case, that is, we will consider $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the identification $\mathbb{R}\backslash 2\pi\mathbb{Z} \ni \omega \mapsto e^{-i\omega} = z \in \mathbb{T}$ and see how the fusion of wavelet theory and the theory of dynamical systems, especially transfer operators, gives rise to interesting results for both theories.
In this trigonometric context, P.E.T. Jorgensen dealt with a transfer operator of the form,

$$(Rf)(z) = \frac{1}{N}\sum_{\omega^N = z} |m_0(\omega)|^2 f(\omega), \qquad f \in L^1(\mathbb{T}), z \in \mathbb{T}, \tag{2.28}$$

where m_0 is given by $m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k$, $z \in \mathbb{T}$. In [**29**], Jorgensen established a one-to-one correspondence between representations of a C^*-algebra and functions which are harmonic for the transfer operator R. We will see his main result below in Theorem 2.2.5.

For $m_0 \in L^\infty(\mathbb{T})$, we study the eigenvalue problem

$$h \in L^1(\mathbb{T}), \qquad h \geqslant 0, \qquad R(h) = h \tag{2.29}$$

for the transfer operator R, and call functions h, which satisfy (2.29), *harmonic* for the transfer operator R. For the *cascade refinement operator* M in $L^2(\mathbb{R})$ given by

$$(M\psi)(x) = \sqrt{N}\sum_{k\in\mathbb{Z}} a_k \psi(Nx - k), \tag{2.30}$$

with $N \geqslant 2$ integral, $a_k \in \mathbb{C}$ satisfying $\sum_{k\in\mathbb{Z}} |a_k|^2 = 1$, $k \in \mathbb{Z}$, $\psi \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$, the eigenvalue problem (2.29) is closely connected to the problem

$$\phi \in L^2(\mathbb{R}), \qquad M\phi = \phi, \tag{2.31}$$

where the nonzero solutions (if existent) are the scaling functions in wavelet theory.
For the purpose of studying the more general eigenvalue problem (2.29) it is useful to take a representation-theoretic viewpoint, instead of insisting on $L^2(\mathbb{R})$ as the Hilbert space for (2.31). We will consider abstract Hilbert spaces $\mathcal{H}$ which admit nonzero solutions $\phi \in \mathcal{H}$ for (2.31). As $\mathcal{H}$ will only be given abstractly, a unitary operator $U : \mathcal{H} \to \mathcal{H}$ which corresponds to the scaling operator needs to be specified by

$$U : \psi \mapsto \frac{1}{\sqrt{N}}\psi\left(\frac{x}{N}\right), \tag{2.32}$$

for the special case when $\mathcal{H} = L^2(\mathbb{R})$. Analogously, we specify a representation π of $L^2(\mathbb{R})$ on $\mathcal{H}$ by

$$U\pi(f) = \pi\left(f\left(z^N\right)\right)U, \qquad f \in L^\infty(\mathbb{T}) \tag{2.33}$$

as a commutation relation for operators on $\mathcal{H}$. In this scope, the problem (2.31) takes the form:

$$U\phi = \pi(m_0)\phi, \qquad \phi \in \mathcal{H}. \tag{2.34}$$

Let $\mathcal{U}_N$ denote the C^*-algebra on two unitary generators U and V satisfying

$$UVU^{-1} = V^N \tag{2.35}$$

(see [**10**] for details). Then U denotes both an element in $\mathcal{U}_N$ and a unitary operator in $\mathcal{H}$. By a representation of (2.35), a realization of U and V as unitary operators on some Hilbert space $\mathcal{H}$ is meant, such that (2.35) holds for those operators. Let $f \in L^\infty(\mathbb{T})$, then $f(V)$ is defined by the spectral theorem (see [**45**] pp. 305), applied to V, and $\pi_V(f) := f(V)$ is a representation of $L^\infty(\mathbb{T})$ in the sense that $\pi_V(f_1, f_2) = \pi_V(f_1)\pi_V(f_2)$, and $\pi_V(f)^* = \pi_V\left(\overline{f}\right)$, with $\overline{f}(z) := \overline{f(z)}$, $z \in \mathbb{T}$, f_1, f_2, $f \in L^\infty(\mathbb{T})$. Considering this setting, (2.35) rewrites as

$$U\pi_V(f)U^{-1} = \pi_V\left(f\left(z^N\right)\right), \tag{2.36}$$

and conversely, if π is a representation of $L^\infty(\mathbb{T})$ on $\mathcal{H}$, and U a unitary operator on $\mathcal{H}$ satisfying $U\pi(f)U^{-1} = \pi\left(f\left(z^N\right)\right)$, then every pair (U, π), is of this form for some V.

For $e_n(z) = z^n$, $n \in \mathbb{Z}$, we set $V := \pi(e_1)$.
As V is unitary, it has a spectral resolution $V = \int_{\mathbb{T}} \lambda E(d\lambda)$ with a projection-valued spectral measure $E(\cdot)$ on $\mathbb{T}$. A vector $\phi \in \mathcal{H}$ is called *cyclic* if $\{\pi(A)\phi : A \in \mathcal{U}_N\}$ is dense in $\mathcal{H}$, the corresponding representation π is called a *cyclic representation.* If, for a cyclic vector $\phi \in \mathcal{H}$, the measure $\|E(\cdot)\phi\|^2$ on $\mathbb{T}$ is absolutely continuous with respect to the Haar measure on $\mathbb{T}$, then the corresponding representation is called *normal.* The normal representations are denoted by $Rep(\mathcal{U}_N, \mathcal{H})$.
Now, we can state the main result of P.E.T. Jorgensen in [**29**]:

Theorem 2.2.5. *(1) Suppose $m_0 \in L^\infty(\mathbb{T})$ and it does not vanish on a subset of $\mathbb{T}$ of positive measure. Then there is a one-to-one correspondence between*

(a) $h \in L^1(\mathbb{T})$, $h \geqslant 0$, and

$$R(h) = h. \tag{2.37}$$

and

(b) $\tilde{\pi} \in Rep(\mathcal{U}_N, \mathcal{H})$, $\phi \in \mathcal{H}$, and the unitary U from $\tilde{\pi}$ satisfying

$$U\phi = \pi(m_0)\phi; \tag{2.38}$$

as equivalence classes under unitary equivalence.

(2) From $(a) \to (b)$, the correspondence is given by

$$\langle \phi, \pi(f)\phi \rangle_{\mathcal{H}} = \int_{\mathbb{T}} f h d\mu, \tag{2.39}$$

with μ the normalized Haar measure on $\mathbb{T}$.
From $(b) \to (a)$, the correspondence is given by

$$h(z) = h_\phi(z) = \sum_{n \in \mathbb{Z}} z^n \langle \pi(e_n)\phi, \phi \rangle_{\mathcal{H}}. \tag{2.40}$$

(3) If there is some h which satisfies (a), and $\tilde{\pi} \in Rep(\mathcal{U}_N, \mathcal{H})$ is the corresponding cyclic representation in (b), then the representation is unique from h and (2.39) up to unitary equivalence, that is:
If there is a $\pi' \in Rep(\mathcal{U}_N, \mathcal{H}')$, $\phi' \in \mathcal{H}'$ also cyclic and satisfying (2.38) and (2.39), then there is a unitary isomorphism W of $\mathcal{H}$ onto $\mathcal{H}'$ such that $W\pi(A) = \pi'(A)W$, $A \in \mathcal{U}_N$, and $W\phi = \phi'$.

Remark 2.2.6. The proof of Theorem 2.2.5 is basically the construction of a generalized multiresolution analysis using the transfer operator (see [**29**]).

The remainder of this section refers to work done by O. Bratteli and P.E.T. Jorgensen [**9**]. It deals with the special case, $N = 2$, for the transfer operator introduced in (2.28). Following (2.14), compactly supported scaling functions ϕ of a multiresolution analysis satisfy the functional equation

$$\phi(x) = \sqrt{2} \sum_{k=0}^{K} a_k \phi(2x - k). \tag{2.41}$$

Then (2.10) implies the conditions

$$\sum_{k\in\mathbb{Z}} \overline{a}_k a_{k+2l} = \delta_l, \qquad l \in \mathbb{Z}, \tag{2.42}$$

and the second standard requirement $\hat{\phi}(0) = 1$ gives the condition

$$\sum_{k\in\mathbb{Z}} a_k = \sqrt{2}. \tag{2.43}$$

We set

$$m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k \tag{2.44}$$

for $z = e^{-it} \in \mathbb{T}$, then condition (2.42) is equivalent to

$$|m_0(z)|^2 + |m_0(-z)|^2 = 2, \tag{2.45}$$

and (2.43) is equivalent to

$$m_0(1) = \sqrt{2}. \tag{2.46}$$

The Fourier transform of (2.41) is

$$\hat{\phi}(t) = \frac{1}{\sqrt{2}} m_0\left(\frac{t}{2}\right)\hat{\phi}\left(\frac{t}{2}\right). \tag{2.47}$$

Since ϕ has compact support, and with (2.41) its support is in $[0, K]$, $\hat{\phi}$ is continuous at 0 and an iteration of (2.47) gives

$$\hat{\phi(t)} = \prod_{k=1}^{\infty}\left(\frac{m_0\left(t2^{-k}\right)}{\sqrt{2}}\right), \tag{2.48}$$

(see (2.27) for comparison). This converges uniformly on compacts since m_0 is a polynomial.
Let $\psi^{(0)}$ be any bounded function of compact support satisfying $\widehat{\psi^{(0)}}(0) = 1$. Now, the cascade approximation operator introduced in (2.30), can be given by iteration

$$\begin{aligned} \psi^{(n+1)}(x) &= \left(M\psi^{(n)}\right)(x) \\ &= \sqrt{2}\sum_{k=0}^{K} a_k \psi^{(n)}(2x-k). \end{aligned} \tag{2.49}$$

Then, we have for the iterates

$$\widehat{\psi^{(n+1)}}(t) = \frac{1}{\sqrt{2}} m_0\left(\frac{t}{2}\right)\widehat{\psi^{(n)}}\left(\frac{t}{2}\right), \tag{2.50}$$

and thus

$$\widehat{\psi^{(n)}}(t) = \prod_{k=1}^{n}\left(\frac{m_0\left(t2^{-k}\right)}{\sqrt{2}}\right)\widehat{\psi^{(0)}}\left(t2^{-n}\right). \tag{2.51}$$

The above equations imply that $\widehat{\psi^{(n)}} \overset{n\to\infty}{\to} \hat{\phi}$, uniformly on compacts, and thus $\psi^{(n)} \overset{n\to\infty}{\to} \phi$. That is, if the coefficients $\{a_k : k = 0, \cdots, K\}$ satisfy (2.43), then the refinement equation

(2.41) possesses a distribution solution ϕ with $\hat{\phi}(0) = 1$ and compact support in $[0, K]$. This solution is defined by (2.48) and can be written as the distribution limit

$$\phi = \lim_{n\to\infty} M^n \psi^{(0)}, \tag{2.52}$$

with $\psi^{(0)}$ any integrable function with compact support satisfying $\widehat{\psi^{(0)}}(0) = 1$.
Set

$$\hat{\phi}_n(t) = \prod_{k=1}^{n} \left(\frac{m_0\left(t2^{-k}\right)}{\sqrt{2}} \right), \tag{2.53}$$

then following Mallat's lemma, Lemma 2.2.4, and its proof, for $f_k := \hat{\phi}_n$, we have

$$\begin{aligned} \int_{-2^n\pi}^{2^n\pi} |\hat{\phi}_n\left(t\right)|^2 dt &= \int_0^{2^{n+1}\pi} |\hat{\phi}_n\left(t\right)|^2 dt \\ &= \int_0^{2^n\pi} |\hat{\phi}_{n-1}\left(t\right)|^2 \frac{1}{2}\left(|m_0\left(2^{-n}t\right)|^2 |m_0\left(2^{-n}t+\pi\right)|^2\right) dt \\ &= \int_0^{2^n\pi} |\hat{\phi}_{n-1}\left(t\right)|^2 dt = \cdots = \int_{-\pi}^{\pi} |\hat{\phi}_0\left(t\right)|^2 dt = 2\pi. \end{aligned} \tag{2.54}$$

The uniform convergence of $\hat{\phi}_n \overset{n\to\infty}{\to} \hat{\phi}$ implies

$$\left\|\hat{\phi}\right\|_2^2 \leqslant 2\pi, \tag{2.55}$$

thus $\phi \in L^2\left(\mathbb{R}\right)$, and $\|\phi\|_2 \leqslant 1$. Since $\hat{\phi}_n(\cdot)\chi_{[-\pi,\pi]}\left(\cdot 2^{-n}\right)$ converges for $n \to \infty$ uniformly on compacts, and has constant L^2-norm equal to $\sqrt{2\pi}$ by (2.54), we get that this sequence converges weakly to $\hat{\phi}$ in $L^2\left(\mathbb{R}\right)$. Thus it converges in the L^2-norm to $\hat{\phi}$ if and only if

$$\left\|\hat{\phi}\right\|_2^2 = 2\pi, \tag{2.56}$$

and, especially, if and only if

$$\|\phi\|_2 = 1. \tag{2.57}$$

This is equivalent to:
The only trigonometric polynomials ξ which satisfy

$$\xi\left(z\right) = \frac{1}{2} \sum_{\omega^2 = z} |m_0\left(\omega\right)|^2 \xi\left(\omega\right) \tag{2.58}$$

are the constants, which is in turn equivalent to:

> The cascade algorithm, with
>
> $$\psi^{(0)}\left(x\right) = \check{\chi}_{[-\pi,\pi]}\left(x\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{itx} dt = \frac{1}{\pi x}\sin\left(\pi x\right), \tag{2.59}$$
>
> converges in L^2-norm to ϕ;

and $\{\phi\left(\cdot - k\right)\}$ is an orthonormal set.

In order to examine the convergence properties of (2.59) under general circumstances the *Ruelle operator* R is introduced by

$$(R\xi)(z) = \frac{1}{2} \sum_{\omega^2 = z} |m_0(\omega)|^2 \xi(\omega). \tag{2.60}$$

The Ruelle operator R can be viewed as an operator on any of the spaces

$$\mathbb{C}\left[z, z^{-1}\right] \subset C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset L^2(\mathbb{T}), \tag{2.61}$$

and its definition (2.60) and the definition of m_0 (2.44), provide that R maps any of these spaces into themselves.
Let $P[n, m]$, $n \leqslant m$, be the subspace of $\mathbb{C}[z, z^{-1}]$ which consists of trigonometric polynomials of the form $\sum_{k=n}^{m} b_k z^k$. Then

$$R(P[n, m]) \subset P\left[-\frac{K-n}{2}, \frac{m+K}{2}\right], \tag{2.62}$$

where $[x]$ is the largest integer $\leqslant x$. By repeated application of R, any $P[n, m]$ will finally be mapped into $P[-K, K,]$, and we have that all spaces

$$P[-K, K] \subset \mathbb{C}\left[z, z^{-1}\right] \subset C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset L^2(\mathbb{T}), \tag{2.63}$$

are invariant under R.
The authors of [**9**], O. Bratteli and P.E.T. Jorgensen, established the following theorem on the convergence of the cascade algorithm:

Theorem 2.2.7. *Let $a_0, \cdots, a_K$ be complex numbers such that (2.42) and (2.43) hold, and let ϕ be the associated scaling function defined in (2.48). The Ruelle operator R is identified with its restriction on $P[-K, K]$ (or $P[n, m]$, for any $n \leqslant -K$ and $m \geqslant K$). Then the following conditions are equivalent:*

(i) *1 is a simple eigenvalue of R and all other eigenvalues λ of R have $|\lambda| < 1$.*

(ii) *If $\psi^{(0)} \in L^2(\mathbb{R})$ has compact support, $\left\{\psi^{(0)}(\cdot - k)\right\}_{k=-\infty}^{\infty}$ is an orthonormal set and $\widehat{\psi^{(0)}}(0) = 1$, then*

$$\lim_{n \to \infty} \left\|\phi - M^n \psi^{(0)}\right\|_2 = 0. \tag{2.64}$$

Proof. See [**9**]. □

3 Harmonic analysis for the Ruelle operator on hypergroups: the polynomial case

In Chapter 2, we both encountered Ruelle's theorem in the classical dynamical systems sense and a Ruelle operator in harmonic analysis on $\mathbb{T}$ which was defined through a preimage given by a polynomial function. In this chapter, we will define a transfer operator which will be based on an orthogonal polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ on the interval $[-1,1]$ by

$$R_{(m_N,P_N)}f(y) = \frac{1}{N}\sum_{P_N(x)=y} m_N(x) f(x),$$

with the weight function $m_N(x) = \sum_{k=0}^{\infty} b_k P_k(x) h(k)$. In our case, the underlying orthogonal polynomials will be the Chebyshev polynomials of the first kind. We will use the orthogonal polynomial theory provided in Chapter 1 as well as the theory of homogeneous Banach spaces. The orthogonal polynomials which we will consider generate polynomial hypergroups.

3.1 Chebyshev polynomials of the first kind: the unweighted Ruelle operator

The Ruelle operator will be defined by the preimages of the Chebyshev polynomials of the first kind, $\{T_n(x)\}_{n=0}^{\infty}$, (see Section 1.1 in Example 1.1.2 (1)) and act on function spaces which are determined by the hypergroup structure induced by the Chebyshev polynomials of the first kind.

We define the Ruelle operator $R = R_{(m_N,T_N)}$ and the corresponding weight function m_N depending on N as follows:

$$\left(R_{(m_N,T_N)}f\right)(y) = \sum_{T_N(x)=y} m_N(x) f(x), \tag{3.1}$$

and

$$m_N(x) = \sum_{k=0}^{\infty} b_k T_k(x) h(k). \tag{3.2}$$

The subscript (m_N, T_N) indicates that the weight function m_N, which depends on the coefficients b_k, and the Nth Chebyshev polynomial $T_N(x)$ are used.
First, we fix $m_N = 1$ and calculate the Ruelle operator acting on T_n for arbitrary N. For some $y \in S := [-1,1]$ the preimage of T_N is $T_N^{-1}(y) = \left\{x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)}\right\}$, where the

superscript indicates the corresponding Chebyshev polynomial $T_N(y)$. We will omit the superscript if the notation is clear. We start with the case that N is even. Then we get for $x_i \in T_N^{-1}(y)$, $i = 1, \ldots, N$ and $0 \leqslant r \leqslant \frac{\pi}{2}$ which is uniquely determined by y,

$$\begin{aligned} x_1 &= \cos(r), \\ x_i &= \cos\left(\frac{i\pi}{N} - r\right), \qquad i = 2, 4, \ldots, N-2, \\ x_{i+1} &= \cos\left(\frac{i\pi}{N} + r\right), \\ x_N &= \cos\left(\frac{N\pi}{N} - r\right) = \cos(\pi - r). \end{aligned} \tag{3.3}$$

Then, by Theorem 1.1.20 (ii), we have for $n \in \mathbb{N}_0$ that

$$T_n\left(x_{\frac{N}{2}+i}\right) = (-1)^n T_n\left(x_{\frac{N}{2}-(i-1)}\right), \qquad i = 1, \ldots, \frac{N}{2}. \tag{3.4}$$

Furthermore, we have

$$\begin{aligned} T_n(x_1) &= \cos(nr), \\ T_n(x_i) &= \cos\left(n\left(\frac{i\pi}{N} - r\right)\right) \\ &= \cos\left(n\left(\frac{i\pi}{N}\right)\right)\cos(nr) + \sin\left(n\left(\frac{i\pi}{N}\right)\right)\sin(nr), \qquad i = 2, 4, \ldots, N-2, \\ T_n(x_{i+1}) &= \cos\left(n\left(\frac{i\pi}{N} + r\right)\right) \\ &= \cos\left(n\left(\frac{i\pi}{N}\right)\right)\cos(nr) - \sin\left(n\left(\frac{i\pi}{N}\right)\right)\sin(nr), \\ T_n(x_N) &= \cos\big(n(\pi - r)\big) = \cos(n\pi)\cos(nr) + \sin(n\pi)\sin(nr). \end{aligned} \tag{3.5}$$

Hence we get

$$\begin{aligned} T_n(x_i) + T_n(x_{i+1}) &= 2\cos\left(n\frac{i\pi}{N}\right)\cos(nr), \qquad i = 2, 4, \ldots, N-2, \\ T_n(x_1) + T_n(x_N) &= (1 + \cos(n\pi))\cos(nr) = \begin{cases} 2\cos(nr), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}. \end{aligned} \tag{3.6}$$

Now, the Ruelle operator applied to T_n is calculated as follows,

$$R_{(1,T_N)}(T_n)(y) = \frac{1}{N}\sum_{i=1}^{N} T_n(x_i) \overset{(3.4)}{=} \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2} 2T_n(x_i), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$
\stackrel{(3.6)}{=} \begin{cases} \frac{1}{N}\Big(\sum_{i\in\{2,4,\dots,\frac{N}{2}-2\}} 4\cos\left(n\frac{i\pi}{N}\right)\cos(nr) \\ \quad +2\cos\left(n\frac{N\pi}{2N}\right)\cos(nr)+2\cos(nr)\Big), & \frac{N}{2}, n \text{ even} \\ \frac{1}{N}\Big(\sum_{i\in\{2,4,\dots,\frac{N}{2}-1\}} 4\cos\left(n\frac{i\pi}{N}\right)\cos(nr)+2\cos(nr)\Big), & \frac{N}{2} \text{ odd}, n \text{ even} \\ 0, & n \text{ odd} \end{cases}
$$

$$
= \begin{cases} \frac{1}{N}\left(\left[\sum_{i\in\{2,4,\dots,\frac{N}{2}-2\}} 4\cos\left(n\frac{i\pi}{N}\right)+2\cos\left(n\frac{N\pi}{N}\right)+2\right]\cos(nr)\right), & \frac{N}{2}, n \text{ even} \\ \frac{1}{N}\left(\left[\sum_{i\in\{2,4,\dots,\frac{N}{2}-1\}} 4\cos\left(n\frac{i\pi}{N}\right)+2\right]\cos(nr)\right), & \frac{N}{2} \text{ odd}, n \text{ even} \\ 0, & n \text{ odd} \end{cases}
$$

$$
= \begin{cases} \cos(nr) = T_n(x_1), & n = Nl, l \in \mathbb{N}_0 \\ 0, & \text{else} \end{cases}.
$$

For the case that N is odd, we have

$$
\begin{aligned}
x_1 &= \cos(r), \\
x_i &= \cos\left(\frac{i\pi}{N}-r\right), \qquad i = 2,4,\dots N-1, \\
x_{i+1} &= \cos\left(\frac{i\pi}{N}+r\right),
\end{aligned} \tag{3.7}
$$

and thus (3.6) can be rewritten as

$$
\begin{aligned}
T_n(x_1) &= \cos(nr), \\
T_n(x_i) &= \cos\left(n\left(\frac{i\pi}{N}-r\right)\right) \\
&= \cos\left(n\left(\frac{i\pi}{N}\right)\right)\cos(nr)+\sin\left(n\left(\frac{i\pi}{N}\right)\right)\sin(nr), \qquad i = 2,4,\dots,N-1, \\
T_n(x_{i+1}) &= \cos\left(n\left(\frac{i\pi}{N}+r\right)\right) \\
&= \cos\left(n\left(\frac{i\pi}{N}\right)\right)\cos(nr)-\sin\left(n\left(\frac{i\pi}{N}\right)\right)\sin(nr).
\end{aligned} \tag{3.8}
$$

Thus we get

$$
\begin{aligned}
R_{(1,T_N)}(T_n)(y) &= \frac{1}{N}\sum_{i=1}^{N} T_n(x_i) \stackrel{(3.8)}{=} \frac{1}{N}\left(\sum_{i\in\{2,4,\dots,N-1\}} 2\cos\left(n\frac{i\pi}{N}\right)\cos(nr)+\cos(nr)\right) \\
&= \frac{1}{N}\left(\left[\sum_{i\in\{2,4,\dots,N-1\}} 2\cos\left(n\frac{i\pi}{N}\right)+1\right]\cos(nr)\right)
\end{aligned}
$$

$$= \begin{cases} \cos(nr), & n = Nl, l \in \mathbb{N}_0 \\ 0, & \text{else} \end{cases}.$$

By Lemma 1.1.40, which we will use throughout this section, we have for the Chebyshev polynomials of the first kind $T_{kl}(x) = T_k(T_l(x))$, $k, l \in \mathbb{N}_0$, thus we have proven the following lemma:

Lemma 3.1.1. *The unweighted Ruelle operator $R_{(1,T_N)}$ satisfies*

$$R_{(1,T_N)}(T_n) = \begin{cases} T_{\frac{n}{N}}, & n = Nl, l \in \mathbb{N}_0 \\ 0, & else \end{cases}. \tag{3.9}$$

Let $f : S \to \mathbb{C}$ be a function. Now, we study the action of $R_{(1,T_N)}$ on various function spaces defined on S. Then the preimages $x_i \in T_N(y)$, $i = 1, \ldots, N$, are real numbers in $[-1, 1]$ (see also Chapter 4). For $N = 2^n$ they can easily be expressed in terms of y by

$$x_{1,2}^{(2)} = \pm\sqrt{\frac{y+1}{2}}, \qquad x_{1,\ldots,2^n}^{(2^n)} = \pm\sqrt{\frac{x_{1,\ldots,2^{n-1}}^{(2^{n-1})}+1}{2}}. \tag{3.10}$$

Since $R_{(1,T_N)}(f)(y) = \frac{1}{N}\sum_{i=1}^N f\left(x_i^{(N)}\right)$ for all $y \in S$, $\|R_{(1,T_N)}\| \leqslant 1$ for the supremum norm which we have from above and $R_{(1,T_N)}\mathbf{1} = 1$, we get the following lemma:

Lemma 3.1.2. *The Ruelle operator $R_{(1,T_N)}$ is a bounded linear operator on $C(S)$, where $C(S)$ is equipped with the supremum norm, and the operator norm satisfies $\|R_{(1,T_N)}\| = 1$.*

We will investigate the action of $R_{(1,T_N)}$ on the Wiener algebra $A(S)$ (defined in Example 1.2.31) with respect to $\{T_n(x)\}_{n=0}^\infty$. We recall that $A(S)$ is a subspace of $C(S)$,

$$A(S) = \{f \in C(S) : \sum_{n=0}^{\infty} |\check{f}(n)| h(n) < \infty\}$$

and that it is a Banach space with the norm $\|f\|_{A(S)} = \sum_{n=0}^\infty |\check{f}(n)| h(n)$. For $f \in A(S)$ we have $f(x) = \sum_{n=0}^\infty \check{f}(n) T_n(x) h(n)$.

Lemma 3.1.3.

$$R_{(1,T_i)} \circ R_{(1,T_j)} = R_{(1,T_{ij})}, \qquad i, j \in \mathbb{N}. \tag{3.11}$$

Proof. By Lemma 1.1.40, we have for the preimage $T_i^{-1} \circ T_j^{-1}(y) = T_{ij}^{-1}(y)$. Thus

$$\begin{aligned} R_{(1,T_i)} \circ R_{(1,T_j)}(f)(y) &= R_{(1,T_i)}\left(\frac{1}{j}\sum_{k=1}^{j} f\left(x_k^{(j)}\right)\right) \\ &= \frac{1}{j} R_{(1,T_i)}\left(\sum_{k=1}^{j} f\left(x_k^{(j)}\right)\right) = \frac{1}{ij}\sum_{k=1}^{ij} f\left(x_k^{(ij)}\right) \\ &= R_{(1,T_{ij})}(f)(y). \end{aligned}$$

□

Lemma 3.1.4. *Let $f, g \in C(S)$. Then for $d\pi(y) = \frac{c}{\sqrt{1-y^2}}dy$*

$$\int_{-1}^{1} R_{(1,T_N)} f(y)\, g(y)\, d\pi(y) = \int_{-1}^{1} f(x)\, g(T_N(x))\, d\pi(x).$$

Proof. We have that $T_n'(x) = nU_{n-1}(x)$ (see (1.12)) and the identity $(T_n(x))^2 - (x^2-1)(U_{n-1}(x))^2 = 1$ (see (1.13)) which directly yields that $\sqrt{1-y^2} = (1-x^2)(U_{n-1}(x))^2$. Thus

$$\begin{aligned}
\int_{-1}^{1} R_{(1,T_N)} f(y)\, g(y)\, d\pi(y) &= \frac{1}{N}\int_{-1}^{1} \sum_{i=1}^{N} f\left(x_i^{(N)}\right) g(y)\, d\pi(y) \\
&= \frac{1}{N}\int_{0}^{1} f(x)\, g(T_N(x)) \frac{NU_{N-1}(x)}{U_{N-1}(x)\sqrt{1-x^2}} dx \\
&\quad + \frac{1}{N}\int_{-1}^{0} f(x)\, g(T_N(x)) \frac{NU_{N-1}(x)}{U_{N-1}(x)\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} f(x)\, g(T_N(x))\, d\pi(x).
\end{aligned}$$

□

Proposition 3.1.5. *For $f \in C(S)$ and $n \in \mathbb{N}_0$ we have, $\widecheck{R_{(1,T_N)} f}(n) = \check{f}(Nn)$.*

Then it is obvious that $R_{(1,T_N)}$ is also bounded on $A(S)$. In fact, if $f \in A(S)$, then $\widecheck{R_{(1,T_N)} f}(n) = \check{f}(Nn)$, and hence $\|R_{(1,T_N)} f\|_{A(S)} = \sum_{n=0}^{\infty} |\check{f}(Nn)|\, h(n) = \sum_{n=0}^{\infty} |\check{f}(Nn)|\, h(Nn) \leqslant \sum_{n=0}^{\infty} |\check{f}(n)|\, h(n) = \|f\|_{A(S)}$. Since $R_{(1,T_N)}\mathbf{1} = 1$ and $\|\mathbf{1}\|_{A(S)} = 1$, we get $\|R_{(1,T_N)}\|_{A(S)} = 1$.

Theorem 3.1.6. *We have $R_{(1,T_N)} \in B(A(S))$ and $\|R_{(1,T_N)}\| = 1$. Moreover, $R_{(1,T_N)} f(y) = \sum_{n=0}^{\infty} \check{f}(Nn)\, T_n(y)\, h(n)$ for each $f \in A(S)$.*

If $f : S \to \mathbb{C}$ is a Borel measurable function, then $R_{(1,T_N)} f$ is Borel measurable, too.

Lemma 3.1.7. *Let $f \in L^p(S, \pi)$. For $1 \leqslant p < \infty$, we have*

$$\int_{-1}^{1} |R_{(1,T_N)} f(y)|^p d\pi(y) = \int_{-1}^{1} |f(x)|^p d\pi(x).$$

Proof. Splitting $S = [-1,1]$ into $[-1,0]$ and $[0,1]$ the substitution of Lemma 3.1.4 directly yields equality. □

We proceed with studying the p-versions of $A(S)$ (see Example 1.2.31 (2)). For $1 \leqslant p < \infty$, we define

$$A^p(S) = \{f \in L^1(S,\pi) : \check{f} \in \ell^p(h)\},$$

and $\|f\|^p = \|f\|_1 + \|\check{f}\|_p$. With this norm $A^p(S)$ is a Banach space. Note that the norm of $A(S)$ and $A^1(S)$ differ by the summand $\|f\|_1$. With Lemma 3.1.7 and Proposition 3.1.5 we get

$$\|R_{(1,T_N)} f\|^p = \|R_{(1,T_N)} f\|_1 + \|\widecheck{R_{(1,T_N)} f}\|_p \leqslant \|f\|_1 + \|\check{f}\|_p = \|f\|^p, \qquad f \in A^p(S).$$

Theorem 3.1.8. *We have for* $1 \leqslant p < \infty$, $R_{(1,T_N)} \in B(A^p(S))$ *and* $\|R_{(1,T_N)}\| = 1$.

Lemma 3.1.7 also implies:

Theorem 3.1.9. *We have* $R_{(1,T_N)} \in B(L^p(S,\pi))$ *and* $\|R_{(1,T_N)}\| = 1$, $1 \leqslant p < \infty$.

It is interesting to consider the action of $R_{(1,T_N)}$ on the Hilbert space $L^2(S,\pi)$. Define the operator $E \in B(L^2(S,\pi))$ by setting

$$E(g) = Eg = g \circ T_N, \qquad g \in L^2(S,\pi).$$

By Lemma 3.1.4 (with $f = 1$) we get $\int_{-1}^{1} |g(y)|\,d\pi(y) = \int_{-1}^{1} |Eg(x)|\,d\pi(x)$ for all $g \in C(S)$. Since $C(S)$ is dense in $L^1(S,\pi)$, we see that $E \in B(L^1(S,\pi))$ and $\|E\| = 1$. Applying Lemma 3.1.4, we conclude:

Proposition 3.1.10. *Let* $f, g \in C(S)$. *Then*

$$\int_{-1}^{1} R_{(1,T_N)} f(y)\,\overline{g(y)}\,d\pi(y) = \int_{-1}^{1} f(x)\,\overline{Eg(x)}\,d\pi(x).$$

In particular, E *is the adjoint operator of* $R_{(1,T_N)}$ *on* $L^2(S,\pi)$.

For each $n \in \mathbb{N}_0$ we have

$$R_{(1,T_N)} \circ R^*_{(1,T_N)}(T_n) = R_{(1,T_N)} \circ E(T_n) = R_{(1,T_N)}(T_{Nn}) = T_n.$$

Since the linear span of $\{T_n : n \in \mathbb{N}_0\}$ is dense in $L^2(S,\pi)$, it follows that

$$R_{(1,T_N)} \circ R^*_{(1,T_N)} = R_{(1,T_N)} \circ E = \text{id}.$$

If $n \in \mathbb{N}_0$ is a multiple of N, then $R^*_{(1,T_N)} \circ R_{(1,T_N)}(T_n) = E\left(T_{\frac{n}{N}}\right) = T_n$. But, if $n \in \mathbb{N}_0$ is not a multiple of N, then $R^*_{(1,T_N)} \circ R_{(1,T_N)}(T_n) = 0$.

Denoting $L^2_0(S,\pi) = \{f \in L^2(S,\pi) : \check{f}(n) = 0 \quad \forall n \in N\mathbb{N}_0\}$ and $L^2_1(S,\pi) = \{f \in L^2(S,\pi) : \check{f}(n) = 0 \quad \forall n \in \mathbb{N}_0 \backslash N\mathbb{N}\}$, we have $L^2(S,\pi) = L^2_0(S,\pi) \oplus L^2_1(S,\pi)$. Furthermore, $\operatorname{Ker}\left(R^*_{(1,T_N)} \circ R_{(1,T_N)}\right) = L^2_0(S,\pi)$ and $\operatorname{Im}\left(R^*_{(1,T_N)} \circ R_{(1,T_N)}\right) = L^2_1(S,\pi)$.

Proposition 3.1.11. $R_{(1,T_N)} \in B(L^2(S,\pi))$. *Then*

(i) $R^*_{(1,T_N)} = E$,

(ii) $E \circ R_{(1,T_N)} = \text{id}$,

(iii) $E \circ R_{(1,T_N)}$ *is a partial isometry with initial space* $Ker\left(E \circ R_{(1,T_N)}\right)^{\perp} = L^2_1(S,\pi)$ *and* $Ker\left(E \circ R_{(1,T_N)}\right) = L^2_0(S,\pi)$. *Moreover,* $E \circ R_{(1,T_N)}$ *is the orthogonal projection onto* $L^2_1(S,\pi)$,

(iv) $\text{id} - E \circ R_{(1,T_N)} = R_{(1,T_N)} \circ E - E \circ R_{(1,T_N)}$ *is the orthogonal projection onto* $L^2_0(S,\pi)$.

Remark 3.1.12. (1) By Proposition 3.1.11 (iv), we see that $R^*_{(1,T_N)} = E$ is hyponormal. (An operator T on a Hilbert space is called *hyponormal* if $T^*T - TT^*$ is positive. Each hyponormal operator is *normaloid*, i.e. $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$.) Hence $R_{(1,T_N)}$ is normaloid. In fact, $\|R^n_{(1,T_N)}\| = \|\left(R^n_{(1,T_N)}\right)^*\| = \|\left(R^*_{(1,T_N)}\right)^n\| = \|R^*_{(1,T_N)}\|^n = \|R_{(1,T_N)}\|$.

(2) $E \circ R_{(1,T_N)}$ is a Markovian projection, see Definition 2.1.20.

The following theorem follows directly from the previous remark and $\|R_{(1,T_N)}\| = 1$.

Theorem 3.1.13. *For the spectral radius of $R_{(1,T_N)} \in B\left(L^2\left(S,\pi\right)\right)$, we have $\rho\left(R_{(1,T_N)}\right) = 1$. And we have that 0 is an eigenvalue to the Chebyshev polynomials $\{T_n\left(x\right)\}_{n=0}^{\infty}$ if n is not a multiple of N and 1 is an eigenvalue to the Chebyshev polynomials if n is a multiple of N.*

3.2 Chebyshev polynomials of the first kind: the weighted Ruelle operator

Now, we use a weight function $m_N\left(x\right)$ as defined in (3.2) and state for the Ruelle operator $R_{(m_N,T_N)}$ with general weight $m_N\left(x\right)$ the following two lemmas corresponding to Lemma 3.1.1:

Lemma 3.2.1. *If we assume for the Ruelle operator $R_{(m_N,T_N)}$ with weight $m_N\left(y\right) = \sum_{k=0}^{\infty} b_k T_k\left(y\right) h\left(k\right)$ that $R_{(m_N,T_N)}\left(\mathbf{1}\right) = 1$, then*

$$b_0 = 1, \qquad b_{Nl} = 0, \qquad l \in \mathbb{N}.$$

Proof. First, we assume that N is even.

$$\begin{aligned}
R_{(m_N,T_N)}\left(\mathbf{1}\right) &= 1 = \frac{1}{N}\sum_{i=1}^{N} m_N\left(x_i\right) = \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{\infty} b_k T_k\left(x_i\right) h\left(k\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} b_k \left(\sum_{i=1}^{N} T_k\left(x_i\right)\right) h\left(k\right)\right) \\
&\overset{(3.4)}{=} \frac{1}{N}\left(\sum_{k=0}^{\infty} b_k \left(\sum_{i=1}^{N/2} T_k\left(x_i\right) + \left(-1\right)^k T_k\left(x_i\right)\right) h\left(k\right)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} b_{2k} \left(\sum_{i=1}^{N/2} 2T_{2k}\left(x_i\right)\right) h\left(2k\right)\right)
\end{aligned}$$

$$= \begin{cases} \frac{1}{N}\left(\sum_{k=0}^{\infty} b_{2k}\left(\left[\sum_{i\in\{2,4,\ldots,N/2-2\}} 4\cos\left(2k\frac{i\pi}{N}\right)\right.\right.\right. \\ \quad \left.\left.\left. +2\cos\left(\frac{2kN\pi}{2N}\right)+2\right]\cos\left(2kr\right)\right)h\left(2k\right)\right), & \frac{N}{2}\text{ even} \\ \frac{1}{N}\left(\sum_{k=0}^{\infty} b_{2k}\left(\left[\sum_{i\in\{2,4,\ldots,N/2-1\}} 4\cos\left(2k\frac{i\pi}{N}\right)+2\right]\right.\right. \\ \quad \left.\left.\cos\left(2kr\right)\right)h\left(2k\right)\right), & \frac{N}{2}\text{ odd} \end{cases}.$$

Now, let N be odd, then

$$R_{(m_N,T_N)}\left(\mathbf{1}\right) \overset{(3.6)}{=} \frac{1}{N}\left(\sum_{k=0}^{\infty} b_k\left(\left[\sum_{i\in\{2,4,\ldots,N-1\}} 2\cos\left(k\frac{i\pi}{N}\right)+1\right]\cos\left(kr\right)\right)h\left(k\right)\right).$$

Since

$$\sum_{i\in\{2,4,\ldots,N/2-2\}} 4\cos\left(2k\frac{i\pi}{N}\right)+2\cos\left(\frac{2kN\pi}{2N}\right)+2 = \begin{cases} N, & k=Nl, l\in\mathbb{N}_0 \\ 0, & \text{else} \end{cases},$$

and

$$\sum_{i\in\{2,4,\ldots,N/2-2\}} 4\cos\left(2k\frac{i\pi}{N}\right)+2 = \begin{cases} N, & k=Nl, l\in\mathbb{N}_0 \\ 0, & \text{else} \end{cases},$$

respectively, we get $b_0 = 1$, $b_{Nl} = 0$ for $l \in \mathbb{N}$. □

Lemma 3.2.2. *The weighted Ruelle operator $R_{(m_N,T_N)}$ satisfies*

$$R_{(m_N,T_N)}\left(T_n\right)$$
$$= \begin{cases} T_{\frac{n}{N}}, & n=Nl, l\in\mathbb{N}_0 \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\left(2+\sum_{k=1}^{\infty} 2b_{2k}T_{2k}\left(x_i\right)\right)h\left(2k\right)\right]T_n\left(x_i\right)\right), & n,N\text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\sum_{k=1}^{\infty} 2b_{2k-1}T_{2k-1}\left(x_i\right)h\left(2k-1\right)\right]T_n\left(x_i\right)\right), & n\text{ odd}, N\text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N}\left[\left(1+\sum_{j=1}^{N-1}\sum_{k=1}^{\infty} b_{k+j}T_{k+j}\left(x_i\right)\right)h\left(k+j\right)\right]T_n\left(x_i\right)\right), & N\text{ odd}, n\in\mathbb{N} \end{cases}.$$

***Proof*.** The lemma follows directly from Lemma 3.2.1 and the previous section as we have

$$R_{(m_N,T_N)}\left(T_n\right)\left(y\right) = \frac{1}{N}\sum_{i=1}^{N} m_N\left(x_i\right)T_n\left(x_i\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{k=0}^{\infty} b_kT_k\left(x_i\right)h\left(k\right)\right)T_n\left(x_i\right)$$

and if N is even, then

$$R_{(m_N,T_N)}\left(T_n\right)\left(y\right) = \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\sum_{k=0}^{\infty} b_k\left(T_k\left(x_i\right)+\left(-1\right)^{n+k}T_k\left(x_i\right)\right)h\left(k\right)\right]T_n\left(x_i\right)\right)$$

$$= \begin{cases} T_n\left(x_1\right), & n = Nl, l \in \mathbb{N}_0 \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\left(2+\sum_{k=1}^{\infty} 2b_{2k}T_{2k}\left(x_i\right)\right)h\left(2k\right)\right]T_n\left(x_i\right)\right), & n \text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\sum_{k=1}^{\infty} 2b_{2k-1}T_{2k-1}\left(x_i\right)h\left(2k-1\right)\right]T_n\left(x_i\right)\right), & n \text{ odd} \end{cases}$$

and if N is odd, we get

$$\begin{aligned} & R_{(m_N,T_N)}\left(T_n\right)(y) \\ & = \begin{cases} T_n\left(x_1\right), & n = Nl, l \in \mathbb{N}_0 \\ \frac{1}{N}\left(\sum_{i=1}^{N}\left[\left(1+\sum_{j=1}^{N-1}\sum_{k=1}^{\infty} 2b_{k+j}T_{k+j}\left(x_i\right)\right)h\left(k+j\right)\right]T_n\left(x_i\right)\right), & \text{else} \end{cases}. \end{aligned}$$

□

Remark 3.2.3. With Lemma 3.2.1 and Lemma 3.2.2, we get that

$$R_{(m_2,T_2)}\left(T_n\right) = \begin{cases} T_{\frac{n}{2}}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

Thus for $N = 2$ the weighted case coincides with the unweighted case when the Ruelle operator acts on $\{T_n\}_{n=0}^{\infty}$.

Let $f : S \to \mathbb{C}$ again be a function in S. We will proceed as in the previous section and use the same notation.
Since

$$R_{(m_N,T_N)}\left(f\right)(y) = \frac{1}{N}\left(\sum_{i=1}^{N} m_N\left(x_i^{(N)}\right) f\left(x_i^{(N)}\right)\right)$$

for all $y \in S$, we have the following lemma.

Lemma 3.2.4. *The Ruelle operator $R_{(m_N,T_N)}$ is a bounded linear operator on $C(S)$, where $C(S)$ is equipped with the supremum norm and the operator norm satisfies $\|R_{(m_N,T_N)}\| = \|m_N\|$.*

The Lemmas 3.1.3 and 3.1.4 can be stated and proven analogously:

Lemma 3.2.5.

$$R_{(m_i,T_i)} \circ R_{(m_j,T_j)} = R_{(m_{ij},T_{ij})}, \qquad i,j \in \mathbb{N}_0.$$

Lemma 3.2.6. *Let $f, g \in C(S)$. Then for $d\pi(y) = \frac{c}{\sqrt{1-y^2}}dy$*

$$\int_{-1}^{1} R_{(m_N,T_N)}f(y)\, g(y)\, d\pi(y) = \int_{-1}^{1} m_N(x)\, f(x)\, g\left(T_N(x)\right) d\pi(x).$$

Proposition 3.2.7. *For $f \in C(S)$ and $n \in \mathbb{N}_0$ we have, $\widetilde{R_{(m_N,T_N)}f}(n) = \widetilde{m_N f}(Nn) = \widetilde{m_N} * \check{f}(Nn)$.*

Then, it is obvious that $R_{(m_N,T_N)}$ is also bounded on $A(S)$. In fact, if $f \in A(S)$, then $\widetilde{R_{(m_N,T_N)}f}(n) = \widetilde{m_N f}(Nn)$, and hence $\|R_{(m_N,T_N)}f\|_{A(S)} = \sum_{n=0}^{\infty} |\widetilde{m_N f}(Nn)|\, h(n) = \sum_{n=0}^{\infty} |\widetilde{m_N f}(Nn)|\, h(Nn) \leqslant \sum_{n=0}^{\infty} |\widetilde{m_N f}(n)| = \|m_N\|_{A(S)}\|f\|_{A(S)}$. Since $R_{(m_N,T_N)}\mathbf{1} = 1$ and $\|\mathbf{1}\|_{A(S)} = 1$, we get $\|R_{(m_N,T_N)}\|_{A(S)} = 1$.

Theorem 3.2.8. *We have* $R_{(m_N,T_N)} \in B(A(S))$ *and* $\|R_{(m_N,T_N)}\| = 1$.

If $f : S \to \mathbb{C}$ is a Borel measurable function, then $R_{(m_N,T_N)}f$ is Borel measureable, too.

Lemma 3.2.9. *Let* $f \in L^p(S,\pi)$. *For* $1 \leqslant p < \infty$, *we have*

$$\int_{-1}^{1} |R_{(m_N,T_N)}f(y)|^p d\pi(y) \leqslant \int_{-1}^{1} |m_N(x)|^p |f(x)|^p d\pi(x).$$

Proof. Splitting $[-1,1]$ into $[-1,0]$ and $[0,1]$ the substitution of Lemma 3.2.6 directly yields equality. □

We proceed with studying the p-versions of $A(S)$. With Lemma 3.2.9 and Proposition 3.2.7 we get

$$\|R_{(m_N,T_N)}f\|^p = \|R_{(m_N,T_N)}f\|_1 + \|\widetilde{R_{(m_N,T_N)}f}\|_p \leqslant \|m_N\|_1\|f\|_1 + \|\widetilde{m_N f}\|_p, \qquad f \in A^p(S).$$

Theorem 3.2.10. *We have for* $1 \leqslant p < \infty$, $R_{(m_N,T_N)} \in B(A^p(S))$.

Lemma 3.2.9 also implies:

Theorem 3.2.11. *We have* $R_{(m_N,T_N)} \in B(L^p(S,\pi))$ *and* $\|R_{(m_N,T_N)}\| = \|m_N\|$, $1 \leqslant p < \infty$.

As in the unweighted case, we again consider the action of $R_{(m_N,T_N)}$ on the Hilbert space $L^2(S,\pi)$ and define the adjoint operator $E \in B(L^2(S,\pi))$ by setting

$$E(g) = Eg = g \circ T_N \qquad \text{for } g \in L^2(S,\pi).$$

With Lemma 3.2.6 we get:

Proposition 3.2.12. *Let* $f, g \in C(S)$. *Then*

$$\int_{-1}^{1} R_{(m_N,T_N)}f(y)\overline{g(y)}d\pi(y) = \int_{-1}^{1} m_N(x) f(x) \overline{Eg(x)} d\pi(x).$$

Using the same argumentation as above, we have the following identities:

(i) $R_{(m_N,T_N)} \circ E(T_n) = R_{(m_N,T_N)}(T_{Nn}) = T_n$, $n \in \mathbb{N}_0$,

(ii) $R_{(m_N,T_N)} \circ E = \text{ id}$, $n \in \mathbb{N}_0$,

(iii) $E \circ R_{(m_N,T_N)}(T_n)$

$$= \begin{cases} T_n, & n = Nl, l \in \mathbb{N}_0 \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\left(2 + \sum_{k=1}^{\infty} 2b_{2k}T_{2k}(x_i)\right)h(2k)\right]T_{nN}(x_i)\right), & n, N \text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N/2}\left[\sum_{k=1}^{\infty} 2b_{2k-1}T_{2k-1}(x_i)\,h(2k-1)\right]T_{nN}(x_i)\right), & n \text{ odd}, N \text{ even} \\ \frac{1}{N}\left(\sum_{i=1}^{N}\left[\left(1 + \sum_{j=1}^{N-1}\sum_{k=1}^{\infty} b_{k+j}T_{k+j}(x_i)\right)h(k+j)\right]T_{nN}(x_i)\right), & N \text{ odd}, n \in \mathbb{N}_0 \end{cases}.$$

3.3 Generalized Chebyshev polynomials: the unweighted Ruelle operator

So far, the function spaces on S are determined by the hypergroup structure induced by the Chebyshev polynomials of the first kind $\{T_n(x)\}_{n=0}^{\infty}$. Now, we consider function spaces based on other orthogonal polynomial sequences $\{P_n(x)\}_{n=0}^{\infty}$, which also induce hypergroup structures on $\mathbb{N}_0$ (or even on S).

Let $\alpha > -1$, $\beta > -1$ and $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ be the Jacobi polynomials, which are orthogonal with respect to $d\pi_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$ on S normalized by $P_n^{(\alpha,\beta)}(1) = 1$. The generalized Chebyshev polynomials $\{T_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ are determined by

$$T_n^{(\alpha,\beta)}(x) = \begin{cases} P_k^{(\alpha,\beta)}(2x^2-1), & n = 2k, k \in \mathbb{N}_0 \\ xP_k^{(\alpha,\beta+1)}(2x^2-1), & n = 2k+1, k \in \mathbb{N}_0 \end{cases}.$$

The orthogonalization measure of $T_n^{(\alpha,\beta)}$ on S is $d\pi^T_{\alpha,\beta}(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}dx$. The $\{T_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ generate a polynomial hypergroup structure if $\alpha \geqslant \beta > -1$, $\alpha+\beta+1 > 0$ (see [**33**]). Since the polynomials $T_n^{(\alpha,\beta)}(x)$ are symmetric, we have:

Lemma 3.3.1. *Let N be even, then the unweighted Ruelle operator $R_{(1,T_N)}$ satisfies*

$$R_{(1,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2} 2P_k^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n = 2k \\ 0, & n = 2k+1 \end{cases}.$$

Moreover, if N is a multiple of 4, then

$$R_{(1,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \begin{cases} \frac{1}{N}\sum_{i=1}^{N/4} 4P_k^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n = 2k \\ 0, & n = 2k+1 \end{cases}.$$

Proof. If N is even, then

$$\begin{aligned}
&R_{(1,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) \\
&= \begin{cases} \frac{1}{N}\sum_{i=1}^{N} P_k^{(\alpha,\beta)}\left(T_2\left(x_i^{(N)}\right)\right), & n = 2k \\ \frac{1}{N}\sum_{i=1}^{N} x_i^{(N)} P_k^{(\alpha,\beta+1)}\left(T_2\left(x_i^{(N)}\right)\right), & n = 2k+1 \end{cases} \\
&= \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2} P_k^{(\alpha,\beta)}\left(T_2\left(x_i^{(N)}\right)\right) + P_k^{(\alpha,\beta)}\left(T_2\left(-x_i^{(N)}\right)\right), & n = 2k \\ \frac{1}{N}\sum_{i=1}^{N/2}\left(x_{N-(i-1)}^{(N)} + x_{N+i}^{(N)}\right) P_k^{(\alpha,\beta+1)}\left(T_2\left(x_i^{(N)}\right)\right), & n = 2k+1 \end{cases} \\
&= \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2} 2P_k^{(\alpha,\beta)}\left(T_2\left(x_i^{(N)}\right)\right) = \frac{1}{N}\sum_{i=1}^{N/2} 2P_k^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n = 2k \\ 0, & n = 2k+1 \end{cases}
\end{aligned}$$

If furthermore, $N/2$ is even, then:

$$R_{(1,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \begin{cases} \frac{1}{N}\sum_{i=1}^{N/4} 4P_k^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n = 2k \\ 0, & n = 2k+1 \end{cases}.$$

□

Since the generalized Chebyshev polynomials are defined via $T_2(x) = 2x^2 - 1$, we get for the special case $N = 2$ that

$$R_{(1,T_2)}\left(T_{2k}^{(\alpha,\beta)}\right)(y) = T_{2k}^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(T_2(x)) = P_k^{(\alpha,\beta)}(y). \tag{3.12}$$

The remainder of this section is restricted to $N = 2$. By (3.12) we have the following relation:

$$R_{(1,T_2)}\left(T_n^{(\alpha,\beta)}\right) = \begin{cases} P_{n/2}^{(\alpha,\beta)}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

The Haar weights of $T_n^{(\alpha,\beta)}(x)$ can be calculated as (see also Example 1.2.27 (4))

$$h_T^{(\alpha,\beta)}(n) = \begin{cases} \frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k(\alpha+1)_k}{k!(\alpha+\beta+1)(\beta+1)_k}, & n = 2k \\ \frac{(2k+\alpha+\beta+2)(\alpha+\beta+2)_k(\alpha+1)_k}{k!(\beta+1)_{k+1}}, & n = 2k+1 \end{cases}.$$

The Haar weights of the Jacobi polynomials, $h_P^{(\alpha,\beta)}(n)$, are defined in Example 1.2.27 (2). In particular, we have that $h_T^{(\alpha,\beta)}(2k) = h_P^{(\alpha,\beta)}(k)$.
The corresponding Wiener spaces are denoted by $A_T^{(\alpha,\beta)}(S)$ and $A_P^{(\alpha,\beta)}(S)$.

Lemma 3.3.2. *Let $f, g \in C(S)$. Then*

$$\int_{-1}^{1} R_{(1,T_2)}f(y)\, g(y)\, d\pi_{\alpha,\beta}^P(y) = \int_{-1}^{1} f(x)\, g\left(2x^2-1\right) d\pi_{\alpha,\beta}^T(x).$$

Proof. Applying Lemma 3.1.4 to f and $\tilde{g}(y) = cg(y)\sqrt{1-y^2}(1-y)^\alpha(1+y)^\beta$ yields the stated equality, where c is an appropriate constant. □

Proposition 3.3.3. *For $f \in C(S)$ and $n \in \mathbb{N}_0$, we denote $\mathcal{F}_{\alpha,\beta}^P f(n) = \int_{-1}^1 f(y)\, P_n^{(\alpha,\beta)}(y)$ $d\pi_{\alpha,\beta}^P(y)$ and $\mathcal{F}_{\alpha,\beta}^T f(n) = \int_{-1}^1 f(x)\, T_n^{(\alpha,\beta)}(x)\, d\pi_{\alpha,\beta}^T(x)$. Then*

$$\mathcal{F}_{\alpha,\beta}^P\left(R_{(1,T_2)}f\right)(n) = \mathcal{F}_{\alpha,\beta}^T(f)(2n).$$

Proof. Let $g(y) = P_n^{(\alpha,\beta)}(y)$ and apply Lemma 3.3.2. □

Theorem 3.3.4. *We have $R_{(1,T_2)} \in B\left(A_T^{(\alpha,\beta)}(S), A_P^{(\alpha,\beta)}(S)\right)$ and $\|R_{(1,T_2)}\| = 1$. Moreover,*

$$R_{(1,T_2)}f(y) = \sum_{k=0}^{\infty} \mathcal{F}_{\alpha,\beta}^T(f)(2k)\, P_n^{(\alpha,\beta)}(y)\, h_P^{(\alpha,\beta)}(k) \qquad \forall f \in A_T^{(\alpha,\beta)}(S).$$

Proof. Let $f \in A_T^{(\alpha,\beta)}(S)$, that means $f(x) = \sum_{n=0}^{\infty} \mathcal{F}_{\alpha,\beta}^T(f)(n)\, T_n^{(\alpha,\beta)}(x)\, h_T^{(\alpha,\beta)}(n)$, where $\|f\|_{A_T^{(\alpha,\beta)}(S)} = \sum_{n=0}^{\infty} |\mathcal{F}_{\alpha,\beta}^T(f)(n)|\, h_T^{(\alpha,\beta)}(n) < \infty$. Proposition 3.3.3 implies that

$$\sum_{k=0}^{\infty} |\mathcal{F}_{\alpha,\beta}^P\left(R_{(1,T_2)}f\right)(k)|\, h_P^{(\alpha,\beta)}(k) = \sum_{k=0}^{\infty} |\mathcal{F}_{\alpha,\beta}^T(f)(2k)|\, h_T^{(\alpha,\beta)}(2k) \leqslant \|f\|_{A_T^{(\alpha,\beta)}(S)}.$$

Hence $R_{(1,T_2)}f \in A_P^{(\alpha,\beta)}(S)$ and $R_{(1,T_2)}f(y) = \sum_{k=0}^{\infty} \mathcal{F}_{\alpha,\beta}^T(f)(2k)\, P_k^{(\alpha,\beta)}(y)\, h_P^{(\alpha,\beta)}(k)$. Since $R_{(1,T_2)}\mathbf{1} = 1$, we also have that $\|R_{(1,T_2)}\| = 1$. □

Using Lemma 3.3.2, we obtain

Lemma 3.3.5. *Let $f \in L^p\left(S, \pi_{\alpha,\beta}^T\right)$. For $1 \leqslant p < \infty$, we get*

$$\int_{-1}^{1} |R_{(1,T_2)}f(y)|^p d\pi_{\alpha,\beta}^P(y) = \int_{-1}^{1} |f(x)|^p d\pi_{\alpha,\beta}^T(x).$$

We denote the corresponding p-versions of the Wiener spaces by $A_T^{p(\alpha,\beta)}(S)$ and $A_P^{p(\alpha,\beta)}(S)$, respectively. Then we have with Lemma 3.3.5 and Proposition 3.3.3 that

$$\|R_{(1,T_2)}f\|^p = \|R_{(1,T_2)}f\|_1 + \|\mathcal{F}_{\alpha,\beta}^P\left(R_{(1,T_2)}f\right)\|_p \leqslant \|f\|_1 + \|\mathcal{F}_{\alpha,\beta}^T(f)\|_p = \|f\|^p, \quad f \in A_T^{p(\alpha,\beta)}(S).$$

Theorem 3.3.6. *We have for $1 \leqslant p < \infty$, $R_{(1,T_2)} \in B\left(A_T^{p(\alpha,\beta)}(S), A_P^{p(\alpha,\beta)}(S)\right)$ and $\|R_{(1,T_N)}\| = 1$.*

Theorem 3.3.7. *We have $R_{(1,T_2)} \in B\left(L^p\left(S, \pi_{\alpha,\beta}^T\right), L^p\left(S, \pi_{\alpha,\beta}^P\right)\right)$ and $\|R_{(1,T_2)}\| = 1$, $1 \leqslant p < \infty$.*

Proof. Follows directly by Lemma 3.3.5. □

Now, we define the right inverse operator of $R_{(1,T_2)}$. Let $g \in L^p\left(S, \pi_{\alpha,\beta}^P\right)$. We put $E(g) = Eg = g \circ T_2$. By Lemma 3.3.2 we get for all $g \in C(S)$

$$\int_{-1}^{1} |g(y)|\, d\pi_{\alpha,\beta}^P(y) = \int_{-1}^{1} |Eg(x)|\, d\pi_{\alpha,\beta}^T(x).$$

Lemma 3.3.2 can be written as:

Proposition 3.3.8. *For $f, g \in C(S)$. Then*

$$\int_{-1}^{1} R_{(1,T_2)}f(y)\, \overline{g(y)}\, d\pi_{\alpha,\beta}^P(y) = \int_{-1}^{1} f(x)\, \overline{Eg(x)}\, d\pi_{\alpha,\beta}^T(x).$$

Remark 3.3.9. Whereas $R_{(1,T_2)}$ maps functions spaces determined by $\{T_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ into function spaces determined by $\{P_n^{(\alpha,\beta)(x)}\}_{n=0}^{\infty}$, the operator E maps into the inverse direction.

It is straightforward to show that

$$R_{(1,T_2)} \circ E\left(P_k^{(\alpha,\beta)}\right) = R_{(1,T_2)}\left(T_{2k}^{(\alpha,\beta)}\right) = P_k^{(\alpha,\beta)},$$

where

$$\begin{aligned} E \circ R_{(1,T_2)}\left(T_{2k}^{(\alpha,\beta)}\right) &= E\left(P_k^{(\alpha,\beta)}\right) = T_{2k}^{(\alpha,\beta)} \qquad \text{and} \\ E \circ R_{(1,T_2)}\left(T_{2k+1}^{(\alpha,\beta)}\right) &= E(0) = 0. \end{aligned}$$

3.4 Generalized Chebyshev polynomials: the weighted Ruelle operator

Now, we consider the Ruelle operator with the weight functions $m_N(x) = \sum_{k=0}^{\infty} b_k T_k(x) h(k)$ and $\tilde{m}_N(x) = \sum_{k=0}^{\infty} \tilde{b}_k T_k^{(\alpha,\beta)}(x) h(k)$ on function spaces determined by the generalized Chebyshev polynomials $\{T_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$. For symmetry reasons we will only consider even N.

Lemma 3.4.1. *Let N be even. We assume for the Ruelle operator $R_{(m_N,T_N)}$ and $R_{(\tilde{m}_N,T_N)}$, respectively, that $R_{(m_N,T_N)}(\mathbf{1}) = 1$ and $R_{(\tilde{m}_N,T_N)}(\mathbf{1}) = 1$. Then*

$$b_0 = 1, \qquad b_{Nk} = 0 \quad \text{and} \quad \tilde{b}_0 = 1 \qquad \tilde{b}_{2k} = 0, \qquad k \in \mathbb{N}.$$

Proof. In Lemma 3.2.1 we have already proven the assertion for b_0 and b_{Nk}, $k \in \mathbb{N}$.

$$\begin{aligned}
R_{(\tilde{m}_N,T_N)}(\mathbf{1}) = 1 &= \frac{1}{N}\sum_{i=1}^{N} \tilde{m}_N\left(x_i^{(N)}\right) = \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{\infty} \tilde{b}_k T_k^{(\alpha,\beta)}\left(x_i^{(N)}\right) h(k) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} \tilde{b}_k \left(\sum_{i=1}^{N} T_k^{(\alpha,\beta)}\left(x_i^{(N)}\right)\right) h(k)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} \tilde{b}_{2k} \left(\sum_{i=1}^{N} T_{2k}^{(\alpha,\beta)}\left(x_i^{(N)}\right)\right) h(2k) + \sum_{k=0}^{\infty} \tilde{b}_{2k+1} \left(\sum_{i=1}^{N} T_{2k+1}^{(\alpha,\beta)}\left(x_i^{(N)}\right)\right) h(2k+1)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} \tilde{b}_{2k} \left(\sum_{i=1}^{N} P_{2k}^{(\alpha,\beta)}\left(T_2\left(x_i^{(N)}\right)\right)\right) h(2k)\right. \\
&\quad \left. + \sum_{k=0}^{\infty} \tilde{b}_{2k+1} \left(\sum_{i=1}^{N} x_i^{(N)} P_{2k+1}^{(\alpha,\beta+1)}\left(T_2\left(x_i^{(N)}\right)\right)\right) h(2k+1)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} \tilde{b}_{2k} \left(\sum_{i=1}^{N/2} 2P_{2k}^{(\alpha,\beta)}\left(x_i^{(N/2)}\right)\right) h(2k)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{\infty} \tilde{b}_{2k} \left(\sum_{i=2}^{N/2} 2P_{2k}^{(\alpha,\beta)}\left(\cos\left(\frac{i\pi}{N/2} - r\right)\right) + 2P_{2k}^{(\alpha,\beta)}\left(\cos(r)\right)\right) h(2k)\right),
\end{aligned}$$

thus we get that $\tilde{b}_0 = 1$ and $\tilde{b}_{2k} = 0$, $k \in \mathbb{N}$. □

Lemma 3.4.2. *Let N be even. Then we have for the weighted Ruelle operator $R_{(m_N,T_N)}$ and $R_{(\tilde{m}_N,T_N)}$, respectively,*

$$R_{(m_N,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y)$$

$$= \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2}\left[\sum_{k=0}^{\infty} 2b_{2k}\left(T_{2k}\left(x_i^{(N)}\right)\right)h\left(2k\right)\right]P_m^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n=2m \\ 0, & n=2m+1 \end{cases}$$

and

$$R_{(\tilde{m}_N,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2} 2P_m^{(\alpha,\beta)}\left(x_i^{N/2}\right), & n=2m \\ 0, & n=2m+1 \end{cases}$$

which coincides with the unweighted case.

Proof. We have

$$R_{(m_N,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \frac{1}{N}\sum_{i=1}^{N} m_N\left(x_i^{(N)}\right)T_n^{(\alpha,\beta)}\left(x_i^{(N)}\right)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=0}^{\infty} b_k T_k\left(x_i^{(N)}\right)h\left(k\right)\right]T_n^{(\alpha,\beta)}\left(x_i^{(N)}\right)$$

$$= \begin{cases} \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=0}^{\infty} b_k T_k\left(x_i^{(N)}\right)h\left(k\right)\right]P_m^{(\alpha,\beta)}\left(T\left(x_i^{(N)}\right)\right), & n=2m \\ \frac{1}{N}\sum_{i=1}^{N} x_i^{(N)}\left[\sum_{k=0}^{\infty} b_k T_k\left(x_i^{(N)}\right)h\left(k\right)\right]P_m^{(\alpha,\beta+1)}\left(T\left(x_i^{(N)}\right)\right), & n=2m+1 \end{cases}$$

$$= \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2}\left[\sum_{k=0}^{\infty} b_k\left(T_k\left(x_i^{(N)}\right)+\left(-1\right)^k T_k\left(-x_i^{(N)}\right)\right)h\left(k\right)\right] \\ \quad P_m^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n=2m \\ \frac{1}{N}\sum_{i=1}^{N/2}\left(x_{N-(i-1)}^{(N)}+x_{N+i}^{(N)}\right)\left[\sum_{k=0}^{\infty} b_k\left(T_k\left(x_i^{(N)}\right)\right.\right. \\ \quad \left.\left.+\left(-1\right)^k T_k\left(-x_i^{(N)}\right)\right)h\left(k\right)\right]P_m^{(\alpha,\beta+1)}\left(x_i^{(N/2)}\right), & n=2m+1 \end{cases}$$

$$= \begin{cases} \frac{1}{N}\sum_{i=1}^{N/2}\left[\sum_{k=0}^{\infty} 2b_{2k}\left(T_{2k}\left(x_i^{(N)}\right)\right)h\left(2k\right)\right]P_m^{(\alpha,\beta)}\left(x_i^{(N/2)}\right), & n=2m \\ 0, & n=2m+1 \end{cases}.$$

and

$$R_{(\tilde{m}_N,T_N)}\left(T_n^{(\alpha,\beta)}\right)(y) = \frac{1}{N}\sum_{i=1}^{N}\tilde{m}_N\left(x_i^{(N)}\right)T_n^{(\alpha,\beta)}\left(x_i^{(N)}\right)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=0}^{\infty}\tilde{b}_k T_k^{(\alpha,\beta)}\left(x_i^{(N)}\right)h\left(k\right)\right]T_n^{(\alpha,\beta)}\left(x_i^{(N)}\right)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=0}^{\infty}\tilde{b}_{2k}T_{2k}^{(\alpha,\beta)}\left(x_i^{(N)}\right)h\left(2k\right)+\sum_{k=0}^{\infty}\tilde{b}_{2k+1}T_{2k+1}^{(\alpha,\beta)}\left(x_i^{(N)}\right)h\left(2k+1\right)\right]T_n^{(\alpha,\beta)}\left(x_i^{(N)}\right)$$

$$= \begin{cases} \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=0}^{\infty}\tilde{b}_{2k}P_{2k}^{(\alpha,\beta)}\left(T_2\left(x_i^{(N)}\right)\right)h(2k)\right]P_m^{(\alpha,\beta)}\left(T\left(x_i^{(N)}\right)\right), & n=2m \\ \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{k=0}^{\infty}\tilde{b}_{2k}P_{2k}^{(\alpha,\beta)}\left(T_2\left(x_i^{(N)}\right)\right)h(2k)\right] & \\ \quad x_i^{(N)}P_m^{(\alpha,\beta+1)}\left(T\left(x_i^{(N)}\right)\right), & n=2m+1 \end{cases}$$

$$= \begin{cases} \frac{1}{N}\sum_{i=1}^{N}2P_m^{(\alpha,\beta)}\left(T_2\left(x_i^N\right)\right) = \frac{1}{N}\sum_{i=1}^{N/2}2P_m^{(\alpha,\beta)}\left(x_i^{N/2}\right), & n=2m \\ 0, & n=2m+1 \end{cases}.$$

□

Remark 3.4.3. With Lemma 3.4.2 and Lemma 3.4.1, we get that

$$R_{(m_2,T_2)}\left(T_n^{(\alpha,\beta)}\right) = \begin{cases} P_{\frac{n}{2}}^{(\alpha,\beta)}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

Thus for $N = 2$ the weighted case coincides with the unweighted case for the Ruelle operator acting on the generalized Chebyshev polynomials $\{T_n^{(\alpha,\beta)}(x)\}$.

3.5 Quadratic polynomials: the unweighted Ruelle operator

We investigate two OPS, $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{P_n(x)\}_{n=0}^{\infty}$, related by the quadratic transformation given by $T_2(x) = 2x^2 - 1$. That is, in this section, we will deal with a more general case compared to the previous sections as we will consider arbitrary symmetric orthogonal polynomial sequences.
Let P_n be given by (see (1.22))

$$\begin{aligned} &P_1(y)P_n(y) = a_nP_{n+1}(y) + c_nP_{n-1}(y), \qquad n \in \mathbb{N}, \\ &P_0(y) = 1, \qquad P_1(y) = y, \end{aligned} \tag{3.13}$$

with $a_n + c_n = 1$, $n \in \mathbb{N}$ and $a_n, c_n > 0$. We call such an OPS a random walk polynomial sequence (RWS). Then by Remark 1.1.36 (1) $\operatorname{supp}\pi_P \subseteq S := [-1,1]$.
We set

$$Q_{2n}(x) = P_n(T_2(x)).$$

Fixing $Q_0(x) = 1$, $Q_1(x) = x$ we have to investigate whether there exist polynomials $Q_{2n+1}(x)$ such that $\{Q_n(x)\}_{n=0}^{\infty}$ is a RWS.
In order to satisfy the recurrence relations, we have

$$\begin{aligned} xQ_{2n}(x) &= \alpha_{2n}Q_{2n+1}(x) + \gamma_{2n}Q_{2n-1}(x) \qquad \text{and then} \\ x^2Q_{2n}(x) &= \alpha_{2n}\alpha_{2n+1}Q_{2n+2}(x) + (\alpha_{2n}\gamma_{2n+1} + \gamma_{2n}\alpha_{2n-1})Q_{2n}(x) + \gamma_{2n}\gamma_{2n-1}Q_{2n-2}(x). \end{aligned}$$

Hence

$$Q_2(x)Q_{2n}(x) = T_2(x)Q_{2n}(x) = \left(2x^2-1\right)Q_{2n}(x)$$

$$= 2\alpha_{2n}\alpha_{2n+1}Q_{2n+2}(x) + \left[2\left(\alpha_{2n}\gamma_{2n+1} + \gamma_{2n}\alpha_{2n-1}\right) - 1\right] Q_{2n}(x) + 2\gamma_{2n}\gamma_{2n-1}Q_{2n-2}(x)$$

and with $y = 2x^2 - 1$

$$Q_2(y)\, Q_{2n}(y) = P_1(y)\, P_n(y) = a_n P_{n+1}(x) + c_n P_{n-1}(y) = a_n Q_{2n+2}(x) + c_n Q_{2n-2}(x).$$

Comparing the coefficients we obtain

$$\begin{aligned} a_n &= 2\alpha_{2n}\alpha_{2n+1} && (3.14)\\ 0 &= 2\left(\alpha_{2n}\gamma_{2n+1} + \gamma_{2n}\alpha_{2n-1}\right) - 1 && (3.15)\\ c_n &= 2\gamma_{2n}\gamma_{2n-1}. && (3.16) \end{aligned}$$

By $Q_1(x)\, Q_1(x) = \alpha_1 Q_2(x) + \gamma_1 Q_0(x)$, we have $\alpha_1 = \frac{1}{2}$, $\gamma_1 = \frac{1}{2}$, and (3.16) implies $\gamma_2 = c_1$ and then $\alpha_2 = 1 - \gamma_2 = 1 - c_1 = a_1$. Now, (3.15) yields $\gamma_3 = \frac{1}{a_1}\left(\frac{1}{2} - \frac{c_1}{2}\right) = \frac{1}{2}$ and $\alpha_3 = \frac{1}{2}$. Continuing in this way, we get

$$\alpha_{2n} = a_n, \qquad \gamma_{2n} = c_n \qquad \text{and} \qquad \alpha_{2n-1} = \frac{1}{2} = \gamma_{2n-1}. \qquad (3.17)$$

Proposition 3.5.1. *Choosing the recurrence coefficients in (3.17), the corresponding RWS $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies $Q_{2n}(x) = P_n(T_2(x))$.*

Remark 3.5.2. The orthogonalization measure π_Q is concentrated on $\operatorname{supp}\pi_Q \subseteq S$. The Haar weights $h_Q(n)$ satisfy $h_Q(n+1) = h_Q(n)\frac{\alpha_n}{\gamma_{n+1}}$, which leads to the following result:

Lemma 3.5.3. *The Haar weights $h_Q(n)$ of the polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ are given by*

$$h_Q(n) = \begin{cases} h_P(k), & n = 2k \\ 2a_k h_P(k), & n = 2k+1, \qquad k \in \mathbb{N}_0 \end{cases},$$

where $h_P(n)$ are the Haar weights of $\{P_n(x)\}_{n=0}^{\infty}$. (We set $a_0 = 1$.)

Proof. The proof follows by induction. □

Now, we search for an explicit representation of the polynomials $Q_{2k+1}(x)$ of odd degree via the polynomials $P_n(x)$.

The Christoffel-Darboux formula (Theorem 1.1.24) provides (note that $P_k(-1) = (-1)^k$)

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k P_k(x)\, h_P(k) &= a_n h_P(n) \frac{P_{n+1}(x)(-1)^n - P_n(x)(-1)^{n+1}}{x+1} \\ &= a_n h_P(n)(-1)^n \frac{P_{n+1}(x) + P_n(x)}{x+1}. \end{aligned}$$

In particular, for $x = 1$: $\sum_{k=0}^{n} (-1)^k h_P(k) = a_n h_P(n)(-1)^n =: W_P(n)$.

Consider

$$\frac{1}{W_P(n)} \sum_{k=0}^{n} (-1)^k P_k(x)\, h_P(k) = \frac{P_{n+1}(x) + P_n(x)}{x+1} =: S_n(x).$$

The polynomials $\{S_n(x)\}_{n=0}^{\infty}$ have the following properties, which are obvious:

(a) $S_n(1) = 1$,

(b) $\int_{-1}^{1} S_n(x) P_m(x)(1+x)\, d\pi_P(x) = 0$, $m < n$,

(c) $\int_{-1}^{1} S_n(x) P_n(x)(1+x)\, d\pi_P(x) \neq 0$.

(b) and (c) imply that the $\{S_n(x)\}_{n=0}^{\infty}$ are orthogonal polynomials with respect to $d\tilde{\pi}(x) = (1+x)\, d\pi_P(x)$. (Concerning this result, compare to Theorem 2.5 in **[57]** which is far more general.)

Proposition 3.5.4. *We have $Q_{2k+1}(x) = xS_k(T_2(x))$ for $k \in \mathbb{N}_0$.*

Proof. We show that $Q_{2k+1}(x)$ and $xS_k(T_2(x))$ are determined by the same recurrence relation. In fact, multiplying the left-hand side by x yields

$$xQ_{2k+1}(x) = \frac{1}{2}Q_{2k+2}(x) + \frac{1}{2}Q_{2k}(x)$$

and the right-hand side multiplied by x is

$$\begin{aligned} x\left(xS_k(T_2(x))\right) &= x^2\frac{P_{k+1}(T_2(x)) + P_k(T_2(x))}{T_2(x)+1} = x^2\frac{Q_{2k+2}(x) + Q_{2k}(x)}{2x^2} \\ &= \frac{1}{2}Q_{2k+2}(x) + \frac{1}{2}Q_{2k}(x). \end{aligned}$$

□

Corollary 3.5.5.

$$\begin{aligned} Q_{2k}(x) &= P_k(T_2(x)), \\ Q_{2k+1}(x) &= \frac{x}{W_P(k)}\sum_{j=0}^{k}(-1)^j P_j(T_2(x))\, h_P(j) = \frac{x}{W_P(k)}\sum_{j=0}^{k}(-1)^j Q_{2j}(x)\, h_P(j) \\ &= \frac{P_{k+1}(T_2(x)) + P_k(T_2(x))}{x}. \end{aligned}$$

In order to guarantee that $|Q_n(x)| \leqslant Q_n(1) = 1$ for all $x \in S$, we study the connection coefficients $d_{n,k}$, defined by

$$Q_n(x) = \sum_{k=0}^{n} d_{n,k}T_k(x).$$

If the $d_{n,k}$ are nonnegative, then $|Q_n(x)| \leqslant \sum_{k=0}^{n} d_{n,k} = 1$ for all $x \in S$. Using a result of Szwarc **[58]**, we have the following result.

Proposition 3.5.6. *If $c_n \leqslant \frac{1}{2}$ for all $n \in \mathbb{N}$, then $Q_n(x) = \sum_{k=0}^{n} d_{n,k}T_k(x)$ with $d_{n,k} \geqslant 0$. In particular, $|Q_n(x)| \leqslant 1$ for all $x \in S$.*

Proof. The recurrence coefficients of $Q_n(x)$ satisfy $\gamma_n \leqslant \frac{1}{2}$, $\alpha_n + \gamma_n = 1$. Thus Corollary 1 in **[58]** yields $d_{n,k} \geqslant 0$. □

Another result of Szwarc yields that the OPS $\{Q_n(x)\}_{n=0}^{\infty}$ generates a polynomial hypergroup on $\mathbb{N}_0$:

Proposition 3.5.7. *If $c_n \leqslant a_n$ and the sequence $(c_n)_{n\in\mathbb{N}}$ is increasing, then $\{Q_n(x)\}_{n=0}^{\infty}$ generates a polynomial hypergroup on $\mathbb{N}_0$.*

***Proof*.** The recurrence coefficients of $Q_n(x)$ satisfy $\gamma_n \leqslant \alpha_n$, and the sequences $(\gamma_{2n})_{n\in\mathbb{N}}$, $(\gamma_{2n+1})_{n\in\mathbb{N}}$, $(\gamma_{2n}+\alpha_{2n})_{n\in\mathbb{N}}$ and $(\gamma_{2n+1}+\alpha_{2n+1})_{n\in\mathbb{N}}$ are increasing. Now, Theorem 1 in **[59]** yields that the linearizion coefficients of the products $Q_m(x)\,Q_n(x)$ are nonnegative. □

Now, we have collected the facts that are needed to study the unweighted Ruelle operator $R_{(1,T_2)}$.

Proposition 3.5.8. *$R_{(1,T_2)}(Q_{2k}) = P_k$ and $R_{(1,T_2)}(Q_{2k+1}) = 0$ for $k \in \mathbb{N}_0$.*

***Proof*.** It suffices to note that $Q_{2k}(x) = Q_{2k}(-x)$ and $Q_{2k+1}(-x) = -Q_{2k+1}(x)$. □

Remark 3.5.9. Let N be even, then, more generally, we get

$$R_{(1,T_N)}(Q_n)(y) = \begin{cases} \frac{1}{N}\left(\sum_{i=1}^{N/2} 2P_k\left(x_i^{(N/2)}\right)\right), & n = 2k \\ = 0, & n = 2k+1 \end{cases}$$

for $k \in \mathbb{N}_0$.

For $f \in C(S)$ we define a right inverse of the Ruelle operator $R_{(1,T_2)}$ by $Ef(x) = f(T_2(x))$. Denote by π_P the orthogonalization measure of $\{P_n(x)\}_{n=0}^{\infty}$ and π_Q the one of $\{Q_n(x)\}_{n=0}^{\infty}$. The existence and uniqueness of π_P and π_Q are guaranteed by the Perron-Favard theorem (see Theorem 1.1.23).

Lemma 3.5.10. *For all $m, n \in \mathbb{N}_0$ we have*

$$\begin{aligned} \int_S R_{(1,T_2)}(Q_m)(y)\,P_n(y)\,d\pi_P(y) &= \int_S Q_m(x)\,P_n(T_2(x))\,d\pi_Q(x) \\ &= \int_S Q_m(x)\,E(P_n)(x)\,d\pi_Q(x). \end{aligned}$$

***Proof*.** Let $n \in \mathbb{N}_0$. If $m = 2k$, then

$$\int_S R_{(1,T_2)}(Q_{2k})(y)\,P_n(y)\,d\pi_P(y) = \int_S P_k(y)\,P_n(y)\,d\pi_P(y) = \begin{cases} \frac{1}{h_P(n)}, & k = n \\ 0, & \text{else} \end{cases},$$

If $m = 2k+1$, then we have $R_{(1,T_2)}(Q_{2k+1}) = 0$, and hence

$$\int_S R_{(1,T_2)}(Q_{2k+1})(y)\,P_n(y)\,d\pi_P(y) = 0.$$

On the other hand,

$$\int_S Q_{2k}(x)\,P_n(T_2(x))\,d\pi_Q(x) = \int_S Q_{2k}(x)\,Q_{2n}(x)\,d\pi_Q(x) = \begin{cases} \frac{1}{h_Q(2n)}, & k = n \\ 0, \text{else} \end{cases}$$

and we know by Lemma 3.5.3 that $h_Q(2n) = h_P(n)$.
Moreover,

$$\int_S Q_{2k+1}(x)\,P_n(T_2(x))\,d\pi_Q(x) = \int_S Q_{2k+1}(x)\,Q_{2n}(x)\,d\pi_Q(x) = 0.$$

This proves the lemma. □

Theorem 3.5.11. *Let $f, g \in C(S)$. Then*

$$\int_S R_{(1,T_2)} f(y)\, g(y)\, d\pi_P(y) = \int_S f(x)\, g(T_2(x))\, d\pi_Q(x) = \int_S f(x)\, E(g)(x)\, d\pi_Q(x).$$

Proof. $R_{(1,T_2)}$ is a continuous operator on $C(S)$ and the linear span of $\{Q_m : m \in \mathbb{N}_0\}$ is $\|\cdot\|_\infty$-dense in $C(S)$. Hence Lemma 3.5.10 implies

$$\int_S R_{(1,T_2)} f(y)\, P_n(y)\, d\pi_P(y) = \int_S f(x)\, E(P_n)(x)\, d\pi_Q(x), \qquad n \in \mathbb{N}_0, f \in C(S).$$

And since $\{P_n : n \in \mathbb{N}_0\}$ is also $\|\cdot\|_\infty$-dense in $C(S)$, we obtain

$$\int_S R_{(1,T_2)} f(y)\, g(y)\, d\pi_P(y) = \int_S f(x)\, Eg(x)\, d\pi_Q(x) \qquad \forall f, g \in C(S).$$

□

Remark 3.5.12. (1) π_Q is the image measure of π_P under the mappings $\psi_{1/2}(y) = \pm\sqrt{\frac{y+1}{2}}$ combined in appropriate way.

(2) In [**37**], Theorem 5 provides a uniquely determined distribution function which in our case rewrites as $d\pi_Q(x) = \frac{|x-1|}{T_2'(x)} d\pi_P(T_2(x))$.

Proposition 3.5.13. *Let $f \in C(S)$ and denote $\mathcal{F}^P f(n) = \int_{-1}^1 f(y)\, P_n(y)\, d\pi_P(y)$ and $\mathcal{F}^Q f(n) = \int_{-1}^1 f(x)\, Q_n(x)\, d\pi_Q(x)$. Then the Ruelle operator $R_{(1,T_2)}$ and its right inverse E, respectively, applied to $f \in C(S)$ give*

$$\mathcal{F}^P\left(R_{(1,T_2)} f\right)(n) = \mathcal{F}^Q f(2n) \quad \text{and} \quad \mathcal{F}^Q(Eg)(2n) = \mathcal{F}^P g(n), \quad \text{respectively.}$$

Proof. Let $g(y) = P_n(y)$ and $f(x) = Q_{2k}(x)$, respectively, and apply Theorem 3.5.11. □

Lemma 3.5.14. *Let $f \in L^p(S, \pi_Q)$. For $1 \leqslant p < \infty$, we have*

$$\int_{-1}^1 |R_{(1,T_2)} f(y)|^p d\pi_P(y) = \int_{-1}^1 |f(x)|^p d\pi_Q(x).$$

Proof. Splitting $S = [-1, 1]$ into $[-1, 0]$ and $[0, 1]$ the substitution of Theorem 3.5.11 directly yields equality. □

Now, we consider the Wiener spaces $A_Q(S)$ and $A_P(S)$, where

$$A_Q(S) = \{f \in C(S) : \sum_{n=0}^{\infty} |\mathcal{F}^Q f(n)|\, \|Q_n\|_\infty h_Q(n) < \infty\} \quad \text{and}$$

$$A_P(S) = \{f \in C(S) : \sum_{n=0}^{\infty} |\mathcal{F}^P f(n)|\, h_P(n) < \infty\}.$$

The norms on $A_Q(S)$ and $A_P(S)$, respectively, are given by

$$\|f\|_{A_Q(S)} = \sum_{n=0}^{\infty} |\mathcal{F}^Q f(n)| \|Q_n\|_\infty h_Q(n) \qquad \text{and}$$

$$\|f\|_{A_P(S)} = \sum_{n=0}^{\infty} |\mathcal{F}^P f(n)| h_P(n).$$

Since we assume that $\{P_n(x)\}_{n=0}^{\infty}$ generates a polynomial hypergroup, we know that for all $n \in \mathbb{N}_0$, $\|P_n\|_\infty = \sup_{x\in S} |P_n(x)| = 1$. Thus $\left(A_P(S), \|\cdot\|_{A_P(S)}\right)$ is a Banach space. In order to prove this, one can use the uniqueness theorem for commutative hypergroups (see **[7]**, p. 87, Theorem 2.2.24), which yields that $\|\cdot\|_{A_P(S)}$ is a norm, and that $\left(A_P(S), \|\cdot\|_{A_P(S)}\right)$ is isometric isomorphic to $\ell^1(h_P)$. In general, $\{Q_n(x)\}_{n=0}^{\infty}$ will not generate a polynomial hypergroup. Therefore, we cannot use that $\|Q_n\|_\infty = 1$. By definition , we have $\|Q_{2k}\|_\infty = 1$. Nevertheless, we can show:

Proposition 3.5.15. $\left(A_Q(S), \|\cdot\|_{A_Q(S)}\right)$ *is a Banach space. Each* $f \in A_Q(S)$ *has a representation* $f(x) = \sum_{n=0}^{\infty} \mathcal{F}^Q f(n) Q_n(x) h(n)$ *for all* $x \in S$.
Moreover, $\left(A_Q(S), \|\cdot\|_{A_Q(S)}\right)$ *is isometric isomorphic to the Banach space* $\ell^1(\lambda_Q)$*, where the weights* λ_Q *are given by* $\lambda_Q(n) = \|Q_n\|_\infty h_Q(n)$ *via the mapping* $f \mapsto \left(\mathcal{F}^Q f(n)\right)_{n\in\mathbb{N}_0}$.

Proof. Let $f \in A_Q(S)$ and assume that $\mathcal{F}^Q f(n) = 0$ for all $n \in \mathbb{N}_0$. Since $A_Q(S) \subseteq L^2(S, \pi_Q)$ and the $\{Q_n(x)\}_{n=0}^{\infty}$ are an orthogonal basis in $L^2(S, \pi_Q)$, f is the zero element in $L^2(S, \pi_Q)$. The continuity of f yields that f is the zero function. Hence $\|\cdot\|_{A_Q(S)}$ is a norm on $A_Q(S)$. Given $f \in A_Q(S)$ the series $\sum_{n=0}^{\infty} \mathcal{F}^Q f(n) Q_n(x) h_Q(n)$ converges uniformly to a continuous function $g \in C(S)$. Since $\mathcal{F}^Q g(n) = \mathcal{F}^Q f(n)$, we have that $g = f$. Finally, it is obvious that $f \mapsto \left(\mathcal{F}^Q f(n)\right)_{n\in\mathbb{N}_0}$, $A_Q(S) \to \ell^1(\lambda_Q)$ is an isometric isomorphism. □

Theorem 3.5.16. *For the Ruelle operator* $R_{(1,T_2)}$ *we have* $R_{(1,T_2)} \in B(A_Q(S), A_P(S))$ *and for its right inverse* E*,* $E \in B(A_P(S), A_Q(S))$*. For the operator norms, we have* $\|R_{(1,T_2)}\| = 1$ *and* $\|E\| = 1$*. Moreover,*

$$R_{(1,T_2)} f(y) = \sum_{k=0}^{\infty} \mathcal{F}^Q f(2k) P_k(y) h_P(k) \quad \forall f \in A_Q(S) \qquad \textit{and}$$

$$Eg(x) = \sum_{k=0}^{\infty} \mathcal{F}^Q(Eg)(2k) Q_{2k}(x) h_Q(2k) \quad \forall g \in A_P(S)$$

Proof. Let $f \in A_Q(S)$, which means

$$f(x) = \sum_{k=0}^{\infty} \mathcal{F}^Q f(k) Q_k(x) h_Q(k),$$

where $\|f\|_{A_Q(S)} = \sum_{k=0}^{\infty} |\mathcal{F}^Q f(k)| \|Q_k\|_\infty h_Q(k) < \infty$. Applying Proposition 3.5.13 and Lemma 3.5.3 gives for $R_{(1,T_2)} f \in C(S)$,

$$\sum_{k=0}^{\infty} \left|\mathcal{F}^P\left(R_{(1,T_2)} f\right)(k)\right| h_P(k) = \sum_{k=0}^{\infty} |\mathcal{F}^Q f(2k)| h_P(k) = \sum_{k=0}^{\infty} |\mathcal{F}^Q f(2k)| h_Q(2k) \leqslant \|f\|_{A_Q(S)}.$$

Thus $R_{(1,T_2)}f \in A_Q(S)$ and $R_{(1,T_2)}f(y) = \sum_{k=0}^{\infty} \mathcal{F}^Q f(2k) P_k(y) h_P(k)$. Since $R_{(1,T_2)}\mathbf{1} = 1$, we have $\|R_{(1,T_2)}\| = 1$. Similarly, for $g \in A_P(S)$, we have

$$g(x) = \sum_{k=0}^{\infty} \mathcal{F}^P g(k) P_k(x) h_P(k),$$

where $\|g\|_{A_P(S)} = \sum_{k=0}^{\infty} |\mathcal{F}^P g(k)| h_P(k) < \infty$. Then for $Eg \in C(S)$

$$\begin{aligned} Eg(x) &= \sum_{k=0}^{\infty} \mathcal{F}^P g(k) P_k(T_2(x)) h_P(k) = \sum_{k=0}^{\infty} \mathcal{F}^P g(k) Q_{2k}(x) h_P(k) \\ &= \sum_{k=0}^{\infty} \mathcal{F}^P g(k) Q_{2k}(x) h_Q(2k) = \sum_{k=0}^{\infty} \mathcal{F}^Q(Eg)(2k) Q_{2k}(x) h_Q(2k). \end{aligned}$$

Thus $Eg \in A_Q(S)$ and since $E\mathbf{1} = 1$, $\|E\| = 1$. □

Remark 3.5.17. By Theorem 3.5.16 we see that $\mathrm{Ker}R_{(1,T_2)} = \{f \in A_Q(S) : \mathcal{F}^Q f(2k) = 0 \ \forall k \in \mathbb{N}_0\}$, $\mathrm{Ker}E = \{g \in A_P(S) : \mathcal{F}^P g(k) = 0 \ \forall k \in \mathbb{N}_0\}$.

Examples 3.5.18. (1) Consider the orthogonal polynomials defined by homogeneous trees. These polynomials $\{R_n(x;a)\}_{n=0}^{\infty}$, $a \geqslant 2$, are determined by the recurrence coefficients

$$a_n = \frac{a-1}{a}, \qquad b_n = 0, \qquad c_n = \frac{1}{a}, \qquad n \in \mathbb{N}$$

and $a_0 = 1, b_0 = 0$. They generate a polynomial hypergroup on $\mathbb{N}_0$, see **[22]**. Putting $Q_{2n}(x) = R_n(T_2(x); a)$ the construction of this section yields an orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ with recurrence coefficients $\alpha_0 = 1$, $\beta_0 = 0$,

$$\alpha_n = \begin{cases} \frac{a-1}{a}, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}, \qquad \gamma_n = 1 - a_n, \qquad \beta_n = 0, \qquad n \in \mathbb{N}.$$

This class of polynomials has already been studied for example in **[21]**. These polynomials are called Karlin-McGregor polynomials and they are an important tool for the analysis of random walks. They generate a polynomial hypergroup on $\mathbb{N}_0$ whose dual spaces are investigated in a recent paper, see **[41]**.

(2) The associated Legendre polynomials $\{L_n^{\nu}(x)\}_{n=0}^{\infty}$ defined by the recurrence coefficients

$$a_n = \frac{\eta_{n+1}}{\eta_n}, \qquad b_n = 0, \qquad c_n = \frac{\eta_{n-1}(n+\nu)^2}{\eta_n\left[4(n+\nu)^2 - 1\right]}, \qquad n \in \mathbb{N},$$

where $\eta_0 = 1$, $\eta_n = \frac{(\nu+1)_n}{2^n(\nu+1/2)_n}\left(1 + \sum_{k=1}^{n} \frac{\nu}{k+\nu}\right)$, $n \in \mathbb{N}$, and $a_0 = 1$, $b_0 = 0$, also generate a polynomial hypergroup on $\mathbb{N}_0$ (see **[22]**). We set $Q_{2n}(x) = L_n^{\nu}(T_2(x))$, then we get an orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ with the recurrence coefficients

$$\alpha_n = \begin{cases} \frac{\eta_{n+1}}{\eta_n}, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}, \qquad \gamma_n = 1 - \alpha_n, \qquad \beta_n = 0, \qquad n \in \mathbb{N},$$

and $\alpha_0 = 1$, $\beta_0 = 0$. Using Theorem 1 in **[59]**, we get that $\{Q_n(x)\}_{n=0}^{\infty}$ generates a polynomial hypergroup on $\mathbb{N}_0$ since the sequences $\{\alpha_{2n}\}_{n=0}^{\infty}$, $\{\alpha_{2n+1}\}_{n=0}^{\infty}$, $\{\alpha_{2n} + \gamma_{2n}\}_{n=0}^{\infty}$, $\{\alpha_{2n+1} + \gamma_{2n+1}\}_{n=0}^{\infty}$ are increasing and $\alpha_n \leqslant \gamma_n$ as $a_n \leqslant c_n$.

(3) For fixed β, q with $-1 < \beta < 1$ and $0 < q < 1$ the q-ultraspherical polynomials $\{P_n^{(\beta,q)}(x)\}_{n=0}^{\infty}$ are determined by the recurrence coefficients

$$c_1 = C_1 A_0, \qquad a_1 = 1 - c_1, \qquad c_n = \frac{C_n A_{n-1}}{a_{n-1}}, \qquad a_n = 1 - c_n, \qquad n = 2, 3, \dots,$$

where

$$A_n = \frac{1 - q^{n+1}}{2\left(1 - \beta q^n\right)}, \quad n \in \mathbb{N}_0, \qquad C_n = \frac{1 - \beta^2 q^{n-1}}{2\left(1 - \beta q^n\right)}, \quad n \in \mathbb{N},$$

and $b_n = 0$, $n \in \mathbb{N}$. They generate a polynomial hypergroup on $\mathbb{N}_0$ (see **[22]**). Then by setting $Q_{2n}(x) = P_n^{(\beta,q)}(T_2(x))$, we get an orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ with recurrence coefficients

$$\alpha_n = \begin{cases} 1 - c_n, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}, \qquad \gamma_n = 1 - \alpha_n, \qquad \beta_n = 0, \qquad n \in \mathbb{N}$$

and $\alpha_0 = 1$, $\beta_0 = 0$. For the same reason as in (2) $\{Q_n(x)\}_{n=0}^{\infty}$ generates a polynomial hypergroup on $\mathbb{N}_0$.

3.6 Quadratic polynomials: the weighted Ruelle operator

In this section, we will investigate the behavior of the weighted Ruelle operator in the case of the OPS $\{Q_n(x)\}_{n=0}^{\infty}$ which we constructed in the previous section. We will restrict our studies to $N = 2$ since this case provides the most interesting results, and consider three kinds of weight functions $m_2(x) = \sum_{k=0}^{\infty} b_k T_k(x) h(k)$, $\tilde{m}_2(x) = \sum_{k=0}^{\infty} \tilde{b}_k T_k^{(\alpha,\beta)}(x) h(k)$ and $\tilde{\tilde{m}}_2(x) = \sum_{k=0}^{\infty} \tilde{\tilde{b}}_k Q_k(x) h_Q(k)$.

Lemma 3.6.1. *Let $N = 2$, then*

$$R_{(m_2,T_2)}(Q_n)(y) = \begin{cases} \frac{1}{2}\left[2b_0 T_0\left(x_1^{(2)}\right) h(0)\right] P_l(y) = P_l(y), & n = 2l \\ \frac{1}{2}\left[\sum_{k=0}^{\infty} 2b_{2k+1} T_{2k+1}\left(x_1^{(2)}\right) h(2k+1)\right] \frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}, & n = 2l+1 \end{cases},$$

$R_{(\tilde{m}_2,T_2)}(Q_n)(y) = \frac{1}{2}\sum_{i=1}^{2} Q_n(x_i)$ *which coincides with the unweighted case, and*

$$R_{\left(\tilde{\tilde{m}}_2,T_2\right)}(Q_n)(y) = \begin{cases} P_l(y), & n = 2l \\ \frac{1}{2}\left[\sum_{k=0}^{\infty} 2\tilde{\tilde{b}}_{2k+1} \frac{P_{k+1}(y)+P_k(y)}{x_1^{(2)}} h_Q(2k+1)\right] \frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}, & n = 2l+1 \end{cases}.$$

Proof. Using Lemma 3.2.1, we get

$$R_{m_2,T_2}(Q_n)(y) = \frac{1}{2}\left(\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty} b_k T_k\left(x_1^{(2)}\right) h(k)\right] Q_n\left(x_1^{(2)}\right)\right)$$

$$= \frac{1}{2}\left(\left[\sum_{k=0}^{\infty} b_k T_k\left(x_1^{(2)}\right) h(k)\right] Q_n\left(x_1^{(2)}\right) + \left[\sum_{k=0}^{\infty} b_k T_k\left(-x_1^{(2)}\right) h(k)\right] Q_n\left(-x_1^{(2)}\right)\right)$$

$$= \begin{cases} \frac{1}{2}\Bigg(\left[\sum_{k=0}^{\infty} b_k T_k\left(x_1^{(2)}\right)h(k)\right] Q_{2l}\left(x_1^{(2)}\right) \\ \quad + \left[\sum_{k=0}^{\infty} b_k T_k\left(-x_1^{(2)}\right)h(k)\right] Q_{2l}\left(-x_1^{(2)}\right)\Bigg), & n=2l \\ \frac{1}{2}\Bigg(\left[\sum_{k=0}^{\infty} b_k T_k\left(x_1^{(2)}\right)h(k)\right] Q_{2l+1}\left(x_1^{(2)}\right) \\ \quad + \left[\sum_{k=0}^{\infty} b_k T_k\left(-x_1^{(2)}\right)h(k)\right] Q_{2l+1}\left(-x_1^{(2)}\right)\Bigg), & n=2l+1 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\Bigg(\left[\sum_{k=0}^{\infty} b_k T_k\left(x_1^{(2)}\right)h(k)\right] P_l\left(T_2\left(x_1^{(2)}\right)\right) \\ \quad + \left[\sum_{k=0}^{\infty} b_k T_k\left(-x_1^{(2)}\right)h(k)\right] P_l\left(T_2\left(-x_1^{(2)}\right)\right)\Bigg), & n=2l \\ \frac{1}{2}\Bigg(\left[\sum_{k=0}^{\infty} b_k T_k\left(x_1^{(2)}\right)h(k)\right] \frac{P_{l+1}\left(T_2\left(x_1^{(2)}\right)\right)+P_l\left(T_2\left(x_1^{(2)}\right)\right)}{x_1^{(2)}} \\ \quad + \left[\sum_{k=0}^{\infty} b_k T_k\left(-x_1^{(2)}\right)h(k)\right] \frac{P_{l+1}\left(T_2\left(-x_1^{(2)}\right)\right)+P_l\left(T_2\left(-x_1^{(2)}\right)\right)}{-x_1^{(2)}}\Bigg), & n=2l+1 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(\left[\sum_{k=0}^{\infty} 2b_{2k} T_{2k}\left(x_1^{(2)}\right)h(2k)\right] P_l(y)\right), & n=2l \\ \frac{1}{2}\left(\left[\sum_{k=0}^{\infty} 2b_{2k+1} T_{2k+1}\left(x_1^{(2)}\right)h(2k+1)\right] \frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}\right), & n=2l+1 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[2b_0 T_0\left(x_1^{(2)}\right)h(0)\right] P_l(y) = P_l(y), & n=2l \\ \frac{1}{2}\left[\sum_{k=0}^{\infty} 2b_{2k+1} T_{2k+1}\left(x_1^{(2)}\right)h(2k+1)\right] \frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}, & n=2l+1 \end{cases},$$

and with Lemma 3.4.1 we obtain

$$R_{(\tilde{m}_2,T_2)}(Q_n)(y) = \frac{1}{2}\left(\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty}\tilde{b}_k T_k^{(\alpha,\beta)}\left(x_i^{(2)}\right)h(k)\right] Q_n\left(x_i^{(2)}\right)\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty}\tilde{b}_{2k} T_{2k}^{(\alpha,\beta)}\left(x_i^{(2)}\right)h(2k) + \sum_{k=0}^{\infty}\tilde{b}_{2k+1} T_{2k+1}^{(\alpha,\beta)}\left(x_i^{(2)}\right)h(2k+1)\right] Q_n\left(x_i^{(2)}\right)\right)$$

$$= \frac{1}{2}\sum_{i=1}^{2} Q_n(x_i).$$

With

$$R_{\left(\tilde{\tilde{m}}_2,T_2\right)}(\mathbf{1}) = 1 = \frac{1}{2}\sum_{i=1}^{2}\tilde{\tilde{m}}_2\left(x_i^{(2)}\right) = \frac{1}{2}\sum_{i=1}^{2}\sum_{k=0}^{\infty}\tilde{\tilde{b}}_k Q_k\left(x_i^{(2)}\right)h_Q(k)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k}Q_{2k}\left(x_i^{(2)}\right)h_Q(2k)+\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k+1}Q_{2k+1}\left(x_i^{(2)}\right)h_Q(2k+1)\right]\\
&= \frac{1}{2}\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k}P_k\left(T_2\left(x_i^{(2)}\right)\right)h_Q(2k)\right.\\
&\quad\left.+\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k+1}\frac{P_{k+1}\left(T_2\left(x_i^{(2)}\right)\right)+P_k\left(T_2\left(x_i^{(2)}\right)\right)}{x_i^{(2)}}h_Q(2k+1)\right]\\
&= \sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k}P_k(y)\,h_Q(2k),
\end{aligned}
$$

we get that $\tilde{\tilde{b}}_0 = 1$ and $\tilde{\tilde{b}}_{2k} = 0$, $k \in \mathbb{N}$. Thus we get for the weighted Ruelle operator

$$
\begin{aligned}
&R_{\left(\tilde{\tilde{m}}_2,T_2\right)}(Q_n)(y) = \frac{1}{2}\left(\sum_{i=1}^{2}\tilde{\tilde{m}}_2\left(x_i^{(2)}\right)Q_n\left(x_i^{(2)}\right)\right)\\
&= \frac{1}{2}\left(\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty}\tilde{\tilde{b}}_kQ_k\left(x_i^{(2)}\right)h_Q(k)\right]Q_n\left(x_i^{(2)}\right)\right)\\
&= \frac{1}{2}\left(\sum_{i=1}^{2}\left[\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k}Q_{2k}\left(x_i^{(2)}\right)h_Q(2k)+\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k+1}Q_{2k+1}\left(x_i^{(2)}\right)h_Q(2k+1)\right]Q_n\left(x_i^{(2)}\right)\right)\\
&= \begin{cases}
\frac{1}{2}\left(\sum_{i=1}^{2}\left(\left[\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k}Q_{2k}\left(x_i^{(2)}\right)h_Q(2k)\right.\right.\right. & \\
\quad\left.\left.\left.+\ \sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k+1}Q_{2k+1}\left(x_i^{(2)}\right)h_Q(2k+1)\right]P_l\left(T_2\left(x_i^{(2)}\right)\right)\right)\right), & n=2l\\
\frac{1}{2}\left(\sum_{i=1}^{2}\left(\left[\sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k}Q_{2k}\left(x_i^{(2)}\right)h_Q(2k)\right.\right.\right. & \\
\quad\left.\left.+\ \sum_{k=0}^{\infty}\tilde{\tilde{b}}_{2k+1}Q_{2k+1}\left(x_i^{(2)}\right)h_Q(2k+1)\right]\right)\left.\frac{P_{l+1}\left(T_2\left(x_i^{(2)}\right)\right)+P_l\left(T_2\left(x_i^{(2)}\right)\right)}{x_1^{(2)}}\right), & n=2l+1
\end{cases}\\
&= \begin{cases}
\frac{1}{2}\left[2\tilde{\tilde{b}}_0h_Q(0)\right]P_l(y) = P_l(y), & n=2l\\
\frac{1}{2}\left[\sum_{k=0}^{\infty}2\tilde{\tilde{b}}_{2k+1}\frac{P_{k+1}(y)+P_k(y)}{x_1^{(2)}}h_Q(2k+1)\right]\frac{P_{l+1}(y)+P_l(y)}{x_1^{(2)}}, & n=2l+1
\end{cases}.
\end{aligned}
$$

□

The following statements for the weighted Ruelle operator hold for both weight functions $m_2(x)$ and $\tilde{\tilde{m}}_2(x)$. For simplicity $m_2(x)$ will denote either one of these weight functions throughout the remainder of this section. We use the same notation as in the previous section.

Lemma 3.6.2. *For all $m, n \in \mathbb{N}_0$ we have*

$$\begin{aligned}\int_S R_{(m_2,T_2)}(Q_m)(y)\, P_n(y)\, d\pi_P(y) &= \int_S m_2(x)\, Q_m(x)\, P_n(T_2(x))\, d\pi_Q(x)\\ &= \int_S m_2(x)\, Q_m(x)\, E(P_n)(x)\, d\pi_Q(x)\,.\end{aligned}$$

Proof. The proof follows from Lemma 3.5.10. □

Theorem 3.6.3. *Let $f, g \in C(S)$. Then*

$$\begin{aligned}\int_S R_{(m_2,T_2)} f(y)\, g(y)\, d\pi_P(y) &= \int_S m_2(x)\, f(x)\, g(T_2(x))\, d\pi_Q(x)\\ &= \int_S m_2(x)\, f(x)\, E(g)(x)\, d\pi_Q(x)\,.\end{aligned}$$

Proof. The proof follows from Theorem 3.5.11. □

Proposition 3.6.4. *Let $f \in C(S)$. Then the Ruelle operator $R_{(m_2,T_2)}$ and its right inverse E, respectively, applied to $f \in C(S)$ give*

$$\mathcal{F}^P\left(R_{(m_2,T_2)} f\right)(n) = \mathcal{F}^Q(m_2 f)(2n) \quad \text{and} \quad \mathcal{F}^Q(Eg)(2n) = \mathcal{F}^P g(n)\,, \quad \text{respectively.}$$

Proof. The proof follows from Proposition 3.5.13. □

We also consider the Wiener spaces $A_Q(S)$ and $A_P(S)$.

Theorem 3.6.5. *For the Ruelle operator $R_{(m_2,T_2)}$, we have $R_{(m_2,T_2)} \in B(A_Q(S), A_P(S))$ and for its right inverse E, $E \in B(A_P(S), A_Q(S))$. For the operator norms, we have $\|R_{(m_2,T_2)}\| = 1$ and $\|E\| = 1$. Moreover,*

$$R_{(m_2,T_2)} f(y) = \sum_{k=0}^{\infty} \mathcal{F}^Q(m_2 f)(2k)\, P_k(y)\, h_P(k) \quad \forall f \in A_Q(S) \qquad \text{and}$$

$$Eg(x) = \sum_{k=0}^{\infty} \mathcal{F}^Q(Eg)(2k)\, Q_{2k}(x)\, h_Q(2k) \quad \forall g \in A_P(S)$$

Proof. The proof follows from Theorem 3.5.16. □

3.7 Cubic polynomials: the unweighted Ruelle operator

Now, we investigate two OPS, $\{C_n(x)\}_{n=0}^{\infty}$ and $\{P_n(x)\}_{n=0}^{\infty}$, related by the cubic transformation given by $T_3(x) = 4x^3 - 3x$.
Using the recurrence formula in (1.22), we let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS satisfying

$$\begin{aligned}&P_1(y)\, P_n(y) = a_n P_{n+1}(y) + b_n P_n(y) + c_n P_{n-1}(y)\,, \qquad n \in \mathbb{N}, \qquad (3.18)\\ &P_0(y) = 1, \qquad P_1(y) = \frac{y}{a_0} - \frac{b_0}{a_0}\end{aligned}$$

with $a_n + b_n + c_n = 1$, $a_0 + b_0 = 1$, $n \in \mathbb{N}$, and $a_n, c_{n+1} > 0$, $b_n \geqslant 0$, $n \in \mathbb{N}_0$. Such an OPS is called a random walk polynomial sequence (RWS). Furthermore, $P_n(1) = 1$. The

orthogonalization measure π_P is concentrated on $S = [-1, 1]$.
Now, we set $C_{3n}(x) = P_n(T_3(x)) = P_n(4x^3 - 3x)$ for $n \in \mathbb{N}$ and investigate under which assumptions there exists $C_{3n+1}(x)$ and $C_{3n+2}(x)$ such that $\{C_n(x)\}_{n=0}^{\infty}$ is a random walk polynomial sequence.
We have that $C_0(x) = 1$, $C_1(x) = \frac{x}{\alpha_0} - \frac{\beta_0}{\alpha_0}$ and we need to construct the polynomials $C_{3n+1}(x)$ and $C_{3n+2}(x)$ such that $C_{3n}(x) = P_n(T_3(x))$ and

$$C_1(x)\, C_n(x) = \alpha_n C_{n+1}(x) + \beta_n C_n(x) + \gamma_n C_{n-1}(x), \tag{3.19}$$

where $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha_0 + \beta_0 = 1$, $n \in \mathbb{N}$ and $\alpha_n, \gamma_{n+1} > 0$, $\beta_n \geqslant 0$ for $n \in \mathbb{N}_0$.
If we choose α_1, β_1 and γ_1 such that (3.19) is satisfied for $n = 1$, then

$$C_2(x) = \frac{1}{\alpha_1}\left[\frac{1}{\alpha_0^2}(x-\beta_0)^2 - \frac{\beta_1}{\alpha_0}(x-\beta_0) - \gamma_1\right]. \tag{3.20}$$

Furthermore, by (3.19) we get for $n = 2$,

$$\begin{aligned}\frac{4}{a_0}x^3 - \frac{3}{a_0}x - \frac{b_0}{a_0} = C_3(x) = \frac{1}{\alpha_2}\Bigg[&\frac{1}{\alpha_0^3\alpha_1}(x-\beta_0)^3 - (\beta_1+\beta_2)\frac{1}{\alpha_0^2\alpha_1}(x-\beta_0)^2 \\ &+ (\beta_1\beta_2 - \gamma_1 - \alpha_1\gamma_2)\frac{1}{\alpha_0\alpha_1}(x-\beta_0) + \frac{\beta_2\gamma_1}{\alpha_1}\Bigg].\end{aligned} \tag{3.21}$$

Comparing the coefficients in (3.21), we obtain

$$\begin{aligned}&\frac{1}{\alpha_0^3\alpha_1\alpha_2} = \frac{4}{a_0}, \qquad -\frac{3\beta_0}{\alpha_0^3\alpha_1\alpha_2} - \frac{\beta_1+\beta_2}{\alpha_0^2\alpha_1\alpha_2} = 0,\\ &\frac{3\beta_0^2}{\alpha_0^3\alpha_1\alpha_2} + \frac{2\beta_0(\beta_1+\beta_2)}{\alpha_0^2\alpha_1\alpha_2} + \frac{\beta_1\beta_2 - \gamma_1 - \alpha_1\gamma_2}{\alpha_0\alpha_1\alpha_2} = -\frac{3}{a_0},\\ &-\frac{\beta_0^3}{\alpha_0^3\alpha_1\alpha_2} - \frac{\beta_0^2(\beta_1+\beta_2)}{\alpha_0^3\alpha_1\alpha_2} - \frac{\beta_0(\beta_1\beta_2 - \gamma_1 - \alpha_1\gamma_2)}{\alpha_0\alpha_1\alpha_2} + \frac{\beta_2\gamma_1}{\alpha_1\alpha_2} = -\frac{b_0}{a_0},\end{aligned} \tag{3.22}$$

In order to gain more insight and to obtain simpler expressions than the above formulas, we now set $a_0 = \alpha_0 = 1$ and $b_0 = \beta_0 = 0$. Thus $P_1(y) = y$ and $C_1(x) = x$. Then we have

$$4 = \frac{1}{\alpha_1\alpha_2} \Leftrightarrow \alpha_2 = \frac{1}{4\alpha_1}, \tag{3.23}$$

$$0 = -\frac{1}{\alpha_1\alpha_2}(\beta_1+\beta_2) \Leftrightarrow \beta_2 = -\beta_1, \tag{3.24}$$

$$\begin{aligned}-3 = (\beta_1\beta_2 - \gamma_1 - \alpha_1\gamma_2)\frac{1}{\alpha_1\alpha_2} \quad &\Leftrightarrow \quad 3\alpha_2 - \frac{\beta_1^2}{\alpha_1} - \frac{\gamma_1}{\alpha_1} = \gamma_2\\ &\Leftrightarrow \quad \frac{3}{4\alpha_1} - \frac{\beta_1^2}{\alpha_1} - \frac{\gamma_1}{\alpha_1} = \gamma_2,\end{aligned} \tag{3.25}$$

$$0 = \beta_2\gamma_1 = -\beta_1\gamma_1 \Rightarrow \beta_1 = \beta_2 = 0 \vee \gamma_1 = 0. \tag{3.26}$$

In (3.26) we choose $\beta_1 = \beta_2 = 0$, as $\gamma_1 > 0$. Then $\gamma_2 = \frac{1}{\alpha_1}\left(\frac{3}{4} - \gamma_1\right)$ and (for $n = 2$) we have to restrict $0 < \gamma_1 < \frac{3}{4}$ or equivalently $\frac{1}{4} < \alpha_1 < 1$, thus $\gamma_2 = 1 - \frac{1}{4\alpha_1}$.
Now, we iterate equation (3.19):

$$C_1(x)\, C_{3n}(x) \quad = \quad \alpha_{3n} C_{3n+1}(x) + \beta_{3n} C_{3n}(x) + \gamma_{3n} C_{3n-1}(x), \tag{3.27}$$

$$\begin{aligned}
C_1^2(x)\,C_{3n}(x) &= \alpha_{3n}\left(\alpha_{3n+1}C_{3n+2}(x)+\beta_{3n+1}C_{3n+1}(x)+\gamma_{3n+1}C_{3n}(x)\right)\\
&\quad+\beta_{3n}\left(\alpha_{3n}C_{3n+1}(x)+\beta_{3n}C_{3n}(x)+\gamma_{3n}C_{3n-1}(x)\right)\\
&\quad+\gamma_{3n}\left(\alpha_{3n-1}C_{3n}(x)+\beta_{3n-1}C_{3n-1}(x)+\gamma_{3n-1}C_{3n-2}(x)\right), \qquad (3.28)\\
C_1^3(x)\,C_{3n}(x) &= \alpha_{3n}\Big[\alpha_{3n+1}\left(\alpha_{3n+2}C_{3n+3}(x)+\beta_{3n+2}C_{3n+2}(x)+\gamma_{3n+2}C_{3n+1}(x)\right)\\
&\qquad+\beta_{3n+1}\left(\alpha_{3n+1}C_{3n+2}(x)+\beta_{3n+1}C_{3n+1}(x)+\gamma_{3n+1}C_{3n}(x)\right)\\
&\qquad+\gamma_{3n+1}\left(\alpha_{3n}C_{3n+1}(x)+\beta_{3n}C_{3n}(x)+\gamma_{3n}C_{3n-1}(x)\right)\Big]\\
&\quad+\beta_{3n}\Big[\alpha_{3n}\left(\alpha_{3n+1}C_{3n+2}(x)+\beta_{3n+1}C_{3n+1}(x)+\gamma_{3n+1}C_{3n}(x)\right)\\
&\qquad+\beta_{3n}\left(\alpha_{3n}C_{3n+1}(x)+\beta_{3n}C_{3n}(x)+\gamma_{3n}C_{3n-1}(x)\right)\\
&\qquad+\gamma_{3n}\left(\alpha_{3n-1}C_{3n}(x)+\beta_{3n-1}C_{3n-1}(x)+\gamma_{3n-1}C_{3n-2}(x)\right)\Big]\\
&\quad+\gamma_{3n}\Big[\alpha_{3n-1}\left(\alpha_{3n}C_{3n+1}(x)+\beta_{3n}C_{3n}(x)+\gamma_{3n}C_{3n-1}(x)\right)\\
&\qquad+\beta_{3n-1}\left(\alpha_{3n-1}C_{3n}(x)+\beta_{3n-1}C_{3n-1}(x)+\gamma_{3n-1}C_{3n-2}(x)\right)\\
&\qquad+\gamma_{3n-1}\left(\alpha_{3n-2}C_{3n-1}(x)+\beta_{3n-2}C_{3n-2}(x)+\gamma_{3n-2}C_{3n-3}(x)\right)\Big]\\
&= \left(\alpha_{3n}\alpha_{3n+1}\alpha_{3n+2}\right)C_{3n+3}(x)\\
&\quad+\left(\alpha_{3n}\alpha_{3n+1}\beta_{3n+2}+\alpha_{3n}\alpha_{3n+1}\beta_{3n+1}+\alpha_{3n}\alpha_{3n+1}\beta_{3n}\right)C_{3n+2}(x)\\
&\quad+\big(\alpha_{3n}\alpha_{3n+1}\gamma_{3n+2}+\alpha_{3n}\beta_{3n+1}^2+\alpha_{3n}^2\gamma_{3n+1}+\alpha_{3n}\beta_{3n}\beta_{3n+1}+\alpha_{3n}\beta_{3n}^2\\
&\qquad+\alpha_{3n-1}\alpha_{3n}\gamma_{3n}\big)C_{3n+1}(x)\\
&\quad+\big(\alpha_{3n}\beta_{3n+1}\gamma_{3n+1}+2\alpha_{3n}\beta_{3n}\gamma_{3n+1}+\beta_{3n}^3+2\alpha_{3n-1}\beta_{3n}\gamma_{3n}\\
&\qquad+\alpha_{3n-1}\beta_{3n-1}\gamma_{3n}\big)C_{3n}(x)\\
&\quad+\big(\alpha_{3n}\gamma_{3n}\gamma_{3n+1}+\beta_{3n}^2\gamma_{3n}+\beta_{3n-1}\beta_{3n}\gamma_{3n}+\alpha_{3n-1}\gamma_{3n}^2+\beta_{3n-1}^2\gamma_{3n}\\
&\qquad+\alpha_{3n-2}\gamma_{3n-1}\gamma_{3n}\big)C_{3n-1}(x)\\
&\quad+\left(\beta_{3n}\gamma_{3n-1}\gamma_{3n}+\beta_{3n-1}\gamma_{3n-1}\gamma_{3n}+\beta_{3n-2}\gamma_{3n-1}\gamma_{3n}\right)C_{3n-2}(x)\\
&\quad+\left(\gamma_{3n-2}\gamma_{3n-1}\gamma_{3n}\right)C_{3n-3}(x). \qquad (3.29)
\end{aligned}$$

For $y=4x^3-3x$ we have

$$\begin{aligned}
C_3(x)\,C_{3n}(x) &= \left(4x^3-3x\right)C_{3n}(x)=P_1(y)\,P_n(y)\\
&= a_nP_{n+1}(y)+b_nP_n(y)+c_nP_{n-1}(y)\\
&= a_nC_{3n+3}(x)+b_nC_{3n}(x)+c_nC_{3n-3}(x)\\
&= 4\alpha_{3n}\alpha_{3n+1}\alpha_{3n+2}C_{3n+3}(x)\\
&\quad+\left(4\alpha_{3n}\alpha_{3n+1}\beta_{3n+2}+4\alpha_{3n}\alpha_{3n+1}\beta_{3n+1}+4\alpha_{3n}\alpha_{3n+1}\beta_{3n}\right)C_{3n+2}(x)\\
&\quad+\big(4\alpha_{3n}\alpha_{3n+1}\gamma_{3n+2}+4\alpha_{3n}\beta_{3n+1}^2+4\alpha_{3n}^2\gamma_{3n+1}+4\alpha_{3n}\beta_{3n}\beta_{3n+1}+4\alpha_{3n}\beta_{3n}^2\\
&\quad+4\alpha_{3n-1}\alpha_{3n}\gamma_{3n}-3\alpha_{3n}\big)C_{3n+1}(x)\\
&\quad+\big(4\alpha_{3n}\beta_{3n+1}\gamma_{3n+1}+8\alpha_{3n}\beta_{3n}\gamma_{3n+1}+4\beta_{3n}^3+8\alpha_{3n-1}\beta_{3n}\gamma_{3n}\\
&\qquad+4\alpha_{3n-1}\beta_{3n-1}\gamma_{3n}-3\beta_{3n}\big)C_{3n}(x)\\
&\quad+\big(4\alpha_{3n}\gamma_{3n}\gamma_{3n+1}+4\beta_{3n}^2\gamma_{3n}+4\beta_{3n-1}\beta_{3n}\gamma_{3n}+4\alpha_{3n-1}\gamma_{3n}^2+4\beta_{3n-1}^2\gamma_{3n}\\
&\qquad+4\alpha_{3n-2}\gamma_{3n-1}\gamma_{3n}-3\gamma_{3n}\big)C_{3n-1}(x)\\
&\quad+\left(4\beta_{3n}\gamma_{3n-1}\gamma_{3n}+4\beta_{3n-1}\gamma_{3n-1}\gamma_{3n}+4\beta_{3n-2}\gamma_{3n-1}\gamma_{3n}\right)C_{3n-2}(x)\\
&\quad+4\gamma_{3n-2}\gamma_{3n-1}\gamma_{3n}C_{3n-3}(x)
\end{aligned}$$

and hence comparison of the coefficients yields

$$\begin{aligned}
a_n &= 4\alpha_{3n}\alpha_{3n+1}\alpha_{3n+2}, && (3.30)\\
0 &= 4\alpha_{3n}\alpha_{3n+1}\left(\beta_{3n+2}+\beta_{3n+1}+\beta_{3n}\right), && (3.31)\\
0 &= \alpha_{3n}\big(4\alpha_{3n+1}\gamma_{3n+2}+4\beta_{3n+1}^2+4\alpha_{3n}\gamma_{3n+1}+4\beta_{3n}\beta_{3n+1}+4\beta_{3n}^2\\
&\quad +4\alpha_{3n-1}\gamma_{3n}-3\big), && (3.32)\\
b_n &= 4\alpha_{3n}\beta_{3n+1}\gamma_{3n+1}+8\alpha_{3n}\beta_{3n}\gamma_{3n+1}+4\beta_{3n}^3+8\alpha_{3n-1}\beta_{3n}\gamma_{3n}\\
&\quad +4\alpha_{3n-1}\beta_{3n-1}\gamma_{3n}-3\beta_{3n}, && (3.33)\\
0 &= \gamma_{3n}\big(4\alpha_{3n}\gamma_{3n+1}+4\beta_{3n}^2+4\beta_{3n-1}\beta_{3n}+4\alpha_{3n-1}\gamma_{3n}+4\beta_{3n-1}^2\\
&\quad +4\alpha_{3n-2}\gamma_{3n-1}-3\big), && (3.34)\\
0 &= 4\gamma_{3n-1}\gamma_{3n}\left(\beta_{3n}+\beta_{3n-1}+\beta_{3n-2}\right), && (3.35)\\
c_n &= 4\gamma_{3n-2}\gamma_{3n-1}\gamma_{3n}. && (3.36)
\end{aligned}$$

For simplification and in order to gain more insight, we assume that $\{C_n(x)\}_{n=0}^\infty$ and $\{P_n(x)\}_{n=0}^\infty$ are symmetric random walk polynomial sequences (SRWS), that is, $\beta_n = 0$ and $b_n = 0$ respectively for $n \in \mathbb{N}_0$, and continue with the determination of α_n and γ_n. By (3.36), $c_1 = 4\gamma_1\gamma_2\gamma_3$, which is equivalent to

$$a_1 = 1-c_1 = 1-4(1-\alpha_1)(1-\alpha_2)(1-\alpha_3) = 1-4(1-\alpha_1)(1-\alpha_2)+4(1-\alpha_1)(1-\alpha_2)\alpha_3.$$

Hence

$$\alpha_3 = \frac{a_1+4(1-\alpha_1)(1-\alpha_2)-1}{4(1-\alpha_1)(1-\alpha_2)} = 1-\frac{1-\alpha_1}{(1-\alpha_2)},$$

and

$$\gamma_3 = \frac{c_1}{4\gamma_1\gamma_2}.$$

We have to guarantee that α_3 (and γ_3) are positive. A straightforward calculation shows that $\alpha_3 > 0$ if and only if $a_1 > 4\alpha_1 + \frac{1}{\alpha_1} - 4$. Then with (3.34)

$$\gamma_4 = \frac{1}{4\alpha_3}(3-4\gamma_3\alpha_2-4\gamma_2\alpha_1) = \frac{1}{\alpha_3}\left(\gamma_1 - \frac{c_1\alpha_2}{4\gamma_1\gamma_2}\right), \qquad \alpha_4 = 1-\gamma_4.$$

(3.30) yields $\alpha_5 = \frac{a_1}{4\alpha_3\alpha_4}$ and $\gamma_5 = 1-\alpha_5$ and subtraction of (3.32) from (3.34) gives

$$\alpha_{3n-2}\gamma_{3n-1} = \alpha_{3n+1}\gamma_{3n+2}, \qquad n \in \mathbb{N}. \tag{3.37}$$

Particularly, $\alpha_{3n+1}\gamma_{3n+2} = \ldots = \alpha_1\gamma_2 = \frac{3}{4} - \gamma_1 = \alpha_1 - \frac{1}{4}$.

Exploring $\{C_n(x)\}_{n=0}^\infty$ more detailed, we fix $\alpha_1 = \frac{1}{2}$. Then $\alpha_1 = \gamma_1 = \frac{1}{2}$, $\alpha_2 = \gamma_2 = \frac{1}{2}$, $\gamma_3 = c_1$, $\alpha_3 = a_1$, $\gamma_4 = \frac{1}{2}$, $\alpha_4 = \frac{1}{2}$, $\alpha_5 = \frac{1}{2}$, $\gamma_5 = \frac{1}{2}$, $\gamma_6 = c_2$, $\alpha_6 = a_2$.
Using (3.34) and (3.32) we obtain $\gamma_{3n+1} = \gamma_{3n+2} = \frac{1}{2} = \alpha_{3n+1} = \alpha_{3n+2}$. Finally (3.36) yields $\gamma_{3n} = c_n$ and $\alpha_{3n} = a_n$.

This three-term recurrence coefficients determine the SRWS $\{C_n(x)\}_{n=0}^\infty$ with unique orthogonalization measure π_C on S.

Now, we can represent the $C_n(x)$ by linear combinations of $T_0(x), \ldots, T_n(x)$.

$C_0(x)=1, C_1(x)=x=T_1(x)$, $C_2(x)=2x^2-1=T_2(x)$, $C_3(x)=4x^3-3x=T_3(x)$. In order to determine $C_4(x)$ note that

$$\frac{1}{2}T_4(x)+\frac{1}{2}T_2(x)=xT_3(x)=xC_3(x)=a_1C_4(x)+c_1C_2(x),$$

and $\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)=\frac{a_1-c_1}{2a_1}$, hence

$$C_4(x)=\frac{1}{2a_1}T_4(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)T_2(x).$$

In the same way we get

$$\begin{aligned}
&\frac{1}{2a_1}\left(\frac{1}{2}T_5(x)+\frac{1}{2}T_3(x)\right)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)\left(\frac{1}{2}T_3(x)+\frac{1}{2}T_1(x)\right)\\
&=\frac{1}{2a_1}xT_4(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)xT_2(x)=xC_4(x)=\frac{1}{2}C_5(x)+\frac{1}{2}C_3(x)\\
&=\frac{1}{2}C_5(x)+\frac{1}{2}T_3(x),
\end{aligned}$$

thus

$$\begin{aligned}
C_5(x) &= \frac{1}{2a_1}T_5(x)+\left(\frac{1}{2a_1}+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)-1\right)T_3(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)T_1(x)\\
&= \frac{1}{2a_1}T_5(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)T_1(x).
\end{aligned}$$

To determine $C_6(x)$ we use

$$\begin{aligned}
\frac{1}{2}T_6(x)+\frac{1}{2}T_0(x) &= T_3(x)T_3(x)=C_3(x)C_3(x)=a_1C_6(x)+c_1C_0(x)\\
&= a_1C_6(x)+c_1T_0(x),
\end{aligned}$$

then

$$C_6(x)=\frac{1}{2a_1}T_6(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)T_0(x).$$

Furthermore, we get $C_7(x)=\frac{1}{2a_12a_2}T_7(x)+\frac{1-2c_2}{2a_12a_2}T_5(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)T_1(x)$, and

$$C_8(x)=\frac{1}{2a_12a_2}T_8(x)+\frac{1-2c_2}{2a_12a_2}T_4(x)+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)T_2(x).$$

Applying $C_3(x)C_6(x)=a_2C_9(x)+c_2T_3(x)$, we obtain

$$C_9(x)=\frac{1}{2a_12a_2}T_9(x)+\left(\frac{1-2c_2}{2a_12a_2}+\frac{1}{a_1}\left(\frac{1}{2}-c_1\right)\right)T_3(x).$$

Moreover, we get as above

$$C_{10}(x) = \frac{1}{2a_12a_22a_3}T_{10}(x)+\frac{1-2c_3}{2a_12a_22a_3}T_8(x)$$

$$+\left(\frac{1-2c_2-2c_3+4c_2c_3}{2a_12a_22a_3}+\frac{1}{2a_1a_3}\left(\frac{1}{2}-c_1\right)\right)T_4(x)$$
$$+\left(\frac{1-2c_2}{2a_12a_22a_3}+\frac{1-2c_3}{2a_1a_3}\left(\frac{1}{2}-c_1\right)\right)T_2(x)$$

and

$$\begin{aligned}C_{11}(x) &= \frac{1}{2a_12a_22a_3}T_{11}(x)+\frac{1-2c_3}{2a_12a_22a_3}T_7(x)\\&+\left(\frac{1-2c_2-2c_3+4c_2c_3}{2a_12a_22a_3}+\frac{1}{2a_1a_3}\left(\frac{1}{2}-c_1\right)\right)T_5(x)\\&+\left(\frac{1-2c_2}{2a_12a_22a_3}+\frac{1-2c_3}{2a_1a_3}\left(\frac{1}{2}-c_1\right)\right)T_1(x)\end{aligned}$$

With $C_3(x)\,C_9(x)=a_3C_{12}(x)+c_3C_6(x)$, we obtain

$$C_{12}(x) = \frac{1}{2a_12a_22a_3}T_{12}(x)+\frac{a_1-c_3}{2a_1a_3}T_6(x)+\left(\frac{1-2c_2}{2a_12a_22a_3}+\frac{1-2c_3}{2a_1a_3}\left(\frac{1}{2}-c_1\right)\right)T_0(x).$$

Using the recurrence formula one could calculate $C_{13}(x), C_{14}(x)$, and so on. We stop at this point and restrict our studies to $C_{3n}(x)$.
We can determine the connection coefficients for $C_{3n}(x)$ directly by using the connection coefficients of P_n.
Let $P_n(x)=\sum_{k=0}^{n}\kappa_{n,k}T_k(x)$, then

$$C_{3n}(x)=P_n(T_3(x))=\sum_{k=0}^{n}\kappa_{n,k}T_k(T_3(x))=\sum_{k=0}^{n}\kappa_{n,k}T_{3k}(x). \tag{3.38}$$

That is, if we write $C_{3n}(x)=\sum_{j=0}^{3n}d_{3n,j}T_j(x)$, then $d_{3n,j}=0$ for $j=3l+1$, $j=3l+2$, where $l=0,\dots,3n-1$ and $d_{3n,3k}=\kappa_{n,k}$ for $k=0,1,\dots,n$. Since the $P_n(x)$ are symmetric, we have

$$\kappa_{2m,k}=0 \qquad \text{for } k=1,3,\dots,2m-1$$

and

$$\kappa_{2m+1,k}=0 \qquad \text{for } k=0,2,\dots,2m.$$

Thus, more precisely, $C_{3n}(x)$ can be represented by

$$\begin{aligned}C_{3n}(x) &= \kappa_{n,n}T_{3n}(x)+\kappa_{n,n-2}T_{3n-6}+\ldots+\kappa_{n,0}T_0(x), && n=2m, m\in\mathbb{N}_0\\ C_{3n}(x) &= \kappa_{n,n}T_{3n}(x)+\kappa_{n,n-2}T_{3n-6}(x)+\ldots+\kappa_{n,1}T_3(x), && n=2m+1, m\in\mathbb{N}_0.\end{aligned}$$

Moreover,

$$C_1(x)\,C_{3n}(x) = a_nC_{3n+1}(x)+c_nC_{3n-1}(x), \tag{3.39}$$
$$C_2(x)\,C_{3n}(x) = T_2(x)\,C_{3n}(x)=a_nC_{3n+2}(x)+c_nC_{3n-2}(x), \tag{3.40}$$
$$C_3(x)\,C_{3n}(x) = a_nC_{3n+3}(x)+c_nC_{3n-3}(x). \tag{3.41}$$

(3.40) can be shown by using (3.39) which yields

$$x^2C_{3n}(x) = a_nxC_{3n+1}(x)+c_nxC_{3n-1}(x)$$

$$
\begin{aligned}
&= a_n \frac{1}{2} C_{3n+2}(x) + a_n \frac{1}{2} C_{3n}(x) + c_n \frac{1}{2} C_{3n}(x) + c_n \frac{1}{2} C_{3n-2} \\
&= \frac{a_n}{2} C_{3n+2}(x) + \frac{1}{2} C_{3n}(x) + \frac{c_n}{2} C_{3n-2}(x).
\end{aligned}
$$

Thus $T_2(x) C_{3n}(x) = (2x^2 - 1) C_{3n}(x) = a_n C_{3n+2}(x) + c_n C_{3n-2}(x)$.

We can determine how the cubic Ruelle operator

$$
R_{(1,T_3)} f(y) = \frac{1}{3} \left(f(x_1) + f(x_2) + f(x_3) \right), \qquad y \in S
$$

acts on C_n, where $y = T_3(x_i)$, $i = 1, 2, 3$, (some x_i are counted twice).
The x_i satisfy the following relations, which are proved by comparison of coefficients in the following way.
We have

$$
\begin{aligned}
T_3(x) - y = 0 &= 4(x - x_1)(x - x_2)(x - x_3) \\
&= 4x^3 - 4(x_1 + x_2 + x_3) + 4(x_1x_2 + x_2x_3 + x_1x_3)x - 4x_1x_2x_3.
\end{aligned}
$$

Hence for given $y \in S$ the x_i satisfy

$$
x_1 + x_2 + x_3 = 0, \tag{3.42}
$$

$$
4(x_1x_2 + x_2x_3 + x_1x_3) = -3, \tag{3.43}
$$

$$
4x_1x_2x_3 = -y. \tag{3.44}
$$

Furthermore, combining (3.42) and (3.43) we obtain

$$
x_1^2 + x_2^2 + x_3^2 = \frac{3}{2}. \tag{3.45}
$$

In fact, we have

$$
0 = (x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_2x_3 + x_1x_3),
$$

and hence $x_1^2 + x_2^2 + x_3^2 = \frac{3}{2}$.

Proposition 3.7.1. *The Ruelle operator $R_{(1,T_3)}$ satisfies*

$$
R_{(1,T_3)}(C_{3n}) = P_n, \qquad R_{(1,T_3)}(C_{3n+1}) = 0 = R_{(1,T_3)}(C_{3n+2})
$$

for all $n \in \mathbb{N}_0$.

Proof. We have $R_{(1,T_3)}(C_{3n})(y) = \frac{1}{3}(C_{3n}(x_1) + C_{3n}(x_2) + C_{3n}(x_3)) = \frac{1}{3}(P_n(T_3(x_1)) + P_n(T_3(x_2)) + P_n(T_3(x_3))) = P_n(y)$ for each $n \in \mathbb{N}_0$.
By formula (3.39), we obtain

$$
\begin{aligned}
&a_n R_{(1,T_3)}(C_{3n+1})(y) + c_n R_{(1,T_3)}(C_{3n-1})(y) \\
&= \frac{1}{3}(C_1(x_1) C_{3n}(x_1) + C_1(x_2) C_{3n}(x_2) + C_1(x_3) C_{3n}(x_3)) \\
&= \frac{1}{3} P_n(y)(x_1 + x_2 + x_3) = 0
\end{aligned}
\tag{3.46}
$$

because of (3.42). By formula (3.40) and (3.45), it follows

$$\begin{aligned} & a_n R_{(1,T_3)}\left(C_{3n+2}\right)(y) + c_n R_{(1,T_3)}\left(C_{3n-2}\right)(y) \\ & = \frac{1}{3}\left(C_2\left(x_1\right) C_{3n}\left(x_1\right) + C_2\left(x_2\right) C_{3n}\left(x_2\right) + C_2\left(x_3\right) C_{3n}\left(x_3\right)\right) \\ & = \frac{1}{3} P_n(y)\left(T_2\left(x_1\right) + T_2\left(x_2\right) + T_2\left(x_3\right)\right) \\ & = \frac{1}{3} P_n(y)\left(2\left(x_1^2 + x_2^2 + x_3^2\right) - 3\right) = 0. \end{aligned} \tag{3.47}$$

Now, we apply induction to prove the statement. For $n = 1$, we have $R_{(1,T_3)}\left(C_1\right)(y) = \frac{1}{3}\left(x_1 + x_2 + x_3\right) = 0$ and $R_{(1,T_3)}\left(C_2\right)(y) = \frac{1}{3}\left(2\left(x_1^2 + x_2^2 + x_3^2\right) - 3\right) = 0$. Suppose that $R_{(1,T_3)}\left(C_{(3n-1)+1}\right)(y) = 0$ and $R_{(1,T_3)}\left(C_{(3n-1)+2}\right)(y) = 0$ holds. Then by (3.47), we get $R_{(1,T_3)}\left(C_{3n+2}\right)(y) = 0$, and by (3.46), $R_{(1,T_3)}\left(C_{3n+1}\right)(y) = 0$. □

Theorem 3.7.2. *For the cubic Ruelle operator $R_{(1,T_3)}$ and the cubic transformation operator $E(g) = Eg = g \circ T_3$, we have:*

$$\int_S R_{(1,T_3)} f(y)\, g(y)\, d\pi_P(y) = \int_S f(x)\, Eg(x)\, d\pi_C(x).$$

***Proof*.** By density arguments (compare to the quadratic case), we have to show for all $m, n \in \mathbb{N}_0$

$$\int_S R_{(1,T_3)}\left(C_m\right)(y)\, P_n(y)\, d\pi_P(y) = \int_S C_m(x)\, P_n\left(T_3(x)\right) d\pi_C(x).$$

For $m = 3k+1, 3k+2$ the integral at the left-hand side is zero because of Proposition 3.7.1. The integral at the right-hand side is zero because of $P_n\left(T_3(x)\right) = C_{3n}(x)$. It remains to consider the case $m = 3k$.

$$\int_S R_{(1,T_3)}\left(C_{3k}\right)(y)\, P_n(y)\, d\pi_P(y) = \int_S P_k(y)\, P_n(y)\, d\pi_P(y) = \begin{cases} \frac{1}{h_P(n)}, & k = n \\ 0, & \text{else} \end{cases},$$

and

$$\int_S C_{3k}(x)\, P_n\left(T_3(x)\right) d\pi_C(x) = \int_S C_{3k}(x)\, C_{3n}(x)\, d\pi_C(x) = \begin{cases} \frac{1}{h_C(3n)}, & k = n \\ 0, & \text{else} \end{cases}.$$

Thus it remains to check that $h_C(3n) = h_P(n)$, which follows immediately from the recurrence coefficients α_k, γ_k and $h_C(m) = \frac{\alpha_{m-1}}{\gamma_m} h_C(m-1)$. □

Now, we can study the action of the cubic Ruelle operator $R_{(1,T_3)}$ on $A(S)$, $L^p(S, \pi_C)$ in a similar way as in the quadratic case.

Proposition 3.7.3. *Let $f \in C(S)$ and denote $\mathcal{F}^P f(n) = \int_{-1}^1 f(y)\, P_n(y)\, d\pi_P(y)$ and $\mathcal{F}^C f(n) = \int_{-1}^1 f(y)\, C_n(y)\, d\pi_C(x)$. Then the Ruelle operator $R_{(1,T_3)}$ and its right inverse E, respectively, applied to $f \in C(S)$ give*

$$\mathcal{F}^P\left(R_{(1,T_3)} f\right)(n) = \mathcal{F}^C f(3n) \quad \textit{and} \quad \mathcal{F}^C(Eg)(3n) = \mathcal{F}^P g(n), \quad \textit{respectively.}$$

Proof. Let $g(y) = P_n(y)$ and $f(x) = C_{3k}(x)$, respectively, and apply Theorem 3.7.2. $\square$

Lemma 3.7.4. *Let $f \in L^p(S, \pi_C)$. For $1 \leqslant p < \infty$, we have*

$$\int_{-1}^{1} |R_{(1,T_3)} f(y)|^p d\pi_P(y) = \int_{-1}^{1} |f(x)|^p d\pi_C(x).$$

Proof. Splitting $S = [-1, 1]$ into $[-1, 0]$ and $[0, 1]$ the substitution of Theorem 3.7.2 directly yields equality. $\square$

Now, we consider the Wiener spaces $A_C(S)$ and $A_P(S)$. Although the $\{C_n(x)\}_{n=0}^{\infty}$ do not generate a polynomial hypergroup, the same argumentation as in Proposition 3.5.15 yields:

Proposition 3.7.5. $\left(A_C(S), \|\cdot\|_{A_C(S)}\right)$ *is a Banach space. Each $f \in A_C(S)$ has a representation $f(x) = \sum_{n=0}^{\infty} \mathcal{F}^C f(n) C_n(x) h(n)$ for all $x \in S$.*

Theorem 3.7.6. *For the Ruelle operator $R_{(1,T_3)}$, we have $R_{(1,T_3)} \in B(A_C(S), A_P(S))$ and for its right inverse E, $E \in B(A_P(S), A_C(S))$. For the operator norms, we have $\|R_{(1,T_3)}\| = 1$ and $\|E\| = 1$. Moreover,*

$$R_{(1,T_3)} f(y) = \sum_{k=0}^{\infty} \mathcal{F}^C f(3k) P_k(y) h_P(k) \quad \forall f \in A_C(S) \qquad \textit{and}$$

$$Eg(x) = \sum_{k=0}^{\infty} \mathcal{F}^C(Eg)(3k) C_{3k}(x) h_C(3k) \quad \forall g \in A_P(S)$$

Proof. The theorem is proven analogously to Theorem 3.5.16. $\square$

Remark 3.7.7. By Theorem 3.20 we see that $\mathrm{Ker} R_{(1,T_3)} = \{f \in A_C(S) : \mathcal{F}^C f(3k) = 0\ \forall k \in \mathbb{N}_0\}$, $\mathrm{Ker} E = \{g \in A_P(S) : \mathcal{F}^P g(k) = 0\ \forall k \in \mathbb{N}_0\}$.

Now, we study the product formulas for the $C_n(x)$

$$\begin{aligned} C_2(x) C_{3n+1}(x) &= g(2, 3n+1; 3n+3) C_{3n+3}(x) + g(2, 3n+1; 3n+1) C_{3n+1}(x) \\ &\quad + g(2, 3n+1; 3n-1) C_{3n-1}(x), \end{aligned}$$

$n \in \mathbb{N}$, where $g(m, n; n+m)$ are the linearizion coefficients introduced in Lemma 1.2.23. Using the formulas for $g(m, n; n+m)$ in Proposition 1.2.24 we get

$$\begin{aligned} g(2, 3n+1; 3n+3) &= \frac{\alpha_{3n+1}\alpha_{3n+2}}{\alpha_1} = \frac{1}{2}, \\ g(2, 3n+1; 3n+1) &= \frac{g(1, 3n+1; 3n)\alpha_{3n}}{\alpha_1} + \frac{g(1, 3n+1; 3n+2)\gamma_{3n+2}}{\alpha_1} \\ &\quad - \frac{g(0, 3n+1; 3n+1)\gamma_1}{\alpha_1} = a_n + \frac{1}{2} - 1 = a_n - \frac{1}{2}, \\ g(2, 3n+1; 3n-1) &= \frac{\gamma_{3n+1}\gamma_{3n}}{\alpha_1} = c_n. \end{aligned}$$

We have $C_1(x) C_{3n+1}(x) = \frac{1}{2} C_{3n+2}(x) + \frac{1}{2} C_{3n}(x)$ and thus by the above calculation

$$C_2(x) C_{3n+1}(x) = \frac{1}{2} C_{3n+3}(x) + \left(a_n + \frac{1}{2} - 1\right) C_{3n+1}(x) + c_n C_{3n-1}(x).$$

In order to get

$$\begin{aligned} C_3(x)\,C_{3n+1}(x) &= g(3,3n+1;3n+4)\,C_{3n+4}(x) + g(3,3n+1;3n+2)\,C_{3n+2}(x) \\ &\quad + g(3,3n+1;3n-2)\,C_{3n-2}(x)\,, \end{aligned}$$

we calculate

$$\begin{aligned} g(3,3n+1;3n+4) &= \frac{\alpha_{3n+1}\alpha_{3n+2}\alpha_{3n+3}}{\alpha_1\alpha_2} = a_{n+1}, \\ g(3,3n+1;3n+2) &= \frac{g(2,3n+1;3n+1)\,\alpha_{3n+1}}{\alpha_2} + \frac{g(2,3n+1;3n+3)\,\gamma_{3n+3}}{\alpha_2} \\ &\quad - \frac{g(1,3n+1;3n+2)\,\gamma_2}{\alpha_2}, \\ &= g(2,3n+1;3n+1) + g(2,3n+1;3n+3)\,2c_{n+1} - \frac{1}{2} \\ &= \left(a_n - \frac{1}{2}\right) + c_{n+1} - \frac{1}{2} = a_n + c_{n+1} - 1, \\ g(3,3n+1;3n) &= \frac{g(2,3n+1;3n-1)\,\alpha_{3n-1}}{\alpha_2} + \frac{g(2,3n+1;3n+1)\,\gamma_{3n+1}}{\alpha_2} \\ &\quad - \frac{g(1,3n+1;3n)\,\gamma_2}{\alpha_2} \\ &= g(2,3n+1;3n-1) + g(2,3n+1;3n+1) - g(1,3n+1;3n) \\ &= c_n + \left(a_n - \frac{1}{2}\right) - \frac{1}{2} = 0, \\ g(3,3n+1;3n-2) &= \frac{\gamma_{3n+1}\gamma_{3n}\gamma_{3n-1}}{\alpha_1\alpha_2} = c_n. \end{aligned}$$

Hence

$$C_3(x)\,C_{3n+1}(x) = a_{n+1}C_{3n+4}(x) + (a_n + c_{n+1} - 1)\,C_{3n+2}(x) + c_nC_{3n-2}(x)\,.$$

We proceed with

$$\begin{aligned} C_3(X)\,C_{3n+2}(x) &= g(3,3n+2;3n+5)\,C_{3n+5}(x) + g(3,3n+2;3n+3)\,C_{3n+3}(x) \\ &\quad + g(3,3n+2;3n+1)\,C_{3n+1}(x) + g(3,3n+2;3n-1)\,C_{3n-1}(x)\,. \end{aligned}$$

We calculate

$$\begin{aligned} g(3,3n+2;3n+5) &= \frac{\alpha_{3n+2}\alpha_{3n+3}\alpha_{3n+4}}{\alpha_1\alpha_2} = a_{n+1}, \\ g(3,3n+2;3n+3) &= \frac{g(2,3n+2;3n+2)\,\alpha_{3n+2}}{\alpha_2} + \frac{g(2,3n+2;3n+4)\,\gamma_{3n+4}}{\alpha_2} \\ &\quad - \frac{g(1,3n+2;3n+3)\,\gamma_2}{\alpha_2} \\ &= g(2,3n+2;3n+2) + g(2,3n+2;3n+4) - \alpha_{3n+2} \\ &= \left(c_{n+1} - \frac{1}{2}\right) + a_{n+1} - \frac{1}{2} = 0, \\ g(3,3n+2;3n+1) &= \frac{g(2,3n+2;3n)\,\alpha_{3n}}{\alpha_2} + \frac{g(2,3n+2;3n+2)\,\gamma_{3n+2}}{\alpha_2} \end{aligned}$$

$$
\begin{aligned}
& -\frac{g\left(1, 3n+2; 3n+1\right)\gamma_2}{\alpha_2} \\
&= 2a_n g\left(2, 3n+2; 3n\right) + g\left(2, 3n+2; 3n+2\right) - \gamma_{3n+2} \\
&= a_n + \left(c_{n+1} - \frac{1}{2}\right) - \frac{1}{2} = a_n + c_{n+1} - 1, \\
g\left(3, 3n+2; 3n-1\right) &= \frac{\gamma_{3n+2}\gamma_{3n+1}\gamma_{3n}}{\alpha_1\alpha_2} = c_n.
\end{aligned}
$$

Thus we have

$$C_3\left(x\right)C_{3n+2}\left(x\right) = a_{n+1}C_{3n+5}\left(x\right) + \left(a_n - c_{n+1} - 1\right)C_{3n+1}\left(x\right) + c_n C_{3n-1}\left(x\right).$$

At this point one could continue the calculation in the same manner as above, but we stop this tedious calculations at this point since we have already seen that the linearization coefficients for $\{C_n\left(x\right)\}_{n=0}^{\infty}$ can be negative. Thus with Theorem 1.2.25, these polynomials do not generate a polynomial hypergroup.

4 The transfer operator on path space and future work

We conclude this thesis with a short outlook on another application of transfer operators, where our transfer operator defined via the preimages of the Chebyshev polynomials of the first kind fits into the concept developed by P.E.T. Jorgensen in [**28**]. Then, we provide a brief outline concerning future work.

4.1 The transfer operator on path-space

Motivated by the fact that various problems of scaling relations have traditionally been addressed with only standard tools from analysis, in [**28**], P.E.T. Jorgensen employs a mix of analysis and path-space methods from probability since many problems in dynamics are governed by transition probabilities W, and P_x, and by an associated transfer operator R_W. We briefly illustrate the work of P.E.T. Jorgensen and show that those concepts are also applicable to the transfer operator we defined in Chapter 3.

Let $(X, \mathcal{B})$ be a measure space and σ a finite-to-one measurable endomorphism on X. That is, for all $B \in \mathcal{B}$ the inverse image

$$\sigma^{-1}(B) := \{x \in X : \sigma(x) \in B\}$$

is again an element of $\mathcal{B}$. We assume that

$$\#\sigma^{-1}(\{x\}) = N \qquad \forall x \in X. \tag{4.1}$$

Let μ be a probability measure on $\mathcal{B}$, and assume that the singletons $\{x\}$, for $x \in X$, are in $\mathcal{B}$, but the measure μ does not need to be atomic.
Considering (4.1), we can label the sets $\sigma^{-1}(\{x\})$ by

$$\mathbb{Z}_N = \{0, \ldots, N-1\} = \mathbb{Z}/N\mathbb{Z}.$$

An enumeration of the inverse images of σ, $\sigma(\tau_i(x)) = x$, determines the branches τ_i which are measurable maps on X. In the following, we will be interested in random walks on these branches τ_i.
Let $W : X \to \mathbb{R}_0^+$ be measurable and satisfy

$$\sum_{y \in X, \sigma(y)=x} W(y) \leqslant 1, \qquad \mu \text{ a.e. } x \in X. \tag{4.2}$$

We define the probability of a transition from x to one of the points $(\tau_i(x))$ in $\sigma^{-1}(x)$ by

$$P(x, \tau_i(x)) := W(\tau_i(x)). \tag{4.3}$$

Hence by (4.2),

$$\sum_i P(x, \tau_i(x)) = \sum_{\sigma(y)=x} W(y) \leqslant 1.$$

The "$\leqslant$" indicates that our random-walk model can also include dissipation.

Definition 4.1.1. (1) The *Ruelle operator* $R = R_W$ associated to these random walks and some nonnegative weight function W on X is defined by

$$Rf(x) = \sum_{y\in X, y\in\sigma^{-1}(x)} W(y) f(y), \qquad x \in X, f \in L^\infty(X),$$

where $\sum_{y\in\sigma^{-1}(x)} W(y) = 1$.

(2) Let Ω be the compact *Cartesian product*

$$\Omega = \mathbb{Z}_N^{\mathbb{N}} = \{0, \ldots, N-1\}^{\mathbb{N}} = \prod_1^\infty \{0, \ldots, N-1\}.$$

Let $C(\Omega)$ denote the algebra of all continuous functions on Ω.

(3) A *path* starting at x is a finite or infinite sequence of points $(z_1, z_2, \ldots)$ such that $\sigma(z_1) = x$ and $\sigma(z_{n+1}) = z_n$ for all n; it can be identified with $(\tau_{\omega_1}x, \tau_{\omega_2}\tau_{\omega_1}x, \ldots, \tau_{\omega_n} \cdot \cdots \tau_{\omega_1}x, \ldots)$. The set of infinite paths starting at x is denoted by Ω_x, the set of paths of length n starting at x by $\Omega_x^{(n)}$, and the set of all infinite paths starting at any point in X by X_∞.

(4) A bounded measurable function V on X^∞ is called a *cocycle* if for any path $(z_1, z_2, \ldots)$,

$$V(z_1, z_2, \ldots) = V(z_2, z_3, \ldots).$$

(5) For $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{Z}_N$,

$$A(i_1, \ldots, i_n) := \{\omega \in \Omega : \omega_1 = i_1, \ldots, \omega_n = i_n\}$$

is a *cylinder set*.

One can construct probability measures P_x on Ω_x, $x \in X$, such that for a function f on Ω_x, only dependent on the first $n+1$ coordinates,

$$P_x[f] = \sum_{(z_1,\ldots,z_n)\in\Omega_x^{(n)}} W(z_1) W(z_2) \cdots W(z_n) f(z_1, \ldots, z_n).$$

As the cylinder sets generate the topology of Ω and its Borel σ-algebra, we will first specify them on cylinder sets in order to obtain the Radon measures on Ω.

Lemma 4.1.2. *Let $(X, \mathcal{B})$ be a measure space, μ a probability measure on $\mathcal{B}$, W, σ and $\tau_0, \ldots, \tau_{N-1}$ as above. Then for every $x \in X$ there is a unique positive Radon probability measure P_x on Ω such that*

$$P_x(A(i_1, \ldots, i_n)) = W(\tau_{i_1}x) W(\tau_{i_2}\tau_{i_1}x) \cdots W(\tau_{i_n} \cdots \tau_{i_1}x).$$

Proof. See [**28**] pp. 44-46 and p. 53 Remark 2.8.3. □

Theorem 4.1.3. *Let X, W, N be as above and*

$$\sum_{y\in X,\sigma(y)=x} W(y) = 1 \qquad a.e.\ x \in X$$

be satisfied. Suppose that $R = R_W$ is the Ruelle operator on $L^\infty(X)$ and $\{P_x : x \in X\}$ the process on Ω (see Lemma 4.1.2). Then there is a $1-1$ correspondence between the bounded harmonic functions h and the cocycles V. That is, for a cocycle $V : X \times \Omega \to \mathbb{C}$, $h(x) = h_V(x) = P_x[V(x,\cdot)]$ is harmonic, i.e. $Rh = h$. Conversely, V can be recovered from a martingale limit.

Proof. See [**28**] pp. 49-50. □

4.2 Inverse branches of Chebyshev polynomials of the first kind

The concept introduced in Section 4.1 carries over to the case that

$$\#\sigma^{-1}(\{x\}) \leqslant N \qquad \forall x \in S, \tag{4.4}$$

for $S = [-1,1]$ and

$$\sigma(x) = T_N(x), \qquad N = 2,3,\ldots.$$

In this case, we are dealing with the inverse branches of the Chebyshev polynomials of the first kind. For each N we can divide S into N disjoint intervals such that the Nth order Chebyshev polynomial of the first kind has N inverse branches $\{\tau_0, \tau_1, \ldots, \tau_{N-1}\}$, which are one-to-one, and every element $\tau_{\omega_i}(x)$ of an infinite path $\cdots\tau_{\omega_i}\cdots\tau_{\omega_1}(x)$ is defined for all i and all $\omega \in \Omega$. In order to show that this is true, we need to proof the following lemma:

Lemma 4.2.1. *If there is a $\tilde{\omega} \in \Omega$ such that $\tau_{\tilde{\omega}_j}\cdots\tau_{\tilde{\omega}_1}(x) = -1$ (and $\tau_{\tilde{\omega}_j}\cdots\tau_{\tilde{\omega}_1}(x) = 1$, respectively) for some $j \in \{1,\ldots,N\}$, then $\tau_{\tilde{\omega}_i}\cdots\tau_{\tilde{\omega}_1}(x) \neq 1$ (and $\tau_{\tilde{\omega}_i}\cdots\tau_{\tilde{\omega}_1}(x) \neq -1$, respectively) for all $i \in \{1,\ldots,N\}$, $i \neq j$.*

Proof. If we enumerate the inverse branches τ_i, with $\tau_0 = \sigma^{-1}|_{\left[-1,\cos\left(\frac{(N-1)\pi}{N}\right)\right]}, \ldots, \tau_{N-1} = \sigma^{-1}|_{\left[\left(\frac{\pi}{N}\right),1\right]}$, then we have

$$\tau_i(x) \begin{cases} \neq 1, & i = 0,\ldots,N-2,\ x \in S \\ \neq 1, & i = N-1,\ x \neq 1 \\ = 1, & i = N-1,\ x = 1 \end{cases} \tag{4.5}$$

and

$$\tau_i(x) \begin{cases} \neq -1, & i = 1,\ldots,N-1,\ x \in S \\ \neq -1, & i = 0,\ x \neq -1 \\ = -1, & i = 0,\ x = -1 \end{cases}. \tag{4.6}$$

For j there is some $\tilde{\omega} \in \Omega$ such that $\tau_{\tilde{\omega}_j} \cdots \tau_{\tilde{\omega}_1}(x) = -1$. Suppose $i \leqslant j$. We assume that there is an $i \in \{1, \ldots, j\}$ such that

$$\tau_{\tilde{\omega}_i} \cdots \tau_{\tilde{\omega}_1}(x) = 1.$$

Then $\tau_{\tilde{\omega}_{i-1}} \cdots \tau_{\tilde{\omega}_1}(x)$ has to equal 1 as well, and by iteration $\tau_{\tilde{\omega}_k} \cdots \tau_{\tilde{\omega}_1}(x) = 1$ for all $k \leqslant j$. By (4.6), this is a contradiction to the assumption $\tau_{\tilde{\omega}_j} \cdots \tau_{\tilde{\omega}_1}(x) = -1$.
Suppose $i \geqslant j$. We assume that there is a i such that $\tau_{\tilde{\omega}_i} \cdots \tau_{\tilde{\omega}_j} \cdots \tau_{\tilde{\omega}_1}(x) = 1$, that is,

$$\tau_{\tilde{\omega}_i} \cdots \tau_{\tilde{\omega}_{j+1}}(-1) = 1. \tag{4.7}$$

Hence $\tau_{\tilde{\omega}_{i-1}} \cdots \tau_{\tilde{\omega}_{j+1}}(-1) = 1$ and by iteration $\tau_{\tilde{\omega}_{j+1}}(-1) = 1$ which is a contradiction to (4.5).
The proof of the case that $\tau_{\tilde{\omega}_j} \cdots \tau_{\tilde{\omega}_1}(x) = 1$ works analogously. □

With the previous lemma, Lemma 4.1.2 is also valid in the current setting.

4.3 Discussion and Outlook

We defined a transfer operator via the preimages of the Chebyshev polynomials of the first kind and studied it acting on classical function spaces from harmonic analysis. Our investigation was guided by the symmetry properties and the permutability of the Chebyshev polynomials of the first kind. We were able to construct a symmetric orthogonal polynomial sequences via a quadratic and a cubic transformation, respectively. In the quadratic case, this orthogonal polynomial sequence generates a polynomial hypergroup provided certain conditions on the recurrence coefficients are satisfied. That is, we investigated the transfer operator with preimage given by the Chebyshev polynomials of the first kind acting on arbitrary symmetric orthogonal polynomials. The framework developed in this thesis gives rise to various applications in the fields of dynamical systems, wavelet theory and iterated function systems which we provided the basic concept for. Using the results of this thesis, we intend to study iterations of the transfer operator, the transformation operator and the corresponding iterated function spaces as well as the corresponding discrete spaces and discrete operators in future. This way, we suppose to find a new class of wavelets corresponding to the work done in **[24]**. Furthermore, various kinds of weight functions could be investigated in future.

For the cubic case, future work will involve the more general case where $0 < \alpha_1 < 1$, instead of using the restriction that $\alpha_1 = \frac{1}{2}$. In future, we not only aim to consider the quadratic and the cubic case, but perform similar constructions for the quintic case and other prime numbers.

In the previous two sections of this chapter, we gave a short glimpse of how future work on transfer operators might look like that are defined via orthogonal polynomials on random walks, especially those ones given by inverse branches of orthogonal polynomials. Iterated function systems and fractals could also be employed, for example in two dimensions like the Sierpinski gasket, referring to **[15]** and **[17]** .

Table of symbols and abbreviations

$\#$	cardinality of a set
$\mathbf{1}$	constant 1 function
$A(i_1,\ldots,i_n)$	$\{\omega \in \Omega : \omega_1 = i_1, \ldots, \omega_n = i_n\}$ cylinder set
$(a)_n$	$(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$ Pochhammer symbol
A_{nk}	weights in the Gauss quadrature formula
$A(S)$	$\{f \in C(S) : \sum_{n=0}^{\infty} \lvert\check{f}(n)\rvert h(n) < \infty\}$ Wiener algebra
$A^p(S)$	$\{f \in L^1(S,\pi) : \check{f} \in \ell^p(h)\}$
$\mathcal{B}$	Borel σ-algebra
$B(x,r)$	open ball centered at x with radius r
$B_n(x,r)$	n-Bowen ball centered at x with radius r
$(U)^c$, $\overline{U}$	closure of a set/space U
χ_S	characteristic function of S
$\mathcal{C}(X)$	space of all nonempty compact subsets of X
$C(X)$	$C(X,\mathbb{R})$ space of all continuous functions $f : X \to \mathbb{R}$; $C(X,\mathbb{C})$ space of all continuous functions $f : X \to \mathbb{C}$; space of all continuous functions on X
$C_{\mathbb{C}}(X)$	$C(X,\mathbb{C})$ space of all continuous complex-valued functions $f : X \to \mathbb{C}$
$C^{\alpha}(X)$	$C^{\alpha}(X,\mathbb{R})$ space of all α-Hölder continuous functions on $f : X \to \mathbb{R}$
$C^{\alpha}_{\mathbb{C}}(X)$	$C^{\alpha}(X,\mathbb{C})$ space of all α-Hölder continuous functions on $f : X \to \mathbb{C}$
$C^{\alpha\perp}_{\mathbb{C}}(X)$	$\{\phi \in C^{\alpha}_{\mathbb{C}}(X) : \langle \mu, \phi \rangle = 0\}$
$C^{\alpha}_{K,s}(X)$	$C^{\alpha}_{K,s}(X,\mathbb{R}) = \{\phi \in C^{\alpha}(X) : \phi \geqslant s, [\log \phi]_{\alpha} \leqslant K\}$
$C_0(K)$	space of all continuous functions on K which vanish at infinity
$C^b(K)$	space of all bounded continuous functions on K
$C_c(K)$	space of all compactly supported continuous functions on K
$\deg(P)$	degree of the polynomial P
δ_x	Dirac measure at x
$\hat{d}(x)$	$\sum_{n=0}^{\infty} d(n)\, P_n^{(\alpha,\beta)}(x)\, h(n)$
$d_n(x,y)$	$\max_{0\leqslant i\leqslant n}\{d(f^i(x), f^i(y))\}$ n-Bowen metric
$d_{n,k}$	connection coefficients
E	transformation operator
$E(\phi\mathcal{F}_n)$	conditional expectation of ϕ given $\mathcal{F}_n$
ϵ_x	point measure of x
$\hat{f}$	Fourier transform of f
$\check{f}(n)$	$\int_{-1}^{1} f(x)\, P_n(x)\, d\pi(x)$ Fourier transform of f w.r.t. P_n
$\mathcal{F}^P f(n)$	Fourier transform of f w.r.t. P_n
${}_qF_p$	${}_qF_p(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n x^n}{(\beta_1)_n\cdots(\beta_q)_n n!}$ hypergeometric function
$\mathcal{F}_0$	standard σ-algebra on $\mathbb{R}$ generated by all open sets
$\mathcal{F}_{\phi}$	$\phi^{-1}(\mathcal{F}_0)$ pull-back σ-algebra
$\mathcal{F}_{\Gamma}$	minimal σ-algebra of all σ-algebras $\mathcal{F}_{\phi}$ for ϕ in a family of functions Γ

$\mathcal{F}_n$	$\mathcal{F}_{ImP_n}$
$\mathcal{F}_\infty$	$\cup_{n=1}^{\infty} \cap_{m\geqslant n} \mathcal{F}_m$
$\mathcal{F}_X$	standard σ-algebra generated by all open sets in X
$\mathcal{G}_n$	$\{\mu \in M^1(X) : P_n^*\mu = \mu\}$
$\mathcal{G}_\infty$	$\cap_{n=1}^{\infty}\mathcal{G}_n$ set of all G-measures
$g(m,n;k)$	linearization coefficients
$h(n)$	Haar function for an OPS
$\mathcal{H}$	abstract Hilbert space
$[\phi]_\alpha$	local Hölder constant
$\tilde{}$	involution (see Lemma 1.2.2, Definition 1.2.14)
K	locally compact Hausdorff space; set
$(K,\omega,\tilde{})$	(discrete) hypergroup
ℓ^1	$\{f : K \to \mathbb{C}$: $f = \sum_{n=1}^{\infty} a_n\epsilon_{x_n}$, $a_n \in \mathbb{C}$, $\sum_{n=1}^{\infty} \lvert a_n\rvert < \infty$, $x_n \in K\}$
ℓ^1_{co}	$\{f \in \ell^1$: $f = \sum_{n=1}^{N} \alpha_n\epsilon_{x_n}$,$\alpha_n \geqslant 0$,$\sum_{n=1}^{N} \alpha_n = 1\}$
$\ell^p(h)$	$\{f : K \to \mathbb{C} : \sum_{n\in K} \lvert f(n)\rvert^p h(n) < \infty\}$
$\ell^\infty(h)$	$\{f : K \to \mathbb{C} : \sup \lvert f(n)\rvert h(n) < \infty\}$
$L^p(S,\pi)$	$\{f : S \to S : f$ measurable, $\int_S \lvert f(x)\rvert d\pi(x) < \infty\}$
$L^\infty(S,\pi)$	$\{f : S \to S : f$ measurable, $\operatorname{ess\,sup}_{x\in S} \lvert f(x)\rvert (x) < \infty\}$
L_x	left-translation operator
$\{L_n(x)\}_{n=0}^{\infty}$	Legendre polynomials
$\{L_n^\nu(x)\}_{n=0}^{\infty}$	associated Legendre polynomials
m_0	(polynomial) wavelet filter
m_N	weight function for the Ruelle operator
M	$\sqrt{N}\sum_{k\in\mathbb{Z}} a_k\psi(Nx-k)$ cascade refinement operator
$M(X)$	space of all (complex) measures
$M^1(X)$	space of all (complex) probability measures
$\mathcal{M}$	moment functional
μ	probability measure
μ_n	moment of order n
$\{\mu_n\}_{n=0}^{\infty}$	moment sequence
$\omega(\cdot,\cdot)$	$\omega(x,y) : K \times K \to M^1(K)$ (see Lemma 1.2.1);
	$\omega(f,g) : K \times K \to \ell^1_{co}$ (see Definition 1.2.14)
Ω	$\mathbb{Z}_N^{\mathbb{N}} = \{0,\ldots,N-1\}^{\mathbb{N}}$
Ω_x	set of all infinite paths starting at x
$\Omega_x^{(n)}$	set of all paths of length n starting at x
OPS	orthogonal polynomial sequence
π	probability measure; orthogonalization measure for an OPS
$\mathcal{P}$	vector space of polynomials with complex coefficients in one variable
$\mathcal{P}_n$	set of all polynomials of degree at most n
$\mathcal{P} = \{P_n\}_{n=1}^{\infty}$	sequence of Markovian projections
$\{P_n^\alpha(x)\}_{n=0}^{\infty}$	ultraspherical polynomials
$\{P_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$	Jacobi polynomials
$P_x[f]$	measure on Ω such that $P_x[f] = P_x^{(n)}$
$P(x,\tau_i(x))$	transition probability form x to τ_i
$Rep(\mathcal{U}_N,\mathcal{H})$	normal representations
R_x	right-translation operator
RPF	Ruelle-Perron-Frobenius (operator)

R	Ruelle operator, Ruelle-Perron-Frobenius operator, transfer operator
$R_{(1,T_N)}$	$\frac{1}{N}\sum_{T_N(x)=y} f(x)$ unweighted Ruelle operator/transfer operator
$R_{(m_N,T_N)}$	$\frac{1}{N}\sum_{T_N(x)=y} m_N(x) f(x)$ weighted Ruelle operator/transfer operator
$\{R_n(x;a)\}_{n=0}^{\infty}$	Karlin-McGregor polynomials
RWS	random walk polynomial sequence
$\operatorname{sgn}(f)$	signum of a function f
SRWS	symmetric random walk polynomial sequence
$\operatorname{supp}(f)$	$\{x \in X : f(x) \neq 0\}$ support of f
τ_i	inverse branches
$\{T_n(x)\}_{n=0}^{\infty}$	Chebyshev polynomials of the first kind
$\{T_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$	generalized Chebyshev polynomials
T_x	generalized translation operator
$\{U_n(x)\}_{n=0}^{\infty}$	Chebyshev polynomials of the second kind
$\mathcal{U}_N$	C^*-algebra on two unitary generators U and V satisfying $UVU^{-1} = V^N$
$x_i^{(N)}$	ith preimage of the Nth order Chebyshev polynomial of the 1st kind
x_{nk}	zeros of an OPS
$(X, \mathcal{B})$	set X with a σ-algebra $\mathcal{B}$ of measurable subsets
(X, d)	metric space
X_∞	set of all infinite paths starting at any point in X
$\mathbb{Z}_N$	$\{0, \ldots, N-1\} = \mathbb{Z}/N\mathbb{Z}$ cyclic group of order N

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