Maxima of Diffusion Processes of Gradient Field Type with Respect to the Level Sets of the Potential

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Abstract

Let $(X_t)_{t\geq 0}$ be a $\mathbb{R}^n$-valued diffusion process of gradient field type where the drift is given by the gradient of a potential Φ. Our aim is to analyze the extreme fluctuations of $(X_t)_{t\geq 0}$ in the following sense. Consider the maximum $M_T := \max_{0\leq s\leq T} q(X_s)$, where the distance function q is generated by the level sets of the potential Φ. We characterize the tail asymptotics of M_T for fixed $T > 0$ as well as the long time behavior of M_T as $T \to \infty$. This approach is adapted to the geometry of the problem and emphasizes the regions where the potential Φ is flat and large fluctuations of Φ are expected. Some examples are also presented.

Keywords: absorption probability, diffusion process, Dirichlet form, Dirichlet problem, eigenvalue asymptotics, extreme value theory, generator, semigroup, tail behavior, Temple's inequality, variational principle.

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1 Introduction

We consider a diffusion process $(X_t)_{t\geq 0}$ with state space $\mathbb{R}^n$, $n \in \mathbb{N}$, of gradient field type, i.e. $(X_t)_{t\geq 0}$ solves a SDE of the form

$$dX_t^i = -\partial_{x_i}\Phi(X_t)dt + \sigma dB_t^i \qquad i = 1, \dots, n\,, \tag{1.1}$$

where $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$, $\sigma > 0$ and $(B_t^i)_{t\geq 0}$, $i = 1, \dots, n$, are independent one-dimensional standard Brownian motions. Note that $(X_t)_{t\geq 0}$ is symmetric (and hence reversible) w.r.t. the measure μ on $\mathbb{R}^n$ with Lebesgue density $\widetilde{\mu}$ given by

$$\widetilde{\mu}(x) = e^{-2\Phi(x)/\sigma^2} \qquad x \in \mathbb{R}^n\,. \tag{1.2}$$

μ is also the stationary (or invariant) measure of the process $(X_t)_{t\geq 0}$ and we assume μ to be finite.

The main interest of this paper is to characterize the extreme fluctuations of $(X_t)_{t\geq 0}$. More precisely, we analyze the asymptotic behavior of the partial maxima of $(X_t)_{t\geq 0}$ w.r.t some distance function q, i.e. we study the random variables

$$M_T := \max_{0\leq s\leq T} q(X_s) \quad T \geq 0\,.$$

The distance function q can be of quite general form, we only assume that q is generated by an *exhausting family* of $\mathbb{R}^n$. This is by definition an increasing family $(O_R)_{R>R_0}$ of open, bounded subsets of $\mathbb{R}^n$ with smooth boundary, such that $\bigcup_{R>R_0} O_R = \mathbb{R}^n$. The set $\{R > R_0 : x \in O_R\}$ is not empty for every $x \in \mathbb{R}^n$ and the distance function is then given by

$$q(x) = \inf\{R > R_0 : x \in O_R\} \qquad x \in \mathbb{R}^n\,. \tag{1.3}$$

The simplest example for an exhausting family of $\mathbb{R}^n$ are the open balls $B_R := \{x \in \mathbb{R}^n : |x| < R\}$, $R > 0$, where $|\cdot|$ is the Euclidean norm. The associated distance function q coincides with the Euclidean norm.

M_T is related to absorption at the boundary of the sets O_R, $R > R_0$, in the following sense. We denote by

$$\tau_R := \inf\{s > 0 : X_s \in \mathbb{R}^n \setminus O_R\} \tag{1.4}$$

the *first exit time* off O_R of the process $(X_t)_{t\geq 0}$, where P_μ is the law of the process $(X_t)_{t\geq 0}$ starting with its stationary measure μ. Then

$$P_\mu(M_T \leq R) = P_\mu(\tau_R > T) \qquad R > R_0,\, T > 0\,. \tag{1.5}$$

Our approach is inspired by the articles of Newell [New62] and Iscoe and McDonald [IM89, IM92]. For more information on the history of this problem for one-dimensional diffusions and some treatment of multi-dimensional processes we refer to Kunz [Kun02] and the references therein. The key idea of our approach is to express the probability $P_\mu(M_T \leq R)$ in terms of the backward semigroup associated to the part of the process $(X_t)_{t\geq 0}$ on the ball O_R, i.e. to the process $(X_t)_{t\geq 0}$ killed when it leaves O_R. The generator of this semigroup, denoted by L_R, is given by the generator of the full process subject to Dirichlet boundary conditions on the domain O_R. The probability $P_\mu(M_T \leq R)$ for large R is essentially determined by the behavior of the bottom eigenvalue λ_R of $-L_R$ in the limit when the domains O_R extend to $\mathbb{R}^n$ as $R \to \infty$.

In Kunz [Kun02] we considered the maximum M_T w.r.t. Euclidean norm. To evaluate the eigenvalue asymptotics and hence the asymptotic behavior of M_T, the potential Φ in the SDE (1.1) was approximated by a rotationally symmetric potential. The associated process can then be regarded as a one-dimensional process and the eigenvalue asymptotics is known in this case, see e.g. Newell [New62]. We gave conditions on the asymmetric part of Φ such that the eigenvalue asymptotics for the process associated to the rotationally symmetric potential was not destroyed. From the point of view of spectral analysis, we gave conditions, when the eigenfunctions corresponding to the bottom eigenvalue λ_R can be suitably approximated by rotationally symmetric functions. If these conditions fail or if we replace the balls $(B_R)_{R>0}$ by an arbitrary exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$ we can no longer use rotationally symmetric functions for the evaluation of the eigenvalue asymptotics.

In Kunz [Kun01], having again the balls $(B_R)_{R>0}$ as exhausting family of $\mathbb{R}^n$, singular perturbation techniques where used to study the asymptotic shape of the eigenfunction ψ_R corresponding to the bottom eigenvalue λ_R. It turns out that the rate of decay of ψ_R near the boundary of B_R depends on the spherical variables and has to be adjusted to the slope of the potential Φ in the particular direction. From this analysis an asymptotic expression for λ_R as $R \to \infty$ can be obtained. It was only assumed that the second derivative in radial direction and the first angular derivatives of Φ vanish faster than the first derivative of Φ in radial direction as $|x| \to \infty$. But these techniques have more or less heuristic character, since it is a-priori assumed that the eigenfunctions ψ_R, $R > R_0$, have an appropriate asymptotic expansion.

In this paper, we choose an exhausting family of $\mathbb{R}^n$ which is more adapted to the geometry of the problem, namely the level sets of the potential Φ itself. Assume for the potential Φ in the SDE (1.1) that

$$\Phi(x) \to \infty \qquad (|x| \to \infty)\,. \tag{1.6}$$

We set

$$O_R^\Phi := \{x \in \mathbb{R}^n : \Phi(x) < R\} \qquad R > R_0 := \min_{x \in \mathbb{R}^n} \Phi(x)\,. \tag{1.7}$$

Note that R_0 is well defined by condition (1.6) and the sets O_R^Φ are bounded for every $R > R_0$. Hence $(O_R^\Phi)_{R>R_0}$ is an exhausting family of $\mathbb{R}^n$ in the sense of the definition. This choice of the exhausting family has the following advantage: large fluctuations of $(X_t)_{t\geq 0}$ are expected in regions where the potential Φ is flat. The level sets of Φ are more extended in this regions and hence stress the directions of large fluctuations of $(X_t)_{t\geq 0}$. Further the choice of the level sets of Φ suggests that we should use test-functions for the evaluation of the eigenvalue asymptotics which are constant on the iso-level sets of Φ, i.e. they are of the form $f \circ \Phi$, where f is a real function. We give conditions when the sharp eigenvalue asymptotics can be obtained by means of test-functions of the shape described above. This yields also to a characterization of the maximum M_T of the process w.r.t. the distance function generated by the level sets $(O_R^\Phi)_{R>R_0}$ of the potential Φ.

As an example we present, apart from the obvious rotationally symmetric case, a diffusion processes of gradient field type, where the asymmetric part of the potential Φ factorizes in radial and spherical component. Further we give the example of a diffusion processes of gradient field type with potential of tetragonal shape. In this case the conditions for getting sharp eigenvalue asymptotics can be explicitly evaluated.

The structure of this paper is as follows. The results are stated in section 2. In section 3 we recall some facts from the theory of Markov processes an operator theory. The proofs of the results are given in section 4 and some examples are presented in section 5.

2 Results

Let $(X_t)_{t\geq 0}$ be a diffusion process of gradient field type solving the SDE (1.1) with potential $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$. Assume that $(X_t)_{t\geq 0}$ is symmetric w.r.t. its stationary measure μ with Lebesgue density $\widetilde{\mu}(x) = e^{-2\Phi(x)/\sigma^2}$, $x \in \mathbb{R}^n$. The existence of a μ-symmetric weak solution of the SDE (1.1) is guaranteed (see section 3 for more details) assuming

$$\int_{\mathbb{R}^n} e^{-2\Phi/\sigma^2} |\nabla\Phi|^2 dx < \infty \,, \tag{2.1}$$

where ∇ denotes the gradient. Further assume that the stationary measure μ is finite, i.e.

$$Z_\sigma := \int_{\mathbb{R}^n} e^{-2\Phi(x)/\sigma^2} dx < \infty \,. \tag{2.2}$$

Let $(O_R)_{R>R_0}$ be an arbitrary exhausting family of $\mathbb{R}^n$. It will turn out that the asymptotic distribution of $M_T := \max_{0\leq t\leq T} q(X_t)$, where the distance function q is generated by $(O_R)_{R>R_0}$ according to (1.3), is determined by spectral properties of the generator of the process $(X_t)_{t\geq 0}$. This generator reads formally

$$Lu = \frac{\sigma^2}{2}\Delta u - \sum_{i=1}^n \partial_{x_i}\Phi \, \partial_{x_i} u = \frac{\sigma^2}{2} e^{2\Phi/\sigma^2} \sum_{i=1}^n \partial_{x_i} \left(e^{-2\Phi/\sigma^2} \partial_{x_i} u \right) \,. \tag{2.3}$$

For $R \in (R_0, \infty]$, the operator L acting on $L^2(O_R, \mu)$ with Dirichlet boundary conditions on O_R is denoted by L_R (set $O_\infty := \mathbb{R}^n$ and no boundary conditions are present, see section 3 for a definition); the operator L_R generates a strongly continuous contraction semigroup $(e^{L_R t})_{t\geq 0}$ on $L^2(O_R, \mu)$. Assume that $-L_\infty$ enjoys the spectral gap property in the sense that

$$\Lambda := \inf \Sigma(-L_\infty) \cap (0, \infty) > 0 \,, \tag{2.4}$$

where Σ denotes the spectrum of the operator. In Proposition 3.2 we state a sufficient condition for (2.4) to hold. For $R \in (R_0, \infty]$, the bottom eigenvalue of the operator $-L_R$ is denoted by $\lambda_R := \inf \Sigma(-L_R)$. Proposition 2.1 of Kunz [Kun02] states that for every $T > 0$ and sufficiently large $R > R_0$

$$(1 - \lambda_R/\Lambda) e^{-\lambda_R T} \leq P_\mu(M_T \leq R) \leq e^{-\lambda_R T} \,. \tag{2.5}$$

Evidently $\lambda_R \to 0$ as $R \to \infty$ and we need an explicit asymptotic expression for the convergence $\lambda_R \to 0$ as $R \to \infty$.

We make use of the following *asymptotic notations*: given two real functions a and b, we write $a(t) \sim b(t)$ and $a(t) \lesssim b(t)$ as $t \to t_0 \in \mathbb{R} \cup \{\pm\infty\}$ if $\lim_{t\to t_0} a(t)/b(t) = 1$ and $\limsup_{t\to t_0} a(t)/b(t) \leq 1$, respectively. By $a(t) \gtrsim b(t)$ we mean that $b(t) \lesssim a(t)$ as $t \to t_0$ and we write further $a(t) = o(b(t))$ as $t \to t_0$ if $\lim_{t\to t_0} |a(t)/b(t)| = 0$.

We have to find a simple function $l : \mathbb{R}^+ \to \mathbb{R}^+$, given in terms of the potential Φ and the diffusion coefficient σ, such that

$$\lambda_R \sim l(R) \qquad (R \to \infty)\,. \tag{2.6}$$

Assume for the moment that such a function l is already given. The following proposition shows that one can replace in (2.5) the bottom eigenvalue λ_R by the asymptotic expression $l(R)$, see Theorems 2.3 and 2.4 of Kunz [Kun02].

Proposition 2.1 *Assume that there exists a function l satisfying (2.6). Then*
(a) for every $T > 0$

$$T\,l(R) \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda)\,l(R) \quad (R \to \infty)\,,$$

(b) for every sequence $R_T \nearrow \infty$ as $T \to \infty$

$$|P_\mu(M_T \leq R_T) - e^{-l(R_T)T}| \to 0 \qquad (T \to \infty)\,.$$

A second exhausting family $(\widetilde{O}_r)_{r>r_0}$ of $\mathbb{R}^n$ is called *compatible* to the exhausting family $(O_R)_{R>R_0}$ if

$$R_r := \inf\{R > R_0 : \widetilde{O}_r \subset O_R\} < \infty \qquad r > r_0\,, \tag{2.7}$$

where $\inf\emptyset := \infty$. The next corollary describes how asymptotic lower bounds for the bottom eigenvalue $\widetilde{\lambda}_r$ associated to $(\widetilde{O}_r)_{r>r_0}$ and hence also for the tail of the maximum $\widetilde{M}_T := \max_{0\leq t\leq T} \widetilde{q}(X_t)$ can be obtained, where $\widetilde{q}$ is the distance function generated by $(\widetilde{O}_r)_{r>r_0}$.

Corollary 2.2 *Assume that there exists a function l satisfying (2.6) with λ_R associated to $(O_R)_{R>R_0}$. Let $(\widetilde{O}_r)_{r>r_0}$ be an exhausting family of $\mathbb{R}^n$ compatible to $(O_R)_{R>R_0}$. Set $\widetilde{l}(r) := l(R_r)$, $r > r_0$. Then $\widetilde{\lambda}_r \gtrsim \widetilde{l}_r$ as $r \to \infty$ and hence for every $T > 0$*

$$T\,\widetilde{l}(r) \lesssim P_\mu(\widetilde{M}_T > r) \qquad (r \to \infty)\,.$$

PROOF. Since $\widetilde{O}_r \subset O_{R_r}$ we have $\widetilde{\lambda}_r \geq \lambda_{R_r}$ for every $r > r_0$. Moreover $\{\tau_{O_{R_r}} \geq T\} \subset \{\tau_{\widetilde{O}_r} \geq T\}$ and we obtain invoking the relation (1.5) that $P_\mu(M_T > R_r) \leq P_\mu(\widetilde{M}_T > r)$ for every $r > r_0$. Further $R_r \to \infty$ as $r \to \infty$ since $(O_R)_{R>R_0}$ and $(\widetilde{O}_r)_{r>r_0}$ are exhausting families of $\mathbb{R}^n$. Hence the result follows from assumption (2.6) on the function l and from the left asymptotic inequality in part (a) of the above Proposition. □

Part (b) of the above proposition allows us to determine the possibly non-degenerated limit distribution of the (properly normalized) maximum M_T as $T \to \infty$ in the spirit of classical extreme value theory, see e.g. chapter 3 of Embrechts et al. [EKM97]. A univariate cumulative distribution function F is said to be in the *domain of attraction* of an extreme value distribution $H \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}$ ($F \in \mathrm{DA}(H)$), where Λ is the Gumbel distribution and Φ_α and Ψ_α are the Frechet and Weibull distribution with index $\alpha > 0$, respectively, if there exist norming sequences $(c_T)_{T>0}$ and $(d_T)_{T>0}$ with $c_T > 0$, $T > 0$, such that

$$\lim_{T\to\infty} F(c_T x + d_T)^T = H(x) \qquad x \in \mathbb{R}. \tag{2.8}$$

For the proof of the next corollary see e.g. Corollary 2.6 of Kunz [Kun02].

Corollary 2.3 *Assume the situation of Proposition 2.1. Set* $F(R) := e^{-l(R)}$, $R > 0$. *If* $F \in \mathrm{DA}(H)$ *for an extreme value distribution* H *with norming constants* $(c_T)_{T>0}$, $(d_T)_{T>0}$ *according to (2.8), then denoting convergence in distribution by* $\xrightarrow{d}$

$$c_T^{-1}(M_T - d_T) \xrightarrow{d} H \qquad (T \to \infty).$$

We come back to the evaluation of the asymptotics of the bottom eigenvalue λ_R in the sense of (2.6). It is not a realistic objective to determine a function l in terms of the parameters of $(X_t)_{t\geq 0}$ satisfying (2.6) for an arbitrary exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$. As mentioned in the introduction, we concentrate on the level sets $(O_R^\Phi)_{R>R_0}$ of Φ defined in (1.7), which are an exhausting family of $\mathbb{R}^n$ assuming (1.6). In this case, the eigenvalue asymptotics is evaluated my means of test-functions of the form $f \circ \Phi$, where f is a real function. In order to apply the generator L defined in (2.3) to these test-functions, we assume

$$\Phi \in C^2(\mathbb{R}^n, \mathbb{R}), \tag{2.9}$$

It is convenient to split up integrals over the level sets O_R^Φ, $R > R_0$, into integrals w.r.t. the iso-level sets of Φ. To this aim assume that there exists $R_1 \geq R_0$ such that

$$\nabla\Phi(x) \neq 0 \qquad x \in \mathbb{R}^n \setminus O_{R_1}^\Phi . \tag{2.10}$$

Hence for a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and $R > R_1$ we can define the following weighted integral over the iso-level set $\partial O_R^\Phi := \{x : \Phi(x) = R\}$

$$m_{\Phi,R}[f] := \int_{\partial O_R^\Phi} \frac{f(\xi)}{|\nabla\Phi(\xi)|}\, d\sigma_{\Phi,R}(\xi)\,, \tag{2.11}$$

where $d\sigma_{\Phi,R}$ is the surface measure on ∂O_R^Φ. The proof of the following lemma is deferred to section 4.

Lemma 2.4 *Assume (1.6) and (2.10). Then for every $f \in C(\mathbb{R}^n, \mathbb{R})$ and $R > R_1$*

$$\int_{O_R^\Phi \setminus O_{R_1}^\Phi} f\, dx = \int_{R_1}^R m_{\Phi,r}[f]\, dr\,.$$

The crucial condition to obtain sharp eigenvalue asymptotics is the relation

$$m_{\Phi,R}[(\Delta\Phi)^2] = o\left(m_{\Phi,R}[|\nabla\Phi|^2]\right) \qquad (R \to \infty)\,. \tag{2.12}$$

If Φ is of polynomial form in x, then $\Delta\Phi$ (as a second order derivative term) is of lower order than the first order derivative term $|\nabla\Phi|$ in the limit $|x| \to \infty$. Hence in this case, condition (2.12) has a good chance to hold. We also need some growth conditions on Φ. Set

$$I(R) := \int_{O_R^\Phi} |\nabla\Phi(x)|^2 e^{2\Phi(x)/\sigma^2}\, dx \qquad R > R_0\,. \tag{2.13}$$

Assume

$$I(R) \nearrow \infty\,, \quad I(R) = o\left(e^{4R/\sigma^2}\right) \qquad (R \to \infty)\,. \tag{2.14}$$

The interpretation of the first condition is that $|\nabla\Phi(x)|$ must not decay too fast to zero as $|x| \to \infty$. By L'Hopital's rule and Lemma 2.4, the second condition also reads as a growth condition on $|\nabla\Phi|$ in the form $m_{\Phi,R}[|\nabla\Phi|^2] = o(e^{2R/\sigma^2})$ as $R \to \infty$. In Lemma 5.1 we state explicit growth conditions on Φ, such that (2.14) holds. The proof of the following theorem is deferred to section 4.

Theorem 2.5 *Assume (2.2) and (2.4). Further suppose that the conditions (1.6), (2.9), (2.10), (2.12), and (2.14) hold. Set*

$$l(R) := \frac{2}{\sigma^2 Z_\sigma} e^{-4R/\sigma^2} I(R) \qquad R > R_0 ,$$

where Z_σ defined in (2.2) is the total mass of μ and $I(R)$ is defined in (2.13). Then the function l satisfies (2.6) with λ_R associated to the exhausting family $(O_R^\Phi)_{R>R_0}$ defined in (1.7) generated by the level sets of Φ.

Remark 2.6 (1) By L'Hopital's rule and Lemma 2.4, the function l can be replaced by

$$l(R) := \frac{2}{\sigma^2 Z_\sigma} e^{-4R/\sigma^2} \int_{R_2}^{R} e^{2r/\sigma^2} m_{\Phi,r}[|\nabla\Phi|^2]\, dr \qquad R > R_1 .$$

(2) Without loss of generality is suffices to prove Theorem 2.5 under the following additional assumptions: $\sigma^2 = 2$ and the potential Φ is normalized in the sense that $\widetilde{\mu}(x) = e^{-\Phi(x)}$ is a probability density on $\mathbb{R}^n$. To recover the general case, the result for normalized potentials has to be applied to the potential $\Phi_\sigma := (2/\sigma^2)\Phi - \ln Z_\sigma$. To reduce the integration over $O_R^{\Phi_\sigma}$ in the term $I(R)$ to the integration over O_R^Φ, R must be replaced by $(\sigma^2/2)(R + \ln Z_\sigma)$. Further the bottom eigenvalue λ_R for the normalized problem has to be multiplied by $2/\sigma^2$, since the same holds for the generator, see (2.3).

3 Preliminaries: Markov Processes and Operator Theory

The behavior of the maximum M_T of a diffusion process $(X_t)_{t\geq 0}$ of gradient field type is related to spectral properties of its generator. Here M_T is the maximum w.r.t. the distance function generated by an exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$. According to Remark 2.6.(2) we assume that $\sigma^2 = 2$ and that the measure μ with Lebesgue density $\widetilde{\mu}(x) = e^{-\Phi(x)}$, $x \in \mathbb{R}^n$, is a probability measure on $\mathbb{R}^n$. Suppose that the diffusion process $(X_t)_{t\geq 0}$ solves the SDE (1.1) and is symmetric w.r.t. μ.

We use the following notations: for $R \in (R_0, \infty]$ we denote by μ_R the restriction of μ to the set O_R (where we set $O_\infty := \mathbb{R}^n$). We write for short $L^2_{\mu_R}$ for $L^2(O_R, \mu_R)$ and $\| \cdot \|_{2,R}$ and $(\cdot, \cdot)_R$ for norm and scalar product in $L^2_{\mu_R}$, respectively. Further the indicator function of a set A is denoted by I_A.

Recall that for a μ-symmetric process $(X_t)_{t \geq 0}$ the associated backward semigroup $(P_t)_{t \geq 0}$ with $P_t f(x) := E_x[f(X_t)]$, $x \in \mathbb{R}^n$, $t \geq 0$, extends (under some regularity conditions) to a strongly contraction semigroup on $L^2_{\mu_\infty}$. We define the following operators and quadratic forms leading to a proper definition of the generator L defined in (2.3): for $R \in (R_0, \infty]$ set

$$\mathcal{E}'_R(u, v) := \sum_{i=1}^{n} \int_{O_R} \partial_{x_i} u \, \partial_{x_i} v \, \mu \, dx \qquad u, v \in C_0^2(O_R) \,, \tag{3.1}$$

where $C_0^2(O_R)$ is the set of two times continuously differentiable functions having value 0 at the boundary of O_R. For every $R \in (R_0, \infty]$, the quadratic form $(\mathcal{E}'_R, C_0^2(B_R))$ is closable in $L^2_{\mu_R}$ and its closure $(\mathcal{E}_R, \mathcal{D}(\mathcal{E}_R))$ is a symmetric Dirichlet form, see e.g. section II.2.(a) in Ma and Röckner [MR92]. Let $(-L_R, \mathcal{D}(L_R))$ be the positive, selfadjoint operator on $L^2_{\mu_R}$ associated to $(\mathcal{E}_R, \mathcal{D}(\mathcal{E}_R))$ and $(e^{L_R t})_{t \geq 0}$ the strongly continuous contraction semigroup on $L^2_{\mu_R}$ generated by the operator L_R for every $R \in (R_0, \infty]$.

Assuming (2.1), there exists a μ-symmetric diffusion process $(X_t)_{t \geq 0}$ solving the SDE (1.1) in the following sense: there exists an increasing sequence $\{\sigma_n : n \in \mathbb{N}\}$ of stopping times with $\sigma_\infty := \lim_{n \to \infty} \sigma_n$ such that $P_\mu(\sigma_\infty < \infty) = 0$ and $(X_t)_{t \geq 0}$ is a weak solution of the SDE (1.1) on the set $\{t < \sigma_n\}$ for every $n \in \mathbb{N}$. Further the $L^2_{\mu_\infty}$-extension of the backward semigroup $(P_t)_{t \geq 0}$ of $(X_t)_{t \geq 0}$ coincides with the semigroup $(e^{L_\infty t})_{t \geq 0}$. This result has been shown by Meyer and Zheng [MZ85], see also section 6.3 of Fukushima et al. [FOT94] and Proposition 3.1 of Kunz [Kun02].

For $R \in (R_0, \infty)$, we denote by $(X_t^R)_{t \geq 0}$ the part of $(X_t)_{t \geq 0}$ on O_R, i.e. the process $(X_t)_{t \geq 0}$ killed when it hits the set $\mathbb{R}^n \setminus O_R$. $(X_t^R)_{t \geq 0}$ is μ_R-symmetric and the backward semigroup $(P_t^R)_{t \geq 0}$ of $(X_t^R)_{t \geq 0}$ is given by $P_t^R f(x) := E_x[f(X_t) I_{\{\tau_R > t\}}]$, $x \in \mathbb{R}^n$, $t \geq 0$, where τ_R is defined in (1.4). $(X_t^R)_{t \geq 0}$ is associated to the Dirichlet form $\mathcal{E}_R$ in the sense that the $L^2_{\mu_R}$-extension of the backward semigroup $(P_t^R)_{t \geq 0}$ of $(X_t^R)_{t \geq 0}$ coincides with the semigroup $(e^{L_R t})_{t \geq 0}$, see e.g. Theorem 4.4.2 and Theorem 4.4.3(i) of Fukushima et al.

[FOT94]. Denoting by $\mathbf{1}$ the constant function with value 1, we deduce that

$$P_\mu(\tau_R > T) = (e^{L_R T}\mathbf{1}, \mathbf{1})_R \qquad T > 0,\, R \in (R_0, \infty)\,. \tag{3.2}$$

Recall the definition of the bottom eigenvalue $\lambda_R := \inf \Sigma(-L_R)$, $R \in (R_0, \infty]$, where $\Sigma(-L_R)$ is the spectrum of the operator $-L_R$ (with respect to $L^2_{\mu_R}$). The term of the RHS in (3.2) can be estimated from above and below in terms of λ_R and the spectral gap Λ defined in (2.4) such that, invoking (1.5), we arrive at (2.5), see e.g. theorem 2.13 of Iscoe and McDonalds [IM94].

To estimate the bottom eigenvalue λ_R of $-L_R$ we use the variation principle for upper bounds and Temple's inequality for lower bounds. For $R \in (R_0, \infty]$ and a function $v \in \mathcal{D}(L_R)$ we define

$$\rho_R(v) := \|v\|_{2,R}^{-2} \mathcal{E}_R(v, v)\,, \quad l_R(v) := \|v\|_{2,R}^{-2} \|L_R v\|_{2,R}^2\,. \tag{3.3}$$

Note that ρ_R is the Rayleigh quotient. We summarize the bounds on λ_R in the following proposition. For a proof see e.g. theorems XIII.2 and XIII.5 of Reed and Simon [RS78], note also Remark 3.6 of Kunz [Kun02].

Proposition 3.1 *Assume (2.4) and let $R \in (R_0, \infty]$. Then for every $v \in \mathcal{D}(L_R)$ with $\rho_R(v) < \Lambda$*

$$\rho_R(v) - \frac{l_R(v) - \rho_R(v)^2}{\Lambda - \rho_R(v)} \le \lambda_R \le \rho_R(v)\,.$$

Finally we state a condition on the potential Φ, such that the spectral gap assumption (2.4) holds. We will make use of the fact, that the operator $-L_\infty$ is unitarily equivalent to the Schrödinger operator $-\Delta + V_\Phi$ on $\mathbb{R}^n$ with potential

$$V_\Phi(x) := \tfrac{1}{4}|\nabla\Phi(x)|^2 - \tfrac{1}{2}\Delta\Phi(x) \qquad x \in \mathbb{R}^n\,. \tag{3.4}$$

We use the notation $\liminf_{|x|\to\infty} V(x) := \lim_{R\to\infty} \inf_{|x|>R} V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$. For a proof of the next proposition see e.g. theorem 3.1 of Berezin and Shubin [BS91] or Proposition 3.7 of Kunz [Kun02].

Proposition 3.2 *Suppose $\Phi \in C^2(\mathbb{R}^n, \mathbb{R})$ and $\liminf_{|x|\to\infty} V_\Phi(x) > 0$. Then the spectral gap property (2.4) holds.*

4 Proofs

Proof of Lemma 2.4. We fix $R > R_1$ and set for $\delta > 0$

$$\Gamma_{R,\delta} := \{x : R \le \Phi(x) \le R + \delta\}.$$

It suffices to show that

$$\frac{1}{\delta}\int_{\Gamma_{R,\delta}} f\,dx = m_{\Phi,R}[f] \qquad (\delta \searrow 0).$$

Since $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$, we obtain, using (2.10) and the implicit function theorem, that $\partial O_R^\Phi = \{x : \Phi(x) = R\}$ is a $n-1$-dimensional C^1-surface, which is orthogonal to the gradient field $\nabla\Phi$. Let $(\xi)_{\xi\in\Xi}$ be a smooth parameterization of ∂O_R^Φ. For every $\xi \in \partial O_R^\Phi$ we define the flow $[0, s^*) \ni s \mapsto T_s\xi \in \mathbb{R}^n$ as the maximal solution of the system of ODEs

$$\dot{z}(s) = |\nabla\Phi(z(s))|^{-1}\nabla\Phi(z(s)) \qquad z(0) = \xi.$$

Note that this is well defined by (2.10) and that the flow $s \mapsto T_s\xi$ has unit speed. Set

$$\phi_\xi(s) := \Phi(T_s\xi) \qquad s \in [0, s^*),\ \xi \in \partial O_R^\Phi.$$

Obviously ϕ_ξ is differentiable at $s = 0$ with $\phi'_\xi(0) = |\nabla\Phi(\xi)| > 0$ by (2.10). Hence we can find for every $\xi \in \partial O_R^\Phi$ and small $\delta > 0$ a constant $S_{\xi,\delta} > 0$ such that $\phi_\xi(S_{\xi,\delta}) = R + \delta$. Since ϕ_ξ is locally invertible near $s = 0$, $S_{\xi,\delta}$ is differentiable w.r.t. δ at $\delta = 0$ with

$$\lim_{\delta\searrow 0}\frac{S_{\xi,\delta}}{\delta} = (\phi_\xi^{-1})'(R) = \frac{1}{\phi'_\xi(\phi_\xi^{-1}(R))} = \frac{1}{\phi'_\xi(0)} = \frac{1}{|\nabla\Phi(\xi)|}. \tag{4.1}$$

Since $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$, the mapping $T : (s, \xi) \mapsto T_s\xi$ is a local diffeomorphism. From assumption (1.6) we deduce that ∂O_R^Φ is compact. Hence by shrinking δ if necessary, T is reduced to a global diffeomorphism $T : \Gamma'_{R,\delta} \to \Gamma_{R,\delta}$, where $\Gamma'_{R,\delta} := \{(s, \xi) : \xi \in \partial O_R^\Phi, s \in [0, S_{\xi,\delta}]\}$. Further the limit in (4.1) can be made uniformly in $\xi \in \partial O_R^\Phi$, since also $(s, \xi) \mapsto \phi_\xi(s) = \Phi(T_s\xi)$ is a local diffeomorphism and ∂O_R^Φ is compact. Using the transformation rule for integrals having in mind that the flow $s \mapsto T_s\xi$ has unit speed we get

$$\frac{1}{\delta}\int_{\Gamma_{R,\delta}} f\,dx = \frac{1}{\delta}\int_{\partial O_R^\Phi}\left(\int_0^{S_{\xi,\delta}} f(T_s\xi)\,ds\right)d\sigma_{\Phi,R}(\xi), \tag{4.2}$$

where $d\sigma_{\Phi,R}$ is the surface measure on ∂O_R^Φ. Using (4.1) we compute by means of the chain rule

$$\lim_{\delta\searrow 0}\frac{1}{\delta}\int_0^{S_{\xi,\delta}} f(T_s\xi)\,ds = f(T_0\xi)\lim_{\delta\searrow 0}\frac{S_{\xi,\delta}}{\delta} = \frac{f(\xi)}{|\nabla\Phi(\xi)|}\,.$$

Using the uniform continuity of f on the compact set $\Gamma_{R,\delta}$ and the fact that the limit in (4.1) is uniformly in $\xi \in \partial O_R^\Phi$, also the above limit is uniformly in $\xi \in \partial O_R^\Phi$. Hence in (4.2), the limit $\delta \searrow 0$ can be interchanged with the integration over ∂O_R^Φ and the result follows. □

The main result of this paper, Theorem 2.5, determines the fine asymptotics of the bottom eigenvalue λ_R as $R \to \infty$ in the sense that a function l is given satisfying (2.6). Here λ_R corresponds to the exhausting family $(O_R^\Phi)_{R>R_0}$ of $\mathbb{R}^n$ generated by the level sets of the potential Φ. To this aim suitable test-functions $(v_R)_{R>R_0}$ must be found such that the bounds on λ_R in Proposition 3.1 get sharp in the limit $R \to \infty$. According to Remark 2.6.(2) we may assume w.l.o.g. that

$$\text{(4.3)} \qquad \sigma = \sqrt{2} \ \text{ and } \ \widetilde{\mu}(x) = e^{-\Phi(x)},\ x \in \mathbb{R}^n, \text{ is a probability density on } \mathbb{R}^n.$$

The quadratic form $\mathcal{E}_R$ and the operator L_R, $R \in (R_0, \infty]$, are taken here w.r.t. the exhausting family $(O_R^\Phi)_{R>R_0}$ of $\mathbb{R}^n$. For a function $v \in \mathcal{D}(\mathcal{E}_R)$ we write for short $\mathcal{E}_R(v)$ for $\mathcal{E}_R(v,v)$ and the norm in $L^2_{\mu_R} := L^2(O_R^\Phi, \mu_R)$ is again denoted by $\|\cdot\|_{2,R}$.

Proof of Theorem 2.5.

Step 1: Construction and properties of the test-functions. Set for $R > R_0$

$$v_R(x) := 1 - e^{\Phi(x)-R} \qquad x \in O_R^\Phi\,.$$

By assumption (2.9), $v_R \in C^2(O_R^\Phi, \mathbb{R})$ for every $R > R_0$ and can be extended continuously by 0 to the boundary of O_R^Φ. Hence $v \in C_0^2(O_R^\Phi, \mathbb{R}) \subset \mathcal{D}(L_R)$ for every $R > R_0$. Further $\nabla v_R(x) = -e^{\Phi(x)-R}\nabla\Phi(x)$ for every $x \in O_R^\Phi$. Plugging this into the quadratic form $\mathcal{E}_R$ defined in (3.1) we obtain for $R > R_0$

$$\text{(4.4)} \qquad \mathcal{E}_R(v_R) = e^{-2R}\int_{O_R^\Phi} |\nabla\Phi|^2 e^{\Phi}\,dx\,.$$

Using the alternative representation of L defined in (2.3) one gets for $R > R_0$

$$\|Lv_R\|_{2,R}^2 = e^{-2R} \int_{O_R^\Phi} (\Delta\Phi)^2 e^\Phi \, dx \tag{4.5}$$

Step 2: It suffices to show

$$\|Lv_R\|_{2,R}^2 = o\left(\mathcal{E}_R(v_R)\right) \quad (R \to \infty)\,. \tag{4.6}$$

We need to prove that $\rho_R(v_R) \to 0$ as $R \to \infty$, where ρ_R is defined in (3.3). Proposition 3.1 is then applicable for large R. We can deduce that $\lambda_R \sim \rho_R(v_R)$ as $R \to \infty$ if we can show

$$\frac{l_R(v_R) - \rho_R(v_R)^2}{\Lambda - \rho_R(v_R)} = o\left(\rho_R(v_R)\right) \quad (R \to \infty)\,. \tag{4.7}$$

By construction, $v_R \nearrow 1$ μ-a.s. as $R \to \infty$ (where v_R is extended to a function on $\mathbb{R}^n$ by setting 0 on $\mathbb{R}^n \setminus O_R^\Phi$). Since μ is a probability measure on $\mathbb{R}^n$, we have

$$\|v_R\|_{2,R}^2 \to 1 \qquad (R \to \infty)\,. \tag{4.8}$$

From the growth condition (2.14) we obtain together with (4.4) that $\mathcal{E}_R(v_R) \searrow 0$ as $R \to \infty$. From this and relation (4.8) we deduce that also $\rho_R(v_R) \to 0$ as $R \to \infty$, having the definition (3.3) of $\rho_R(v_R)$ in mind. Further (4.8) allows to replace in (4.7) in the limit $R \to \infty$ the terms $l_R(v_R)$ and $\rho_R(v_R)$ (defined in (3.3)) by $\|Lv_R\|_{2,R}^2$ and $\mathcal{E}_R(v_R)$, respectively. But then (4.7) follows from (4.6) using again $\rho_R(v_R) \to 0$ as $R \to \infty$. Hence we get $\lambda_R \sim \mathcal{E}_R(v_R)$ as $R \to \infty$ with $\mathcal{E}_R(v_R)$ in the form (4.4). To obtain the general asymptotic expression $l(R)$ without assuming the simplifying condition (4.3) see Remark 2.6.(2).

Step 3: Condition (4.6) holds. We need to show that

$$q_R := \mathcal{E}_R(v_R)^{-1} \|Lv_R\|_{2,R}^2 = \left(\int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi \, dx \right)^{-1} \int_{O_R^\Phi} (\Delta\Phi)^2 e^\Phi \, dx \to 0 \qquad (R \to \infty)\,,$$

where we used the representations (4.4) and (4.5). The growth condition (2.14) implies that $\int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi \, dx \nearrow \infty$ as $R \to \infty$ and hence L'Hopital's rule can be applied to the quotient q_R. Thus we calculate making use of Lemma 2.4

$$\lim_{R\to\infty} q_R = \lim_{R\to\infty} \left(e^R m_{\Phi,R}[|\nabla\Phi|^2]\right)^{-1} e^R m_{\Phi,R}[(\Delta\Phi)^2] = 0\,.$$

The last step follows from the crucial condition (2.12). □

5 Examples

We give some examples of diffusion processes of gradient field type for which the sharp eigenvalue asymptotics can be evaluated by Theorem 2.5 and hence the asymptotics of the maximum M_T of the process w.r.t. the distance function generated by the level sets $(O_R^\Phi)_{R>R_0}$ of the potential Φ is given by Proposition 2.1 and Corollary 2.3.

The following lemma provides a method to check whether the growth conditions (2.14) hold. Assume that the open balls $(B_\rho)_{\rho>0}$ around the origin and the level sets $(O_R^\Phi)_{R>R_0}$ (which are an exhausting family of $\mathbb{R}^n$ by assumption (1.6)) are compatible to each other in the sense of (2.7). Set for sufficiently large $R > 0$

$$\rho_*(R) := \sup\{\rho > 0 : B_\rho \subset O_R^\Phi\}\,, \qquad \rho^*(R) := \inf\{\rho > 0 : O_R^\Phi \subset B_\rho\}\,.$$

Note that $\rho_*(R) \nearrow \infty$ as $\rho \to \infty$ by condition (1.6) and $\rho^*(R) < \infty$ for every $R > R_0$. Further we define for a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ the terms $f_*(\rho) := \min_{|x|=\rho} f(x)$ and $f^*(\rho) := \max_{|x|=\rho} f(x)$.

Lemma 5.1 *Assume $n \geq 2$.*

(a) The first growth condition in (2.14) holds, if there exist $C > 0$ and $\rho_0 > 0$ such that

$$|\nabla\Phi|_*(\rho) \geq C\rho^{-n/2} \qquad \rho > \rho_0\,. \tag{5.1}$$

(b) Assume that Φ_* is differentiable and Φ_*' is finally strictly positive. The second growth condition in (2.14) holds, if Φ satisfies

$$\rho^{n-1}\frac{|\Delta\Phi|^*(\rho)}{(\Phi_*)'(\rho)} = o(e^{\Phi_*(\rho)}) \qquad (\rho \to \infty)\,. \tag{5.2}$$

Proof. Recall the definition (2.13) of $I(R)$.

(a) Choose $\rho_1 > \rho_0$ such that $\Phi(x) > 0$ for every $|x| \geq \rho_1$ (possible by assumption (1.6)). Denoting by γ_n the volume of the unit sphere in $\mathbb{R}^n$, we estimate for R with $\rho_*(R) > \rho_1$

$$I(R) \geq \int_{B_{\rho_*(R)}} |\nabla\Phi|^2 e^\Phi\, dx \geq \gamma_n \int_{\rho_1}^{\rho_*(R)} r^{n-1}|\nabla\Phi|_*^2(r)\, dr \geq C\gamma_n \int_{\rho_1}^{\rho_*(R)} r^{-1}\, dr\,.$$

The last expression raises to infinity as $R \to \infty$ since also $\rho_*(R) \nearrow \infty$ as $R \to \infty$.

(b) Applying Lemma 2.4 to the term $I(R)$, it suffices by L'Hopital's rule to show

$$e^{-R} m_{\Phi,R}[|\nabla\Phi|^2] \to 0 \qquad (R \to \infty)\,. \tag{5.3}$$

Since O_R^Φ has C^1 boundary, we are allowed to apply Stoke's formula (having in mind that the outer normal to ∂O_R^Φ is given by $|\nabla\Phi|^{-1}\nabla\Phi$)

$$m_{\Phi,R}[|\nabla\Phi|^2] = \int_{\partial O_R^\Phi} \nabla\Phi(\xi)\cdot\tfrac{\nabla\Phi(\xi)}{|\nabla\Phi(\xi)|}\,d\sigma_{\Phi,R}(\xi) = \int_{O_R^\Phi} \mathrm{div}(\nabla\Phi(x))\,dx = \int_{O_R^\Phi} \Delta\Phi(x)\,dx\,.$$

Estimating further we obtain

$$\left|\int_{O_R^\Phi} \Delta\Phi(x)\,dx\right| \le \int_{B_{\rho^*(R)}} |\Delta\Phi(x)|\,dx \le \gamma_n \int_0^{\rho^*(R)} r^{n-1}|\Delta\Phi|^*(r)\,dr\,.$$

Using the fact that $\rho^*(\Phi_*(\rho)) = \rho$ and that $\Phi_*^{-1}(\rho) \nearrow \infty$ as $\rho \to \infty$, (5.3) is proved by showing

$$\lim_{\rho\to\infty} \frac{\int_0^\rho r^{n-1}|\Delta\Phi|^*(r)\,dr}{e^{\Phi_*(\rho)}} = \lim_{\rho\to\infty} \frac{\rho^{n-1}|\Delta\Phi|^*(\rho)}{\Phi_*'(\rho)e^{\Phi_*(\rho)}} = 0\,.$$

Here we used L'Hopital's rule once again and assumption (5.2). □

The following lemma is used for the asymptotic evaluation of integrals over exponential terms.

Lemma 5.2 *Let $A, \gamma > 0$ and $\delta \in \mathbb{R}$. Then*

$$\int_1^R r^\delta e^{Ar^\gamma}\,dr \sim (\gamma A)^{-1} R^{\delta-\gamma+1} e^{AR^\gamma} \quad (R \to \infty)\,.$$

PROOF. Apply L'Hopital's rule to the quotient. □

5.1 Rotationally symmetric case

Assume that the potential Φ in the SDE (1.1) has the property that there exist $\rho_0 > 0$ and $\phi \in C^2([\rho_0,\infty),\mathbb{R})$ such that

(5.4) $$\Phi(x) = \phi(|x|) \qquad |x| > \rho_0\,.$$

Assume further that

(5.5) $$\liminf_{\rho\to\infty} \phi'(\rho) > 0\,.$$

The crucial condition (2.12) in this context has the form of the regularity condition

$$\phi''(\rho) = o(\phi'(\rho)) \quad (\rho \to \infty)\,. \tag{5.6}$$

Note that $\phi(\rho) \nearrow \infty$ as $\rho \to \infty$ at least linearly by (5.5). Moreover ϕ^{-1} exists on $[R_2, \infty)$ for some $R_2 > 0$ large enough and also $\phi^{-1}(R) \nearrow \infty$ as $R \to \infty$. Further we obtain for the level sets

$$O_R^\Phi = B_{\phi^{-1}(R)} = \{x : |x| < \phi^{-1}(R)\}\,. \tag{5.7}$$

A simple calculation yields

$$|\nabla\Phi(x)| = \phi'(|x|)\,, \quad \Delta\Phi(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|) \qquad |x| > \rho_0\,. \tag{5.8}$$

By γ_n we denote the volume of the unit sphere in $\mathbb{R}^n$. In the rotationally symmetric case, Theorem 2.5 yields to the following corollary.

Corollary 5.3 *Let $n \geq 2$. Assume that the potential Φ in the SDE (1.1) is of the form (5.4). Suppose that (5.5) and (5.6) holds.*
(a) Set

$$l(R) := \frac{2\gamma_n}{\sigma^2 Z_\sigma} e^{-4R/\sigma^2} \int_{\phi^{-1}(R_0)}^{\phi^{-1}(R)} e^{2\phi(t)/\sigma^2} t^{n-1} \phi'(t)^2\, dt \qquad R > R_2\,.$$

Then l satisfies (2.6) with λ_R associated to the exhausting family $(O_R^\Phi)_{R>R_0}$.
(b) Consider the exhausting family $(B_\rho)_{\rho>0}$ of $\mathbb{R}^n$ ($B_\rho = \{x : |x| < \rho\}$). Set $\widetilde{l}(\rho) := l(\phi(\rho))$, $\rho > \rho_0$. Then $\widetilde{l}(\rho) \sim \widetilde{\lambda}_\rho$ as $\rho \to \infty$, where $\widetilde{\lambda}_\rho$ is associated to the exhausting family $(B_\rho)_{\rho>0}$.

Remark 5.4 Assume the situation of part (b) of the above theorem. In Theorem 5.4 of Kunz [Kun02] we could show under slightly weaker conditions that $\widetilde{l}_1(\rho) \sim \widetilde{\lambda}_\rho$ as $\rho \to \infty$ where

$$\widetilde{l}_1(\rho) := \frac{\sigma^2\gamma_n}{2Z_\sigma}\left(\int_{\rho_0}^{\rho} t^{1-n} e^{2\phi(t)/\sigma^2}\, dt\right)^{-1} \qquad \rho > \rho_0\,.$$

It was only assumed that $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ with V_Φ defined in (5.9). In particular we did not need to assume the regularity condition (5.6). This is due to the fact that in [Kun02] we used test-functions which are more adapted to the rotationally symmetric case. If $\phi(\rho) = \rho^\alpha$, $\rho > \rho_0$, then the two asymptotics expressions $\widetilde{l}(\rho)$ and $\widetilde{l}_1(\rho)$ coincide.

Using Lemma 5.2 we get as $\rho \to \infty$

$$\begin{aligned}\widetilde{l}(\rho) &= \frac{2\gamma_n}{\sigma^2 Z_\sigma} e^{-4\rho^\alpha/\sigma^2} \int_{\rho_0}^{\rho} e^{2t^\alpha/\sigma^2} t^{n-1} (\alpha t^{\alpha-1})^2 \, dt \sim \frac{\gamma_n \alpha}{Z_\sigma} \rho^{n+\alpha-2} e^{-2\rho^\alpha/\sigma^2} \,, \\ \widetilde{l}_1(\rho) &\sim \frac{\sigma^2 \gamma_n}{2Z_\sigma} \left(\frac{\sigma^2}{2\alpha} \rho^{-n-\alpha+2} e^{2\rho^\alpha/\sigma^2} \right)^{-1} \sim \widetilde{l}(\rho) \,.\end{aligned}$$

PROOF. We have to check that the conditions of Theorem 2.5 are satisfied. Since $\phi(\rho) \nearrow \infty$ as $\rho \to \infty$ at least linearly by (5.5), the conditions (1.6), (2.2), and (2.10) hold. We show the spectral gap condition (2.4) by means of Proposition 3.2. The function V_Φ defined in (3.4) reads here setting $\rho = |x|$ and using the representations in (5.8)

$$V_\Phi(x) = \frac{\phi'(\rho)^2}{4} - \frac{1}{2} \left(\phi''(\rho) + \frac{n-1}{\rho} \phi'(\rho) \right) = \phi'(\rho) \left(\frac{\phi'(\rho)}{4} - \frac{\phi''(\rho)}{2\phi'(\rho)} - \frac{n-1}{2\rho} \right) . \tag{5.9}$$

Using (5.5) and (5.6), it is seen that $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ and hence the spectral gap condition (2.4) holds by Proposition 3.2. To show the crucial condition (2.14) we compute

$$m_{\Phi,R}[|\nabla\Phi|^2] = \int_{|\xi|=\phi^{-1}(R)} |\phi'(\xi)| \, d\sigma(\xi) = \gamma_n (t^{n-1}\phi'(t))_{t=\phi^{-1}(R)} \,, \tag{5.10}$$

$$\begin{aligned} m_{\Phi,R}[(\Delta\Phi)^2] &= \int_{|\xi|=\phi^{-1}(R)} \frac{(\phi''(|\xi|) + \frac{n-1}{|\xi|}\phi'(|\xi|))^2}{|\phi'(\xi)|} \, d\sigma(\xi) \\ &= \gamma_n \left[t^{n-1} \left(\tfrac{\phi''(t)^2}{\phi'(t)} + \tfrac{2(n-1)}{t}\phi''(t) + \tfrac{(n-1)^2}{t^2}\phi'(t) \right) \right]_{t=\phi^{-1}(R)} . \end{aligned} \tag{5.11}$$

Since $\phi^{-1}(R) \nearrow \infty$ as $R \to \infty$ we have using (5.6)

$$\lim_{R\to\infty} \frac{m_{\Phi,R}[(\Delta\Phi)^2]}{m_{\Phi,R}[|\nabla\Phi|^2]} = \lim_{R\to\infty} \left((\tfrac{\phi''(t)}{\phi'(t)})^2 + \tfrac{2(n-1)}{t}\tfrac{\phi''(t)}{\phi'(t)} + (\tfrac{n-1}{t})^2 \right)_{t=\phi^{-1}(R)} = 0 \,.$$

To check the growth conditions (2.14) it suffices to prove the conditions (5.1) and (5.2) of Lemma 5.1. Obviously $\phi'(\rho) > \rho^{-n/s}$ for sufficiently large $\rho > 0$ by (5.5) and hence condition (5.1) holds. Using that $\phi(\rho) \nearrow \infty$ as $\rho \to \infty$ at least linearly by (5.5) and condition (5.6), we obtain

$$\rho^{n-1} \left| \frac{\phi''(\rho) + \frac{n-1}{\rho}\phi'(\rho)}{\phi'(\rho)} \right| e^{-\phi(\rho)} = \rho^{n-1} \left| \frac{\phi''(\rho)}{\phi'(\rho)} + \frac{n-1}{\rho} \right| e^{-\phi(\rho)} \to 0 \qquad (\rho \to \infty) \,.$$

Hence also condition (5.2) is satisfied and the growth conditions (2.14) by Lemma 5.1. This finishes the proof of part (a). Part (b) is obvious having (5.7) in mind. □

5.2 Non-Symmetric Processes

We illustrate the method for the following specific potential Φ in the SDE (1.1). To avoid trivialities, we assume $n \geq 2$. Using polar coordinates $\mathbb{R}^n \setminus \{0\} \ni x = \rho e_\theta$ where $\rho = |x| > 0$ and $(e_\theta)_{\theta \in \Theta}$ is a smooth parameterization of the unit sphere S^{n-1} in $\mathbb{R}^n$ (the same symbol is used for functions in Cartesian as well as in polar coordinates), we suppose that there exists $\rho_0 > 1$ and a function $p \in C^2(S^{n-1}, [0, \infty))$ with $\min_{\theta \in \Theta} p(\theta) = 0$ such that

$$\Phi(\rho, \theta) = \rho^\alpha + p(\theta)\rho^\beta \qquad \rho > \rho_0,\, \theta \in \Theta\,, \tag{5.12}$$

where $\alpha \geq 1$ and $\beta \in \mathbb{R}$.

The essential feature in the definition of Φ is that the asymmetric part factorizes in radial and spherical component. It is possible to replace in the definition (5.12) of the potential Φ the terms ρ^α and ρ^β by functions $\phi(\rho)$ and $\psi(\rho)$, see also Example 5.2 in Kunz [Kun02]. In this case quite technical compatibility and asymptotic growth conditions have to be imposed. We omit these cumbersome calculations in the present paper.

The expressions $|\nabla\Phi|$ and $\Delta\Phi$ read in polar coordinates (∇ and Δ denote the gradient and the Laplace operator w.r.t. the Cartesian coordinates, respectively)

$$|\nabla\Phi|^2(\rho, \theta) = (\alpha\rho^{\alpha-1} + \beta p(\theta)\rho^{\beta-1})^2 + |\nabla_\theta p(\theta)|^2 \rho^{2\beta-2}\,, \tag{5.13}$$

$$\Delta\Phi(\rho, \theta) = \alpha(\alpha + n - 2)\rho^{\alpha-2} + \beta(\beta + n - 2)p(\theta)\rho^{\beta-2} + \Delta_\theta p(\theta)\rho^{\beta-2}\,. \tag{5.14}$$

Here ∇_θ and Δ_θ denote the gradient and Laplace operator w.r.t. the angular coordinates θ, respectively. We obtain the following estimates

$$|\nabla\Phi|(\rho, \theta) \geq \alpha\rho^{\alpha-1} \qquad \rho > \rho_0,\, \theta \in \Theta\,. \tag{5.15}$$

Since $p \in C^2(S^{n-1})$ and S^{n-1} is compact, there exists a constant $\kappa > 0$ such that

$$|\Delta\Phi|(\rho, \theta) \leq \kappa\rho^{\max\{\alpha,\beta\}-2} \qquad \rho > \rho_0,\, \theta \in \Theta\,. \tag{5.16}$$

Theorem 5.5 *Assume that the potential Φ in the SDE (1.1) is of the form (5.12). The assertion of Theorem 2.5 holds in the following situations:*
(i) $\alpha \in [1, 2)$ and $\beta < 2$,
(ii) $\alpha \geq 2$ and $\beta < 1 + \sqrt{\alpha(\alpha - 1) + 1}$.

Remark 5.6 be imposed. (a) To obtain lower asymptotic bounds on the bottom eigenvalue $\widetilde{\lambda}_\rho$ associated to the exhausting family $(B_\rho)_{\rho>0}$ of $\mathbb{R}^n$, where B_ρ is the open ball around the origin with radius ρ, we can use Corollary 2.2. Set

$$p^* := \max_{\theta\in\Theta} p(\theta)\,, \qquad \overline{\phi}(\rho) := \rho^\alpha + p^*\rho^\beta \quad \rho > \rho_0\,. \tag{5.17}$$

Invoking (2.7) we get $R_\rho := \inf\{R > R_0 : B_\rho \subset O_R^\Phi\} = \overline{\phi}(\rho)$ and obtain that $\widetilde{l}(\rho) \lesssim \widetilde{\lambda}_\rho$ as $\rho \to \infty$ where for $\rho > \rho_0$

$$\widetilde{l}(\rho) := l(R_\rho) = \frac{2}{\sigma^2 Z_\sigma} e^{-4\overline{\phi}(\rho)/\sigma^2} \int_{O^\Phi_{\overline{\phi}(\rho)}} |\nabla\Phi|^2 e^{2\Phi/\sigma^2}\,dx \geq \frac{2}{\sigma^2 Z_\sigma} e^{-4\overline{\phi}(\rho)} \int_{B_\rho} |\nabla\Phi|^2 e^{2\Phi/\sigma^2}\,dx\,.$$

It can be seen by Laplace's method that the last integral has exponential decay of order $e^{-2\rho^\alpha/\sigma^2}$ as $\rho\to\infty$. Hence we obtain

$$\ln \widetilde{l}(\rho) \gtrsim -\frac{2}{\sigma^2}(\rho^\alpha + 2p^*\rho^\beta) \qquad (\rho\to\infty)\,.$$

In Kunz [Kun02] the fine eigenvalue asymptotics of $\widetilde{\lambda}_\rho$ was evaluated for the potential of the form (5.12) in the two-dimensional case for $\alpha \geq 1$ and $\beta \in (0, 2\alpha)$ in the sense that $\widetilde{\lambda}_\rho \sim \widetilde{l}_1(\rho)$ as $\rho\to\infty$ with

$$\widetilde{l}_1(\rho) := C\rho^{-\beta/\varpi^*} \left(\int_{\rho_0}^\rho r^{-1} e^{2r^\alpha/\sigma^2}\,dr\right)^{-1} \sim \frac{\alpha C}{\sigma^2}\rho^{\alpha-\beta/\varpi^*} e^{-2\rho^\alpha/\sigma^2} \qquad (\rho\to\infty)\,,$$

where the constants $C, \varpi^* > 0$ depend on the curvature of p in its zero points. We used Lemma 5.2 for the last asymptotic evaluation. It is seen that the exponential decay of $\widetilde{l}(\rho)$ and $\widetilde{l}_1(\rho)$ differ by the factor $e^{-4p^*\rho^\beta/\sigma^2}$ as $\rho\to\infty$. This effect is due to the fact that the ball B_ρ is compared with the domain $O^\Phi_{\overline{\phi}(\rho)}$ which is in general much bigger.

(b) The upper bounds on β in situation (i) and (ii) of Theorem 5.5 are used to insure the crucial condition (2.14) by means of crude estimates, see (5.15) - (5.23). These upper bounds do not seem to be crucial and may be extended by a more careful analysis for specific expressions of the function p.

Proof. Set $m := \max\{\alpha, \beta\}$. We have to check the conditions of Theorem 2.5. The conditions (1.6), (2.2), and (2.10) obviously hold by inequality (5.15) since $\alpha \geq 1$. The spectral gap condition (2.4) is shown by means of Proposition 3.2. The function V_Φ defined in (3.4) can be estimated using (5.15) and (5.16)

$$V_\Phi(\rho,\theta) \geq \frac{\alpha^2}{4}\rho^{2(\alpha-1)} - \frac{\kappa}{2}\rho^{m-2} \qquad \rho > \rho_0,\, \theta\in\Theta\,.$$

Since $\alpha \geq 1$ and $m < 2\alpha$ in both situations (i) and (ii), we obtain $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ and hence the spectral gap condition (2.4) holds by Proposition 3.2. To show the growth conditions (2.14), we prove the conditions (5.1) and (5.2) of Lemma 5.1. Condition (5.1) holds by the inequality (5.15) since $\alpha \geq 1$. To establish condition (5.2), we use again the estimations (5.15) and (5.16)

$$\rho^{n-1}\frac{|\Delta\Phi|^*(\rho)}{(\Phi_*)'(\rho)}\,e^{-\Phi_*(\rho)} \leq \frac{\kappa}{\alpha}\rho^{n-1}\rho^{m-2}\rho^{-(\alpha-1)}\,e^{-\rho^\alpha} \to 0 \qquad (\rho\to\infty)\,.$$

It remains to show the crucial condition (2.14). We need to parameterize the iso-level sets $\partial O_R^\Phi = \{x : \Phi(x) = R\}$. Recall the definition of $\overline{\phi}(\rho)$ in (5.17). Note that $\rho \mapsto \Phi(\rho,\theta)$ is strictly monotone increasing for every $\theta\in\Theta$. Thus

$$\gamma_R(\theta) := \Phi(\cdot,\theta)^{-1}(R) \qquad R > \overline{\phi}(\rho_0),\, \theta\in\Theta$$

exists. From the inequality $\phi(\rho) \leq \Phi(\rho,\theta) \leq \overline{\phi}(\rho)$, $\rho > \rho_0$, $\theta\in\Theta$, we deduce for the inverse functions

$$\overline{\phi}^{-1}(R)) \leq \gamma_R(\theta) \leq R^{1/\alpha} \qquad R > \overline{\phi}(\rho_0),\, \theta\in\Theta\,. \tag{5.18}$$

Recalling the definition (2.11) of $m_{\Phi,R}[\cdot]$ we estimate

$$m_{\Phi,R}[|\nabla\Phi|^2] \;\geq\; \min_{\theta\in\Theta}|\nabla\Phi|(\gamma_R(\theta),\theta)\cdot \mathrm{Vol}(\partial O_R^\Phi)\,, \tag{5.19}$$

$$m_{\Phi,R}[(\Delta\Phi)^2] \;\leq\; \frac{\max_{\theta\in\Theta}(\Delta\Phi)^2(\gamma_R(\theta),\theta)}{\min_{\theta\in\Theta}|\nabla\Phi|(\gamma_R(\theta),\theta)}\cdot \mathrm{Vol}(\partial O_R^\Phi)\,. \tag{5.20}$$

Hence the crucial condition (2.14) holds if we can show

$$J(R) := \frac{\max_{\theta\in\Theta}|\Delta\Phi|(\gamma_R(\theta),\theta)}{\min_{\theta\in\Theta}|\nabla\Phi|(\gamma_R(\theta),\theta)} \to 0 \quad (R\to\infty)\,. \tag{5.21}$$

Using the estimations (5.15), (5.16), and (5.18) we obtain that for every $\theta\in\Theta$

$$|\nabla\Phi|(\gamma_R(\theta),\theta) \geq \alpha\gamma_R(\theta)^{\alpha-1} \geq \alpha\overline{\phi}^{-1}(R)^{\alpha-1}\,, \tag{5.22}$$

$$|\Delta\Phi|(\gamma_R(\theta),\theta) \leq \kappa\gamma_R(\theta)^{m-2} \leq \kappa\begin{cases} R^{(m-2)/\alpha} & m\geq 2\,,\\ \overline{\phi}^{-1}(R)^{m-2} & m<2\,.\end{cases} \tag{5.23}$$

In the case $m<2$, corresponding to situation (i), the term $J(R)$ in (5.21) can be further estimated:

$$J_R \leq \frac{\kappa}{\alpha}\overline{\phi}^{-1}(R)^{m-2-(\alpha-1)} \to 0 \qquad (R\to\infty)\,,$$

since $\overline{\phi}^{-1}(R) \nearrow \infty$ as $R \to \infty$ and $m - 2 - (\alpha - 1) < 1 - \alpha \leq 0$. Hence the crucial condition (2.14) holds in this case.

In the case $m \geq 2$, corresponding to situation (ii), the term $J(R)$ is estimated using (5.22) and (5.23)

$$J_R \leq (\kappa/\alpha) R^{(m-2)/\alpha} \left(\overline{\phi}^{-1}(R)\right)^{-(\alpha-1)} \qquad R > \overline{\phi}(\rho_0)\,.$$

Instead of showing $\lim_{R\to\infty} J_R = 0$, it suffices to prove $\lim_{\rho\to\infty} J_{\overline{\phi}(\rho)}$, since also $\overline{\phi}(\rho) \nearrow \infty$ as $\rho \to \infty$. We estimate

$$\begin{aligned} J_{\overline{\phi}(\rho)} &\leq \frac{\kappa}{\alpha}(\rho^\alpha + p^* \rho^\beta)^{(m-2)/\alpha} \rho^{-(\alpha-1)} \\ &= \frac{\kappa}{\alpha}\rho^{m(m-2)/\alpha+1-\alpha}(\rho^{\alpha-m} + p^*\rho^{\beta-m})^{(m-2)/\alpha} \to 0 \qquad (\rho \to 0)\,. \end{aligned}$$

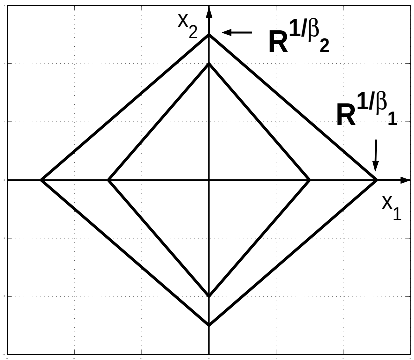
This convergence holds, since the last term in braces is bounded and the upper bound on β in the situation (ii) ensures that $m(m-2)/\alpha + 1 - \alpha < 0$. Hence the crucial condition (2.14) holds also in this case. □

5.3 Potential of tetragonal shape

We consider the following two-dimensional situation where the potential Φ appearing in the SDE (1.1) satisfies the relation

(5.24) $$|x_1|\Phi(x_1, x_2)^{-1/\beta_1} + |x_2|\Phi(x_1,x_2)^{-1/\beta_2} = 1\,, \qquad x_1, x_2 \in \mathbb{R} \setminus \{0\}\,.$$

where $0 < \beta_1 \leq \beta_2$. Obviously $\Phi(x_1, 0) = |x_1|^{\beta_1}$, $\Phi(0, x_2) = |x_2|^{\beta_2}$ for $x_1, x_2 \in \mathbb{R} \setminus \{0\}$ and the iso-level set $\partial O_R^\Phi = \{x : \Phi(x) = R\}$ for large $R > 0$ is a tetragon with edges $(0, \pm R^{1/\beta_2})$ and $(\pm R^{1/\beta_1}, 0)$, see figure. In this setting we are able to compute the terms $m_{\Phi,R}[|\nabla\Phi|^2]$ and $m_{\Phi,R}[(\Delta\Phi)^2]$ explicitly.



Note the similarities with the potential of the form (5.12) in the last example. In contrast to the preceding example it will turn out that we do not need growth restriction on the asymmetric part of the potential (growing with r^β) as in Theorem 5.5.

To overcome the problem that $\Phi \notin C^2(\mathbb{R}^2,\mathbb{R})$, one can smooth the edges of Φ in such a way that the terms resulting from this smoothing procedure do not matter in the limit $R \to \infty$. We will check the conditions of Theorem 2.5. The conditions (2.2) and (1.6) obviously hold by the definition of Φ. Further $\Phi(x) \nearrow \infty$ as $|x| \to \infty$, i.e. condition (2.10) holds. We begin to calculate $|\nabla\Phi|$ and $\Delta\Phi$. By the symmetry of the potential Φ it is sufficient to work only in the positive quadrant. Set

$$\delta = 1/\beta_1 - 1/\beta_2\,, \qquad h(R,x) := 1/\beta_1 - \delta y^{-1/\beta_2}x\,, \quad R,x>0\,.$$

The relation (5.24) can be rewritten in the positive quadrant $x_1+x_2\Phi^{\delta} = \Phi^{1/\beta_1}$, $x_1,x_2>0$. Applying the partial derivatives w.r.t. x_1, x_2 to this equation we obtain after some obvious transformations

$$\partial_{x_1}\Phi = \Phi^{1-1/\beta_1}h(\Phi,x_2)^{-1}\,, \quad \partial_{x_2}\Phi = \Phi^{1-1/\beta_2}h(\Phi,x_2)^{-1}\,. \tag{5.25}$$

Hence

$$|\nabla\Phi| = \frac{\Phi}{h(\Phi,x_2)}\sqrt{\Phi^{-2/\beta_1}+\Phi^{-2/\beta_2}} = \frac{\Phi^{1-1/\beta_1-1/\beta_2}}{h(\Phi,x_2)}\sqrt{\Phi^{2/\beta_1}+\Phi^{2/\beta_2}}\,. \tag{5.26}$$

Applying again the partial derivative w.r.t. x_1 to the first equation in (5.25) and w.r.t. x_2 to the second equation, respectively, and substituting the arising terms $\partial_{x_i}\Phi$, $i=1,2$, according to (5.25), we get after some transformations

$$\partial^2_{x_i}\Phi = \Phi^{1-2/\beta_i}h(\Phi,x_2)^{-3}(\kappa_i - K_i\Phi^{-1/\beta_2}x_2) \qquad i=1,2\,,$$

where $\kappa_1 = 1/\beta_1 - 1/\beta_1^2$, $K_1=\delta-\delta^2$, $\kappa_2 = 1/\beta_1(1+1/\beta_1-2/\beta_2)$, and $K_2=\delta+\delta^2$. Hence

$$(\Delta\Phi)^2 = \frac{\Phi^{2-4/\beta_2}}{h(\Phi,x_2)^6}\Big\{\Phi^{-2\delta}(\kappa_1 - K_1\Phi^{-1/\beta_2}x_2) + (\kappa_2 - K_2\Phi^{-1/\beta_2}x_2)\Big\}^2\,. \tag{5.27}$$

We need the following estimates: since $1/\beta_2 \le h(R,x_2) \le 1/\beta_1$ for $x_2 \in [0,R^{1/\beta_2}]$, we deduce from (5.26) and (5.27)

$$\min_{\partial O_R^\Phi}|\nabla\Phi|^2 \gtrsim \beta_1^2 R^{2-2/\beta_2}\,, \quad \max_{\partial O_R^\Phi}|\Delta\Phi| \lesssim \beta_2^3\kappa_2 R^{1-2/\beta_2} \qquad (R\to\infty)\,. \tag{5.28}$$

We claim that the spectral gap property (2.4) holds if $\beta_2 \ge 1$. To this aim we estimate the function V_Φ defined in (3.4) on the iso-level set ∂O_R^Φ using (5.28)

$$\min_{\partial O_R^\Phi} V_\Phi \gtrsim \frac{\beta_1^2}{4}R^{2(1-1/\beta_2)} - \frac{\beta_2^3\kappa_2}{2}R^{1-2/\beta_2} = R^{2(1-1/\beta_2)}\left(\frac{\beta_1^2}{4} - \frac{\beta_2^3\kappa_2}{2R}\right) \qquad (R\to\infty)\,.$$

Since $\beta_2 \geq 1$ and $\Phi(x) \nearrow \infty$ as $|x| \to \infty$, we obtain that $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ and hence the spectral gap property (2.4) holds by Proposition 3.2.

In order to show the crucial condition (2.14), we compute $m_{\Phi,R}[|\nabla\Phi|^2]$ and $m_{\Phi,R}[(\Delta\Phi)^2]$. Fix $R > 0$ sufficiently large. Note that the iso-level set ∂O_R^Φ in the positive quadrant is the line joining the points $(0, R^{1/\beta_2})$ and $(R^{1/\beta_1}, 0)$, see figure. This line can be parameterized by

$$\gamma_R(x_2) = (R^{1/\beta_1} - R^\delta x_2, x_2)\,, \quad x_2 \in [0, R^{1/\beta_2}]\,.$$

Setting $d_R := \sqrt{R^{2/\beta_1} + R^{2/\beta_2}}$ we obtain for the infinitesimal curvature $|\gamma_R'(x_2)|dx_2 = (d_R/R^{1/\beta_2})dx_2$. To obtain the values of $|\nabla\Phi|$ and $(\Delta\Phi)^2$ on the iso-level set ∂O_R^Φ in the positive quadrant we simply have to set $\Phi = R$ constant in (5.26) and (5.27) using x_2 to parameterize ∂O_R^Φ. Recalling the definition (2.11) of $m_{\Phi,R}[\cdot]$, we obtain invoking (5.26) and (5.27)

$$\begin{aligned}
m_{\Phi,R}[|\nabla\Phi|^2] &= \int_{\partial O_R^\Phi} |\nabla\Phi(\xi)|\, d\sigma_{\Phi,R}(\xi) \\
&= 4\,R^{1-1/\beta_1-1/\beta_2}\sqrt{R^{2/\beta_1}+R^{2/\beta_2}} \int_0^{R^{1/\beta_2}} \frac{1}{h(R,x_2)}\frac{d_R}{R^{1/\beta_2}}dx_2 \\
&= 4\,R^{1-1/\beta_1-2/\beta_2}\, d_R^2\, R^{1/\beta_2} \int_0^1 \frac{dz}{1/\beta_1 - \delta z} \\
&= \frac{4}{\delta}\ln\left(\frac{\beta_2}{\beta_1}\right) R^{1-1/\beta_1-1/\beta_2}(R^{2/\beta_1}+R^{2/\beta_2}) \\
&\sim \frac{4}{\delta}\ln\left(\frac{\beta_2}{\beta_1}\right) R^{1+\delta} \qquad (R\to\infty)\,.
\end{aligned} \tag{5.29}$$

Similarly

$$\begin{aligned}
m_{\Phi,R}[(\Delta\Phi)^2] &= \int_{\partial O_R^\Phi} \frac{(\Delta\Phi)^2}{|\nabla\Phi|}(\xi)\, d\sigma_{\Phi,R}(\xi) \\
&\sim 4\frac{R^{2-4/\beta_2}}{R^{1-1/\beta_1-1/\beta_2}\, d_R} \int_0^{R^{1/\beta_2}} \frac{(\kappa_2 - K_2 R^{-1/\beta_2}x_2)^2}{h(R,x_2)^5}\frac{d_R}{R^{1/\beta_2}}dx_2 \\
&= 4\,R^{1+1/\beta_1-4/\beta_2}\, R^{1/\beta_2} \int_0^1 \frac{(\kappa_2 - K_2 z)^2}{(1/\beta_1-\delta z)^5}dz \\
&= K\,R^{1+\delta-2/\beta_2} \qquad (R\to\infty)\,,
\end{aligned} \tag{5.30}$$

where $K > 0$ is a constant. From (5.29) and (5.30) we see that crucial condition (2.12) holds for every choice of $0 < \beta_1 \leq \beta_2$.

The term $I(R)$ defined in (2.13) reads in our situation using (5.29) and Lemma 2.4

$$\begin{aligned} I(R) &= \frac{4}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) \int_0^R e^{2r/\sigma^2} r^{1+\delta}(1+r^{-2\delta})\, dr \\ &\sim \frac{2\sigma^2}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) R^{1+\delta} e^{2R/\sigma^2} \qquad (R\to\infty)\,. \end{aligned} \tag{5.31}$$

The last step follows from Lemma 5.2. Hence the growth conditions (2.14) are obviously satisfied. Setting

$$l(R) := \frac{4}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) R^{1+\delta} e^{-2R/\sigma^2} \qquad R>0\,,$$

we obtain by Theorem 2.5 and (5.31), that $\lambda_R \sim l(R)$ as $R\to\infty$, where λ_R is the bottom eigenvalue associated to the exhausting family $(O_R^\Phi)_{R>0}$ of $\mathbb{R}^2$.

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