On Riesz-Fischer sequences and lower frame bounds

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Abstract

We investigate the consequences of the lower frame condition and the lower Riesz basis condition without assuming the existence of the corresponding upper bounds. We prove that the lower frame bound is equivalent to an expansion property on a subspace of the underlying Hilbert space $\mathcal{H}$, and that the lower frame condition alone is not enough to obtain series representations on all of $\mathcal{H}$. We prove that the lower Riesz basis condition for a complete sequence implies the lower frame condition and ω-independence; under an extra condition the statements are equivalent.

1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space. Recall that a sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *frame* if

$$\exists A, B > 0 : A||f||^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}. \tag{1.1}$$

The sequence $\{f_i\}_{i=1}^{\infty}$ is a *Riesz basis* if $\overline{span}\{f_i\}_{i\in I} = \mathcal{H}$ and there exist $A, B > 0$ such that for all finite scalar sequences $\{c_i\}$,

$$A \sum |c_i|^2 \leq ||\sum c_i f_i||^2 \leq B \sum |c_i|^2. \tag{1.2}$$

A Riesz basis is a frame; and if $\{f_i\}_{i=1}^{\infty}$ is a frame, there exists a dual frame $\{g_i\}_{i=1}^{\infty}$ such that

$$f = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i\rangle g_i, \ \forall f \in \mathcal{H}. \tag{1.3}$$

In this note we investigate the consequences of the lower bounds in (1.1) and (1.2) without assuming the existence of the upper bounds. Note that the lower condition in (1.1) implies that every $f \in \mathcal{H}$ is uniquely determined by the inner products $\{\langle f, f_i\rangle\}_{i=1}^{\infty}$: if $\langle f, f_i\rangle = \langle g, f_i\rangle \ \forall i \in \mathbb{N}$, then $f = g$. That is, in principle we can recover every $f \in \mathcal{H}$ based on knowledge of the sequence $\{\langle f, f_i\rangle\}_{i=1}^{\infty}$. We prove that we actually obtain a representation of the type (1.3) for certain $f \in \mathcal{H}$. The question whether the representation can be extended to work for all $f \in \mathcal{H}$ has been open for some time. We present an example where it can not be extended.

2 Some definitions and basic results

For convenience we will index all sequences by the natural numbers $\mathbb{N}$.

Definition 2.1 Let $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$. We say that

(i) $\{f_i\}_{i=1}^{\infty}$ is a *Riesz-Fischer sequence* if there exists a constant $A > 0$ such that $A \sum |c_i|^2 \leq ||\sum c_i f_i||^2$ for all finite scalar sequences $\{c_i\}$;

(ii) $\{f_i\}_{i=1}^{\infty}$ *satisfies the lower frame condition* if there exists a constant $A > 0$ such that $A||f||^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2$ for all $f \in \mathcal{H}$.

(iii) $\{f_i\}_{i=1}^{\infty}$ is a *Bessel sequence* if there exists a constant $B > 0$ such that $\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B||f||^2$ for all $f \in \mathcal{H}$.

(iv) $\{f_i\}_{i=1}^{\infty}$ is *minimal* if for all $j \in \mathbb{N}$, $f_j \notin \overline{\text{span}}\{f_i\}_{i\neq j}$.

(v) $\{f_i\}_{i=1}^{\infty}$ is *ω-independent* if $\sum_{i=1}^{\infty} c_i f_i = 0$ implies that $c_i = 0$ for all $i \in \mathbb{N}$.

(vi) $\{f_i\}_{i=1}^{\infty}$ is *complete* if $\overline{\text{span}}\{f_i\}_{i=1}^{\infty} = \mathcal{H}$.

(vii) $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are *biorthogonal* if $\langle f_i, g_j\rangle = \delta_{i,j}$ (Kronecker's δ).

For a given family $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, our analysis is based on the *synthesis operator*

$$T : \mathcal{D}(T) := \{\{c_i\}_{i=1}^{\infty} \in \ell^2 | \sum_{i=1}^{\infty} c_i f_i \text{ converges}\} \to \mathcal{H}, \ T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i \quad (2.4)$$

and the *analysis operator*

$$U : \mathcal{D}(U) := \{f \in \mathcal{H} \mid \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 < \infty\} \to \ell^2, \quad Uf := \{\langle f, f_i\rangle\}_{i=1}^{\infty}. \quad (2.5)$$

The Lemma below is proved in [7].

Lemma 2.2 *Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$. Then*

(i) $\{f_i\}_{i=1}^{\infty}$ has a biorthogonal sequence if and only if $\{f_i\}_{i=1}^{\infty}$ is minimal; and if a biorthogonal sequence exists, it is unique if and only if $\{f_i\}_{i=1}^{\infty}$ is complete.

(ii) $\{f_i\}_{i=1}^{\infty}$ is a Riesz-Fischer sequence if and only if the associated analysis operator is surjective.

We collect two other characterizations of Riesz-Fischer sequences. Apparently, they have not been stated explicitely before; they can be proved using methods developed in [7].

Proposition 2.3
(i) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. The Riesz-Fischer sequences in $\mathcal{H}$ are precisely the families $\{Ve_i\}_{i=1}^{\infty}$, where V is an operator on $\mathcal{H}$ (having $\{e_i\}_{i=1}^{\infty}$ in the domain), which has a bounded inverse $V^{-1} : R(V) \to \mathcal{H}$.

(ii) The Riesz-Fischer sequences in $\mathcal{H}$ are precisely the families for which a biorthogonal Bessel sequence exists.

Example 2.4 Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis and consider $\{g_i\}_{i=1}^{\infty} := \{e_i + e_{i+1}\}_{i=1}^{\infty}$. Then $\{g_i\}_{i=1}^{\infty}$ is complete and minimal; it is also a Bessel sequence, but not a frame. A straightforward calculation shows that the biorthogonal system is given by

$$f_i = \sum_{k=1}^{i} (-1)^k e_k \text{ if } i \text{ is even}, \quad f_i = \sum_{k=1}^{i} (-1)^{k+1} e_k \text{ if } i \text{ is odd}.$$

$\{f_i\}_{i=1}^{\infty}$ is a Riesz-Fischer sequence by Proposition 2.3.

3 The lower frame condition

Lemma 3.1 *For an arbitrary sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$, the associated analysis operator U is closed. Furthermore, $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition if and only if U has closed range and is injective.*

Proof: That U is closed follows by a standard argument. To prove that $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition if and only if U has closed range and is injective, note that the existence of a lower frame bound implies injectivity of U. Since U is closed, U^{-1} is closed. Thus, by the closed graph theorem, U has closed range if and only if U^{-1} is continuous on $R(U)$, which is obviously equivalent to the existence of a lower frame bound. □

Recall that a frame is a Riesz basis if and only if it is ω-independent. The Theorem below generalizes this result to the case where $\{f_i\}_{i=1}^{\infty}$ only satisfies the lower frame condition. It connects the concepts listed in Definition 2.1:

Theorem 3.2 *Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ with associated synthesis operator T. Consider the following statements:*

(i) $\{f_i\}_{i=1}^{\infty}$ *is a complete Riesz-Fischer sequence.*

(ii) $\{f_i\}_{i=1}^{\infty}$ *is minimal and satisfies the lower frame condition.*

(iii) $\{f_i\}_{i=1}^{\infty}$ *is ω-independent and satisfies the lower frame condition.*

Then the implications (i) ⇒ (ii) ⇒ (iii) hold. In general (iii) does not imply any of the other statements, but if T is closed and surjective, then all statements are equivalent.

Proof: (i) ⇒ (ii). By Lemma 2.2 (ii), the analysis operator U is surjective, and since $\{f_i\}_{i=1}^{\infty}$ is complete, it is also injective. From Lemma 3.1 it follows that $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition. That $\{f_i\}_{i=1}^{\infty}$ is minimal follows easily from the definition of Riesz-Fischer sequences.
(ii) ⇒ (iii): Suppose $\sum_{i=1}^{\infty} c_i f_i = 0$ with not all c_i zero. Then there is some j such that $c_j \neq 0$ and hence $f_j = -\sum_{i \neq j} \frac{c_i}{c_j} f_i$, implying $f_j \in \overline{\text{span}}\{f_i\}_{i \neq j}$, contradicting minimality of $\{f_i\}_{i=1}^{\infty}$.
We now show that (iii) does not imply (ii). In Theorem 3.5 we will show that in an arbitrary Hilbert space there exists a ω-independent sequence $\{f_i\}_{i=1}^{\infty}$ which satisfies the lower frame condition and for which there is a

$f \in \mathcal{H}$ such that no sequence of scalars $\{a_i\}$ satisfies $f = \sum_{i=1}^{\infty} a_i f_i$. Then $\{f_i\}_{i=1}^{\infty} \cup \{f\}$ satisfies the lower frame condition and is ω-linearly independent, but is not minimal, since $\{f_i\}_{i=1}^{\infty}$ is already complete. Clearly, this argument also shows that (i) can not be satisfied. On the other hand, if T is closed and surjective, it is proved in [1] that there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that $f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i$ for all $f \in \mathcal{H}$. Assuming (iii), it follows that $\langle f_i, g_j \rangle = \delta_{i,j}$, i.e., $\{g_i\}_{i=1}^{\infty}$ is a biorthogonal Bessel sequence; thus, via Proposition 2.3, $\{f_i\}_{i=1}^{\infty}$ is a Riesz-Fischer sequence, and completeness of it follows from the lower frame bound. $\square$

Riesz-Fischer sequences can also be characterized by the following property, involving lower frame bounds for the subspaces spanned by finite subsets.

Proposition 3.3 *Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and let $\{I_n\}_{n=1}^{\infty}$ be a family of finite subsets of $\mathbb{N}$ such that*

$$I_1 \subseteq I_2 \subseteq \cdots \uparrow \mathbb{N}.$$

Denote by $A_{I_n}^{opt}$ the optimal lower frame bound for $\{f_i\}_{i \in I_n}$ in $\operatorname{span}\{f_i\}_{i \in I_n}$. Then $\{f_i\}_{i=1}^{\infty}$ is a Riesz-Fischer sequence if and only if it is (finitely) linearly independent and $\inf_{n \in \mathbb{N}} A_{I_n}^{opt} > 0$.

The proof for this follows the same lines as [3, Proposition 1.1], where this was proved under the additional condition that $\{f_i\}_{i=1}^{\infty}$ was a frame for $\mathcal{H}$. Under this extra condition, the characterization was first proved by Kim and Lim [4] as a consequence of a series of Theorems.

The Proposition below characterizes sequences satisfying the lower frame condition in terms of an expansion property.

Proposition 3.4 *Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$. Then $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition if and only if there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ such that*

$$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle g_i, \ \forall f \in \mathcal{D}(U). \tag{3.6}$$

Proof: Assume that $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition. Then $U^{-1} : R(U) \to \mathcal{H}$ is bounded. Define a linear operator $V : l^2(\mathbb{N}) \to \mathcal{H}$ by $V = U^{-1}$ on $R(U)$, by $V = 0$ on $R(U)^{\perp}$ and extending it linearly. Then V is bounded.

Let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis for $\ell^2(\mathbb{N})$ and let $g_i := Ve_i$. Then $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence and by construction, for all $f \in \mathcal{D}(U)$ we have

$$f = VUf = \sum_{i=1}^{\infty} \langle f, f_i \rangle g_i.$$

On the other hand, if $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence with bound B and (3.6) is satisfied, then for all $f \in \mathcal{D}(U)$,

$$||f||^2 = ||\sum_{i=1}^{\infty} \langle f, f_i \rangle g_i||^2 \leq B \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2,$$

meaning that the lower frame condition is satisfied. □

Note that when $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition, the Bessel sequence $\{g_i\}_{i=1}^{\infty}$ constructed in the proof of Proposition 3.4 belongs to $\mathcal{D}(U)$. Observe, that the equality (3.6) might hold for all $f \in \mathcal{H}$ without $\mathcal{D}(U)$ being equal to $\mathcal{H}$. For instance, if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis and we define $f_i := ie_i,\ i \in \mathbb{N}$, then

$$\mathcal{D}(U) = \{f = \sum_{i=1}^{\infty} c_i e_i \mid \sum_{i=1}^{\infty} |ic_i|^2 < \infty\}$$

which is only a subspace of $\mathcal{H}$. Nevertheless,

$$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle \frac{1}{i} e_i, \quad \forall f \in \mathcal{H}. \tag{3.7}$$

Note that $\{ie_i\}_{i=1}^{\infty}$ is a Riesz-Fischer sequence, but not a Riesz basis. For several families of elements having a special structure, the Riesz-Fischer property *implies* the upper Riesz basis condition; let us just mention families of complex exponentials in $L^2(-\pi, \pi)$, cf. [7], [8], [5]. As far as we know, no example of a norm-bounded family in a general Hilbert space satisfying the Riesz-Fischer property but not the upper Riesz basis condition has been known. Theorem 3.5 will provide such an example.

As we have seen in Proposition 3.4, the lower frame condition on $\{f_i\}_{i=1}^{\infty}$ is enough to obtain a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that (3.6) holds. In (3.7) we have seen that the representation (3.6) might hold for all $f \in \mathcal{H}$, even

if $\mathcal{D}(U)$ is a proper subspace of $\mathcal{H}$; one could hope that the representation always hold on $\mathcal{H}$. Our next purpose is to prove that this is not the case. We need to do some preparation before the proof, but we state the result already now.

Theorem 3.5 *In every separable, infinite dimensional Hilbert space $\mathcal{H}$ there exists a norm-bounded Riesz-Fischer sequence $\{f_i\}$ for which*

(1) $\{f_i\}$ has lower frame bound 1 and no finite upper frame bound.

(2) $\mathcal{D}(U)$ is dense in $\mathcal{H}$, and $\{f_i\} \subseteq \mathcal{D}(U)$.

(3) $\{f_i\}$ is ω-independent.

(4) $\{f_i\}$ is not a (Schauder) basis for $\mathcal{H}$.

(5) There is an $f \in \mathcal{H}$ so that there is no sequence of scalars $\{a_i\}$ for which

$$f = \sum_i a_i f_i.$$

(6) There is no family of functions $\{g_i\}$ so that for every $f \in \mathcal{H}$ we have

$$f = \sum_i \langle f, f_i \rangle g_i.$$

Moreover, (4),(5) and (6) hold for all permutations of $\{f_i\}$.

Our proof of Theorem 3.5 is constructive, and the result was used in the proof of Theorem 3.2 to show that in general (iii) does not imply (i).

The idea in the construction proving Theorem 3.5 is to consider a Hilbert space $\mathcal{H}$ which is a direct sum of subspaces of increasing order. Before we go into details with the construction, we need some preliminary results. Given $n = 2, 3, \ldots$, let $\mathcal{H}_n$ be a Hilbert space of dimension n and let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathcal{H}_n$. Let P_n be the orthogonal projection onto the unit vector $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i$, i.e.,

$$P_n \left(\sum_{i=1}^n a_i e_i \right) = \frac{\sum_{i=1}^n a_i}{n} \sum_{i=1}^n e_i.$$

Let $\mathcal{H}_n^1 = (I - P_n)\mathcal{H}_n$. For all $1 \leq j \leq n-1$ let

$$f_j^n = e_j - e_n.$$

Note that $\{f_j^n\}_{j=1}^{n-1}$ is a linearly independent family which spans $\mathcal{H}_n^1$. Our first lemma will identify the frame bounds and the dual frame for subfamilies of $\{f_j^n\}_{j=1}^{n-1}$.

Lemma 3.6 *Given any $n = 2, 3, ...$ and any $I \subset \{1, 2, \cdots, n-1\}$, the family $\{f_j^n\}_{j\in I}$ is a linearly independent frame for its span with lower frame bound 1 (which is optimal for $|I| > 1$) and upper frame bound at least $\frac{|I|+3}{2}$. The dual frame for $\{f_j^n\}_{j=1}^{n-1}$ is given by*

$$g_j = \frac{n-1}{n} e_j - \frac{1}{n} \sum_{i\neq j} e_i, \ j = 1, 2, .., n-1.$$

Proof: Given $f \in \text{span } \{f_j^n\}_{j\in I}$, there are scalars $\{a_j\}$ so that

$$f = \sum_{j\in I} a_j f_j^n = \sum_{j\in I} a_j e_j - \left(\sum_{i\in I} a_i \right) e_n. \tag{3.8}$$

Note that $||f||^2 = \sum_{j\in I} |a_j|^2 + |\sum_{i\in I} a_i|^2$ and $\langle f, f_j^n \rangle = a_j + \sum_{i\in I} a_i$. Thus

$$\sum_{j\in I} |\langle f, f_j^n \rangle|^2 = \sum_{j\in I} |a_j + \sum_{i\in I} a_i|^2$$

$$= \sum_{j\in I} \left[a_j + \sum_{i\in I} a_i \right] \overline{\left[a_j + \sum_{i\in I} a_i \right]}$$

$$= \sum_{j\in I} |a_j|^2 + 2 \sum_{j\in I} \text{Re} \left(a_j \overline{\left[\sum_{i\in I} a_i \right]} \right) + (|I|)| \sum_{i\in I} a_i|^2.$$

Here we observe that

$$\sum_{j\in I} \text{Re} \left(a_j \overline{\left[\sum_{i\in I} a_i \right]} \right) = \text{Re} \sum_{j\in I} a_j \overline{\left[\sum_{i\in I} a_i \right]} = \text{Re} \left[\left(\sum_{j\in I} a_j \right) \overline{\left[\sum_{i\in I} a_i \right]} \right] = |\sum_{i\in I} a_i|^2.$$

Thus

$$\sum_{j\in I} |\langle f, f_j^n \rangle|^2 = \sum_{j\in I} |a_j|^2 + (|I| + 2)| \sum_{i\in I} a_i|^2 = \|f\|^2 + (|I| + 1)| \sum_{i\in I} a_i|^2.$$

So the choice $A = 1$ is a lower frame bound. If $|I| > 1$, we can choose $\{a_i\}_{i\in I}$ such that $\sum_{i\in I} a_i = 0$, so the choice $f = \sum_{i\in I} a_i f_i$ with exactly those coefficients shows that $A = 1$ is actually the optimal lower bound in this

case. If $|I| = 1$, say, $I = \{j\}$, then the relation (3.8) between f and $\{a_i\}_{i\in I}$ gives that

$$(|I|+1)|\sum_{i\in I} a_i|^2 = 2|a_j|^2 = 2\frac{||f||^2}{||f_j^n||^2} = ||f||^2,$$

so the optimal lower bound is $A = 2$ in this case.

Now we fix $i \in I$ and compute

$$\sum_{j\in I} |\langle e_i - e_n, f_j^n\rangle|^2 = 4 + |I| - 1 = |I| + 3 = \frac{|I|+3}{2}||e_i - e_n||^2.$$

It follows that the optimal upper bound is at least $\frac{|I|+3}{2}$.

Since our family $\{f_j^n\}_{j=1}^{n-1}$ is linearly independent, the dual frame $\{g_j\}_{j=1}^{n-1}$ is the family of dual functionals for the (Schauder) basis $\{f_j^n\}_{j=1}^{n-1}$. We will now compute this family explicitly. Because of symmetry, it suffices to find g_1^n, which we now do. Write

$$g_1^n = \sum_{i=1}^n a_i e_i$$

and observe that g_1^n is uniquely determined by the following 3 conditions:

(i) $1 = \langle g_1^n, e_1 - e_n\rangle = a_1 - a_n$.

(ii) For all $2 \le i \le n-1$, $\quad 0 = \langle g_1^n, f_i^n\rangle = a_i - a_n$.

(iii) Since g_1^n is in the orthogonal complement of the vector $\sum_{i=1}^n e_i$, the coefficients satisfy

$$\sum_{i=1}^n a_i = 0.$$

Now, by (i) and (ii) we have

$$g_1^n = (1 + a_n)e_1 + a_n \sum_{i=2}^n e_i.$$

By (iii),

$$1 + a_n + (n-1)a_n = 0.$$

Hence, $1 = -na_n$, and so $a_n = -1/n$. Finally,

$$a_1 = 1 + a_n = \frac{n-1}{n}. \quad \square$$

Recall that the *basis constant* for a sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ is defined as

$$K := \sup \left\{ \frac{||\sum_{i=1}^{m} c_i f_i||}{||\sum_{i=1}^{n} c_i f_i||} : 1 \leq m \leq n < \infty,\ c_1, \cdots, c_n \in \mathbb{C}, \sum_{i=1}^{n} c_i f_i \neq 0 \right\}$$

(for finite sequences $\{f_i\}_{i=1}^{N}$, we replace "$n < \infty$" by "$n \leq N$"). To make the calculations in the next Lemma easier, we will work with $\mathcal{H}_{2n+1}^{1}$.

Lemma 3.7 *Let $n = 2, 3, \cdots$ and σ be a permutation of $\{1, 2, \cdots, 2n\}$. Then there is a sequence of scalars $\{a_i\}_{i=1}^{2n}$ so that*

$$\| \sum_{i=1}^{n} a_i f_{\sigma(i)}^{2n+1} \|^2 = n + 1$$

while

$$\| \sum_{i=1}^{2n} a_i f_{\sigma(i)}^{2n+1} \|^2 = 2.$$

In particular, the basis constant for $\{f_{\sigma(i)}^{2n+1}\}_{i=1}^{2n}$ is at least $\sqrt{\frac{n+1}{2}}$.

Proof: Let

$$a_i = \frac{1}{\sqrt{n}} \ \text{ for } 1 \leq i \leq n, \quad a_i = -\frac{1}{\sqrt{n}} \ \text{ for } n+1 \leq i \leq 2n.$$

Then,

$$\| \sum_{i=1}^{n} a_i f_{\sigma(i)}^{2n+1} \|^2 = \sum_{i=1}^{n} |a_i|^2 + | \sum_{i=1}^{n} a_i |^2 = 1 + n.$$

Also, $\sum_{i=1}^{2n} a_i = 0$ implies

$$\sum_{i=1}^{2n} a_i f_{\sigma(i)}^{2n+1} = \sum_{i=1}^{2n} a_i e_{\sigma(i)}.$$

Hence,

$$\| \sum_{i=1}^{2n} a_i f_{\sigma(i)}^{2n+1} \|^2 = \sum_{i=1}^{2n} |a_i|^2 = 2. \quad \square$$

It is proved in [6] that $\{f_i\}_{i=1}^{\infty}$ can only be a basis if the basis constant is finite. We are now ready for the proof of Theorem 3.5.

Proof of Theorem 3.5: Using the notation above we consider the Hilbert space

$$\mathcal{H} = \left(\sum_{n=2}^{\infty} \oplus \mathcal{H}_n^1 \right)_{\ell_2}.$$

We refer to [6] for details about such constructions. Let the sequence $\{f_i\}_{i=1}^{\infty}$ be any enumeration of $\{f_j^n\}_{j=1,n=2}^{n-1,\ \infty}$. Since $\{f_j^n\}_{j=1}^{n-1}$ spans $\mathcal{H}_n^1$ and is linearly independent for each $n = 2, 3, \cdots$, (1) and (3) follow. (2) is clear.

We now prove that $\{f_i\}_{i=1}^{\infty}$ can not be a Schauder basis; since $\{f_i\}_{i=1}^{\infty}$ is defined as an arbitrary enumeration of the elements in $\{f_j^n\}_{j=1,n=2}^{n-1,\ \infty}$, this will prove (4). The basis constant for $\{f_i\}_{i=1}^{\infty}$ is larger than or equal to the basis constant for any subsequence. But for each $n \in \mathbb{N}$, a permutation of the family $\{f_j^{2n+1}\}_{j=1}^{2n}$ is a subsequence of $\{f_i\}_{i=1}^{\infty}$, and by Lemma 3.7 its basis constant is at least $\sqrt{\frac{n+1}{2}}$; thus the basis constant for $\{f_i\}_{i=1}^{\infty}$ is infinite, and it can not be a basis. This proves (4).

We now prove (5). It clearly follows from (3) that whenever $f \in \mathcal{H}$, if there is a sequence of scalars $\{a_j\}$ so that $f = \sum_j a_j f_j$, then $\{a_j\}$ is unique; since $\{f_j\}$ is not a Schauder basis, this gives (5).

For the proof of (6), we observe that corresponding to $\{f_j^n\}_{j=1,n=2}^{n-1,\ \infty}$, the dual functionals $\{g_j^n\}_{j=1,n=2}^{n-1,\ \infty}$ are by Lemma 3.6 given by

$$g_j^n = \frac{n-1}{n} e_j - \frac{1}{n} \sum_{i \neq j} e_i, \quad \text{for} \quad 1 \leq j \leq n-1, n = 2, 3, ...$$

This family is the *only* candidate to satisfy (6). In fact, suppose that a sequence $\{h_j^n\}_{j=1,n=1}^{n-1,\ \infty}$ satisfies $f = \sum_{j,n} \langle f, f_j^n \rangle h_j^n$ for all $f \in \mathcal{H}$. Now, for all $n \neq m$ and all $1 \leq i \leq n-1$ we have that

$$\langle g_j^m, f_i^n \rangle = 0.$$

Also,

$$\langle g_j^m, f_i^m \rangle = 0, \text{ for all } 1 \leq i \neq j \leq m-1, \text{ while } \langle g_j^m, f_j^m \rangle = 1.$$

Putting this altogether,

$$g_j^m = \sum_{i,n} \langle g_j^m, f_i^n \rangle h_i^n = \langle g_j^m, f_j^m \rangle h_j^m = h_j^m.$$

That is, $h_j^m = g_j^m$, for all $2 \leq m \in \mathbb{N}$ and all $1 \leq j \leq m-1$. Now we observe that this family *does not* work for reconstruction. For $n \in \mathbb{N}$, $\{g_j^n\}_{j=1}^{n-1}$ are the dual functionals to $\{f_j^n\}_{j=1}^{n-1}$. Since $\{f_j^n\}_{j=1,n=1}^{n-1,\ \infty}$ is not a basis, we conclude that $\{g_j^n\}_{j=1,n=1}^{n-1,\ \infty}$ is not a basis. Since $\{g_j^n\}_{j=1,n=1}^{n-1,\ \infty}$ is clearly an ω-independent family, this means that there exists $f \in \mathcal{H}$ which can not be written $f = \sum_{j,n} c_j^n g_j^n$ for any choice of coefficients $\{c_j^n\}$. This proves (6). □

To conclude the paper we observe that if every subfamily of $\{f_i\}_{i\in I}$ satisfies the lower frame condition with a common bound A, then there exists a subfamily of $\{f_i\}_{i\in I}$ which satisfies the lower Riesz basis condition. The proof is similar to the proof of Theorem 3.2 in [2].

Proposition 3.8 *Suppose that $\{f_i\}_{i\in I}$ satisfies (1.1) and that every subfamily $\{f_i\}_{i\in J},\ J \subseteq I$ satisfies*

$$A||f||^2 \leq \sum_{i\in J} |\langle f, f_i\rangle|^2,\ \forall f \in \overline{\text{span}}\{f_i\}_{i\in J}. \tag{3.9}$$

Then $\{f_i\}_{i\in I}$ contains a complete subfamily $\{f_i\}_{i\in J}$ for which

$$A\sum_{i\in J} |c_i|^2 \leq ||\sum_{i\in J} c_i f_i||^2 \tag{3.10}$$

for all finite sequences $\{c_i\}_{i\in J}$.

In Proposition 3.8 the conclusion $A\sum_{i\in J}|c_i|^2 \leq ||\sum_{i\in J} c_i f_i||^2$ actually holds for all sequences $\{c_i\} \in \ell^2$ for which $\sum c_i f_i$ is convergent.

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