Pricing Derivatives of American and Game Type in Incomplete Markets

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Abstract

In this paper the *neutral valuation* approach is applied to American and game options in incomplete markets. Neutral prices occur if investors are utility maximizers and if derivative supply and demand are balanced. Game contingent claims are derivative contracts that can be terminated by both counterparties at any time before expiration. They generalize American options where this right is limited to the buyer of the claim. In turns out that as in the complete case, the price process of American and game contingent claims corresponds to a Snell envelope or to the value of a Dynkin game, respectively.

Key words: American options, game contingent claims, neutral derivative pricing, incomplete markets, Dynkin game

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1 Introduction

In recent years various suggestions have been made how to price European-type contingent claims in incomplete markets. By contrast, there is only very little corresponding literature dealing with American options. Pricing the latter is conceptually more involved: In addition to the uncertainty caused by the underlyings, one has to take the seller's ignorance of the buyer's exercise strategy into account. If we fix a stopping time as exercise time, then the American option reduces to a European claim. It is obvious that the American option should be worth at least as much as the most valuable of these implied European claims.

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In the financial literature the price of an American option is often just *defined* as the supremum of all European style claims corresponding to arbitrary stopping times of the buyer. Consequently, the problem of pricing American options is reduced to the simpler problem of pricing European contingent claims. However, this concept already implies by definition that an American option is not worth more than the highest priced of its implied European-style derivatives, i.e. the *right* to choose the exercise time has no value in itself. To us, this is not entirely obvious because in the American case the seller faces the disadvantage not to know the preferred stopping time of the buyer.

In complete markets, arbitrage arguments suffice to derive unique prices for American contingent claims. Here, it turns out that the fair price is indeed the supremum of the implied European option values (cf. Bensoussan (1984) and Karatzas (1988)). Analogous results are shown in varying degrees of generality for the superhedging price in incomplete markets (cf. Karatzas and Kou (1998), Kramkov (1996), Föllmer and Kabanov (1998), and Föllmer and Kramkov (1997)). This price denotes the smallest initial capital that allows to construct a portfolio which dominates the payoff process of the option. Although superhedging is an interesting concept from a theoretical point of view, it yields only trivial upper bounds in many models of practical importance (cf. e.g. Eberlein and Jacod (1997), Frey and Sin (1999), Cvitanić et al. (1999)). This is somewhat unsatisfactory.

Utility-based indifference pricing is a concept which has been applied explicitly to American options. Here, one takes the perspective of a particular counterparty and fixes the number of shares of the claim (say, 1 for an option buyer or -1 for an option seller). The *indifference premium* is a price such that the optimal expected utility among all portfolios containing the prespecified number of options coincides with the optimal expected utility among all portfolios without option. Put differently, the investor is indifferent to including the option into the portfolio. Taking the perspective of the option buyer, it turns out that the indifference price is indeed the supremum of the indifference prices of the implied European claims (cf. Davis and Zariphopoulou (1995)). Surprisingly, this is not true for the option seller: Unless exponential utility is chosen, it may happen that a reasonable indifference premium for an American option exceeds the indifference price of all implied European claims (cf. Kühn (2002)).

In this paper we show that the concept of *neutral derivative pricing*, as suggested in Kallsen (2001), can be adapted quite naturally to American options. Neutral prices occur if traders maximize their expected utility and if derivative supply and demand are balanced. More precisely, a derivative price process is called *neutral* if the optimal portfolio contains no contingent claim. We will see that the neutral price of an American option coincides as in the complete case with the supremum of the neutral prices of all implied European claims.

Both utility-based indifference pricing and neutral pricing rely on expected utility maximization and indifference to trading the option. Let us point out the differences between the two concepts. Indifference pricing takes an asymmetric point of view. Moreover, it depends decisively on the fixed number of claims under consideration. As far as options are concerned, intermediate trades are not allowed. Therfore, this approach is particularly well suited for over-the-counter trades: Suppose that the buyer wants to purchase a specific contingent claim. Then he has to pay the seller at least her indifference price in order to prompt her to enter the contract.

The concept of neutral pricing, on the other hand, takes a symmetric point of view. It assumes that options are traded in arbitrary positive and negative amounts. It tries to mimic the economic reasoning in complete markets by substituting utility maximizers for arbitrage traders. Neutral prices are the unique prices such that neither buyer nor seller takes advantage from trading the claim. For motivation of neutral derivative pricing, references, and connections to other approaches in the literature we refer the reader to Kallsen (2001).

As mentioned above, neutral pricing relies on utility maximization for portfolios containing derivatives. This is a non-trivial issue in the presence of American-type contingent claims. The point is that short positions in the claim may suddenly be terminated if the buyer exercises the option. Therefore, investment in American claims corresponds to investment under specific short-selling constraints (cf. Section 3).

In the present paper, American options are treated as special cases of game contingent claims. The latter naturally generalize American contingent claims by giving both counterparties the right to cancel the contract prematurely. This generalization requires some mathematical but no additional conceptual efforts. By contrast, it makes the neutral pricing approach even more transparent.

A *game contingent claim (GCC)*, as introduced in Kifer (2000), is a contract between a seller A and a buyer B which can be terminated by A and exercised by B at any time $t \in [0, T]$ up to a maturity date T when the contract is terminated anyway. More precisely, the contract may be specified in terms of stochastic processes $(L_t)_{t\in[0,T]}$, $(U_t)_{t\in[0,T]}$ with $L_t \leq U_t$ for $t \in [0, T]$ and $L_T = U_T$. If A terminates the contract at time t before it is exercised by B, she has to pay B the amount U_t. If B exercises the option before it is terminated by A, he is paid L_t. For motivation and examples for this kind of derivatives we refer the reader to Kifer (2000).

With American options the right to terminate the contract is restricted to the buyer B. Formally, they can be interpreted as game contingent claims by setting $U_t := m$ for $t \in [0, T)$, where $m \in \mathbb{R} \cup \{\infty\}$ exceeds the maximal payoff of the American option, e.g. $m = \infty$ in the unbounded case. This allows us to consider both kinds of options in a common framework.

Similarly as American options correspond to optimal stopping problems, GCC's incorporate a Dynkin game: If seller A selects stopping time τ^U as cancellation time and buyer B chooses stopping time τ^L as exercise time, then A pledges to pay B at time $\tau^L \wedge \tau^U$ the amount

$$R(\tau^L, \tau^U) = L_{\tau^L} 1_{\{\tau^L \leq \tau^U\}} + U_{\tau^U} 1_{\{\tau^U < \tau^L\}}.$$

In complete markets with a unique equivalent martingale measure $P^\star$, the random payoff $R(\tau^L, \tau^U)$ has the unique fair value $E_{P^\star}(R(\tau^L, \tau^U))$ at time 0. In analogy to American options, the buyer may want to choose his stopping time so as to maximize $E_{P^\star}(R(\tau^L, \tau^U))$ whereas the seller tries to minimize the same value. This is precisely the situation of a zerosum *Dynkin stopping game*. It is well-known that such a game has a unique value in the sense that

$$\inf_{\tau^U} \sup_{\tau^L} E_{P^\star}(R(\tau^L, \tau^U)) = \sup_{\tau^L} \inf_{\tau^U} E_{P^\star}(R(\tau^L, \tau^U)) \tag{1.1}$$

(cf. Lepeltier and Maingueneau (1984)). Kifer (2000) shows by hedging arguments that this value is in fact the unique no-arbitrage price of the GCC.

In incomplete markets these arguments fail because perfect replication is usually impossible. But it turns out that the price process of a GCC corresponds again to the value of a Dynkin game if we apply the neutral pricing approach. The unique equivalent martingale measure in Equation (1.1) is replaced with a properly chosen *neutral pricing measure*.

The paper is organized as follows. Section 2 summarizes and states some facts on utility maximization. These are needed in the subsequent section to address the derivative pricing problem for game contingent claims. The appendix contains some auxiliary results from stochastic calculus.

Throughout, we use the notation of Jacod and Shiryaev (1987) (henceforth JS) and Jacod (1979, 1980). The transposed of a vector x is denoted as $x^\top$ and its components by superscripts. Increasing processes are identified with their corresponding Lebesgue-Stieltjes measure. Stochastic integrals are written in dot notation, i.e. $\varphi^\top \cdot S_t$ means $\int_0^t \varphi_s^\top dS_s$.

2 Utility maximization

The derivative pricing approach in Section 3 relies on assumptions concerning investors who maximize their expected utility. Therefore, we discuss two kinds of portfolio optimization problems in this section, based on the classical *utility of terminal wealth* and on *local utility* as in Kallsen (1999), respectively.

Our mathematical framework for a frictionless market model is as follows: Fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ in the sense of JS, I.1.2. In this section we consider traded *securities* $1, \dots, d$ whose price processes are expressed in terms of multiples of a *numeraire* security 0. Put differently, these securities are modelled by their discounted *price process* $S := (S^1, \dots, S^d)$. We assume that S is a $\mathbb{R}^d$-valued semimartingale.

2.1 Utility of terminal wealth

In this subsection we consider an investor who tries to maximize utility from terminal wealth. Her *initial endowment* is denoted by $\varepsilon \in (0, \infty)$. *Trading strategies* are modelled by $\mathbb{R}^d$-valued, predictable stochastic processes $\varphi = (\varphi^1, \dots, \varphi^d) \in L(S)$, where φ_t^i denotes the number of shares of security i in the investor's portfolio at time t. A strategy φ belongs to the set $\mathfrak{S}$ of all *admissible* strategies if its discounted *wealth process* $V(\varphi) := \varepsilon + \varphi^\top \cdot S$ is nonnegative (no debts allowed).

Trading constraints are expressed in terms of subsets of the set of all trading strategies. More specifically, we consider a process Γ whose values are convex cones in $\mathbb{R}^d$. The *constrained set* of trading strategies $\mathfrak{S}(\Gamma)$ is the subset of admissible strategies φ which satisfy $(\varphi^1, \dots, \varphi^d)_t \in \Gamma_t$ pointwise on $\Omega \times [0, T]$. Important examples are $\Gamma := \mathbb{R}^d$ (no constraints) and $\Gamma := (\mathbb{R}_+)^d$ (no short sales).

The investor's preferences are modelled by a strictly concave *utility* function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ which is continuously differentiable on $(0, \infty)$ and satisfies $\lim_{x\to 0} u'(x) = \infty$, $\lim_{x\to\infty} u'(x) = 0$, and $\limsup_{x\to\infty} \frac{xu'(x)}{u(x)} < 1$ (i.e. it is of reasonable asymptotic elasticity in the sense of Kramkov and Schachermayer (1999), Definition 2.2). Her aim is to make the best out of her money in the following sense:

Definition 2.1 We say that $\varphi \in \mathfrak{S}(\Gamma)$ is an *optimal strategy for terminal wealth under the constraints* Γ if it maximizes $\widetilde{\varphi} \mapsto E(u(V_T(\widetilde{\varphi})))$ over all $\widetilde{\varphi} \in \mathfrak{S}(\Gamma)$. (By convention, we set $E(u(V_T(\widetilde{\varphi}))) := -\infty$ if $E(-u(V_T(\widetilde{\varphi})) \vee 0) = \infty$.)

Optimal portfolios are characterized by the following result. Many references to related statements in the literature can be found in Kallsen (2001), Section 2.2 and Schachermayer (2001).

Lemma 2.2 *Let $\varphi \in \mathfrak{S}(\Gamma)$ with finite expected utility. Then we have equivalence between:*

1. *φ is optimal for terminal wealth under the constraints Γ.*
2. *$u'(V_T(\varphi))((\psi - \varphi)^\top \cdot S_T)$ is integrable and has non-positive expectation for any $\psi \in \mathfrak{S}(\Gamma)$ with $E(u(V_T(\psi))) > -\infty$.*

PROOF. 2⇒1: Let $\psi \in \mathfrak{S}(\Gamma)$ with $E(u(V_T(\psi))) > -\infty$. Since u is concave, we have

$$\begin{aligned} E(u(\varepsilon + \psi^\top \cdot S_T)) &\leq E(u(\varepsilon + \varphi^\top \cdot S_T)) + E(u'(\varepsilon + \varphi^\top \cdot S_T)((\psi - \varphi)^\top \cdot S_T)) \\ &\leq E(u(\varepsilon + \varphi^\top \cdot S_T)), \end{aligned}$$

which yields the assertion.

1⇒2: Let $\psi \in \mathfrak{S}(\Gamma)$ with $E(u(V_T(\psi))) > -\infty$. Define $\widetilde{\psi} := \varphi + \frac{1}{2}(\psi - \varphi)$ and $\psi^{(\lambda)} := \varphi + \lambda(\psi - \varphi)$ for $\lambda \in [0, 1]$. Since $\mathfrak{S}(\Gamma)$ is convex and u is concave, we have that $\widetilde{\psi} \in \mathfrak{S}(\Gamma)$ and $E(u(V_T(\widetilde{\psi}))) > -\infty$. From

$$\begin{aligned} -\infty &< E(u(\varepsilon + \psi^\top \cdot S_T)) \\ &\leq E(u(\varepsilon + \varphi^\top \cdot S_T)) + E(u'(\varepsilon + \varphi^\top \cdot S_T)((\psi - \varphi)^\top \cdot S_T)) \end{aligned}$$

and $E(u(V_T(\varphi))) < \infty$ it follows that $E((u'(\varepsilon + \varphi^\top \cdot S_T)((\psi - \varphi)^\top \cdot S_T))^-) < \infty$. Similarly,

$$\begin{aligned} -\infty &< E(u(\varepsilon + \psi^\top \cdot S_T)) \\ &\leq E(u(\varepsilon + \widetilde{\psi}^\top \cdot S_T)) + \frac{1}{2} E(u'(\varepsilon + \widetilde{\psi}^\top \cdot S_T)((\psi - \varphi)^\top \cdot S_T)) \end{aligned}$$

implies that $E((u'(\varepsilon + \widetilde{\psi}^\top \cdot S_T)((\psi - \varphi)^\top \cdot S_T))^-) < \infty$.

Let $\lambda \in (0, \frac{1}{2}]$. By optimality of φ, we have $0 \geq E(u(\varepsilon + \psi^{(\lambda),\top} \cdot S_T)) - E(u(\varepsilon + \varphi^\top \cdot S_T))$, which equals $\lambda E(\xi^{(\lambda)}((\psi - \varphi)^\top \cdot S_T))$ for some random variable $\xi^{(\lambda)}$ with values in

$[u'(\varepsilon+\varphi^\top\cdot S_T), u'(\varepsilon+\widetilde{\psi}^\top\cdot S_T)]$ or $[u'(\varepsilon+\widetilde{\psi}^\top\cdot S_T), u'(\varepsilon+\varphi^\top\cdot S_T)]$, respectively. Note that $(\xi^{(\lambda)}((\psi-\varphi)^\top\cdot S_T))^- \leq (u'(\varepsilon+\varphi^\top\cdot S_T)((\psi-\varphi)^\top\cdot S_T))^- + (u'(\varepsilon+\widetilde{\psi}^\top\cdot S_T)((\psi-\varphi)^\top\cdot S_T))^- \in L^1(\Omega,\mathscr{F},P)$.

Since $\psi^{(\lambda),\top}\cdot S_T \to \varphi^\top\cdot S_T$, we have that $\xi^{(\lambda)} \to u'(\varepsilon+\varphi^\top\cdot S_T)$ almost surely for $\lambda\to 0$. Fatou's lemma yields $E(u'(\varepsilon+\varphi^\top\cdot S_T)((\psi-\varphi)^\top\cdot S_T)) \leq \liminf_{\lambda\to 0} E(\xi^{(\lambda)}((\psi-\varphi)^\top\cdot S_T))$. It follows that $E(u'(\varepsilon+\varphi^\top\cdot S_T)((\psi-\varphi)^\top\cdot S_T)) \leq 0$ as claimed. □

Suppose that φ is an optimal strategy for terminal wealth without constraints (i.e. for $\Gamma=\mathbb{R}^d$). If the probability space is finite, then $\frac{u'(V_T(\varphi))}{E(u'(V_T(\varphi)))}$ is the density of some equivalent martingale measure (EMM) $P^\star$ (cf. Kallsen (2001), Corollary 2.7). In addition, this measure solves some dual minimization problem (cf. Schachermayer (2001), Theorem 2.3). In general markets, the density process of $P^\star$ is replaced with a supermartingale which may not be the density process of a probability measure, let alone an EMM (cf. Kramkov and Schachermayer (1999), Section 5). Nevertheless, in many models of practical importance the *dual measure* $P^\star$ exists and it is at least a σ*-martingale measure*, i.e. $S^1,\dots,S^d$ are σ-martingales relative to $P^\star$. Since it plays a key role in the neutral pricing approach, we call $P^\star$ *neutral pricing measure for terminal wealth*.

Definition 2.3 Suppose that φ is an optimal strategy for terminal wealth without constraints (i.e. for $\Gamma=\mathbb{R}^d$) and, moreover, has finite expected utility. If $\frac{u'(V_T(\varphi))}{E(u'(V_T(\varphi)))}$ is the density of some σ-martingale measure $P^\star$, we call $P^\star$ *dual measure* or *neutral pricing measure for terminal wealth*.

In some cases the neutral pricing measure for terminal wealth can be computed explicitly:

Example 2.4 Suppose that $S^1,\dots,S^d$ are positive processes of the form $S^i = S^i_0\mathscr{E}(L^i)$ for $i=1,\dots,d$, where L is a $\mathbb{R}^d$-valued Lévy process with characteristic triplet (b,c,F) relative to some truncation function $h:\mathbb{R}^d\to\mathbb{R}^d$ (i.e. a PIIS in the sense of JS, II.4.1). In the last couple of years, processes of this type have become popular for securities models, since they are mathematically tractable and provide a good fit to real data (cf. Eberlein and Keller (1995), Eberlein et al. (1998), Madan and Senata (1990), Barndorff-Nielsen (1998)). Suppose that u is of power or logarithmic type, i.e. $u(x)=\frac{x^{1-p}}{1-p}$ for some $p\in\mathbb{R}_+\setminus\{0,1\}$ or $u(x)=\log(x)$, which corresponds to the case $p=1$. Assume that there exists some $\gamma\in\mathbb{R}^d$ such that $F(\{x\in\mathbb{R}^d : 1+\gamma^\top x\leq 0\})=0$, $\int|\frac{x}{(1+\gamma^\top x)^p}-h(x)|F(dx)<\infty$, and

$$b-pc\gamma+\int\Big(\frac{x}{(1+\gamma^\top x)^p}-h(x)\Big)F(dx)=0.$$

Let $Z:=\mathscr{E}(-p\gamma^\top L^c+((1+\gamma^\top x)^{-p}-1)*(\mu^L-\nu^L))$, where L^c denotes the continuous martingale part of L and μ^L,ν^L the random measure of jumps of L and its compensator. In the proof of Kallsen (2000), Theorem 3.2 it is shown that Z is the density process of the dual measure $P^\star$, which is even an equivalent martingale measure in this case. Relative to $P^\star$, L is a Lévy process with characteristic triplet $(b^\star,c,F^\star)$, where $\frac{dF^\star}{dF}(x)=(1+\gamma^\top x)^{-p}$ and $b^\star=-\int(x-h(x))F^\star(dx)$.

Example 2.5 In the case of logarithmic utility $u(x) = \log(x)$, the neutral pricing measure for terminal wealth can be calculated explicitly for a large number of semimartingale models (cf. Goll and Kallsen (2001), Section 6).

2.2 Local utility

Secondly, we turn to portfolio optimization based on local utility. We assume that S is a $\mathbb{R}^d$-valued special semimartingale. Denote by (b, c, F, A) differential characteristics of S in the sense of Definition A.1, but relative to the truncation function $h(x) = x$. This choice of truncation function is possible because S is special. It is typically straightforward to obtain the differential characteristics from other local descriptions of S e.g. in terms of stochastic differential equations or one-step transition densities in the discrete-time case.

In this subsection, the family of trading strategies under consideration is the set $\mathfrak{S}'$ of all predictable $\mathbb{R}^d$-valued processes $\varphi = (\varphi^1, \dots, \varphi^d)$ satisfying the integrability condition

$$E\left(\left(|\varphi^\top b| + \varphi^\top c\varphi + \int((\varphi^\top x)^2 \wedge |\varphi^\top x|)F(dx)\right) \cdot A_T\right) < \infty.$$

Similarly to above, we denote by $\mathfrak{S}'(\Gamma)$ the set of all trading strategies in $\mathfrak{S}'$ meeting the cone constraints Γ. In order to avoid technical proofs, we assume that there exist polyhedral cones $K_1, \dots, K_n \subset \mathbb{R}^d$ and predictable sets $D_1, \dots, D_n$ such that $\Gamma_t(\omega) = \bigcap_{\{i \in \{1,\dots,n\}:(\omega,t)\in D_i\}} K_i$ for $(\omega, t) \in \Omega \times [0, T]$. The *utility function* $u : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following conditions: u is twice continuously differentiable, the derivatives u', u'' are bounded with $\lim_{x\to\infty} u'(x) = 0$, moreover $u(0) = 0$, $u'(0) = 1$, $u'(x) > 0$ and $u''(x) < 0$ for any $x \in \mathbb{R}$. For any $\psi \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ the random variable

$$\gamma_t(\psi) := \psi^\top b_t + \frac{u''(0)}{2}\psi^\top c_t\psi + \int(u(\psi^\top x) - \psi^\top x)F_t(dx)$$

is termed *local utility* of ψ in t.

Definition 2.6 We call a strategy $\varphi \in \mathfrak{S}'(\Gamma)$ *locally optimal under the constraints* Γ if

$$E(\gamma(\varphi) \cdot A_T) \geq E(\gamma(\psi) \cdot A_T)$$

for any $\psi \in \mathfrak{S}'(\Gamma)$.

For motivation of local optimality we refer the reader to Kallsen (1999). Intuitively, a locally optimal strategy maximizes the expected utility of the gains over infinitesimal time intervals, or put differently, the expected utility of consumption among all strategies whose financial gains are immediately consumed.

Locally optimal portfolios can be determined by pointwise solution of equations in $\mathbb{R}^d$:

Theorem 2.7 *A trading strategy $\varphi \in \mathfrak{S}'(\Gamma)$ is locally optimal under the constraints Γ if and only if*

$$b_t + u''(0)c_t\varphi_t + \int x(u'(\varphi_t^\top x) - 1)F_t(dx) \in \Gamma_t^\circ \tag{2.1}$$

$(P \otimes A)$*-almost everywhere, where* $\Gamma_t^\circ := \{y \in \mathbb{R}^d : x^\top y \leq 0$ *for any* $x \in \Gamma_t\}$ *denotes the polar cone of* Γ_t*.*

PROOF. In view of Farkas' lemma (cf. Rockafellar and Wets (1998), Lemma 6.45), Theorem 2.7 follows from Kallsen (1999), Theorem 3.5. Strictly speaking, Kallsen (1999) considers a narrower set-up where A and Γ are deterministic. As is pointed out in Kallsen (2002), the statements in Kallsen (1999) remain valid for $A \in \mathscr{A}_{\text{loc}}^+$. Moreover, a careful inspection of the proofs of Proposition 3.10 and Theorem 3.5 in that paper reveals that these results hold for random constraints of the above type as well. □

Neutral pricing of European contingent claims is discussed in Kallsen (2002) in the context of local utility. A key role is played by the corresponding neutral pricing measure, which is defined as follows:

Definition 2.8 Suppose that there exists a locally optimal strategy $\varphi \in \mathfrak{S}'$ without constraints (i.e. for $\Gamma = \mathbb{R}^d$). Moreover, assume that the local martingale $Z := \mathscr{E}(u''(0)\varphi^\top \cdot S^c + \frac{u'(\varphi^\top x)-1}{1+V} * (\mu^S - \nu^S))$ is a martingale, where μ^S, ν^S are the random measure of jumps of S and its compensator, $V_t := \int (u'(\varphi_t^\top x) - 1)\nu^S(\{t\} \times dx)$ for $t \in [0, T]$, and S^c denotes the continuous local martingale part of S. Then the probability measure $P^\star \sim P$ defined by $\frac{dP^\star}{dP} = Z_T$ is called *neutral pricing measure for local utility*.

Since the determination of the optimal strategy φ reduces to solving Equation (2.1) with $\Gamma_t^\circ = \{0\}$, the neutral pricing measure for local utility is often easier to obtain than the neutral pricing measure for terminal wealth. For concrete examples cf. Kallsen (2002), Section 5.

3 Neutral pricing

In this section we turn to the valuation of game contingent claims. Let us briefly review the idea of neutral pricing. For references and connections to similar approaches in the literature we refer the reader to Kallsen (2001).

In complete models there exist unique arbitrage-free derivative values. The assertion that real market prices have to coincide with these values can be easily justified. It suffices to assume the existence of traders (from now on called *derivative speculators*) who exploit favourable market conditions once they detect them. The existence of derivative speculators explains why the market price cannot deviate too strongly from the right value: If it did, the huge demand for (resp. supply of) the mispriced security would push its price immediately closer to the rational value. The only assumption on the preferences of the speculators is that they do not reject riskless profits – which most people may agree on. The elegance of this approach comes at a price. It only works in complete models, or more exactly, for attainable claims.

We extend this reasoning to incomplete markets by imposing stronger assumptions on the preferences of derivative speculators. We suppose that they trade by maximizing a specific kind of utility. The role of the unique arbitrage-free price will now be played by the *neutral* derivative value. This is the unique price such that the speculators' optimal portfolio contains no contingent claim. Similarly as in the complete case we argue that the speculators' presence should prevent the market price from deviating too strongly from the neutral value.

The general setting is as in the previous section. We distinguish two kinds of securities: *underlyings* $1, \dots, m$ and *derivatives* $m+1, \dots, m+n$. We assume that the derivatives are game contingent claims with discounted *exercise process* L^i and discounted *cancellation process* U^i, where L^i and U^i are semimartingales with $L^i < U^i$ as well as $L^i_- < U^i_-$ on $[\![0, T[\![$ and $L^i_T = U^i_T$ for $i = m+1, \dots, m+n$. European and American options are treated as special cases of game contingent claims as it is explained in Remark 2 below. We call semimartingales $S^{m+1}, \dots, S^{m+n}$ *derivative price processes* if $L^i \leq S^i \leq U^i$ for $i = m+1, \dots, m+n$. As noted above, we are interested in derivative price processes that have a neutral effect on the market in the sense that they do not cause supply of or demand for contingent claims by derivative speculators.

Speculators may not be able to hold arbitrary amounts of game contingent claims because these contracts can be cancelled. If the market price approaches the upper cancellation value U^i, it may happen that all options vanish from the market because they are terminated by the sellers. So a long position in the option is no longer feasible. Conversely, all derivative contracts may be exercised by the claim holders if the market price coincides with the exercise value L^i. This terminates short positions in the claim. However, as long as the derivative price stays above the exercise value, nobody will exercise the option because selling it on the market yields a higher reward. Similarly, there is no danger that the seller of a GCC cancels the contract as long as the cancellation value exceeds the market price. Summing up, the derivative speculators are facing trading constraints Γ given by

$$\begin{aligned} \Gamma_t \quad := \quad & \{x \in \mathbb{R}^{m+n} : \text{ For } i = m+1, \dots, m+n \text{ we have } x^i \geq 0 \text{ if } S^i_{t-} = L^i_{t-} \\ & \text{and } x^i \leq 0 \text{ if } S^i_{t-} = U^i_{t-}\}. \end{aligned}$$

In the following subsections, we treat neutral pricing separately for utility of terminal wealth and for local utility, respectively.

3.1 Terminal wealth

We start by assuming that derivative speculators are identical investors trying to maximize expected utility from terminal wealth. Moreover, we suppose that the neutral pricing measure for terminal wealth $P^\star$ in the sense of Definition 2.3 exists for the underlyings' market $S^1, \dots, S^m$. As explained above, we look for neutral derivative prices in the following sense:

Definition 3.1 We call derivative price processes $S^{m+1}, \dots, S^{m+n}$ *neutral for terminal wealth* if there exists a strategy $\overline{\varphi}$ in the extended market $S^1, \dots, S^{m+n}$ which is optimal for terminal wealth under the constraints Γ and satisfies $\overline{\varphi}^{m+1} = \dots = \overline{\varphi}^{m+n} = 0$.

The following main result of this paper treats existence and uniqueness of neutral derivative price processes. Moreover, it shows that they are recovered as the value of a Dynkin game relative to the neutral pricing measure $P^\star$.

Theorem 3.2 *Suppose that* $L^{m+1}, \dots, L^{m+n}$ *and* $U^{m+1}, \dots, U^{m+n}$ *are bounded. Then there exist unique neutral derivative price processes. These are given by*

$$\begin{aligned} S^i_t &= \operatorname{ess\,inf}_{\tau^U \in \mathscr{T}_t} \operatorname{ess\,sup}_{\tau^L \in \mathscr{T}_t} E_{P^\star}(R^i(\tau^L, \tau^U)|\mathscr{F}_t) \\ &= \operatorname{ess\,sup}_{\tau^L \in \mathscr{T}_t} \operatorname{ess\,inf}_{\tau^U \in \mathscr{T}_t} E_{P^\star}(R^i(\tau^L, \tau^U)|\mathscr{F}_t) \end{aligned} \tag{3.1}$$

for $t \in [0,T]$, $i = m+1, \dots, m+n$, *where* $\mathscr{T}_t$ *denotes the set of* $[t,T]$*-valued stopping times and*

$$R^i(\tau^L, \tau^U) := \begin{cases} L^i_{\tau^L} & \text{if } \tau^L \le \tau^U \\ U^i_{\tau^U} & \text{otherwise.} \end{cases}$$

Moreover, the extended market $S^1, \dots, S^{m+n}$ *satisfies condition NFLVR in the sense of*

Definition 3.3 We say that the market $S = (S^1, \dots, S^{m+n})$ satisfies the condition *no free lunch with vanishing risk (NFLVR)* if 0 is the only non-negative element of the $L^\infty(\Omega, \mathscr{F}, P)$-closure of the set $C := \{f \in L^\infty(\Omega, \mathscr{F}, P) : f \le \psi \cdot S_T \text{ for some } \psi \in \mathfrak{S}(\Gamma)\}$. (Note that this is a straightforward extension of the usual NFLVR condition in Delbaen and Schachermayer (1994), Definition 2.8 to markets containing game contingent claims.)

PROOF OF THEOREM 3.2. *Step 1:* By Lepeltier and Maingueneau (1984), Théorème 9 and Corollaire 12, there exist right-continuous adapted processes $S^{m+1}, \dots, S^{m+n}$ satisfying Equation (3.1). Fix $i \in \{m+1, \dots, m+n\}$. Define stopping times $T^k_1 := \inf\{t \in \mathbb{R}_+ : S^i_t \ge U^i_t - \frac{1}{k}\}$ for any $k \in \mathbb{N}$ and $T_1 := \sup_{k \in \mathbb{N}} T^k_1$. By Lepeltier and Maingueneau (1984), Théorème 11 and Dellacherie and Meyer (1982), Theorem VI.3, $(S^i)^{T^k_1}$ is a $P^\star$-supermartingale for any $k \in \mathbb{N}$. Obviously, $(S^i_{T^k_1})_{k \in \mathbb{N}}$ converges for $k \to \infty$ $P^\star$-almost surely to $R := U_{T_1} 1_{\cup_{k \in \mathbb{N}}\{T^k_1 = T_1\}} + U_{T_1 -} 1_{\cap_{k \in \mathbb{N}}\{T^k_1 < T_1\}}$. Define an adpated right-continuous process $\overline{S}^i$ by

$$\overline{S}^i_t := \begin{cases} S^i_t & \text{if } t < T_1 \text{ or } t = 0 \\ U_{T_1 -} & \text{if } 0 \neq t \ge T_1 \text{ and } T^k_1 < T_1 \text{ for any } k \in \mathbb{N} \\ U_{T_1} & \text{if } 0 \neq t \ge T_1 \text{ and } T^k_1 = T_1 \text{ for some } k \in \mathbb{N}, \end{cases}$$

i.e. $\overline{S}^i = \sum_{k \in \mathbb{N}} (S^i)^{T^k_1} 1_{]\!]T^{k-1}_1, T^k_1]\!]} + R 1_{(\cup_{k \in \mathbb{N}}[\![0, T^k_1]\!])^C}$ (with the convention $]\!]T^{-1}_1, T^0_1]\!] := [\![T^0_1]\!]$).

Let $s, t \in [0,T]$ with $s \le t$. If $s \in (\cup_{k \in \mathbb{N}}[\![0, T^k_1]\!])^C$, then $\overline{S}^i_s = R = \overline{S}^i_t$ and hence $E_{P^\star}(\overline{S}^i_t|\mathscr{F}_s) = \overline{S}^i_s$. Now, let $s \in]\!]T^{k-1}_1, T^k_1]\!]$ for some $k \in \mathbb{N}$. Then $\overline{S}^i_s = (S^i)^{T^l_1}_s \ge E_{P^\star}((S^i)^{T^l_1}_t|\mathscr{F}_s) = E_{P^\star}(\overline{S}^i_{T^l_1 \wedge t}|\mathscr{F}_s)$ for $l \ge k$. Moreover, dominated convergence yields that $E_{P^\star}(\overline{S}^i_{T^l_1\wedge t}|\mathscr{F}_s) \to E_{P^\star}(\overline{S}^i_t|\mathscr{F}_s)$ in measure for $l \to \infty$. Hence $\overline{S}^i_s \geq E_{P^\star}(\overline{S}^i_t|\mathscr{F}_s)$. Altogether, it follows that $\overline{S}^i$ is a $P^\star$-supermartingale. Hence, $(S^i)^{T_1}$ is a semimartingale.

For $l \in \mathbb{N} \setminus \{0,1\}$ define $T_l := \sup_{k\in\mathbb{N}} T^k_l$ where $T^k_l := \inf\{t \geq T_{l-1} : S^i_t \leq L^i_t + \frac{1}{k}\}$ for $l = 2,4,6,\ldots$ and $T^k_l := \inf\{t \geq T_{l-1} : S^i_t \geq U^i_t - \frac{1}{k}\}$ for $l = 3,5,7,\ldots$ Similarly to above, one shows by induction that $(S^i)^{T_l}$ is a semimartingale for any $l \in \mathbb{N}$.

Step 2: We keep the notation from the previous step. Fix $l \in \mathbb{N}$. For $t_0 \in [0,T]$ and $k \in \mathbb{N}$ define stopping times $\tau_{t_0,k} := \inf\{t \geq t_0 : (S^i)^{T_l}_t \leq (L^i)^{T_l}_t + \frac{1}{k}\} \wedge T$. From Lepeltier and Maingueneau (1984), Théorème 11 it follows that $1_{\rrbracket t_0,\tau_{t_0,k}\rrbracket}\cdot(S^i)^{T_l}$ is a $P^\star$-submartingale for any $t_0 \in [0,T]$, $k \in \mathbb{N}$. In particular, we have

$$b^\star + \int (x - h(x))F^\star(dx) \geq 0 \tag{3.2}$$

$(P \otimes A)$-almost everywhere on $\rrbracket t_0, \tau_{t_0,k}\rrbracket$ (cf. Lemma A.2), where $(b^\star, c^\star, F^\star, A)$ denote $P^\star$-differential characteristics of the semimartingale $(S^i)^{T_l}$ in the sense of Definition A.1. Since $\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\} \cap \rrbracket 0,T\rrbracket = \cup_{t_0\in\mathbb{Q}\cap[0,T]} \cup_{k\in\mathbb{N}} \rrbracket t_0, \tau_{t_0,k}\rrbracket$, it follows that Equation (3.2) holds $(P\otimes A)$-almost everywhere on $\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\}$. Therefore, $1_{\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\}} \cdot (S^i)^{T_l}$ is a $P^\star$-σ-submartingale (cf. Kallsen and Shiryaev (2001), Lemma 2.5 and Lemma A.2). Analogously, it follows that $1_{\{(S^i)^{T_l}_- < (U^i)^{T_l}_-\}} \cdot (S^i)^{T_l}$ is a $P^\star$-σ-supermartingale, and hence $1_{\{(L^i)^{T_l}_- < (S^i)^{T_l}_- < (U^i)^{T_l}_-\}} \cdot (S^i)^{T_l}$ is a $P^\star$-σ-martingale.

Step 3: We keep the notation from the previous steps. Let $T_\infty := \lim_{l\to\infty} T_l$. Since L^i, U^i are $P^\star$-special semimartingales with integrable L^i_0, U^i_0, they are locally in class $\mathscr{H}^1$ in the sense of Definition A.3 and relative to $P^\star$ (cf. Dellacherie and Meyer (1982), VII.99). Denote by $(\sigma_k)_{k\in\mathbb{N}}$ a corresponding localizing sequence. Fix $k \in \mathbb{N}$. By Proposition A.5, applied to $(L^i)^{T_l\wedge\sigma_k}$, $(S^i)^{T_l\wedge\sigma_k}$, and $(U^i)^{T_l\wedge\sigma_k}$, it follows that $\sup_{l\in\mathbb{N}} \|(S^i)^{T_l\wedge\sigma_k}\|_{\mathscr{H}^1} < \infty$, which in turn implies that $(S^i)^{T_\infty\wedge\sigma_k}$ is a semimartingale (cf. Proposition A.6). Therefore, $(S^i)^{T_\infty}$ is a local semimartingale and hence a semimartingale. In particular, it has left-hand limits at T_∞. Since $L^i_{t-} < U^i_{t-}$ for $t < T$, this is only possible if $T_\infty = T$. Consequently, S^i is a semimartingale.

Step 4: Let Z denote the density process of $P^\star$ and φ an optimal strategy for terminal wealth in the market $S^1,\ldots,S^m$. We want to show that the $\mathbb{R}^{m+n}$-valued process $\overline{\varphi} := (\varphi, 0) \in \mathfrak{S}(\Gamma)$ is an optimal strategy for terminal wealth under the constraints Γ, now referring to the extended market $S := (S^1,\ldots,S^{m+n})$. Since $ZE(u'(V_T(\varphi)))$ coincides with the optimal solution $\widehat{Y}(y)$ to the dual problem in Kramkov and Schachermayer (1999), Theorem 2.2, we have that $(\varphi^\top \cdot (S^1,\ldots,S^m))Z$ is a martingale. This implies that $\overline{\varphi}^\top \cdot S = \varphi^\top \cdot (S^1,\ldots,S^m)$ is a $P^\star$-martingale (cf. JS, III.3.8).

Consider a strategy $\psi \in \mathfrak{S}(\Gamma)$ in the extended market. Denote by $(b^\star, c^\star, F^\star, A)$ $P^\star$-differential characteristics of S in the sense of Definition A.1. The same argument as in Step 2 shows that $b^{\star,i} + \int(x^i - h^i(x))F^\star(dx) \geq 0$ $(P\otimes A)$-almost everywhere on $\{L^i_- < S^i_-\}$ and ≤ 0 on $\{S^i_- < U^i_-\}$ for $i = m+1,\ldots,m+n$. Since $S^1,\ldots,S^m$ are $P^\star$-σ-martingales, we have $b^{\star,i} + \int(x^i - h^i(x))F^\star(dx) = 0$ for $i = 1,\ldots,m$. From the form of the constraints Γ it follows that $\psi^i(b^{\star,i} + \int(x^i - h^i(x))F^\star(dx)) \leq 0$ for $i = m+1,\ldots,m+n$, which yields that $\psi^\top(b^\star + \int(x - h(x))F^\star(dx)) \leq 0$ $(P \otimes A)$-almost everywhere. In view of Kallsen and Shiryaev (2001), Lemma 2.5 and Lemma A.2, this implies that $\psi^\top \cdot S$ is a $P^\star$-σ-supermartingale. By Goll and Kallsen (2001), Proposition 7.9, this process and hence also $(\psi - \overline{\varphi})^\top \cdot S$ is even a $P^\star$-supermartingale. In particular, we have $E(u'(V_T(\overline{\varphi}))((\psi - \overline{\varphi})^\top \cdot S)) = E(u'(V_T(\overline{\varphi})))E_{P^\star}((\psi - \overline{\varphi})^\top \cdot S) \leq 0$. Due to Lemma 2.2, $\overline{\varphi}$ is an optimal strategy for terminal wealth under the constraints Γ. Hence, $S^{m+1}, \dots, S^{m+n}$ are neutral price processes for terminal wealth.

Step 5: For the uniqueness part assume that $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative price processes corresponding to some optimal strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m, 0, \dots, 0)$ in the extended market $\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$. Since $\widetilde{\varphi}$ does not contain any derivative, we have that $(\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m)$ is an optimal strategy for the small market $S^1, \dots, S^m$ with the same expected utility. Similarly, the expected utility of φ in the small market and of $\overline{\varphi} = (\varphi, 0)$ in the extended market $\widetilde{S}$ tally. Since φ is optimal in the small market $S^1, \dots, S^m$, it follows that $\overline{\varphi} \in \mathfrak{S}'(\Gamma)$ is optimal in the extended market $\widetilde{S}$ under the constraints Γ. Hence we may w.l.o.g. assume that $\widetilde{\varphi} = \overline{\varphi}$.

Fix $i \in \{m+1, \dots, m+n\}$. Firstly, we show that $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-submartingale for any predictable subset D of $\{L^i_- < \widetilde{S}^i_-\}$. Since $\widetilde{S}^i$ is bounded, we have that $1_D \cdot \widetilde{S}^i$ is locally bounded. Hence, there exists an increasing sequence $(T_k)_{k \in \mathbb{N}}$ of stopping times with $P^\star(T_k = T) \to 1$ and $\sup_{t \in [0,T]} |(1_D \cdot \widetilde{S}^i)^{T_k}_t| \leq k$. Fix $k \in \mathbb{N}$, $s, t \in [0, T]$ with $s \leq t$, and $F \in \mathscr{F}_s$. Define an admissible strategy $\psi \in \mathfrak{S}(\Gamma)$ in the market $\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$ by $\psi^j := 0$ for $j \neq i$ and $\psi^i = -\frac{\varepsilon}{4k} 1_{D \cap [\![0,T_k]\!] \cap (F \times]s,t])}$. Lemma 2.2 and the fact that $\overline{\varphi}^\top \cdot \widetilde{S} = \varphi^\top \cdot (S^1, \dots, S^m)$ is a $P^\star$-martingale yield that

$$\begin{aligned}
&-\frac{\varepsilon}{4k} E_{P^\star}(((1_D \cdot \widetilde{S}^i)^{T_k}_t - (1_D \cdot \widetilde{S}^i)^{T_k}_s) 1_F) \\
&= E_{P^\star}((\psi - \overline{\varphi})^\top \cdot \widetilde{S}_T) + E_{P^\star}(\overline{\varphi}^\top \cdot \widetilde{S}_T) \\
&= (E(u'(V_T(\overline{\varphi}))))^{-1} E(u'(V_T(\overline{\varphi}))((\psi - \overline{\varphi})^\top \cdot \widetilde{S}_T)) \\
&\leq 0.
\end{aligned}$$

Therefore, $(1_D \cdot \widetilde{S}^i)^{T_k}$ is a $P^\star$-submartingale, which implies that $1_D \cdot \widetilde{S}^i$ is a local $P^\star$-submartingale. Similarly, it follows that $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-supermartingale for any predictable subset D of $\{\widetilde{S}^i_- < U^i_-\}$.

Define stopping times $\tau_{t_0,k} := \inf\{t \geq t_0 : S^i_t \leq \widetilde{S}^i_t + \frac{1}{k}\}$ for any $t_0 \in [0, T]$, $k \in \mathbb{N}$. Note that $\{S^i_- > \widetilde{S}^i_-\} \cap]\!]0, T]\!] = \cup_{t_0 \in \mathbb{Q} \cap [0,T]} \cup_{k \in \mathbb{N}}]\!]t_0, \tau_{t_0,k}]\!]$. Fix $t_0 \in [0, T]$, $k \in \mathbb{N}$. Since $\{L^i_- < S^i_-\} \cap \{\widetilde{S}^i_- < U^i_-\} \supset \{S^i_- > \widetilde{S}^i_-\}$, we have that $1_{]\!]t_0, \tau_{t_0,k}]\!]} \cdot S^i$ and hence also $((S^i)^{\tau_{t_0,k}}_t)_{t \in [t_0,T]}$ is a $P^\star$-σ-submartingale. By Goll and Kallsen (2001), Proposition 7.9, this process is even a $P^\star$-submartingale. Similarly, it follows that $((\widetilde{S}^i)^{\tau_{t_0,k}}_t)_{t \in [t_0,T]}$ is a $P^\star$-supermartingale. Since $(S^i)^{\tau_{t_0,k}}_T \leq (\widetilde{S}^i)^{\tau_{t_0,k}}_T + \frac{1}{k}$, we have that $(S^i)^{\tau_{t_0,k}}_{t_0} \leq (\widetilde{S}^i)^{\tau_{t_0,k}}_{t_0} + \frac{1}{k}$ P-almost surely for any $k \in \mathbb{N}$. Consequently, $S^i_{t_0} \leq \widetilde{S}^i_{t_0}$ P-almost surely. Since this holds for any $t_0 \in \mathbb{Q} \cap [0, T]$, we have that $S^i \leq \widetilde{S}^i$ by right-continuity. Similarly, it is shown that $\{S^i < \widetilde{S}^i\}$ is evanescent, which yields the uniqueness of neutral price processes for terminal wealth.

Step 6: The NFLVR property of the price process S is shown in the usual way: Let $\psi \in \mathfrak{S}(\Gamma)$. In Step 4 it is shown that $\psi^\top \cdot S$ is a $P^\star$-supermartingale and hence $E_{P^\star}(f) \leq 0$ for any $f \in C$. Since $P^\star \sim P$, this is also true for any f in the $L^\infty(\Omega, \mathscr{F}, P)$-closure of C. Therefore $f = 0$ P-almost surely for any such f with $f \geq 0$. $\square$

Remarks.

1. If $\sup_{t\in[0,T]} |L_t^i|$ and $\sup_{t\in[0,T]} |U_t^i|$ are $P^\star$-integrable instead of bounded for $i = m+1, \ldots, m+n$, we still have the existence of neutral derivative prices for terminal wealth. As Kifer (2000) points out, the results of Lepeltier and Maingueneau (1984) hold also if L^i, U^i satisfy the above integrability conditions. The existence follows now from Steps 1–4 in the proof of Theorem 3.2.

2. European options with bounded discounted terminal payoff R^i at time T may be considered as special cases of game contingent claims by letting

$$L_t^i := \begin{cases} \text{ess inf } R^i - 1 & \text{if } t < T \\ R^i & \text{if } t = T \end{cases}$$

   and

$$U_t^i := \begin{cases} \text{ess sup } R^i + 1 & \text{if } t < T \\ R^i & \text{if } t = T. \end{cases}$$

   If we assume the absence of arbitrage, the price of the European claim will never leave the interval $[\text{ess inf } R^i, \text{ess sup } R^i]$. Therefore, the additional right to cancel the contract prematurely is worthless. Equation (3.1) reduces to

$$S_t^i = E_{P^\star}(R^i | \mathscr{F}_t)$$

   for European options.

   American options with bounded exercise process L^i and final payoff L_T^i are treated similarly by defining

$$U_t^i := \begin{cases} \text{ess sup } (\sup_{t\in[0,T]} L_t^i) + 1 & \text{if } t < T \\ L_T^i & \text{if } t = T. \end{cases}$$

   The neutral price process S^i in Equation (3.1) now has the form of a Snell envelope:

$$S_t^i = \text{ess sup}_{\tau \in \mathscr{T}_t} E_{P^\star}(L_\tau^i | \mathscr{F}_t).$$

   Moreover, an inspection of the proof reveals that we can slightly weaken the conditions on L^i in the American option case. It is enough to assume that L^i is a càdlàg, adapted process instead of a semimartingale.

3. In general, neutral derivative prices for terminal wealth depend on the utility function u, the time horizon T, the initial endowment ε, and the numeraire. In the setting of

Example 2.4, the density process of $P^\star$ does not depend on T and ε. Therefore, neutral prices do not depend on the time horizon and the initial endowment of derivative speculators in this case.

Logarithmic utility is even more agreeable in this respect: As it is discussed in Goll and Kallsen (2001), Section 6, the neutral prices relative to $P^\star$ depend neither on T, ε, nor on the chosen numeraire. Moreover, the density process of $P^\star$ can be calculated explicitly even in very complex models.

3.2 Local utility

In this subsection, we suppose that derivative speculators maximize their local utility. Similarly to above, we assume that the neutral pricing measure for local utility $P^\star$ exists for the underlyings' market $S^1, \dots S^m$ (cf. Definition 2.8).

Definition 3.4 We call derivative price processes $S^{m+1}, \dots, S^{m+n}$ *neutral for local utility* if there exists a strategy $\overline{\varphi}$ in the extended market $S^1, \dots, S^{m+n}$ which is locally optimal under the constraints Γ and satisfies $\overline{\varphi}^{m+1} = \dots = \overline{\varphi}^{m+n} = 0$.

The following result corresponds to Theorem 3.2 in the local utility setting.

Theorem 3.5 *Suppose that L^i, U^i are special semimartingales and that $\sup_{t\in[0,T]} |L^i_t|$ and $\sup_{t\in[0,T]} |U^i_t|$ are $P^\star$-integrable for $i = m+1, \dots, m+n$. Then there exist unique neutral derivative price processes. These are given by*

$$\begin{aligned} S^i_t &= \operatorname{ess\,inf}_{\tau^U \in \mathscr{T}_t} \operatorname{ess\,sup}_{\tau^L \in \mathscr{T}_t} E_{P^\star}(R^i(\tau^L, \tau^U)|\mathscr{F}_t) \\ &= \operatorname{ess\,sup}_{\tau^L \in \mathscr{T}_t} \operatorname{ess\,inf}_{\tau^U \in \mathscr{T}_t} E_{P^\star}(R^i(\tau^L, \tau^U)|\mathscr{F}_t) \end{aligned} \tag{3.3}$$

for $t \in \mathbb{R}_+$, $i = m+1, \dots, m+n$, where $\mathscr{T}_t$ and $R^i(\tau^L, \tau^U)$ are defined as in Theorem 3.2. Moreover, the extended market $S^1, \dots, S^{m+n}$ satisfies condition NFLVR in the sense of Definition 3.3.

PROOF. Steps 1–3 and 6 are shown literally as in the proof of Theorem 3.2. Only Steps 4 and 5 have to be modified slightly.

Step 4: Since $L^i \leq S^i \leq U^i$, we have that S^i is a special semimartingale for $i = m+1, \dots, m+n$ (cf. Kallsen (2002), Proposition 3.7). Similarly as in Step 4 of the proof of Theorem 3.2 we want to show that $\overline{\varphi} := (\varphi, 0) \in \mathfrak{S}'(\Gamma)$ is a locally optimal strategy for $S = (S^1, \dots, S^{m+n})$, where φ denotes an optimal strategy in the small market $S^1, \dots, S^m$. Denote by (b, c, F, A) the P-differential characteristics of S relative to $h(x) = x$. In view of Theorem 2.7 we have to show that

$$b + u''(0)c\overline{\varphi} + \int x(u'(\overline{\varphi}^\top x) - 1)F(dx) \in \Gamma^\circ. \tag{3.4}$$

Note that $\Gamma^\circ_t = \{y \in \{0\}^m \times \mathbb{R}^n$: For $i = m+1, \dots, m+n$ we have $y^i \geq 0$ if $L^i_{t-} < S^i_{t-}$ and $y^i \leq 0$ if $S^i_{t-} < U^i_{t-}\}$. From the Girsanov-Jacod-Mémin theorem it follows that the

$P^\star$-differential characteristics $(b^\star, c^\star, F^\star, A)$ of S relative to some truncation function $h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ satisfy the equation

$$b_t^{\star,i} + \int (x^i - h^i(x)) F_t^\star(dx) = b_t^i + u''(0) c_t^{i\cdot} \overline{\varphi}_t + \int x^i \Big(\frac{u'(\overline{\varphi}_t^\top x)}{1+V_t} - 1 \Big) F_t(dx)$$
$$= \frac{1}{1+V_t} \Big(b_t^i + u''(0) c_t^{i\cdot} \overline{\varphi}_t + \int x^i (u'(\overline{\varphi}_t^\top x) - 1) F_t(dx) \Big) \qquad (3.5)$$

for $i = 1, \dots, m+n$, where V_t is defined as in Definition 2.8 (cf. Kallsen (2002), Steps 3 and 4 on page 122 for the arguments in detail). Since φ is optimal in the small market, Theorem 2.7 yields that expression (3.5) equals 0 for $i = 1, \dots, m$. The same argument as in Step 2 of the proof of Theorem 3.2 shows that the left-hand side of Equation (3.5) is non-negative on $\{L_{t-}^i < S_{t-}^i\}$ (resp. non-positive on $\{S_{t-}^i < U_{t-}^i\}$) for $i = 1, \dots, m+n$. Together, it follows that Condition (3.4) holds. Therefore, $S^{m+1}, \dots, S^{m+n}$ are neutral price processes for local utility.

Step 5: For the uniqueness part assume that $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative price processes corresponding to some locally optimal strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m, 0, \dots, 0)$ in the extended market $\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$. As in Step 5 of the proof of Theorem 3.2 we may w.l.o.g. assume that $\widetilde{\varphi} = \overline{\varphi}$.

In this step, we denote by (b, c, F, A) the P-differential characteristics of $\widetilde{S}$ relative to $h(x) = x$. Since $\overline{\varphi}$ is an optimal strategy, Theorem 2.7 yields that Condition (3.4) holds $(P \otimes A)$-almost everywhere. As in the previous step, we express this condition in terms of the $P^\star$-differential characteristics $(b^\star, c^\star, F^\star, A)$ of $\widetilde{S}$ relative to some truncation function $h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$. Fix $i \in \{m+1, \dots, m+n\}$. Then the $P^\star$-drift $b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx)$ of $\widetilde{S}^i$ it is non-negative on $\{L_{t-}^i < \widetilde{S}_{t-}^i\}$ resp. non-positive on $\{\widetilde{S}_{t-}^i < U_{t-}^i\}$. Due to Kallsen and Shiryaev (2001), Lemma 2.5 and Lemma A.2, this means that $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-submartingale for any predictable subset D of $\{L_{t-}^i < \widetilde{S}_{t-}^i\}$ and $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-supermartingale for any predictable subset D of $\{\widetilde{S}_{t-}^i < U_{t-}^i\}$. The uniqueness of neutral price processes follows now as in the second half of Step 5 in the proof of Theorem 3.2. □

Remark 2 following Theorem 3.2 holds accordingly in this setting.

A Appendix

In this appendix we state some auxiliary results from stochastic calculus. Firstly, we consider the σ-supermartingale property in terms of semimartingale characteristics. Secondly, we turn to the $\mathscr{H}^1$-norm of semimartingales.

Definition A.1 Let X be a $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν) relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. By JS, II.2.9 there exists some predictable process $A \in \mathscr{A}_{\text{loc}}^+$, some predictable $\mathbb{R}^{d \times d}$-valued process c whose values are non-negative, symmetric matrices, and some transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that

$$B = b \cdot A, \;\; C = c \cdot A, \;\; \nu = A \otimes F.$$

We call (b, c, F, A) *differential characteristics* of X.

One should observe that the differential characteristics are not unique: E.g. $(2b, 2c, 2F, \frac{1}{2}A)$ yields another version. Typical choices for A are $A_t := t$ (e.g. for Lévy processes, diffusions, Itô processes etc.) and $A_t := \sum_{s \leq t} 1_{\mathbb{N}\setminus\{0\}}(s)$ (discrete-time processes). Especially for $A_t = t$, one can interpret b_t or rather $b_t + \int (x - h(x))F_t(dx)$ as a drift rate, c_t as a diffusion coefficient, and F_t as a local jump measure. As the following result shows, a non-positive or vanishing drift corresponds to a σ-supermartingale or σ-martingale, respectively. These processes play an important role in the context of fundamental theorems of asset pricing (cf. Delbaen and Schachermayer (1998), Kabanov (1997), Cherny and Shiryaev (2001)). For background on σ-localization and the related classes of processes we refer the reader to Goll and Kallsen (2001).

Lemma A.2 *Let X be a semimartingale in $\mathbb{R}^d$ with differential characteristics (b, c, F, A). Fix $i \in \{1, \dots, d\}$. Then X^i is a σ-supermartingale if and only if $\int |x^i - h^i(x)|F(dx) < \infty$ and*

$$b^i + \int (x^i - h^i(x))F(dx) \leq 0$$

$(P \otimes A)$-almost everywhere. If we replace ≤ 0 with $= 0$ or ≥ 0, we obtain corresponding statements for σ-martingales and σ-submartingales, respectively.

PROOF. We use the notation of Goll and Kallsen (2001), Section 7 (henceforth GK).

$\Rightarrow$: This is shown in the first part of the proof of GK, Proposition 7.9.

$\Leftarrow$: From JS, II.2.29, II.2.13, I.3.10 it follows that X is a local supermartingale if we have, in addition, $\int |x^i - h^i(x)|F(dx) \in L(A)$, i.e. if $X \in \mathscr{D}_{\text{loc}}$. Since $X \in \mathscr{D}_\sigma$ (cf. GK, Lemma 7.6), X belongs to the σ-localized class of the set of local supermartingales, which coincides with the set of σ-supermartingales (cf. GK, Lemma 7.4). □

In the proof of Theorem 3.2 we make use of the $\mathscr{H}^1$-norm in the sense of Emery (1978), Protter (1977, 1978, 1992), Dellacherie and Meyer (1982). Note that we treat the value X_0 differently from e.g. Dellacherie and Meyer (1982) because we use the conventions of JS as far as starting values of $[X, X]$, ΔX etc. are concerned.

Definition A.3 For any real-valued semimartingale X we define

$$\begin{aligned} \|X\|_{\mathscr{H}^1} \;\; := \;\; \inf \Big\{ & E\Big(|X_0| + \mathrm{Var}(A)_\infty + \sqrt{[M, M]_\infty}\Big) : \\ & X = X_0 + M + A \text{ with } M \in \mathscr{M}_{\text{loc}}, A \in \mathscr{V} \Big\}, \end{aligned}$$

where $\mathrm{Var}(A)$ denotes the variation process of A. By $\mathscr{H}^1$ we denote the set of all real-valued semimartingales X with $\|X\|_{\mathscr{H}^1} < \infty$.

Proposition A.4 *Let X be a non-negative semimartingale. Then $1_{\{X_-=0\}}\cdot X\in\mathscr{V}^+$.*

PROOF. This is shown by applying the Itô-Meyer formula to $X^-=-(X\wedge 0)$. Indeed, since $X^-=0$, Jacod (1979), (5.49) yields that $0=-\frac{1}{2}1_{\{X_-=0\}}\cdot X+\frac{1}{2}L^0+\sum_{t\leq\cdot}\frac{1}{2}1_{\{X_-=0\}}\Delta X_t$, where L^0 denotes the local time of X in 0 in the sense of Jacod (1979), (5.47). Since L^0 is increasing and $\Delta X\geq 0$ on $\{X_-=0\}$, it follows that $1_{\{X_-=0\}}\cdot X$ is increasing as well. $\square$

Proposition A.5 *Let L,X,U be real-valued semimartingales with $L\leq X\leq U$ and such that $1_{\{L_-<X_-\}}\cdot X$ is a σ-submartingale and $1_{\{X_-<U_-\}}\cdot X$ is a σ-supermartingale. Then $\|X\|_{\mathscr{H}^1}\leq c(\|L\|_{\mathscr{H}^1}+\|U\|_{\mathscr{H}^1})$ for some $c\in\mathbb{R}_+$ which is independent of L,X,U.*

PROOF. In this proof, we write $Y^*_\infty:=\sup_{t\in\mathbb{R}_+}|Y_t|$ for any semimartingale Y and $\mathrm{Var}(Y)$ for the variation process of any $Y\in\mathscr{V}$.

Step 1: W.l.o.g. L,U are special because otherwise $\|L\|_{\mathscr{H}^1}=\infty$ or $\|U\|_{\mathscr{H}^1}=\infty$ (cf. JS, I.4.23). By Kallsen (2002), Proposition 3.7, X is special as well. Denote by $X=X_0+M^X+A^X$, $U=U_0+M^U+A^U$, $L=L_0+M^L+A^L$ the canonical decompositions of the special semimartingales X,L,U into a local martingale and a process of finite variation, respectively.

Step 2: By JS, I.3.13, there exist predictable processes H^X,H^L,H^U such that $A^X=H^X\cdot A$, $A^L=H^L\cdot A$, $A^U=H^U\cdot A$, where $A:=\mathrm{Var}(A^X)+\mathrm{Var}(A^L)+\mathrm{Var}(A^U)\in\mathscr{V}^+$ is predictable. Since $1_{\{L_-<X_-\}}\cdot X=1_{\{L_-<X_-\}}\cdot M^X+(1_{\{L_-<X_-\}}H^X)\cdot A$ is a σ-submartingale, we have that $H^X\geq 0$ $(P\otimes A)$-almost everywhere on $\{L_-<X_-\}$. Similarly, it follows that $H^X\leq 0$ $(P\otimes A)$-almost everywhere on $\{X_-<U_-\}$. Proposition A.4 yields that $1_{\{L_-=X_-\}}\cdot(M^X-M^L)+(1_{\{L_-=X_-\}}(H^X-H^L))\cdot A=1_{\{L_-=X_-\}}\cdot(X-L)\in\mathscr{V}^+$. From JS, I.3.17 and the uniqueness of the special semimartingale decomposition it follows that $(1_{\{L_-=X_-\}}(H^X-H^L))\cdot A\in\mathscr{V}^+$, which implies that $H^X\geq H^L$ $(P\otimes A)$-almost everywhere on $\{L_-=X_-\}$. Similarly, we have $H^X\leq H^U$ $(P\otimes A)$-almost everywhere on $\{X_-=U_-\}$. Altogether, it follows that $|H^X|\leq|H^L|+|H^U|$ $(P\otimes A)$-almost everywhere. Consequently, we have $\mathrm{Var}(A^X)=|H^X|\cdot A\leq|H^L|\cdot A+|H^U|\cdot A=\mathrm{Var}(A^L)+\mathrm{Var}(A^U)$.

Step 3: Since $M^X=X-A^X-X_0\geq L-A^X-X_0=L_0-X_0+A^L-A^X+M^L$ and $M^X\leq U_0-X_0+A^U-A^X+M^U$, we have that $|M^X|\leq|L_0|+|U_0|+|A^L|+|A^U|+|A^X|+|M^L|+|M^U|$ and hence $M^{X,*}_\infty\leq|L_0|+|U_0|+2\mathrm{Var}(A^L)_\infty+2\mathrm{Var}(A^U)_\infty+M^{L,*}_\infty+M^{U,*}_\infty$ by Step 2. By the Burkhölder-Davis-Gundy inequality (cf. Jacod (1979), (2.34)), it follows that there exists some constant $c_1\geq 2$ such that $E(M^{L,*}_\infty)\leq c_1E(\sqrt{[M^L,M^L]_\infty})$ and likewise for U. By Dellacherie and Meyer (1982), VII.98, there exists some constant $c_2\geq 1$ such that $E(|L_0|+\mathrm{Var}(A^L)_\infty+\sqrt{[M^L,M^L]_\infty})\leq c_2\|L\|_{\mathscr{H}^1}$ and likewise for U. Together, it follows that $E(M^{X,*}_\infty)\leq c_1c_2(\|L\|_{\mathscr{H}^1}+\|U\|_{\mathscr{H}^1})$.

Step 4: From Step 2 we conclude that $|X_0|+\mathrm{Var}(A^X)\leq|L_0|+\mathrm{Var}(A^L)+|U_0|+\mathrm{Var}(A^U)$, which implies that $E(|X_0|+\mathrm{Var}(A^X)_\infty)\leq c_2(\|L\|_{\mathscr{H}^1}+\|U\|_{\mathscr{H}^1})$. By the Burkhölder-Davis-Gundy inequality (cf. Jacod (1979), (2.34)), there exists some constant $c_3\geq 1$ such that $E(\sqrt{[M^X,M^X]_\infty})\leq c_3E(M^{X,*}_\infty)$. Altogether, it follows that $\|X\|_{\mathscr{H}^1}\leq E(|X_0|+\mathrm{Var}(A^X)_\infty+\sqrt{[M^X,M^X]_\infty})\leq(c_2+c_1c_2c_3)(\|L\|_{\mathscr{H}^1}+\|U\|_{\mathscr{H}^1})$. $\square$

Proposition A.6 *Let X be an adapted real-valued process and $(T_n)_{n\in\mathbb{N}}$ an increasing sequence of stopping times such that X^{T_n} is a semimartingale for any $n \in \mathbb{N}$. If we have $\sup_{n\in\mathbb{N}} \|X^{T_n}\|_{\mathscr{H}^1} < \infty$, then X^{T_∞} is a semimartingale, where $T_\infty := \sup_{n\in\mathbb{N}} T_n$.*

PROOF. It is easy to see that $(X^{T_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{H}^1$. Due to completeness (cf. Dellacherie and Meyer (1982), VII.98) there is a limit in $\mathscr{H}^1$ which coincides with X on the set $[\![0, T_\infty[\![$. □

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