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Tail behaviour of the busy period of a GI/GI/1 queue with subexponential service times

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Abstract

This paper considers a stable GI/GI/1 queue with subexponential service time distribution. Under natural assumptions we derive the tail behaviour of the busy period of this queue. We extend the results known for the regular variation case under minimal conditions. Our method of proof is based on a large deviations result for subexponential distributions.

MSC: 60K25; 90B22

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1. Introduction and the main result

This paper studies the tail behaviour of the distribution function (d.f.) Z of a generic busy period random variable (r.v.) $\tilde{T}$ in a stable GI/GI/1 queue in the case that the d.f. B of a generic service time r.v. X is subexponential. We assume throughout that $E[X]=\rho E[Y]$ for some $0<\rho<1$, where Y denotes a generic inter-arrival time; let A denote its d.f. Subject to certain other conditions on the d.f.s A and B, we prove that

$$\frac{\bar{Z}(t)}{\bar{B}((1-\rho)t)} := \frac{P(\tilde{T}>t)}{P(X>(1-\rho)t)} \to \mathrm{e}^{B'}, \quad t\to\infty, \tag{1.1}$$

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where

$$B' = \sum_{n=1}^{\infty} \frac{P(S_n > 0)}{n} \tag{1.2}$$

and

$$S_n = \sum_{i=1}^{n} (X_i - Y_i) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i := S_n^X - S_n^Y, \tag{1.3}$$

where $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots$ are two independent sequences of independent identically distributed (i.i.d.) r.v.s.

Define the number of customers served in the generic busy period by

$$\tilde{N} = \inf\{n \geqslant 1 \colon S_n \leqslant 0\}, \tag{1.4}$$

where, because $\rho < 1$, this r.v. $\tilde{N}$ is a proper r.v. on $\mathbb{N}$. The busy period can then be expressed as

$$\tilde{T} = S_{\tilde{N}}^X. \tag{1.5}$$

Note that

$$S_k^X > S_k^Y \text{ for } k = 1, \ldots, \tilde{N} - 1 \quad \text{and} \quad S_{\tilde{N}-1}^Y < S_{\tilde{N}}^X = \tilde{T} \leqslant S_{\tilde{N}}^Y. \tag{1.6}$$

The length of the busy period is of prime interest in any queueing system. A first textbook account of busy periods for GI/GI/1 queues can be found in Prabhu (1965), with emphasis on random walks and the Baxter–Spitzer identity (see also, e.g. Feller (1971, Chapter XII.7), and, for the queueing setting, e.g. Asmussen (2003, Chapter VIII).

The tail behaviour of the busy period in the M/GI/1 queue under Cramér-type assumptions has been studied earlier in Abate and Whitt (1997). Our concern is with heavy-tailed distributions. For these, De Meyer and Teugels (1980) gave the first results akin to (1.1), though they did not express them in that form: instead, assuming (as they did) that the tail $\bar{B}$ of the service time is a regularly varying function, they used an Abelian theorem, relating the regular variation of $\bar{B}$ to the regular variation of its Laplace transform. The suggestion that the arguments of $\bar{B}$ and $\bar{Z}$ should be related as shown in (1.1) is more recent, in Asmussen and Teugels (1996). Zwart (2001) generalized De Meyer and Teugel's result to $\bar{B}$ satisfying an extended regular variation condition (see Bingham et al., 1987), linking a large busy period to a large maximum virtual waiting time during the busy period. In Jelenković and Momčilović (2004) the asymptotic relationship (1.1) is shown under log-concavity of $\bar{B}$ and some further condition guaranteeing the existence of all moments.

We prove (1.1) under minimal conditions on the service and inter-arrival time d.f.s; we formulate these conditions separately for clarity of presentation. Our method of proof includes the heavy-tailed (regular variation) as well as the moderate-tailed regime, establishing the result for all service time distributions with a tail heavier than $e^{-\sqrt{x}}$; this is a natural boundary as pointed out already in Asmussen et al. (1999) in the case of Poisson arrivals; see also Foss and Korshunov (2000).

Our methods are based on a more general result of Baltrūnas (2001), where he proved (under a log-concavity condition) a large deviations theorem for a random walk with negative drift, aiming at estimating the finite time ruin probability in the insurance risk model. In the present paper, we use a similar approach to obtain results on the busy period of a GI/GI/1 queue.

Condition A. *The inter-arrival time d.f. A is such that for every increasing function $g(n)$ satisfying $g(n)/n \to 0$ and $g(n)/\sqrt{n} \to \infty$ as $n \to \infty$, there is a positive constant c_A and an integer n_g such that for $n \geqslant n_g$,*

$$P(|S_n^Y - nE[Y]| > g(n)) \leqslant \exp(-c_A[g(n)]^2/n). \tag{1.7}$$

We verify in Lemma 2.2 below that Condition A is satisfied when Y has any finite exponential moment.

Conditions B. *The service time d.f. B is absolutely continuous with density b so that its hazard function $Q = -\log \bar{B}$ has a hazard rate $q = Q' = b/\bar{B}$ satisfying*

(i) $r := \limsup_{t\to\infty} tq(t)/Q(t) < \frac{1}{2}$.

(ii) $\beta := \liminf_{t\to\infty} tq(t) \geqslant \begin{cases} 2 & \text{if } r = 0, \\ c_B/(1-r) & \text{if } 0 < r < 1 \text{ for some } c_B > 2 + \sqrt{2}. \end{cases}$

We show in Lemma 3.8 below that Conditions B(i)–(ii) imply the following:

(iii) $\lim_{n\to\infty} \sqrt{n} \sup_{t \geqslant t_n} Q(t)/t = 0$ *for some sequence* $\{t_n\}$ *with* $\lim_{n\to\infty} t_n = \infty$.

Remark 1.1. (a) Condition (i) is satisfied for all d.f.s whose right tail is heavier than a Weibull tail with exponent $\frac{1}{2}$, i.e. $Q(t) = o(\sqrt{t})$ as $t \to \infty$. Lemma 3.6 below and (ii) imply that $\liminf_{t\to\infty} Q(t)/\log t \geqslant \beta$ and, hence, that (ii) is a moment condition on the service time and limits the pathological cases, which have been prominent in the subexponential area. In non-pathological cases (if e.g. $\lim_{t\to\infty} tq(t)$ exists), the case $r \neq 0$ corresponds to d.f.s with finite moments of all order, hence (ii) is satisfied, whereas d.f.s with infinite moments correspond to $r = 0$ and then (ii) requires a finite second moment.

(b) Recently, Baltrūnas (2002) derived the second-order behaviour of the busy period under the additional assumption that the hazard rate of the integrated service time distribution satisfies some regular variation condition.

We shall ultimately prove the following result.

Theorem 1.2. *Assume that the inter-arrival time d.f. $A(t) = P(Y \leqslant t)$, $t \geqslant 0$, satisfies Condition A, that the service time d.f. $B(t) = P(X \leqslant t)$, $t \geqslant 0$, satisfies Conditions* B(i)–(ii), *and that $\rho = E[X]/E[Y] < 1$. Then limit* (1.1) *holds.*

Our paper is organized as follows. The necessary basic properties of the busy period and renewal processes are collected in Section 2, and we state there a result that relates busy period, queue length and random walks. It is a key intermediate step in proving Theorem 1.2. In Section 3, we summarize some results on subexponential distributions; these results show that Conditions B above ensure that the service time distribution has finite second moment and belongs to $\mathscr{S}^*$. The essential large deviations property is derived in Section 4. The main result, Theorem 1.2, is finally proved in Section 5.

2. Busy periods and renewal processes

Our analysis that leads to (1.1) is based on a simple observation that relates the number $\tilde{N}$ of customers served in a busy period of duration $\tilde{T}$ via the renewal process v^Y generated by the inter-arrival times and defined by

$$v^Y(t) = \max\{k \in \mathbb{N}\colon\ S_k^Y \leqslant t\}, \tag{2.1}$$

so $v^Y(t) = n$ if and only if $S_n^Y \leqslant t < S_{n+1}^Y$. Since, the busy period $\tilde{T}$ is determined by the index where the first partial sum $S_n^X \leqslant S_n^Y$ (see (1.6)), namely $S_{\tilde{N}-1}^Y < S_{\tilde{N}}^X = \tilde{T} \leqslant S_{\tilde{N}}^Y$, we must have $v^Y(\tilde{T}) = \tilde{N} - 1$, i.e.

$$\{v^Y(\tilde{T}) = k\} = \{\tilde{N} = k + 1\}, \quad k \in \mathbb{N}_0. \tag{2.2}$$

Forming the union over $k \geqslant n$ gives

$$\{\tilde{N} \geqslant n + 1\} = \{v^Y(\tilde{T}) \geqslant n\} = \{\tilde{T} \geqslant S_n^Y\} \tag{2.3}$$

and thus

$$P(\tilde{T} \geqslant S_n^Y) = P(\tilde{N} \geqslant n + 1). \tag{2.4}$$

Recall the Baxter–Spitzer identity for random walks $\{S_n\}$ with negative mean increment, relating the probability of the first passage time event on the right-hand side of (2.4) to terms of the sequence $\{s_n\}_{n\in\mathbb{N}} := \{P(S_n > 0)/n\}_{n\in\mathbb{N}}$; see Feller (1971, Chapter XII.7). We present a new relationship in (2.5) below, when the r.v.s S_n are such as to make the sequence $\{s_n\}$ subexponential (see Definitions 3.1 and 3.2). The equality in (2.5) comes from (2.4); the rest is proved in Section 5 and also appeals to Proposition 3.10 which quotes the appropriate result on the Baxter–Spitzer identity for random walks with negative drift when the positive tail of the increments has a subexponential distribution in $\mathscr{S}^*$. The symbol $\sim$ means that the quotient of the right- and left-hand sides converges to 1.

Theorem 2.1. *In a stable* GI/GI/1 *queue in which the service time d.f. B has finite mean and belongs to $\mathscr{S}^*$, the busy period $\tilde{T}$, the number of customers $\tilde{N}$ served in a busy period, and the random walk $\{S_n\}$ at* (1.3) *of partial sums of differences between service and inter-arrival times, satisfy the relations*

$$P(\tilde{T} \geqslant S_n^Y) = P(\tilde{N} \geqslant n + 1) \sim \mathrm{e}^{B'} \frac{P(S_n > 0)}{n}, \quad n \to \infty, \tag{2.5}$$

where $B' = \sum_{n=1}^{\infty} P(S_n > 0)/n$ is as in (1.2).

Using large deviations arguments as in Asmussen (2003, Chapter XIII), we give a simple sufficient condition for Condition A to hold; obviously, Condition A holds for exponentially distributed Y; i.e. for a Poisson arrival stream.

Lemma 2.2. *If* $E[e^{\varepsilon Y}] < \infty$ *for some* $\varepsilon > 0$, *then Condition A is satisfied.*

Proof. Let the d.f. A have mean $1/\lambda$. Then $E[v^Y(t)] \sim \lambda t$ for large t. Now by definition, for any $g(t) < \lambda t$, $\{v^Y(t) < \lfloor \lambda t - g(t) \rfloor\} = \{S^Y_{\lfloor \lambda t - g(t) \rfloor} \geqslant t\}$, so by Markov's inequality, for any θ lying in $(0, \varepsilon)$ and t such that $g(t) < \lambda t$,

$$P(v^Y(t) < \lfloor \lambda t - g(t) \rfloor) = P(S^Y_{\lfloor \lambda t - g(t) \rfloor} \geqslant t) \leqslant e^{-\theta t}(E[e^{\theta Y}])^{\lfloor \lambda t - g(t) \rfloor}. \tag{2.6}$$

Write $K(\theta) = \log E[e^{\theta Y}]$, and let $g(\cdot)$ satisfy the constraints of Condition A. Then for all sufficiently large t, since $K(\theta)$ is analytic in $|\theta| < \varepsilon$, there certainly exists a root $\theta := \theta(t)$ lying in $(0, \varepsilon)$ of $K'(\theta) = \lambda^{-1} + \lambda^{-2} g(t)/t$ because $K'(0) = 1/\lambda$, and $\theta(t) \sim g(t)/[\lambda^2 t K''(0)] \to 0$ for $t \to \infty$. Then since $K(\theta) = \theta(\lambda^{-1} + \frac{1}{2}\theta K''(0)) + O(\theta^3)$,

$$e^{-\theta t}(\exp K(\theta))^{\lfloor \lambda t - g(t) \rfloor} \sim \exp\left(-\frac{[g(t)]^2/t}{2\lambda^3 K''(0)} \right), \tag{2.7}$$

which with the inequality at (2.6) gives half of (1.7). The other half is proved similarly using $E[e^{-\theta Y}]$ and $\{v^Y(t) > \lfloor \lambda t + g(t) \rfloor\} = \{S^Y_{\lfloor \lambda t + g(t) \rfloor} \leqslant t\}$. □

3. Some subexponential properties

We start by recalling some definitions concerning subexponential d.f.s and subexponential sequences. Throughout this section, F denotes the d.f. of a non-negative r.v. X, $Q(x) := -\log \bar{F}(x)$, $x \geqslant 0$, denotes its *hazard function*, and (where applicable) q is the density of Q, i.e. the *hazard rate*, when F is absolutely continuous.

Definition 3.1. (a) F is *subexponential* ($F \in \mathscr{S}$) if

$$\lim_{t \to \infty} \frac{\overline{F^{2*}}(t)}{\bar{F}(t)} = 2. \tag{3.1}$$

(b) If F has finite mean, it is in the class $\mathscr{S}^*$ if

$$\lim_{t \to \infty} \int_0^t \frac{\bar{F}(t-u)}{\bar{F}(t)} \bar{F}(u)\, du = 2 \int_0^\infty \bar{F}(u)\, du. \tag{3.2}$$

As shown in Klüppelberg (1988), when $F \in \mathscr{S}^*$ it follows that $F \in \mathscr{S}$ and hence $F \in \mathscr{L}$, i.e.

$$\lim_{x \to \infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} = 1 \quad \text{locally uniformly in } y \in \mathbb{R}. \tag{3.3}$$

A discrete analogue of $\mathscr{S}^*$ is the following class.

Definition 3.2 (Chover et al., 1973; Embrechts and Hawkes, 1982). The summable non-negative sequence $\{h_n\}_{n\in\mathbb{N}_0}$ is in the class $\mathscr{S}_D^*$ if both

$$\lim_{n\to\infty} h_{n+1}/h_n = 1 \tag{3.4}$$

and the terms $h_n^{\oplus 2} := \sum_{i=0}^{n} h_i h_{n-i}$, $n\in\mathbb{N}$, of its second convolution power satisfy

$$\lim_{n\to\infty} h_n^{\oplus 2}/h_n = 2\sum_{i=0}^{\infty} h_i < \infty. \tag{3.5}$$

Note that (3.4) is an analogue of (3.3), which in the discrete case does not hold automatically.

Lemma 3.4 below states that approximating the right-continuous decreasing function $\bar F(x)$, $x>0$, by any discrete skeleton $\{\bar F(cn)\}_{n\in\mathbb{N}}$ for some $c>0$, does not destroy the subexponential property (3.2); the numerator and sum in (3.5) approximate, respectively, the integrals on the left- and right-hand sides of (3.2).

As a prelude to the proof of Lemma 3.4 we rewrite the defining property (3.2).

Lemma 3.3. *$F\in\mathscr{S}^*$ if and only if for any $a>0$*

$$\int_a^{t-a} \bar F(t-u)\bar F(u)\,\mathrm{d}u \sim 2\bar F(t-a)\int_a^{\infty}\bar F(u)\,\mathrm{d}u, \qquad t\to\infty. \tag{3.6}$$

Proof. First note that if $F\in\mathscr{S}^*$, and hence $F\in\mathscr{L}$, (3.2) gives

$$\begin{aligned}
2\int_0^{\infty}\bar F(u)\,\mathrm{d}u &= \lim_{t\to\infty}\left(\frac{\bar F(t-a)}{\bar F(t)}\,\frac{1}{\bar F(t-a)}\int_a^{t-a}\bar F(t-u)\bar F(u)\,\mathrm{d}u\right.\\
&\qquad\left. +\frac{2}{\bar F(t)}\int_0^{a}\bar F(t-u)\bar F(u)\,\mathrm{d}u\right)\\
&= \lim_{t\to\infty}\frac{1}{\bar F_a(t-2a)}\int_0^{t-2a}\bar F_a(t-2a-u)\bar F_a(u)\,\mathrm{d}u + 2\int_0^{a}\bar F(u)\,\mathrm{d}u,
\end{aligned}$$

where $\bar F_a(u)=\bar F(u+a)$, implying that (3.6) holds. But (3.6) is equivalent to $F_a\in\mathscr{S}^*$, and hence $F_a\in\mathscr{L}$, equivalently $F\in\mathscr{L}$. Now reverse the algebra from (3.2) to (3.6). □

Lemma 3.4. *If the d.f. $F\in\mathscr{S}^*$, then for every $c>0$ the sequence $\{f_n\}\equiv\{\bar F(cn)\}\in\mathscr{S}_D^*$.*

Proof. Since $F\in\mathscr{S}^*\subset\mathscr{L}$ we have $f_n/f_{n+1}\to 1$ as $n\to\infty$; i.e. (3.4) holds. We must show that (3.5) holds for $\{f_n\}$, i.e. that $f_n^{\oplus 2}\sim 2f_n(\sum_{i=0}^{r-1} f_i+\sum_{i=r}^{\infty} f_i)$ for any positive integer r. First, since $F\in\mathscr{S}^*$, F has a finite first moment, so by monotonicity $\sum_{i=0}^{\infty} f_i<\infty$, and $\sum_{i=r}^{\infty} f_i\to 0$ for $r\to\infty$. Next, for any fixed integer $r<\frac{1}{2}n$,

$$f_n^{\oplus 2} = \sum_{i=0}^{n} f_{n-i}f_i = 2\sum_{i=0}^{r-1} f_{n-i}f_i + \sum_{i=r}^{n-r} f_{n-i}f_i, \qquad n\in\mathbb{N}. \tag{3.7}$$

By choosing r sufficiently large, $\sum_{i=0}^{r-1} f_i$ may be taken arbitrarily close to the corresponding infinite sum, and since $f_{n-i}/f_n \geqslant 1$, twice this infinite sum is certainly a lower bound on the limit. For $c > 0$, by monotonicity,

$$\int_{rc}^{(n+1)c-rc} \bar{F}(nc-u+c)\bar{F}(u)\,\mathrm{d}u \leqslant c\sum_{i=r}^{n-r} f_{n-i}f_i \leqslant \int_{rc}^{(n+1)c-rc} \bar{F}(nc-u)\bar{F}(u-c)\,\mathrm{d}u.$$

By (3.6) the left-hand integral is asymptotically equivalent for $n \to \infty$ to $2\bar{F}((n-r)c)\int_{rc}^{\infty}\bar{F}(u)\,\mathrm{d}u$, when the right-hand integral equals

$$\int_{(r-1)c}^{(n-1)c-(r-1)c} \bar{F}((n-1)c-v)\bar{F}(v)\,\mathrm{d}v \sim 2\bar{F}((n-1-r)c)\int_{(r-1)c}^{\infty}\bar{F}(u)\,\mathrm{d}u.$$

Hence the last sum in (3.7) is asymptotically equivalent to $2f_n\sum_{i=r}^{\infty} f_i$ within bounds which are at most of order $2f_{n-r}f_r$. Thus sum (3.5) can be attained within bounds of order $2f_r$, which is arbitrarily small for r large enough. Hence $f_n^{\oplus 2} \sim 2f_n\sum_{i=0}^{\infty} f_i$ as required. □

Lemma 3.5. *If F and G are d.f.s on the positive half-line of independent r.v.s X and Y, say, and if $F \in \mathscr{S}^*$, then the d.f. H of $U = X - Y$, whose right-hand tail is given by*

$$\bar{H}(x) = \int_0^{\infty} \bar{F}(x+u)\,\mathrm{d}G(u), \qquad x \in \mathbb{R},$$

satisfies

$$\lim_{x\to\infty}\frac{\bar{H}(x)}{\bar{F}(x)} = 1, \qquad \lim_{x\to\infty}\int_0^x \frac{\bar{H}(x-u)}{\bar{H}(x)}\bar{H}(u)\,\mathrm{d}u = 2E[(X-Y)_+]$$

and for the left-hand tail,

$$H(-x) = P(U \leqslant -x) \leqslant P(Y \geqslant x).$$

Proof. The relation $\bar{F} \sim \bar{H}$ follows by using the fact that $F \in \mathscr{L}$ together with the dominated convergence theorem. The rest is all but trivial: for all $x \in \mathbb{R}$, $P(U \leqslant -x) = P(Y \geqslant x + X) \leqslant P(Y \geqslant x)$ since $X \geqslant 0$ a.s. □

We now give some properties of tails of d.f.s that relate to Conditions B and lead up to Lemma 3.8. These properties imply in particular that all service time distributions satisfying Conditions B have finite second moment and are in $\mathscr{S}^*$.

In relation to Condition B(ii), define the *moment index*

$$\kappa = \sup\{k\colon E[X^k] < \infty\}. \tag{3.8}$$

Lemma 3.6. (a) $\kappa = \liminf_{t\to\infty} Q(t)/\log t =: \tilde{\kappa}$.
(b) $\kappa \geqslant \liminf_{t\to\infty} tq(t) = \beta$ *as in Condition* B(ii).
(c) $[\bar{F}(\cdot)]^v = \exp(-vQ(\cdot))$ *is integrable for all* $v > 1/\kappa$.

Proof. (a) From the definition of $\tilde{\kappa}$ it follows that for any ε in $(0,\tilde{\kappa})$ there exists finite $t_\varepsilon > 0$ such that for all $t > t_\varepsilon$, $\tilde{\kappa} - \varepsilon < Q(t)/\log t$, equivalently, that $\bar{F}(t) < 1/t^{\tilde{\kappa}-\varepsilon}$, so that

$$\int_{t_\varepsilon}^{\infty} u^{\tilde{\kappa}-2\varepsilon-1}\bar{F}(u)\,\mathrm{d}u \leqslant \int_{t_\varepsilon}^{\infty} u^{-1-\varepsilon}\,\mathrm{d}u < \infty,$$

so $E(X^k) < \infty$ for all $k < \tilde{\kappa}$, and therefore $\kappa \geqslant \tilde{\kappa}$. Conversely, from the definition of κ, for any δ in $(0,\kappa)$ we have $\bar{F}(t) \leqslant E[X^{\kappa-\delta}]/t^{\kappa-\delta}$, equivalently, $Q(t) \geqslant -\log E[X^{\kappa-\delta}] + (\kappa-\delta)\log t$. Thus, for all $t > 1$,

$$\frac{Q(t)}{\log t} \geqslant (\kappa-\delta) - \frac{\log E[X^{\kappa-\delta}]}{\log t},$$

so $\tilde{\kappa} = \liminf_{t\to\infty} Q(t)/\log t \geqslant \kappa$. (a) is proved.

(b) Given ε in $(0,\beta)$, the definitions imply that $q(t) > (\beta-\varepsilon)/t := \beta_\varepsilon/t$ for $t > t'_\varepsilon$ for some finite $t'_\varepsilon > 0$. By integration, this inequality implies for $t > t'_\varepsilon$ that $Q(t) > Q(t'_\varepsilon) + \beta_\varepsilon \log(t/t'_\varepsilon)$, so for such t, we have $Q(t)/\log t > \beta_\varepsilon + [Q(t'_\varepsilon) - \beta_\varepsilon \log t'_\varepsilon]/\log t \to \beta_\varepsilon$ as $t \to \infty$. Consequently, $\kappa \geqslant \beta_\varepsilon$ and, since ε is arbitrarily small, we have in fact that $\kappa \geqslant \beta$.

(c) Given any $v > 1/\kappa$, there exists $k < \kappa$ such that $kv > 1$. Since $k < \kappa$, $\infty > E[X^k] \geqslant t^k \bar{F}(t)$ for all $t > 0$, i.e. $\bar{F}(t) < E[X^k]t^{-k}$. Then for any $u > 0$,

$$\int_u^{\infty} [\bar{F}(t)]^v\,\mathrm{d}t \leqslant (E[X^k])^v \int_u^{\infty} t^{-kv}\,\mathrm{d}t = (E[X^k])^v \frac{u^{1-kv}}{kv-1} < \infty. \qquad \square$$

Versions of the following results can be found in Baltrūnas (2001) and Klüppelberg (1987, 1989); see also Cline (1986) and Goldie and Klüppelberg (1998). We include the proofs here, since (to our knowledge) they have not appeared in this generality elsewhere; it also keeps the paper self-contained.

Proposition 3.7. *Let* $\rho \in (0,1)$ *and* $v \geqslant 1$. *Then the following are equivalent*:

(a) $Q(ty) \leqslant y^\rho Q(t)$ *for all* $t \geqslant v$ *and* $y \geqslant 1$.
(b) $Q(t)/t^\rho$ *decreases on* $t \geqslant v$.
(c) $Q(t)$ *is absolutely continuous on* $t \geqslant v$ *with Lebesgue density* $q(t) \to 0$ *as* $t \to \infty$, *and* $tq(t)/Q(t) \leqslant \rho$ *for all* $t \geqslant v$.

Proof. (a) $\Leftrightarrow$ (b): Set $g(t) = t^{-\rho}Q(t)$, $t \geqslant v$; then $g(t) \geqslant g(ty)$ for $y > 1$ if and only if (a) holds.

(a) $\Leftrightarrow$ (c): From (a) we conclude

$$0 \leqslant \frac{Q(t)}{t} \leqslant \left(\frac{t}{v}\right)^{\rho-1} \frac{Q(v)}{v} \leqslant \frac{Q(v)}{v}, \qquad t \geqslant v. \tag{3.9}$$

Then $Q(t)/t$ decreases for $t \geqslant v$ and we obtain

$$\limsup_{t\to\infty} \frac{Q(t)}{t} \leqslant \frac{Q(v)}{v} \limsup_{t\to\infty} \left(\frac{t}{v}\right)^{\rho-1} = 0.$$

For fixed $y > 0$ and $t \geqslant v$ we obtain from (3.9)

$$\frac{Q(t+y)-Q(t)}{Q(t)} \leqslant \left(1+\frac{y}{t}\right)^{\rho} - 1 \leqslant \rho\frac{y}{t}, \tag{3.10}$$

giving by (3.9)

$$0 \leqslant Q(t+y) - Q(t) \leqslant \rho y \frac{Q(t)}{t} \leqslant y\frac{Q(v)}{v}.$$

Thus, Q is absolutely continuous (see e.g. Royden, 1968, pp.104–106), with density q on $[v,\infty)$ satisfying $0 \leqslant \limsup_{t\to\infty} q(t) \leqslant \limsup_{t\to\infty} \rho Q(t)/t = 0$, so (c) holds.

Conversely, (c) implies that for $y \geqslant 1$ and $t \geqslant v$,

$$\log\frac{Q(ty)}{Q(t)} = \int_t^{ty} \frac{q(u)}{Q(u)}\,\mathrm{d}u \leqslant \int_t^{ty} \frac{\rho}{u}\,\mathrm{d}u = \log y^{\rho},$$

so (a) holds.

In relation to Condition B(i), the parameter ρ in Proposition 3.7 must satisfy $\rho \geqslant r$. Quite generally, given any d.f. F on $(0,\infty)$ for which a density f exists, the quantity $r := \limsup_{t\to\infty} tq(t)/Q(t)$ is well defined (though possibly infinite); when $r < \infty$, for any $\varepsilon > 0$ the index

$$t_\varepsilon := \inf\{t\colon\ uq(u) < (r+\varepsilon)Q(u) \quad (\text{all } u > t)\} \tag{3.11}$$

is well defined and finite. In particular, if $r < 1$, there exists positive ε such that $r_\varepsilon := r + \varepsilon < 1$.

Lemma 3.8. (a) *If* $r < 1$, *then* $F \in \mathscr{S}$.

(b) *If also* $\int_0^\infty \exp(-(2-2^r)Q(u))\,\mathrm{d}u < \infty$, *then* $F \in \mathscr{S}^*$. *In particular this integrability condition is satisfied if* $\kappa(2-2^r) > 1$, *i.e.* $r < [\log(2-\kappa^{-1})]/\log 2$.

(c) *If Conditions* B(i)–(ii) *are satisfied*, *then Condition* B(iii) *is met in particular by* $t_n = n^{1/[2(1-r)]}$, *for which* $\lim_{n\to\infty} \sqrt{n}Q(t_n)/t_n = 0$.

Proof. The proof is a consequence of Proposition 3.7, setting $\rho = r_\varepsilon < 1$.

(a) From Proposition 3.7(a) we conclude that for $t_\varepsilon < y \leqslant \frac{1}{2}t$,

$$\begin{aligned} Q(t) - Q(t-y) &\leqslant \left(1 - \left(1-\frac{y}{t}\right)^{r_\varepsilon}\right) Q(t) \\ &\leqslant \left(1 - \left(1-\frac{y}{t}\right)^{r_\varepsilon}\right)\left(\frac{t}{y}\right)^{r_\varepsilon} Q(y) \\ &= \left[\left(\frac{t}{y}\right)^{r_\varepsilon} - \left(\frac{t}{y}-1\right)^{r_\varepsilon}\right] Q(y) \\ &\leqslant (2^{r_\varepsilon}-1)Q(y), \end{aligned} \tag{3.12}$$

the last inequality coming from $x^r + 1 \leqslant (x-1)^r + 2^r$ for $x \geqslant 2$ as follows from the concavity of x^r in $x > 0$ when $r \in (0,1)$. Moreover, by (3.12),

$$\frac{[\bar{F}(t)]^2}{\bar{F}(2t)} = \exp\left(-\left(2 - \frac{Q(2t)}{Q(t)}\right) Q(t)\right) \leqslant \exp(-(2-2^{r_\varepsilon})Q(t)). \tag{3.13}$$

Now, we use the following decomposition for $t_\varepsilon < v < \frac{1}{2}t$:

$$\frac{\overline{F^{2*}}(t)}{\bar{F}(t)} = 2\int_0^v \frac{\bar{F}(t-y)}{\bar{F}(t)}\,\mathrm{d}F(y) + 2\int_v^{t/2} \frac{\bar{F}(t-y)}{\bar{F}(t)}\,\mathrm{d}F(y) + \frac{[\bar{F}(\frac{1}{2}t)]^2}{\bar{F}(t)}.$$

Letting first $t \to \infty$ and then $v \to \infty$, and using $F \in \mathscr{L}$, the first integral tends to 1 and the last term vanishes by (3.13). The second integral can be bounded using (3.12):

$$\int_v^{t/2} \frac{\bar{F}(t-y)}{\bar{F}(t)}\,\mathrm{d}F(y) \leqslant \int_v^{t/2} [\bar{F}(y)]^{-(2^{r_\varepsilon}-1)}\,\mathrm{d}F(y) = \frac{\bar{F}(v)^{2-2^{r_\varepsilon}} - \bar{F}(\frac{1}{2}\,t)^{2-2^{r_\varepsilon}}}{2-2^{r_\varepsilon}}.$$

Letting first $t \to \infty$ and then $v \to \infty$, we see that this term tends to 0.

(b) Writing for $t \geqslant 2t_\varepsilon$

$$\int_0^t \frac{\bar{F}(t-y)}{\bar{F}(t)}\bar{F}(y)\,\mathrm{d}y = 2\int_0^{t/2} \frac{\bar{F}(t-y)}{\bar{F}(t)}\bar{F}(y)\,\mathrm{d}y$$

and noting that

$$2\int_{t_\varepsilon}^{\infty} [\bar{F}(y)]^{2-2^r}\,\mathrm{d}y < \infty,$$

the integrability result now follows from the dominated convergence theorem. That κ as stated suffices follows from Lemma 3.6(c).

(c) Proposition 3.7(b) implies that for some bound $K > 0$, $\sup_{t \geqslant t_n} Q(t)/t \leqslant Q(t_n)/t_n \leqslant K$. This implies in particular that Condition B(iii) is met by t_n as stated. □

Remark 3.9. Conditions B imply that $B \in \mathscr{S}^*$, since under those conditions we have $2 - 2^r > \frac{1}{2} > 1/\kappa$, so the required integrability condition at (b) above is met.

The next result is the key to proving Theorem 2.1.

Proposition 3.10 (Chover et al., 1973). *Let the probability distribution $\{v_n\}_{n\in\mathbb{N}_0}$ with generating function $\hat{v}(z) = \sum_{n=1}^{\infty} v_n z^n$ ($|z| \leqslant 1$) satisfy the conditions*

(i) $\lim_{n\to\infty} v_n^{\oplus 2}/v_n = c$ *exists and is finite,*
(ii) $\lim_{n\to\infty} v_{n+1}/v_n = 1/R$ *for some* $1 \leqslant R < \infty$, *and*
(iii) $d = \hat{v}(R)$ *is finite.*

Assume that the function $\Psi(w)$ is analytic in a region containing the range of $\hat{v}(z)$ for $|z| \leqslant R$. Then $c = 2d$ and there exists a sequence $\Psi_v \equiv \{(\Psi_v)_n\}$ satisfying

$$\hat{\Psi}_v(z) \equiv \sum_{n=0}^{\infty} (\Psi_v)_n z^n = \Psi(\hat{v}(z)), \quad |z| \leqslant R \tag{3.14}$$

and for which

$$\lim_{n\to\infty} (\Psi_v)_n/v_n = \Psi'(d). \tag{3.15}$$

If in fact $\Psi(w) = \sum_{n=0}^{\infty} c_k w^k$ *for* $|w| \leqslant 1$, *where* $\sum_{n=0}^{\infty} |c_k| < \infty$, *then*

$$\hat{\Psi}_v(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} c_k v_n^{\oplus k} \right) z^n. \tag{3.16}$$

Remark 3.11. (a) Conditions (i) and (ii) with $R=1$ imply that the sequence $\{v_n\}_{n\in\mathbb{N}_0}$ belongs to the class $\mathscr{S}_D^*$ as defined in Definition 3.2.

(b) Notice that $\Psi(w)$ is analytic in a region containing the range of $\hat{v}(z)$ for $|z| \leqslant R$ and that this range includes the origin because $\hat{v}(0)=0$, so $\Psi(w)$ is analytic at $w=\hat{v}(R)$, but this does not mean that $\Psi(w)$ has a power series expansion about $w=0$ valid within the circle $|w| \leqslant \hat{v}(R)$. However, if this condition (on Ψ rather than on $\hat{v}(\cdot)$) is satisfied, then $\sum |c_k|(R+\varepsilon)^k < \infty$ for sufficiently small positive ε. This would imply that Kesten's subexponential inequality (e.g. Embrechts et al., 1997, Lemma 1.3.5(c)) could be applied, and then Eq. (3.16) would be readily established for such Ψ.

4. A large deviations result

In this section, we establish a large deviations result for sums of i.i.d. r.v.s with mean zero and subexponential right tail, because this enables us to estimate the quantity

$$P(S_n > 0) = P(S_n^X \geqslant S_n^Y) = P\left(S_n - nE[S_1] \geqslant n|E[S_1]|\right)$$

that appears in (2.5). Theorem 4.1 below is in the spirit of a result of Nagaev (1977); the methods we use are similar to those in Pinelis (1985) (see also Baltrūnas, 1995).

We are interested in showing that, when the d.f.s A and B of inter-arrival and service times satisfy Conditions A and B of Section 1, then

$$\lim_{n\to\infty} \frac{P(S_n > 0)}{nP(S_1 > n|E[S_1]|)} = \lim_{n\to\infty} \frac{P(S_n - nE[S_1] > n|E[S_1]|)}{nP(S_1 > n|E[S_1]|)} = 1.$$

Define

$$F(t) = P(X - Y \leqslant t) = \int_0^{\infty} B(t+u)\,\mathrm{d}A(u), \qquad t \in \mathbb{R}, \tag{4.1}$$

set

$$\mu := E[S_1] = E[X - Y] = -(1-\rho)EY < 0, \tag{4.2}$$

and define the centred r.v. $U = X - Y - \mu$ with distribution tail

$$P(U > t) = \bar{F}(t+\mu). \tag{4.3}$$

Then

$$P(S_n > 0) = P(S_n - \mu n > |\mu| n) = P\left(\sum_{k=1}^{n} U_k > |\mu| n \right). \tag{4.4}$$

With this notation we can formulate the following result.

Theorem 4.1 (Large deviations property for subexponential r.v.s.). *Let* $U, U_1, U_2, \ldots$ *be i.i.d. r.v.s with d.f.* $P(U \leqslant t) = F(t+\mu)$, $t \in \mathbb{R}$, *as in* (4.3) *and assume that Conditions B hold. Then for any sequence* $\{t_n\}$ *satisfying Condition* B(iii), *the sums* $S_n^U = \sum_{i=1}^n U_i$, $n \geqslant 1$, *satisfy*

$$\lim_{n\to\infty} \sup_{t \geqslant t_n} \left| \frac{P(S_n^U > t)}{n\bar{B}(t)} - 1 \right| = 0. \tag{4.5}$$

Remark 4.2. (a) Note that we can rewrite (4.5) as

$$P(S_n^U > t) \sim n\bar{B}(t), \quad t \geqslant t_n, \quad n \to \infty. \tag{4.6}$$

(b) It follows from Lemma 3.4 that $B \in \mathscr{S}^*$, and thus that the sequence $\{\bar{B}(|\mu|n)\}$ belongs to the class $\mathscr{S}_D^*$. From this it follows that, under the assumptions of Theorem 4.1, the sequence $\{s_n\} = \{P(S_n > 0)/n\}$ is a subexponential sequence in $\mathscr{S}_D^*$ and has finite sum B' as in (1.2). This result appears in Baltrūnas (2001) under three sets of conditions: (i) his Proposition 2.1 has $\kappa > 1$ but the more stringent condition that $\limsup_{t\to\infty} tq(t) < \infty$, with a reference to Baltrūnas (1995) for proof; (ii) his Proposition 2.2 has $r < \frac{1}{2}$, $\kappa > 2$ and q eventually decreasing to 0, with proof similar to Theorem 1 of Nagaev (1977); and (iii) his Proposition 2.3, which concludes that the positive tail of F is in $\mathscr{S}^*$, has q eventually decreasing to 0, $r < 1$ and $\kappa(1-r) > 1$.

Before proving Theorem 4.1, we establish some preliminary results.

The next lemma follows immediately from the definition of F in (4.1), monotonicity and dominated convergence.

Lemma 4.3. (a) $\bar{F}(t) \leqslant \bar{B}(t)$, $t \geqslant 0$.

(b) *If* $B \in \mathscr{L}$, *then* $\bar{F}(t) \sim \bar{B}(t)$ *as* $t \to \infty$.

Lemma 4.4. *Suppose that* $r := \limsup_{t\to\infty} tq(t)/Q(t) < 1$. *Then for all sufficiently large* t *and* $s = s(t) = Q(t)/t$, *there is a finite constant* C *such that*

$$\int_{t/Q(t)}^{t} \exp\left(\frac{Q(t)}{t} u\right) \mathrm{d}F(u) = \int_{1/s}^{t} \mathrm{e}^{su} \, \mathrm{d}F(u) \leqslant C < \infty. \tag{4.7}$$

Proof. Use integration by parts and Lemma 4.3 to write for $0 < 1/s < t < \infty$

$$\begin{aligned} \int_{1/s}^{t} \mathrm{e}^{su} \, \mathrm{d}F(u) &= -\mathrm{e}^{st}\bar{F}(t) + \mathrm{e}\bar{F}(1/s) + s\int_{1/s}^{t} \mathrm{e}^{su}\bar{F}(u) \, \mathrm{d}u \\ &\leqslant 0 + \mathrm{e} + s\int_{1/s}^{t} \mathrm{e}^{su}\bar{B}(u) \, \mathrm{d}u \\ &= \mathrm{e} + s\int_{1/s}^{t} \exp(su - Q(u)) \, \mathrm{d}u \\ &=: \mathrm{e} + J(t). \end{aligned}$$

Choose $r_\varepsilon < 1$ and t_ε as in (3.11). From (3.10) we conclude, setting $\rho = r_\varepsilon$, for $u \leqslant t$

$$Q(t) - Q(u) \leqslant \left(\left(\frac{t}{u}\right)^{r_\varepsilon} - 1\right) Q(u) \leqslant r_\varepsilon \left(\frac{t}{u} - 1\right) Q(u). \tag{4.8}$$

Then for t such that $1/s = t/Q(t) > t_\varepsilon$, we have for u satisfying $1/s \leqslant u \leqslant t$,

$$\begin{aligned} su - Q(u) &\leqslant -(1 - r_\varepsilon)\frac{Q(u)}{ut}\, u(t-u) \\ &\leqslant -(1 - r_\varepsilon)\frac{Q(t)}{t^2}\, u(t-u), \end{aligned} \tag{4.9}$$

where in the last step we have used the fact that $Q(v)/v$ is decreasing. From this, we obtain for all t such that $1/s = t/Q(t) > t_\varepsilon$,

$$\begin{aligned} J(t) &\leqslant s\int_{1/s}^{t} \exp(-(1-r_\varepsilon)(s/t)u(t-u))\,\mathrm{d}u \\ &\leqslant s\int_{0}^{t} \exp(-(1-r_\varepsilon)(s/t)u(t-u))\,\mathrm{d}u \\ &\leqslant 2s\int_{0}^{t/2} \exp(-\tfrac{1}{2}(1-r_\varepsilon)su)\,\mathrm{d}u \\ &\leqslant \frac{4}{1-r_\varepsilon} < \infty. \qquad \square \end{aligned}$$

For $t > 1$ define $y \in (0, t)$ (note that y depends on Q and t) by

$$y = \sup\left\{u > 1\colon\ \frac{2\log u}{Q(u)} \leqslant (1 - r_\varepsilon)\,\frac{t-u}{t}\right\}. \tag{4.10}$$

To see that y is indeed well defined, observe that in the defining inequality, the right-hand side $\downarrow 0$ as $u \to t$ while the left-hand side, which has $\limsup$ as $u \to \infty$ equal to $2/\kappa$, tends to 0 as $u \downarrow 1$ and is positive for $u > 1$. Indeed, positivity and continuity imply that we must in fact have

$$\frac{y}{t} = 1 - \frac{2\log y}{(1-r_\varepsilon)Q(y)}.$$

Lemma 4.5. *Under Conditions* B, *there exists positive* δ *such that* $y > \delta t$ *for all* t *sufficiently large. For finite* κ *we find* $0 < \delta < 1 - 2/c_B$, *while if* $\kappa = \infty$ *we can choose* δ *in* $(0, 1)$ *arbitrarily close to* 1.

Proof. When

$$\limsup_{u\to\infty} \frac{2\log u}{Q(u)} = \frac{2}{\kappa} < 1 - r_\varepsilon = \lim_{t\to\infty} (1 - r_\varepsilon)\left(1 - \frac{u}{t}\right),$$

we must have

$$\frac{y}{t} > 1 - \frac{2}{\kappa(1-r_\varepsilon)} - \varepsilon'$$

for sufficiently small positive ε' for all t sufficiently large. If $\kappa=\infty$ then $y>\delta t$ for $\delta\in(0,1)$ and all sufficiently large t, while when $\kappa<\infty$ and Condition B(ii) holds, we conclude by appealing to Lemma 3.6(b) that for any given δ in $(0,1-2/c_B)$, $y/t>\delta$ for all sufficiently large t. $\square$

Proof of Theorem 4.1. For given $y>-\mu>0$ let ξ be the number of summands U_k in $\sum_{k=1}^n U_k := S_n^U$ such that $U_k>y$. Then ξ is a binomial r.v. with parameters n and $\bar F(y+\mu)$, and we can write

$$P(S_n^U>t)=P(S_n^U>t,\xi=0)+P(S_n^U>t,\xi=1)+P(S_n^U>t,\xi\geqslant 2)$$
$$=: I_0+I_1+I_2.$$

We show that there is a sequence $\{t_n\}$ for which, as $n\to\infty$, $t_n\to\infty$ and for $t\geqslant t_n$, $I_2=\mathrm{o}(n\bar B(t))=I_0$ and $I_1=n\bar B(t)[1+\mathrm{o}(1)]$.

First we estimate I_2. Using Lemma 4.3(b) we obtain

$$\begin{aligned}I_2&\leqslant P(\xi\geqslant 2)\\&=\mathrm{O}(1)[nP(U_1>y)]^2\\&=\mathrm{O}(1)[n\bar B(y)]^2\\&=\mathrm{O}(1)n\bar B(t)n\exp(-2Q(y)+Q(t)).\end{aligned}$$

To bound the exponential term here, apply the inequality at (4.8) for $t_\varepsilon\leqslant y\leqslant t$ together with (4.10) to conclude that

$$Q(t)-2Q(y)\leqslant -Q(y)\left(1-r_\varepsilon\frac{t-y}{y}\right)=-2\log y\frac{1-r_\varepsilon(t/y-1)}{(1-r_\varepsilon)(1-y/t)}.$$

The coefficient of $-2\log y$ here exceeds 1 for all $0<r_\varepsilon\leqslant\frac12$ provided

$$\frac{1}{r_\varepsilon}>\left(\frac{t}{y}-1\right)^2,$$

i.e. provided $y/t>\sqrt2-1$, which is certainly satisfied when $c_B>2+\sqrt2$ by Lemma 4.5. Moreover, by Lemma 4.5 $-\log y\leqslant -2(\log\delta+\log t)$, i.e., thus, for such y and t as above, and consequently, for $t\geqslant t_n$ with $n/t_n^2\to 0$ as $n\to\infty$,

$$I_2\leqslant \mathrm{O}(1)n\bar B(t)n/t^2=\mathrm{o}(n\bar B(t)).$$

To estimate I_0 define for given $y>-\mu$ the truncated r.v. $V=UI_{\{U<y\}}$ with moment generating function

$$\tilde f(s)=\int_{-\infty}^y \mathrm{e}^{sv}\,\mathrm{d}P(V\leqslant v)=E[\mathrm{e}^{sU}\mid U\leqslant y]P(U\leqslant y),\quad s\in\mathbb{R}.$$

Next, we introduce the Esscher transform

$$F_s(u)=P(V^s\leqslant u)=\frac{1}{\tilde f(s)}\int_{-\infty}^u \mathrm{e}^{sv}\,\mathrm{d}P(V\leqslant v),\quad -\infty<u\leqslant y$$

of V with parameter s (see e.g. Asmussen, 2003, Chapter XIII; Feller, 1971, Section XVI.7). By introducing i.i.d. r.v.s V_k^s with d.f.s F_s and defining partial sums $S_n^{V^s} = \sum_{k=1}^n V_k^s$, $n \in \mathbb{N}$, we can write for every $u > 0$ (the integral vanishes for $u > ny$),

$$
\begin{aligned}
I_0 &= P(S_n^U > u, \xi = 0) \\
&= \int_u^\infty \mathrm{d}P(S_n^U \leqslant v, \xi = 0) \\
&= \int_u^\infty \mathrm{e}^{-sv}\mathrm{e}^{sv}\,\mathrm{d}P(S_n^U \leqslant v \mid \xi = 0)P(\xi = 0) \\
&= [\tilde{f}(s)]^n \int_u^\infty \mathrm{e}^{-sv}\,\mathrm{d}P(S_n^{V^s} \leqslant v).
\end{aligned}
$$

The exponential function is monotone, so

$$
I_0 \leqslant [\tilde{f}(s)]^n \mathrm{e}^{-su} P\left(\sum_{k=1}^n V_k^s > u\right), \quad u > 0. \tag{4.11}
$$

To bound $\tilde{f}(s)$, recall that the r.v. U has zero mean and finite second moment, and $P(U \leqslant u) = F(u+\mu)$. We have (for $sy > 1$ without loss of generality)

$$
\tilde{f}(s) = \int_{-\infty}^y \mathrm{e}^{su}\,\mathrm{d}P(U \leqslant u) = \left(\int_{-\infty}^{1/s} + \int_{1/s}^y\right)\mathrm{e}^{sv}\,\mathrm{d}F(v+\mu) =: J_0 + J_1,
$$

where by partial integration and the first inequality of Lemma 4.3(a), for $1/s > -\mu$,

$$
\begin{aligned}
0 < J_1 &= \int_{1/s}^y \mathrm{e}^{su}\,\mathrm{d}F(u+\mu) = -\int_{1/s}^y \mathrm{e}^{su}\mathrm{d}\bar{F}(u+\mu) \\
&= \mathrm{e}\bar{F}(1/s+\mu) - \mathrm{e}^{sy}\bar{F}(y+\mu) + s\int_{1/s}^y \exp(sv - Q_F(v+\mu))\,\mathrm{d}v \\
&\leqslant s\int_{1/s}^y \exp(sv - Q_B(v))\,\mathrm{d}v,
\end{aligned}
$$

with $Q_B = Q$. The last inequality holds because $\mathrm{e}^{su}\bar{F}(u+\mu)$ increases on $u \geqslant 1/s = 1/s(t) = t/Q(t)$ and $t \geqslant t_\varepsilon$.

A second-order Taylor expansion, together with Condition B(ii), which guarantees the existence of a second moment, gives $J_0 = 1 + \mathrm{O}(1)s^2$.

For J_1, recall that $J_1 \leqslant s\int_{1/s}^y \exp(su - Q(u))\,\mathrm{d}u$. Since $su > 1$ for $1/s < u < y$, we have

$$
\begin{aligned}
J_1 &\leqslant s^3 \int_{1/s}^y u^2 \exp(su - Q(u))\,\mathrm{d}u = s^3\int_{1/s}^y \exp(su - [Q(u) - 2\log u])\,\mathrm{d}u, \\
&\leqslant s^2 s_1 \int_{1/s_1}^y \exp(s_1 u - Q_1(u))\,\mathrm{d}u \quad \text{if } s_1 \geqslant s,
\end{aligned} \tag{4.12}
$$

where for $t \geqslant 1$ we define

$$
Q_1(t) = Q(t) - 2\log t, \quad s_1 = s_1(y) = \frac{Q_1(y)}{y}, \quad q_1(t) = \frac{\mathrm{d}}{\mathrm{d}t}Q_1(t) = q(t) - \frac{2}{t}.
$$

Note immediately from the first inequality at (4.9) and the definition of y that

$$s_1 - s = s_1(y) - s(t) = \frac{Q(y)}{y} - \frac{Q(t)}{t} - \frac{2\log y}{y} \geqslant 0,$$

as required for the second inequality at (4.12). By Condition B(ii), where $\beta > 2$, $q_1(t)$ is ultimately positive, and therefore for sufficiently large t, it can be regarded as the hazard rate of a hazard function. In particular, if $\limsup_{t\to\infty} tq_1(t)/Q_1(t) < 1$, then the uniform boundedness property of Lemma 4.4 holds for the last integral at (4.12), implying that

$$J_1 = \mathrm{O}(1)s^2. \tag{4.13}$$

Now

$$\frac{tq_1(t)}{Q_1(t)} = \frac{tq(t) - 2}{Q(t) - 2\log t} = \frac{[tq(t)/Q(t)] - 2/Q(t)}{1 - 2/[Q(t)/\log t]}$$

for which for $t \to \infty$, lim sup of the numerator equals r and lim inf of the denominator equals $1 - 2/\kappa$, so for (4.13) to hold it suffices that $\kappa > 2/(1-r)$, which is true under Condition B(ii) by Lemma 3.6(b).

Returning to (4.11), it now follows under our assumptions that for some $c^* > 0$

$$I_0 \leqslant \exp(c^* ns^2)\exp(-su)P\left(\sum_{k=1}^{n} V_k^s \geqslant u\right), \quad u > 0. \tag{4.14}$$

We have

$$E[(V^s)^2] \leqslant \frac{1}{\tilde{f}(s)}\left(\int_{-\infty}^{1/s} u^2 \mathrm{e}^{su}\,\mathrm{d}F(u) + \int_{1/s}^{y} u^2\mathrm{e}^{su}\,\mathrm{d}F(u)\right).$$

Since F has finite second moment,

$$\int_{-\infty}^{1/s} u^2\mathrm{e}^{su}\,\mathrm{d}F(u) < \infty.$$

Next, we estimate

$$\int_{1/s}^{y} u^2\mathrm{e}^{su}\,\mathrm{d}F(u) \leqslant s^{-2}\bar{F}(1/s) + s\int_{1/s}^{y} u^2\mathrm{e}^{su}\bar{F}(u)\,\mathrm{d}u + 2\int_{1/s}^{y} u\mathrm{e}^{su}\bar{F}(u)\,\mathrm{d}u$$

$$\leqslant s^{-2}\bar{F}(1/s) + s\int_{1/s}^{y} u^2\mathrm{e}^{su}\bar{F}(u)\,\mathrm{d}u + 2s\int_{1/s}^{y} u^2\mathrm{e}^{su}\bar{F}(u)\,\mathrm{d}u.$$

Using (4.12) and (4.13), we obtain

$$\int_{1/s}^{y} u^2\mathrm{e}^{su}\,\mathrm{d}F(u) < \infty.$$

Hence, $E[(V^s)^2] < \infty$.

Observe that

$$P\left(\sum_{k=1}^{n} V_k^s > t\right) \leqslant (n/t^2)E[(V^s)^2] = \mathrm{O}(1)n/t^2.$$

Moreover, by Lemma 4.3(a), $e^{-st} = \exp(-Q(t)) = \bar{F}(t) \leqslant \bar{B}(t)$. Hence, for t satisfying condition B(iii) so that $ns^2 \to 0$ for $t \geqslant t_n$ and $n \to \infty$,

$$I_0 = O(1)\exp(c^* ns^2)\exp(-st)n/t^2 = O(1)\exp(c^* ns^2)\bar{B}(t)n/t^2 = o(n\bar{B}(t)).$$

Finally, we estimate I_1. For $y \geqslant 1/s$ and arbitrary $\varepsilon > 0$ we have

$$\begin{aligned} I_1 &= nP\left(S_n^U > t, U_n \geqslant y, \max_{k \leqslant n-1} U_k < y\right) \\ &= nP(S_n^U > t, t - \varepsilon/s(\tfrac{1}{2}t) > U_n \geqslant y, \max_{k \leqslant n-1} U_k < y) \\ &\quad + nP(S_n^U > t, U_n \geqslant t - \varepsilon/s(\tfrac{1}{2}t), \max_{k \leqslant n-1} U_k < y) \\ &=: I_{11} + I_{12} \quad \text{say.} \end{aligned}$$

First,

$$I_{11} \leqslant n \int_{1/s}^{t-\varepsilon/s(t/2)} P\left(S_{n-1}^U \geqslant t - u, \max_{k \leqslant n-1} U_k < y\right) \mathrm{d}F(u).$$

Using partial integration and (4.14) we obtain

$$\begin{aligned} I_{11} &\leqslant O(1)n \int_{1/s}^{t-\varepsilon/s(t/2)} P\left(\sum_{i=1}^{n-1} V_i^s > t - u\right) e^{-s(t-u)} \mathrm{d}F(u) \\ &\leqslant O(1)nP\left(\sum_{i=1}^{n-1} V_i^s \geqslant \frac{\varepsilon}{s(\frac{1}{2}t)}\right) e^{-st} \int_{1/s}^{t-\varepsilon/s(t/2)} e^{su}\,\mathrm{d}F(u) \\ &\leqslant O(1)nn\frac{s^2(\frac{1}{2}t)}{\varepsilon^2}\bar{B}(t)E[(V^s)^2] \\ &= O(1)n^2s^2(t)\bar{B}(t) = o(n\bar{B}(t)) \end{aligned}$$

by Condition B(iii) (see Lemma 3.8(c)).

To estimate I_{12}, start by noting that I_{12} is bounded by $nP(X_1 \geqslant t - \varepsilon/s(\frac{1}{2}t))$. From our assumptions we obtain for ε small enough

$$P(X_1 \geqslant t - \varepsilon/s(\tfrac{1}{2}t)) = \bar{B}(t)\exp\left(\int_{t-\varepsilon/s(t/2)}^{t} q(u)\,\mathrm{d}u\right) \sim \bar{B}(t), \quad t \to \infty.$$

On the other hand, we have

$$\begin{aligned} I_{12} &= nP(S_n^U \geqslant t, U_n \geqslant t - \varepsilon/s(\tfrac{1}{2}t), \max_{k \leqslant n-1} U_k < y) \\ &= n\int_{-\infty}^{\infty} P(U_n \geqslant \max(t-u, t-\varepsilon/s(\tfrac{1}{2}t)))\,\mathrm{d}P\left(S_{n-1}^U \leqslant u, \max_{k \leqslant n-1} U_k < y\right) \end{aligned}$$

$$\geqslant n \int_{-\varepsilon/s(t/2)}^{\infty} P(U_n \geqslant \max(t-u, t-\varepsilon/s(\tfrac{1}{2}t)))\, \mathrm{d}P\left(S_{n-1}^U < u, \max_{k \leqslant n-1} U_k < y\right)$$

$$\geqslant nP(U_n \geqslant t + \varepsilon/s(\tfrac{1}{2}t))P\left(\max_{k \leqslant n-1} U_k < y\right)$$

$$- nP(U_n \geqslant t)P\left(S_{n-1}^U \leqslant -\varepsilon/s(\tfrac{1}{2}t), \max_{k \leqslant n-1} U_k < y\right)$$

$$=: I'_{12} - I''_{12} \quad \text{say}.$$

We first estimate

$$I''_{12} \leqslant n\bar{B}(t)P\left(|S_{n-1}^U| \geqslant \varepsilon/s(\tfrac{1}{2}t), \max_{k \leqslant n-1} U_k < y\right)$$

$$\leqslant n\bar{B}(t)E[S_{n-1}^U]^2 s^2(\tfrac{1}{2}t)/\varepsilon^2$$

$$= \mathrm{O}(1)n\bar{B}(t)ns^2(\tfrac{1}{2}t)$$

$$= \mathrm{o}(n\bar{B}(t)).$$

To estimate I'_{12} note that we have for each fixed y, since the U_k are unbounded to the right,

$$1 \geqslant P\left(\max_{k \leqslant n-1} U_k \leqslant y\right) = \exp((n-1)\log(1-\bar{F}(y)))$$

$$\geqslant \exp(-2n\bar{F}(y)) \to 1, \quad n \to \infty,$$

since $y \geqslant \delta t$ and $n\bar{F}(y) \leqslant n\bar{F}(\delta t) \sim n\bar{B}(\delta t) = \mathrm{O}(nt^{-\kappa}) \to 0$ as in the estimate of I_2. Using Lemma 4.3 we have that

$$I'_{12} \sim n\bar{B}(t + \varepsilon/s(\tfrac{1}{2}t)) \sim n\bar{B}(t), \quad t \to \infty.$$

This proves Theorem 4.1. □

Corollary 4.6. *Assume that Conditions* B *hold. Then*

$$P(S_n > 0)/n \sim P(X_1 > |\mu|n), \quad n \to \infty. \tag{4.15}$$

Proof. By (4.4) we have

$$P(S_n > 0) = P(S_n^U > |\mu|n) \sim n\bar{B}(|\mu|n), \quad n \to \infty.$$

Set $t = |\mu|n$. Then by Condition B(i), we have $Q(|\mu|n)/\sqrt{|\mu|n} \to 0$ and hence, $\{t_n\} = \{|\mu|n\}$ satisfies B(iii) of Theorem 4.1. □

5. Proof of main results

We first describe the asymptotic behaviour of the tail $P(\tilde{N} \geqslant n)$ of the number of customers served in a busy period as $n \to \infty$ (the next result is an analogue of

Theorem 3.2 in Baltrūnas (2001) with the same proof). This enables us to describe the asymptotic behaviour of the tail $P(\tilde{T} > t)$ of the length of a busy period.

Theorem 5.1. *Assume that Conditions* B *hold. Then*

$$P(\tilde{N} > n) \sim e^{B'} P(S_n > 0)/n \sim e^{B'} P(X > |\mu| n), \quad n \to \infty. \tag{5.1}$$

Proof. By Remark 4.2(b) the sequence $\{s_n\} = \{P(S_n > 0)/n\}_{n\in\mathbb{N}}$ belongs to the class $\mathscr{S}_D^*$. Apply Proposition 3.10 with the function $\Psi(w) = e^w$: since $B' = \sum_n s_n < \infty$ and $\{s_n\} \in \mathscr{S}_D^*$, the first relation at (5.1) follows. Corollary 4.6 implies the second relation at (5.1). □

In using asymptotic properties of the renewal process it is convenient to rephrase Condition A in a form that refers not to partial sums but to the counting function $v^Y(t)$. Elementary manipulation shows that Condition A for non-negative i.i.d. r.v.s $\{Y_i\}$ with finite first two moments can be expressed, equivalently, in terms of the counting process $v^Y(t)$ based on first passage times of the partial sums S_n^Y; the idea of the proof is presented in the proof of Lemma 2.2.

Condition A′. *For any monotone function $\gamma(t)$ for which $\gamma(t)/\sqrt{t} \to \infty$ and $\gamma(t)/t \to 0$ as $t \to \infty$, the counting function $v^Y(t)$ of a renewal process whose lifetimes have finite first and second moments satisfies*

$$P(|v^Y(t) - \lambda t| > \gamma(t)) \leqslant \exp(-\tilde{c}_A[\gamma(t)]^2/t) \tag{5.2}$$

for some finite constant $\tilde{c}_A$.

Finally, we prove our main result.

Proof of Theorem 1.2. Note first by (2.3), Theorem 5.1 and using $B \in \mathscr{L}$, that

$$P(\tilde{T} \geqslant S_n^Y) = P(\tilde{N} \geqslant n+1) \sim e^{B'} \bar{B}(|\mu| n), \quad n \to \infty, \tag{5.3}$$

equivalently,

$$z_n := \frac{P(\tilde{T} \geqslant S_n^Y)}{e^{B'} P(X > |\mu| n)} \to 1, \quad n \to \infty. \tag{5.4}$$

Also, recall from the definition of $v^Y(t)$ at (2.1) that $S^Y_{v^Y(t)} \leqslant t < S^Y_{v^Y(t)+1}$ a.s., so that

$$P(\tilde{T} \geqslant S^Y_{v^Y(t)+1}) \leqslant P(\tilde{T} \geqslant t) \leqslant P(\tilde{T} \geqslant S^Y_{v^Y(t)}). \tag{5.5}$$

Since $v^Y(t)/(\lambda t) \to 1$ a.s. for $t \to \infty$, Anscombe's theorem (see e.g. Embrechts et al., 1997, Lemma 2.5.8) can be applied to the random sequence $\{z_{v^Y(t)}\}$ to conclude that $z_{v^Y(t)} \to 1$ a.s. or, equivalently,

$$P(\tilde{T} > S^Y_{v^Y(t)}) \sim e^{B'} P(X > |\mu| v^Y(t)), \quad t \to \infty. \tag{5.6}$$

Since $B \in \mathscr{S}$, applying Anscombe's theorem again implies that $P(\tilde{T} > S_{v^Y(t)}) \sim P(\tilde{T} > S_{v^Y(t)+1})$. Using this property with the sandwich relation (5.5) together with (5.6) now yields

$$P(\tilde{T} > t) \sim \mathrm{e}^{B'} P(X > |\mu| v^Y(t)) = \mathrm{e}^{B'} \sum_{j=0}^{\infty} \Delta(j) P(v^Y(t) = j), \tag{5.7}$$

where $\Delta(j) = \bar{B}(|\mu| j)$, the last identity coming from the independence of X and $v^Y(t)$.

In the right-hand side of (5.7), $v^Y(t)$ satisfies a central limit theorem, whereas under Conditions B the d.f. B is heavy-tailed, so it is plausible and also true that the weighted sum should be approximately like $\Delta(E[v^Y(t)])$. The usual approach is to partition the range of summation into three parts $[0, \chi_0 - 1]$, $[\chi_0, \chi_1 - 1]$ and $[\chi_1, \infty]$ for suitably chosen integers χ_0 and χ_1, i.e. to write

$$\begin{aligned}[1 + \mathrm{o}(1)]\mathrm{e}^{-B'} P(\tilde{T} > t) &= \left(\sum_{j=0}^{\chi_0 - 1} + \sum_{j=\chi_0}^{\chi_1 - 1} + \sum_{j=\chi_1}^{\infty} \right) \Delta(j) P(v^Y(t) = j) \\ &=: D_0 + D_1 + D_2\end{aligned} \tag{5.8}$$

with the property that almost all mass of the distribution of $v^Y(t)$ is concentrated on the second of these intervals and hope that D_0 and D_2 are negligeably small.

Consider first the choice $\chi_0 = \lfloor \lambda(t - t^{(1/2)+\delta}) \rfloor$, $\chi_1 - 1 = \lceil \lambda(t + t^{(1/2)+\delta}) \rceil$ for some positive δ to be determined, so for the ratio of the largest element $\Delta(j)$ in D_1 to $\Delta(\lfloor \lambda t \rfloor)$ we have

$$\begin{aligned}1 \leqslant \frac{\bar{B}(|\mu| \lambda(t - t^{(1/2)+\delta}))}{\bar{B}(|\mu| \lambda t)} &\sim \exp(Q((1-\rho)t) - Q((1-\rho)(t - t^{(1/2)+\delta}))) \\ &\leqslant \exp(K t^{r_\varepsilon}[1 - (1 - t^{-(1/2)+\delta})^{r_\varepsilon}]) \to 1, \quad t \to \infty,\end{aligned}$$

where we have used Proposition 3.7(a) twice, and the limit holds provided that $r_\varepsilon - \frac{1}{2} + \delta < 0$; similarly, under the same condition,

$$1 \geqslant \frac{\bar{B}(|\mu| \lambda(t + t^{(1/2)+\delta}))}{\bar{B}(|\mu| \lambda t)} > \exp(\mathrm{o}(t^{r_\varepsilon})[1 - (1 + c t^{-(1/2)+\delta})^{-r_\varepsilon}]) \to 1.$$

Moreover,

$$P(|v^Y(t) - \lambda t| \leqslant t^{(1/2)+\delta}) > 1 - \exp(-\tilde{c}_A t^{2\delta}) \to 1$$

by Condition A′, so since $\Delta(j)$ is decreasing in j, we have immediately that

$$D_1 \sim \bar{B}((1-\rho)t) \quad \text{and} \quad D_2 = \mathrm{o}(\bar{B}((1-\rho)t)). \tag{5.9}$$

We also have $D_0 = \mathrm{o}(\bar{B}((1-\rho)t))$ if $\exp(Q((1-\rho)t) - \tilde{c}_A t^{2\delta}) \to 0$ for $t \to \infty$; this certainly holds, again by Proposition 3.7(a), if $2\delta > r_\varepsilon$. Such δ exists consistent also with $\delta < \frac{1}{2} - r_\varepsilon$ only if $r_\varepsilon < \frac{1}{3}$, but as the following argument shows, by using a finer partitioning of the interval $[0, \chi_0 - 1]$ it is enough to have $\delta < \frac{1}{2} - r_\varepsilon$ and $r < \frac{1}{2}$ (so that there is positive ε such that $r_\varepsilon < \frac{1}{2}$).

Suppose $r < \frac{1}{2}$ and define $\sigma_i = \frac{1}{2} + (\frac{1}{2})^{i+1}$, $i \geqslant 1$, noting that there is a smallest finite integer i_r such that $\sigma_{i_r} < 1 - r$. Define integers $n_0 = 0$ and $n_i = \lfloor \lambda(t - t^{\sigma_i}) \rfloor$ for $i = 1, \ldots, i_r$

so that $\delta := \sigma_{i_r} - \frac{1}{2} < \frac{1}{2} - r$ satisfies the earlier constraint that ensures that both (5.9) holds and $\bar{B}((1-\rho)(t - t^{(1/2)+\delta})) \sim \bar{B}((1-\rho)t)$. Identify $\chi_0 = n_{i_r}$, and for $i = 1, \ldots, i_r$ define

$$D_{0i} = \sum_{j=n_{i-1}}^{n_i - 1} \Delta(j) P(v^Y(t) = j) \leqslant \Delta(\lambda t) = \frac{\Delta(n_{i-1})}{\Delta(\lambda t)} P(v^Y(t) < n_i).$$

Then using Proposition 3.7(a) and Condition A′ gives

$$D_{0i} < \Delta(\lambda t) \exp(K t^{r_\varepsilon} t^{\sigma_{i-1}-1} - \tilde{c}_A t^{2\sigma_i - 1}) = \Delta(\lambda t) \mathrm{o}(1)$$

when $r_\varepsilon + \sigma_{i-1} - 1 = r_\varepsilon - \frac{1}{2} + (\frac{1}{2})^i < 2\sigma_i - 1 = (\frac{1}{2})^i$, and this inequality holds by assumption. Thus, $D_0 \leqslant \sum_{i=1}^{i_r} D_{0i} = \mathrm{o}(\Delta(\lambda t))\mathrm{o}(\bar{B}((1-\rho)t))$. $\square$

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References

Abate, J., Whitt, W., 1997. Asymptotics for M/GI/1 low-priority waiting-time tail probabilities. Queueing Systems 25, 173–233.

Asmussen, S., 2003. Applied Probability and Queues, 2nd Edition. Springer, New York.

Asmussen, S., Teugels, J.L., 1996. Convergence rates for M/GI/1 queues and ruin problems with heavy tails. J. Appl. Probab. 33, 1181–1190.

Asmussen, S., Klüppelberg, C., Sigman, K., 1999. Sampling at sub-exponential times, with queueing applications. Stochastic Process Appl. 79, 265–286.

Baltrūnas, A., 1995. On the asymptotics of one-sided large deviation probabilities. Lithuanian Math. J. 35, 11–17.

Baltrūnas, A., 2001. Some asymptotic results for transient random walks with applications to insurance risk. J. Appl. Probab. 38, 108–121.

Baltrūnas, A., 2002. Second order tail behaviour of the busy period distribution of a GI/GI/1 queue with subexponential service times, submitted for publication.

Bingham, N.H., Goldie, C.M., Teugels, J.L., 1987. Regular Variation. Cambridge University Press, Cambridge.

Chover, J., Ney, P., Wainger, S., 1973. Functions of probability measures. J. Analyse Math. 26, 255–302.

Cline, D.B.H., 1986. Convolution tails, product tails and domains of attraction. Probab. Theory Related Fields 72, 529–557.

De Meyer, A., Teugels, J.L., 1980. On the asymptotic behaviour of the distributions of the busy period and service time in M/GI/1. J. Appl. Probab. 17, 802–813.

Embrechts, P., Hawkes, J., 1982. A limit theorem for the tails of discrete infinitely divisible laws with application to fluctuation theory. J. Aust. Math. Soc. (Ser. A) 32, 412–422.

Embrechts, P., Klüppelberg, C., Mikosch, T., 1997. Modelling Extremal Events for Insurance and Finance. Springer, Berlin.

Feller, W., 1971. An Introduction to Probability Theory and its Applications, Vol. 2, 2nd Edition. Wiley, New York.

Foss, S., Korshunov, D., 2000. Sampling at a random time with a heavy-tailed distribution. Markov Process. Related Fields 6, 543–568.

Goldie, C.M., Klüppelberg, C., 1998. Subexponential distributions. In: Adler, R., Feldman, R., Taqqu, M.S. (Eds.), A Practical Guide to Heavy Tails: Statistical Techniques for Analysing Heavy Tailed Distributions. Birkhäuser, Boston, MA, pp. 435–459.

Jelenković, P., Momčilović, P., 2004. Large deviations of square root insensitive random sums. Math. Oper. Res., in press.

Klüppelberg, C., 1987. Subexponentielle Verteilungen und Charakterisierungen verwandter Klassen. Ph.D. Thesis. Universität Mannheim.

Klüppelberg, C., 1988. Subexponential distributions and integrated tails. J. Appl. Probab. 25, 132–141.

Klüppelberg, C., 1989. Estimation of ruin probabilities by means of hazard rates. Insurance: Math. Econom. 8, 279–285.

Nagaev, A.V., 1977. On a property of sums of independent random variables. Teor. Veroyat. Prim. 22, 335–346 (in Russian). (Transl. = Theory Probab. Appl. 22, 326–338).

Pinelis, I.F., 1985. Asymptotic equivalence of the probabilities of large deviations for sums and maxima of independent random variables. Limit theorems of the probability theory. Trudy Inst. Mat. 5, 144–173 (in Russian).

Prabhu, N.U., 1965. Queues and Inventories. Wiley, New York.

Royden, H.L., 1968. Real Analysis, 2nd Edition. Macmillan, New York.

Zwart, A.P., 2001. Tail asymptotics for the busy period in the GI/GI/1 queue. Math. Oper. Res. 26, 485–493.