FRAMES AND THE FEICHTINGER CONJECTURE

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ABSTRACT. We show that the conjectured generalization of the Bourgain-Tzafriri *restricted-invertibility theorem* is equivalent to the conjecture of Feichtinger, stating that every bounded frame can be written as a finite union of Riesz basic sequences. We prove that any bounded frame can at least be written as a finite union of linear independent sequences. We further show that the two conjectures are implied by the *paving conjecture*. Finally, we show that Weyl-Heisenberg frames over rational lattices are finite unions of Riesz basic sequences.

1. INTRODUCTION

The purpose of this paper is to relate a large number of conjectures appearing in different branches of analysis. We state the exact conjectures later in this Section, but in order to proceed we need some definitions.

A *frame* for a Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}$ so that there are constants $A, B > 0$ satisfying:

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. If we can choose $A = B$ we say that $\{f_i\}_{i\in I}$ is a *B-tight frame*. If at least the upper frame condition is satisfied we call $\{f_i\}_{i\in I}$ a *Bessel sequence*, with *Bessel constant B*. For any Bessel sequence $\{f_i\}_{i\in I}$ it is immediate that

$$\sup_{i\in I} \|f_i\| < \infty.$$

A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is *bounded* if

$$0 < \inf_{i\in I} \|f_i\| \le \sup_{i\in I} \|f_i\| < \infty.$$

A bounded unconditional basis for $\mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$. It is known that $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_i\}_{i\in I}$ is complete in $\mathcal{H}$ and

The first author was supported by NSF DMS 0102686. The last author thanks PIMS for support.

there are constants $A, B > 0$ so that for all finite families of scalars $\{a_i\}_{i\in I'\subset I}$ we have

$$A\sum_{i\in I'}|a_i|^2 \le \|\sum_{i\in I'} a_i f_i\|^2 \le B\sum_{i\in I'}|a_i|^2.$$

In this case we call A a *lower Riesz basis bound* of $\{f_i\}_{i\in I}$ and B an *upper Riesz basis bound*. Riesz bases are special kinds of frames. More precisely, a sequence $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if it is a frame for $\mathcal{H}$ which fails to be a frame for $\mathcal{H}$ if any of its elements are removed. For a Riesz basis the Riesz basis bounds and the frame bounds coincide. If $\{f_i\}_{i\in I}$ is a Riesz basis for its closed linear span we call it a *Riesz basic sequence*. For the basic properties of frames, Bessel sequences, Riesz sequences and Riesz basic sequences we refer the reader to [Ca2, Ch, Y].

We now formulate the conjectures we will deal with in this paper. We begin with the original conjecture by Feichtinger:

Conjecture 1.1 (Feichtinger)**.** *Every bounded frame can be written as a finite union of Riesz basic sequences.*

Given $N \in \mathbb{N}$, let ℓ_2^N denote $\mathbb{C}^N$ equipped with ℓ_2-norm. We now state a conjecture concerning frames for ℓ_2^N, which we refer to as the *finite Feichtinger conjecture*.

Conjecture 1.2 (Finite Feichtinger Conjecture)**.** *For every $B, C > 0$ there is a natural number $M = M(B,C)$ and an $A = A(B,C) > 0$ so that whenever $\{f_i\}_{i\in I}$ is a frame for ℓ_2^N ($N \in \mathbb{N}$) with upper frame bound B and $\|f_i\| \ge C$ for all $i \in I$, then I can be partitioned into $\{I_j\}_{j=1}^M$ so that for each $1 \le j \le M$, $\{f_i\}_{i\in I_j}$ is a Riesz basic sequence with lower Riesz basis bound A and upper Riesz basis bound B.*

The corresponding conjectures for Bessel sequences are:

Conjecture 1.3. *Every bounded Bessel sequence can be written as a finite union of Riesz basic sequences.*

Conjecture 1.4. *For every $B > 0$ there exists a natural number $M = M(B)$ and an $A = A(B)$ so that every Bessel sequence $\{f_i\}_{i=1}^n$ with Bessel constant $B > 0$ and $\|f_i\| = 1$, for all $1 \le i \le n$, can be written as a union of M Riesz basic sequences each with lower Riesz basis bound A.*

Throughout this paper, $\{e_i\}$ will denote an orthonormal basis for whatever Hilbert space we are working in. In 1987, Bourgain and Tzafriri [BT1] proved the following fundamental result known as the *Restricted-Invertibility Theorem*:

Theorem 1.5 (Bourgain-Tzafriri)**.** *There is a universal constant $c > 0$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$ for $1 \le i \le n$,* *then there exists a subset* $\sigma \subset \{1, 2, \cdots, n\}$ *of cardinality* $|\sigma| \geq \frac{cn}{\|T\|^2}$ *so that*

$$\|\sum_{j\in\sigma} a_j T e_j\|^2 \geq c \sum_{j\in\sigma} |a_j|^2,$$

for all choices of scalars $\{a_j\}_{j\in\sigma}$*.*

Theorem 1.5 gave rise to the following conjecture which is still open today:

Conjecture 1.6. *For every* $B > 0$ *there is a natural number* $M = M(B)$ *and an* $A = A(B) > 0$ *so that if* $T : \ell_2^n \to \ell_2^n$ *is a linear operator for which* $\|Te_i\| = 1$ *for all* $1 \leq i \leq n$*, and* $\|T\| \leq \sqrt{B}$*, then there is a partition* $\{I_j\}_{j=1}^M$ *of* $\{1, 2, \cdots, n\}$ *so that for each* $1 \leq j \leq M$ *and all choices of scalars* $\{a_i\}_{i\in I_j}$ *we have:*

$$\|\sum_{i\in I_j} a_i T e_i\|^2 \geq A \sum_{i\in I_j} |a_i|^2.$$

We will show that Conjectures 1.1,1.2 and 1.6 are equivalent in the sense that all three have positive answers or all three have negative answers. We will also show that these conjectures are equivalent to the corresponding conjectures about Bessel sequences and that all of these are true if the well known *Paving Conjecture* holds. Given a subset I of the integers, we denote by P_I the orthogonal projection in ℓ_2 onto the subspace spanned by $\{e_i\}_{i\in I}$.

Conjecture 1.7 (The Paving Conjecture [KS])**.** *For any* $\varepsilon > 0$*, there is a constant* $M = M(\varepsilon)$ *such that for every integer* n *and every linear operator* S *on* ℓ_2^n *whose matrix with respect to* $\{e_i\}_{i=1}^n$ *has zero diagonal, one can find a partition* $\{\sigma_j\}_{j=1}^M$ *of* $\{1, \cdots, n\}$*, such that*

$$\|P_{\sigma_j} S P_{\sigma_j}\| \leq \varepsilon \|S\| \quad \textit{for all } j = 1, 2, \cdots, M.$$

The paving conjecture is known to be equivalent to the Kadison-Singer conjecture [KS] (See also [BT2] for a deep analysis of the paving conjecture). In an interesting paper [W], Weaver gives several reformulations of the Kadison-Singer conjecture and thus of the Paving conjecture. One of these, in terms of frames, is the following:

Conjecture 1.8. *There exists a universal constant* $B \geq 2$ *and a natural number* M *such that the following holds. Let* $\{f_i\}_{i=1}^N$ *be a* B*-tight frame for* ℓ_2^n *with* $\|f_i\| \leq 1$*, for all* $i = 1, 2, \cdots, N$*. Then there is a partition* $\{I_j\}_{j=1}^M$ *of* $\{1, 2, \cdots, N\}$ *such that for all* $1 \leq j \leq M$ *we have*

$$\sum_{i\in I_j} |\langle f, f_i\rangle|^2 \leq (B-1)\|f\|^2, \quad \textit{for all } f \in \ell_2^n.$$

It is possible that all these conjectures have negative answers in general. In this case it becomes important to know the strongest results available. In this direction we will show that conjectures 1.2, 1.4 and 1.6 are true - up to a logarithmic factor. We also show that these conjectures hold for $\{f_i\}_{i=1}^n$ if the off-diagonal elements of the Grammian matrix $(\langle f_i, f_j\rangle)_{i,j}$ satisfies for some $\gamma > 0$:

$$|\langle f_i, f_j\rangle| \leq \frac{1}{\log^{1+\gamma} n}, \quad \text{for all } i \neq j.$$

We also prove that any Bessel sequence can at least be decomposed into a finite union of linearly independent sets. Finally we consider frames with a special structure, namely, Weyl-Heisenberg frames, and give a sufficient condition for the decomposition into a finite number of Riesz basic sequences.

2. Equivalence of the conjectures

To simplify the proof of the main result of this section, we first prove an elementary proposition.

Proposition 2.1. *Fix a natural number M and assume for every natural number n we have a partition $\{I_i^n\}_{i=1}^M$ of $\{1, 2, \cdots, n\}$. Then there are natural numbers $\{n_1 < n_2 < \cdots\}$ so that if $j \in I_i^{n_j}$ for some $i \in \{1, \cdots, M\}$, then $j \in I_i^{n_k}$, for all $k \geq j$. Hence, if $I_i = \{j | j \in I_i^{n_j}\}$ then*

(1) $\{I_i\}_{i=1}^M$ is a partition of $\mathbb{N}$.

(2) If $I_i = \{j_1 < j_2 < \cdots\}$ then for every natural number k we have $\{j_1, j_2, \cdots, j_k\} \subset I_i^{n_k}$.

Proof: For each natural number n, 1 is in one of the sets $\{I_i^n\}_{i=1}^M$. Hence, there are natural numbers $n_1^1 < n_2^1 < n_3^1 < \cdots$ and an $1 \leq i \leq M$ so that $1 \in I_i^{n_j^1}$, for all $j \in \mathbb{N}$. Now, for every natural number n_j^1, 2 is in one of the sets $\{I_i^{n_j^1}\}_{i=1}^M$. Hence, there is a subsequence $\{n_j^2\}$ of $\{n_j^1\}$ and an $1 \leq i \leq M$ so that $2 \in I_i^{n_j^2}$, for all $j \in \mathbb{N}$. Continuing by induction, we get a subsequence $\{n_j^{\ell+1}\}_{j=1}^\infty$ of $\{n_j^\ell\}_{j=1}^\infty$ and an $1 \leq i \leq M$ so that $\ell + 1 \in I_i^{n_j^{\ell+1}}$, for all $j \in \mathbb{N}$. Letting $\{n_j\}_{j=1}^\infty$ be $\{n_j^j\}_{j=1}^\infty$ gives the conclusion of the proposition. $\square$

We can now state the main result of this section.

Theorem 2.2. *Conjectures 1.1, 1.2, 1.3, 1.4 and 1.6 are all equivalent in the sense that either all four of these conjectures have positive answers or all four have negative answers.*

Proof: Conjecture 1.3 $\Rightarrow$ Conjecture 1.1: This is obvious.

Conjecture 1.1 $\Rightarrow$ Conjecture 1.4: We will prove the contrapositive. So we assume that Conjecture 1.4 fails. Then there is a constant $B > 0$ so that for every $M \in \mathbb{N}$ and for every $A > 0$ there is an $n = n(M, A) \in \mathbb{N}$, a finite dimensional Hilbert space H and a Bessel sequence $\{f_i\}_{i=1}^n$ in H with Bessel constant B and $\|f_i\| = 1$, for all $1 \leq i \leq n$, and whenever we partition $\{1, 2, \cdots, n\}$ into sets $\{I_j\}_{j=1}^M$, then there exists some $1 \leq \ell \leq M$ and a set of scalars $\{a_i\}_{i\in I_\ell}$ with

$$\|\sum_{i\in I_\ell} a_i f_i\|^2 \le A \sum_{i\in I_\ell} |a_i|^2.$$

Now, for each $k \in \mathbb{N}$, we can choose a finite dimensional Hilbert space H_k and letting $M = k$ and $A = 1/k$ above we can choose $n_k = n(k, 1/k)$ and $\{f_i^k\}_{i=1}^{n_k}$ satisfying the above conditions. Let $H = (\sum \oplus H_k)_{\ell_2}$ and consider $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty}$ as elements of H. This family is now a Bessel sequence with Bessel constant B and $\|f_i^k\| = 1$, for all $1 \le i \le n_k, k \in \mathbb{N}$. Assume we can partition $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty}$ into M sets of Riesz basic sequences each with lower Riesz basis bound A. But, for all k with $k \ge M$ and $1/k \le A$, $\{f_i^k\}_{i=1}^{n_k}$ cannot be partitioned into M sets each with lower Riesz basis bound $\ge A$, and hence $\{f_i^k\}_{i=1,k=1}^{n_k\ ,\infty}$ cannot be partitioned this way. This shows that Conjecture 1.1 fails.

Conjecture 1.4 ⇒ Conjecture 1.2. Given $\{f_i\}$ as in Conjecture 1.2, the sequence $\{\frac{f_i}{\|f_i\|}\}$ is a Bessel sequence in ℓ_2^N with Bessel constant $\frac{B}{C^2}$. So Conjecture 1.2 follows from Conjecture 1.4, since every frame is automatically bounded from above.

Conjecture 1.2 ⇒ Conjecture 1.6. This is obvious.

Conjecture 1.6 ⇒ Conjecture 1.3. Let $\{f_i\}_{i=1}^\infty$ be a bounded Bessel sequence for an infinite dimensional Hilbert space H with Bessel constant B. Without loss of generality we may assume that $\|f_i\| = 1$, for all $1 \le i < \infty$. For each n, choose an n-dimensional Hilbert space H_n containing the span of $\{f_i\}_{i=1}^n$ and let $\{e_i^n\}_{i=1}^n$ be an orthonormal basis for H_n. Define $T_n : H_n \to H_n$ by $Te_i^n = f_i$. Then $\|T_n\| \le \sqrt{B}$ and so by assuming that Conjecture 1.6 has a positive answer, we can find a partition $\{I_j^n\}_{j=1}^M$, $M = M(B)$, of $\{1, 2, \cdots, n\}$ so that for every $1 \le j \le k$, $\{f_i\}_{i\in I_j^n}$ is a Riesz basic sequence with lower Riesz basis bound $A = A(B)$. By Proposition 2.1, we can partition $\mathbb{N}$ into sets $\{I_i\}_{i=1}^M$ so that if $I_i = \{j_1 < j_2 < \cdots\}$, then for every natural number k we have that $\{j_1, j_2, \cdots, j_k\} \subset I_i^{n_k}$. It follows that $\{f_{j_\ell}\}_{\ell=1}^k$ is a Riesz basic sequence with the same lower Riesz basis bound A for all $k \in \mathbb{N}$. Hence, $\{f_j\}_{j\in I_i}$ has lower Riesz basis bound A, for all $1 \le i \le M$. Also, B is an upper Riesz basis bound for all these sets. This shows that Conjecture 1.3 has a positive answer. □

3. The Paving Conjecture

Kadison and Singer raised the problem, which is still open, whether every pure state on $\mathbb{D}$, the C^*-algebra of the diagonal operators on ℓ_2, admits a unique extension to a (pure) state on $\mathcal{L}(\ell_2)$, the C^*-algebra of all bounded linear operators on ℓ_2. The problem of Kadison and Singer reduces to (and is equivalent to) the Paving Conjecture [KS] (see also [DS]).

Proposition 3.1. *The Paving Conjecture implies Conjecture 1.4.*

Proof. Let $\{f_i\}_{i=1}^n$ be a unit norm Bessel sequence in ℓ_2^n with Bessel constant B. Define the linear operator T on ℓ_2^n by setting $Te_i = f_i$ for all i. Then $\|T\| \le \sqrt{B}$. Consider the operator $S = T^*T - I$. Then the (i,j)-entry of the matrix of S is

$$\langle Se_i, e_j\rangle = \begin{cases} \langle f_i, f_j\rangle, & i \ne j, \\ 0, & i = j. \end{cases}$$

By the Paving Conjecture, for any $\varepsilon > 0$ there exists a number $M = M(\varepsilon)$ and a partition $\{\sigma_k\}_{k=1}^M$ of the set $\{1, \cdots, n\}$ such that

$$\|P_{\sigma_k} S P_{\sigma_k}\| \le \varepsilon \|S\| \quad \text{for all } x \in \ell_2^n \text{ and all } k.$$

Applying this with $\varepsilon = \frac{1}{2(B+1)}$, and noting that $\|S\| \le B + 1$, we obtain:

$$\|P_{\sigma_k} S P_{\sigma_k} x\| \le \frac{1}{2}\|x\| \quad \text{for all } k.$$

Now,

$$\begin{aligned} \langle P_{\sigma_k} S P_{\sigma_k} x, x\rangle &= \langle P_{\sigma_k}(T^*T - I)P_{\sigma_k} x, x\rangle \\ &= \langle (T^*T - I)P_{\sigma_k} x, P_{\sigma_k} x\rangle \\ &= \langle T^*TP_{\sigma_k} x, P_{\sigma_k} x\rangle - \langle P_{\sigma_k} x, P_{\sigma_k} x\rangle \\ &= \langle TP_{\sigma_k} x, TP_{\sigma_k} x\rangle - \|P_{\sigma_k} x\|^2 \\ &= \|TP_{\sigma_k} x\|^2 - \|P_{\sigma_k} x\|^2. \end{aligned}$$

Hence

$$\frac{\left|\|TP_{\sigma_k} x\|^2 - \|P_{\sigma_k} x\|^2\right|}{\|x\|} = \frac{|\langle P_{\sigma_k} S P_{\sigma_k} x, x\rangle|}{\|x\|} \le \|P_{\sigma_k} S P_{\sigma_k} x\| \le \frac{1}{2}\|x\|.$$

In particular,

$$\left|\|TP_{\sigma_k} x\|^2 - \|P_{\sigma_k} x\|^2\right| \le \frac{1}{2}\|P_{\sigma_k} x\|^2 \quad \text{for all } x \in \ell_2^n.$$

Thus,

$$\frac{1}{2}\|P_{\sigma_k} x\|^2 \le \|TP_{\sigma_k} x\|^2 \le \frac{3}{2}\|P_{\sigma_k} x\|^2 \quad \text{for all } x \in \ell_2^n.$$

By the definition of T, this implies that $\{f_i\}_{i\in\sigma_k}$ is a Riesz basic sequence with lower Riesz basis bound 1/2 and upper Riesz basis bound 3/2. The proposition is proved. □

The Paving Conjecture is known to be true for various classes of operators T on ℓ_2^n; see [BT2] for references. In particular, the Paving Conjecture is proved for the operators whose matrices have small entries, $O(1/\log^{1+\gamma} n)$ for some $\gamma > 0$.

Theorem 3.2 (Bourgain-Tzafriri)**.** *Let $\varepsilon > 0$ and S be a linear operator on ℓ_2^n whose matrix has zero diagonal and all entries are bounded by $1/\log^{1+\gamma} n$ for some $\gamma > 0$. Then S satisfies the conclusion of the Paving Conjecture: there exists a partition $\{\sigma_k\}_{k\le M}$ of the set $\{1, \cdots, n\}$, where $M = M(\gamma, \varepsilon)$, and such that*

$$\|P_{\sigma_k} S P_{\sigma_k}\| \le \varepsilon \|S\| \quad \textit{for all } k.$$

Actually, the partition $\{\sigma_k\}$ constructed by Bourgain and Tzafriri is random, i.e. σ_k is the image of the interval $\{1, \cdots, n/M\}$ under a random permutation π of the interval $\{1, \cdots, n\}$; for such a partition, the conclusion holds with probability close to one.

Theorem 3.2 implies the positive answer to Conjecture 1.4 for sequences which are in a certain sense "well separated". It is clear that similar statements hold for conjectures 1.2 and 1.6 as well.

Corollary 3.3. *Let $\{f_i\}_{i=1}^n$ be a Bessel sequence with Bessel constant $B > 0$ and with $\|f_i\| = 1$ for all i. Assume that*

$$|\langle f_i, f_j\rangle| \le \frac{1}{\log^{1+\gamma} n} \quad \textit{for all } i \ne j.$$

Then the sequence $\{f_i\}_{i=1}^n$ can be written as a union of $M = M(B, \gamma)$ Riesz basic sequences each with lower Riesz basis bound 1/2 and upper Riesz basis bound 3/2.

Proof. This follows from Theorem 3.2 with an argument analogous to that of Proposition 3.1. $\square$

4. Positive results

It is possible that all these conjectures have negative answers. In this case it will be of interest to know the strongest results available. We will look at some positive results now.

First we will show that bounded Bessel sequences can be decomposed into a finite union of linearly independent sets. For this, we need a result of Christensen and Lindner [CL].

Proposition 4.1. *Let $M \in \mathbb{N}$, I a finite subset of $\mathbb{N}$ and let $\{f_i\}_{i\in I}$ be a sequence of nonzero elements in a Hilbert space. The following are equivalent:*

(1) I can be partitioned into M disjoint sets $I_1, I_2, \cdots, I_M$ so that each family $\{f_i\}_{i\in I_j}$ $(j = 1, 2, \cdots, M)$ is linearly independent.

(2) For any nonempty subset $J \subset I$ we have

$$\frac{|J|}{\dim \operatorname{span}\{f_j\}_{j\in J}} \le M.$$

We now can show:

Theorem 4.2. *Every Bessel sequence $\{f_i\}_{i\in I}$ with Bessel bound B and $\|f_i\| \geq C > 0$, for every $i \in I$, can be decomposed into $\lceil B/C^2 \rceil$ linearly independent sets.*

Proof: We proceed by way of contradiction. Assume that $\{f_i\}_{i\in I}$ is a sequence, with Bessel bound B and $\|f_i\| \geq C > 0$, which cannot be decomposed into $\lceil B/C^2 \rceil$ linearly independent sets. By Proposition 2.1, with the same reasoning as used for the implication "Conjecture 1.6 $\Longrightarrow$ Conjecture 1.3", we can assume that I is finite. By Proposition 4.1 there is a finite subset $J \subset I$ so that

$$\frac{|J|}{\dim \operatorname{span}\{f_j\}_{j\in J}} > \left\lceil \frac{B}{C^2} \right\rceil .$$

Now, $\{f_j/\|f_j\|\}_{j\in J}$ is a frame for its span with upper frame bound $B_J \leq \lceil \frac{B}{C^2} \rceil$. Denote the corresponding frame operator by S, i.e.

$$S : \operatorname{span}\,\{f_j\}_{j\in J} \to \operatorname{span}\,\{f_j\}_{j\in J}, \quad f \mapsto \sum_{j\in J} \langle f, f_j\rangle f_j .$$

It is known that S has exactly $\dim \operatorname{span}\{f_j\}_{j\in J}$ eigenvalues (counted with multiplicity), that all these eigenvalues are positive and less than or equal to B_J, and that their sum equals $|J|$, see [Ch, Th. 1.2.1]. Thus it follows that the largest eigenvalue λ_{max} must satisfy:

$$\lambda_{max} \geq \frac{|J|}{\dim \operatorname{span}\{f_j\}_{j\in J}} > \left\lceil \frac{B}{C^2} \right\rceil .$$

But, $\lambda_{max} \leq B_J$ and so $B_J > \lceil \frac{B}{C^2} \rceil$, which is a contradiction. $\square$

Next, we will show that, up to a logarithmic factor, the generalized Bourgain-Tzafriri invertibility theorem (and hence the finite Feichtinger conjecture) is true. Namley, we can iterate Theorem 1.5 to obtain:

Proposition 4.3. *There is a universal constant $c > 0$ and a function $d = d(\|T\|)$ so that whenever $T : \ell_2^n \to \ell_2^n$ ($n \geq 2$) is a linear operator for which $\|Te_i\| = 1$, for $1 \leq i \leq n$, then there is a partition $\{I_j\}_{j=1}^{\lfloor d \log n \rfloor}$ of $\{1, 2, \cdots, n\}$ so that for each $1 \leq j \leq \lfloor d \log n \rfloor$ and all choices of scalars $\{a_i\}_{i\in I_j}$ we have:*

$$\| \sum_{i\in I_j} a_i T e_i \|^2 \geq c \sum_{i\in I_j} |a_i|^2. \tag{4.1}$$

Proof: Let $0 < c < 1$ be as in Theorem 1.5, $0 < b = \frac{c}{\|T\|^2} < 1$ and let $d > 0$ be such that $\lfloor d \log n \rfloor > \frac{-\log n}{\log(1-b)}$ for all $n \geq 2$. By Theorem 1.5 we can find a set $I_1 \subset \{1, 2, \cdots, n\}$ with $|I_1| \geq bn$ and satisfying inequality (4.1). Let $J_1 = \{1, 2, \cdots, n\} \setminus I_1$. Choose a (possibly into) isometry $U_1 : \operatorname{Rng} T_{|\ell_2^{J_1}} \to \operatorname{span}\,\{e_i\}_{i\in J_1}$. Then $U_1 T : \ell_2^{J_1} \to \ell_2^{J_1}$ satisfies Theorem 1.5, so there is a set

$I_2 \subset J_1$ with $|I_2| \geq b(n - |I_1|)$ and satisfying inequality (4.1). Continuing, there are disjoint sets $\{I_j\}_{j=1}^{\lfloor d \log n \rfloor}$ with

$$|I_j| \geq b(n - |I_1| - |I_2| - \cdots - |I_{j-1}|)$$

and each I_j satisfies inequality (4.1). Denoting $a_k := \sum_{j=1}^{k} |I_j|$, we have $a_k \geq bn + (1-b)a_{k-1}$ for any $k \geq 2$, and with $a_1 \geq bn$ this shows

$$\sum_{j=1}^{\lfloor d \log n \rfloor} |I_j| = a_{\lfloor d \log n \rfloor} \geq bn \sum_{j=0}^{\lfloor d \log n \rfloor - 1} (1-b)^j = n(1 - (1-b)^{\lfloor d \log n \rfloor}) > n - 1$$

by the definition of d. Hence $\sum_{j=1}^{\lfloor d \log n \rfloor} |I_j| = n$, completing the proof. □

We can obtain a slightly more general result, namely

Theorem 4.4. *There is a universal constant $c > 0$ and a $D = D(B)$ so that whenever $\{f_i\}_{i=1}^{k}$ is a frame for an n-dimensional Hilbert space $\mathcal{H}$ with $\|f_i\| = 1$ for all $1 \leq i \leq k$ and upper frame bound B, then there is a partition $\{I_j\}_{j=1}^{\lfloor D \log n \rfloor}$ of $\{1, 2, \cdots, k\}$ so that for each $1 \leq j \leq \lfloor D \log n \rfloor$, $\{f_i\}_{i \in I_j}$ is a Riesz basic sequence with lower Riesz basis bound c.*

Proof: By Theorem 4.2, $\{f_i\}_{i=1}^{k}$ can be decomposed into $\lceil B \rceil$ linearly independent sets, each of which has dimension at most n. In particular, $k \leq \lceil B \rceil n$. Then if d denotes the constant appearing in Proposition 4.3, it suffices to choose D such that

$$\lfloor d \log(\lceil B \rceil n) \rfloor = \lfloor d \log \lceil B \rceil + d \log n \rfloor \leq \lfloor D \log n \rfloor \quad \text{for all } n \geq 2.$$

□

Using Theorem 1.5, Casazza [Ca1] showed:

Theorem 4.5. *There is a function $g : \mathbb{R}^3 \to \mathbb{R}^+$ with the following property: Let $\{f_i\}_{i=1}^{k}$ be a frame for an n-dimensional Hilbert space H_n with frame bounds A, B, $\|f_i\| = 1$, for all $1 \leq i \leq k$, and let $0 < \varepsilon < 1$. Then there is a subset $\sigma \subset \{1, 2, \cdots, n\}$ with $|\sigma| \geq (1-\varepsilon)n$ so that $\{f_i\}_{i \in \sigma}$ is a Riesz basis for its span with lower Riesz basis constant $g(\varepsilon, A, B)$.*

Vershynin [V1], [V2] removed the assumption in Theorem 4.5 that the frame elements be bounded above and below and got the conclusion that there is a "large" subset of the frame which is an *unconditional basis* for its span.

5. Weyl-Heisenberg Frames

In this section we show that the Feichtinger conjecture is true for certain Weyl-Heisenberg frames. If $g \in L^2(\mathbb{R})$, $a, b > 0$ we define for all $m, n \in \mathbb{Z}$:

$$E_{mb}g(t) = e^{2\pi i m b t} g(t)$$

and

$$T_{na}g(t) = g(t - na).$$

If $\{E_{mb}T_{na}g\}_{n,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call it a *Weyl-Heisenberg* or *Gabor frame*. Our purpose is to show that whenever ab is rational, a Weyl-Heisenberg frame can be written as a finite union of Riesz basic sequences. In [G], Gröchenig shows that frames with a certain "localization property" can always be written as finite unions of Riesz basic sequences. This includes the case of Weyl-Heisenberg frames when g lies in a certain *modulation space*. The latter assumption is not required in our approach at a cost of having to work with rational lattices.

Theorem 5.1. *Let $g \in L^2(\mathbb{R})$ and $0 < ab < 1$ with ab rational. If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Weyl-Heisenberg frame for $L^2(\mathbb{R})$ then it can be written as a finite union of Riesz basic sequences.*

Proof: After a change of variables we may assume that $b = 1$ and $a = \frac{p}{q}$ with $p, q \in \mathbb{N}$. We first reduce the problem to the case of integer oversampling. Notice that

$$\begin{aligned}
\{E_m T_{\frac{1}{q}n} g\}_{m,n\in\mathbb{Z}} &= \cup_{k=0}^{p-1} \{E_m T_{\frac{1}{q}(np+k)} g\}_{m,n\in\mathbb{Z}} \\
&= \cup_{k=0}^{p-1} \{E_m T_{\frac{p}{q}n+\frac{k}{q}} g\}_{m,n\in\mathbb{Z}}.
\end{aligned}$$

Since each of the families $\{E_m T_{\frac{p}{q}n+\frac{k}{q}} g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence (and a frame for $k = 0$) we conclude that $\{E_m T_{\frac{1}{q}n} g\}_{m,n\in\mathbb{Z}}$ is a frame. In the rest of the proof we show that $\{E_m T_{\frac{1}{q}n} g\}_{m,n\in\mathbb{Z}}$ is a finite union of Riesz basic sequences. Since $\{E_m T_{\frac{p}{q}n} g\}_{m,n\in\mathbb{Z}} \subset \{E_m T_{\frac{1}{q}n} g\}_{m,n\in\mathbb{Z}}$ the conclusion of the theorem follows from here.

We use a result by Ron and Shen [RS] (see also [G, Th. 7.4.3]), stating that $\{E_{qm}T_n g\}_{m,n\in\mathbb{Z}}$ is a Riesz basic sequence. Now,

$$\begin{aligned}
\{E_m T_{\frac{n}{q}} g\}_{m,n\in\mathbb{Z}} &= \cup_{k=0}^{q-1} \{E_m T_{\frac{nq+k}{q}} g\}_{m,n\in\mathbb{Z}} \\
&= \cup_{k=0}^{q-1} \{E_m T_{n+\frac{k}{q}} g\}_{m,n\in\mathbb{Z}} \\
&= \cup_{k=0}^{q-1} \cup_{j=0}^{q-1} \{E_{qm+j} T_{n+\frac{k}{q}} g\}_{m,n\in\mathbb{Z}}.
\end{aligned}$$

By the commutator relations between the translation and modulation operators, $\{E_{qm+j} T_{n+\frac{k}{q}} g\}_{m,n\in\mathbb{Z}}$ is a Riesz basic sequence (with the same Riesz basis bounds as $\{E_{qm}T_n g\}_{m,n\in\mathbb{Z}}$) for all j, k, from which the result follows. $\square$

Remark 5.2. *The proof of Theorem 5.1 shows that a frame $\{E_m T_{\frac{p}{q}n} g\}_{m,n\in\mathbb{Z}}$ is a union of q^2 Riesz basic sequences.*

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