An Algorithm for Computing Invariants of Linear Actions of Algebraic Groups up to a given Degree

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Abstract

We propose an algorithm for computing invariant rings of algebraic groups which act linearly on affine space, provided that degree bounds for the generators are known. The groups need not be finite nor reductive, in particular, the algorithm does not use a Reynolds operator. If an invariant ring is not finitely generated the algorithm can be used to compute invariants up to a given degree.

Introduction

There are several efficient algorithms for computing invariant rings of finite matrix groups or of linear actions of reductive algebraic groups on affine space. For finite groups Gregor Kemper provided efficient algorithms, cf. [Kemper1996], [Kemper1997], and [Kemper and Steel1997], together with an implementation in Maple and the Magma system (cf. [Bosma et.al.1997]). Other approaches can be found, e.g., in [Bayer1998], [Decker et. al1997]), [Heydtmann1997], or [Sturmfels1993]. Invariant rings of reductive groups can be computed by Derksen's algorithm, cf. [Derksen1997]. In general, these approaches require the use of a Reynolds operator to obtain algebra generators.

We propose an algorithm which computes the invariant ring of an arbitrary algebraic group which acts linearly on affine space without using a Reynolds operator, provided that a degree bound for the generators is known and all variables of the coordinate ring have weight > 0. In particular, the algorithm can handle unipotent groups which play an important role in the construction of moduli spaces for singularities, cf. [GP1993]. For degree bounds for finite, respectively reductive algebraic groups we refer, e.g., to [Kemper1999] or [DK1995]. If the ring is not finitely generated, as might happen if the group is not reductive (cf. [Nagata1958]), the algorithm can be used to compute invariants up to a given degree.

1 Invariant Rings

Let $\mathbf{K}$ be a field and G be an algebraic group defined by the radical ideal $I_G \subseteq \mathbf{K}[s_1, s_2, \ldots, s_m]$. The algebraic group action of G on the affine space $\mathbf{K}^n$ is given on the ring level by

$$\Psi : \mathbf{K}[t_1, t_2, \ldots, t_n] \to \mathbf{K}[s_1, s_2, \ldots, s_m]/I_G \otimes \mathbf{K}[t_1, t_2, \ldots, t_n],$$

$$t_i \mapsto \psi_i(s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n)$$

where $\psi_1, \psi_2, \ldots, \psi_n \in \mathbf{K}[s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n]$. For $\sigma \in G$ and $\mathbf{t} \in \mathbf{K}^n$ the group action is given by $\sigma \cdot f(\mathbf{t}) := \Psi(f)(\sigma, \mathbf{t})$. We consider polynomials as functions by allowing them to take values in the algebraic closure of $\mathbf{K}$ if the field $\mathbf{K}$ is finite. A polynomial $f \in \mathbf{K}[t_1, t_2, \ldots, t_n]$ is **invariant** w.r.t. G if $\sigma \cdot f(t_1, t_2, \ldots, t_n) = f(t_1, t_2, \ldots, t_n)$ for all $\sigma \in G$. The **invariant ring** $\mathbf{K}[t_1, t_2, \ldots, t_n]^G$ of G is the subring of $\mathbf{K}[t_1, t_2, \ldots, t_n]$ containing all polynomials invariant under G. Note that the invariant ring $\mathbf{K}[t_1, t_2, \ldots, t_n]^G$ is isomorphic to

$$\begin{array}{ccc} \mathbf{K}[t_1, t_2, \ldots, t_n]^G & \xrightarrow{\simeq} & \mathbf{K}[\psi_1, \psi_2, \ldots, \psi_n] \cap \mathbf{K}[t_1, t_2, \ldots, t_n] \\ f & \mapsto & [f]\,, \end{array}$$

where the rings on the right hand side are considered to be subrings of $\mathbf{K}[s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n]/I_G$. We obtain generators for the invariant ring of G by computing generators for the intersection on the right hand side.

To test if a polynomial is invariant w.r.t. the action of G, given by $\psi_1, \psi_2, \ldots, \psi_n$, can be done as follows (cf. [Vasconcelos1998], Prop. 7.4.3). Note that I_G is a radical ideal.

Lemma 1 *A polynomial $f \in \mathbf{K}[t_1, t_2, \ldots, t_n]$ is invariant w.r.t. G iff*

$$f - \Psi(f) \in \langle I_G \rangle \subset \mathbf{K}[s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n].$$

Proof. For $f \in \mathbf{K}[t_1, t_2, \ldots, t_n]^G$ the polynomial $f - \Psi(f)$ vanishes on the variety $G \times \mathbf{K}^n$ by assumption. Hence $f - \Psi(f)$ is contained in the ideal of $G \times \mathbf{K}^n$ which is precisely the ideal I_G. Conversely, $f - \Psi(f) \in I_G$ implies that $f - \Psi(f)$ vanishes identically on $\{\sigma\} \times \mathbf{K}^n$ for every $\sigma \in G$, i.e., $f(t_1, t_2, \ldots, t_n) = \Psi(f)(\sigma, t_1, t_2, \ldots, t_n) = \sigma \cdot f(t_1, t_2, \ldots, t_n)$. Therefore f is invariant w.r.t. G. $\Box$

2 The Algorithm

In this section let $I_G \subset \mathbf{K}[s_1, s_2, \dots, s_m]$ be a radical ideal defining an algebraic group G. We make use of homogenization of polynomials and ideals w.r.t. a new variable X, which we denote by "^{h}" and refer, e.g., to [Vasconcelos1998] for computational properties. We assume that the polynomials $\psi_1, \psi_2, \dots, \psi_n \in \mathbf{K}[s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n]$ which define a linear action of G on $\mathbf{K}^n$, are homogeneous of the same degree. This can be achieved by homogenizing $\psi_1, \psi_2, \dots, \psi_n$ w.r.t. a new variable s (not X) and adding the equation $s - 1$ to I_G.

The algorithm is based on the following observation.

Proposition 1 *Let $\psi_1, \psi_2, \dots, \psi_n \in \mathbf{K}[s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n]$ be homogeneous polynomials of degree δ defining a linear action of the algebraic group G on $\mathbf{K}^n$ and let $I = \left\langle \langle \psi_1^{\alpha_1} \psi_2^{\alpha_2} \dots \psi_n^{\alpha_n} : |\alpha| = d \rangle \cup I_G^h \right\rangle \subset \mathbf{K}[s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n]$. If $GB = \{f_1, f_2, \dots, f_k\}$ is a Gröbner basis of the ideal $J = I \cap \mathbf{K}[t_1, t_2, \dots, t_n, X]$ we have, as $\mathbf{K}-$vectorspaces,*

$$\langle f_i(t_1, t_2, \dots, t_n, 1) : 1 \leq i \leq k, \deg(f_i) = d \cdot \delta \rangle_{\mathbf{K}} = \mathbf{K}[t_1, t_2, \dots, t_n]_d^G.$$

Proof. If $\deg(f_i) < d\delta$ then $f_i \in \left\langle I_G^h \right\rangle$ and therefore $f_i \notin I \cap \mathbf{K}[t_1, t_2, \dots, t_n, X]$, a contradiction. Hence $\deg(f_i) \geq d\delta$ for $1 \leq i \leq k$. Since GB is a Gröbner Basis and $\deg(f_i) \geq d\delta$ the $\mathbf{K}-$vectorspace $\langle f_i(t_1, t_2, \dots, t_n, 1) : \deg(f_i) = d\delta \rangle_{\mathbf{K}}$ is the dehomogenization of $J_{d\delta}$. Let $f \in \mathbf{K}[t_1, t_2, \dots, t_n]_d^G$ be a homogeneous invariant of degree d and note that $\Psi(f) \in I$. By Lemma 1 $f - \Psi(f) \in \langle I_G \rangle$ which implies $fX^{d \cdot (\delta - 1)} - \Psi(f) \in \left\langle I_G^h \right\rangle \subset I$, and therefore $fX^{d \cdot (\delta - 1)} \in I$. In particular, $fX^{d \cdot (\delta - 1)} \in J$ is of degree $d\delta$ as required.

Conversely, assume that f_1 is of minimal degree. If $\deg(f_1) > d\delta$ then $\deg_{(t_1, t_2, \dots, t_n)}(f_1) > d$ because the action is linear in $t_1, t_2, \dots, t_n$, so there are no invariants of degree d in this case. Now assume $\deg(f_1) = d\delta$ and note that the dehomogenization $f_1' = f_1(t_1, t_2, \dots, t_n, 1)$ is a homogeneous polynomial of degree d. The condition $f_1 \in I$ implies the existence of $p \in \mathbf{K}[t_1, t_2, \dots, t_n]$ and $g \in \left\langle I_G^h \right\rangle$ s.t. p is homogeneous of degree d and $f_1 = \Psi(p) + g$. Therefore $f_1 - \Psi(p) \in \left\langle I_G^h \right\rangle$ and $f_1(\mathbf{t}, 1) = f_1'(\mathbf{t}) = \Psi(p)(\sigma, \mathbf{t})$ for all $\sigma \in G$ and $\mathbf{t} \in \mathbf{K}^n$. In particular,

$$f_1'(\mathbf{t}) = \Psi(p)(id, \mathbf{t}) = id \cdot p(\mathbf{t}) = p(\mathbf{t})$$

so $f_1' = p$ and the claim follows from Lemma 1. $\square$

In the $j-$th iteration the algorithm computes a $\mathbf{K}-$basis $[f_{i_1}], [f_{i_2}], \dots, [f_{i_r}]$ of the degree d_j part of the intersection of $\mathbf{K}[\psi_1, \psi_2, \dots, \psi_n] \cap \mathbf{K}[t_1, t_2, \dots, t_n]$ as subrings of $\mathbf{K}[s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n]/I_G^h$, where $f_i \in \mathbf{K}[t_1, t_2, \dots, t_n]_{d_j}^G$ and d_j is some degree.

Algorithm 1 Invariants$(I_G, \langle \psi_1, \psi_2, \dots, \psi_n \rangle, degrees)$
In: *radical ideal $I_G \subset \mathbf{K}[s_1, s_2, \dots, s_m]$ of an algebraic group G and polynomials $\psi_1, \psi_2, \dots, \psi_n \in \mathbf{K}[s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n]$ defining a linear action of G on*

$\mathbf{K}^n$, *a list degrees of positive integers,* $<$ *a graded lex. order s.t.* $s_\rho > t_\tau > X$.
Out: $\mathbf{K}-$*vectorspace basis of* $\left\{f \in \mathbf{K}[t_1, t_2, \ldots, t_n]_d^G : d \in degrees\right\}$.
begin
$\{\psi_1', \psi_2', \ldots, \psi_n'\}$:= *homogenization of* $\{\psi_1, \psi_2, \ldots, \psi_n\}$ *w.r.t. 's' s.t.* $\deg(\psi_i) = \deg(\psi_j)$.
$\delta := \deg(\psi_1')$;
$I_G := I_G \cup \{s - 1\}$;
I_G^h := *homogenization of* I_G *w.r.t. new variable* X.
$B := \{\}$;
for $j := 1$ ***to*** $|degrees|$ ***do***
  $I := \left\langle \left\langle \psi_1'^{\alpha_1} \psi_2'^{\alpha_2} \ldots \psi_n'^{\alpha_n} : |\alpha| = degrees[j] \right\rangle \cup I_G^h \right\rangle$;
  $\{f_1, f_2, \ldots, f_k\} :=$*GröbnerBasis*$_<(I) \cap \mathbf{K}[t_1, t_2, \ldots, t_n, X]$;
  for $i := 1$ ***to*** k ***do***
    if $\deg(f_i) = degrees[j] \cdot \delta$ ***then*** $B := B \cup \{f_i(t_1, t_2, \ldots, t_n, 1)\}$;
  end;
end;
return(B);
end **Invariants**.

Remark 1 *(a) It suffices to compute the Gröbner Basis of* I *up to degree* $d\delta$.
(b) Since the algorithm computes a $\mathbf{K}-$*vectorspace basis of invariants of degree* d_j *in the* $j-$*th loop it can be used to compute primary invariants for finite groups as described in [Kemper1997] or [Decker et. al1997]. In particular, one should make use of the Hilbert series if it is known (e.g., by Molien's Theorem, cf. [Sturmfels1993]).*

The ideal operations in the algorithm are performed by Gröbner bases computations, cf. [Buchberger1985]. For the elimination of variables we refer, e.g., to [Vasconcelos1998].

Theorem 1 *The algorithm* **Invariants** *is correct.*

Proof. Fix a graded lex. order where $s_\rho > t_\tau > X$ and let $j > 0$. We show that any homogeneous invariant of degree $d = degrees[j]$ is contained in the linear span of $B^{(j)}$, where $B^{(j)}$ denotes the set B in the $j-$th iteration of the for-loop. Let $GB = \{f_1, f_2, \ldots, f_r\}$ be the Gröbner basis of $I \subset \mathbf{K}[s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n, X]$ in the $j-$th iteration s.t. $GB \cap \mathbf{K}[t_1, t_2, \ldots, t_n, X] = \{f_1, f_2, \ldots, f_k\} =: GB'$. By elimination theory (cf., e.g., [Vasconcelos1998], Proposition 2.1.1), GB' is a Gröbner basis of the ideal $I \cap \mathbf{K}[t_1, t_2, \ldots, t_n, X]$. By Proposition 1 we have

$$f \in \mathbf{K}[t_1, t_2, \ldots, t_n]_d^G \Longleftrightarrow f = \sum_{i=1}^{k} \lambda_i f_i(t_1, t_2, \ldots, t_n, 1)$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbf{K}$. Hence f is contained in the linear span of $B^{(j)}$. $\square$

In the two examples below we apply the algorithm to finite reductive/nonreductive and infinite reductive/nonreductive groups. The invariant rings of the first example can be computed, e.g., with Kemper's algorithms, or with the `SINGLUAR` library `finvar.lib`.

Example 1 *(a) Let* $\mathbf{K} = \mathbf{F_5}, \sigma_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$ *and* $G = \langle \sigma_1, \sigma_2 \rangle$*. Since* $|G| = 96$ *the group* G *is reductive. By use of the Hilbert series we only compute invariants of degree* 8 *and* 12 *and obtain*

$$\begin{aligned} h_1 &= t_1^8 - 2t_1^7t_2 - 2t_1^6t_2^2 + t_1^5t_2^3 - t_1^4t_2^4 + 2t_1^3t_2^5 + 2t_1^2t_2^6 - t_1t_2^7 + t_2^8, \\ h_2 &= t_1^{12} + 2t_1^{11}t_2 + t_1^{10}t_2^2 + 2t_1^8t_2^4 - 2t_1^7t_2^5 - t_1^5t_2^7 + 2t_1^4t_2^8 - t_1^2t_2^{10} + t_1t_2^{11} + t_2^{12}. \end{aligned}$$

Since h_1, h_2 *are algebraically independent we conclude, by using the Hilbert series, that* $\mathbf{K}[t_1, t_2]^G = \mathbf{K}[h_1, h_2]$.
(b) Let $\mathbf{K} = \mathbf{F_3}$ *and consider the linear action of*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2(\mathbf{K}).$$

The ideal I_G *of* G *is given by* $\langle s_1(s_1 - 1)(s_1 - 2) \rangle \subset \mathbf{K}[s_1]$ *and the action is defined by the two polynomials* $t_1 + s_1t_2, t_2$*. In the application of the algorithm the action is homogenized w.r.t.* s*, the equation* $s - 1$ *is added to* I_G *and* I_G *is homogenized w.r.t.* X*. We have* $I_G^h = \langle s_1(s_1 - 1)(s_1 - 2), s - X \rangle$ *and a the new action equals* $st_1 + s_1t_2, st_2$*. The algorithm computes the following fundamental invariants (degree bound* $= 3$*).*

$$t_2, t_2^2, t_2^3, t_1^3 - t_1t_2^2$$

Note that G *does not admit a Reynolds operator.*

We apply the algorithm to an infinite reductive and an infinite nonreductive group.

Example 2 *(a) The action of* $SL_2(\mathbf{C})$ *on* $V \oplus V \oplus S^2V$*, where* V *is the usual* $2-$*dimensional representation of* $SL_2(\mathbf{C})$ *(cf. Example 6.2 of [Derksen1997]). The ideal of* $SL_2(\mathbf{C})$ *equals* $\langle s_1s_4 - s_2s_3 - 1 \rangle \subset \mathbf{C}[s_1, s_2, s_3, s_4]$ *and the action is given by the representation*

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \mapsto \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \oplus \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \oplus \begin{pmatrix} s_1^2 & 2s_1s_2 & s_2^2 \\ s_1s_3 & s_1s_4 + s_2s_3 & s_2s_4 \\ s_3^2 & 2s_3s_4 & s_4^2 \end{pmatrix}$$

Derksen's algorithm delivers an ideal basis of the nullcone having degrees $2, 2, 3, 3, 3$ *and containing the polynomial* $t_1t_3t_7 - 2t_2t_3t_6 + t_2t_4t_5$ *which is not invariant.*

By applying **Invariants** *with upper bound* 3 *we obtain fundamental invariants*

$$-t_1t_4 + t_2t_3, -t_5t_7 + t_6^2, t_3^2t_7 - 2t_3t_4t_6 + t_4^2t_5,$$
$$t_1t_3t_7 - t_1t_4t_6 - t_2t_3t_6 + t_2t_4t_5, t_1^2t_7 - 2t_1t_2t_6 + t_2^2t_5$$

having degrees $2, 2, 3, 3, 3$.
(b) Consider the linear action of the nonreductive group

$$G = \left\{ \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix} : B_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix}, b_1 + 2b_2 + 3b_3 = 0 \right\} \subset GL_6(\mathbf{C})$$

on $\mathbf{C}^6$. *The ideal of* G *is* $I_G = \langle s_1 + 2s_2 + 3s_3 \rangle \subset \mathbf{C}[s_1, s_2, s_3]$ *and the action is given by* $\{\psi_1, \psi_2, \ldots, \psi_6\} = \{t_1 + s_1t_2, t_2, t_3 + s_2t_4, t_4, t_5 + s_3t_6, t_6\}$. *A variant of* **Invariants**, *where only those elements not contained in* $\mathbf{C}[B]$ *are added to* B, *yields the invariants* $t_2, t_4, t_6, 3t_2t_4t_5 + 2t_2t_3t_6 - t_1t_4t_6$ *of degree* ≤ 4.

3 Performance and Limitations

3.1 Performance

We provide running times of the algorithm in the computer algebra system `SINGULAR` (cf. [GPS1997]) on a Sun Ultra 60 (300MHz, 384 MB). The implementation and the examples can be found at the homepage of the author. The algorithm **Invariants** has been called with degree bounds, or with a list of degrees as mentioned in the examples. By * we denote that the Reynolds operator must be applied to the output and by '−' we denote that the algorithm cannot handle the current group. By **Derksen** and `finvar` we denote the running time of Derksen's algorithm and of the algorithm `invariant_ring()` from `finvar.lib`, cf. [Heydtmann1997] respectively.

Ex.	**Invariants**	**Derksen**	`finvar`
1(a)	0.46	9.59*	5.47
1(b)	0.06	−	0.19
2(a)	0.29	0.19*	−
2(b)	0.94	−	−

3.2 Limitations

Theoretical Limitations
The algorithm cannot handle nonlinear actions and variables of weight 0. The variables $t_1, t_2, \ldots, t_n$ must be of weight > 0 and the polynomials $\psi_1, \psi_2, \ldots, \psi_n$ defining the group action must be homogeneous of degree 1 in the variables

$t_1, t_2, \ldots, t_n$. E.g., the algorithm cannot handle the nonlinear $\mathbf{G}_a-$action on $\mathbf{K}^7$, given by

$$(\lambda, (t_1, t_2, \ldots, t_7)) \mapsto \left(t_1, t_2, t_3, \lambda \cdot t_1^2 + t_4, \lambda \cdot t_2^2 + t_5, \lambda \cdot t_3^2 + t_6, \lambda \cdot t_1^2 t_2^2 t_3^2 + t_7\right)$$

where the degree of t_1, t_2, t_3 equals 0, the degree of t_4, t_5, t_6, t_7 equals 1. Note that the invariant ring is not finitely generated (cf [A'Campo-Neuen1994]).

For $\mathbf{G}_a$-actions, as in the example above, there are algorithms for computing invariants up to a given degree, cf. [Maubach2000] or the `ainvar.lib` library of `SINGULAR`.

Practical Limitations

For invariant subrings of $\mathbf{K}[t_1, t_2, \ldots, t_n]$ where the degree of a (minimal) generator equals d and $\binom{n+d-1}{d-1} > 600$, the computation may become infeasible. (e.g., action of $S_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$ on $\mathbf{C}^4$, a minimal generator has degree 15, computation aborted after 2 weeks). The algorithm seems to be better suited for infinite groups.

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