The Geometry of Sequential Computation I: A Simplicial Geometry of Interaction

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1 General motivations and background

This is the first of a two-part study aimed at the giving account of a certain extensional fully-abstract model of higher-type functional sequential computation, generated from a geometrical representation of the normalization of linear sequent proofs. Our approach is based on the representation of linear proofs as certain kinds of concurrent processes — and eventually as a kind of higher-dimensional automata, modeled as Simplicial sets in an affine geometry. This representation draws significantly upon the ideas of Eric Goubault, Vaughan Pratt and others ([GJ92, Gou93, Pra91, vG91]), using geometric simplices for modeling higher-dimensional automata. The novelty in our approach is using this essentially concurrent structure to anchor a second-level sequential representation—that of the global *history* of the (development) of the proof-objects, encoded as a certain kind of homological data correlated with the simplicial structure. In the process-theoretic metaphor, these could be viewed as set of *sequential* traces of these automata, along with a minimal synchronization structure.

The representation gives us finer control over the normalization process, since global manipulations at the syntactic level are strictly localized to lie within a certain neighborhood of the unit interval in Euclidean space. Thus global operations available at the syntactic level are controlled so as not to obscure deeper operational phenomena, like sequentiality. This allows us to eventually project the relevant denotationally-equivalent sets of computations into a global topological space, preserving the property of sequentiality in at every step.

Thus the main principle behind our approach is the use of (local) geometrical structure of Simplicial sets in enriching the (global) topological space of denotations. In other words, we show that global topological continuity based upon local geometric structure is a good way to represent sequential computation — thus remaining faithful to the traditional approach epitomized in the works of Scott ([Sco76]), Plotkin ([KP93]) and Berry ([Ber78]). Additionally, our approach opens up a new way of looking at linear proofs as geometrically structured processes, along with the possibility of importing notions like equivalence and causal dependence from traditional process theory into the study of proofs.

2 Basic notions

We shall fix some basic terminology and notation. Let K be a countably infinite set of (occurrence) tags, and P a (finite) set of linear propositional atoms. The set of K-labeled occurrences of the atoms, $\mathcal{P}_K$, is given by $P \times K$. The set of all LL-formulae constructed over $\mathcal{P}_K$ shall be denoted as $\mathcal{F}_K$. The subset of $\mathcal{F}_K$ consisting of literals, *i.e.* occurrences of atoms and their negations is denoted as $\mathcal{L}_K$. For any number n, we denote by $[n]$ the set $\{0,\ldots,n\}$. The cardinality of a set M would be denoted by $|M|$.

The first concept we need is that of the geometric n-simplex in Euclidean space. The main concepts and terminology is taken from the excellent book by Gelfand and Manin on Homological algebra ([GM96]).

Definition 2.0.1 *The (affine) n-dimensional simplex is a topological space*

$$\Delta_n = \{(x_0,\ldots,x_n) \in \mathbb{R}^n \mid \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}.$$

The point e_i for which $x_i = 1$ is called the i-th vertex *of Δ_n; the set of vertices is ordered as $e_0 < \cdots < e_n$. For any subset $M \subseteq [n]$, we define the M-th* face *of Δ_n to be the set of points $(x_0,\ldots,x_n) \in \Delta_n$ with $x_i = 0$ for $i \notin M$.*

It is standard to conceive of a face $M \subseteq [n]$, with $|M| = m$ of an n-dimensional simplex Δ_n, as the (unique) monotone increasing map $f : [m-1] \to [n]$, with image M.[1] Our next important notion is the concept of *gluing*, which is used to build more complicated spaces from the basic building blocks of n-dimensional simplexes.

Definition 2.0.2 *Given a set of sets $\{X_{(i)}\}$ ($i \in \mathbb{N}$), where the members of each $X_{(i)}$ are meant to index i-simplices, and for any family of monotone increasing maps $\{f : [m] \to [n]\}_F$, by* gluing data *we mean a corresponding family of maps $\{X(f) : X_{(n)} \to X_{(m)}\}_F$, satisfying the conditions*

$$X(id) = id \qquad\qquad X(g \circ f) = X(f) \circ X(g) \tag{2.1}$$

We would be interested in the generalization of the preceding notions in the form detailed below.

Definition 2.0.3 *The* Simplicial Category Δ *is the category with objects all finite non-empty ordered sets of the form $[n], n \geq 0$, and with morphisms $\alpha : [n] \to [m]$ all non-decreasing functions α (such that $0 \leq i \leq j \leq n$ implies $\alpha(i) \leq \alpha(j)$).*

It is easy to see that Δ is isomorphic to the category of "standard" (affine) n-dimensional simplexes and affine maps (*i.e.* one that preserves weighted averages). The next definition is central to our work.

Definition 2.0.4 *A* Simplicial Set *is a contravariant functor $X : \Delta^{op} \to$ **Sets**. We shall write X_n for $X([n])$ and $X(\alpha) : X_m \to X_n$ for the action of X on $\alpha : [n] \to [m]$. The elements of X_n would be known as n-dimensional* faces*, consistent with earlier usage.*

Spelled out, the condition of functoriality entails that

$$X(id) = id \qquad\qquad X(g \circ f) = X(f) \circ X(g) \tag{2.2}$$

which is identical to the constraints on the gluing map set out earlier. We have the natural extension of this idea to the category of simplicial sets — as the category of functors defined in Definition 2.0.4,

[1] Note that according to this definition there is an unique n-dimensional simplex: however, in the sequel, we shall use the term n-simplex in a looser sense to mean any object in general that may be identified as the M-th face of some p-dimensional simplex, with $|M| = n+1$ — or in general, the image of the unique n-dimensional simplex under any affine map of the m-dimensional ($m > n+1$) Euclidean space onto itself.

and we would denote this as $\Delta - \mathbf{Sets}$. Being a functor category, morphisms of simplicial sets are the usual natural transformations; of course, this is a Topos ...

The generalization to *non-decreasing* maps in Definition 2.0.3 introduces what are known as *degenerate* simplices into the scheme of representations. Given a simplicial set X, a degenerate simplex $x \in X_n$ is one such that there exists a surjective non-decreasing map $f : [n] \to [m]$, with $m < n$, and an element $y \in X_m$, such that $X(f) : y \mapsto x$. We shall see in the sequel, that degenerate simplices are an essential aspect of our scheme of representation, and give us an elegant way to record information at intermediate stages of the development of a proof.

Example 2.0.1 *Let us see what the standard affine n-simplex Δ_n looks like in this form. We define the simplicial set Δ^n as follows.*

$$\begin{aligned} \Delta^n_i &= \{f : [i] \to [n] \mid f \text{ is non-decreasing } \} \\ \Delta^n(f : [i] \to [j]) &: (g : [j] \to [n]) \mapsto (g \circ f : [i] \to [n]) \end{aligned}$$

Note that for $p > n$, Δ^n_p consists only of degenerate simplices according to this formulation, while Δ^n_n consists of exactly one non-degenerate simplex.

In analogy to this, we shall define the *dimension* of a simplicial set X to be the least integer n such that for all $p > n$, X_p consists only of degenerate faces. If no such least integer exists, we would say that the dimension of X is infinite. Henceforth, we shall use the symbol Δ^n solely to denote the simplicial set derived from the standard affine simplex Δ_n as described above.

Given a simplicial set X, and some n-dimensional face $x \in X_n$, we shall write x/m for the set

$$\{y \in X_m \mid \exists(\alpha : [m] \to [n]) \in \Delta.\, X(\alpha) : x \mapsto y\}.$$

In other words, x/m denotes the set of m-dimensional faces of x. The next important concept we need is that of an n-chain.

Definition 2.0.5 *An n-dimensional chain (or simply, an n-chain) of a simplicial set X, is a formal linear combination*

$$\sum_{x \in X_n} a_x x \tag{2.3}$$

where $a_x \in \mathbb{Z}$, and $a_x \neq 0$ for only a finite number of simplices x. Thus, an n-chain is essentially an element of the free abelian group $C_n(X)$ generated by the set of n-simplices of X. We have the generalization of this construction, as n-chains with coefficients in an Abelian group A—$C_n(X, A)$*: these are formal linear combinations as in Equation 2.3, with $a_x \in A$. Thus, we have $C_n(X) = C_n(X, \mathbb{Z})$.*

Dually, we have the notion of *n-dimensional co-chains* with coefficients in A, $C^n(X, A)$, defined as the group of functions on X_n with values in the Abelian group A. In the sequel we would prefer to work with co-chains mainly, as they yield a somewhat more compact theory.

An important notion for us is that of a *coefficient system* (for homology or cohomology). We define the notion of a coefficient system for cohomology as it is that that we would be more directly concerned with. Consider a simplicial set X; and the category $[X]$ with objects $\bigcup_n X_n$, and such that there is exactly one morphism $\bar{\alpha} : x \to y$ exactly when $X(\alpha) : x \mapsto y$ for some face map $\alpha : [m] \to [n]$.

Definition 2.0.6 *A cohomological coefficient system $\mathcal{C}$ on a simplicial set is a (contravariant) functor $\mathcal{C} : [X]^{op} \to$ **Ab-Grps**, where **Ab-Grps** is the category of Abelian groups. As usual, we shall write $\mathcal{C}(\bar{\alpha})$ for the action of $\mathcal{C}$ on a morphism $\bar{\alpha} \in [X]$.*

Definition 2.0.7 *Given such a cohomological coefficient system $\mathcal{C}$ on a simplicial set X, we have the notion of the group of n-*dimensional co-chains (of X) with coefficients in $\mathcal{C}$, $C^n(X, \mathcal{C})$, *defined as the set of functions*

$$f : X_n \to \bigcup_{x \in X_n} \mathcal{C}(x)$$

with $f(x) \in \mathcal{C}(x)$.

Note that Definition 2.0.6 may be generalized to contravariant functors into any Abelian category. We shall say that a morphism $f : A \to B$ in **Ab-Grps** is *trivial* if the kernel of f is the whole of A. A cohomological coefficient system $\mathcal{C}$ would be said to be *discrete* if very morphism in the image of $\mathcal{C}$ is trivial.

The set of all faces of a simplicial set X would be denoted as $|X| \equiv \bigcup_n X_n$; the set of its non-degenerate faces would be denoted by the symbol $|X|^{\blacktriangle}$. An important notion for us is that of a *graded co-chain* of a simplicial set X, with coefficients in the Abelian group A: this is simply a function $\gamma : |X| \to A$. The set of all graded co-chains on X with coefficients in A would be denoted as $C^\infty(X, A)$. We have the obvious generalization of this notion to that of graded co-chains w.r.t. a cohomological coefficient system $\mathcal{C}$; we shall denote the set of all such chains by the symbol $C^\infty(X, \mathcal{C})$.

We note a few algebraic preliminaries at this point. Elementary notions like free groups, rings, quotients and modular arithmetic, and modules over rings are assumed, as is the idea of presentations by generators and relations. For any set A, we shall denote the free Abelian group generated by A as A^g. [2] We shall write the group operation multiplicatively, (usually by juxtaposition); the inverse operation on A^g would be denoted by $(\mathbf{.})^{-1}$, while unit for the group operation would be denoted by 1. The free Abelian group generated by the empty set $\emptyset$ is the *zero group*; we shall denote it as $\mathbf{1}$.

We denote by $\mathbb{Z}/\mathbf{2}$, the (commutative) ring of integers *modulo* 2; this ring has two elements 0 and 1, with addition and multiplication being defined modulo 2. For a given set A, the free $\mathbb{Z}/\mathbf{2}$-module generated by A^g would be denoted as $\overline{A}$. Such an $\overline{A}$ is, by definition, an Abelian group. We would denote the module operation additively; the identity element for this would be denoted by 0. We note that $\overline{A}$ also has the structure of a *Hopf-algebra*— with the evident algebra and co-algebra structures, and with the antipode $\overline{A} \to \overline{A}$ given (on the basis) by $x \mapsto x^{-1}$. The only part if this structure that we would use in the sequel is the algebra multiplication

$$\nabla : \overline{A} \otimes \overline{A} \to \overline{A} : x \otimes y \mapsto xy$$

where, obviously, the monomial $xy \in A^g$—the group operation being denoted by juxtaposition as stated earlier.

If A is the empty set $\emptyset$, then the module $\overline{A}$ is the $\mathbb{Z}/\mathbf{2}$-module $\{0, 1\}$ and we would denote it as $\mathbf{1}$ too (depending on context ...). On the other hand, the null $\mathbb{Z}/\mathbf{2}$-module $\{0\}$ would be denoted by $\mathbf{0}$. With the additive and multiplicative conventions for denoting module and group operations respectively, we may refer to elements of the module $\overline{A}$ as *polynomial*, while elements of the group A^g would be known as *monomials* (for any set A). For any $x, y \in \overline{A}$, we write $x \in y$ if x is a summand of y.

[2] As usual, this is (isomorphic to) the set of functions $s : A \to \mathbb{Z}$ non-zero at only a finite number of arguments, and with the group operation being the point-wise addition of the functions.

3 The Multiplicative Fragment

The following account of the interpretation of proofs in the Multiplicative fragment of Linear Logic would be divided into two sections: Statics and Dynamics. In the Statics part we would formulate an interpretation of the proofs and state the full-completeness theorem. In the Dynamics part, we would give an account of cut-elimination or normalization—which is essentially the Geometry of Interaction for this fragment.

3.1 Statics

Consider a mapping from the set $\mathcal{F}_K$ of multiplicative linear formulae over some set of occurrence identifiers K, and a set of propositional atoms $\mathcal{P}$. We define a mapping π from $\mathcal{F}_K$ to the module $H = \overline{\mathcal{P}_K}$ inductively as follows. Let $*$ denote the composite

$$H \times H \xrightarrow{\delta} H \otimes H \xrightarrow{\nabla} H$$

where δ is the universal $\mathbb{Z}/\mathbf{2}$-bilinear function yielded by the definition of the tensor product $H \otimes H$. (Thus, we would have $(a+b)*c = ac+bc$, and $a*(b+c) = ab+ac$). The mapping π is given as follows.

$$\begin{aligned} \pi(a) =& a \\ \pi(a^{\perp}) =& a^{-1} \\ \pi(A \otimes B) =& \pi(A) * \pi(B) \\ \pi(A \parr B) =& \pi(A) + \pi(B) \end{aligned}$$

where $a \in \mathcal{P}_K$, and $A, B \in \mathcal{F}_K$.

Consider the simplicial sets Δ^m and Δ^n. We say that their *product*, $\Delta^m \otimes \Delta^n$, is the simplicial set Δ^{m+n}: there are the evident injective maps of simplicial sets $\imath_{m,m+n} : \Delta^m \to \Delta^{m+n}$ and $\imath_{n,m+n} : \Delta^n \to \Delta^{m+n}$ defined as follows. $\imath_{m,m+n}$ consists of an $\mathbb{N}$-indexed set of maps $\imath^i_{m,m+n} : \Delta^m_i \to \Delta^{m+n}_i$ defined by

$$\imath^i_{m,m+n} : (f : [i] \to [m]) \mapsto (f' : [i] \to [m+n]) \qquad \text{s.t. } f'(k) = f(k) \text{ for all } k \leq i$$

while $\imath_{n,m+n}$ consists of an $\mathbb{N}$-indexed set of maps $\imath^i_{n,m+n} : \Delta^n_i \to \Delta^{m+n}_i$ defined by

$$\imath^i_{n,m+n} : (f : [i] \to [n]) \mapsto (f' : [i] \to [m+n]) \qquad \text{s.t. } f'(k) = \begin{cases} f(k) + m & \text{if } f(k) > 0 \\ f(k) & \text{if } f(k) = 0. \end{cases}$$

Given the simplicial set Δ^n, a face $f : [i] \to [n]$ is said to a *principal* face if $f : 0 \to 0$. Consider a non-degenerate principal face $f : [i] \to [n]$ of Δ^n. We define the *Diagonal* of f, as the (non-degenerate, non-principal) face $\triangle(f) : [i-1] \to [n] : k \mapsto f(k+1)$ for $0 \leq k < i$. Geometrically, this is the largest face of f not containing the vertex 0. Any non-degenerate m-face $f : [m] \to [n]$ of Δ^n may be uniquely identified by its image; we shall write $Im(f)$ for this image. Given a set of (non-degenerate) principal 1-faces P of cardinality p, the *Span* of P, symbolized as $Span(P)$, is defined as the unique non-degenerate principal p-face f, such that $Im(f) = \bigcup\{Im(g) \mid g \in P\}$. Note that any non-degenerate principal face x is uniquely identified by the set of 1-faces P with $Span(P) = x$. For any $(n > 1)$-dimensional, the (abused) notation $Span(x)$ would denote the set P of principal, non-degenerate 1-faces such that $Span(P) = x$.

Given non-degenerate principal faces $x, y \in \Delta^n$, we define their product as $Span(Span(x) \cup Span(y))$. Note the two different senses in which *Span* is being used in this expression.

Consider again the simplicial set Δ^n, and a face $x \in \Delta^n_p$: a degenerate face $y \in \Delta^n_q$, for $q > p$, is said to be an *image* of x—symbolized as $x \leqslant y$—if there exists a surjective non-decreasing $f : [q] \to [p]$ such that $\Delta^n(f) : x \mapsto y$. If we wish to emphasize the face that $y \in \Delta^n_q$, we would write $x \leqslant_q y$, to be understood as "y is the image of x at the q-th stage"; the set of all images of a non-degenerate face x at the q-th stage would be denoted as $[x]_q$. $\leqslant$ is obviously a transitive relation: we shall denote the reflexive closure of this relation by $\leqq$. The set of all images of x, along with x itself, would be symbolized as $\overline{\overline{\vee}}(x)$. It is easy to verify the following properties.

Lemma 3.1.1 *Given faces $x, y \in \Delta^n_p, z \in \Delta^n_q$ such that $x \leqslant z, y \leqslant z$, then we must have that $x = y$. For any degenerate face $y \in \Delta^n_q$, there exists an unique non-degenerate face $x \in \Delta^n_p$, for a certain $p < q$, such that $x \leqslant y$. We say that a non-degenerate face x is the* root *of y—symbolized as $x = \overline{\overline{\wedge}}(y)$—if x is the unique face with $x \leqq y$, as required by the previous condition. Moreover, for faces x and y, $x \leqq y \Rightarrow \overline{\overline{\wedge}}(x) = \overline{\overline{\wedge}}(y)$.*

For any non-degenerate face $x \in \Delta^n_p$, we define the equivalence relation $\overset{x}{=}$ on any Δ^n_q, with $q > p$, by: $y \overset{x}{=} z$ iff $\overline{\overline{\wedge}}(z) = \overline{\overline{\wedge}}(y) = x$. If we wish to elide the reference to x, we would use the relation symbol $y \doteq z$ for $\exists x. y \overset{x}{=} z$. Note that the set of all $y \in \Delta^n_q$ such that $y \in [x]_q$ for some x, is precisely the domain of the equivalence relation $\overset{x}{=}$ on Δ^n_q.

In the sequel, all (graded) co-chains would be with respect to some specified cohomological coefficient system $\mathcal{C}$. Such a system is said to be *equivariant* if $x \doteq y \Rightarrow \mathcal{C}(x) = \mathcal{C}(y)$; a graded co-chain γ on Δ^n, with respect to an equivariant cohomological coefficient system, would be said to be *equivariant* if it has the same value on all faces related by the equivalence $\doteq$; hence, for any non-degenerate face x, we may refer to the unique value of γ on any and all the images of x at some stage p, as $\gamma([x]_p)$. Consider a graded co-chain γ on a simplicial set Δ^n, and a number $q \geq n$: γ would be said to have the dimension q, if

$$\forall x \in X. \forall p > q. \gamma([x]_p) = \gamma([x]_q)$$

where x is non-degenerate of course. In other words, the values of γ on the various faces do not change after the q-th "stage"; we shall indicate this by writing $\gamma^{(q)}$. Given any graded co-chain γ on Δ^n, we can obtain a co-chain of dimension $q > n$ by restriction: we denote this by $\gamma|_q$, and this has the same action as γ on any $x \in \Delta^n_{p \leq q}$, while its value on other faces is given by the defining condition above.

In the sequel, we would be working only with equivariant co-chains of some finite dimension. Given an equivariant co-chain $\gamma^{(p)}$ on Δ^n, and a non-degenerate face $x \in \Delta^n_q$, we say that γ is *undefined* on x if $\mathcal{C}(x) = \mathbf{0}$—and thus $\gamma(x) = 0$ trivially; in this context the *history* of x w.r.t. γ is the sequence

$$\langle (\mathcal{C}([x]_m), \gamma([x]_m)) \mid m \geq q, \ \gamma \text{ is defined on } [x]_m \rangle$$

The *trace* of x w.r.t γ, symbolized as $Tr_\gamma(x)$, is the subsequence of its history (w.r.t. γ) obtained by including only the *unique* members of its history (preserving the sequence order of course). The set of *traces* of γ, symbolized as, $\eth(\gamma)$, is defined as the set of pairs

$$\{(x, Tr_\gamma(x)) \mid x \in \Delta^n, \ x \text{ non-degenerate}\}$$

Now given co-chains $\gamma^{(p)}$ and $\delta^{(q)}$ on Δ^n (with possibly different underlying cohomological coefficient systems), we define the equivalence $\gamma \simeq \delta$ as follows. For a subset P of $\eth(\gamma)$, we write $P \subseteq_1 \eth(\gamma)$ if P consists purely of principal 1-faces. Then we define $\gamma \sim \delta$ iff there exists an isomorphism

$f : \eth(\gamma) \to \eth(\delta)$ such that

$$f : (x,t) \mapsto (x,s) \Rightarrow t = s \tag{3.1}$$
$$\forall P \subseteq_1 \eth(\gamma).\, \forall Q \subseteq_1 \eth(\delta).\, f : P \mapsto Q \Rightarrow f : Span(P) \mapsto Span(Q) \tag{3.2}$$
$$\forall P \subseteq_1 \eth(\gamma).\, \forall Q \subseteq_1 \eth(\delta).\, f : P \mapsto Q \Rightarrow f : \triangle(P) \mapsto \triangle(Q) \tag{3.3}$$

where by abuse of notation, the operations $Span(.)$ and $\triangle(.)$ are meant to be operating only on the corresponding set of first components of their arguments. $\sim$ is reflexive and symmetric; we shall denote its transitive closure as $\simeq$. We shall denote the $\simeq$-equivalence class of a chain $\gamma^{(n)}$ as $[\gamma^{(n)}]_{\simeq}$, or simply elide the subscript when the relation is clear from the context.

Remark 3.1.1 *The introduction of the relation $\simeq$ allows for the identification of interpretations which differ only in the order of independent operations. It also allows for the possibility of "delays" where the $\mathcal{C}$ and γ values do not change along a sequence of stages; such delays should not obviously yield an interpretation of a different proof. More importantly, it also allows us to assume that particular terminal operations in a proof are always available at the terminal stage of the corresponding co-chain interpretation—since, adding delays in order to propagate these operations to the terminal stage would not take us out of the $\simeq$-class of the interpretation. This principle is implicit in all that follows.*

We shall abbreviate "cohomological cohomological system" by CCS. All such systems in the sequel would be assumed to be *discrete*. Extending earlier usage, a non-degenerate face x would be said to be *defined* w.r.t. the co-chain γ if, at some stage m, we have $\mathcal{C}([x]_m) \neq \mathbf{0}$. We would write $x\!\downarrow$ to signify x is defined. Our interpretation would be defined by induction on the structural complexity of the proofs. Consider a proof Π within this fragment of linear logic; let $|\Pi|$ denote the set of occurrences of propositional atoms appearing in Π. The interpretation of Π, would be the $\simeq$-class of a graded equivariant co-chain γ^m, on a simplicial set Δ_n, $n \leq m$.

The interpretation would be developed on the basis of a certain property—Property I say, which would itself be inductively established along the process of the interpretation. In the context of an interpreting object $[\gamma^{(m)}]$ as stated above, Property I may be stated as follows.

Definition 3.1.1 (Property I) *Let K be the set of occurrence-identified propositional atoms appearing in the the interpreted proof Π; then, for all defined non-degenerate faces x, we would have $\mathcal{C}([x]_m) = \overline{K}$; additionally, for every formula A appearing in the conclusion of the Π, we would have either of the following possibilities.*

$$\gamma([x]_m) = \begin{cases} \pi(A) + \pi(A^{\perp}) & \text{if } A \text{ is atomic} \\ \pi(A) & \text{otherwise} \end{cases}$$

for some non-degenerate principal face $x \in \Delta^n$.

Under the assumption of Property I, we shall write $[\![A]\!]_{\gamma}$ for the set x of non-degenerate, principal faces such that $\gamma^{(m)}([x]_m) = \pi(A)$ (or $\pi(A) + \pi(A^{\perp})$, respectively, as the case may be)—where $\gamma^{(m)}$ is the interpretation of some proof Π, and A is a conclusion of Π.

Procedure 3.1.1 *We shall adopt the convention, that the action of γ on any face in its domain which is not explicitly specified, or is not available on the basis of the equivariant property, is undefined. For every CCS $\mathcal{C}$ constructed or assumed in the sequel, we would assume that the value of $\mathcal{C}$ on all 0-faces is the zero module $\mathbf{0}$.*

Axiom *The Axiom Rule*

$$\overline{a \quad a^{\perp}}$$

where a is a propositional symbol occurrence, is interpreted by the following structure. Consider the simplicial set Δ^1 derived from the standard affine 1-dimensional simplex. Let $i_0 : [0] \rightarrow [1] : 0 \mapsto 0$, and $i_1 : [0] \rightarrow [1] : 0 \mapsto 1$, be the two elements of Δ^1_0. The CCS we shall use has the value $\overline{A}$ where $A = \{a\}$. The interpretation of this proof is the $\simeq$-class of following 1-dimensional graded co-chain $\gamma^{(1)}$ on Δ^1:

$$\gamma(i_0) = 0$$
$$\gamma(i_1) = 0$$
$$\gamma(i) = a + a^{-1}$$

for the only non-degenerate $i \in \Delta^1_1$. Note the fulfillment of Property I.

The Tensor *We have to interpret a proof Π, the last rule of which is the Tensor rule.*

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}$$

Let us assume that the interpretation of the proof Π_A of the sequent $\vdash \Gamma, A$ is the co-chain $[\gamma_A^{(p)}]$, with respect to the simplicial set Δ^n (and underlying CCS $\mathcal{C}_A$), while that of the proof Π_B of the sequent $\vdash B, \Delta$ is the co-chain $[\gamma_B^{(q)}]$, with respect to the simplicial set Δ^m (and underlying CCS $\mathcal{C}_B$). By the inductive hypothesis—Property I, on "less complex" constituent proofs Π_A and Π_B, we have the sets $[\![A]\!]$ and $[\![B]\!]$. Let $\mathcal{C}_A([y]_p) = \overline{K_A}$, for any defined *(non-degenerate) face $y \in \Delta^n$; similarly, let $\mathcal{C}_B([y]_q) = \overline{K_B}$, for any* defined *(non-degenerate) face $y \in \Delta^m$; let $K = K_A \cup K_B$. Let $s = \max(p, q) + 1$. Consider the co-chain $\delta^{(s)}$ on Δ^{n+m} (with underlying CCS $\mathcal{C}$), defined as follows (where all faces referred to, are to be understood as principal and non-degenerate). For faces $x \in \Delta^{n+m}$ such that $\exists y{\downarrow} \in \Delta^n. \imath_{n,m+n} : y \mapsto x$ and $y \notin [\![A]\!]$ we have*

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_A([y]_r) & \delta([x]_r) = \gamma_A([y]_r) & \text{if } r \leq p \\ \mathcal{C}([x]_r) = \mathcal{C}_A([y]_p) & \delta([x]_r) = \gamma_A([y]_p) & \text{if } p < r < s; \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_A([y]_p) & \text{if } r = s; \end{array} \tag{3.4}$$

for faces $x \in \Delta^{n+m}$ such that $\exists y{\downarrow} \in \Delta^m. \imath_{m,m+n} : y \mapsto x$ and $y \notin [\![B]\!]$ we have

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_B([y]_r) & \delta([x]_r) = \gamma_B([y]_r) & \text{if } r \leq q \\ \mathcal{C}([x]_r) = \mathcal{C}_B([y]_q) & \delta([x]_r) = \gamma_B([y]_q) & \text{if } q < r < s; \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_B([y]_q) & \text{if } r = s; \end{array} \tag{3.5}$$

for faces $x \in \Delta^{n+m}$ such that $\exists y \in [\![A]\!]. \imath_{n,m+n} : y \mapsto x$ we have

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_A([y]_r) & \delta([x]_r) = \gamma_A([y]_r) & \text{if } r \leq p \\ \mathcal{C}([x]_r) = \mathcal{C}_A([y]_p) & \delta([x]_r) = \gamma_A([y]_p) & \text{if } p < r < s \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_A([y]_p) - A & \text{if } r = s; \end{array} \tag{3.6}$$

for faces $x \in \Delta^{n+m}$ such that $\exists y \in [\![B]\!]. \imath_{m,m+n} : y \mapsto x$ we have

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_B([y]_r) & \delta([x]_r) = \gamma_B([y]_r) & \text{if } r \leq q \\ \mathcal{C}([x]_r) = \mathcal{C}_B([y]_p) & \delta([x]_r) = \gamma_B([y]_p) & \text{if } q < r < s \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_B([y]_q) - B & \text{if } r = s; \end{array} \tag{3.7}$$

finally, for faces $x \in \Delta^{n+m}$ *such that* $\exists y \in [\![A]\!]. \exists z \in [\![B]\!]. x = y \otimes z$ *we have*

$$\mathcal{C}([x]_s) = \overline{K} \qquad \delta([x]_s) = \pi(A \otimes B) \tag{3.8}$$

The interpretation of Π *is the class* $[\delta^{(s)}]$. *Note that Property I is preserved in this process.*

The Par *We have to interpret a proof* Π, *the last rule of which is the Par rule.*

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

Let us assume that the interpretation of the proof Π' *of the sequent* $\vdash \Gamma, A$ *is the co-chain* $[\gamma^{(p)}]$ *(with the underlying CCS* $\mathcal{C}$*), with respect to the simplicial set* Δ^n. *By virtue of the inductive hypothesis again, we have the sets* $[\![A]\!]$ *and* $[\![B]\!]$. *The co-chain* $\delta^{(p+1)}$ *on* Δ^n *(with underlying CCS* $\mathcal{D}$*) is constructed as follows. For all non-degenerate* $x \in \Delta^n$, *we have*

$$\begin{array}{lll} \mathcal{D}([x]_r) = \mathcal{C}([x]_r) & \delta([x]_r) = \gamma([x]_r) & \text{if } r \leq p \\ \mathcal{D}([x]_{p+1}) = \mathcal{D}([x]_p) & \delta([x]_{p+1}) = \delta([x]_p) & \text{if } x \notin ([\![A]\!] \cup [\![B]\!]) \\ \mathcal{D}([x]_{p+1}) = \mathcal{D}([x]_p) & \delta([x]_{p+1}) = \flat(\pi(A) + \pi(B)) & \text{if } x \in ([\![A]\!] \cup [\![B]\!]) \end{array} \tag{3.9}$$

where $\flat(\pi(A)+\pi(B)) = 0$ *if* $B = A^\perp$ *upon ignoring occurrence identifiers; otherwise,* $\flat(\pi(A)+\pi(B)) = \pi(A) + \pi(B)$. *The interpretation of* Π *is then the class* $[\delta^{(p+1)}]$. *Note again that Property I is preserved in this process.*

The Cut *We have to interpret a proof* Π, *the last rule of which is the Cut rule.*

$$\frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

We interpret this by a process very similar to that used for the "times" rule above. Let us assume that the interpretation of the proof Π_A *of the sequent* $\vdash \Gamma, A$ *is the co-chain* $[\gamma_A^{(p)}]$, *with respect to the simplicial set* Δ^n *(and underlying CCS* $\mathcal{C}_A$*), while that of the proof* Π_B *of the sequent* $\vdash A^\perp, \Delta$ *is the co-chain* $[\gamma_B^{(q)}]$, *with respect to the simplicial set* Δ^m *(and underlying CCS* $\mathcal{C}_B$*). By the inductive hypothesis—Property I, on "less complex" constituent proofs* Π_A *and* Π_B, *we have the sets* $[\![A]\!]$ *and* $[\![A^\perp]\!]$. *Let* $\mathcal{C}_A([y]_p) = \overline{K_A}$, *for any* defined *(non-degenerate) face* $y \in \Delta^n$; *similarly, let* $\mathcal{C}_B([y]_q) = \overline{K_B}$, *for any* defined *(non-degenerate) face* $y \in \Delta^m$; *let* $K = K_A \cup K_B$. *Let* $s = \max(p, q) + 1$. *Consider the co-chain* $\delta^{(s)}$ *on* Δ^{n+m} *(with underlying CCS* $\mathcal{C}$*), defined as follows (where all faces referred to, are to be understood as principal and non-degenerate). For faces* $x \in \Delta^{n+m}$ *such that* $\exists y{\downarrow} \in \Delta^n . \imath_{n,m+n} : y \mapsto x$ *and* $y \notin [\![A]\!]$ *we have*

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_A([y]_r) & \delta([x]_r) = \gamma_A([y]_r) & \text{if } r \leq p \\ \mathcal{C}([x]_r) = \mathcal{C}_A([y]_p) & \delta([x]_r) = \gamma_A([y]_p) & \text{if } p < r < s; \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_A([y]_p) & \text{if } r = s; \end{array} \tag{3.10}$$

for faces $x \in \Delta^{n+m}$ *such that* $\exists y{\downarrow} \in \Delta^m . \imath_{m,m+n} : y \mapsto x$ *and* $y \notin [\![B]\!]$ *we have*

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_B([y]_r) & \delta([x]_r) = \gamma_B([y]_r) & \text{if } r \leq q \\ \mathcal{C}([x]_r) = \mathcal{C}_B([y]_q) & \delta([x]_r) = \gamma_B([y]_q) & \text{if } q < r < s; \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_B([y]_q) & \text{if } r = s; \end{array} \tag{3.11}$$

for faces $x \in \Delta^{n+m}$ *such that* $\exists y \in [\![A]\!].\, \imath_{n,m+n} : y \mapsto x$ *we have*

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_A([y]_r) & \delta([x]_r) = \gamma_A([y]_r) & \textit{if } r \le p \\ \mathcal{C}([x]_r) = \mathcal{C}_A([y]_p) & \delta([x]_r) = \gamma_A([y]_p) & \textit{if } p < r < s \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_A([y]_p) - A & \textit{if } r = s; \end{array} \tag{3.12}$$

for faces $x \in \Delta^{n+m}$ *such that* $\exists y \in [\![B]\!].\, \imath_{m,m+n} : y \mapsto x$ *we have*

$$\begin{array}{lll} \mathcal{C}([x]_r) = \mathcal{C}_B([y]_r) & \delta([x]_r) = \gamma_B([y]_r) & \textit{if } r \le q \\ \mathcal{C}([x]_r) = \mathcal{C}_B([y]_p) & \delta([x]_r) = \gamma_B([y]_p) & \textit{if } q < r < s \\ \mathcal{C}([x]_r) = \overline{K} & \delta([x]_r) = \gamma_B([y]_q) - B & \textit{if } r = s; \end{array} \tag{3.13}$$

finally, for faces $x \in \Delta^{n+m}$ *such that* $\exists y \in [\![A]\!].\, \exists z \in [\![B]\!].\, x = y \otimes z$ *we have*

$$\mathcal{C}([x]_s) = \overline{K} \qquad \delta([x]_s) = 0 \tag{3.14}$$

The interpretation of Π *is the class* $[\delta^{(s)}]$*. Note that Property I is preserved in this process.Note also the crucial difference from the case of the "times" rule, in that the labeling on product faces is set to* 0*, in the* $\mathbb{Z}/\mathbf{2}$*-module* $\mathbf{1}$*. Thus, we are interpreting cut as a form of parallel composition, but hiding (or internalizing) the labels on the composed faces (or ports).*

Constants *The introduction rule for the constant* $\mathbf{1}$*, which is the neutral element for* $\otimes$ *is stated as:*

$$\frac{}{\vdash \mathbf{1}}$$

Recall the simplicial set Δ^1 *used in the interpretation of the Axiom Rule above. We use the same symbols* i_0 *and* i_1 *for the two elements of* Δ^1_0*. Our interpreting co-chain* $\gamma^{(1)}$ *has is defined w.r.t. the CCS* $\mathcal{C}$*, with* $\mathcal{C}(i \in \Delta^1_1)$ *being the* $\mathbb{Z}/\mathbf{2}$*-module* $\mathbf{1}$*:*

$$\begin{aligned} \gamma(i_0) &= 0 \\ \gamma(i_1) &= 0 \\ \gamma(i) &= 1 \end{aligned}$$

for the only non-degenerate $i \in \Delta^1_1$*. The interpretation is the class* $[\gamma^{(1)}]$*.*

The introduction rule for the constant $\perp$*, which is the neutral element for* $\parr$*, is stated as:*

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

Let us assume that the proof up to the sequent $\vdash \Gamma$ *is interpreted as the co-chain* $[\gamma^{(p)}]$ *on* Δ^n *(w.r.t the CCS* $\mathcal{C}$*). Consider the co-chain* $\delta^{(p+1)}$ *(w.r.t. the CCS* $\mathcal{D}$*) on the simplicial set* $\Delta^{n+1} = \Delta^n \otimes \Delta^1$*, given as follows. Let* z *be the* 1*-face of* Δ^{n+1} *given by* $z : [1] \to [n+1] : 0 \mapsto 0\ ,\ 1 \mapsto n+1$*. Let* $\mathcal{C}([y]_p) = K$ *for any non-degenerate defined face* $y \in \Delta^n$ *(it must be the same for all defined faces, at stage* p*). For all non-degenerate faces* $x \in \Delta^{n+1}$*, such* $\exists y \in \Delta^n.\, \imath_{n,n+1} : y \mapsto x$*, we have* $\mathcal{D}([x]_r) = \mathcal{C}([y]_r)$ *and* $\delta([x]_r) = \gamma([y]_r)$ *for* $r \le p+1$*; for the face* z *we have* $\mathcal{D}([z]_{p+1}) = K$*, and* $\delta([z]_{p+1}) = 0$*. Then our interpreting class is* $[\delta^{(p+1)}]$*.*

This completes the account of the interpretation. In the next section we provide an account of the normalization operation on proofs in this fragment.

3.2 Dynamics

The "dynamics" in the Geometry of Interaction paradigm is commonly understood to refer to the process of cut-elimination in LL-proof structures. In the multiplicative fragment, Cut-elimination is relatively uncomplicated, and epitomized in the following basic rule that moves the cut between a tensor and par to "simpler" cuts between their components and finally to cuts between axiom links which are trivially eliminated.

$$\frac{\dfrac{\vdash A,\Gamma \quad \vdash B,\Delta}{\vdash A\otimes B,\Gamma,\Delta} \quad \dfrac{\vdash B^{\perp},A^{\perp},\Lambda}{\vdash A^{\perp}\parr B^{\perp},\Lambda}}{\vdash \Gamma,\Delta,\Lambda} \quad \Rightarrow \quad \frac{\vdash A,\Gamma \quad \dfrac{\vdash B,\Delta \quad \vdash B^{\perp},A^{\perp},\Lambda}{\vdash A^{\perp},\Delta,\Lambda}}{\vdash \Gamma,\Delta,\Lambda} \tag{3.15}$$

An pure axiom cut is eliminated in the obvious way.

$$\frac{\overline{\vdash A,A^{\perp}} \quad \overline{\vdash A,A^{\perp}}}{\vdash A,A^{\perp}} \quad \Rightarrow \quad \overline{\vdash A,A^{\perp}}$$

We would need a few geometrical concepts for representing these operations. For the simplicial set Δ^n $(n \geq 1)$, any principal non-degenerate 1-face, *i.e.* an increasing map $x : [1] \to [n]$, may be uniquely identified by the vertex $x(1)$. Hence, we may refer to such a 1-face simply by this value—as the face $k \equiv x(1)$. Now given a pair of 1-faces k, l of the simplicial set Δ^n, we define a *partial* simplicial morphism $\Delta^n \to \Delta^{n-1}$ that we would symbolize as $[k \doteqdot l]$, as follows.

Let $\widehat{l{\leftarrow}k}$ denote the map $[n] \to [n-1]$ defined by:

$$\widehat{l{\leftarrow}k}(i) = \begin{cases} i & \text{for } 0 \leq i < k \\ l & \text{for } i = k \\ i-1 & \text{for } k < i \leq n \end{cases} \tag{3.16}$$

The morphism $[k \doteqdot l]$ as we know, would be in the form of a family of (partial) maps

$$\{[k \doteqdot l]_p : \Delta^n_p \to \Delta^{n-1}_p \mid p \in \mathbb{N}\}$$

and thus, for any non-degenerate face $f \in \Delta^n_p : [p] \to [n]$, with $m < n$, *and such that* $l \notin Im(f)$, we define the action of $[k \doteqdot l]_p$ on f to be the unique non-degenerate face $g : [p] \to [n-1]$ of Δ^{n-1}_p whose image is given as $Im(\widehat{l{\leftarrow}k} \circ f)$. The action of $[k \doteqdot l]_p$ on all other non-degenerate faces in Δ^n_p is undefined. As for degenerate faces, suppose x is a such a face in Δ^n_p: let the root of x be $\overline{\overline{\wedge}}(x) \in \Delta^n_q$ $(q < p)$, and such that $x = \Delta^n(f)(\overline{\overline{\wedge}}(x))$ for some (surjective) non-decreasing map $f : [p] \to [q]$. Then if $[k \doteqdot l]_q$ is defined on $\overline{\overline{\wedge}}(x)$, and has the value x', then we set $[k \doteqdot l]_p(x) = \Delta^n(f)(x')$; in all other cases (*i.e.* $[k \doteqdot l]_q$ is undefined on $\overline{\overline{\wedge}}(x)$) the action of $[k \doteqdot l]_p$ on such a degenerate face x is undefined. It is quite easy to see that $[k \doteqdot l]$ as defined, is one-one.

Now in the same context in which the partial simplicial morphism $[k \doteqdot l] : \Delta^n \to \Delta^{n-1}$ was defined, we define a total map from the set of graded co-chains on Δ^n to that of graded co-chains on Δ^{n-1}. Thus, using the same symbol as before, we have $[k \doteqdot l] : C^{\infty}(\Delta^n, \mathcal{C}) \to C^{\infty}(\Delta^{n-1}, \mathcal{D})$ defined as follows: We have

$$\mathcal{D}(x \in \Delta^{n-1}_p) = \begin{cases} \mathcal{C}(y \in \Delta^n_p) & \text{where } [k \doteqdot l]_p(y) = z \text{ and } x \doteq z \\ \mathbf{0} & \text{if no such } y \text{ exists} \end{cases} \tag{3.17}$$

and

$$[k \doteqdot l](\gamma)(x \in \Delta^{n-1}_p) = \begin{cases} \gamma(y \in \Delta^n_p) & \text{where } [k \doteqdot l]_p(y) = z \text{ and } x \doteq z \\ 0 & \text{if no such } y \text{ exists} \end{cases} \tag{3.18}$$

for all $p \geq 0$.

Procedure 3.2.1 *For an interpreting co-chain $[\gamma^p]$ of a proof Π w.r.t. some Δ^n and CCS $\mathcal{C}$, and for a certain formula A occurring in Π at some stage, we shall write $x \in [\![A]\!]^m_\gamma$ to signify that for the principal non-degenerate face x of Δ^n, $m \leq p$ is the least stage such that $\gamma([x]_m) = \pi(A)$ (respectively, $\gamma([x]_m) = \pi(A) + \pi(A^\perp)$ in case A is atomic).*

The Atomic Cut *The base case of the inductive procedure is that of atomic cuts; that is, elimination of cuts of the form*

$$\frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \tag{3.19}$$

where A is a propositional symbol. Let the interpretation of the proof containing this cut be the co-chain $[\gamma^{(m)}]$ on the simplicial set Δ^n. The cut would be represented by a non-degenerate 2-face a spanned by 1-faces a_1, $a_2 \in \{1, \ldots, n\}$ (in other words, for some numbers $p, q \leq m$ we have $a_1 \in [\![A]\!]^p$, $a_2 \in [\![A^\perp]\!]^q$, $\mathcal{C}([a]_m) = \mathbf{1} \ \ldots$). The elimination of this cut yields the simplicial set Δ^{n-1}, along with the interpreting co-chain δ^n which is simply $[a_1 \doteqdot a_2](\gamma^{(m)})$ (and where the underlying CCS is changed according to Equation 3.17 above).

The Multiplicative Cut *We consider the situation illustrated in Equation 3.15. Let the proof with cut be represented by the co-chain $[\gamma^m]$ (with underlying CCS $\mathcal{C}$) on the simplicial set Δ^n. Referring again to the procedures through which the effect of the cut, as also of the particular $\otimes$ and $\parr$ operations which are involved in the cut-formulae, are represented, we can claim to have the following data. The cut is represented by a set of non-degenerate principal faces*

$$\overline{x} = \{x_{ij} = x_i \otimes x_j \mid 0 \leq i \leq l, 0 \leq j < k\}$$

all drawn representing simplicial set Δ^n, and such that for certain $p, q \in \mathbb{N}$, $x_i \in [\![A \otimes B]\!]^p$ and $x_j \in [\![A^\perp \parr B^\perp]\!]^q$. Let I denote the set of faces $\{x_i \mid 0 \leq i \leq l\}$; let J denote the faces $\{x_j \mid 0 \leq j \leq k\}$. The normalization process consists of the following set of operations: for each of the x_{ij}, we set $\mathcal{C}([x_{ij}]_{r \geq m}) = \mathbf{0}$ and $\gamma([x_{ij}]_{r \geq m}) = 0$. Now consider x_i: since this is in the interpretation of $A \otimes B$, it must itself be a product $x_i^1 \otimes x_i^2$ with $x_i^1 \in [\![A]\!]$ and $x_i^2 \in [\![B]\!]$. Let $\mathcal{C}([x_i^1]_{p-1}) = \overline{D}$ and $\mathcal{C}([x_i^2]_{p-1}) = \overline{E}$; let $\mathcal{C}([x_j]_{m-1}) = \overline{F}$. Set $K = D \cup E \cup F$. Now we must have that either $x_j \in [\![A^\perp]\!]$ or $x_j \in [\![B^\perp]\!]$. In the former case, we consider the face x of x_{ij}, formed as the product $x_i^1 \otimes x_j$, and set $\mathcal{C}([x]_{r \geq m}) = K$ and $\gamma([x]_{r \geq m}) = 0$; in the latter case, we do exactly the same but with $x \equiv x_i^2 \otimes x_j$. Note that $\mathcal{C}([x_i]_p)$ must be $\overline{D \cup E}$, for all $x_i \in I$. For all $x_i \in I$, we set $\mathcal{C}([x_i]_{r \geq p}) = \mathbf{0}$ and $\gamma([x_i]_{r \geq p}) = 0$; for all faces $y \notin I$ such that $\mathcal{C}([y]_{p+r}) = \overline{D \cup E \cup S}$ for some set S and $r > 0$, we set $\mathcal{C}([y]_{p+r}) = K \cup S$. Finally, for all faces $x_j \in J$, we set $\gamma([x_j]_{q \leq r \leq m}) = \gamma([x_j]_{q-1})$. The values of $\mathcal{C}$ and γ remain constant for all stages $r \geq m$. The effect of this elimination is interpreted by the class $[\gamma^m]$.

4 The Multiplicative-Exponential fragment

The exponential operations in this fragment is handled through the algebraic device of quotienting operations on groups. Consider a $(\mathbb{Z}/(2))$-module K, and a subset $J \subseteq K$; let the smallest subgroup of K containing J be denoted as $Comp_K(J)$. We would be interested in the quotient group $K/Comp_K(J)$; let us denote this by $K/(J)$. This is the group consisting of the cosets $a + Comp_K(J)$, for all $a \in K$; the identity, which we would denote as $0_{K/(J)}$ is obviously the sub-group $Comp_K(J)$ itself, and addition is given as

$$(a + Comp_K(J)) + (b + Comp_K(J)) = (a + b) + Comp_K(J)$$

We shall denote the coset containing the element a, as $(\!|a|\!) \equiv a + Comp_K(J)$. Given a MELL formula A, where the set of propositional atoms occurring in A—denoted as $|A|$, is a subset of K, we would use overload the symbol $(\!|A|\!)$ to signify the coset $(\!|\pi(\sharp A)|\!)$ in $K/(J)$ (for some $J \subseteq K$).

4.1 Statics

For any MELL formula A, we would denote by $\sharp(A)$, the formula obtained by erasing all exponential modalities from the formula. We define a function μ on MELL formulas, which we shall call the *multiplicity*. We have,

$$
\begin{aligned}
\mu(a) &= 1 \\
\mu(A \otimes B) &= \mu(A) \times \mu(B) \\
\mu(A \parr B) &= \mu(A) + \mu(B) \\
\mu(?A) &= \mu(!A) = \mu(A)
\end{aligned}
$$

where a is either a literal, or a constant.

The basic features of the interpretation remain the same as before. The interpreting objects are $\simeq$-classes of equivariant graded co-chains $\gamma^{(m)}$ of finite dimension, on some simplicial set Δ^n, and with respect to a specified CCS $\mathcal{C}$. As before, the interpretation would be developed on the basis of a certain property—Property E say, which would itself be inductively established along the process of the interpretation. In the context of an interpreting object $[\gamma^{(m)}]$ as stated above, Property E may be stated as follows.

Definition 4.1.1 (Property E) *Let K be the set of occurrence-identified propositional atoms appearing in the interpreted proof Π; then, for all defined non-degenerate faces x, we would have $\mathcal{C}([x]_m) = \overline{K}/(J)$ for some subset $J \subseteq \overline{K}$ (in particular, J might be the trivial sub-group $\mathbf{0}$—in which case $\overline{K}/(J) \cong \overline{K}$); additionally, for every formula A appearing in the conclusion of the Π, we would have either of the following possibilities.*

$$
\gamma([x]_m) = \begin{cases} (\!| A |\!) + (\!| A^{\perp} |\!) & \textit{if } A \textit{ is atomic} \\ (\!| A |\!) & \textit{otherwise} \end{cases}
$$

for some non-degenerate principal face $x \in \Delta^n$.

Note that in particular, $(\!| A |\!)$ or $(\!| A^{\perp} |\!)$ might be the identity element 0 in the corresponding quotient group. As earlier, the second alternative in Property E is essentially to deal with the Axiom Rule, and various exponential operations in this context.

In analogy to our practice in the multiplicative case, we shall use the symbol $[\![A]\!]$ to denote the set of (non-degenerate principal) faces x which satisfy either of the defining conditions of Property E above.

Procedure 4.1.1 *The various clauses of the interpretation are described below.*

Axiom *The interpretation of the Axiom Rule remains exactly the same as in Procedure 3.1.1. Note the fulfillment of the second alternative in Property E.*

Dereliction *We have to interpret a proof Π the last rule of which is the Dereliction rule.*

$$
\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}
$$

Let us assume that the interpretation of the proof up to the sequent $\vdash \Gamma, A$ is the co-chain $[\gamma^{(m)}]$ (with the underlying CCS $\mathcal{C}$), with respect to the simplicial set Δ^n. Consider any $x \in [\![A]\!]$;

suppose $\mathcal{C}([x]_m) = K/(J)$. We modify γ by setting $\mathcal{C}([x]_{m+1})$ and $\gamma([x]_{m+1})$ according to the following conditions:

$$\begin{array}{lll} \mathcal{C}([x]_{m+1}) = K/(\{\sharp A\}) & \gamma([x]_{m+1}) = (\!| A^{\perp} |\!) & \textit{if } \Gamma = \{A^{\perp}\}, \textit{ with } A \textit{ atomic} \\ \mathcal{C}([x]_{m+1}) = K/(\{\sharp A, \sharp A^{\perp}\}) & \gamma([x]_{m+1}) = 0 & \textit{if } \Gamma = \{?A^{\perp}\}, \textit{ with } A \textit{ atomic} \\ \mathcal{C}([x]_{m+1}) = K/(\{\sharp A\}) & \gamma([x]_{m+1}) = (\!| A |\!) \ = \ 0 & \textit{otherwise} \end{array}$$

for every $x \in [\![A]\!]$; everything else remains unchanged. The interpretation is the class $[\gamma^{(m+1)}]$. Note again the fulfillment of Property E.

Weakening *We have to interpret a proof Π the last rule of which is the Weakening rule.*

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}$$

Let $\mu(A) = m$. Let us assume that the proof up to the sequent $\vdash \Gamma$ is interpreted as the co-chain $[\gamma^{(p)}]$ on Δ^n (w.r.t the CCS $\mathcal{C}$). Consider the co-chain δ (w.r.t. the CCS $\mathcal{D}$) on the simplicial set $\Delta^{n+m} = \Delta^n \otimes \Delta^m$ as follows. Let z_k be the 1-face of Δ^{n+m} given by $z_k : [1] \to [n+1] : 0 \mapsto 0 \ , \ 1 \mapsto n+k$. By Property E, any of the values $\mathcal{C}([y]_p)$ for non-degenerate defined face $y \in \Delta^n$, must be of the form $\overline{M}/(J)$, for some module M; let $C = |?A|$, and $K = M \cup C$. Let $q = \max(p, n+m)$. For all non-degenerate faces $x \in \Delta^{n+m}$, such $\exists y \in \Delta^n . \imath_{n,n+m} : y \mapsto x$, we set

$$\begin{array}{lll} \mathcal{D}([x]_r) = \mathcal{C}([y]_r) & \delta([x]_r) = \gamma([y]_r) & \textit{for } 0 \leq r < p \\ \mathcal{D}([x]_r) = \overline{K}/(J) & \delta([x]_r) = \gamma([y]_r) & \textit{for } p \leq r \leq q \end{array}$$

for any z_k we set

$$\mathcal{D}([z_k]_r) = K/(\{\sharp A\}) \qquad \delta([z]_r) = 0 \qquad \textit{for } p \leq r \leq q. \tag{4.1}$$

Then our interpreting class is $[\delta^{(q)}]$, where

Contraction *We have to interpret a proof Π the last rule of which is the Contraction rule.*

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

Let us assume the following harmless condition on the assignment of occurrence identifiers in this context, which simplifies the representation to some extent: namely, we shall assume that the occurrence of the $?A$ at the consequence of the proof is assigned the same occurrence identifier as is assigned to either *of the occurrences of $?A$ in the antecedent of the rule. Let us assume that the interpretation of the proof up to the sequent $\vdash \Gamma, ?A, ?A$ is the the co-chain $[\gamma^{(m)}]$ (with the underlying CCS $\mathcal{C}$), with respect to the simplicial set Δ^n. Let the first occurrence of $?A$ be denoted as $?A_{(1)}$ and the second as $?A_{(2)}$. As mentioned earlier, we have, by Property E, that any of the values $\mathcal{C}([y]_m)$ for non-degenerate defined face $y \in \Delta^n$, must be of the form $K/(J)$, for some module K. We modify γ at its $(m+1)$-th stage as follows. We set*

$$\mathcal{C}([x]_{m+1}) = K/(\{\pi(\sharp A_{(1)}), \pi(\sharp A_{(2)})\}) \qquad \gamma([x]_{m+1}) = 0$$

for all $x \in [\![A_{(1)}]\!] \cup [\![A_{(2)}]\!]$. Everything else in γ is remains the same. The interpretation of Π is the class $[\gamma^{(m+1)}]$.

Promotion *We have to interpret a proof* Π *the last rule of which is the "Of Course" rule.*

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$$

Let us assume that the interpretation of the proof up to the conclusion $\vdash ?\Gamma, A$ *is the co-chain* $[\gamma^{(m)}]$ *(with the underlying CCS* $\mathcal{C}$*), with respect to the simplicial set* Δ^n*. Again as earlier, we have by Property E, that any of the values* $\mathcal{C}([y]_m)$ *for non-degenerate defined face* $y \in \Delta^n$*, must be of the form* $K/(J)$*, for some module* K*. Let us define the subset* $\overline{\Gamma}$ *of* K *as follows.*

$$\overline{\Gamma} = \{\pi(\sharp C) \mid C \in ?\Gamma\} \cup \{\pi(\sharp A)\}$$

We modify γ *at its* $(m+1)$*-th stage as follows. We set*

$$\mathcal{C}([x]_{m+1}) = K/(\overline{\Gamma}) \qquad\qquad \gamma([x]_{m+1}) = 0$$

for all $x \in [\![A]\!]$*. The interpretation of* Π *is the class* $[\gamma^{(m+1)}]$*.*

Tensor *The interpretation described in Procedure 3.1.1 with only the following few changes. We refer to the numbered equations in the rules for Tensor interpretation in the same procedure. Using the same symbols as in the earlier context, consider Equation 3.4: let the value of* $\mathcal{C}([y]_p) = L/(J)$ *for some* L *and* J *(using Property E). Then we modify Equation 3.4 to read*

$$\mathcal{C}([x]_s) = \overline{K}/(J) \qquad\qquad \delta([x]_s) = \gamma_A([y]_p)$$

Similarly for Equation 3.5; let the value of $\mathcal{C}([y]_q) = M/(I)$ *for some* M *and* I *(using Property E). Then we modify Equation 3.5 to read*

$$\mathcal{C}([x]_s) = \overline{K}/(I) \qquad\qquad \delta([x]_s) = \gamma_B([y]_q)$$

We also modify Equation 3.6 as follows. Again, let the value of $\mathcal{C}([y]_p) = L/(J)$ *for some* L *and* J*.*

$$\begin{array}{lll} \mathcal{C}([x]_s) = \overline{K}/(J - \{\pi(\sharp A)\}) & \delta([x]_s) = (\!| A^\perp |\!) & \textit{if } \gamma_A([y]_p) = (\!| A |\!) + (\!| A^\perp |\!) \\ \mathcal{C}([x]_s) = \overline{K} & \delta([x]_s) = 0 & \textit{otherwise} \end{array}$$

We modify Equation 3.7 analogously; let the value of $\mathcal{C}([y]_q) = M/(I)$ *for some* M *and* I*:*

$$\begin{array}{lll} \mathcal{C}([x]_s) = \overline{K}/(I - \{\pi(\sharp B)\}) & \delta([x]_s) = (\!| B^\perp |\!) & \textit{if } \gamma_B([y]_q) = (\!| B |\!) + (\!| B^\perp |\!) \\ \mathcal{C}([x]_s) = \overline{K} & \delta([x]_s) = 0 & \textit{otherwise} \end{array}$$

Finally, we make the following small modification in Equation 3.8.

$$\mathcal{C}([x]_s) = \overline{K} \qquad\qquad \delta([x]_s) = \pi(\sharp(A \otimes B))$$

Par *Here again, we introduce a small modification to the earlier interpretation, in order to take into account the use of the* $\sharp(.)$ *operator in co-chain values. By Property E, let us assume that* $\mathcal{D}([x]_p) = \overline{K}/(J)$*; then Equation 3.9 in the earlier context is now modified to read*

$$\mathcal{D}([x]_{p+1}) = \overline{K} \qquad \delta([x]_{p+1}) = \flat(\pi(\sharp A) + \pi(\sharp B)) \qquad \textit{if } x \in ([\![A]\!] \cup [\![B]\!])$$

Cut *Exactly the same modifications are made, in Equations 3.10, 3.11, 3.12 and 3.13, as in the corresponding equations in the case of the Tensor rule above. Of course, no corresponding change to Equation 3.14 is necessary.*

Constants *The rule for the introduction of* $\mathbf{1}$ *remains exactly the same. In the procedure for* $\perp$*, we signify by* K *the module of which every value* $\mathcal{C}([y]_p)$ *is a quotient (by Property E as usual).*

This completes the description of the interpretation of proofs in the MELL fragment.

4.2 Dynamics

The representation of cut-elimination in this fragment is really the heart of the Geometry of Interaction—since it is here that non-local syntactic effects are the most evident. Our representation requires certain operations on faces of the interpreting simplexes (and co-chain values at these faces); these are described in the sequel. We have the following definition of what are known as *(exponential) boxes* in the LL literature.

Given any $(\mathbb{Z}/\mathbf{2})$-module $H \equiv \overline{K}$ for some set K, we would call K the *sub-basis* of H, and denote it by $\underline{H}$. By a small abuse of notation, given any module of the form $H \equiv \overline{K}/(J)$, we would use the same term "sub-basis", and symbol $\underline{H}$, for K. For such H above, we would denote the subset J as $(H)^{\vee}$.

Definition 4.2.1 *Given a co-chain $\gamma^{(m)}$ (w.r.t. a CCS $\mathcal{C}$) on a simplicial set Δ^n, interpreting a proof Π, and a principal and non-degenerate face x, a* box *at p ($p \leq m$) is the* least *principal (non-degenerate) face $R \subseteq [n]$ (we are representing a principal face by its image) satisfying*

$$\begin{aligned}
&x \subseteq R \text{ and } \forall y.(\underline{\mathcal{C}([y]_p)} = \underline{\mathcal{C}([x]_p)} \Rightarrow y \subseteq R)\\
&\gamma([x]_p) = 0\\
&(\mathcal{C}([x]_p))^{\vee} \supset \bigcup \{(\mathcal{C}([y]_p))^{\vee} \mid y \text{ is defined at } p\}
\end{aligned}$$

where p is the least *number satisfying these conditions. A face $y \subseteq R$ is said to be an* auxiliary door *if y is defined at stage p and $\emptyset \neq (\mathcal{C}([y]_p))^{\vee} \subset (\mathcal{C}([x]_p))^{\vee}$. The face x is known as the* principal door *of the box R.*

Hence a box is the least face that includes all sub-faces involved in its global history up to a certain stage.

The next notion is an operation on proof representations, designed to capture the effect of eliminating a Weakening cut. We operate on a simplicial set Δ^n, and the operation is parameterized by two variables—first, a (finite) sequence of principal, non-degenerate faces of Δ^n (represented by their images, as subsets of $[n]$), and second, a single face C; we shall call them *Weaken*, and *Cut* respectively. Symbolizing these parameters by $W \equiv \{W_1, \ldots, W_k\}$ and C respectively, let

$$l = k + |[n] - (\bigcup_{i=1}^{k} W_i \cup C)| \tag{4.2}$$

consider the partial map $f : |\Delta^n|^{\blacktriangle} \rightharpoonup |\Delta^l|^{\blacktriangle}$ given by

$$f(x) = \begin{cases} \{0, i\} \cup (x - W_i) & \text{if } W_i \subseteq x \text{ and } \forall j \neq i.\, x \cap W_j = \emptyset \\ x & \text{if } x \cap (\bigcup_{i=1}^{k} W_i \cup C) = \emptyset \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{4.3}$$

Note that this map is one-one.

On the basis of f as defined above, we may obtain a mapping $\bar{f} : C^{\infty}(\Delta^n, \mathcal{C}) \rightarrow C^{\infty}(\Delta^l, \mathcal{D})$. Thus given a $\gamma^{(m)}$ (w.r.t. a CCS $\mathcal{C}$) on Δ^n, we have, for all principal, non-degenerate faces $x \in \Delta^l$, with $f(y) = x$ for some principal non-degenerate face $y \in \Delta^n$

$$\begin{aligned}
\mathcal{D}([x]_p) &= \mathcal{C}([y]_p)\\
\bar{f}(\gamma)([x]_p) &= \gamma([y]_p)
\end{aligned}$$

for all $p \geq 0$; all other faces in Δ^l are undefined.

While the operation above was devised to handle the effect of eliminating a weakening-cut, the next is intended for handling a contraction cut. We operate on a simplicial set Δ^n, and the operation is parameterized by a non-degenerate principal face R, represented as a subset $R \subseteq [n]$. Let us assume that R is enumerated as an increasing sequence, *i.e.* $R = \{r_0 < r_1 < \cdots < r_k\}$, where (obviously) $r_0 = 0$. Consider the map $f : [n+k] \rightarrow [n]$ defined by

$$f(i) = \begin{cases} i & \text{if } i \leq n \\ r_{n-i} & \text{if } n < i \leq n+k. \end{cases}$$

We may obtain a map $\bar{f} : |\Delta^{n+k}|^{\blacktriangle} \rightarrow |\Delta^n|^{\blacktriangle}$, on the basis of f, as follows. First, given any injective map $g : [h] \rightarrow [n+k]$ $(h \leq n+k)$, which may not be increasing on its whole domain, we shall denote by $\hat{g}$, the unique increasing map which has the same image as g (such an unique map always exists). Now for any face x given by an increasing map $x : [h] \rightarrow [n+k]$ $(h \leq n+k)$, we have

$$\bar{f}(x) = \widehat{f \circ x} \tag{4.4}$$

Pairs of faces x, y identified by the map $\bar{f}$ (there may be a maximum of only two) would be said to be related on the basis of a relation $\stackrel{R}{=}$. As before, $\bar{f}$ gives us a map $\bar{\bar{f}} : C^\infty(\Delta^n, \mathcal{C}) \rightarrow C^\infty(\Delta^{n+k}, \mathcal{D})$. Thus given a $\gamma^{(m)}$ (w.r.t. a CCS $\mathcal{C}$) on Δ^n, we have, for all non-degenerate faces $x \in \Delta^{n+k}$, such that $\bar{f}(x) = y$ for some non-degenerate face $y \in \Delta^n$

$$\mathcal{D}([x]_p) = \mathcal{C}([y]_p) \tag{4.5}$$

$$\bar{\bar{f}}(\gamma)([x]_p) = \gamma([y]_p) \tag{4.6}$$

for all $p \geq 0$.

Procedure 4.2.1 *For any formula A, we shall denote by $|A|$, the set of propositional atoms (occurrences) in A; and by generalization, $|\Gamma|$, for sequents Γ, and so on. A frequent abuse of notation in the sequel: identities between co-chain values may be written, even though the underlying module (as assigned by the CCS) may be different; since all modules involved are free, this simply means that the relevant co-chain values have the same syntactic expression, even though in different modules;*[3] *The various operations involved in the elimination of exponential cuts are as follows.*

Weakening Cut *We would refer to the Figure 1. Let the proof with cut be represented by the co-chain $[\gamma^m]$ (with underlying CCS $\mathcal{C}$) on the simplicial set Δ^n. Let the exponential box involved in the cut be introduced at stage $p < m$, and represented by the face $R \subseteq [n]$. Let the set of auxiliary doors of the box (enumerated in some order) be signified by the set $W = \{W_1, \ldots, W_k\}$ and the set of principal doors by the set $P = \{P_1, \ldots, P_l\}$; let $\mathfrak{C} = R - (\bigcup W)$. For any $x \in P$, let $C = \mathcal{C}([x]_p)$; let $C' = \mathcal{C}([y]_{m-1}) - |?A^\perp|$ for any $y \in [\![?A^\perp]\!]$; let $C'' = |?\Gamma|$. Let G denote the set of $\overline{\text{faces } g \in \Delta^n \text{ such that}}$ for some set S, and for some stage $l > p$, $\underline{\mathcal{C}([g]_l)} = C \cup S$. Recall the simplicial set Δ^l and the co-chain $\delta = \bar{f}(\gamma)$ constructed on the $\overline{\text{basis}}$ of Equations 4.2 and 4.3 above (the relevant operation being parameterized by W and $\mathfrak{C}$ of course). We make the following modifications to δ; for all w s.t. $\exists w' \in W. f(w') = w$, we set*

$\mathcal{D}([w]_r) = \mathbf{0}$	$\delta([w]_r) = 0$	*for* $0 \leq r < m$
$\mathcal{D}([w]_r) = \overline{C' \cup C''}$	$\delta([w]_r) = 0$	*for* $r \geq m$

[3]This notion may be formalized on the basis of universal properties of free constructions, but we shall not do that here.

for all $x \in \Delta^l$, *s.t.* $\exists g \in G. x = f(g)$, *we set*

$$\mathcal{D}([x]_{p+r}) = \overline{E}/(F) \qquad \delta([x]_{p+r}) = \gamma([x]_{p+r}) \qquad \text{for } 0 < r < m-p$$

where $F = (\mathcal{C}([x]_{p+r}))^{\vee}$, *and* E *is given by*

$$E = \begin{cases} (\underline{\mathcal{C}([x]_{p+r})} - C) \cup (C' \cup C'') & \text{if } C \subseteq \underline{\mathcal{C}([x]_{p+r})} \\ \underline{\mathcal{C}([x]_{p+r})} & \text{otherwise.} \end{cases}$$

Contraction Cut *We would refer to the Figure 3. Let the proof with cut be represented by the co-chain* $[\gamma^m]$ *(with underlying CCS* $\mathcal{C}$*) on the simplicial set* Δ^n. *Let the exponential box involved in the cut be introduced at stage* $p < m$, *and represented by the face* $R \subseteq [n]$. *Let the set of auxiliary doors of the box (enumerated in some order) be signified by the set* $W = \{W_1, \ldots, W_k\}$ *and the set of principal doors by the set* $P = \{P_1, \ldots, P_h\}$. *Let* x *be any face s.t.* $x \in [\![?A^{(1)}]\!] \cup [\![?A^{(2)}]\!]$; *let* $C' = \underline{\mathcal{C}([x]_{m-1})}$; *let* $C = \underline{\mathcal{C}([y]_p)}$, *for any* $y \in P$. *Let* G *denote the set of faces* g *such that for some set* S, *and for some stage* $l > p$, $\underline{\mathcal{C}([g]_l)} = C \cup S$. *Let us assume a* renaming operator π *on* C *(in some meta-theory) which assigns "fresh" occurrence identifiers to every propositional occurrence in the set* C, *and is the identity on the complement of* C. *By obvious extensions, we have the effect of* π *on* $\overline{C}$, *and on quotients of the form* $\overline{C}/(I)$. *Recall the simplicial set* Δ^{n+k} *and the co-chain* $\delta = \bar{\bar{f}}(\gamma)$ *(with CCS* $\mathcal{D}$*) described in the Equations 4.4 through 4.6 (the parameter for the operation described being the box* R*). We put* $R' = \{0\} \cup \{n+1, \ldots, n+k\}$. *We define a new co-chain* β *(w.r.t. CCS* $\mathcal{B}$*) on* Δ^{n+k} *as follows. For any non-degenerate face* x, *we have*

$$\mathcal{B}([x]_r) = \mathcal{D}([x]_r) \qquad \beta([x]_r) = \delta([x]_r) \qquad \text{if } x \subseteq [n]$$
$$\mathcal{B}([x]_r) = \pi(\mathcal{D}([y]_r)) \qquad \beta([x]_r) = \pi(\delta([y]_r)) \qquad \text{if } x \not\subseteq [n],\ x \overset{R}{=} y \subseteq [n]$$

for $0 \le r \le p$. *Denoting the first occurrence of* $?A$ *as* $?A^{(1)}$ *and the second as* $?A^{(2)}$, *we set*

$$\mathcal{B}([x]_{m-1}) = \mathcal{D}([x]_{m-2}) \qquad \beta([x]_{m-1}) = \delta([x]_{m-2}) \qquad \forall x. x \in [\![?A^{(1)}]\!] \cup [\![?A^{(2)}]\!]$$
$$\mathcal{B}([x]_m) = \mathbf{0} \qquad \beta([x]_m) = 0 \qquad \forall x. x = y \otimes z$$

with $z \in P$ *and* $y \in [\![?A^{(2)}]\!]$; *now let* $B = C' \cup C \cup \pi(C)$; *we set*

$$\mathcal{B}([x]_m) = \overline{B} \qquad \beta([x]_m) = 0 \qquad \forall x. \exists w \in P. x \overset{R}{=} w$$
$$\mathcal{B}([x]_m) = \overline{B} \qquad \beta([x]_m) = 0 \qquad \forall x. x \in [\![?A^{(2)}]\!]$$
$$\mathcal{B}([x]_m) = \overline{B} \qquad \beta([x]_m) = 0 \qquad \forall x. x = y \otimes z$$

such that $\exists w. \in P\, z \overset{R}{=} w$ *and* $y \in [\![?A^{(2)}]\!]$; *we also set*

$$\mathcal{B}([x]_m) = \overline{B}/(I) \qquad \beta([x]_m) = \delta([x]_m) \qquad \forall x \subseteq R. \mathcal{D}([x]_m) = \overline{C}/(I)$$
$$\mathcal{B}([x]_m) = \overline{B}/(\pi(I')) \qquad \beta([x]_m) = \pi(\delta([x]_m)) \qquad \forall x \subseteq R'. \mathcal{D}([x]_m) = \overline{C}/(I')$$
$$\mathcal{B}([x]_m) = \overline{B}/(J) \qquad \beta([x]_m) = \delta([x]_m) \qquad \forall x. \mathcal{D}([x]_m) = \overline{C'}/(J)$$

again, for all $x \subseteq R$ *and for all* $z \subseteq R'$ *s.t.* $z \overset{R}{=} x$, *we put*

$$\mathcal{B}([z]_{m+1}) = \overline{B}/(S_1 \cup S_2) \qquad \beta([z]_{m+1}) = 0 \qquad \text{where } \mathcal{B}([x]_p) = \overline{B}/(S_1)$$
$$\mathcal{B}([x]_{m+1}) = \overline{B}/(S_1 \cup S_2) \qquad \beta([x]_{m+1}) = 0 \qquad \text{where } \mathcal{B}([z]_p) = \overline{B}/(S_2)$$

finally, for all y s.t. $\bar{f}(y) = x$ for some $x \in G$

$$\mathcal{B}([y]_{p+r}) = \overline{E}/(F \cup \pi(F)) \qquad \beta([y]_{p+r}) = \delta([y]_{p+r}) \qquad 0 < r < m-p$$

where $F = (\mathcal{D}([y]_{p+r}))^{\vee}$, and E is given by

$$E = \begin{cases} \underline{\mathcal{D}([y]_{p+r})} \cup B & \text{if } C \subseteq \underline{\mathcal{D}([y]_{p+r})} \\ \underline{\mathcal{D}([y]_{p+r})} & \text{otherwise.} \end{cases}$$

*Note the abuse of notation in the fourth equation of this item: what we really mean here is that x should be the image under $\bar{f}$, of the relevant faces in Δ^n (*i.e.* those contained in $[\![?A^{(1)}]\!] \cup [\![?A^{(2)}]\!]$). There are similar abuses after that, and in the sequel.*

Dereliction Cut *We would refer to the Figure 2. Let the proof with cut be represented by the co-chain $[\gamma^m]$ (with underlying CCS $\mathcal{C}$) on the simplicial set Δ^n. Let the exponential box involved in the cut be introduced at stage $p < m$, and represented by the face $R \subseteq [n]$. Let the set of principal doors of box R denoted by the set $P = \{P_1, \ldots, P_h\}$. For any $x \in P$, let $C = \underline{\mathcal{C}([x]_p)}$; let $C' = \underline{\mathcal{C}([y]_{m-1})}$ for any $y \in [\![?A^{\perp}]\!]$. Let G denote the set of faces g such that for some set S, and for some stage $l > p$, $\underline{\mathcal{C}([g]_l)} = C \cup S$. We modify γ, to obtain a new co-chain δ (w.r.t CCS $\mathcal{D}$) as follows: for all $y \in [\![A^{\perp}]\!]$ we put*

$$\mathcal{D}([y]_{m-1}) = \mathcal{C}([y]_{m-2}) \qquad \delta([y]_{m-1}) = \gamma([y]_{m-2})$$

let $D = C \cup C'$; for any $x \in [\![A^{\perp}]\!]$ and any $y \in [\![A]\!]$, we set

$$\mathcal{D}([z]_m) = \overline{D} \qquad \delta([z]_m) = 0 \qquad \forall z.\, z = x \otimes y$$

for all other faces $x \subseteq R$, s.t. $\mathcal{C}([x]_{p-1}) = C/(J_1)$ and y s.t. $\mathcal{C}([y]_{m-1}) = C'/(J_2)$

$$\begin{aligned} \mathcal{D}([x]_m) &= \overline{D}/(J_1) & \delta([x]_m) &= \gamma([x]_m) \\ \mathcal{D}([y]_m) &= \overline{D}/(J_2) & \delta([y]_m) &= \gamma([y]_m) \end{aligned}$$

finally, for all $x \in G$, we set

$$\mathcal{D}([x]_{p+r}) = \overline{E}/(F) \qquad \delta([x]_{p+r}) = \gamma([x]_{p+r}) \qquad 0 < r < m-p$$

where $F = (\mathcal{C}([x]_{p+r}))^{\vee}$, and E is given by

$$E = \begin{cases} \underline{\mathcal{C}([x]_{p+r})} \cup C' & \text{if } C \subseteq \underline{\mathcal{C}([x]_{p+r})} \\ \underline{\mathcal{C}([x]_{p+r})} & \text{otherwise} \end{cases}$$

Commuting Cut *We would refer to the Figure 4. Let the proof with cut be represented by the co-chain $[\gamma^m]$ (with underlying CCS $\mathcal{C}$) on the simplicial set Δ^n. Let the exponential box with promoted formula B, be introduced at stage $p < m$, and represented by the face R_1; let the box with promoted formula A, be introduced at stage $q < m$, and represented by the face R_2. Let the set of principal doors of the box R_1 be denoted by the set $P_1 = \{P_1^1, \ldots, P_h^1\}$. Let the set of principal doors of the box R_2 be denoted by the set $P_2 = \{P_1^2, \ldots, P_r^2\}$. For any $x \in P_1$, let $C_1 = \underline{\mathcal{C}([x]_p)}$; for any $x \in P_2$ let $C_2 = \underline{\mathcal{C}([x]_q)}$. Let G_1 denote the set of faces g such that for some set S_1, and for some stage $l_1 > p$, $\underline{\mathcal{C}([g]_{l_1})} = C_1 \cup S_1$; Let G_2 denote the set of faces g such that for some stage $l_2 > q$, $\underline{\mathcal{C}([g]_{l_2})} = \overline{C_2 \cup S_2}$. We modify γ, to obtain a new co-chain δ (w.r.t CCS $\mathcal{D}$) as follows: let $D = \overline{C_1 \cup C_2}$; we set*

$$\mathcal{D}([z]_m) = \overline{D} \qquad \delta([z]_m) = 0 \qquad \forall z.\, z = x \otimes y,\ x \in [\![B^{\perp}]\!],\ y \in P_1$$

for all other defined faces $x \subseteq R_1$, *s.t.* $(\mathcal{C}([x]_{p-1}))^{\vee} = J_1$ *and* $y \subseteq R_2$ *s.t.* $(\mathcal{C}([y]_{q-1}))^{\vee} = J_2$

$$\mathcal{D}([x]_m) = \overline{D}/(J_1) \qquad \delta([x]_m) = \gamma([x]_m)$$
$$\mathcal{D}([y]_m) = \overline{D}/(J_2) \qquad \delta([y]_m) = \gamma([y]_m)$$

let $J = \{\pi(\sharp X) \mid X \in ?\Gamma \cup ?\Delta \cup \{A\}\}$; *for* $x \in [\![A]\!]$ *we set*

$$\mathcal{D}([x]_{m+1}) = \overline{D}/(J) \qquad \delta([x]_{m+1}) = 0$$

finally, for all $g_1 \in G_1$ *and* $g_2 \in G_2$ *we set*

$$\mathcal{D}([g_1]_{p+r}) = \overline{E_1}/(F_1) \qquad \delta([g]_{p+r}) = \gamma([g]_{p+r}) \qquad 0 < r < m-p$$
$$\mathcal{D}([g_2]_{q+r}) = \overline{E_2}/(F_2) \qquad \delta([g]_{q+r}) = \gamma([g]_{q+r}) \qquad 0 < r < m-q$$

where $F_1 = (\mathcal{C}([g_1]_{p+r}))^{\vee}$, *and* $F_2 = (\mathcal{C}([g_2]_{q+r}))^{\vee}$; *and* E_1 *and* E_2 *are given by*

$$E_1 = \begin{cases} \underline{\mathcal{C}([g_1]_{p+r})} \cup C_2 & \textit{if } C_1 \subseteq \underline{\mathcal{C}([g_1]_{p+r})} \\ \underline{\mathcal{C}([g_1]_{p+r})} & \textit{otherwise} \end{cases}$$

$$E_2 = \begin{cases} \underline{\mathcal{C}([g_2]_{q+r})} \cup C_1 & \textit{if } C_2 \subseteq \underline{\mathcal{C}([g_2]_{q+r})}g \\ \underline{\mathcal{C}([g_2]_{q+r})} & \textit{otherwise} \end{cases}$$

5 Conclusions

The framework developed in the previous pages would be applied in the next part of our study to the generation of extensional domains for the higher-type sequential computation. This application depends on the well-known encoding, via the Curry-Howard isomorphism, of such functional terms as proof nets in the exponential-multiplicative fragment. However, the move to the intuitionistic level offers considerable simplification in the representation: boxes may now be uniquely identified by a certain *output* type, and tensor structure may simply be represented in the juxtaposition of boxes. The par structure is used to represent abstraction, and this may simply adjuncted to the output type information on the boxes. A potential complication is that both tensor and par used in such a way need to be non-commutative; but, as we would see, this doesn't complicated the formalism really. The most important simplification is that the global history of the proof, which was represented as a graded co-chain in the current work, would not really be required in that form for the purposes of normalization; the part of that data which is relevant, would be reconstructed purely on the basis of the co-chain value at the final stage.

On the critical side, it would be seen that the model developed in this work is really *static* in some sense: the implementation of the cut is interactional only notionally, and the two component participating in the cut preserve their individual structures without any kind of structural interaction or in particular, feedback. This is in contradistinction to the original model of the Geometry of Interaction developed by J.-Y. Girard and others ([Gir89, DR93, DR95]), mainly in the framework of C^*-algebras, and subsequent ones by S. Abramsky and others ([AJ94a, AJ94b]), mainly in the framework of two-person Games. The import of this deficiency is that there is nothing really like an Execution formula, that can represent the normalization process in an analytical form. We would see the effect of this in our subsequent study, where interactional processes actually occurring in the process of normalization would need a separate kind of reconstruction, seemingly unrelated to the "dynamics" in this part of the study.

On the positive side, the process-theoretic aspect of our representation carries independent interest for insights it offers into the process-structure of proof-objects. Questions as the structural and other equivalences imported into the study of proofs through this process would surely be of interest for the development of a finer-grained framework for the semantics of proofs, in the constructive tradition of proof theory. Finally, recent work by Fiore, Plotkin and Power ([FPP97]) has highlighted the role of cuboidal sets in instantiating models of Axiomatic Domain Theory. The idea of a cuboidal set is a natural generalization of the simplicial set that would occur if, within the current framework, one would wish to accomodate the additives. Though speculative at this stage, it would be reasonable to imagine that the simplicial and cuboidal toposes could play a significant role in developing intutionistic universes (with general recursion) that has a *sequential* realizability structure as its basis.

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$$
\dfrac{
\begin{array}{c} 1 \\ \vdots \\ \vdots \\ \dfrac{\vdash \Delta}{\vdash \Delta, ?A^{\perp}} \end{array}
\qquad
\begin{array}{c} 2 \\ \vdots \\ \vdots \\ \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \\ \vdots \\ 3 \\ \vdots \\ \vdash !A, \Gamma' \end{array}
}{\vdash \Delta, \Gamma'}
\quad \Longrightarrow \quad
\begin{array}{c} 1 \\ \vdots \\ \vdots \\ \dfrac{\vdash \Delta}{\vdash \Delta, ?\Gamma} \\ \vdots \\ 3 \\ \vdots \\ \vdash \Delta, \Gamma' \end{array}
$$

Figure 1: A Weakening Cut

$$
\dfrac{
\begin{array}{c} 1 \\ \vdots \\ \vdots \\ \dfrac{\vdash \Delta, A^{\perp}}{\vdash \Delta, ?A^{\perp}} \end{array}
\qquad
\begin{array}{c} 2 \\ \vdots \\ \vdots \\ \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \\ \vdots \\ 3 \\ \vdots \\ \vdash !A, \Gamma' \end{array}
}{\vdash \Delta, \Gamma'}
\quad \Longrightarrow \quad
\begin{array}{c}
\dfrac{
\begin{array}{c} 1 \\ \vdots \\ \vdots \\ \vdash \Delta, A^{\perp} \end{array}
\qquad
\begin{array}{c} 2 \\ \vdots \\ \vdots \\ \vdash A, ?\Gamma \end{array}
}{\vdash \Delta, ?\Gamma} \\ \vdots \\ 3 \\ \vdots \\ \vdash \Delta, \Gamma'
\end{array}
$$

Figure 2: A Dereliction Cut

$$
\dfrac{\dfrac{\begin{array}{cc}1 & 4\\ \vdots & \vdots\\ \vdash \Delta, ?A^{\perp}, & ?A^{\perp}\end{array}}{\vdash \Delta, ?A^{\perp}} \qquad \begin{array}{c}2\\ \vdots\\ \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\\ \vdots\\ 3\\ \vdots\\ \vdash !A, \Gamma'\end{array}}{\vdash \Delta, \Gamma'}
\quad \Longrightarrow \quad
\begin{array}{c}
\dfrac{\begin{array}{c}2\\ \vdots\\ \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\end{array} \quad \begin{array}{cc}1 & 4\\ \vdots & \vdots\\ (\vdash ?A^{\perp}, \Delta, & ?A^{\perp})\end{array} \quad \begin{array}{c}2\\ \vdots\\ \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\end{array}}{\vdash \Delta, ?\Gamma, ?\Gamma}\\
\vdots\\
\text{Contractions}\\
\vdots\\
\vdash \Delta, ?\Gamma\\
\vdots\\
3\\
\vdots\\
\vdash \Delta, \Gamma'
\end{array}
$$

Figure 3: A Contraction Cut

$$
\dfrac{\begin{array}{c}1\\ \vdots\\ \dfrac{\vdash A, ?\Gamma, ?B^{\perp}}{\vdash !A, ?\Gamma, ?B^{\perp}}\\ \vdots\\ 2\\ \vdots\\ \vdash \Gamma', ?B^{\perp}\end{array} \qquad \begin{array}{c}3\\ \vdots\\ \dfrac{\vdash B, ?\Delta}{\vdash !B, ?\Delta}\\ \vdots\\ 4\\ \vdots\\ \vdash !B, \Delta'\end{array}}{\vdash \Gamma', \Delta'}
\quad \Longrightarrow \quad
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}1\\ \vdots\\ \vdash A, ?\Gamma, ?B^{\perp}\end{array} \qquad \begin{array}{c}3\\ \vdots\\ \dfrac{\vdash B, ?\Delta}{\vdash !B, ?\Delta}\end{array}}{\vdash A, ?\Gamma, ?\Delta}}{\vdash !A, ?\Gamma, ?\Delta}\\
\vdots\\
2,4\\
\vdots\\
\vdash \Gamma', \Delta'
\end{array}
$$

Figure 4: A Commuting Cut