The Geometry of Sequential Computation II: Full abstraction for PCF

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1 Introduction

In this study, our earlier representation of linear (MELL) proofs using affine simplices is used to develop an approach to a certain version of the problem of full abstraction for a denotational semantics for PCF. In this set of papers our interest is fundamentally in the the problem of syntax-independent representations of sequentiality, and the semantics of sequentially computable functions derived therefrom: thus we look at order-extensional models over the *free interpretation*—that is, the free continuous algebra generated by the constants of PCF—and not the standard interpretation—which has *de facto* come to characterize the problem of full abstraction for PCF.

We stay as far as possible, within the classical ideas of Scott, Berry and others, by conceiving our denotational space as structured essentially as a certain kind of topological space (specifically, a consistently complete, algebraic cpo); the generalization we use, is to look at this space not just as a set of points as in traditional topology, but as a *variable* set, with its domain of variation structured by the geometry of *simplicial sets.* This kind of generalization is well-known, coming down to us from the work of Lawvere and others on toposes (particularly, Grothendieck toposes). However, we would not go the full way along the construction of a topos, since we require the geometry only in a very rudimentary way. The geometry used to enrich our points is then used to control the size of the function spaces by cutting them down to consist of only those kinds of continuous maps, which satisfy a criterion akin to the Kahn-Plotkin ([KP93]) notion of sequential functions, but framed purely in terms of our underlying geometry.

An interesting aspect of this model is that it shares certain features with two other important current approaches to the full abstraction problem—the Games model ([AJM94, AJ94a]) and the Strong Stability/Sequential algorithms framework ([BE91, BE93, Ehr94]): with respect to the former, it would be seen that the applicative structure in our model is defined on the basis of a structure reminiscent of strategies, especially with regard to the alternation and well-bracketing criteria; on the other hand, the domains in our model is equipped with a notion of *Cells* and hence admits the notion of a Coherent subset, of the Strong Stability approach. Finally, since we are more interested in models over the free interpretation, our attempt does *not* need to use an observable quotient in the eventual step in order to eliminate intensional structure. On the other hand, this entails that there is a great deal of syntactic determination in the model—a dialectic that is hard to avoid in this game.

2 Basic Definitions

We shall use the symbol $\mathbb{N}$ for the set of natural numbers $\{0, 1, \dots\}$. We recall a few notions about the infinite dimensional Euclidean Space $\mathbb{E}^\infty$; most of the basic ideas are taken from the book by Munkres on Algebraic Topology ([Mun84]). Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}^\infty$ the $\mathbb{N}$-fold product of $\mathbb{R}$ with itself. An element of $\mathbb{R}^\infty$ may be thought of as the tuple $\langle x_i \rangle_{i\in\mathbb{N}}$. Let $\mathbb{E}^\infty$ be defined as the subset of $\mathbb{R}^\infty$ consisting of tuples $\langle x_i \rangle_{i\in\mathbb{N}}$ with x_i non-zero for only a finite number of indices i. $\mathbb{E}^\infty$ is a vector space with component-wise addition and and the usual scalar multiplication. We would denote its zero vector as $\vec{0}$. A basis for $\mathbb{E}^\infty$ is the set

$$\mathcal{B}(\mathbb{E}^\infty) = \{\epsilon_i \mid i \in \mathbb{N}, b_i : \mathbb{N} \to \mathbb{R}\}$$

where ϵ_i maps only the natural number i to 1 and all other natural numbers to 0. The space $\mathbb{E}^\infty$ is a topological space: the "standard" topology on this space is induced by the metric:

$$\|x - y\| = \max_{i\in\mathbb{N}}\{|x_i - y_i|\}$$

, and it is straightforward to show that with this topology, every finite dimensional subspace of $\mathbb{E}^\infty$ is homeomorphic to the space $\mathbb{R}^n$ for some finite n.

For notational economy in the sequel, we define the sequence $\langle \vec{b}_i \rangle_{i\in\mathbb{N}}$ of vectors in $\mathbb{E}^\infty$:

$$\langle \vec{b}_i \mid \vec{b}_0 = \vec{0}, \vec{b}_{i+1} = \epsilon_i \rangle_{i\geq 0}$$

Given a linearly independent set $\mathbf{P} = \{\vec{P}_0, \dots \vec{P}_k\}$ of points (equivalently, position vectors) in $\mathbb{E}^\infty$, the k-plane spanned by the points $\mathbf{P}$ is the set of points satisfying

$$\vec{x} = \sum_{i=0}^{k} a_i \vec{P}_i \qquad \text{with} \sum_{i=0}^{k} a_i = 1 \qquad \text{s.t. } \forall i.\, 0 \leq a_i \leq 1 \tag{2.1}$$

The standard affine n-simplex Δ^n is the n-plane spanned by the set of points $\{\vec{b}_0, \dots, \vec{b}_n\}$. A finite sub-sequence $\overline{k} = \langle k_1, \dots k_n \rangle$ of the sequence of natural numbers $\mathbb{N}$ (in the usual order) would be said to be *continuous*, iff the map $k_{i+1} = k_i + 1$ $(0 < i < n)$.

Given a subset $K \subseteq \mathbb{N}$, or alternatively, an increasing map $K : [k] \to \mathbb{N}$ (where $k = |K|$), we shall say that the k-plane spanned by the set of points $\{\vec{b}_i \mid i \in K\}$ (respectively, the set $\{\vec{b}_{K(i)} \mid i \in [k]\}$), denoted as $\Delta(K)$, is the K*-face*—presumably of some standard n-simplex, with $n \geq k$. The ordered set $\{\vec{b}_i \mid i \in K\}$ would be called the *Span* of the face $\Delta(K)$. The smallest vector in *Span*(K) would be denoted as $\wedge(K)$ and the largest as $\vee(K)$. A face K would be said to be *principal*, if $0 \in K$; we would usually represent a principal face K by the set $\angle(K) \equiv K - \{0\}$; by a similar abuse of notation, we would say that *Span*(K) for a principal face $\Delta(K)$, is the ordered set of vectors $\{\vec{b}_i \mid i \in \angle(K)\}$. Thus, for principal faces, the operator $\wedge$ would now denote the least vector in this notion of the Span.[1]

We denote by Δ^∞, the subspace of $\mathbb{E}^\infty$ given as

$$\Delta^\infty = \bigcup_{n\geq 0} \Delta^n$$

—in other words, the collection of simplices of the form Δ^n (for finite n); thus, every finite sub-collection of this union is contained in some $\mathbb{R}^n$ (for some finite n), though not the entire collection—a consideration made possible precisely by the infinite dimensionality of the space $\mathbb{E}^\infty$ (*cf.*[Mun84, pg. 14]). We induce a coarser topology on Δ^∞ by taking as its basis

$$\mathcal{B}(\Delta^\infty) = \{\mathit{Int}(\Delta(K)) \mid K \subseteq \mathbb{N}\}$$

[1]This abused representation of principal faces would in practice never create confusions with regard to non-principal faces, since we would consider only principal faces in the sequel.

where, as the interior operator is defined w.r.t. the standard topology on $\mathbb{E}^\infty$, and acts as $Int(\Delta(K)) = \Delta(K) - Boundary(\Delta(K))$.

Let the set of Ground Types in PCF be denoted as $\mathbb{T} = \{\mathbb{B}, \mathbb{N}\}$ and the set of constants (excluding the recursion combinator Y, for any Type) as $\mathbb{C}$; let us assume some denumerably infinite of set, say $\mathbb{N}$, which are implicitly meant to be a set of occurrence identifiers. For a general Type $\bar{A}$ represented as $A_0 \Rightarrow \ldots \Rightarrow A_{n-1} \Rightarrow A$, where A is a Ground Type, we would say that the *output type* of $\bar{A}$ is A. Let K be the set defined as $K = K_v \cup K_c$ where

$$K_v = \mathbb{T} \times \mathbb{N} \qquad K_c = \{\langle c, t, n\rangle \mid c \in \mathbb{C} \text{ with output type } t \text{, or a special symbol } \varnothing\}$$

and let $\overline{K}$ denote the free $(\mathbb{Z}/\mathbf{2})$-module[2] generated by K^*, the free monoid over K. Note that $\overline{K}$ has a standard bi-algebra structure; we shall denote by the (infix) operator symbol $*$, the composite

$$\overline{K} \times \overline{K} \xrightarrow{\delta} \overline{K} \otimes \overline{K} \xrightarrow{\nabla} \overline{K}$$

where δ is the universal $\mathbb{Z}/\mathbf{2}$-bilinear function yielded by the definition of the tensor product $\otimes$, and ∇ is the algebra multiplication.

We shall consider the category of set-valued sheaves[3] Δ^∞—that is, the category of functors $\mathcal{O}(\Delta^\infty)^{op} \to \mathbf{Sets}$, satisfying the usual sheaf conditions. Our object of interest, and what we shall call the *universal type*, is the unique sheaf χ satisfying the following conditions:

1. for any basic open $a \in \mathcal{B}(\Delta^\infty)$, $\chi(a) = \overline{K}$ (the latter now considered as a set); and
2. for an open set a, s.t. $a = b \cup c$, we shall require that $\chi(a) = \chi(b) \times \chi(c)$; moreover, the restriction maps $\chi(b) \leftarrow \chi(a) \to \chi(c)$ are given by the Cartesian projections.

The terminal object in this category would be denoted by as $\bar{1}$, mapping every open set to the terminal object in **Sets**.

In the sequel, we would use abuse notation by using the same notation for the subset $K \subseteq \mathbb{N}$, instantiating a face $\Delta(K)$, and the basic open set $Int(\Delta(K))$. We shall reserve the special face-designator $\underline{0}$ for (the face formally corresponding to) the empty set $\emptyset$.

Consider a global section (simply, section) $\nu : \bar{1} \to \chi$ of our universal type. Any basic open derived from a face $K \subseteq \mathbb{N}$ such that the value of ν_K is not 0, would be said to be a *box* (of the section ν); if the face K is principal, then the corresponding box would be said to be a principal box. Such a box would be said to be *contiguous* if it is principal and $\angle(K)$ is a continuous sub-sequence of the natural numbers. A (finite) set of contiguous boxes $\overline{B}$ would be said to be *contiguous*, if it is pairwise disjoint and can be uniquely ordered into a sequence $\langle B_1, \ldots, B_n\rangle$, s.t. $\vee(B_i)$ is one lower in the enumeration order than $\wedge(B_{i+1})$ (for $0 \le i < n$). We define the order $\leqslant$ on boxes of a specified section ν: $B_1 \leqslant B_2$ iff $B_1 \subseteq B_2$. The set of boxes strictly lesser than a box x in the $\leqslant$ order, would be denoted as $\downarrow(x)$. We would use the symbol $\underline{\downarrow}(x)$ for the set of maximal boxes in $\downarrow(x)$.

A section $\nu : \bar{1} \to \chi$ would be said to be *principal* if the set of its boxes is finite and consists only of principal boxes. The data for a principal global section ν may be represented as a sum $\sum_{i=0}^{n} a_i x_i$, where x_i is a box of ν and a_i the value of ν_{x_i}. Henceforth, we would consider only principal sections. A box of the (principal) section ν is said to be *well-structured* iff it satisfies the following inductively specified condition:

[2] $(\mathbb{Z}/\mathbf{2})$-modules are essentially *sets*; however, understanding them as such kinds of modules allows us greater economy in the formulation of certain notions—in particular, the map $*$ used in the sequel.

[3] It is possible to give an entirely equivalent account using the notion of *co-chains* used in Part I of our study; we prefer to use the idea of a sheaf since it indicates a context of generalization with regard to the simplicial topos, which we would wish to pursue in later treatments of this theme of sequential computation.

1. any 1-box is well-structured;

2. a contiguous box x with $Span(x) = \{\vec{b}_r, \ldots, \vec{b}_{r+p+1}\}$, is well-structured, if the set $\underline{\downarrow}(x)$ is a contiguous set of well-structured boxes $\{x_1, \ldots, x_n\}$, with $\wedge(x_1) = \vec{b}_r$ and $\vee(x_n) = \vec{b}_{r+p}$ for some $r, p \in \mathbb{N}$

(we assume that the unique sequencing of the maximal boxes of $\downarrow(x)$ is $\langle x_1, \ldots, x_n \rangle$). The (principal) section $\nu = \sum_{i=0}^{n} a_i x_i$ is said to be *well-structured* if the set of its boxes is $\leqslant$-directed with its $\leqslant$-lub well-structured, and for all $i \neq j$, $\downarrow(x_i) \cap \downarrow(x_j) = \emptyset$. Under these conditions, any well-structured section ν has a maximal box, which we shall denote as $\underline{\nu}$. In the sequel, we would assume that the value of $\underline{\downarrow}(x)$ for any box x of a well-structured section is an ordered set (or sequence) $\langle x_1, \ldots, x_n \rangle$, with $\vee(x_i)$ being lower in the enumeration order than $\wedge(B_{i+1})$ (for $0 \leq i < n$).

A well-structured section ν is said to be *typable* if the conditions 1–2 below are satisfied.

1. for any box $x \in \nu$, an *unique* value of $\nu(x)$, which is either of the form a for some $a \in K$, or of the form $s_x * s_1 * \cdots * s_n$ where $s_x \in K$ and each s_i is of the form of $s_{i_1} + \cdots + s_{i_k}$, with each $s_{i_j} \in K_v$. We shall write $s_x = Head(x)$, $\nu(x) = Coeff(x)$ and $s_i = Coeff_i(x)$;

2. for each s_{i_j}, there is an *unique* box $y \leq x$ such that $Head(y) = s_{i_j} \notin K_c$.

An important condition that we would stipulate is that the form of $\nu(x)$ in terms $*$-factors would be assumed as *given* in every instance; this is to avoid problems with non-unique such factorizations—though it should be noted, that taking the second condition into account, we could always obtain an unique factorization. A box x, such that $Head(x)$ is of the form $\langle \varnothing, \_, \_ \rangle$, would be said to be a *weakening* box.

For any typable section ν, we may define, by mutual recursion, the following parameters for any box $x \in \nu$, which we shall call *Spine* (σ_ν) and *Type* (τ_ν). For any box $x \in \nu$, we shall write $Child_\nu(x)$ for the sub-sequence of $\underline{\downarrow}(x)$ including only (and all) its *non-weakening* members. Then,

$$\sigma_\nu(x) = \begin{cases} Head(x) & \text{if } x \text{ is } \leqslant\text{-minimal} \\ \tau_\nu(x_1) \Rightarrow \ldots \Rightarrow \tau_\nu(x_n) \Rightarrow Head(x) & \text{otherwise} \end{cases}$$

$$\tau_\nu(x) = \begin{cases} Head(x) & \text{if } x \text{ is } \leqslant\text{-minimal} \\ \sigma_\nu(y_1) \Rightarrow \ldots \Rightarrow \sigma_\nu(y_m) \Rightarrow Head(x) & \text{otherwise} \end{cases}$$

where $\langle x_1, \ldots, x_n \rangle = Child_\nu(x)$ and y_i is the unique box $\leqslant x$ such that $Head(y_i)$ is a summand in $Coeff_i(x)$.

Note the non-determinism in these definitions, expressed in the last line of the previous paragraph: since $Head(y_i)$ is a summand, we may obtain a set of values instead of a single value when we evaluate the σ for any other summand of $Coeff_i(x)$. Thus we would say that the section ν is *well-typed* (or sometimes *well-formed*) if the definition of the parameters *Spine* and *Type* yields a single well-defined value for any box of ν.

With regard to a calculus with non-logical constants, we would impose an additional constraint on well-typed sections to take into account the type restrictions on constants. Thus: for any box x of a well-typed section ν, with $Head(x) = \langle c, t, n \rangle \in K_c$, such that $c : A_0 \Rightarrow \ldots \Rightarrow A_{n-1} \Rightarrow t$, we must have that $\underline{\downarrow}(x) = \langle x_0, \ldots, x_{n-1} \rangle$ with $\tau_\nu(x_i) = A_i$ (for $0 \leq i < n$).

For any well-typed section ν, we would say that the *Type* of ν, $\tau(\nu)$ is the value of $\tau_\nu(\underline{\nu})$. For any Type A, we would write $\|A\|$ for the set of (well-typed) sections ν with $\tau(\nu) = A$.

3 Type Extensions

In this section we define certain relations on $\|A\|$ (for any Type A) that would facilitate the interpretation of their sections in syntactic terms. We also take the first steps in defining the extensions of the Types, and explore their order-theoretic structure.

For a section $\nu \in \|A\|$, with maximal box $\underline{\nu} \equiv x$, with $\nu(x) = s_x * s_1 * \cdots * s_n$, and a natural number $0 < k \leq n$, we define the operation $\mathit{Trunc}_k(x)$ to result in the modified section ν', with the coefficient of its maximal box x set to $s_x * s_{k+1} * \cdots * s_n$; all other data for the section are unchanged. Obviously, the modified section would no longer be an element of $\|A\|$.

Given a principal face y, with $\mathit{Span}(y) = \{\vec{b}_k, \ldots, \vec{b}_{k+p}\}$, and some integer n, we define $\mathit{Shift}_n(y)$ to be the principal face with $\mathit{Span} = \{\vec{b}_{k+n}, \ldots, \vec{b}_{k+p+n}\}$. Similarly, we define the operation $\mathit{Exp}_n(y)$ to be the principal face with $\mathit{Span} = \{\vec{b}_k, \ldots, \vec{b}_{k+p+n}\}$. Consider a section $\nu \equiv \sum_{i=0}^{n} a_i.x_i$, and a specific index $k \in [n]$; let $\mathit{Span}(x_k) = \{\vec{b}_m, \ldots, \vec{b}_{m+p}\}$. We define the section $(\nu \backslash k)$ to be

$$\sum_{i \in I} a_i.\mathit{Shift}_{(-m)}(x_i) \qquad\qquad \text{where, } \forall i \in I.\, x_k \leqslant x_i.$$

and this formalizes the operation of instantiating the box corresponding to the index k as a well-formed section in its own right. At several points in the sequel we would, by abuse of notation, regard a box of a section as a section by itself without any further clarification. Obviously in such cases, an application of the $(.\backslash.)$ operation would be implicit.

Consider now a section $\nu \equiv \sum_{i=0}^{n} a_i.x_i$ with $\vee(\underline{\nu}) = \vec{b}_l$, a specific index $k \in [n]$ and another section $\mu \equiv \sum_{j=0}^{m} b_j.y_j$ such that $\tau(\mu) = \tau_\nu(x_k)$. Let I denote the subset of $[n]$, s.t. $\vee(x_i)$ is lesser than $\wedge(x_k)$ (in the enumeration order) for any $i \in I$; let H denote the subset of $[n]$ such that $x_k \leqslant x_h$, $x_k \neq x_h$ for any $h \in H$; finally, let J denote the subset of $[n]$ s.t. $\wedge(x_j)$ is greater than $\vee(x_k)$ for any $j \in J$. Let the cardinality of $\mathit{Span}(\underline{\mu})$ be r, and that of $\mathit{Span}(x_k)$ be s; let $q = r - s$. Then we define the section $\nu[k \leftarrow \mu]$ as

$$\sum_{i \in I} a_i.x_i + \sum_{j \in J} a_j.\mathit{Shift}_q(x_j) + \sum_{h \in H} a_h.\mathrm{Exp}_q(x_h) + \mathit{Coeff}(\underline{\mu}).\underline{\mu}$$

—which formalizes the operation of replacing the box x_k in ν by the maximal box in μ.

We define the following equivalence on $\|A\|$. Consider sections $\nu = \sum_0^n a_i x_i$ and $\mu = \sum_0^n b_j y_j$ both in $\|A\|$. We define an equivalence relation $\stackrel{\pi}{=}$ as follows: $\nu \stackrel{\pi}{=} \mu$ iff there exists a permutation π on the set $[n]$, such that for all $i \in [n]$:

$$\begin{aligned} \tau_\nu(x_i) &= \tau_\mu(y_{\pi(i)}) \text{ and} \\ \mathit{Child}_\nu(x_i) &= \pi(\mathit{Child}_\mu(y_{\pi(i)})) \end{aligned}$$

where we define $\pi(x_i) \triangleq y_{\pi(i)}$, and extend this to act on sets and sequences as usual. For any type A, we denote $(\|A\| / \stackrel{\pi}{=})$ as $[\![A]\!]$, and call it the *extension* of the Type A.

Now given sections $\nu \equiv \sum_{i=0}^{n} a_i.x_i$ and $\mu \equiv \sum_{j=0}^{m} b_j.y_j$, and an element $k \in \overline{K}$, we define the operation $\nu \boxtimes \mu$ as follows: let $\mathit{Span}(\underline{\nu}) = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathit{Span}(\underline{\mu}) = \{\vec{b}_1, \ldots, \vec{b}_m\}$; let z denote the principal face with $\mathit{Span}(z) = \{\vec{b}_1, \ldots, \vec{b}_{n+m+1}\}$; We define the non-well-formed section $\nu \boxtimes \mu$ as:

$$\nu \boxtimes \mu = 1.z + \sum_{i=0}^{n} a_i.x_i + \sum_{j=0}^{m} b_j.\mathit{Shift}_n(y_j).$$

Thus, the operator $\boxtimes$ essentially juxtaposes the maximal boxes of its arguments, preserving the coefficient values of all sub-boxes. Now given a section ν (typically derived as the result of a $\boxtimes$

operation), and $\alpha \in \overline{K}$, we define the operation $\alpha.\nu$ as the operation which sets the *Coeff* of the maximal box of ν to α.

We define a couple operations on type-extensions and sections. Note that any Type A may be written in the form $A_0 \Rightarrow \ldots \Rightarrow A_{n-1} \Rightarrow \bar{A}$, with $\bar{A}$ atomic; we shall write $Out(A)$ for the type (-occurrence) $\bar{A}$. For any Type A we may define the constant section $\eta(A)$ in $[\![A]\!]$ as follows.

$$\eta(A) = \begin{cases} A^{(k)}.\vec{b}_1 & \text{if } A \text{ is atomic} \\ (A_n^{(k)} a_0 \ldots a_{n-1}).\eta(A_0) \boxtimes \cdots \boxtimes \eta(A_{n-1})) & \text{if } A \text{ is of the form } A_0 \Rightarrow \ldots \Rightarrow A_{n-1} \Rightarrow A_n \end{cases}$$

where $A^{(k)}$ is an element of K_v of the form $\langle A, k\rangle$ for some $k \in \mathbb{N}$, and in the second clause, $a_i = Head(x_i)$ where x_i the maximal box in $\chi(A_i)$ (for $0 \leq i < n$); note that we may assume A_n as atomic without loss in generality. The constant section $\perp_A$ in $[\![A]\!]$ is defined to be $\eta(A)$, but with $Head(x)$ set to $\langle \Omega, A, k\rangle$, where x is the maximal box of $\eta(A)$, and k is some occurrence identifier in $\mathbb{N}$.

On the basis of the previous notions, we define a partial order $\sqsubseteq_A$ on any $[\![A]\!]$ as follows:

$$\nu \sqsubseteq_A \mu \Leftrightarrow ((\nu \backslash k) = \perp_B \ \& \ \mu = \nu[k \leftarrow \rho])$$

for some natural number k, type B and section ρ with $\tau(\rho) = B$. Obviously, we have effaced the difference between sections and their $\stackrel{\pi}{=}$-equivalence classes; this should be taken as implicit in this and other statements in the sequel.

We note the following simple proposition.

Proposition 3.1 *For any Type A, the poset $([\![A]\!], \sqsubseteq_A)$ is a meet-semi-lattice with least element $\perp_A$; it is also consistently complete.*

Proof: Consider sections $\nu, \mu \in [\![A]\!]$; writing $\smile$ for the binary predicate of consistency, it is straightforward to verify that $\nu \smile \mu$ iff there is a section $\pi \equiv \sum_0^n a_i.x_i \in [\![A]\!]$, and indices $k_1, k_2 \in [n]$, such that $(\pi \backslash k_1) = \perp_{A_1}$ and $(\pi \backslash k_2) = \perp_{A_2}$, (for some types A_1 and A_2) and $\nu = \pi[k_1 \leftarrow \nu']$ and $\mu = \pi[k_2 \leftarrow \mu']$ for some sections $\nu' \in [\![A_1]\!]$ and $\mu' \in [\![A_2]\!]$. In this case, we can easily derive that $\nu \sqcup_A \mu = (\pi[k_1 \leftarrow \nu'])[k_2 \leftarrow \mu']$, while $\nu \sqcap_A \mu = \pi$. The meet semi-lattice property is immediate.

We shall denote the directed completion of $[\![A]\!]$ as $[\![\underline{A}]\!]$, and use the same symbol $\sqsubseteq_A$ for its associated partial order. The compact elements of $[\![\underline{A}]\!]$ are in bijective correspondence with the elements of $[\![A]\!]$ and we would denote them by the same symbols. The non-compact elements x would be represented as the ideals $\downarrow_A(x)$ (this use of $\downarrow$ is subscripted by the Type to distinguish it from the previous use in the context of the containment order on boxes). The partial order in $[\![\underline{A}]\!]$ is given as the subset order among ideals as is standard. We note the following property of the completion, which may be easily verified.

Proposition 3.2 *For any Type A, the poset $([\![\underline{A}]\!], \sqsubseteq_A)$ is a consistently complete algebraic CPO (hence a Scott Domain); it is also prime algebraic, and in fact, a dI-Domain.*

We topologize the CPO $[\![\underline{A}]\!]$ by specifying its basis of Scott-open filters, *viz.*:

$$\mathcal{B}([\![\underline{A}]\!]) = \{\uparrow_A(\nu) \mid \nu \in [\![A]\!],\ \nu \text{ sub-maximal}\} \cup \{\emptyset\} \qquad \text{where} \qquad \uparrow_A(\nu) = \{\theta \in [\![\underline{A}]\!] \mid \nu \sqsubseteq_A \theta\}$$

We would use the same symbol to denote the CPO and the topological space based on it.

4 Applicative Structure

In this section, we define the various operations relevant to instantiating a typed combinatory algebra structure on the collection of spaces of the form $[\![A]\!]$. We shall define first, a certain $(\mathbb{Z}/\mathbf{2})$-algebra denoted as $[\mathbb{N}]$, and called the *bracket algebra* over $\mathbb{N}$. Let $\mathcal{B} \equiv \{[,]\}$; then the sub-basis of this algebra is the set $\mathcal{N} = \mathbb{N} \times \mathcal{B}$—we would denote elements $\langle n, [\rangle$ or $\langle n,]\rangle$, as n-annotated brackets, *viz.* $[_n$ or $]_n$ respectively. Thus, $[\mathbb{N}]$ is the free $(\mathbb{Z}/\mathbf{2})$-module generated by $\mathcal{N}$, and is (up to isomorphism) the set of sequences of $\mathbb{N}$-annotated brackets. As usual, this has a standard bi-algebra structure, and we would denote the composite

$$[\mathbb{N}] \times [\mathbb{N}] \xrightarrow{\delta} [\mathbb{N}] \otimes [\mathbb{N}] \xrightarrow{\nabla} [\mathbb{N}]$$

by the operator symbol $*$.

Consider sections $\nu = \sum_{i \in I} a_i.x_i \in [\![A \Rightarrow B]\!]$ and $\mu = \sum_{j \in J} b_j.y_j \in [\![A]\!]$ where the index sets $I, J \subset \mathbb{N}$ are chosen to be disjoint. An element l of $[\mathbb{N}]$ is said to be an *interaction sequence between ν and μ* iff it is a sum of sequences that are well-bracketed (upon ignoring annotations), successive opening (respectively, closing) brackets are alternately annotated by valid indexes in ν and μ respectively, and matching pairs of opening and closing brackets are annotated by the same natural number. A sequence of the form $[_n\centerdot \ldots \centerdot]_n$ in the basis of $[\mathbb{N}]$ would be said to be a *context*, and a context of the form $[_n\centerdot]_n$ would be known as a *minimal* context. We would, by abuse of notation write expressions like $i \leqslant j$ for indices $i, j \in I$ (or in J) to actually mean $x_i \leqslant x_j$ in the appropriate section.

A specific interaction sequence between ν and μ is said to be a *dialogue* if and only if it is generated by the following recursive function $\Psi(\nu, \mu)$: $\Psi(\nu, \mu)$ is defined on the basis of a subsidiary function ψ; we define $\Psi(\nu, \mu) = \psi(\underline{\nu}, \underline{\mu}, 1, 0)$ where the definition of $\psi()$ is as follows:

$$\psi(x_i, y_j, n, m) = [_i\centerdot]_i$$

if the unique box z s.t. $\mathit{Coeff}_n(x_i) = s_n = \mathit{Head}(z)$ is weakening box, with $x, z \in \nu$ if $m = 0$, and in μ otherwise; then again, if $\mathit{Head}(x_i) = \langle \Omega, \_, \_\rangle$, then we define

$$\psi(x_i, y_j, n, m) = [_i\centerdot]_i$$

in the general case $\mathit{Coeff}_n(x) = s_n = s_n^1 + \cdots + s_n^p$, we define

$$\begin{aligned}
\psi(x_i, y_j, n, m) = & [_{k_1} * \psi(y_j, z_1^1, 1, m^+) * \cdots * \psi(y_j, z_{m_1}^1, m_1, m^+) *]_{k_1} \\
& + [_{k_2} * \psi(y_j, z_1^2, 1, m^+) * \cdots * \psi(y_j, z_{m_2}^2, m_2, m^+) *]_{k_2} \\
& \vdots \\
& + [_{k_p} * \psi(y_j, z_1^p, 1, m^+) * \cdots * \psi(y_j, z_{m_p}^p, m_p, m^+) *]_{k_p}
\end{aligned}$$

where z^k is the unique box s.t. $\mathit{Head}(z^k) = s_n^k$, and $k_1, \ldots, k_p$ are the indices (in ν (respectively, μ) if $m = 0$ (respectively $m = 1$)) of the boxes $z^1, \ldots, z^p$, and s.t. $\underline{\downarrow}(z^k) = \{z_1^k, \ldots, z_{m_k}^k\}$ $(0 < k \leq p)$—if z^k is not $\leqslant$-minimal, while $m_k = 1$ and $z_{m_k}^k = \underline{0}$ otherwise; finally, we have

$$\psi(x_i, y_j, n, m) = [_i\centerdot]_i$$

if $y_j = \underline{0}$.

We define a function Φ from dialogues to (appropriately typed) sections. Given $\nu = \sum_{i \in I} a_i.x_i \in [\![A \Rightarrow B]\!]$ and $\mu = \sum_{j \in J} b_j.x_j \in [\![A]\!]$ (where again, the index sets $I, J \subset \mathbb{N}$ are chosen to be disjoint) and the dialogue $X \equiv X_0 + \cdots + X_n$ between ν and μ, we define $\Phi(X, \nu, \mu)$ on the basis of a composed sequence of applications of a subsidiary operation ϕ (on sections) as follows. First let the partial order $\preceq_X$ on the set $\{X_i \mid 0 \leq i \leq n\}$ be defined as: $X_i \preceq_X X_j$ iff $X_i = s.[_p.s_i$ and $X_j = s.[_q.s_j$ for some sequences s, s_i, s_j in the basis of $[\mathbb{N}]$, and such that $p \leqslant q$ in the appropriate section. From the procedure by which the dialogue X is generated, it is a simple exercise to verify the following proposition.

Proposition 4.1 *The partial order $\preceq_X$ equips X with the structure of a forest—with the root of any component tree, a $\preceq$-maximal element.*

Let $\overline{X} = \langle X_{\pi_0}, \ldots, X_{\pi_n} \rangle$ denote any particular linearization of X—*i.e.* a sequence based on some permutation π on the indexing set $[n]$ s.t. the partial order $\preceq_X$ implies the sequential order of $\overline{X}$.

The basic operation involved in the definition of Φ is that of replacement at some box (correspondingly, index) of a section—symbolized earlier in the form $\rho[k \leftarrow \sigma]$. In its invocation in the sequel, we would assume that the indices identify sub-boxes uniquely, even when they have been substituted into sections of which they formed no part originally. We would also extend the definition of the operation $\rho[k \leftarrow \sigma]$ to yield the embedding section ρ in case the particular box indexed by k does not happen to be a box of ρ. In this latter case, we would say that the index k has been *weakened*—a description which would be clarified in the sequel.

We may now define Φ as follows:

$$\Phi(\overline{X}, \nu, \mu) = \prod_{i=0}^{n} \phi(X_{\pi_i}, \nu_i, \mu)$$

where we use the symbol $\prod$ for the sequential composition of operations (in the order of indexing), and $\nu_0 = \mathit{Trunc}_1(\nu)$, and $\nu_{i+1} = \phi(X_{\pi_i}, \nu_i, \mu)$ $(0 \leq i < n))$. ϕ is defined recursively on the form of its first argument:

$$\phi([_b. \overbrace{[_{b_1}. \cdots .]_{b_1}}^{B_1} . \cdots . \overbrace{[_{b_k}. \cdots .]_{b_k}}^{B_k} .]_b, \rho, \sigma) \\ = \begin{cases} \rho & \text{if } k = 0 \text{ or } \exists C.\, \rho = \bot_C \\ \rho\,[b \leftarrow \sigma] & \text{if either } x_b \text{ is } \leqslant\text{-minimal, or } \exists C.\, \sigma = \bot_C \\ \prod_{j=1}^{k} \phi(B_j, \rho_j, \beta_j) & \text{otherwise} \end{cases}$$

where, in the latter case, $\underline{\downarrow}(x_b) = \langle \beta_1, \ldots, \beta_k \rangle$ (in the appropriate section); $\rho_1 = \rho\,[b \leftarrow \mathit{Trunc}_k(\sigma)]$, while $\rho_{j+1} = \phi(B_j, \rho_j, \beta_j)$ (for $1 \leq j < k$).

For sections $\nu \in [\![A \to B]\!]$ and $\mu \in [\![A]\!]$, we would write $\nu.\mu$ for the section $\Phi(s, \nu, \mu)$ where $s = \Psi(\nu, \mu)$.

We are now in a position to define the admissible maps between spaces $[\![\underline{A}]\!]$ and $[\![\underline{B}]\!]$. These are going to be continuous maps, obtained on the basis of a dialogue with a specific invariant section in $[\![A \to B]\!]$, which we shall call the *realizer* for the map. Formally, a continuous map $f : [\![\underline{A}]\!] \to [\![\underline{B}]\!]$ is said to be *sequentially realized* if there is an unique section $\hat{f} \in [\![A \Rightarrow B]\!]$, such that the image of any compact $y \in [\![\underline{A}]\!]$ is given by $\Phi(s, \hat{f}, y)$, and $s = \Psi(\hat{f}, y)$ (or in other words, $f(x) = \hat{f}.x$). By universal properties of the completion, we have the extension of this definition to non-compact elements $x \in [\![\underline{A}]\!]$; thus

$$\hat{f}(\bigsqcup_{i \in I} \{x_i\}) = \bigsqcup_{i \in I} \{\hat{f}.x_i\} \tag{4.2}$$

where $\{x_i \mid i \in I\}$ is the directed set of compact approximants of x. We shall say that $\hat{f}$ *sequentially realizes* the map f; and in general, given any section $x \in [\![A \Rightarrow B]\!]$, we denote the *unique* map sequentially realized by x as $[\![x]\!]$. We note a few important properties of the sequentially realizable functions.

Proposition 4.2 *The map $[\![x]\!]$ sequentially realized by the section $x \in [\![A \Rightarrow B]\!]$ is continuous for any choice of x.*

Proof: Monotonicity follows from the way the dialogue between a realizer and its argument is defined. For arguments y_1, y_2, with $y_1 \sqsubseteq_A y_2$, we know that there exists an index k, such that $(y_1 \backslash k) = \bot_C$ and $y_2 = y_1[k \leftarrow z]$ for some section z and type C with $z \in \|C\|$. Now consider the $\leqslant$-minimal index l, with $k \leqslant l$ in y_1 such that the the minimal context $[_l \bullet]_l$ occurs in the dialogue $\Psi(x, y_1)$. We have two possibilities that may arise: (1) that no such l exists: — in which case there is an instance of weakening in some application occurring at some super-face of the box indexed by k; in this case, from the structure of the procedures ψ and subsequently ϕ, we can see that the dialogue sequence at the this super-face may not contribute to any syntactic extension of the result term—whatever be the substitution z at the index k; (2) there is such an l: — in which case, if $k < l$, then although the dialogue sequence would have no extension at l, the box substituted by the procedure ϕ at the point l would be $\sqsubseteq$-greater in the case of y_2; on the other hand, if $k = l$, then in fact the dialogue sequence at the point l would get extended in the case of y_2 and we would eventually obtain a $\sqsubseteq$-greater result. Now preservation of finite bounded $\sqsubseteq$-lubs ($\sqcup$) follow from the way these lubs are defined in the proof of Proposition 3.1: we apply the preceding argument of monotonicity at each of the non-overlapping pair of indices k_1 and k_2 in the binary case and extend the argument by induction to the case of bounded sets of finite cardinality > 2. Finally, preservation of lubs of infinite $\sqsubseteq$-directed sets fall out of the definition of the sequentially realizable maps on the non-compact elements of our domains.

We derive two interesting corollaries to Proposition 4.2, essentially through the argument through which the property of monotonicity is derived. The first is essentially a version of the Kahn-Plotkin sequential functions in the context of our representation.

Corollary 4.3 *For a function $f \equiv [\![x]\!]$ ($x \in [\![A \Rightarrow B]\!]$), an argument $y \in [\![A]\!]$, and an index $h \in f(y)$, such that $(f(y) \backslash h) = \bot_C$ (for some type C), it is either the case that $(f(z) \backslash h) = \bot_C$ for all z, $y \sqsubseteq_A z$, or there is an unique index $k \in y$ with $(y \backslash k) = \bot_D$ (for some type D) and such that $\bot_C \sqsubseteq_C (f(z) \backslash h)$ implies that $\bot_D \sqsubseteq_D (z \backslash k)$—for any z, $y \sqsubseteq_A z$.*

Proof: Note that we assume in the statement of this Corollary, that the identical parts (*i.e.* common boxes) of sections related by the order $\sqsubseteq$ are identically indexed. The two cases of the corollary correspond to the respective cases in the proof of Proposition 4.2. In other words, the first case corresponds to an instance of weakening at some position l, $k \leqslant l$, and any extension of the simplicial structure at index l would be weakened away and have no effect of the substitution done in procedure ϕ at the position l. The other case is respectively the case when there is no such weakening, and in this case we may always identify the unique index (box) $k \in y$, associated with the index h through the functions ψ and ϕ.

Corollary 4.4 *For sections $x, y \in [\![A \to B]\!]$ with $x \sqsubseteq_{A \Rightarrow B} y$, and any argument $z \in [\![A]\!]$, we have that $[\![x]\!](z) \sqsubseteq_B [\![y]\!](z)$—in other words, $[\![x]\!] \leq_{A \Rightarrow B} [\![y]\!]$ (using the symbol $\leq_{A \Rightarrow B}$ for the point-wise order). In fact, we have that $[\![x \sqcup_{A \Rightarrow B} y]\!](z) = [\![x]\!](z) \sqcup_B [\![y]\!](z)$.*

Proof: Follows by exactly similar reasoning as in Proposition 4.2 applied now to the appropriate indices in x and y.

We may state the fact that is the key to our extensionality result.

Proposition 4.5 *The maps $\widehat{(\bullet)}$ and $[\![\bullet]\!]$ instantiate an order-preserving bijective correspondence between the sequentially realized maps $[\![\underline{A}]\!] \to [\![\underline{B}]\!]$ (with the point-wise order) and the set $[\![A \Rightarrow B]\!]$.*

Proof: It is quite easy to see that (extensionally) distinct maps in $[\![\underline{A}]\!] \to [\![\underline{B}]\!]$ must be realized by distinct sections in $[\![A \Rightarrow B]\!]$: for, if this were not the case, then since the procedures Ψ and Φ are deterministic and depend only on the values of the realizing and the argument sections, we would obtain the same extensional map. Moving to the directed completion poses no further complication in this argument, since the CPOs are algebraic and maps which are distinct on compact elements would be also distinct on the set of non-compact one. For order-preservation, we would invoke Corollary 4.4. The direct argument in the other direction is not so easy, and we would postpone the proof since it is more elegantly obtainable on the basis of a bijective correspondence to another extensional model set up in the last section (*cf.* Theorem 5.1). However, a brief sketch of the direct argument is as follows. Assuming that $f \leq g$ (in the point-wise order), we have that for all x in their domains, $f(x) \sqsubseteq_B g(x)$. Arguing by contraposition, assume that $\hat{f} \not\sqsubseteq \hat{g}$. This implies either of two cases: (1) that $\hat{g} < \hat{f}$, or (2) $\hat{f}$ and $\hat{g}$ are incomparable in the order $\leq$. The first may be discounted, since it leads to a contradiction, in view of the fact of monotonicity. For the second, we argue by induction over the least $\leqslant$-depth at which $\hat{f}$ and $\hat{g}$ differ. By case examination of the form of the value of the section at the box at this depth, we eventually prove that the two sections must agree at every depth (index) at which $\hat{f}$ is not $\perp$. Thus the only possibility is that $\hat{g}$ is a derived by a substitution at such an index, and hence the proposition.

We would extend the class of maps in $[\![\underline{A}]\!] \to [\![\underline{B}]\!]$ by extending the definition of sequential realizability to non-compact sections of $[\![\underline{A \Rightarrow B}]\!]$, in the obvious manner. Given any non-compact section $x \in [\![\underline{A \Rightarrow B}]\!]$ and any section $y \in [\![\underline{A}]\!]$ we have the general definition of the map $[\![x]\!]$ sequentially realized by x.

$$[\![x]\!](y) = \bigsqcup_{i \in I} \bigsqcup_{j \in J} \{[\![x_i]\!](y_j)\} \tag{4.3}$$

where the sets $\{x_i \mid i \in I\}$ and $\{y_j \mid j \in J\}$ are the directed sets of compact approximants of x and y respectively. We define a map f in $[\![\underline{A}]\!] \to [\![\underline{B}]\!]$ to be sequentially realizable iff there exist an element $\hat{f} \in [\![\underline{A \Rightarrow B}]\!]$ such that $f = [\![\hat{f}]\!]$. Correspondingly we extend the definition of the application operation . to non-compact arguments. It is straightforward to show that the Propositions 4.2 and 4.5 and Corollaries 4.3 and 4.4 continue to hold. Thus, anticipating Theorem 5.1 in the next section, we can state the following.

Theorem 4.1 *The maps $\widehat{(\cdot)}$ and $[\![\cdot]\!]$ instantiate an order-isomorphism between the sequentially realized maps $[\![\underline{A}]\!] \to [\![\underline{B}]\!]$ (with the point-wise order) and the set $[\![\underline{A \Rightarrow B}]\!]$.*

5 The Model and its Full-abstraction

In this section we use the applicative structure defined in the previous section, to instantiate a semantics for PCF, and finally prove the various properties that imply the Full-abstraction condition.

We define a bijection $([\cdot]_A, \nabla_A(\cdot))$ (we would suppress the subscript when clear from the context) between the set of Y-free normal form closed terms of the Type A and sections in the corresponding extension $[\![A]\!]$. They would furnish an useful tool for subsequent proofs, facilitating familiar syntactic arguments in most cases. We assume that every variable-occurrence in the terms are type-annotated with an unique type-occurrence (occurrence identifiers are drawn from the set of natural numbers). The definition of the map $([\cdot]_A$ is actually defined on the class of open normal-form terms for technical convenience.

$$\begin{aligned}
[x : A] &= \eta(A)\\
[(\lambda x : A.N) : A{\Rightarrow}B] &= Out(A) \between [N : B]_B\\
[f M_0 \dots M_n : B_0{\Rightarrow}\dots{\Rightarrow}B_k{\Rightarrow}A] &\\
&= \alpha.([M_0 : A_0] \boxtimes \cdots \boxtimes [M_n : A_n] \boxtimes \eta(B_0) \boxtimes \cdots \boxtimes \eta(B_k))
\end{aligned}$$

where, in Clause 2, the infix operator $\between$ modifies the value of *Coeff* of its right argument by inserting its left argument (an element of K_v) after the former's *Head*; in Clause 3, $\alpha = \bar{\alpha}\beta_0 \dots \beta_{k-1}$ with $\bar{\alpha}$ of the form $\langle A, p\rangle$, for p an appropriate occurrence identifier, if f is a variable (respectively, $\langle f, A, p\rangle$, if f is a constant function of type A) (note that we have assumed that A is atomic, without loss of generality). We would extend the definition of $[.]_A$ to terms of A which are not in normal form (though $\mathbf{Y}$-free) by defining in general $[M]_A = [M']_A$ where M' is the normal form of M. We would also regard $[.]_A$ as applying to finite Böhm trees of Type A; it may be noted that for both terms and trees, the function is monotonic, w.r.t. their respective standard orders.

We define next the map $\nabla_A(.)$ from (compact) global sections of the type extension $[\![\underline{A}]\!]$ to (closed) terms of PCF, which would subsequently evidence the condition of definability for our model. Consider a global section $\nu \in [\![A]\!]$, such that $\underline{\downarrow}(\underline{\nu}) = \{\nu_0, \dots, \nu_k\}$ (ordered in the usual fashion), and *Coeff*$(\underline{\nu})$ is of the form $s * s_0 * \dots s_m$, with each $s_i = s_0^i + \cdots + s_{k_i}^i$ $(0 \leq i \leq m)$; then, we define

$$\nabla(\nu) = \lambda x_0 : A_0 \dots x_m : A_m . y \nabla(\nu_0) \dots \nabla(\nu_k) \tag{5.1}$$

where we have elided Type-subscripts for economy, and:

1. y is a "fresh" variable of Type $\sigma_\nu(\underline{\nu})$, or the constant function f if *Head*$(\underline{\nu})$ $(= s)$ is of the form $\langle f, \tau_\nu(\underline{\nu}), \_\rangle$;
2. $A_i = \sigma_\nu(B_0^i)$ such that B_0^i is unique box of ν with $\nu(B_0^i) = s_0^i$;
3. and as a global constraint, we take care that the head variable of the term obtained in the invocation $\nabla(B_{k_i}^i)$ is the variable symbol x_i $(\forall i. \forall j. 0 \leq i \leq m \;\&\; 0 \leq j \leq k_i)$.

In the sequel, we would regard the result of the function $\nabla_A(.)$ as either a term or its Böhm tree, depending on the the context of use. It is quite straightforward again to deduce the following proposition.

Proposition 5.1 *For any $\nu \in [\![A]\!]$, the type of $\nabla_A(\nu)$ is A, and the Böhm tree $\nabla_A(\nu)$ is in its maximal η-expansion. Also $[\nabla_A(\nu)]_A = \nu$ and $\nabla_A([N]_A) = N$ for any section ν and Böhm tree N in its maximal η-expansion.*

Proof: Straightforward induction over the $\leqslant$-structure of the boxes of ν, and properties of the function $[.]$. For the second part, concerning maximal η-expansion of the result, note that the structure of the $\overline{K}$-values for a box of any well-typed section (Section 2), explicitly imposes the maximal η-expansion condition in conjunction with the procedure ∇:—any such box, in its entirety, instantiates the head variable of the corresponding term it represents, and the *Head* value of such a box may only have an atomic Type; thus, any box that would be instantiate a functional variables of higher-type, may only be represented in the appropriate applicative form—which is to imply that the instantiated variable would be in the maximal η-expansion. Note that for higher-type constants, this condition is explicitly imposed as an adjunct to the two typability conditions in Section 2. The last part, expressing that the functions ∇_A and $[.]_A$ are inverses, follows in a straightforward fashion from induction over maximal height of the $\leqslant$-order among the boxes in ν, (correspondingly, on the height of the Böhm tree N).

Our next step is to extend the definitions of $[\mathbf{.}]_A$ and $\nabla()$ to non-compact elements of their respective domains. In the case of the former, we shall now regard its domain of application as Böhm trees partially ordered by the ordering inherited from Ω-match ordering on the corresponding terms; this is, of course, a Scott-domain. We would use the same symbol $[\mathbf{.}]_A$ for the continuous extension of the earlier map to infinite ($\mathbf{Y}$-free obviously) closed Böhm trees, defined by taking the $\sqcup$ of its compact approximants. Analogously, we use the same symbol $\nabla()$ for the continuous extension of the earlier-defined function to non-compact elements—again by taking least upper bounds in its co-domain of Böhm trees. With these extensions it easy to verify the following Lemma.

Lemma 5.1 *The pair of maps $(\nabla_A(\mathbf{.}), [\mathbf{.}]_A)$ instantiate a continuous isomorphism between $[\![\underline{A}]\!]$ and the domain of maximally η-expanded Böhm trees of Type A.*

Proof: Straightforward extension of the arguments of Proposition 5.1 to the respective directed completions on the basis of least upper bounds.

An important idea that we would need to use in the sequel is that application in our typed algebra of sections mirrors that in the continuous algebra of Böhm trees.

Lemma 5.2 *For sections $\nu \in [\![\underline{A{\Rightarrow}B}]\!]$ and $\mu \in [\![\underline{A}]\!]$, we have that*

$$\nabla(\nu)\nabla(\mu) = \nabla(\nu.\mu)$$

where the left-hand juxtaposition signifies application in the applicative algebra of Böhm trees.

Proof: We deal with the compact case first. The result in this case follows by straightforward induction over the length of a leftmost-outermost β-reduction sequence of the tree $(\nabla(\nu)\nabla(\mu))$ (we assume that any δ-reductions are postponed till the after all the β-reductions have been done—*cf.* [BCL85, Prop. 3.1.2]). Note that trees in the range of $\nabla()$ are $\mathbf{Y}$-free. The basic idea is that every step in such a reduction strategy is mimicked by the function $\psi()$ in the previous section—essentially, a β-reduction step, resulting in the substitution of an argument at the head variable of some sub-tree K of the function term n, is echoed by ψ by the opening of a context $[_k$ in the dialogue sequence, where k is the box index corresponding to the sub-tree K. This context is closed by $]_k$ precisely when the reduction sequence initiated by the initial substitution is complete. The sequence of substitutions indicated by the contexts is actually implemented by the operation $\phi()$. The leftmost-outermost order is reflected precisely by the partial order $\prec_X$ on the evaluation $(\phi())$ of summands of the dialogue corresponding to contracted variables (multiple occurrences of bound-variables). Finally, the δ-reductions implemented through the operator $\Delta()$ in our definition, are taken care of by the map $\mathbf{I}_B()$: as may be easily seen, the former has been defined to agree with the latter precisely. The extension to the non-compact case follows by routine application of the continuity of $\nabla(\mathbf{.})$, that of the application operation . for sections (*cf.* Equation 4.3 and following remarks).

This proposition would furnish an important tool to prove facts about the semantic function $\mathbf{P}[\![\mathbf{.}]\!]_\rho$ in the sequel.

On the basis of the above facts, we have the following properties of the sequentially realizable maps.

Lemma 5.3 *The identity map is sequentially realizable, and the sequentially realizable maps are closed under composition.*

Proof: Given sequentially realizable maps $f : [\![\underline{A}]\!] \to [\![\underline{B}]\!]$, realized by $\bigsqcup_I\{f_i \mid i \in I\}$, and $g : [\![\underline{B}]\!] \to [\![\underline{C}]\!]$, realized by $\bigsqcup_J\{g_j \mid j \in J\}$, we claim that the composed map $g \circ f : [\![\underline{A}]\!] \to [\![\underline{C}]\!]$ is sequentially realized by the section $[h : A \Rightarrow C]$ where

$$h = \bigsqcup_J \bigsqcup_I \{[\lambda x : A.\nabla(g_j)(\nabla(f_i)x)]\}$$

(such a normal form would always exist, since recall that the recursion combinator Y would not occur in either of the terms $\nabla(f_i)$ or $\nabla(g_j)$). The result follows directly from Lemma 5.2 and the well-known *syntactic continuity property* of Böhm trees (*cf.* [BCL85, Theorem 3.5.5]). Moreover, composition is obviously left-strict. Again, the identity map in any $[\![\underline{A}]\!] \to [\![\underline{A}]\!]$ would obviously be realized by the section $[(\lambda x : A.x) : A \Rightarrow A]$.

We define the category P to be the sub-category of topological spaces consisting of objects of the form $[\![\underline{A}]\!]$ (for A a PCF Type), and sequentially realizable maps.

Our semantics would be defined over the category $\mathbf{P}$ formed by closing P under products and exponents *conservatively*; *i.e.* the least Cartesian closed sub-category of the category of sets containing P, such that exponents of the objects existing in the latter coincide with the exponents in the new Cartesian closed category. Let us define the binary operator $\Rightarrow$ on sets as follows.

$$X \Rightarrow Y = \begin{cases} [\![\underline{A \Rightarrow B}]\!] & \text{if } X = [\![\underline{A}]\!] \;\&\; Y = [\![\underline{B}]\!] \\ \{f : X \to Y \mid f \text{ a set-theoretic map}\} & \text{otherwise} \end{cases} \tag{5.2}$$

where obviously A and B are PCF Types. Let us denote by $\star \equiv \{*\}$, the terminal object of the category of sets. The category $\mathbf{P}$ is defined inductively: the set objects of $\mathbf{P}$ is the smallest set satisfying:

$$\begin{array}{rcl} & \star \in Obj(\mathbf{P}) & \\ X \in Obj(P) & \Rightarrow & X \in Obj(\mathbf{P}) \\ X, Y \in Obj(\mathbf{P}) & \Rightarrow & (X \times Y) \in Obj(\mathbf{P}) \\ X, Y \in Obj(\mathbf{P}) & \Rightarrow & (X \Rightarrow Y) \in Obj(\mathbf{P}) \end{array}$$

The morphisms in $\mathbf{P}$ are give by the following rules.

$$\begin{array}{rcl} f \in \mathbf{P}(X \times Y, Z) & \Leftrightarrow & \exists! g \in \mathbf{P}(X, Y \Rightarrow Z).\, f(\langle x, y\rangle) = [\![g(x)]\!](y) \\ f \in \mathbf{P}(X, Y \times Z) & \Leftrightarrow & \exists! g \in \mathbf{P}(X, Y).\exists! h \in \mathbf{P}(X, Z).\, f = \langle g, h\rangle \\ \mathbf{P}(X, Y) = X \Rightarrow Y & - & \text{in all other cases} \end{array}$$

where the operator $[\![.]\!]$ in the second clause is defined as earlier, if $(Y \Rightarrow Z) \in Obj(P)$, and is the identity otherwise. Following standard notation, we would refer to the unique morphism g in the second clause as $\Lambda(f)$. In the sequel, we would confuse an element $k \in [\![\underline{A}]\!]$ with the corresponding map $\kappa : \star \to [\![\underline{A}]\!] : * \mapsto k$.

The Cartesian closure of $\mathbf{P}$ is a straightforward consequence of the definition.

Lemma 5.4 *The category* $\mathbf{P}$ *is Cartesian closed.*

Proof: More or less immediate from the defining conditions. The Cartesian projections $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ are the usual in case X or $Y \notin P$; otherwise, for $X, Y \in P$ we have

$$\Lambda(\pi_0) = [\![[\lambda x : A.\lambda y : B.x]]\!] \quad \text{and} \quad \Lambda(\pi_1) = [\![[\lambda x : A.\lambda y : B.y]]\!]$$

where $X \equiv [\![\underline{A}]\!]$ and $Y \equiv [\![\underline{B}]\!]$. The function sl eval() : $(X \Rightarrow Y) \times X \to Y$ is inherited from the Cartesian closed structure of sets: in case $X, Y \in P$, we have explicitly

$$eval(\langle f, x\rangle) = [\![f]\!](x)$$

and the Curry function $\Lambda()$ is as defined earlier. The uniqueness of the value of $\Lambda(f)$ for any f is easy to see, as are the defining identities for any exponent object.

An important fact to note is that $\mathbf{P}$ is *order enriched.*

Lemma 5.5 *The category* $\mathbf{P}$ *is enriched over the category of dI-domains.*

Proof: A simple induction over the *length* of the referring expression $\mathbf{P}(X,Y)$ for the Homset: the length is defined as the sum of the lengths of X and Y; the length of X is 0 if X refers to an object of P; length of $X \times Y$ is one more than the sum of the lengths of X and Y, while length of $X \Rightarrow Y$ is the same as the sum of the lengths of X and Y. All Homsets of $\mathbf{P}$ are ordered in the point-wise order. For the base case of the induction, we know that Homsets inherited from P obviously inherit the order on their co-domains, which are all *dI*-domains (*cf.* Proposition 3.2, Theorem 4.1). In the general case, we propagate this order along the inductive structure of our category. Thus, Cartesian products are component-wise ordered, and are obviously *dI*-domains whenever their components are. Exponents inherit this order point-wise. Hence the argument.

In the sequel, we would use the operator $\widehat{(\centerdot)}$ as earlier if its argument is in $Mor(P)$, and otherwise define it as the identity.

The interpretation $\mathbf{I}$ of the ground Types $\mathbb{B}$ and $\mathbb{N}$ is given by the spaces $[\![\mathbb{B}]\!]$ and $[\![\mathbb{N}]\!]$ respectively. The basic set of constant and function symbols are interpreted by the specific morphisms sequentially realized by the corresponding sections (derived from the corresponding terms in maximally η-expanded form); thus, for instance, the constant function `succ` is interpreted by the specific morphism in $[\![\mathbb{N}]\!] \rightarrow [\![\mathbb{N}]\!]$ sequentially realized by the section $[\lambda x : \mathbb{N}.\mathtt{succ}(x)]$ in $[\![\underline{\mathbb{N}\Rightarrow\mathbb{N}}]\!]$; the function $\mathtt{cond}^N$ is interpreted by the specific morphism in $[\![\mathbb{B}]\!] \rightarrow [\![\underline{\mathbb{N}\Rightarrow\mathbb{N}\Rightarrow\mathbb{N}}]\!]$ sequentially realized by the section $[\lambda x : \mathbb{B}.\lambda y : \mathbb{N}.\lambda z : \mathbb{N}.\mathtt{cond}^N xyz]$ in $[\![\underline{\mathbb{B}\Rightarrow\mathbb{N}\Rightarrow\mathbb{N}\Rightarrow\mathbb{N}}]\!]$, and so forth. A constant k of any ground Type $\mathbb{T}$ is interpreted by the corresponding map $k : \star \rightarrow [\![\mathbb{T}]\!]$. Thus, our interpretation is *not* the *standard* one, but the *free* interpretation—*i.e.* the free continuous algebra generated by the constants of base Types, which is also the *initial* interpretation.

The semantics of PCF in Cartesian categories enriched over the category of Scott-domains is quite standard, and is in terms of a semantic functions $\mathbf{P}[\![\centerdot]\!]_\rho$ parameterized by a sequence ρ containing all free variables (appropriately type-annotated) in the argument term. We sketch the more important details of this scheme; precise technical details may be looked up in any standard reference ([BCL85, Ber79, Ber80]).

We define the denotations (extensions) of the types as: $\mathbf{P}[\![A]\!] = [\![\underline{A}]\!]$. In general, the denotation of any term $M : A$ with $Fvar(M) \subseteq \langle x_i : A_i \rangle_{0 \leq i \leq n}$, is a morphism $\mathbf{P}[\![A_0]\!] \times \cdots \times \mathbf{P}[\![A_n]\!] \rightarrow \mathbf{P}[\![A]\!]$ in $\mathbf{P}$. We would denote the projection on the i-th component of this product as π_i. Given morphisms $m : X \rightarrow (Y \Rightarrow Z)$ and $n : X \rightarrow Y$, we would write $m * n$ for the composition $eval \circ \langle m, n \rangle$. We may define our semantic function by structural induction.

$$
\begin{aligned}
\mathbf{P}[\![f : A]\!]_\rho &= \widehat{\mathbf{I}(f)} : \star \rightarrow \mathbf{P}[\![A]\!] \\
\mathbf{P}[\![x : A]\!]_\rho &= \pi_i \quad \text{where } x \text{ is the } i\text{-th member of } \rho \\
\mathbf{P}[\![\Omega : A]\!]_\rho &= \bot : \star \rightarrow \mathbf{P}[\![A]\!] \\
\mathbf{P}[\![\mathbf{Y} : (A \Rightarrow A) \Rightarrow A]\!]_\rho &= \Big(\bigsqcup_{n \geq 0} [\lambda g : A \Rightarrow A.g^n(\Omega^A)]\Big) : \star \rightarrow \mathbf{P}[\![(A \Rightarrow A) \Rightarrow A]\!] \\
\mathbf{P}[\![MN]\!]_\rho &= \mathbf{P}[\![M]\!]_\rho * \mathbf{P}[\![N]\!]_\rho \\
\mathbf{P}[\![\lambda x : A.M]\!]_\rho &= \Lambda(\mathbf{P}[\![M]\!]_{\rho \cup x:A})
\end{aligned}
$$

where, in the fourth clause, $g^0 = id$ and $g^{n+1} = g \circ g^n$; and f is any constant or constant function of the indicated Type, in the first clause. Also, we use the notation $\rho \cup x : A$ to signify that $x : A$ is appended to the sequence ρ.

The proof that $\mathbf{P}[\![.]\!]_\rho$ defines a least fixed point (according to established usage; *cf.* [BCL85, Defn. 4.2.2]), or *continuous* model (*cf.* [Ong95, Pg. 306]) is quite standard, and we refer the reader to the references quoted earlier (*ibid.*; in particular [BCL85, Theorem 6.2.2]). It is also a well-known fact that the model defined in this fashion over the free interpretation is *computationally adequate* and thus *sound*.

The Proposition 5.1 paves the way for our definability result.

Proposition 5.2 *Any compact element f in a domain $[\![\underline{A}]\!]$ of the model* **P** *is* definable—i.e. *the denotation of an unique term $\bar{f}$ of the calculus.*

Proof: Immediate, by Proposition 5.1: the unique term $\bar{f}$ posited in the proposition is simply $\nabla(f)$; it is easily seen that $\mathbf{P}[\![\bar{f}]\!] = f$ (confusing the distinction between the element f and the corresponding map from the terminal object).

The critical fact that falls out of the definability condition is that of *order-extensionality*. In order to prove it, we use the map $\nabla(.)$ and its property above to import the order-extensionality of the model $\mathcal{BT}_\eta$ consisting of (finite and infinite) closed Böhm trees in maximal η-expansion, over the free (initial) interpretation $\mathbf{I}^0$. This is a well-known result in the literature and we cite it below for reference.

Lemma 5.6 *The model $\mathcal{BT}_\eta$ is order-extensional—in other words, for Böhm trees f, g of Type $A{\Rightarrow}B$, we have*

$$f \sqsubseteq g \quad \Leftrightarrow \quad f(x) \sqsubseteq g(x)$$

for all trees x of Type A (the notation $f(x)$ denotes the usual operation of application among Böhm trees).

Proof: The proof is rather technical, and well-documented in [BCL85, Ber79, Ber80]; we would refer the reader to these references. A brief sketch has also been provided in the proof of Proposition 4.5.

Finally we may state the order-extensionality proposition.

Theorem 5.1 *The model* **P** *is order-extensional—in other words, for any $f, g : X \to Y$, we have*

$$f \leq g \quad \Leftrightarrow \quad \forall x \in X.\, f(x) \sqsubseteq g(x)$$

where we use the symbol $\leq$ for the point-wise order, and $\sqsubseteq$ for the order on the object Y.

Proof: We need consider only the case that X and Y are objects in P; the general case is argued by a straightforward induction over the length of the referring expression for $\mathbf{P}(X, Y)$ as in the proof of Lemma 5.5. The base case for the induction, with $X, Y \in Obj(P)$ goes as follows. The key idea in the proof is importing the order-extensionality property from $\mathcal{BT}_\eta$. Considering the more difficult part of proposition of order-extensionality (the $\Leftarrow$ direction), we consider the case that $f(x) \leqslant g(x)$ for all x in the domain of the functions f, g in our model. Now,

$$\begin{aligned}
f(x) \sqsubseteq g(x) &\Rightarrow \nabla(f(x)) \sqsubseteq \nabla(g(x)) && \text{– continuity of } \nabla() \\
&\Rightarrow \nabla(\hat{f}.x) \sqsubseteq \nabla(\hat{g}.x) && \text{– since } f(x) = \hat{f}.x \\
&\Rightarrow \nabla(\hat{f})\nabla(x) \sqsubseteq \nabla(\hat{g})\nabla(x) && \text{– property of } \nabla\text{; } \textit{cf.} \text{ Lemma 5.2} \\
&\Rightarrow \nabla(\hat{f}) \sqsubseteq \nabla(\hat{g}) && \text{– order-extensionality of } \mathcal{BT}_\eta\text{; } \textit{cf.} \text{ Lemma 5.6} \\
&\Rightarrow \hat{f} \sqsubseteq \hat{g} && \text{– applying } [.] \text{ to both sides; } \textit{cf.} \text{ Lemma 5.1} \\
&\Rightarrow f \leq g && \text{– monotonicity; } \textit{cf.} \text{ Corollary 4.4}
\end{aligned}$$

where we have signified the Ω-match order on Böhm trees by $\sqsubseteq$. This proves the more difficult part of the order-extensionality property; the converse is almost immediate from monotonicity properties (*cf.* Corollary 4.4) and we shall not labor it. The rest of the induction argument is straightforward, noting that the order on all function spaces and other objects is point-wise.

Hence finally, on the basis of the Proposition 5.2 and Theorem 5.1, we may assert our main result.

Theorem 5.2 *The model* $\mathbf{P}$ *is order-extensional, and fully abstract with respect to the interpretation* $\mathbf{I}$.

Proof: It is a well-known result, that an extensional model of PCF over the free interpretation is fully abstract iff it is order-extensional and and all compact elements of the domains of interpretation are definable (*cf.* [BCL85, Theorem 5.3.3]).

6 Conclusions

This brings to a conclusion our attempt to study the intensional and extensional aspects of higher-type sequential computation in the framework of affine geometry. The basic point of this exercise was to demonstrate that enriching the topological spaces of traditional denotational theory with the (local) geometrical structure of simplicial sets, gives a nice way to constrain the class of computations that we would like to model in such spaces to intrinsically *sequential* computations. The key advantage of this additional geometric structure is to give a formal method to *localize* computations to certain localities (cells, faces) of the domain, and to *phase* the resultant computations so that certain basic causal dependencies are respected. These two basic operative principles are at the heart of any kind of sequential computation and in our case, they are evidenced respectively, in the generation of the dialogue sequences (recursing within and out of the simplicial sub-structure in a "call-return" fashion; *cf.* [ADLR94, DR94]) and in the eventual evaluation of these sequences by Φ, (weakly) constrained by the $\preceq$ order on the evidently parallel sets of such sequences.

The principal gaps in this framework are two-fold. First, as we have pointed out in the earlier part, the normalization process on the basis of dialogue sequences are only tenuously related to the cut-elimination process of the previous part of our study. It would be extremely desirable if such sequences could be analytically generated on the basis of something like the Execution formula in other approaches drawing upon the Geometry of Interaction ([Gir89, DR93, DR95, AJ94a, AJ94b]). Second, the topological structure of our spaces are induced in a facile fashion from the syntactic notion of Ω-match, and the sequentially realizable maps are quite unrelated to the morphisms in the initial category of $(\mathbb{Z}/\mathbf{2})$-sheaves: both are really aspects of the general problem of the high degree of syntactic determination in our model and it would certainly be desirable to have a more synthetic topological structure. It is an aspect of our approach that observational quotienting is not required for extensionality w.r.t. the free interpretation: however, the reader would note that our semantics is still not fully abstract with respect to the standard interpretation—though it contains only sequentially computable functions. This has to to do with the intensionality of our representations at the ground types, and to eliminate *this* (thus obtaining a fully abstract model over the standard interpretation) one *would* have to use some device as observational quotients.

Finally, it would appear that the categorical structure of the simplicial topos (and related toposes, such as that of cuboidal sets, *cf.* [FPP97]) could be a viable basis to derive models of Axiomatic and Synthetic domain theory, with a *sequential* realizability structure at its basis (see also [FR]) . As remarked earlier, this has a good localization structure (it is a Grothendieck topos) and could presumably furnish a nice framework for the study of inductive structure within a constructive universe.

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