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Partitioning Bispanning Graphs into Spanning Trees

Matthias Baumgart

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Abstract

Given a weighted bispanning graph $\mathcal{B} = (V, P, Q)$ consisting of two edge-disjoint spanning trees P and Q such that $w(P) < w(Q)$ and Q is the only spanning tree with weight $w(Q)$, it is conjectured that there are $|V| - 1$ spanning trees with pairwise different weight where each of them is smaller than $w(Q)$. This conjecture due to Mayr and Plaxton is proven for bispanning graphs restricted in terms of the underlying weight function and the structure of the bispanning graphs. Furthermore, a slightly stronger conjecture is presented and proven for the latter class.

1 Introduction

Let $G = (V, E)$ be a weighted graph. One of the fundamental problems in computer science and graph theory is to compute a minimum spanning tree (MST) of G, *i.e.*, an acyclic spanning subgraph of minimum weight in G. The history of minimum spanning tree algorithms dates back at least to Borůvka [1] in 1926. The most popular textbook algorithms are those by Kruskal [7] and Prim [9]. Depending on the order in which the vertices are visited it is possible to obtain different minimum spanning trees provided that there is more than one MST. If there are different minimum spanning trees then it is possible to transform any MST T into another MST by performing exactly one edge swap which means that we remove one edge of T and insert one of the remaining edges. In this context, the so-called tree graph of a graph G can be defined. The vertex set of this graph is the set of all spanning trees of G with an edge between two spanning trees if and only if they are related by an edge swap. Regarding this tree graph, various questions arise. Some of them were discussed by Kano [5] who proposed four conjectures concerning distances (regarding the number of edge swaps) between different spanning trees in tree graphs motivated by a paper by Kawamoto, Kajitani and Shinoda [6]. One of his conjectures was that any minimum spanning tree can be transformed into a kth smallest spanning tree (a so-called k-MST) by at most $k - 1$ edge swaps. This conjecture was proven by Mayr and Plaxton [8]. Furthermore, they formulated a new conjecture which unified Kano's remaining three conjectures. The present paper addresses this new conjecture and its equivalent formulation that each weighted bispanning graph $\mathcal{B} = (V, P, Q)$ such that $w(P) < w(Q)$ and Q is the only spanning tree with weight $w(Q)$ has at least $|V| - 1$ spanning trees with pairwise different weights strictly less than $w(Q)$. We show that this is true if the spanning tree P is the only spanning tree of weight $w(P)$ or if the given bispanning graph has no minor isomorphic to the complete graph K_4.

This paper is organized as follows. In Section 2, we give some definitions and introduce different conjectures concerning tree graphs. The content of the subsequent sections is the analysis of bispanning graphs. The first step of this analysis is the restriction on special weight functions in Section 3. Afterwards, we turn our main focus onto the structure of bispanning graphs. In Section 4, we use techniques from matroid theory, and in Section 5 we show that it suffices to consider weighted bispanning graphs that are 2-vertex-connected and 3-edge-connected. We conclude this paper with an analysis of the complete graph on four vertices which is the smallest of these bispanning graphs.

2 Preliminaries

Throughout this paper, we assume that graphs $G = (V, E)$ are always connected undirected graphs where multiple edges are allowed. We denote the number of vertices by n and the number of edges by m, respectively. Let $w\colon E \to \mathbb{R}$ be a weight function. For any subset $E' \subseteq E$ we define the weight of E', denoted by $w(E')$, as the sum of the weights of all edges in E', that is,

$$w(E') =_{\mathrm{def}} \sum_{e \in E'} w(e) \ .$$

A *spanning tree* T of G is any subset of E for which the graph $G = (V, T)$ is acyclic and connected. We denote by $\mathcal{T}(G)$ the set of all spanning trees of G. Given a graph $G = (V, E)$ and a weight function $w\colon E \to \mathbb{R}$, we denote by $\mathcal{W}(G)$ the set of different weights of spanning trees of G and by $\mathcal{W}_i(G)$ the ith smallest element of $\mathcal{W}(G)$. Analogously, we denote by $\mathcal{T}_i(G)$ the set of spanning trees T where $w(T) = \mathcal{W}_i(G)$. We define the order $\mathrm{ord}(G, T)$ of a spanning tree T with respect to G as the number $i \in \mathbb{N}$ such that $T \in \mathcal{T}_i(G)$. We denote the number of spanning trees with weight $w(T)$ by $\sigma(G, T)$, that is, $\sigma(G, T) = |\mathcal{T}_{\mathrm{ord}(G,T)}(G)|$.

Let $G = (V, E)$ be a graph, T a spanning tree of G and $f \in E \setminus T$. We denote by $Cyc(T, f)$ the *fundamental cycle* of G defined by f with respect to T. Given a pair of distinct edges e, f such that $e \in Cyc(T, f)$, we define (e, f) to be a single *edge swap*. We denote by $L_k(G, T)$ the set of all those spanning trees T' of G such that T can be transformed into T' by at most k edge swaps. The following four conjectures were proposed by Kano [5].

Conjecture 1. *If T is a 1-MST of G then $L_{i-1}(G, T)$ contains an i-MST for all $1 \leq i \leq |\mathcal{W}(G)|$.*

Conjecture 2. *If T is an i-MST in $L_i(G, T)$ then T is an i-MST of G.*

Conjecture 3. *If T is an i-MST of G then T is an i-MST in $L_{i-1}(G, T)$.*

Conjecture 4. *Let $G(i, j)$ denote the graph with vertex set $\mathcal{T}_i(G)$ where an edge exists between each pair of i-MSTs T and T' if T can be transformed into T' by at most j edge swaps in G. Then $G(i, i)$ is connected.*

Conjecture 1 was proven by Mayr and Plaxton [8]. Moreover, they gave the following conjecture and showed that this one implies Conjectures 2 through 4.

Conjecture 5. *If T is a j-MST of G then $L_{i-1}(G, T)$ contains an i-MST for all $1 \leq j < i \leq |\mathcal{W}(G)|$.*

In their proof of Conjecture 1, Mayr and Plaxton used so-called *bispanning graphs*.

Definition 2.1. *A graph $G = (V, E)$ is called a* bispanning graph *if E is the union of two (edge) disjoint spanning trees P and Q. We denote a bispanning graph by the triple $\mathcal{B} = (V, P, Q)$.*

More precisely, Mayr and Plaxton showed that Conjecture 1 holds if and only if there is no weighted bispanning graph $\mathcal{B} = (V, P, Q)$ such that $1 = \operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q) < n$ and $\sigma(\mathcal{B}, Q) = 1$ holds. Indeed there is no such bispanning graph.

Theorem 2.2 ([8])**.** *There is no weighted bispanning graph $\mathcal{B} = (V, P, Q)$ such that $1 = \operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q) < n$ and $\sigma(\mathcal{B}, Q) = 1$.*

If it is possible to prove a stronger version of Theorem 2.2, *i.e.*, there is no weighted bispanning graph $\mathcal{B} = (V, P, Q)$ such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q) < n$ and $\sigma(\mathcal{B}, Q) = 1$ hold, we immediately get a proof of Conjecture 5 [8]. Hence, we would be done by proving the following conjecture implying Conjectures 2 through 5.

Conjecture 6. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$. Then it holds that $\operatorname{ord}(\mathcal{B}, Q) \geq n$.*

A powerful tool for proving Conjecture 1 in [8] are contractions and deletions of edges.

Definition 2.3. *Let $G = (V, E)$ be a graph and $e, f \in E$ be two edges. We denote by $G[e, f]$ the graph we obtain by contracting the edge e and deleting the edge f.*

Furthermore, the following lemma is very useful. For a proof we refer the reader to [8].

Lemma 2.4. *Let T be a spanning tree of a weighted graph $G = (V, E)$, and let $e \in E$. If $e \notin T$, let $G' = G[\emptyset, e]$ and $T' = T$. Otherwise, let $G' = G[e, \emptyset]$ and $T' = T[e, \emptyset]$. In either case, the following statements hold:*

1. *T' is a spanning tree of G'.*
2. $\operatorname{ord}(G', T') \leq \operatorname{ord}(G, T)$.
3. $\sigma(G', T') \leq \sigma(G, T)$.

3 Assuming singularity of P

In this section, we will prove Conjecture 6 under the assumption that the spanning tree P is also unique, *i.e.*, the weight function is restricted to satisfy $\sigma(\mathcal{B}, P) = 1$.

Theorem 3.1. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$, $\sigma(\mathcal{B}, Q) = 1$, and $\sigma(\mathcal{B}, P) = 1$. Then it holds that $\operatorname{ord}(\mathcal{B}, Q) \geq n$.*

Proof. This Theorem is proven by induction over the number of vertices of $\mathcal{B}$. Clearly, if $n = 2$ then $\mathcal{B}$ consists of two parallel edges with distinct weights, thus $\operatorname{ord}(\mathcal{B}, Q) \geq 2$.

We assume $n > 2$ and consider an arbitrary symmetric exchange (p, q) with $p \in P$ and $q \in Q$, that is, $P \setminus \{p\} \cup \{q\}$ and $Q \setminus \{q\} \cup \{p\}$ are spanning trees. It is not difficult to see that there always exists a symmetric exchange [2, 3]: If we choose an arbitrary edge $p \in P$ then $P \setminus \{p\}$ decomposes into two connected components C_1 and C_2. Color the vertices of these two components with distinct colors, e.g., associate with each vertex of C_1 the color blue and with each vertex of C_2 the color red. Note that the two vertices of edge p have different color. Now we consider the fundamental cycle $Cyc(Q,p)$ of $\mathcal{B}$ defined by p with respect to Q and choose an edge $q \in Q$ such that the two vertices of q also have different color. Clearly, such an edge must exist, and $P \setminus \{p\} \cup \{q\}$ as well as $Q \setminus \{q\} \cup \{p\}$ are spanning trees. Since $\sigma(\mathcal{B}, Q) = 1$ (or $\sigma(\mathcal{B}, P) = 1$), the edges p and q must have different weight. Now we distinguish different cases.

1. If $w(q) < w(p)$ holds then we consider the bispanning graph $\mathcal{B}' = \mathcal{B}[q,p]$. The elements of $\mathcal{W}(\mathcal{B}')$ in strictly increasing order are (for the sake of readability, we associate by a spanning tree T also its weight $w(T)$)

$$(A'_1, \ldots, A'_\alpha, P', B'_1, \ldots, B'_\beta, Q', C'_1, \ldots, C'_\gamma)$$

   with $P' = P[q,p]$ and $Q' = Q[q,p]$. By Lemma 2.4, it holds that $\sigma(\mathcal{B}', Q') = 1$, thus, applying the induction hypothesis, we obtain $\mathrm{ord}(\mathcal{B}', Q') \geq n-1$. Observe that each spanning tree of $\mathcal{B}'$ together with the edge q forms a spanning tree of $\mathcal{B}$. Thus, there are at least $n-2$ spanning trees with the distinct weights

$$(A_1, \ldots, A_\alpha, P' + w(q), B_1, \ldots, B_\beta)$$

   with $A_i = A'_i + w(q)$, $1 \leq i \leq \alpha$, and $B_j = B'_j + w(q)$, $1 \leq j \leq \beta$ such that each of these weights is strictly smaller than $w(Q)$ (see Figure 1(a)). Since $\sigma(\mathcal{B}, P) = 1$ none of these spanning tree weights can map into the weight $w(P)$, thus, we obtain $\mathrm{ord}(\mathcal{B}, Q) \geq n$.

2. We assume $w(p) < w(q)$ and

   (a) $w(P) - w(p) < w(Q) - w(q)$. Consider the bispanning graph $\mathcal{B}' = \mathcal{B}[q,p]$ and the elements of $\mathcal{W}(\mathcal{B}')$ in strictly increasing order

$$(A'_1, \ldots, A'_\alpha, P', B'_1, \ldots, B'_\beta, Q', C'_1, \ldots, C'_\gamma)\ .$$

   Again, we have $\sigma(\mathcal{B}', Q') = 1$ by Lemma 2.4. Applying the induction hypothesis, we have $\mathrm{ord}(\mathcal{B}', Q') \geq n-1$. Analogous to the previous case, each spanning tree of $\mathcal{B}'$ can be combined with the contracted edge q obtaining the following weights

$$(A_1, \ldots, A_\alpha, P' + w(q), B_1, \ldots, B_\beta)$$

   with $A_i = A'_i + w(q)$, $1 \leq i \leq \alpha$, and $B_j = B'_j + w(q)$, $1 \leq j \leq \beta$ (see Figure 1(a)). Each of these distinct weights is strictly smaller than $w(Q)$ and none of them can map into $w(P)$. Hence, we also count the weight of spanning tree P resulting in $\mathrm{ord}(\mathcal{B}, Q) \geq n$.

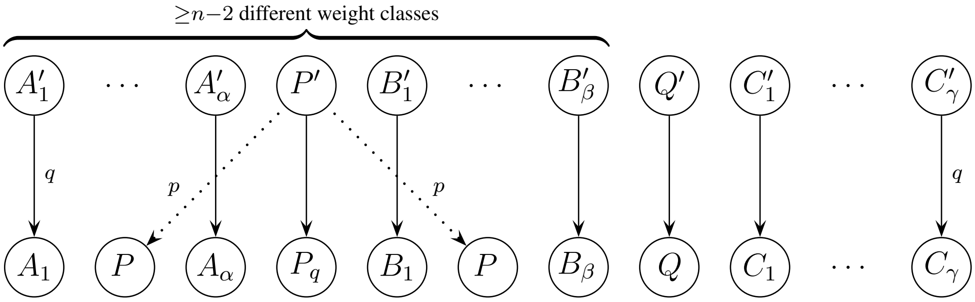
   (b) $w(Q) - w(q) < w(P) - w(p)$. We contract p and delete q, that is, we consider the bispanning graph $\mathcal{B}' = \mathcal{B}[p,q]$. The increasing sequence of weights from $\mathcal{W}(\mathcal{B}')$ is

$$(A'_1, \ldots, A'_\alpha, Q', B'_1, \ldots, B'_\beta, P', C'_1, \ldots, C'_\gamma)\ .$$

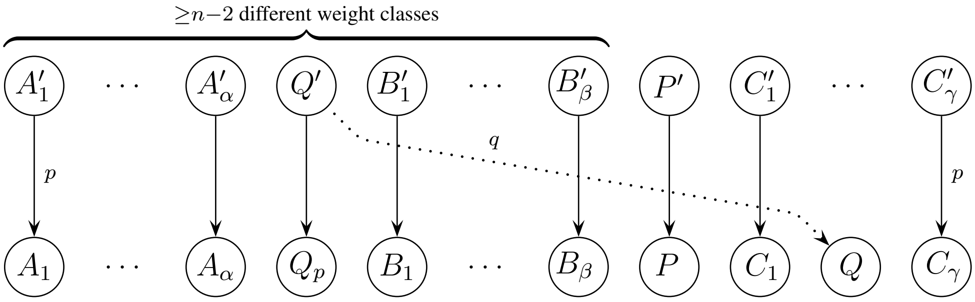
   Now, it holds that $\sigma(\mathcal{B}', P') = 1$. Thus, by induction hypothesis, we have $\mathrm{ord}(\mathcal{B}', P') \geq n-1$. Combining these spanning trees with the contracted edge p we get at least $n-1$ different spanning trees with weights

$$(A_1, \ldots, A_\alpha, Q' + w(p), B_1, \ldots, B_\beta, P)$$

   with $A_i = A'_i + w(p)$, $1 \leq i \leq \alpha$, and $B_j = B'_j + w(p)$, $1 \leq j \leq \beta$. This situation is illustrated in Figure 1(b). Since $w(P) < w(Q)$, we obtain $\mathrm{ord}(\mathcal{B}, Q) \geq n$ and the theorem follows. □



(a) Constructing new classes of spanning trees by adding the edge q (solid lines) to the classes of $\mathcal{B}[q,p]$. Depending on whether $w(q) < w(p)$ or $w(p) < w(q)$ (together with $w(P') < w(Q')$) hold we get either the right spanning tree P or the left one.



(b) Constructing new classes of spanning trees by adding the edge p (solid lines) to the classes of $\mathcal{B}[q,p]$.

Figure 1: Mapping of spanning tree weight classes after contracting and deleting of two edges.

4 Strongly base orderable matroids

In the previous section, we proved that Conjecture 6 holds for weighted bispanning graphs $\mathcal{B} = (V,P,Q)$ restricted to weight functions $w\colon E \to \mathbb{R}$ with the property that both spanning trees, P and Q, have unique weight. An opposed approach is to analyze specially structured bispanning graphs $\mathcal{B} = (V,P,Q)$ where the weight function is only required to satisfy $w(P) < w(Q)$ (equivalent to $\operatorname{ord}(\mathcal{B},P) < \operatorname{ord}(\mathcal{B},Q)$) and $\sigma(\mathcal{B},Q) = 1$. For this reason, we will now introduce the concept of matroids.

Definition 4.1. *A pair $(E,\mathcal{I})$ consisting of a finite set E and a nonempty family $\mathcal{I}$ of subsets of E is called a* matroid *if $\mathcal{I}$ satisfies*

(i) $\emptyset \in \mathcal{I}$,

(ii) $I_1 \in \mathcal{I}$ *and* $I_2 \subseteq I_1$ *imply* $I_2 \in \mathcal{I}$, *and*

(iii) $I_1, I_2 \in \mathcal{I}$ *and* $|I_1| < |I_2|$ *imply* $I_1 \cup \{x\} \in \mathcal{I}$ *for some* $x \in I_2 \setminus I_1$.

The following proposition is well known from graph theory.

Proposition 4.2. *Let $G = (V,E)$ be a graph and let $\mathcal{I}$ be the family of all subsets of E such that each $I \in \mathcal{I}$ is a forest of G. Then $M = (E,\mathcal{I})$ is a matroid which is called the cycle matroid of G.*

A subset I of E is called *independent* if $I \in \mathcal{I}$, and *dependent* otherwise. An independent subset B of E is called a base if there is no subset B' of E such that $B \subset B'$. Thus, the bases of the cycle matroid of a connected graph G are the spanning trees of G. In the following, we consider bispanning graphs $\mathcal{B} = (V, P, Q)$ where the cycle matroid of $\mathcal{B}$ is strongly base orderable.

Definition 4.3. *A matroid $M = (E, \mathcal{I})$ is called* strongly base orderable *if there exists a bijection $\varphi\colon B \to B'$ for each two bases B, B' such that for each subset X of B the set $(B \setminus X) \cup \varphi(X)$ is a base, too.*

Lemma 4.4. *Let $M = (E, \mathcal{I})$ be a strongly base orderable matroid. Then for each two bases B and B' there exists a bijection $\varphi\colon B \to B'$ such that for all subsets X of B the sets $(B \setminus X) \cup \varphi(X)$ and $(B' \setminus \varphi(X)) \cup X$ are bases.*

Proof. By the definition of strongly base orderable matroids, there exists a bijection $\varphi\colon B \to B'$ such that for each $X \subseteq B$ the set $(B \setminus X) \cup \varphi(X)$ is a base. Let X be a subset of B of minimal cardinality such that $(B' \setminus \varphi(X)) \cup X$ is not a base, that is, $(B' \setminus \varphi(X)) \cup X$ contains exactly one cycle C since otherwise X is not minimal. Furthermore, because of this minimality we have $X \subset C$. Let $X' = C \setminus X$ and let $\tilde{X} = \varphi^{-1}(X') \subseteq B$ be the elements of B that map onto an element of X'. Clearly, it holds that $X \cap \tilde{X} = \emptyset$. Furthermore, the set $(B \setminus \tilde{X}) \cup \varphi(\tilde{X})$ contains the cycle C in contradiction to the property of M to be strongly base orderable. □

Note that the bases B and B' need not to be disjoint. In this case, the relation for all elements in $B \cap B'$ is the identity. However, in our analysis we only need the special case of disjoint bases denoted by P and Q.

Theorem 4.5. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$. Let $M = (P \cup Q, \mathcal{I})$ be the cycle matroid of $\mathcal{B}$. If M is strongly base orderable then it holds that $\operatorname{ord}(\mathcal{B}, Q) \geq n$.*

Proof. Since M is strongly base orderable, there exists a bijection $\varphi\colon Q \to P$ such that for each subset Q' of Q the set $(Q \setminus Q') \cup \varphi(Q')$ is a spanning tree. Let $w\colon (P \cup Q) \to \mathbb{R}$ be the weight function of $\mathcal{B}$. Clearly, it holds that $w(Q') \neq w(\varphi(Q'))$ for each $Q' \subseteq Q$ since otherwise we have $\operatorname{ord}(\mathcal{B}, Q) > 1$. Let $\delta(q) = w(\varphi(q)) - w(q)$ be the difference of the weights of an edge q and its image $\varphi(q)$ with respect to the weight function w. Thus, the function δ measures the increase of the spanning tree weight after changing the edge q by its image $\varphi(q)$. In general, we define for a subset Q' of Q

$$\delta(Q') = \sum_{q \in Q'} \delta(q) = \sum_{q \in Q'} \big(w(\varphi(q)) - w(q)\big) . \tag{4.1}$$

If we choose $Q' = Q$ equation (4.1) implies

$$\delta(Q) = \sum_{q \in Q} \delta(q) = \sum_{q \in Q} \big(w(\varphi(q)) - w(q)\big) = w(P) - w(Q) < 0 \iff w(P) < w(Q) .$$

Now, we arrange the elements of Q in such a way that $\delta(q_1) \leq \ldots \leq \delta(q_{n-1})$ holds and consider the $n-1$ different sets $Q_i = \{q_1, \ldots, q_i\}$ for $1 \leq i \leq n-1$. Clearly, for each $1 \leq i \leq n-1$ the set $(Q \setminus Q_i) \cup \varphi(Q_i)$ is a spanning tree since M is a strongly base orderable matroid. Furthermore, each set Q_i satisfies

$$\delta(Q_i) = \sum_{q \in Q_i} \delta(q) < 0$$

that is, if we remove the edges Q_i from Q and add the edges $\varphi(Q_i)$, the weight of the resulting spanning tree is smaller than $w(Q)$. Thus, if we show that these spanning tree weights are distinct, the claim follows. Let i and j be two indices such that $\delta(Q_i) = \delta(Q_j)$. In this case, it holds that

$$\begin{aligned}
\delta(Q') &= \sum_{q\in Q'} \delta(q) = \sum_{q\in Q'} \big(w(\varphi(q)) - w(q)\big) \\
&= \sum_{q\in Q_j} \big(w(\varphi(q)) - w(q)\big) - \sum_{q\in Q_i} \big(w(\varphi(q)) - w(q)\big) \\
&= \delta(Q_j) - \delta(Q_i) = 0
\end{aligned}$$

which implies that the weight of the spanning tree $(Q \setminus Q') \cup \varphi(Q')$ is equal to $w(Q)$ contradicting $\sigma(\mathcal{B}, Q) = 1$. Hence, there are at least $n - 1$ spanning trees with distinct weight such that each of them is smaller than $w(Q)$ implying $\operatorname{ord}(\mathcal{B}, Q) \geq n$ which proves the theorem. □

Observe that because of Lemma 4.4 we only counted spanning trees T of a bispanning graph $\mathcal{B} = (V, P, Q)$ such that the remaining edges $(P \cup Q) \setminus T$ also form a spanning tree. There are strong indications that these spanning trees are sufficient to prove Conjecture 6. This approach will be pursued in the next section.

A questions which arises now is how to distinguish bispanning graphs whose cycle matroid is strongly base orderable from bispanning graphs that do not have this property? The following two theorems are well known from matroid theory and give an answer to this question. For a proof we refer the reader to [11].

Theorem 4.6. *Transversal matroids are strongly base orderable.*

Theorem 4.7. *Let $G = (V, E)$ be a finite graph. Then its cycle matroid $M = (E, \mathcal{I})$ is transversal if and only if G contains no minor isomorphic to K_4 or C_k^2 $(k > 2)$.*

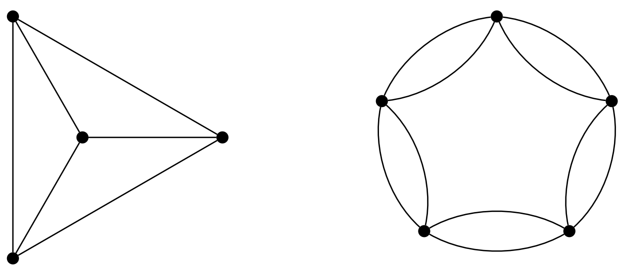


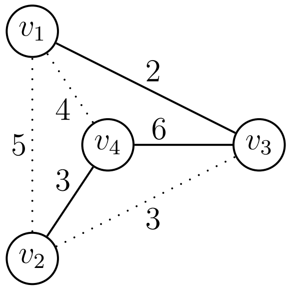
Figure 2: K_4 and C_5^2.

Since a bispanning graph cannot contain a minor isomorphic to C_k^2 $(k > 2)$ we have to focus on graphs which contain a K_4 minor.

Corollary 4.8. *Let $\mathcal{B} = (V, P, Q)$ be a finite bispanning graph. Then its cycle matroid is transversal if and only if $\mathcal{B}$ contains no minor isomorphic to K_4.*

5 Partitioning bispanning graphs

In this section, we want to merge the ideas of the previous two sections. In Section 3, we have seen that Conjecture 6 holds under the assumption that the weight function is required to satisfy $\sigma(\mathcal{B}, P) = 1$. In general, the weight of spanning tree P is not unique even if the number of vertices is small (see Figure 3).



This picture shows the complete graph on four vertices which is the smallest bispanning graph $\mathcal{B} = (V, P, Q)$ without multiple edges. The edges of Q are dotted. The weights are $w(P) = 11$ and $w(Q) = 12$, thus, it holds that $\mathrm{ord}(\mathcal{B}, P) < \mathrm{ord}(\mathcal{B}, Q)$. One can easily check that $\sigma(\mathcal{B}, Q) = 1$ and $\sigma(\mathcal{B}, P) = 4$ since

$$w(P) = 11 = \underbrace{2+4+5}_{(v_1, V\setminus\{v_1\})} = \underbrace{3+3+5}_{(v_2, V\setminus\{v_2\})} = \underbrace{2+3+6}_{(v_3, V\setminus\{v_3\})} .$$

Figure 3: Example of a bispanning graph $\mathcal{B}(V, P, Q)$ such that $\sigma(\mathcal{B}, Q) = 1$ and $\sigma(\mathcal{B}, P) = 4$.

The main observation in this case is that given a spanning tree $T \neq P$ where $w(T) = w(P)$ the remaining edges $E \setminus T$ contain at least one cycle. In Figure 3 the remaining edges cannot form a spannning tree since each spanning tree with weight $w(P)$ contains a cut. To avoid the problem of $\sigma(\mathcal{B}, P) > 1$ we introduce the concept of partitioning bispanning graphs into spanning trees which was already indicated in Section 4. This approach leads to a somewhat stronger conjecture compared to Conjecture 6.

Definition 5.1. *Let $\mathcal{B} = (V, P, Q)$ be a bispanning graph. A spanning tree T of $\mathcal{B}$ is called a* partition spanning tree *if and only if its complement $E \setminus T$ is a spanning tree, too.*

Let $\mathcal{B} = (V, P, Q)$ be a bispanning graph. We denote by $\mathcal{T}'(\mathcal{B})$ the set of all partition spanning trees of $\mathcal{B}$. Given a weight function $w\colon (P \cup Q) \to \mathbb{R}$ we denote by $\mathcal{W}'(\mathcal{B})$ the set of different weights of spanning trees of $\mathcal{B}$ and by $\mathcal{W}'_i(\mathcal{B})$ the ith smallest element of $\mathcal{W}'(\mathcal{B})$. Moreover, $\mathcal{T}'_i(\mathcal{B})$ is the set of partition spanning trees T where $w(T) = \mathcal{W}'_i(\mathcal{B})$. We define the order $\mathrm{ord}'(\mathcal{B}, T)$ of a partition spanning tree T with respect to $\mathcal{B}$ as the number $i \in \mathbb{N}$ such that $T \in \mathcal{T}'_i(\mathcal{B})$. The number of partition spanning trees with weight $w(T)$ is denoted by $\sigma'(\mathcal{B}, T)$, that is, $\sigma'(\mathcal{B}, T) = |\mathcal{T}'_{\mathrm{ord}'(\mathcal{B},T)}(\mathcal{B})|$.

Conjecture 7. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\mathrm{ord}(\mathcal{B}, P) < \mathrm{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$. Then it holds that $\mathrm{ord}'(\mathcal{B}, Q) \geq n$.*

If Conjecture 7 holds then it implies immediately Conjecture 6 since $\mathrm{ord}'(\mathcal{B}, T) \leq \mathrm{ord}(\mathcal{B}, T)$ for all partition spanning trees T of $\mathcal{B}$. But how many partition spanning trees does a given bispanning graph have? The following lemma, the so-called "symmetric subset exchange axiom", is well known from matroid theory. For a proof we refer the reader to [10].

Lemma 5.2. *If B and B' are bases of a Matroid $M = (E, \mathcal{I})$ and B_1 is partitioned into X and Y, then B' can be partitioned into X' and Y' such that $X \cup Y'$ and $Y \cup X'$ are bases.*

Hence, we can give a lower bound on the number of partition spanning trees with weight less than $w(Q)$.

Lemma 5.3. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$. Then there are at least 2^{n-2} weighted partition spanning trees T in $\mathcal{B}$ such that $w(T) < w(Q)$.*

Proof. There are 2^{n-1} subsets P' of P. According to Lemma 5.2, for each P' there is a subset Q' of Q such that $(P \setminus P') \cup Q'$ as well as $(Q \setminus Q') \cup P'$ are partition spanning trees. Clearly, the weight of at least one of them is less than $w(Q)$. Because of symmetry we have to divide by two and the claim follows. □

5.1 Strictly 2-edge-connected bispanning graphs

In this section, we consider strictly 2-edge-connected bispanning graphs and show that each of these bispanning graphs can be reduced to some 3-edge-connected bispanning graph under the assumption that it is only necessary to count partition spanning trees.

Theorem 5.4. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph with edge connectivity $\lambda = 2$, $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$, and $\sigma(\mathcal{B}, Q) = 1$. Then there are two edges $p \in P$ and $q \in Q$ such that $\operatorname{ord}'(\mathcal{B}[q, p], Q[q, p]) < \operatorname{ord}'(\mathcal{B}, Q)$.*

Proof. Since $\lambda = 2$, there exists a cut $(V', V \setminus V')$ in G with exactly two edges between V' and $V \setminus V'$. Clearly, one of these edges belongs to P and the other one belongs to Q since otherwise either P or Q is not a spanning tree. We denote by p the edge which belongs to P and by q the edge which belongs to Q, respectively. Now, we consider the bispanning graph $\mathcal{B}' = \mathcal{B}[q, p]$ and observe that each partition spanning tree of this graph can be combined either with p or with q yielding a partition spanning tree of $\mathcal{B}$. Depending on the weight of p and q the set of different partition spanning tree weights of $\mathcal{B}'$ in increasing order is either

$$(A'_1, \ldots, A'_\alpha, P', B'_1, \ldots, B'_\beta, Q', C'_\alpha, \ldots, C'_1) \tag{5.1}$$

or

$$(A'_1, \ldots, A'_\mu, Q', B'_1, \ldots, B'_\nu, P', C'_\mu, \ldots, C'_1) \tag{5.2}$$

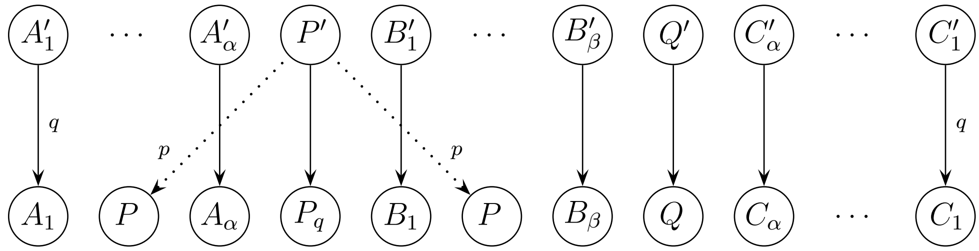
where, for the sake of readability, we associate with a tree T also its weight $w(T)$.

If (5.1) holds, we combine each partition spanning tree with the edge q resulting in $\alpha + \beta + 1$ partition spanning trees with distinct weight where each of them is smaller than $w(Q)$. Since none of these weights can map into $w(P)$, there are $\alpha + \beta + 2$ implying $\operatorname{ord}'(\mathcal{B}', Q') < \operatorname{ord}'(\mathcal{B}, Q)$.
In the case that (5.2) holds, we combine each partition spanning tree with the edge p. Since $w(P' \cup \{p\}) < w(Q' \cup \{q\})$, we arrive at $\operatorname{ord}'(\mathcal{B}', Q') < \operatorname{ord}'(\mathcal{B}, Q)$ and the claim follows. Both cases are illustrated in Figure 4. □

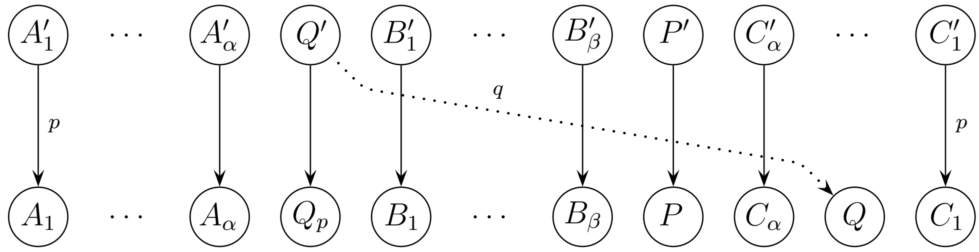
Analogous to Theorem 5.4, we can show that it is possible to omit multiple edges.

Corollary 5.5. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$, $\sigma(\mathcal{B}, Q) = 1$, and (p, q) a pair of multiple edges. Then $\operatorname{ord}'(\mathcal{B}[q, p], Q[q, p]) < \operatorname{ord}'(\mathcal{B}, Q)$ holds.*

Note that Corollary 5.5 is true even if the edge connectivity of $\mathcal{B}$ is greater than two. The consequence of Theorem 5.4 and Corollary 5.5 is that we only have to consider 3-edge-connected bispanning graphs without multiple edges to prove Conjecture 7.



(a) Constructing new classes of partition spanning trees by adding the edge q (solid lines) to the classes of $\mathcal{B}[q, p]$. Since P is the only partition spanning tree with weight $w(P)$ none of them can map into $w(P)$ Depending on whether $w(q) < w(p)$ or $w(p) < w(q)$ (together with $w(P') < w(Q')$) hold we get either the right spanning tree P or the left one.



(b) Constructing new classes of partition spanning trees by adding the edge p (solid lines) to the classes of $\mathcal{B}[p, q]$.

Figure 4: Mapping of partition spanning tree weight classes.

5.2 Breaking up bispanning graphs with articulation vertices

In the previous subsection, we have seen that we can turn our attention to 3-edge-connected bispanning graphs. Now, we want to show that it is also sufficient to consider only 2-vertex-connected bispanning graphs, i.e., graphs that do not contain any articulation vertex.

Definition 5.6. *Let $G = (V, E)$ be a graph and $v \in V$ be a vertex. We call v an articulation vertex (or cut vertex) if the removal of v (and its adjacent edges) will disconnect the graph.*

In the following theorem, we assume that the bispanning graph $\mathcal{B}$ contains exactly one articulation vertex. This is no restriction since it is possible to decompose each bispanning graph into its 2-vertex-connected components and apply the theorem inductively to every two adjacent components.

Theorem 5.7. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$ which consists of exactly one articulation vertex $v \in V$. Let $\mathcal{B}_1 = (V_1, P_1, Q_1)$ and $\mathcal{B}_2 = (V_2, P_2, Q_2)$ be the two 2-vertex-connected components. Then both components are weighted bispanning graphs and if Conjecture 7 holds for both of them, then Conjecture 7 holds for $\mathcal{B}$.*

In the proof of Theorem 5.4, we combined partition spanning trees with the edges p and q. Now we have to combine partition spanning trees of $\mathcal{B}_1$ with partition spanning tree of $\mathcal{B}_2$. Since this is somewhat more difficult, we take a look at the following two lemmas because they will simplify the proof of Theorem 5.7.

Lemma 5.8. *Let $X = (a_1, \ldots, a_\alpha, p_1, b_1, \ldots, b_\beta, q_1)$ and $Y = (c_1, \ldots, c_\mu, p_2, d_1, \ldots, d_\nu, q_2)$ be strictly increasing sequences of numbers. Let S be the set of all possible sums of two elements $x \in X$ and $y \in Y$. Then there are at least $\alpha + \beta + \mu + \nu + 2$ distinct $s \in S$ such that $s < q_1 + q_2$.*

Proof. We consider the following chain of distinct sums

$$\underbrace{(a_1 + c_1) < (a_1 + c_2) < \ldots < (a_1 + c_\mu)}_{\mu \text{ pairs}} < (a_1 + p_2) <$$
$$\underbrace{(a_1 + d_1) < (a_1 + d_2) < \ldots < (a_1 + d_\nu)}_{\nu \text{ pairs}} <$$
$$< \underbrace{(a_1 + q_2) < (a_2 + q_2) < \ldots < (a_\alpha + q_2)}_{\alpha \text{ pairs}} < (p_1 + q_2) <$$
$$< \underbrace{(b_1 + q_2) < (b_2 + q_2) < \ldots < (b_\beta + q_2)}_{\beta \text{ pairs}} < (q_1 + q_2) .$$

Hence, there are $\alpha + \beta + \mu + \nu + 2$ distinct sums which are smaller than $q_1 + q_2$. □

It is easy to see that Lemma 5.8 helps us to prove the case if $\text{ord}'(\mathcal{B}_1, P_1) < \text{ord}'(\mathcal{B}_1, Q_1)$ and $\text{ord}'(\mathcal{B}_2, P_2) < \text{ord}'(\mathcal{B}_2, Q_2)$ holds. On the other hand, the next lemma will help us to analyze the remaining case that either $\text{ord}'(\mathcal{B}_1, Q_1) < \text{ord}'(\mathcal{B}_1, P_1)$ or $\text{ord}'(\mathcal{B}_2, Q_2) < \text{ord}'(\mathcal{B}_2, P_2)$ holds. The case $\text{ord}'(\mathcal{B}_1, Q_1) < \text{ord}'(\mathcal{B}_1, P_1)$ and $\text{ord}'(\mathcal{B}_2, Q_2) < \text{ord}'(\mathcal{B}_2, P_2)$ is impossible since it implies $w(Q) < w(P)$.

Lemma 5.9. *Let $X = (a_1, \ldots, a_\alpha, q_1, b_1, \ldots, b_\beta, p_1)$ and $Y = (c_1, \ldots, c_\mu, p_2, d_1, \ldots, d_\nu, q_2)$ be strictly increasing sequences of numbers with the following restrictions. Let E_X, E_Y, F_X, and F_Y defined depending on X and Y as*

$$\begin{aligned} E_X &= \{p_1 - x \mid x \in X \setminus \{p_1\}\} \qquad & E_Y &= \{q_2 - y \mid y \in Y \setminus \{q_2\}\} \\ F_X &= \{b_i - q_1 \mid 1 \le i \le \beta\} \qquad & F_Y &= \{d_j - p_2 \mid 1 \le j \le \nu\} . \end{aligned}$$

We assume X and Y satisfy $F_X \subseteq E_X$, $F_Y \subseteq E_Y$, $F_X \cap F_Y = \emptyset$, and $p = p_1 + p_2 < q_1 + q_2 = q$. If $p = x + y$ holds if and only if $x = p_1$ and $y = p_2$ where $x \in X$ and $y \in Y$ then the set of all possible sums S of two elements $x \in X$ and $y \in Y$ consists of at least $\alpha + \beta + \mu + \nu + 2$ distinct elements $s \in S$ such that $s < q_1 + q_2$.

Proof. We consider the following chain of pairwise different sums

$$\underbrace{(a_1 + c_1) < (a_1 + c_2) < \ldots < (a_1 + c_\mu)}_{\mu \text{ pairs}} <$$
$$< \underbrace{(a_1 + p_2) < (a_2 + p_2) < \ldots < (a_\alpha + p_2)}_{\alpha \text{ pairs}} < (q_1 + p_2) <$$
$$< \underbrace{(b_1 + p_2) < (b_2, p_2) < \ldots < (b_\beta + p_2)}_{\beta \text{ pairs}} < (p_1 + p_2) .$$

Obviously, all of these sums are less than $q_1 + q_2$, that is, we have already found $\alpha + \beta + \mu + 2$ distinct sums. Now we consider the sums that are formed by q_1 and d_j where $1 \le j \le \nu$. It is easy to see that these sums are distinct where each of them is greater than $q_1 + p_2$ and smaller than $q_1 + q_2$. Since $p = p_1 + p_2$ is unique, they can only conflict with some pair $b_i + p_2$ in the chain given above. Therefore, we assume $q_1 + d_j = b_i + p_2 \iff d_j - p_2 = b_i - q_1$ for arbitrarily chosen $1 \le i \le \beta$ and $1 \le j \le \nu$. But this is a contradiction to our assumption $F_X \cap F_Y = \emptyset$ since $b_j - q_1 \in F_X$ and $d_i - p_2 \in F_Y$. This proves the lemma. □

Proof of Theorem 5.7. As described above, we have to distinguish between the following cases. Either it holds that

$$\operatorname{ord}'(\mathcal{B}_1, P_1) < \operatorname{ord}'(\mathcal{B}_1, Q_1) \text{ and } \operatorname{ord}'(\mathcal{B}_2, P_2) < \operatorname{ord}'(\mathcal{B}_2, Q_2) \tag{5.3}$$

or it holds that

$$\operatorname{ord}'(\mathcal{B}_1, P_1) < \operatorname{ord}'(\mathcal{B}_1, Q_1) \text{ and } \operatorname{ord}'(\mathcal{B}_2, P_2) > \operatorname{ord}'(\mathcal{B}_2, Q_2)\ . \tag{5.4}$$

In the first case (5.3), the ordered sequences of all partition spanning tree weights of $\mathcal{B}_1$ and $\mathcal{B}_2$ are

$$\begin{aligned} X &= (A_1, \ldots, A_\alpha, P_1, B_1, \ldots, B_\beta, Q_1, \tilde{A}_\alpha, \ldots, \tilde{A}_1) \text{ and} \\ Y &= (C_1, \ldots, C_\mu, P_2, D_1, \ldots, D_\nu, Q_2, \tilde{C}_\mu, \ldots, \tilde{C}_1)\ . \end{aligned}$$

Note that $\sigma(\mathcal{B}_1, Q_1) = \sigma(\mathcal{B}_2, Q_2) = 1$ since otherwise $\sigma(\mathcal{B}, Q) > 1$. If Conjecture 7 holds then we have $\operatorname{ord}'(\mathcal{B}_1, Q_1) \geq n_1$ and $\operatorname{ord}'(\mathcal{B}_2, Q_2) \geq n_2$ with $n_1 = |V_1|$ and $n_2 = |V_2|$. It is easy to see that if we combine a partition spanning tree of $\mathcal{B}_1$ with an partition spanning tree of $\mathcal{B}_2$ we get a partition spanning tree of $\mathcal{B}$. The number of vertices in $\mathcal{B}$ is $|V| = n = n_1 + n_2 - 1$. Thus, if we can construct $n_1 + n_2 - 2$ partition spanning trees with distinct weights where each weight is smaller than $w(Q)$, we are done. Applying Lemma 5.8 we obtain

$$\alpha + \beta + 1 \geq n_1 - 1 \text{ and } \mu + \nu + 1 \geq n_2 - 1\ .$$

Hence, it holds that $\alpha + \beta + \mu + \nu + 2 \geq n_1 + n_2 - 2$ which implies $\operatorname{ord}'(\mathcal{B}, Q) \geq n_1 + n_2 - 1 = n$. All combinations are illustrated in Figure 5 where it is easy to see that all of them lead to partition spanning trees of different weights since there are no crossing lines.

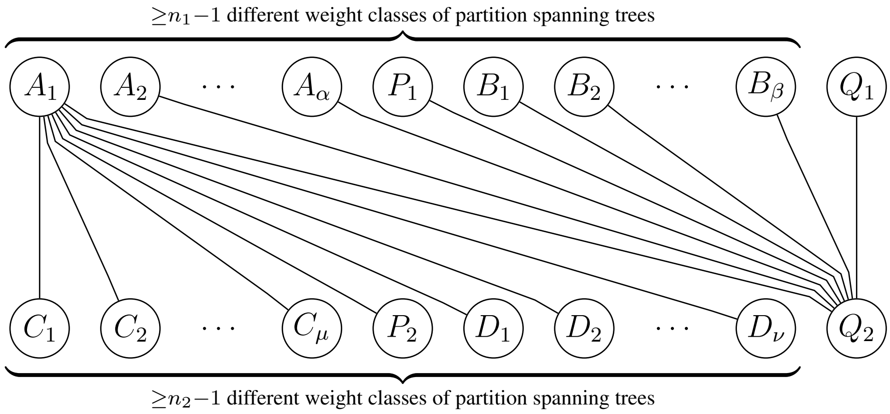


Figure 5: Combinations if $\operatorname{ord}'(\mathcal{B}_1, P_1) < \operatorname{ord}'(\mathcal{B}_1, Q_1)$ and $\operatorname{ord}'(\mathcal{B}_2, P_2) < \operatorname{ord}'(\mathcal{B}_2, Q_2)$ hold.

Assume now that (5.4) holds. Again, we consider the ordered sequences of all partition spanning tree weights of $\mathcal{B}_1$ and $\mathcal{B}_2$ which are

$$\begin{aligned} X &= (A_1, \ldots, A_\alpha, Q_1, B_1, \ldots, B_\beta, P_1, \tilde{A}_\alpha, \ldots, \tilde{A}_1) \\ Y &= (C_1, \ldots, C_\mu, P_2, D_1, \ldots, D_\nu, Q_2, \tilde{C}_\mu, \ldots, \tilde{C}_1)\ . \end{aligned}$$

Analogous to the previous case, we have $\sigma(\mathcal{B}_1, Q_1) = \sigma(\mathcal{B}_2, Q_2) = 1$ and $\mathrm{ord}'(\mathcal{B}_2, Q_2) \geq n_2$. Because of symmetry, we obtain $\mathrm{ord}'(\mathcal{B}_1, P_1) \geq n_1$ and $\alpha + \beta + 1 \geq n_1 - 1$. On the other hand, it holds that $\mu + \nu + 1 \geq n_2 - 1$. Let E_X, E_Y, F_X, and F_Y be defined as follows (for the sake of readability, we write T instead of $w(T)$)

$$
\begin{aligned}
E_X &= \{P_1 - x \mid x \in X \setminus \{P_1\}\} \qquad & E_Y &= \{Q_2 - y \mid y \in Y \setminus \{Q_2\}\} \\
F_X &= \{B_i - Q_1 \mid 1 \leq i \leq \beta\} \qquad & F_Y &= \{D_j - P_2 \mid 1 \leq j \leq \nu\} \,.
\end{aligned}
$$

It holds that $F_X \subseteq E_X$ and $F_Y \subseteq F_Y$ (again because of symmetry). Assuming $F_X \cap F_Y \neq \emptyset$, there exist indices $1 \leq i \leq \beta$ and $1 \leq j \leq \nu$ such that $B_i - Q_1 = D_j - P_2 = Q_2 - D_{n-j} \iff Q_1 + Q_2 = B_i + D_{n-j}$ resulting in a contradiction to $\sigma(\mathcal{B}, Q) = 1$. Hence, it holds that $F_X \cap F_Y = \emptyset$. Furthermore, P is the only partition spanning tree with weight $w(P)$. Therefore, we can apply Lemma 5.9 and construct $\alpha + \beta + \mu + \nu + 2 \geq n_1 + n_2 - 2$ partition spanning trees with distinct weights smaller than $w(Q)$. Hence, we arrive at $\mathrm{ord}'(\mathcal{B}, Q) \geq n_1 + n_2 - 1 = n$ proving the theorem.

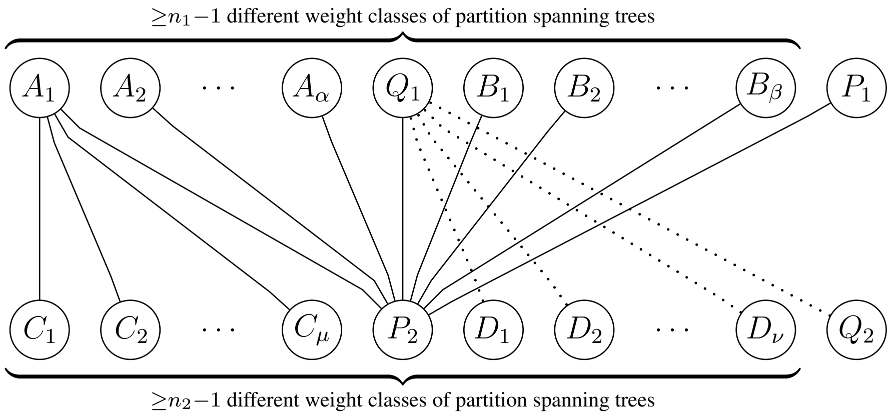


Figure 6: Combinations if $\mathrm{ord}'(\mathcal{B}_1, Q_1) < \mathrm{ord}'(\mathcal{B}_1, P_1)$ and $\mathrm{ord}'(\mathcal{B}_2, P_2) < \mathrm{ord}'(\mathcal{B}_2, Q_2)$ hold.

In Figure 6 all considered combinations are illustrated. In this figure, there are only crossings of dotted and solid lines. Therefore, only these combinations can conflict with each other. By definition, the combination (Q_1, Q_2) forms a spanning tree of unique weight. On the other hand, no combination (P_2, B_i), $1 \leq i \leq \beta$ can conflict with a combination (Q_1, D_j), $1 \leq j \leq \nu$ since otherwise we can construct a partition spanning tree $T \neq Q$ with weight $w(Q)$ contradicting $\sigma(\mathcal{B}, Q) = 1$. □

5.3 Partitioning the complete graph on four vertices

We have seen that it is possible to separately analyze the 2-vertex-connected and 3-edge-connected bispanning components of an arbitrary bispanning graph. The following theorem is well known from graph theory [4].

Theorem 5.10. *A 2-connected simple graph in which the degree of every vertex is at least 3 has a minor isomorphic to the complete graph K_4.*

Hence, each of these 2-vertex-connected and 3-edge-connected bispanning graphs $\mathcal{B} = (V, P, Q)$ has a K_4 minor and therefore their cycle matroids are *not* strongly base orderable. Nevertheless, we will prove that Conjecture 7 holds even if the given weighted bispanning graph $\mathcal{B} = (V, P, Q)$ is isomorphic to the complete graph on four vertices with a weight function which is required to satisfy $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$.

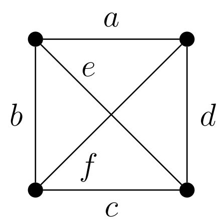


Figure 7: K_4

Theorem 5.11. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph on four vertices such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$. Then it holds that $\operatorname{ord}'(\mathcal{B}, Q) \geq 4$.*

Proof. If the bispanning graph $\mathcal{B} = (V, P, Q)$ on four vertices has any multiple edges, the cycle matroid of $\mathcal{B}$ is strongly base orderable and we are done. Therefore, we consider the complete graph on four vertices. It is easy to see that there is up to isomorphism only one assignment of the edges to two disjoint spanning tree. In the following, we set $Q := \{a, b, c\}$ and $P := \{d, e, f\}$ according to Figure 7. This graph has 12 different partition spanning trees which are illustrated in Figure 8, where each pair of complementary trees is in a box with gray background.

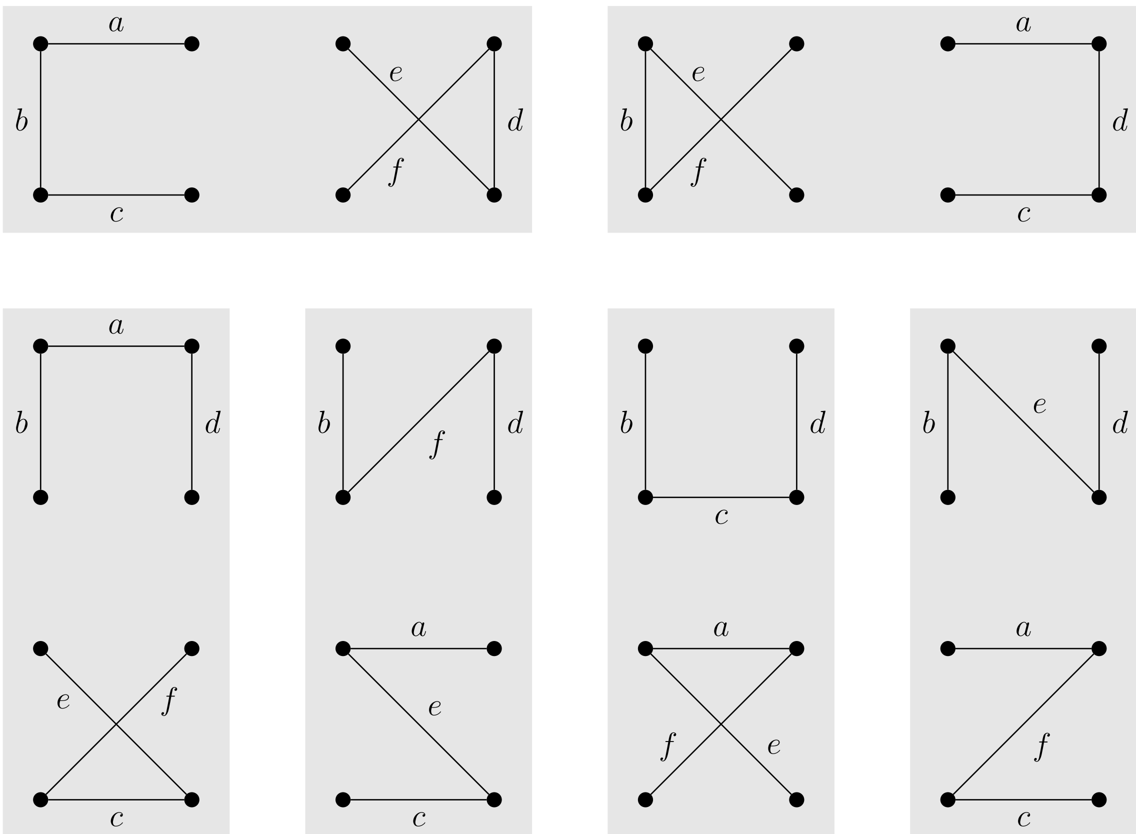


Figure 8: All partition spanning trees of the complete graph K_4.

Note that at least one partition spanning tree of each pair has a weight smaller than $w(Q)$. First, we consider the complementary spanning trees $\{a, c, d\}$ and $\{b, e, f\}$ and assume both trees have equal weight. Observe that at least $\{a, e, f\}$ *and* $\{b, c, d\}$ or $\{a, c, f\}$ *and* $\{b, d, e\}$ must have different weight since otherwise it holds that $w(c) = w(e)$ implying $\sigma(\mathcal{B}, Q) > 1$. Thus, there must be three distinct partition spanning trees (including P) with weight smaller than $w(Q)$.

Hence, we assume the trees $\{a, c, d\}$ and $\{b, e, f\}$ have different weight (where at least one of these weights is strictly smaller than $w(Q)$). Analogously, if one of the remaining four pairs of partition spanning trees consists of trees with equal weight, we are done.

The remaining case is that each pair consists of different weighted spanning trees implying the following two matrices of inequations (for the sake of readability, we associate with an edge q also its weight $w(q)$):

$$\begin{pmatrix} a+b+d & \cdots & \neq & \cdots & b+d+f \\ \vdots & \ddots & & ⋰ & \vdots \\ \neq & & ? & & \neq \\ \vdots & ⋰ & & \ddots & \vdots \\ c+e+f & \cdots & \neq & \cdots & a+c+e \end{pmatrix} \text{ and } \begin{pmatrix} b+c+d & \cdots & \neq & \cdots & b+d+e \\ \vdots & \ddots & & ⋰ & \vdots \\ \neq & & ? & & \neq \\ \vdots & ⋰ & & \ddots & \vdots \\ a+e+f & \cdots & \neq & \cdots & a+c+f \end{pmatrix}.$$

There are two question marks inside these matrices. If at least one of them can be replaced by "$\neq$" then it is easy to see that there are three partition spanning trees with weight less than $w(Q)$. Hence we assume both question marks can be replaced by "$=$". This means both matrices consist of at least one weight less than $w(Q)$. If these weights are different we are done. Otherwise, we have to distinguish two different cases.

1. The case $c+e+f = b+d+f = b+c+d = a+c+f$ (equivalent to $a+b+d = a+c+e = a+e+f = b+d+e$ because of complementary spanning trees) implies $a = e$ and $c = f$ resulting in $w(Q) = w(\{b, e, f\})$ which is a contradiction to $\sigma(\mathcal{B}, Q) = 1$.

2. The case $a+b+d = a+c+e = b+c+d = a+c+f$ (equivalent to $c+e+f = b+d+f = a+e+f = b+d+e$) can either conflict with $a+c+d$ or with $b+e+f$.

   (a) We assume there is a conflict with spanning tree $\{a, c, d\}$ that means we have $a+b+d = a+c+e = b+c+d = a+c+f = a+c+d$ which implies $a = b = c$ and $d = e = f$. But then Q must be a maximum spanning tree and P a minimum spanning tree, respectively. In this case, the partition spanning trees $\{a, c, e\}$,$\{c, e, f\}$, and $\{d, e, f\}$ are sufficient to prove the claim.

   (b) On the other hand we assume $a+b+d = a+c+e = b+c+d = a+c+f = b+e+f$ which implies $a = c$, $e = f$, $a+c = b+e$, and $c+d = e+f$. Clearly, we have $a = c \neq e = f$ since otherwise $\sigma(\mathcal{B}, Q) > 1$. If $a = c < e = f$ holds then $a+c = b+e$ and $c+d = e+f$ imply $b < a = c$ and $d > e = f$ resulting in a contradiction to $w(Q) > w(P)$. On the other hand $a = c > e = f$ together with $a+c = b+e$ and $c+d = e+f$ imply $b > a = c > e = f > d$, that is, Q is a maximum spanning tree and P a minimum spanning tree, respectively. Thus, the claim follows as seen above. ☐

Since Conjecture 7 is at least as strong as Conjecture 6 (in fact we only consider a smaller class of spanning trees) we obtain the following corollary.

Corollary 5.12. *Let $\mathcal{B} = (V, P, Q)$ be a weighted bispanning graph on four vertices such that $\operatorname{ord}(\mathcal{B}, P) < \operatorname{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$. Then it holds that $\operatorname{ord}(\mathcal{B}, Q) \geq 4$.*

6 Summary

In this paper, we discussed a conjecture of Mayr and Plaxton [8] and its equivalent formulation viz. that all weighted bispanning graphs $\mathcal{B} = (V, P, Q)$ satisfying $\text{ord}(\mathcal{B}, P) < \text{ord}(\mathcal{B}, Q)$ and $\sigma(\mathcal{B}, Q) = 1$ have $|V| - 1$ distinct spanning trees with pairwise different weights strictly smaller than $w(Q)$. We have shown that this conjecture is true when we restrict ourselves to special weight functions (see Section 3) or to specially structured bispanning graphs (see Section 4). Furthermore, we formulate a slightly stronger conjecture where we count only so-called partition spanning trees of bispanning graphs. Under this stronger conjecture, it is sufficient to analyze 2-vertex-connected and 3-edge-connected bispanning graphs. The complete graph K_4, which is the smallest graph of this class, has the desired property. Unfortunately, the given proof consists of a tedious case analysis. It remains open to find a shorter proof which might be extended to other 2-vertex-connected and 3-edge-connected bispanning graphs.

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