Zentrum Mathematik
Lehrstuhl für Mathematische Statistik der
Technischen Universität München

Stochastic processes beyond semimartingales with application to interest rates, credit risk and volatility modeling

Holger Maria Fink

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat)

genehmigten Dissertation.

Vorsitzender:	Univ.-Prof. Dr. Herbert Spohn
Prüfer der Dissertation:	
	1. Univ.-Prof. Dr. Claudia Klüppelberg
	2. Univ.-Prof. Dr. Christoph Kühn, Johann Wolfgang Goethe-Universität, Frankfurt am Main
	3. Univ.-Prof. Dr. Christian Bender, Universität des Saarlandes, Saarbrücken (nur schriftliche Beurteilung)

Die Dissertation wurde am 07.12.2011 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 15.03.2012 angenommen.

Zusammenfassung

Der erste Teil dieser Dissertation analysiert die bedingten Verteilungen fraktionaler Prozesse, wie der fraktionalen Brownschen Bewegung und der (multivariaten) fraktionalen Lévy Prozesse, welche über Molchan-Golosov Kerne eingeführt werden. Mithilfe einer Formel für die bedingte Erwartung der fraktionalen Brownschen Bewegung und klassischen Eigenschaften der Gaussverteilung wird die bedingte charakteristische Funktion der fraktionalen Brownschen Bewegung und verwandter Prozesse (wie fraktionaler Analogien bekannter affiner Prozesse, inklusive fraktionaler Ornstein-Uhlenbeck und fraktionaler Cox-Ingersoll-Ross Modelle) hergeleitet. In einem nächsten Schritt wird eine bivariate fraktionale Brownsche Bewegung betrachtet, wobei eine bestimmte Abhängigkeitsstruktur vorausgesetzt wird. Danach werden die vorherigen Ergebnisse auf fraktionale Lévy Prozesse verallgemeinert.

Motiviert durch empirische Untersuchungen, welche Evidenz von Langfristabhängigkeit in makroökonomischen Variablen, wie Zinsen, BIP oder Angebot- und Nachfrageraten, zeigen, werden verschiede fraktionale Vasicek Marktmodelle (inklusive Kreditrisiko) vorgeschlagen und untersucht. Mit den vorherigen Ergebnissen werden (analytische) Preisformeln für (ausfallbare) Nullkuponanleihen und allgemeine Kreditderivate bewiesen.

Lévy Prozesse mit ganzzahligen Werten können herangezogen werden, um hochfrequente Finanzdaten zu modellieren. Dabei werden Preisticks nach oben und unten separat beschrieben. Allerdings spiegelt dieser Ansatz nicht die sogenannten 'stilisierten Eigenschaften' wieder, die oft in empirischen Untersuchungen nachgewiesen werden. Im zweiten Teil dieser Dissertation werden die ganzzahligen Lévy Modelle erweitert, so dass sie Effekte, wie Volatilitätsclustering und statistische Hebelwirkung, beinhalten. Dies wird durch eine bestimmte Art von Zeittransformation erreicht. Die Eigenschaften dieser Modellklasse wird analysiert und ein Anwendungsbeispiel vorgeschlagen.

Abstract

The first part of this thesis analyses the conditional distributions of fractional processes like fractional Brownian motion (fBm) and (multivariate) Molchan-Golosov fractional Lévy processes (MG-fLps) which will be introduced by Molchan-Golosov kernels. Using a simple prediction formula for the conditional expectation of a fBm and its Gaussianity, the conditional characteristic function of fBm and related processes (like fractional analogies of prominent affine processes including fractional Ornstein-Uhlenbeck and fractional Cox-Ingersoll-Ross models) is derived. In a next step a bivariate fBm is considered by assuming a certain dependence structure and afterwards the previous results are generalized to MG-fLps including multivariate fBm and fractional subordinators, i.e. almost surely increasing MG-fLps.

Motivated by empirical evidence of long range dependence in macroeconomic variables like interest rates, domestic gross products or supply and demand rates, various fractional Vasicek bond market models (including credit risk) driven by fBms and MG-fLps are proposed. Using the results on conditional characteristic functions, tractable pricing formulas of (defaultable) zero coupon bonds and general credit derivatives are proven.

Integer-valued Lévy processes, which model up and down ticks separately, can be used as price processes for low latency financial data. However this approach does not reflect the so-called 'stylized facts' that can be found in empirical studies. In the second part of this thesis these models are extended to allow for diurnal features, volatility clustering and statistical leverage. These features are obtained using different time-changes for up and down ticks. The properties of this model class is analysed and an application is provided.

Acknowledgements

First I want to thank my supervisor Prof. Dr. Claudia Klüppelberg for her steady support and never ending help over the last years. I am particular grateful to her for many opportunities to attend international conferences which gave me the possibility of an intellectual exchange with brilliant mathematicians from all over the world.

Furthermore I want to express my gratitude to Prof. Neil Shephard who gave me the opportunity to visit the Oxford-Man Institute of Quantitative Finance. Working with him in Oxford was very encouraging and fruitful.

Special thanks go to Prof. Dr. Martina Zähle, Prof. Dr. Francesca Biagini and Prof. Tyrone Duncan for the collaboration and many interesting discussions especially about fractional processes. Also I am very grateful to Prof. Dr. Christian Bender, Prof. Dr. Alexander Lindner, Prof. Dr. Robert Stelzer and to Prof. Dr. Christoph Kühn for their encouragement and very helpful suggestions which led to a significant improvement of my research.

Also I want to thank Dr. Klaus Böcker who gave me the opportunity to do an internship at UniCredit Group. Our development of a Bayesian risk model was a great opportunity to broaden my mathematical knowledge. Moreover I am grateful to Denise Lichtig who was an outstanding coordinator of the elite graduate program TopMath.

Last but not least my thanks go to my parents and my friends. Without all your patience and support, this thesis would not have been realized. I am deeply indebted to all of you.

Financial support through a scholarship of the German National Academic Foundation and through the Elite Network of Bavaria and TopMath is gratefully acknowledged.

Contents

Introduction

The so called *classical models* of mathematical finance belong to the repertoire of nearly every researcher and practitioner in finance. They include the Black-Scholes model for stock option pricing (developed by Black and Scholes [22] and Merton [93]) and the Vasicek model (cf. Vasicek [130]) for interest rates or (by extension) for credit markets. In mathematical terms, the Black-Scholes model describes the stock price process $S = (S(t))_{t\geq 0}$ under the real-world measure by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad t \geq 0, \quad S(0) \in \mathbb{R}^+, \tag{0.1}$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. The model is driven by a Brownian motion $B = (B(t))_{t\geq 0}$ which lies in the class of semimartingales. Consequently, an extensive theory for stochastic integration is available, giving sense to (0.1), cf. Protter [108]. The main advantage of the Black-Scholes model is that it provides easily tractable pricing formulas for plain vanilla options and even more complicated payoffs. Since practitioners have to rely on fast calculations, this explains its popularity and manifold use in different areas of mathematical finance.

However this comes at a high price: Empirical studies of logarithmic stock price returns show that they are clearly not Gaussian but heavy tailed. As consequence the Black-Scholes model underestimates the probability of large price moves. This drawback can be avoided by using a Lévy process instead of a Brownian motion to describe the dynamics of (0.1) (cf. early work of Mandelbrot [87] and Press [107], followed by Eberlein and Keller [50] or Eberlein, Keller and Prause [51].

Another disadvantage of (0.1) is the constant volatility of logarithmic stock price returns, a property which is also not supported by empirical studies. Many models were suggested to deal with this situation by considering volatility to be stochastic itself, cf. the subordination approach of Clark [33], the discrete time (G)ARCH setting of Engle [54] or the general survey of Shephard [123]. In particular, Barndorff-Nielsen and Shephard [11]

proposed an extension of the Black-Scholes model by introducing stochastic volatility σ through an Ornstein-Uhlenbeck process $\sigma = (\sigma(t))_{t\geq 0}$ driven by a Lévy subordinator $L = (L(t))_{t\geq 0}$, i.e. an a.s. increasing Lévy process:

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dB_t + \rho d(L_{\lambda t} - E[L_{\lambda t}])), \quad t \geq 0, \quad S(0) \in \mathbb{R}^+, \\ d\sigma_t^2 &= -\lambda \sigma_t^2 dt + dL_{\lambda t}, \quad t \geq 0, \quad \sigma(0) \in \mathbb{R}^+, \end{aligned} \tag{0.2}$$

for $\lambda > 0$ and $\rho < 0$. This setting also allows for statistical leverage, i.e. negative correlation between volatility and stock price.

Another approach has been developed by Klüppelberg, Lindner and Maller [80] who introduced a continues time GARCH model (COGARCH), cf. Klüppelberg, Lindner and Maller [81] for a comparison with the setting of (0.2). Further steps lead to continuous time ARMA (CARMA) processes (cf. Brockwell [26]) and fractional integrated CARMA (FICARMA) processes (cf. Brockwell and Marquardt [27] and Marquardt [92]), allowing for stronger autocorrelation in the volatility process.

In real markets, prices are usually quoted only up to a specific tick size (depending on the individual financial object), making them basically (after a straightforward transformation) integer-valued, while (0.1) and (0.2) are not. Therefore, especially for high tick (also called low latency) data other approaches might be more suitable. This kind of data is recorded and provided usually less than 1 millisecond after leaving the exchange, cf. Russel and Engle [114] and Bauwens and Hautsch [13] for recent reviews on the importance of low latency data.

Of course it is possible to use the continuous Black-Scholes model in a low latency setting. However other approaches might be more appropriate. Delattre and Jacod [37] considered diffusion processes with round-off errors and their stochastic properties. Rosenbaum [112] used their results to propose a model for asset prices with focus on estimating integrated volatility. Statistical approaches, working directly with the discrete-valued observations, have been considered e.g. by Müller and Czado [97] who used an autoregressive probit model to describe absolute price changes, and Haug and Czado [70] who applied an ARMA approach.

Barndorff-Nielsen, Pollard and Shephard [10] recently suggested to describe the price process $P = (P(t))_{t\geq 0}$ of low latency data by the difference of two discrete-valued Lévy subordinators $L^+ = (L^+(t))_{t\geq 0}$ and $L^- = (L^-(t))_{t\geq 0}$, i.e.

$$P(t) = P(0) + L^+(t) - L^-(t), \quad t \geq 0, \quad P(0) \in \mathbb{N}. \tag{0.3}$$

This approach has the advantage that up and down ticks can be analysed separately which simplifies estimation.

Turning to interest rate markets, a simple classical model has been proposed by Vasicek [130]. The instantaneous short rate $r = (r(t))_{t\geq 0}$ is described by a stochastic differential equation of the Gaussian Ornstein-Uhlenbeck type:

$$dr(t) \quad = \quad (k - ar(t))dt + \sigma dB(t), \quad t \geq 0, \quad r(0) \in \mathbb{R}^+, \tag{0.4}$$

for $k, a, \sigma > 0$. Therefore the model implies not only a stationary but also mean-reverting short rate with respect to the long-term mean k/a, properties, which are supported by empirical observations. Note, that as it is common for short rate models, (0.4) describes the dynamics of r directly under a risk-neutral measure. Similar to the Black-Scholes model (0.1), the Vasicek approach (0.4) provides fast and efficient pricing formulas for zero coupon bonds, the building blocks of interest rate markets (cf. Brigo and Mercurio [24]).

As a Gaussian process, r can also take negatives values which is one of the major disadvantages of the Vasicek model. Of course one can always shift and scale the model to make the probability of a negative r as small as possible. Another way of avoiding this issue was suggested by Cox, Ingersoll and Ross [35] who used a square root process to model the short rate. In such a setting, the random term $dB(t)$ is multiplied with the square root of the short rate process. Consequently, if r gets close to zero, the influence of the Brownian term vanishes and the mean reversion pushes r away from zero again, ensuring that the short rate always stays positive.

Similar to the Black-Scholes situation, (0.4) assumes a constant volatility parameter. The introduction of a time dependent but deterministic volatility into (0.4) leads to the Hull-White model (cf. Hull and White [74])

$$dr(t) \quad = \quad (k(t) - a(t)r(t))dt + \sigma(t)dB(t), \quad t \geq 0, \quad r(0) \in \mathbb{R}^+, \tag{0.5}$$

where $a(\cdot), k(\cdot), \sigma(\cdot)$ are positive deterministic functions. In the literature this is also often called the extended Vasicek model and from now on we will always imply time-dependent parameters when speaking of a Vasicek model. Another very important extension has been developed by Heath, Jarrow and Morton [72]) who modeled not only the short rate, but the whole forward curve.

Further approaches include Eberlein and Raible [52] who described the dynamics of the term structure by general Lévy processes and Duffie [44] and Duffie, Filipovic and

Schachermayer [45] who used not only Brownian motion but general affine Markov processes.

However all these interest rate models have one drawback. Empirical observations (cf. Henry and Zaffaroni [73] and Backus and Zin [7]) suggest that the Markov structure inherent in these models might not be able to capture the situation at the real markets. In fact there is evidence of long range dependence in the short rate (Section 4 of Backus and Zin [7]).

The last observation is the starting point of Part I of this thesis. We aim to replace the Brownian motion in (0.5) by fractional Brownian motion or, more general, fractional Lévy processes to capture the long range dependence effect in the data. Doing that, we leave the Markov setting, implying that the whole past paths will enter prices of financial derivatives now. As mentioned earlier it is very important for practitioners to have tractable pricing formulas. Therefore we extensively consider conditional distributions of such fractional processes and show that they still allow for fairly explicit calculations. We propose various settings for interest rate and credit markets and motivate our approach by deriving a fractional Brownian Vasicek model from the fractional Heath-Jarrow-Morton approach of Ohashi [100] who ruled out arbitrage in this fractional market.

The starting point of Part II is the simple Lévy model (0.3) of Barndorff-Nielsen et al. [10]: a major drawback is the fact that this leads to independent and stationary returns. In reality, stock prices often show strong diurnal features and time-varying volatility. Since the classical ideas of Barndorff-Nielsen and Shephard [11] cannot be applied, we develop a new approach to model stochastic volatility in integer-valued markets that allow for volatility clustering and statistical leverage. Our model setting will also include fractional Lévy processes as driving processes of the volatility. We consider no-arbitrage properties, market completeness, estimation methods and provide an application to the prices of the Euro-Dollar IMM FX futures contract.

Outline of the thesis

The first two chapters state the main preliminaries for the content of this thesis: *Chapter 1* includes a short introduction into the notation and gives brief outlines of the main mathematical concepts used later. *Chapter 2* provides preliminaries regarding (unique) solutions

of fractional stochastic differential equations. *Section 2.1* outlines the fractional Brownian case which has been investigated by Buchmann and Klüppelberg [28]. The case of the fractional Lévy processes defined via Mandelbrot-Van Ness kernels has been developed by Fink and Klüppelberg [59] and is stated in *Section 2.2*. It will also provide motivation why later in this thesis we will use Molchan-Golosov instead of Mandelbrot-Van Ness kernels.

The chapters that follow constitute the new work for this thesis, are all based on individual papers and therefore can mostly be considered self-contained. A brief outline follows.

Part I of this thesis mainly focuses on fractional processes and possible applications in interest rate, credit and stochastic volatility models.

Chapter 3 is based on Fink, Klüppelberg and Zähle [60] entitled *Conditional distributions of processes related to fractional Brownian motion* and the work of Biagini, Fink and Klüppelberg [18], entitled *A fractional credit model with long range dependent hazard rate*. In *Section 3.1*, the conditional characteristic functions of integrals driven by univariate fractional Brownian motion (including the conditional expectation of geometric fractional Brownian motion, cf. Valkeila [129]) are obtained by discrete approximations of the σ-algebra we condition on, well-known facts about the multivariate Gaussian distribution and early results of Gripenberg and Norros [64] and Pipiras and Taqqu [104]. Afterwards we extend these results to solutions of stochastic differential equations driven by a fractional Brownian motion. The case of d-dimensional fractional Brownian motion with independent entries follows directly and is briefly outlined in *Section 3.2* for the sake of completeness.

As an application we present an interest rate model where the short rate is described by fractional Brownian Vasicek dynamics in *Section 3.3*. After a short motivation in *Section 3.3.1*, we derive the model from the fractional Heath-Jarrow-Morton approach of Ohashi [100] (which is based on previous work of Guasoni, Rásonyi and Schachermayer [65, 66]) in *Sections 3.3.2* and *3.3.3*. Even in this non-Markovian setting, zero coupon bond prices can still be calculated explicitly which we do in *Section 3.3.4*.

In *Sections 3.4* and *3.5*, based on [18], we derive the conditional distribution of a bivariate fractional Brownian motion (where the dependence is given as in Elliott and van der Hoek [53]) by using special properties of the Wick product. Invoking these results we introduce credit risk in the interest rate market of *Section 3.3* and calculate the prices of

defaultable bonds (*Sections 3.5.1* and *3.5.2*) and general credit derivatives (*Section 3.5.3*).

Chapter 4, which consists of Fink [58], entitled *Conditional characteristic functions of Molchan-Golosov fractional Lévy processes with application to credit risk*, generalizes the results from *Chapter 3* to the situation of multivariate Molchan-Golosov fractional Lévy processes. In *Section 4.1.1* these kind of processes are defined by a Molchan-Golosov kernel extending the univariate definition of Tikanmäki and Mishura [127] and in *Section 4.1.2* integration is specified by an L^2 approach. Prediction results are presented in *Section 4.2.1*, while *Section 4.2.2* considers important examples including Ornstein-Uhlenbeck and Cox-Ingersoll-Ross type processes. As a first application we present in *Section 4.3* a credit model (similar to *Section 3.5*) driven by almost surely increasing fractional processes avoiding the problem of potential negative interest and hazard rates. In a second application we briefly review the Black-Scholes market with fractional volatility introduced by Bender and Marquardt [16] in *Section 4.4.1* Afterwards we show in *Section 4.4.2* that our prediction results can be used to explicitly derive the whole price process of a European call option in this model.

In *Chapter 5* we consider prices of zero coupon bonds in two different fractional Vasicek markets - one driven by a fractional Brownian motion and one driven by a Molchan-Golosov fractional Poisson process. The derivates with respect to the individual parameters are calculated, we carry out a numerical analysis to study the impact of parameter changes on prices and compare our results to classical Markov cases.

Part II of the thesis considers financial models especially designed for high tick data using differences of discrete-valued Lévy subordinators.

Chapter 6 is based on Fink and Shephard [61], entitled *Integer-valued volatility clustering and statistical leverage for low latency financial data.* First we briefly review the basic model developed in Barndorff-Nielsen, Pollard and Shephard [10] in *Section 6.1* and extend it afterwards to mirror stylized facts of financial time series like volatility clustering and statistical leverage by a certain kind of time change (*Sections 6.2.1 - 6.2.4*). We obtain the classical stochastic volatility model of Barndorff-Nielsen and Shephard [11] as a limit in *Section 6.2.5* and consider no-arbitrage properties in *Section 6.2.6.* We briefly outline different ways of estimating our model in *Section 6.3* and provide an example using price

quotes of the Euro-Dollar IMM FX futures contract on 11th December 2009 in *Section 6.4*.

Chapter 7 provides a brief outlook and concludes this thesis.

Chapter 1

Preliminaries - Basics and fractional processes

In this section we will state the necessary preliminaries for our further work. The short introduction into the notation of this thesis is followed by brief sections about fractional calculus, Riemann-Stieltjes integration and p-variation. Afterwards we introduce general Lévy processes (including Brownian motion) which will be used to derive their fractional counterparts by integration. This can either be done by a Mandelbrot-Van Ness (cf. Mandelbrot and Van Ness [89]) or a Molchan-Golosov kernel (cf. Molchan and Golosov [95] and Kleptsyna, LeBreton and Roubaud [79], Norros, Valkeila and Virtamo [99] and Decreusefond and Üstünel [36]). Both approaches lead to the same process in distribution, a fractional Brownian motion, in the Gaussian case. However we will see that the resulting processes differ for Lévy processes. Since we shall only state properties and results which will be needed later, further literature is provided for the interested reader.

1.1 Notation

If nothing else is mentioned we shall always assume a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Denote by $L^2(\Omega)$ the space of square integrable random variables. For a family of random variables $(X(i))_{i\in I}$, I some index set, let $\sigma\overline{\{X(i), i \in I\}}$ denote the completion of the generated σ-algebra. For $A \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, the spaces of integrable, square integrable and continuous functions $f : A \to \mathbb{R}^n$ are denoted by $L^1(A, \mathbb{R}^n)$, $L^2(A, \mathbb{R}^n)$ and $\mathcal{C}^0(A, \mathbb{R}^n)$. In the case $n = 1$ we shall just write $L^1(A)$, $L^2(A)$ and $\mathcal{C}^0(A)$. Furthermore, $\mathrm{Lip}(A)$ and $C^1(A)$ are the spaces of real functions on A, which are Lipschitz continuous on compacts

and continuously differentiable, respectively. Moreover for $T > 0$, $\|\cdot\|_{L^2,T}$ ($\|\cdot\|_{L^2,\infty}$) is the L^2-norm and $\langle\cdot,\cdot\rangle_{L^2,T}$ ($\langle\cdot,\cdot\rangle_{L^2,\infty}$) the corresponding Euclidian scalar product on $L^2([0,T])$ ($L^2(\mathbb{R})$). $\mathbb{R}_+$ ($\mathbb{R}_-$) are the positive (negative) real half lines. For a matrix A, $A^\top$ shall be the adjoint. The gamma function shall be denoted by Γ. For $d \in \mathbb{N}$ let $\mathbb{S}^{d\times d}$ denote the space of all positive semidefinite symmetric matrices of dimension d. The expression $\overset{d}{=}$ means equality of finite dimensional distributions. For $x \in \mathbb{R}$, define $(x)_+ := \max(0,x)$. Let $a \in \mathbb{R}^n$, $n \in \mathbb{N}$, then denote by δ_a the Dirac measure with respect to a.

1.2 Fractional calculus

Fractional integrals and derivatives are at the core of fractional processes like fractional Brownian motion or fractional Lévy processes. Both possible integration kernels, the Mandelbrot-Van Ness and the Molchan-Golosov, can be represented using fractional calculus. In this section we will briefly introduce the main concepts and state the necessary results which will be needed later in this thesis.

A very detailed survey on fractional calculus can be found in Samko, Kilbas and Marichev [117]. The main concept is also closely related to Riemann-Stieltjes integration and stochastic calculus and we refer the interested reader to Zähle [135, 136].

Fractional integrals and derivatives can be introduced on compacts and on the whole real line. There is also a distinction between the left- and right-sided integrals and derivatives. However in this thesis only the right-sided types will appear and therefore we will drop this specification in the definition.

1.2.1 On compacts

We will work on the compact interval $[0,T]$ for some fixed $T > 0$.

Definition 1.2.1. *For $0 < \alpha < 1$ and $f \in L^1([0,T])$ define the* fractional Riemann-Liouville integral of f of order α *with finite time horizon by*

$$(I^\alpha_{T-}f)(s) = \frac{1}{\Gamma(\alpha)}\int_s^T f(r)(r-s)^{\alpha-1}dr, \quad 0 < s < T, \tag{1.1}$$

where Γ denotes the Gamma-function.

For $f \in L^1([0,T])$ this always exists almost everywhere, cf. (7) of Zähle [135]. We shall also need the fractional derivative with finite time horizon for $\alpha \in (0,1)$ which can be introduced as an inverse operation to fractional integration, although its existence is

much more sophisticated. However it is sufficient for this thesis to work with a class of functions for which this operation is well-defined.

Definition 1.2.2. *For* $0 < \alpha < 1$ *let* $g \in L^1([0,T])$ *such that there exists* $\psi_g \in L^1([0,T])$ *satisfying*

$$g(s) = I^{\alpha}_{T-}(\psi_g(\cdot))(s), \quad 0 < s < T. \tag{1.2}$$

Then define the fractional Riemann-Liouville derivative of g of order α *by*

$$(D^{\alpha}_{T-}g)(u) = \frac{1}{\Gamma(1-\alpha)}\Big(\frac{g(u)}{(T-u)^{\alpha}} + \alpha\int_u^T \frac{g(u)-g(s)}{(s-u)^{\alpha+1}}ds\Big), \quad 0 < u < T. \tag{1.3}$$

Remark 1.2.3. In the literature the above definition of the fractional derivative is called the *Weyl representation*. Further it can be shown that the function ψ_g is unique in $L^1([0,T])$ if it exists. We refer to Section 1 of Zähle [135].

As it is convention in the literature, we shall often write $I^{-\alpha}_{T-} = D^{\alpha}_{T-}$. For $\alpha = 0$ we set $I^{\alpha}_{T-} = D^{\alpha}_{T-} = id$.

1.2.2 On the real line

Fractional integrals and derivatives on the whole real line are defined as follows.

Definition 1.2.4. *For* $0 < \alpha < 1$ *and* $f \in L^1(\mathbb{R})$ *the* fractional Riemann-Liouville integral of f of order α *is defined by*

$$(\mathcal{I}^{\alpha}_{-}f)(s) = \frac{1}{\Gamma(\alpha)}\int_s^{\infty} f(t)(t-s)^{\alpha-1}dt, \quad s \in \mathbb{R}.$$

Remark 1.2.5. Samko, Kilbas and Marichev [117], p.94, showed that the fractional integral above exists for almost all $x \in \mathbb{R}$.

Again we shall also need the fractional derivatives. Also, we will only take derivatives when existence will is ensured and use the so called *Marchaud fractional derivatives*, cf. Section 5.4 of Samko, Kilbas and Marichev [117].

Definition 1.2.6. *For* $0 < \alpha < 1$ *let* $g \in L^1(\mathbb{R})$ *such that there exists* $\psi_g \in L^1(\mathbb{R})$ *such that*

$$g(s) = \mathcal{I}^{\alpha}_{-}(\psi_g(\cdot))(s), \quad s \in \mathbb{R}. \tag{1.4}$$

Then define the fractional Marchaud derivative of g of order α *by*

$$(\mathcal{D}^{\alpha}_{-}g)(u) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty} \frac{g(u)-g(u+s)}{s^{\alpha+1}}ds, \quad u \in \mathbb{R}.$$

1.3 Riemann-Stieltjes integration and p-variation

Let $a, b \in \mathbb{R}, a < b$. Besides fractional calculus we will also need Riemann-Stieltjes integration in this thesis. That is, for functions $f, h : [a, b] \mapsto \mathbb{R}$, we take the limit of

$$S(f, g, \kappa, \rho) := \sum_{i=1}^{n} f(y_i)[h(x_i) - h(x_{i-1})] \tag{1.5}$$

where $\kappa = (x_i)_{i=0,\dots,n}$ is a partition and $\rho = (y_i)_{i=1,\dots,n}$ an intermediate partition of $[a, b]$, i.e.

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \quad x_{i-1} \leq y_i \leq x_i, \quad \text{for all} \quad i \in \{1, \dots, n\},$$

while letting $\mathbf{mesh}(\kappa) := \sup_{i=1,\dots,n} |x_i - x_{i-1}|$ go to zero. Using the Banach-Steinhaus Theorem one can prove that, if for a right-continuous h and all continuous f the Riemann-Stieltjes sums of (1.5) converge, h is already of bounded variation. However, we can weaken this assumption on the integrator by restricting the space of possible integrands.

We recall the definition of p-variation over a compact interval $[a, b] \subset \mathbb{R}$:

Definition 1.3.1. *Let $f : [a, b] \mapsto \mathbb{R}$. We define for $0 < p < \infty$ the* p-variation *of f as*

$$v_p(f, [a, b]) := \sup_{\kappa} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p, \tag{1.6}$$

where the supremum is taken over all subdivisions κ of $[a, b]$. If $v_p(f, [a, b]) < \infty$, then we say that f is of bounded p-variation *on $[a, b]$.*

Mostly it will be enough for our purposes to work with continuous functions. However we want to mention that the concept is far more general.

We will further call a stochastic process $X = (X(t))_{t \in A}$, $A \subset \mathbb{R}$, of bounded p-variation, if it is a.s. of bounded p-variation on compacts. Also for $A \subset \mathbb{R}$ we define

$$\mathfrak{W}_p^{\text{con}}(A) \quad := \quad \{f \in \mathcal{C}^0(A) : v_p(f, [s, t]) < \infty \, \forall \, [s, t] \subseteq \mathbb{R}\}. \tag{1.7}$$

Exploiting the concept of p-variation we now state an existence theorem for Riemann-Stieltjes integrals proven by Young [134]:

Theorem 1.3.2. *Let $[a, b] \subset \mathbb{R}$ be a compact interval, $f \in \mathfrak{W}_q^{con}([a, b])$ and $h \in \mathfrak{W}_p^{con}([a, b])$ for some $p, q > 0$ with $p^{-1} + q^{-1} > 1$. Then $\int_a^b f_s dh_s$ exists in the Riemann-Stieltjes sense.*

As in the classical Riemann-Stieltjes calculus a chain rule can be proven; see Zähle [135] Theorem 3.1.

Theorem 1.3.3 (Chain rule)**.** *Let $[a,b]$ be a compact interval and $g \in \mathfrak{W}_p^{con}([a,b])$ for some $p \in (0,2)$. Furthermore, let $F \in \mathcal{C}^1(\mathbb{R})$ with $F' \in Lip(\mathbb{R})$. Then the Riemann-Stieltjes integral $\int_a^b (F' \circ g)_s dg_s$ exists and we have*

$$(F \circ g)(b) - (F \circ g)(a) = \int_a^b (F' \circ g)_s dg_s. \tag{1.8}$$

At last we state a density formula, which we have not found in the literature; for a proof we refer to Fink [57], Theorem 4.3.2.

Theorem 1.3.4 (Density formula)**.** *Let $[a,b] \subset \mathbb{R}$ be a compact interval, $f, h \in \mathfrak{W}_q^{con}([a,b])$ and $g \in \mathfrak{W}_p^{con}([a,b])$ for some $q > 0$ and $p > 1$ with $p^{-1} + q^{-1} > 1$. For all $x \in [a,b]$ we define $\phi(x) := \int_a^x h_s dg_s$. Then we have $\phi \in \mathfrak{W}_p^{con}([a,b])$ and*

$$\int_a^b f_s d\phi_s = \int_a^b f_s h dg_s. \tag{1.9}$$

1.4 Lévy processes

In this section we will provide the necessary preliminaries on multivariate Lévy processes. For more details we refer to the very exhaustive work of Sato [120]. We start this section by the general definition of a Lévy process.

Definition 1.4.1. *For $n \in \mathbb{N}$, let $\mathbf{L} = (\mathbf{L}(t))_{t \geq 0} = (L^1(t), \dots, L^n(t))^\top_{t \geq 0}$ be a stochastic process in $\mathbb{R}^n$ with the following conditions satisfied:*

(i) $\mathbf{L}(0) = 0$ *a.s.*

(ii) $\mathbf{L}(t_n) - \mathbf{L}(t_{n-1}), \dots, \mathbf{L}(t_2) - \mathbf{L}(t_1)$ *are independent for* $0 \leq t_1 \leq \dots \leq t_n$, $n \in \mathbb{N}$;

(iii) $\mathbf{L}(t) - \mathbf{L}(s) \stackrel{d}{=} \mathbf{L}(u) - \mathbf{L}(v)$ *for* $s \leq t$ *and* $v \leq u$ *with* $t - s = u - v$;

(iv) $\mathbf{L}$ *is stochastically continuous, i.e. for all* $\varepsilon > 0$ *and all* $s > 0$

$$\lim_{t \to s} \mathcal{P}(|\mathbf{L}(t) - \mathbf{L}(s)| > \varepsilon) = 0.$$

Then $\mathbf{L}$ *is called a* Lévy process.

Remark 1.4.2. (i) In this work also Lévy processes on a compact time set will appear, i.e. $\mathbf{L} = (\mathbf{L}(t))_{t \in [0,T]} = (L^1(t), \dots, L^n(t))^\top_{t \in [0,T]}$ for $T > 0$.

(ii) A one-dimensional *two-sided Lévy process* $L = (L(t))_{t\in\mathbb{R}}$ is defined by the following procedure: given two independent copies of the same one-dimensional Lévy process, L^1 and L^2, we take

$$L(t) := L^1(t)1_{\{t\geq 0\}} + L^2(-t-)1_{\{t<0\}}, \quad t \in \mathbb{R}. \tag{1.10}$$

Remark 1.4.3. Every Lévy process has a unique càdlàg modification, i.e. almost all sample paths have left limits and are right-continuous, and we shall always work with this modification from now on.

Mostly we will always consider a given multivariate Lévy process

$$\mathbf{L} = (\mathbf{L}(t))_{t\in[0,T]} = (L^1(t), \ldots, L^n(t))^\top_{t\in[0,T]},$$

for $n \in \mathbb{N}$ and $T > 0$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{P})$ satisfying the usual conditions of right-continuity and completeness. The filtration (after possible augmentation) is assumed to be generated by $\mathbf{L}$ (cf. Theorem 2.1.9 of Applebaum [4]).

Then $\mathbf{L}$ can be described in terms of the *characteristic triple* (γ, Σ, ν) by its characteristic function $E[\exp\{i\langle u, \mathbf{L}(t)\rangle\}] = \exp\{t\psi(u)\}$, $t \in [0, T]$, with

$$\psi(u) \quad = \quad i\langle\gamma, u\rangle - \frac{1}{2}u^\top\Sigma u + \int_{\mathbb{R}^n}(\exp\{i\langle u, x\rangle\} - 1 - i\langle u, x\rangle\mathbf{1}_{\{\|x\|<1\}})\nu(dx),$$

for $u \in \mathbb{R}^n$. Here we have $\gamma \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite, and the measure ν satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n}(\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

Also we will only consider finite second moment Lévy processes. Therefore we also have that

$$\int_{\mathbb{R}^n}\|x\|^2\nu(dx) < \infty.$$

Remark 1.4.4. A one-dimensional Lévy process with characteristic triple given by $(\gamma, \Sigma, 0)$, $\gamma \in \mathbb{R}$ and $\Sigma > 0$, is a Brownian motion with drift γ and variance Σ.

Integration with respect to Lévy processes shall be understood in the usual $L^2(\Omega)$-sense (e.g. see Rajput and Rosinski [109] or Sato [121]). For fixed $T > 0$ consider simple functions of the form

$$f(\cdot) = \sum_{k=1}^{m} a_k\mathbf{1}_{[t_k,t_{k+1})}(\cdot),$$

where $0 \leq t_1 \leq \cdots \leq t_m \leq T$ and $a_k \in \mathbb{R}^{n\times n}$ for $1 \leq k \leq n$. The integral with respect to f is defined by

$$\begin{aligned}\int_0^T f(s)d\mathbf{L}(s) &:= \sum_{k=1}^m a_k(\mathbf{L}(t_{k+1}) - \mathbf{L}(t_k)) \\ &= \sum_{k=1}^m \begin{pmatrix} \sum_{j=1}^n (a_k)_{1j}(L^j(t_{k+1}) - L^j(t_k)) \\ \vdots \\ \sum_{j=1}^n (a_k)_{nj}(L^j(t_{k+1}) - L^j(t_k)) \end{pmatrix}.\end{aligned}$$

Then we have the following theorem which will be crucial in Section 4. It is the multivariate version of Theorem 2.7 of Rajput and Rosinski [109] and can be obtained using Proposition 2.17 of Sato [121].

Theorem 1.4.5. *For $f \in L^2([0,T],\mathbb{R}^{n\times n})$ the integral $\int_0^T f(t)d\boldsymbol{L}(t)$ exists as an $L^2(\Omega)$-limit of approximating step functions in $L^2([0,T],\mathbb{R}^{n\times n})$. Moreover, we have for $u \in \mathbb{R}^n$*

$$E\Big[\exp\Big\{i\Big\langle u, \int_0^T f(t)d\boldsymbol{L}(t)\Big\rangle\Big\}\Big] = \exp\Big\{\int_0^T \psi(f(t)^\top u)dt\Big\}.$$

We will end this section with a well-known property of the multivariate normal distribution which will be important in Section 3.1.

Lemma 1.4.6. *Let $Z \sim N(\mu,\Sigma)$; i.e. $Z = (z_1,\ldots,z_d)^T$ is multivariate normally distributed with mean $\mu \in \mathbb{R}^d$ and variance-covariance matrix $\Sigma \in \mathbb{S}^{d\times d}$. For $k \in \{1,\ldots,d-1\}$, set $X = (z_1,\ldots,z_k)^T$ and $Y = (z_{k+1},\ldots,z_d)^T$. Partition*

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

with $\mu_1 \in \mathbb{R}^k$, $\mu_2 \in \mathbb{R}^{d-k}$, $\Sigma_{11} \in \mathbb{S}^{k\times k}$, $\Sigma_{22} \in \mathbb{S}^{(d-k)\times(d-k)}$ and $\Sigma_{12}^T = \Sigma_{21} \in \mathbb{R}^{(d-k)\times k}$. Then we have

$$X|\{Y = y\} \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y-\mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

1.5 Fractional Brownian motion

We will now briefly review the definition and basic properties of fractional Brownian motion, already assuming its existence. For general background we refer to Samorodnitsky and Taqqu [119]. Questions related to integration and integrand spaces are considered in Pipiras and Taqqu [103, 104].

Definition 1.5.1. *For $H \in (0,1)$ let $B^H = (B^H(t))_{t\in\mathbb{R}}$ be a Gaussian process with the following properties:*

(i) $B^H(0) = 0$ *a.s.,*

(ii) $E(B^H(t)) = 0$*,*

(iii) $Cov(B^H(t)B^H(s)) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]$ *for all* $t, s \in \mathbb{R}$*.*

Then B^H is called a fractional Brownian motion (fBm) with Hurst index H*.*

Therefore by our definition, fBm is always standard; i.e. that $E[B^H(1)^2] = 1$. It follows e.g. by Corollary 7.2.3 of [119] that fBm has stationary increments and is *self-similar with index* H, i.e. $(B^H(ct))_{t\in\mathbb{R}} \overset{d}{=} c^H(B^H(t))_{t\in\mathbb{R}}$ for every $c > 0$.

Remark 1.5.2. It is appropriate in our context to use fractional calculus, which suggests to replace H by the *fractional (integration) parameter* $\kappa = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$. We also recall that $\kappa = 0$ refers to standard Brownian motion and we shall write $B^0 = B$.

Remark 1.5.3. In this thesis will also work with fBm on a compact time set, i.e. $B^\kappa = (B^\kappa(t))_{t\in[0,T]}$ for $T > 0$.

An important fact about the paths of fBm follows in the next lemma.

Lemma 1.5.4 (Version of Theorem 3.1 of Decreusefond and Üstünel [36])**.** *For $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ and every $0 < \alpha < \kappa + \frac{1}{2}$ there exists a modification of B^κ whose sample paths are a.s. locally Hölder continuous of order α. Moreover B^κ has a.s. continuous sample paths.*

We shall from now on always work with this modification.

One of the most prominent facts of fBm is that for $\kappa \in (0, \frac{1}{2})$ its increments exhibit a feature called long range dependence. In the literature there are various definitions of this property and we refer to Samorodnitsky [118] for a detailed overview. As outlined there, early considerations of Mandelbrot, e.g. Mandelbrot [88] and Mandelbrot and Wallis [90], were motivated by studies on the flow of water in the Nile river carried out by Hurst [75, 76].

In the context of this thesis we will define long range dependence by the rate of decrease of the autocovariance function, as also has been done by Marquardt [92]:

Definition 1.5.5. *Let $X = (X(t))_{t\in\mathbb{R}}$ be a stationary process and let*

$$\gamma_X(h) := Cov(X(t+h), X(t)), \quad h \in \mathbb{R},$$

be its autocovariance function. The process X exhibits long range dependence *if $d \in (0, \frac{1}{2})$ and $c_\gamma > 0$ exists such that*

$$\lim_{h\to\infty} \frac{\gamma_X(h)}{h^{2d-1}} = c_\gamma. \tag{1.11}$$

Consider for $\kappa \in (0, \frac{1}{2})$ the covariance between two increments of a fBm. Then we get by stationarity

$$\begin{aligned} \gamma_{B^\kappa}(h) &:= Cov(B^\kappa(k) - B^\kappa(k-1), B^\kappa(k+h) - B^\kappa(k+h-1)) \\ &= \frac{1}{2}[|h+1|^{2\kappa+1} - |h-1|^{2\kappa+1} - 2|h|^{2\kappa+1}], \quad h, k \in \mathbb{N}. \end{aligned} \tag{1.12}$$

Corollary 1.5.6. *For $\kappa \in (0, \frac{1}{2})$ the increments of fBm exhibit long range dependence.*

Proof. This follows by equation (1.12) with $d := \kappa = H - \frac{1}{2}$ since we have

$$\gamma_{B^\kappa}(h) \sim Ch^{2\kappa-1}, \quad \text{as } h \to \infty \text{ for some fixed } C > 0.$$

□

Remark 1.5.7. *For $\kappa = 0$ we have $\gamma_{B^\kappa}(h) = 0$ for all $h \in \mathbb{N}$ since the increments of a Brownian motion are independent.*

Fractional Brownian motion can be introduced by various integral representations and we will review two concepts. For details we refer to the quoted literature.

Proposition 1.5.8 (Version of Proposition 7.2.6 of Samorodnitsky and Taqqu [119])**.** *Let B be a two-sided standard Brownian motion and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then the process given by*

$$\frac{\sqrt{\Gamma(2\kappa+2)\sin(\pi(\kappa+\frac{1}{2}))}}{\Gamma(\kappa+1)} \int_{-\infty}^{\infty} \Big((t-s)_+^\kappa - (-s)_+^\kappa\Big) dB(s) \tag{1.13}$$

$$= \sqrt{\Gamma(2\kappa+2)\sin((\kappa+1/2)\pi)} \int_{-\infty}^{\infty} \mathcal{I}_-^\kappa \mathbf{1}_{[0,t)}(s) dB(s), \quad t \in \mathbb{R}, \tag{1.14}$$

is a fBm.

Remark 1.5.9. The integrand in (1.13) is often called *Mandelbrot-Van Ness kernel* since this representation was introduced in Mandelbrot and Van Ness [89].

Proposition 1.5.10 (Version of Proposition 3.1 of Pipiras and Taqqu [104])**.** *Let $T > 0$, B be a standard Brownian motion on $[0, T]$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then the process given by*

$$\Big(\frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)}\Big)^{\frac{1}{2}} \int_0^T s^{-\kappa} I_{T-}^\kappa((\cdot)^\kappa \mathbf{1}_{[0,t)}(\cdot))(s) dB(s), \quad t \in [0, T], \tag{1.15}$$

is a fBm.

Remark 1.5.11. Integrands of the type as in (1.15) are often called *Molchan-Golosov kernels*. For more details we refer to Molchan and Golosov [95], Kleptsyna, LeBreton and Roubaud [79], Norros, Valkeila and Virtamo [99] and Decreusefond and Üstünel [36].

Remark 1.5.12. Both kernels can be used to introduce fractional Lévy processes. However in contrast to fBm this leads to two different processes in general. Details will follow in Section 1.6.

If we want to consider integrals with respect to fBm on compacts, possible spaces of integrands have been introduced by Pipiras and Taqqu [104]: Define for $\kappa \in (0, \frac{1}{2})$

$$\widetilde{\Lambda}_T^\kappa := \Big\{ f : [0,T] \to \mathbb{R} \Big| \int_0^T [s^{-\kappa} I_{T-}^\kappa((\cdot)^\kappa f(\cdot))(s)]^2 ds < \infty \Big\},$$

and for $\kappa \in (-\frac{1}{2}, 0)$

$$\widetilde{\Lambda}_T^\kappa := \Big\{ f : [0,T] \to \mathbb{R} \Big| \exists \phi_f \in L^2([0,T]) : f(s) = s^{-\kappa} I_{T-}^{-\kappa}((\cdot)^\kappa \phi_f(\cdot))(s) \Big\}.$$

In the light of Lemma 4.3 of Bender and Elliott [14] we shall adjust these spaces such that they are closed with respect to multiplication with an indicator function. Therefore define for $\kappa \in (-\frac{1}{2}, \frac{1}{2})$

$$\Lambda_T^\kappa := \Big\{ f : [0,T] \to \mathbb{R} \Big| \forall [s,t] \subseteq [0,T] : f \mathbf{1}_{[s,t]} \in \widetilde{\Lambda}_T^\kappa \Big\}.$$

By Remark 4.2 of Pipiras and Taqqu [104] follows that for $\kappa \in (0, \frac{1}{2})$ the inclusion $L^2([0,T]) \subset \Lambda_T^\kappa$ holds. For $\kappa = 0$ both spaces coincide and are equal to $L^2([0,T])$. For $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ and $f, g \in \Lambda_T^\kappa$ define the scalar product

$$\langle f, g \rangle_{\kappa,T} := \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^T s^{-2\kappa} [I_{T-}^\kappa((\cdot)^\kappa f(\cdot))(s)][I_{T-}^\kappa((\cdot)^\kappa g(\cdot))(s)] ds, \tag{1.16}$$

where we set $\langle f, g \rangle_{\kappa,T} = \langle f, g \rangle_{L^2,T}$ for $\kappa = 0$. Denote the corresponding norm by $\| \cdot \|_{\kappa,T}$. For $\kappa = 0$ we have $\| \cdot \|_{0,T} = \| \cdot \|_{L^2,T}$.

If c is a step function, $\int_0^T c(s) dB^\kappa(s)$ can be reduced to a finite sum. We then have the isometry

$$\Big\| \int_0^T c(s) dB^\kappa(s) \Big\|_2 = \| c(\cdot) \|_{\kappa,T} \tag{1.17}$$

and, by using approximating sequences of step functions, integration for general $c \in \Lambda_T^\kappa$ is defined in the $L^2(\Omega)$-sense, while (1.17) still holds true, cf. Pipiras and Taqqu [104], Theorems 4.1 and 4.2.

Remark 1.5.13. In Section 3.4 also integrals over the whole real line will appear, details will be provided in Remark 3.4.2.

Let $\overline{\mathrm{sp}}_{[0,T]}(B^\kappa)$ be the closure in $L^2(\Omega)$ of all possible linear combinations of the increments of fBm on $[0,T]$. Assume we want to calculate an expression for the prediction

$$X_t(s,\kappa) := E[B^\kappa(t)|B^\kappa(v), v \in [0,s]], \quad 0 \le s \le t.$$

If $X_t(s,\kappa) \in \overline{\mathrm{sp}}_{[0,s]}(B^\kappa)$, we would hope that there exists some function $c \in \Lambda_T^\kappa$ such that $X_t(s,\kappa) = \int_0^s c(v)dB^\kappa(v)$. This is not clear immediately because it has been shown in [104] that, while for $\kappa \in (-\frac{1}{2}, 0]$ the space $(\Lambda_T^\kappa, \langle,\rangle_{\kappa,T})$ is complete, i.e. a Hilbert space, for $\kappa \in (0, \frac{1}{2})$ this is not true.

However, it has been derived by Gripenberg and Norros [64], Theorem 3.1, that such a suitable c still exists for $\kappa \in (0, \frac{1}{2})$ and a fBm defined via a Mandelbrot-Van Ness kernel. An explicit formula for c has been calculated. In fact Theorem 7.1 of Pipiras and Taqqu [104] shows that the same formula holds true for $\kappa \in (-\frac{1}{2}, 0]$ and a Molchan-Golosov fBm:

Lemma 1.5.14. *Let $0 \le s \le t \le T$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then*

$$E[B^\kappa(t)|B^\kappa(v), v \in [0,s]] = B^\kappa(s) + \int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)\,, \tag{1.18}$$

where for $v \in (0,t)$,

$$\begin{aligned} \Psi^\kappa(s,t,v) &= v^{-\kappa}(I_{s-}^{-\kappa}(I_{t-}^{\kappa}(\cdot)^\kappa \mathbf{1}_{[s,t]}(\cdot)))(v) \\ &= \frac{\sin(\pi\kappa)}{\pi} v^{-\kappa}(s-v)^{-\kappa} \int_s^t \frac{z^\kappa(z-s)^\kappa}{z-v} dz \end{aligned} \tag{1.19}$$

and for $v \in \{0,s\}$, we have that $\Psi^\kappa(s,t,v) = 0$.

If we write now

$$E[B^\kappa(t) - B^\kappa(s)|B^\kappa(v), v \in [0,s]] = \int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)$$

it is immediately clear, that this prediction formula can be extended to integrals of fBm, which has been done in Lemma 1 of Duncan [46]:

Proposition 1.5.15. *For $0 \le s \le t \le T$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ let $c \in \Lambda_T^\kappa$. Then*

$$E\left[\int_0^t c(v)dB^\kappa(v)\Big|B^\kappa(v), v \in [0,s]\right] = \int_0^s c(v)dB^\kappa(v) + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v),$$

where for $v \in (0, s)$,

$$\begin{aligned}\Psi_c^\kappa(s,t,v) &= v^{-\kappa}(I_{s-}^{-\kappa}(I_{t-}^{\kappa} z^\kappa c(z)\mathbf{1}_{[s,t]}(z)))(v) \\ &= \frac{\sin(\pi\kappa)}{\pi} v^{-\kappa}(s-v)^{-\kappa}\int_s^t \frac{z^\kappa(z-s)^\kappa}{z-v} c(z)dz. \end{aligned} \tag{1.20}$$

and for $v \in \{0, s\}$, *we have that* $\Psi_c^\kappa(s,t,v) = 0$.

1.6 Fractional Lévy processes

Fractional Lévy processes can be introduced as natural generalizations of the integral representations of fractional Brownian motion (fBm). In the literature there are many possible approaches to define such processes and a readable overview of the two main concepts can be found in Tikanmäki and Mishura [127]. Both ways are mainly based on the idea of integrating memory into a Lévy process by choosing an appropriate kernel function. This can either be done by integration over the whole real line by a Mandelbrot-Van Ness kernel like in Marquardt [92] or on a compact interval by a Molchan-Golosov kernel (cf. Molchan and Golosov [95]) which has been done by Tikanmäki and Mishura [127]. In contrast to the Brownian case this does not necessarily lead to the same fractional Lévy process. Chapter 2 will consider fractional Lévy processes and stochastic differential equations based on the Mandelbrot-Van Ness kernel. In Chapter 4 we are aiming on fractional bond and stochastic volatility models. Therefore we will choose and generalize the Molchan-Golosov kernel approach (since it allows for fractional subordinators) to the multivariate case.

We shortly review the two main concepts, see Marquardt [92], Section 3, and Tikanmäki and Mishura [127] for details and more background. For notational convenience we work (similar to fBm) with the fractional integration parameter $d \in (-\frac{1}{2}, \frac{1}{2})$ instead of the Hurst index H, where $d = H - \frac{1}{2}$. Furthermore we only consider processes with existing second moments.

1.6.1 Definition by Mandelbrot-Van Ness kernel

We will restrict ourselves to $d \in (0, \frac{1}{2})$ here. Analogously to Mandelbrot and Van Ness [89] for fBm we choose (like Marquardt [92]) the following definition.

Definition 1.6.1. *Let* $L = (L(t))_{t\in\mathbb{R}}$ *be a zero-mean two-sided Lévy process with*

$E[L(1)^2] < \infty$ and without Brownian component. For $d \in (0, \frac{1}{2})$ we define

$$L^d(t) := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(ds), \quad t \in \mathbb{R}. \tag{1.21}$$

We call $L^d = (L^d(t))_{t\in\mathbb{R}}$ a Mandelbrot-Van Ness fractional Lévy process (MVN-fLp) *and L the* driving Lévy process *of L^d.*

The integrals above exist in the $L^2(\Omega)$-sense; see Marquardt [92], Theorem 3.5, for details.

Recall that, by the Lévy-Itô decomposition, every Lévy process can be represented as the sum of a Brownian motion and an independent jump process. The Brownian motion gives rise to a fBm, which has been studied extensively; see for instance Samorodnitsky and Taqqu [119] for general background, or Buchmann and Klüppelberg [28] in the context of Chapter 2.

The next result ensures that there is in fact a modification of (1.21), which equals a pathwise improper Riemann integral, and gives first properties.

Proposition 1.6.2 (Marquardt [92], Theorem 3.7, Theorem 4.1 and Theorem 4.4)**.** *Let L^d be a MVN-fLp with $d \in (0, \frac{1}{2})$. Then the following assertions hold:*

(i) L^d has a modification, which equals the improper Riemann integral

$$\frac{1}{\Gamma(d)} \int_{\mathbb{R}} [(t-s)_+^{d-1} - (-s)_+^{d-1}] L(s) ds, \quad t \in \mathbb{R}. \tag{1.22}$$

Furthermore, (1.22) *is continuous in t.*

(ii) For $s, t \in \mathbb{R}$ we have

$$Cov(L^d(t), L^d(s)) = \frac{E[L(1)^2]}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} (|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}). \tag{1.23}$$

(iii) L^d has stationary increments and is symmetric, i.e.

$$(L^d(-t))_{t\in\mathbb{R}} \stackrel{d}{=} (-L^d(t))_{t\in\mathbb{R}}.$$

Remark 1.6.3. Equation (1.23) of Proposition 1.6.2 shows that for $d \in (0, \frac{1}{2})$ the increments of a MVN-fLp exhibit long range dependence.

From now on, we always work with the modification of Proposition 1.6.2 (i).

Next we want to define integration with respect to MVN-fLps.

Remark 1.6.4. As has been shown in Marquardt [92], Theorem 4.10, MVN-fLps may not be semimartingales, and integration in the $L^2(\Omega)$-sense has been developed in Marquardt [92], Section 5. Theorem 4.4 in Marquardt [92] also shows that MVN-fLps are only Hölder continuous up to the fractional integration parameter d and not to the Hurst index H as in the case for fBm.

Therefore, pathwise Riemann-Stieltjes integration by Hölder continuity does not make sense for sde's. On the other hand, using an approach like Young [134] based on p-variation of the sample paths, integration in a pathwise Riemann-Stieltjes sense can be defined; for details see Section 1.3. This means we have a chain rule and a density formula provided the integrator is of bounded p-variation for $p \in [1, 2)$. Let L^d be a MVN-fLp of bounded p-variation, $d \in (0, \frac{1}{2})$ and $p \in [1, 2)$.

Then we define for every stochastic process with sample paths $H \in \mathfrak{W}_q^{\text{con}}(\mathbb{R})$ a.s. and for $p, q > 0$ with $p^{-1} + q^{-1} > 1$ the integral

$$\int_a^b H(s) dL^d(s), \quad -\infty \leq a \leq b \leq \infty, \tag{1.24}$$

pathwise in the Riemann-Stieltjes sense.

As stated in Theorem 1.3.2 the integral in (1.24) always exists on finite intervals $[a, b]$. We will consider also improper integrals, where $a = \infty$ or $b = -\infty$. The existence of the tail integral has then to be justified.

Example 1.6.5. (i) If the driving Lévy process is of finite activity, then the corresponding MVN-fLp is of bounded p-variation for all $p \geq 1$, cf. Theorem 2.25 of Marquardt [92].

(ii) Theorem 2.1 of Bender, Lindner and Schicks [15] states: A MVN-fLp is of finite variation if and only if the Lévy measure ν of the driving Lévy process satisfies

$$\int_{-1}^{1} |x|^{\frac{1}{1-d}} \nu(dx) < \infty.$$

1.6.2 Definition by Molchan-Golosov kernel

To avoid confusion with the Mandelbrot-Van Ness concept we call this class of processes *Molchan-Golosov fractional Lévy processes (MG-fLps)*. We shall only state a version of the definition of Tikanmäki and Mishura [127] here.

Definition 1.6.6. *For $T > 0$ let $L = (L(t))_{t\in[0,T}$ be a zero-mean Lévy process with $E[L(1)^2] < \infty$ and without Brownian component. For $d \in (-\frac{1}{2}, \frac{1}{2})$ define for $s, t \in [0, T]$ the kernel function*

$$z^d(t,s) := \mathbf{1}_{\{s\leq t\}} c_d s^{-d} I^d_{T-}((\cdot)^d \mathbf{1}_{[0,t)}(\cdot))(s)$$

where

$$c_d = \Big(\frac{(2d+1)\Gamma(d+1)\Gamma(1-d)}{\Gamma(1-2d)}\Big)^{\frac{1}{2}}.$$

We call $L^d = (L^d(t))_{t\in[0,T]}$ with

$$L^d(t) = \int_0^t z^d(t,s)dL(s), \quad t \in [0,T],$$

a Molchan-Golosov fractional Lévy processes (MG-fLps) *and L the* driving Lévy process *of L^d.*

The integrals above exist in the $L^2(\Omega)$-sense, cf. Remark 4.1.2 for more details. We have for its autocovariance function by Proposition 3.5 of Tikanmäki and Mishura [127]:

Proposition 1.6.7. *For $0 \leq s, t \leq T$ we have*

$$Cov(L^d(t), L^d(s)) = \frac{E[L(1)^2]}{2}(t^{2d+1} + s^{2d+1} - |t-s|^{2d+1}).$$

Remark 1.6.8. In contrast Proposition 1.6.2, (iii), the increments of a MG-fLp do not necessarily have to be stationary, cf. Proposition 3.11 of Tikanmäki and Mishura [127]. Therefore the processes obtained by a Mandelbrot-Van Ness and Molchan-Golosov kernel are (even when scaled with the same coefficients) in general not the same in distribution.

Remark 1.6.9. In Chapter 4 we extend Definition 1.6.6 to the multivariate case, including Brownian components. Further concepts as integration will by stated and proven.

Chapter 2

Preliminaries - Fractional stochastic differential equations

Stochastic differential equations are at the core of many theoretical and practical applications. In this section we want to recall the results of Buchmann and Klüppelberg [28] on fractional Brownian stochastic differential equations and state the extension of their work to the fractional Lévy case which has been developed by Fink and Klüppelberg [59]. We want to remark that for the rest of the section we shall only consider the case of long range dependence, i.e. $\kappa, d \in (0, \frac{1}{2})$.

2.1 Fractional Brownian stochastic differential equations

Buchmann and Klüppelberg [28] provided an extensive theory on sde's driven by fractional Brownian motion. They used the Hölder continuity of the fBm paths to explicitly solve general pathwise sde's by monotone transformations of fractional Ornstein-Uhlenbeck type processes. We want to briefly recall their results and implications for fractional Brownian sde's.

Definition 2.1.1. *Let B^κ be a fBm, $\kappa \in (0, \frac{1}{2})$ and $\lambda > 0$. Then*

$$\mathcal{O}^{d,\lambda}(t) := \int_{-\infty}^{t} e^{-\lambda(t-s)} dB^\kappa(s)\,, \quad t \in \mathbb{R}, \tag{2.1}$$

is called fractional Ornstein-Uhlenbeck process (fOUp).

The integral in the definition above is understood in the pathwise Riemann-Stieltjes sense, cf. Cheridito, Kawaguchi and Maejima [32], Mikosch and Norvaiša [94] or Buchmann and Klüppelberg [28].

The following theorem is a stronger version of Proposition A.1 c) of [32] and can be found as Theorem 2.3 of [28].

Theorem 2.1.2. *Let $\lambda > 0$ and $\kappa \in (0, \frac{1}{2})$. Then the unique stationary pathwise solution of*

$$d\mathcal{O}^{\kappa,\lambda}(t) = -\lambda \mathcal{O}^{\kappa,\lambda}(t)dt + dB^{\kappa}(t), \quad t \in \mathbb{R}. \tag{2.2}$$

is given a.s. by the fOUp (2.1).

Before stating the main theorems on existence and uniqueness of pathwise solutions to fractional Brownian sde's we have to fix a few definitions. First we will make use of the following Hölder continuity spaces:

$$\begin{aligned} \mathcal{C}^{\beta-}(\mathbb{R}) \quad := \quad & \{f : \mathbb{R} \to \mathbb{R} : \forall K \subset \mathbb{R},\, K \text{ compact:} \\ & f|K \text{ is Hölder continuous of all orders } d < \beta\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}^{\beta+}(\mathbb{R}) \quad := \quad & \{f : \mathbb{R} \to \mathbb{R} : \forall K \subset \mathbb{R},\, K \text{ compact:} \\ & f|K \text{ is Hölder continuous of some order } d(K) > \beta\}\,. \end{aligned}$$

Definition 2.1.3. *Let $\kappa \in (0, \frac{1}{2})$. Suppose that $I \subseteq \mathbb{R}$ is a non-empty interval and $\mu, \sigma \in \mathcal{C}^0(I)$. We refer to a stochastic process $X := (X(t))_{t\in\mathbb{R}}$ as a pathwise solution of the sde*

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB^{\kappa}(t), \quad t \in \mathbb{R}, \tag{2.3}$$

if for almost all sample paths the following conditions are satisfied:
$X \in \mathcal{C}^{(1/2+\kappa)-}(\mathbb{R})$ and X takes only values in I such that for $s \leq t$

(S1) $\sigma \circ X$ is a.s. Riemann-Stieltjes integrable with respect to B^{κ} on $[s, t]$;

(S2) the following integral equation holds:

$$X(t) - X(s) = \int_s^t \mu(X(u))du + \int_s^t \sigma(X(u))dB^{\kappa}(u).$$

The space of all solutions of (2.3) is denoted by $\mathcal{S}^{\kappa}(I, \mu, \sigma, B^{\kappa})$.

Of special importance are the following technical definitions.

Definition 2.1.4 (Version of Definition 3.2 and 3.3 of [28])**.** *(i) A triple (I, μ, σ) is called* proper *if and only if it satisfies the following properties:*

(P1) $I = (a, b) \subseteq \mathbb{R}$ *is an open interval, where* $-\infty \leq a < b \leq \infty$ *and* $\mu, \sigma \in \mathcal{C}^0(I)$.

(P2) There exists a strictly decreasing ψ*, absolutely continuous with respect to Lebesgue measure such that* $\psi = \mu/\sigma$ *on* $I \setminus Z(\sigma)$*, where* $Z(\sigma)$ *are the zeros of* σ*, and*

$$\lim_{x \nearrow b} \psi(x) = -\lim_{x \searrow a} \psi(x) = -\infty\,.$$

(P3) There exists $\lambda > 0$ *such that* $\sigma\psi' \equiv \lambda$ *holds on* I *Lebesgue-a.e.*

(ii) We call the triple (I, μ, σ) κ-proper *if in addition to (P1)-(P3) also the following condition is satisfied:*

(P4) The inverse function $\psi^{-1} : \mathbb{R} \to \psi^{-1}(\mathbb{R}) = I$ *is differentiable and*

$$(\psi^{-1})' \in \mathcal{C}^{\left(\frac{1/2-\kappa}{1/2+\kappa}\right)+}(\mathbb{R}).$$

(iii) The interval I *is called the* state space*, the unique constant* $\lambda > 0$ *in (P3) is called the* friction coefficient (FC) *and the unique function* $f : \mathbb{R} \to I = f(\mathbb{R})$*,* $f(x) := \psi^{-1}(-\lambda x)$*, is called the* state space transform (SST) *for* (I, μ, σ).

Solutions to sde's with different starting values will be constructed by changing the starting value of the fOUp:

Definition 2.1.5. *Let* $\kappa \in (0, \frac{1}{2})$ *and* $\lambda > 0$*. We define the Ornstein-Uhlenbeck operator by*

$$\mathfrak{O}^\lambda(B^\kappa, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \quad \longrightarrow \quad \mathcal{C}^0(\mathbb{R}), \quad (\tau, z) \longmapsto \mathcal{O}^{\kappa,\lambda}(t) - e^{-\lambda(t-\tau)}\mathcal{O}^{\kappa,\lambda}(\tau) + e^{-\lambda(t-\tau)}z. \tag{2.4}$$

Assume further that (I, μ, σ) is κ-proper with SST f and FC λ. Then we set

$$X^{f,\lambda}(B^\kappa, \cdot, \cdot) : \mathbb{R} \times I \quad \longrightarrow \quad \mathcal{C}^0(\mathbb{R}), \quad (\tau, z) \longmapsto f(\mathfrak{O}^\lambda(B^\kappa, \tau, f^{-1}(z))(t)) \tag{2.5}$$

where $\mathfrak{O}^\lambda(t)$ as in (2.4). Then the following theorem states one of the main results of [28].

Let $\mathrm{M}(\Omega, I)$ denote all mappings from Ω into I.

Theorem 2.1.6 (Version of Theorem 3.4 and 3.5 of [28])**.** *Let* $\kappa \in (0, \frac{1}{2})$*. If* (I, μ, σ) *is* κ*-proper with SST* f *and FC* $\lambda > 0$*, then*

$$\{X^{f,\lambda}(B^\kappa, \tau, W) : \tau \in \mathbb{R}, W \in M(\Omega, I)\} \subseteq \mathcal{S}^\kappa(I, \mu, \sigma, B^\kappa).$$

Furthermore, if we assume that $Z(\sigma) = \emptyset$*, then we have*

$$\{X^{f,\lambda}(B^\kappa, \tau, W) : \tau \in \mathbb{R}, W \in M(\Omega, I)\} = \mathcal{S}^\kappa(I, \mu, \sigma, B^\kappa).$$

2.2 Fractional Lévy stochastic differential equations

As explained in Remark 1.6.4 and in contrast to fBm, MVN-fLps are only Hölder continuous up to the fractional integration parameter d and not to the Hurst index $H = \kappa + \frac{1}{2}$. Therefore the approach of Section 2.1 has to be modified and detailed motivation is provided in Section 2.2.2. The following results have been developed in Fink and Klüppelberg [59]. We shall only state the main results and examples of [59] and refer to the paper for details.

2.2.1 Fractional Lévy Ornstein-Uhlenbeck processes

Similar to Section 2.1 fractional Lévy Ornstein-Uhlenbeck processes (fLOUps) are introduced in [59] as improper Riemann-Stieltjes integrals and stationary solutions of the corresponding pathwise Langevin equation

$$d\mathcal{L}^{d,\lambda}(t) = -\lambda\mathcal{L}^{d,\lambda}(t)dt + dL^d(t), \quad t \in \mathbb{R}. \tag{2.6}$$

The definition follows.

Definition 2.2.1. *Let L^d be a MVN-fLp, $d \in (0, \frac{1}{2})$ and $\lambda > 0$. Then*

$$\mathcal{L}^{d,\lambda}(t) := \int_{-\infty}^{t} e^{-\lambda(t-s)} dL^d(s)\,, \quad t \in \mathbb{R}, \tag{2.7}$$

is called fractional Lévy Ornstein-Uhlenbeck process (fLOUp).

Theorem 2.2.2. *Let L^d be a MVN-fLp, $d \in (0, \frac{1}{2})$ and $\lambda > 0$. Then the unique stationary pathwise solution of* (2.6) *is given a.s. by the corresponding fLOUp $\mathcal{L}^{d,\lambda}$.*

The following Ornstein-Uhlenbeck operator will be used to obtain solutions to sde's with different starting values. The operator here modifies the starting value of the fLOUp and the next lemma shows that this modified process still solves the Langevin equation.

Definition 2.2.3. *Let L^d be a MVN-fLp, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $\mathcal{L}^{d,\lambda}$ the corresponding fLOUp. We define the Ornstein-Uhlenbeck operator by*

$$\mathfrak{L}^{\lambda}(L^d, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \quad \longrightarrow \quad \mathcal{C}^0(\mathbb{R}), \quad (\tau, z) \longmapsto \mathcal{L}^{d,\lambda}(t) - e^{-\lambda(t-\tau)}\mathcal{L}^{d,\lambda}(\tau) + e^{-\lambda(t-\tau)}z. \tag{2.8}$$

It is immediate from this definition that $\mathfrak{L}^{\lambda}(L^d, \tau, z)(\tau) = z$ a.s. for $(\tau, z) \in \mathbb{R}^2$.

The next lemma shows that L^d transformed by the OU operator still satisfies the Langevin equation, its proof follows directly by the definition.

Lemma 2.2.4. *Let L^d be a MVN-fLp, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $\mathcal{L}^{d,\lambda}$ be the corresponding fLOUp. For a continuous process $l := (l(t))_{t\in\mathbb{R}}$ the identity $l(t) = \mathfrak{L}^\lambda(L^d, \tau, l(\tau))(t)$ holds for all $\tau, t \in \mathbb{R}$ if and only if*

$$dl(t) = -\lambda l(t)dt + dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.9}$$

2.2.2 Solutions of fractional sde's by state space transforms and proper triples

In this section we consider sde's driven by MVN-fLps. Similar to Section 2.1, using pathwise integration we must solve for almost all $\omega \in \Omega$ a deterministic integral equation. Consequently, we build on an extensive theory starting with the seminal work by Young [134]. We also recall that for Brownian motion the pathwise approach goes back to Doss [42] and Sussman [126] leading to the Fisk-Stratonovich integral. Required is that μ is Lipschitz-continuous and $\sigma \in C^2(\mathbb{R})$ with bounded first and second derivatives. Readable accounts on the history can be found in Karatzas and Shreve [78] and in Ikeda and Watanabe [77].

Regularity assumptions of sample paths of the driving process like Hölder continuity or bounded p-variation for $p < 2$ have been considered by Lyons [86]. We shall work in the framework of p-variation, however, go beyond the work of Lyons, who proves only existence and uniqueness theorems under certain Lipschitz assumptions on the coefficient functions and gives no analytical form of the solution.

The approach by Zähle [136] is indeed comparable to [59], where explicit solutions can be given under differentiability and Lipschitz conditions on the coefficient functions. Most of her results can be applied to sde's driven by a MVN-fLp of bounded p-variation for $p < 2$. The contribution of [59] is two-fold. Firstly, the assumptions are easy to verify and, secondly, [59] is able to present analytic solutions to sde's of the form

$$dX(t) = (\alpha|X(t)|^\gamma + \beta X(t))dt + \sigma|X(t)|^\gamma dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.10}$$

In this situation we cannot apply the results of Zähle [136], since the volatility coefficient does not match the required differentiability assumption. Lyons [86] provides us at least with an existence theorem, but gives no closed form solution.

Similar to Section 2.1 (cf. Buchmann and Klüppelberg [28]) we have to establish certain regularity conditions on the coefficient functions μ and σ.

Definition 2.2.5. *A triple (I, μ, σ) is called* strongly proper *if and only if it is proper and satisfies the following condition:*

(P4) The inverse function $\psi^{-1} : \mathbb{R} \to \psi^{-1}(\mathbb{R}) = I$ is differentiable and $(\psi^{-1})' \in Lip(\mathbb{R})$.*

Condition (P4*) differs from the κ-proper assumption required in Definition 2.1.4, because we work with p-variation instead of Hölder continuity.

Again we need to specify, what will be understood to be a solution to a sde.

Definition 2.2.6. *Let L^d be a MVN-fLp of bounded p-variation, $p \in [1, 2)$ and $d \in (0, \frac{1}{2})$. Suppose that $I \subseteq \mathbb{R}$ is a non-empty interval and $\mu, \sigma \in \mathcal{C}^0(I)$. We refer to a stochastic process $X := (X(t))_{t\in\mathbb{R}}$ as a pathwise solution of the sde*

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dL^d(t), \quad t \in \mathbb{R}, \tag{2.11}$$

if for almost all sample paths the following conditions are satisfied: $X \in \mathfrak{W}_p^{con}(\mathbb{R})$ and the image of X is a subset of I such that for $s \leq t$

(S1) $\sigma \circ X$ is a.s. Riemann-Stieltjes integrable with respect to L^d on $[s, t]$;

(S2) the following integral equation holds:

$$X(t) - X(s) = \int_s^t \mu(X(u))du + \int_s^t \sigma(X(u))dL^d(u).$$

The space of all solutions of (2.11) is denoted by $\mathcal{S}(I, \mu, \sigma, L^d)$.

We consider now a sde as given in (2.11). If we assume that the triple (I, μ, σ) is strongly proper with SST f and FC λ, we define the following operator

$$X^{f,\lambda}(L^d, \cdot, \cdot) : \mathbb{R} \times I \quad \longrightarrow \quad \mathcal{C}^0(\mathbb{R}), \quad (\tau, z) \longmapsto f(\mathfrak{L}^\lambda(L^d, \tau, f^{-1}(z))(t)) \tag{2.12}$$

with OU operator $\mathfrak{L}^\lambda(t)$ as in Definition 2.2.3. We also remark that

$$X^{f,\lambda}(L^d, \tau, f(\mathcal{L}^{d,\lambda}(\tau)))(t) = f(\mathcal{L}^{d,\lambda}(t)), \quad t \in \mathbb{R}. \tag{2.13}$$

Next we state the existence theorem. Let $\mathrm{M}(\Omega, I)$ denote all mappings from Ω into I.

Theorem 2.2.7. *Let L^d be a MVN-fLp of bounded p-variation, $p \in [1, 2)$ and $d \in (0, \frac{1}{2})$. If (I, μ, σ) is strongly proper with SST f and FC $\lambda > 0$, then*

$$\{X^{f,\lambda}(L^d, \tau, W) : \tau \in \mathbb{R}, W \in M(\Omega, I)\} \subseteq \mathcal{S}(I, \mu, \sigma, L^d).$$

The following result ensures uniqueness under natural conditions.

Theorem 2.2.8. *Let L^d be a MVN-fLp of bounded p-variation, $p \in [1,2)$ and $d \in (0, \frac{1}{2})$. Let also (I, μ, σ) be strongly proper with SST f and FC $\lambda > 0$. Furthermore, assume that $Z(\sigma) = \emptyset$. Then*

$$\{X^{f,\lambda}(L^d, \tau, W) : \tau \in \mathbb{R}, W \in M(\Omega, I)\} = \mathcal{S}(I, \mu, \sigma, L^d).$$

And the next corollary covers the important case of a stationary solution.

Corollary 2.2.9. *Let L^d be a MVN-fLp of bounded p-variation, $p \in [1,2)$ and $d \in (0, \frac{1}{2})$ and $\mathcal{L}^{d,\lambda}$ be the corresponding fLOUp. Furthermore, let (I, μ, σ) be strongly proper with SST f and FC $\lambda > 0$. Set $X(t) = f(\mathcal{L}^{d,\lambda}(t))$ for $t \in \mathbb{R}$. Then the following assertions hold:*

(i) X is a stationary pathwise solution of the sde

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.14}$$

(ii) If $Z(\sigma) = \emptyset$, then X is the unique stationary pathwise solution of (2.14).

2.2.3 Examples by means of strongly proper triples

This section is dedicated to examples, which we illustrate by simulations. For those we consider as driving Lévy process a compensated Poisson process L^θ with intensity $\theta > 0$; i.e.

$$L^\theta(t) := P^\theta(t) - t\theta, \quad t \in \mathbb{R},$$

where P^θ is a Lévy process with drift $\gamma = 0$ and Lévy measure $\nu(dx) = \theta\delta_1(dx)$ without Brownian component. Of course, we consider this process to be defined on the whole of $\mathbb{R}$ using (1.10).

In a first step we simulate sample paths of L^θ and compute the corresponding MVN-fLp L^d by a Riemann-Stieltjes approximation; i.e. we approximate

$$\begin{aligned} L(t)^d \;\approx\; & \frac{1}{\Gamma(d+1)} \left\{ \sum_{k=-n^2}^{0} \left[\left(t - \frac{k}{n}\right)^d - \left(-\frac{k}{n}\right)^d \right] \left(L^{a,b}\left(\frac{k+1}{n}\right) - L^{a,b}\left(\frac{k}{n}\right) \right) \right. \\ & \left. + \sum_{k=1}^{[nt]} \left(t - \frac{k}{n}\right)^d \left(L^{a,b}\left(\frac{k+1}{n}\right) - L^{a,b}\left(\frac{k}{n}\right) \right) \right\}, \quad t \in \mathbb{R}. \end{aligned}$$

From Theorem 2.55 of Marquardt [92] we know that the quality of this approximation is

$$O(n^{d-\frac{1}{2}}) + O(n^{-\frac{1}{2}}) + O(n^{\frac{1+2d-2d^2}{2d-3}}).$$

Furthermore, the Poisson MVN-fLp is of finite variation by Theorem 2.25 of Marquardt [92] or by Theorem 2.1 of Bender, Lindner and Schicks [15].

Now we use a version of the explicit Euler method for the sde

$$d\mathcal{L}^{d,\lambda}(t) = -\lambda\mathcal{L}^{d,\lambda}(t)dt + dL^d(t), \quad t \in \mathbb{R},$$

to compute sample paths of the fLOUp. We want to remark that all these computations are pathwise. Probability comes in only through the underlying paths of the driving Lévy processes.

Next we study some examples of solutions to the sde (2.11) given by strongly proper triples. We will mainly draw from structural results of Buchmann and Klüppelberg [28] taking into account that their κ-proper condition has to be replaced by Assumption (P4*) in Definition 2.2.5.

For the rest of this section let L^d be a MVN-fLp of bounded p-variation, $p \in [1, 2)$ and $d \in (0, \frac{1}{2})$.

Example 2.2.10. As a first example we consider for parameters $\alpha, \beta \in \mathbb{R}$ and $\sigma > 0$ a sde of the form

$$dX(t) = (\alpha|X(t)|^\gamma + \beta X(t))dt + \sigma|X(t)|^\gamma dL^d(t)\,, \quad t \in \mathbb{R}\,.$$

We analyse this sde by taking the volatility coefficient $\sigma : \mathbb{R} \to [0, \infty)$ defined by $\sigma(x) := \sigma_0|x|^\gamma$ for $\sigma_0 > 0$ and $\gamma \in \mathbb{R}$ as given. The question is now, what drift functions μ and intervals I lead to strongly proper triples (I, μ, σ) as defined in Definition 2.2.5. More precisely, we want to find elements in the set

$$\mathcal{K}_\sigma^I := \left\{(\lambda, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : (I, \mu, \sigma) \text{ is proper with FC } \lambda \right\}.$$

Using Proposition 5.5 of [28] we see that only $\gamma \in [0, 1]$ leads to a non-empty $\mathcal{K}_\sigma^I$. We consider the cases $\gamma = 0$, $\gamma = 1$ and $\gamma \in (0, 1)$ separately.

Take first $\gamma = 0$. Then for a triple (I, μ, σ) to be proper we must have that $\sigma\psi' \equiv -\lambda$ with $\psi = \mu/\sigma$. This results in

$$dX(t) = (\alpha + \beta X(t))dt + \sigma dL^d(t), \quad t \in \mathbb{R},$$

with state space $I = \mathbb{R}$ and SST is affine, more precisely, $f(x) = \alpha + \beta x$ for $x \in \mathbb{R}$.

If $\gamma = 1$, by Proposition 5.6 of [28], the state space can only be either $I = (-\infty, 0)$ or $I = (0, +\infty)$. For $I = (0, \infty)$ we get

$$\mathcal{K}_\sigma^{(0,\infty)} \quad = \quad \left\{(|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : \mu(x) = \alpha x + \beta x \log x, \alpha \in \mathbb{R}, \beta < 0, x \in (0, \infty)\right\}$$

and the state space transform is $f(x) = \exp\left\{\sigma_0 x - \frac{\alpha}{\beta}\right\}$ for $x \in (0, \infty)$. Simple calculation ensures that condition (P4*) of Definition 2.2.5 is satisfied, and every element of $\mathcal{K}_\sigma^{(0,\infty)}$ leads to a strongly proper triple. An example to a sde of this kind for $\alpha = 0$ can be found in (2.20). The case $I = (-\infty, 0)$ can be treated analogously.

Finally, we consider $\gamma \in (0, 1)$. Proposition 5.8 of [28] shows that the only possible state space is the whole real line $\mathbb{R}$ and

$$\mathcal{K}_\sigma^{\mathbb{R}} = \left\{((1-\gamma)|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : \mu(x) = \alpha|x|^\gamma + \beta x, \alpha \in \mathbb{R}, \beta < 0, x \in \mathbb{R}\right\}.$$

Furthermore the SST is given by

$$f(x) = \operatorname{sign}\left((1-\gamma)\sigma_0 x - \frac{\alpha}{\beta}\right)\left|(1-\gamma)\sigma_0 x - \frac{\alpha}{\beta}\right|^{\frac{1}{1-\gamma}}.$$

The derivative of f can easily be calculated yielding that only $\gamma \in [\frac{1}{2}, 1)$ leads to strongly proper triples. An example for such a sde (with $\alpha = 0$) is a fractional Cox-Ingersoll-Ross type model, which is investigated in detail in Section 2.2.4; cf. the sde in (2.17).

Example 2.2.11. We consider the following sde's with affine drift.

$$dX(t) = (\alpha + \beta X(t))dt + \sigma(X(t))dL^d(t), \quad t \in \mathbb{R},$$

i.e. $\mu : \mathbb{R} \to \mathbb{R}$ is defined by $\mu(x) := \alpha + \beta x$ for $\alpha, \beta \in \mathbb{R}$. To find suitable volatility coefficients and state spaces we consider the set

$$\Lambda_\mu^I := \left\{\lambda \in \mathbb{R}^+ : \exists \sigma \in \mathcal{C}^0(I) \text{ with } (I, \mu, \sigma) \text{ is proper with FC } \lambda\right\}$$

and, if there is a $\lambda \in \Lambda_\mu^I$, we investigate

$$\mathcal{H}_{\mu,\lambda}^I = \left\{\sigma \in \mathcal{C}^0(I) : (I, \mu, \sigma) \text{ is proper with FC } \lambda\right\}.$$

Proposition 5.1 of [28] implies that there exist I, σ with (I, μ, σ) being proper if and only if $\beta < 0$. In this case it also follows that $I = \mathbb{R}$ and $\Lambda_\mu^I = (0, |\beta|]$. A FC $\lambda = |\beta|$ leads again to an affine model, namely $\sigma(x) = \sigma_0 x$ for some $\sigma_0 > 0$.

If we choose a FC $\lambda = (1-\delta)|\beta| \in (0, |\beta|)$ for some $\delta \in (0, 1)$, then by Proposition 5.3 of [28], every $\sigma \in \mathcal{H}_{\mu,(1-\delta)|\beta|}^{\mathbb{R}}$ is of the form

$$\sigma(x) = \sigma_1 |\alpha + \beta x|^\delta 1_{\{x \le -\alpha/\beta\}} + \sigma_2 |\alpha + \beta x|^\delta 1_{\{x \ge -\alpha/\beta\}}$$

for some $\sigma_1, \sigma_2 > 0$. Setting $f_i := |\beta|^{\frac{\delta}{1-\delta}} \sigma_i^{\frac{1}{1-\delta}} (1-\delta)^{\frac{1}{1-\delta}}$ for $i = 1, 2$ the SST takes the form

$$f(x) = \left(\frac{\alpha}{\beta} - f_1 |x|^{\frac{1}{1-\delta}}\right) 1_{\{x \le 0\}} + \left(\frac{\alpha}{\beta} + f_2 |x|^{\frac{1}{1-\delta}}\right) 1_{\{x \ge 0\}}.$$

Calculating the derivative of f we see that a possible proper triple is strongly proper if and only if $\delta \in [\frac{1}{2}, 1)$. An example of such a sde is for parameters $\alpha \in \mathbb{R}$ and $\beta < 0$ given by

$$dX(t) = (\alpha + \beta X(t))dt + \sigma\sqrt{|\alpha + \beta X(t)|}dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.15}$$

Example 2.2.12. Consider the sde

$$dX(t) = -\sigma_1 \sin(\sigma_2 X(t))\cos(\sigma_2 X(t))dt - \sin^2(\sigma_2 X(t))dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.16}$$

This example provides a bounded state space model. Define the triple (I, μ, σ) by $I := (0, \frac{\pi}{\sigma_2})$, $\mu(x) := -\sigma_1 \sin(\sigma_2 x)\cos(\sigma_2 x)$ and $\sigma(x) := -\sin^2(\sigma_2 x)$ where $\sigma_1, \sigma_2 > 0$. It can be shown that this triple is in fact strongly proper with FC $\lambda = \sigma_1\sigma_2$. More precisely, we have $\psi(x) = \sigma_1 \cot(\sigma_2 x)$ and, therefore,

$$X(t) := \frac{1}{\sigma_1}\operatorname{arccot}\big(-\sigma_2 \mathcal{L}^{d,\sigma_1\sigma_2}(t)\big)\,, \quad t \in \mathbb{R},$$

is the unique stationary solution of the sde (2.16).

2.2.4 Fractional Cox-Ingersoll-Ross models

Whenever positive phenomena are modeled for instance interest rates, volatilities or default rates in finance, the Cox, Ingersoll and Ross (CIR) [35] model is the most prominent model. It is the solution to

$$dX(t) = (a - \gamma X(t))dt + \sigma\sqrt{X(t)}dB(t)\,, \quad X_0 = x_0 \geq 0,$$

where $B = (B(t))_{t\in[0,\infty)}$ denotes standard Brownian motion, $a, \gamma \in \mathbb{R}$ and $\sigma > 0$. General existence and uniqueness theorems of Brownian sde's cannot be applied here, because the square root is clearly not Lipschitz continuous. However, Ikeda and Watanabe [77], p. 221, showed that for any $X_0 = x \geq 0$ there exists a unique non-negative solution. We shall consider analogous sde's driven by MVN-fLps.

Within the framework of strongly proper triples, Examples 2.2.10 and 2.2.11 show that the theory of this chapter only covers CIR models with mean reversion to $a = 0$. Consider for $\sigma, \gamma > 0$ a solution to the pathwise sde

$$dX(t) = -\gamma X(t)dt + \sigma\sqrt{|X(t)|}dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.17}$$

Define $\widetilde{\sigma}(x) := \sigma|x|^{\frac{1}{2}}$, choose $\widetilde{\mu}(x) = -\gamma x$ and take $I = \mathbb{R}$. Example 2.2.10 implies that $(I, \widetilde{\mu}, \widetilde{\sigma})$ is strongly proper with SST

$$f(x) = \operatorname{sign}(x)\frac{\sigma^2}{4}x^2,$$

and, by Theorem 2.2.7 and Corollary 2.2.9, a stationary solution of (2.17) is given by $(f(\mathcal{L}^{d,\lambda}(t)))_{t\in\mathbb{R}}$ with $\lambda = \gamma/2$. Obviously, this CIR model takes also negative values.

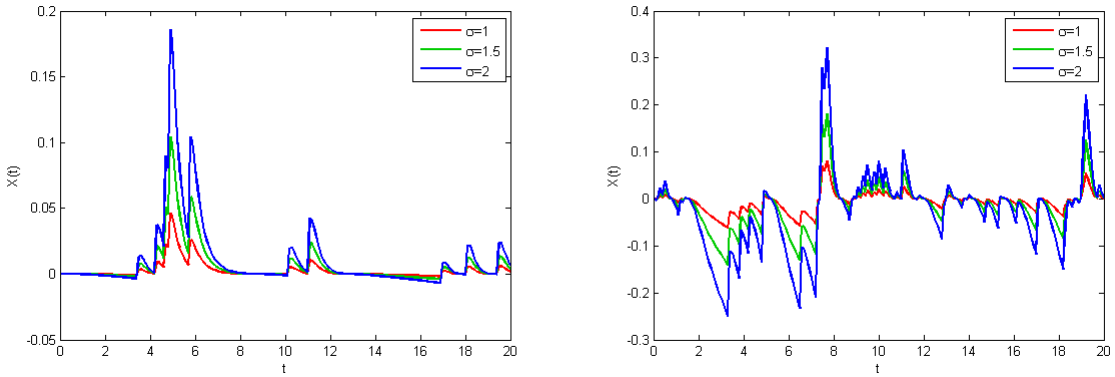

Figure 2.1: *Sample paths of a solution of the Cox-Ingersoll-Ross model* (2.17) *with* $X(0) = 0$ *for varying* σ*, fixed* $\lambda = 2.5$ *and* $d = 0.35$*, using two different MVN-fLp sample paths, left* $\theta = 0.5$*, right* $\theta = 2.5$*.*

A natural non-negative transformation of the fLOUp is given by $Z(t) := (\sigma\mathcal{L}(t)^{d,\lambda})^2$ for $t \in \mathbb{R}$, and, using the chain rule from Theorem 1.3.3 and the existence of all appearing Riemann-Stieltjes integrals, we get

$$\begin{aligned} dZ(t) &= 2\sigma^2\mathcal{L}^{d,\lambda}(t)d\mathcal{L}^{d,\lambda}(t) = -2\lambda\sigma^2(\mathcal{L}^{d,\lambda}(t))^2dt + 2\sigma^2\mathcal{L}^{d,\lambda}(t)dL^d(t) \\ &= -2\lambda Z(t)dt + 2\sigma\sqrt{Z(t)}dL^d(t), \quad t \in \mathbb{R}. \end{aligned}$$

Defining now $\kappa(z) := -2\lambda z$ and $\iota(z) := 2\sigma\sqrt{z}$ we have Z as a solution to

$$dZ(t) = \kappa(Z(t))dt + \iota(Z(t))dL^d(t)\,, \quad t \in \mathbb{R}.$$

However the triple $((0,\infty), \kappa, \iota)$ is not strongly proper, because Assumption (P2) of Definition 2.1.4 is violated.

We can now formulate the following general result:

Proposition 2.2.13. *Let* $\mathfrak{L}^{\frac{\lambda}{2}}(L^d, \cdot, \cdot)$ *be the OU operator from Definition 2.2.3. Then for* $\tau \in \mathbb{R}$ *and* $z \geq 0$ *the process*

$$X^{\lambda,\tau,z}(t) := \left(\frac{\sigma}{2}\mathfrak{L}^{\frac{\lambda}{2}}(L^d, \tau, z)(t)\right)^2, \quad t \in \mathbb{R},$$

solves the sde's

$$dX(t) = -\lambda X(t)dt + \sigma\sqrt{|X(t)|}dL^d(t) \quad \textit{and} \tag{2.18}$$

$$dX(t) = -\lambda X(t)dt + \sigma\sqrt{X(t)}dL^d(t)\,, \quad t \in \mathbb{R}. \tag{2.19}$$

In fact, any solution to (2.19) also solves (2.18).

This result is not surprising, because Theorem 2.2.8 does not hold for the sde (2.18). However, the reverse is not true: a solution of (2.18) does not necessarily solve (2.19), because it can be negative. Also the constant process $X(t) := 0$, $t \in \mathbb{R}$, solves both, (2.19) and (2.18).

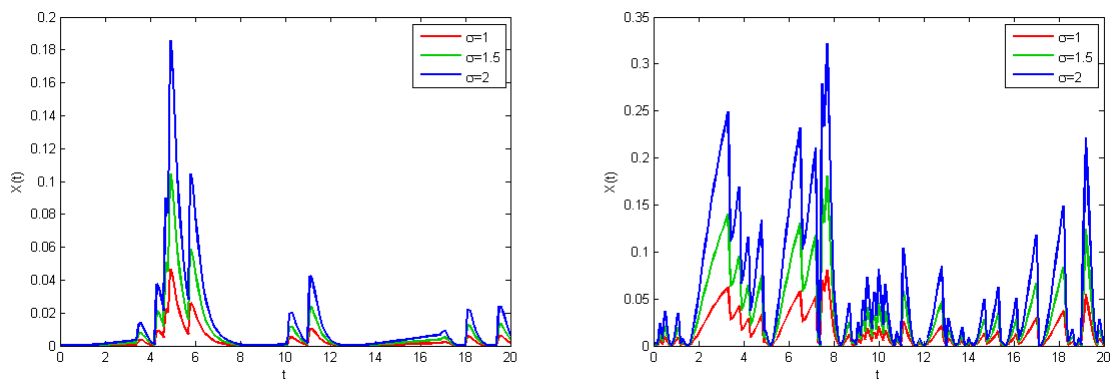

Figure 2.2: *Sample paths of squared fLOUps for varying σ, fixed $\lambda = 2.5$ and $d = 0.35$, using the same sample paths as in Figure 2.1: left $\theta = 0.5$, right $\theta = 2.5$.*

Considering a squared Ornstein-Uhlenbeck process leads in the case of a driving Brownian motion to a CIR model with mean reversion to positive values. This approach does not work for pathwise integrals, neither in the MVN-fLp case nor for fBm, since the Itô term in the chain rule vanishes by bounded p-variation for some $p < 2$.

A positive process based on Theorem 2.2.7 is given as solution to

$$dY(t) = -\lambda\sqrt{Y(t)}\log\left(Y(t)\right)dt + \sigma|Y(t)|dL^d(t), \quad t \in \mathbb{R}, \quad Y_0 = y_0, \tag{2.20}$$

for $\lambda, \sigma > 0$. Example 2.2.10 states that the triple for state space $I = (0, \infty)$ is strongly proper and the SST can be calculated as $f(x) = e^{\sigma x}$.

Remark 2.2.14. As can be seen by the examples of this section, it is not straightforward to create positive processes using MVN-fLps. Lévy subordinators as driving processes in

Definition 1.6.1 (cf. Marquardt [92]) are also excluded because (except for the trivial case) they do not have zero-mean.

In Section 4, Definition 4.1.1, we will introduce (multivariate) fractional Lévy processes defined by Molchan-Golosov kernels in the sense of Tikanmäki and Mishura [127]. To avoid confusion between the two concepts we shall call these objects Molchan-Golosov fractional Lévy processes. They will allow for fractional subordinators, i.e. a.s. increasing fractional processes, in the sense of Bender and Marquardt [16], cf. Example 4.1.4.

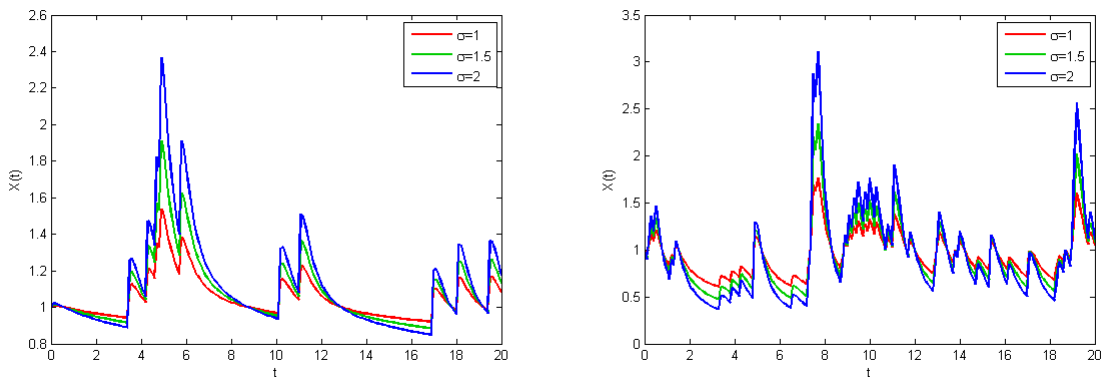

Figure 2.3: *Sample paths of a solution of (2.20) for varying* σ*, fixed* $\lambda = 2.5$ *and* $d = 0.35$*, using the same sample paths as in Figure 2.1, left* $\theta = 0.5$*, right* $\theta = 2.5$*.*

Part I

Conditional distributions of fractional processes

Chapter 3

Fractional Brownian motion and related processes

In this chapter we will consider conditional distributions and characteristic functions of fractional Brownian motion and related processes in increasing generality. First we will cover the one-dimensional case by an approximation of the information up until a time $t \geq 0$, i.e. an approximation of the σ-algebra we condition on, using finite observation samples. These results can be directly used to analyze the situation of a multivariate fractional Brownian motion with independent entries. In a next step we will assume a certain dependence structure between two fractional Brownian motions in the two-dimensional case similar to the situation of Elliott and van der Hoek [53] and apply the properties of the Wick product to solve the prediction problem for the long range dependence case, i.e. positive fractional parameters. A more general result about Molchan-Golosov fractional Lévy processes (as we will also call the multivariate extension of Definition 1.6.6), including (multivariate) fractional Brownian motion, will follow in Chapter 4.

As application we shall consider a fractional bond market where the short rate process is described by fractional Vasicek dynamics. Our setup includes the classical Brownian model introduced by Vasicek [130]. Since fractional Brownian motions are no longer semimartingales in general, arbitrage opportunities may arise, cf. Delbaen and Schachermayer [38, 39]. Further details on this fact can be found in Bender, Sottinen and Valkeila [17] or Cheridito [31].

Therefore we derive the Vasicek model from the fractional Heath-Jarrow-Morton (HJM) approach of Ohashi [100] based on previous work of Guasoni, Rásonyi and Schachermayer [65, 66]: Similar to the classical Brownian HJM setting of Heath et al. [72] the

dynamics of the whole forward curve are described by fractional Brownian motions under a measure $\mathcal{P}$.

However since - in general - fractional Brownian motions are not semimartingales, arbitrage opportunities may occur. However in a more realistic setting with proportional transaction costs, Ohashi showed that arbitrage strategies cannot be constructed anymore. The existence of an average risk-neutral measure $\mathcal{Q}$ can be proven and we can formally calculate prices of defaultable bonds or more general contingent claims under this measure as suggested in Sottinen and Valkeila [125]. On the other hand it is of course always possible to directly define prices via conditional expectations leading in general to an arbitrage-free model.

Afterwards credit risk is introduced by assuming a certain dependence structure (cf. Elliott and van der Hoek [53]) between the short and hazard rate. This will allow us to price defaultable zero bonds as special contingent claims in the fractional Vasicek market model. In a next step we will price a call option on a bond by invoking a Girsanov-like measure change (cf. Norros, Valkeila and Virtamo [99], Theorem 4.1, and Theorem 3.3 of Duncan, Hu and Pasik-Duncan [48]). For general options Fourier methods will be applied.

Remark 3.0.15. Recall that for $\mu \in \mathbb{R}$ and $\sigma > 0$ a random variable X is normally distributed with expectation μ and variance σ^2 if and only if

$$E[e^{iuX}] = \exp\{iu\mu - \frac{u^2}{2}\sigma^2\}$$

for $u \in \mathbb{R}$. We will repeatedly use this relationship without further comment.

3.1 One-dimensional fractional Brownian motion

Calculating conditional distributions by conditional characteristic functions means essentially predicting exponentials. A possible way to approach this problem for fBm driven integrals has been considered in Duncan [46] by transforming the exponential function to a Wick exponential. While this idea works well, Proposition 2 of that paper is not correct. This can be seen immediately, because its result suggests that the prediction is deterministic. The correct version with proof can be found in Duncan and Fink [47]. A similar prediction formula for the conditional expectation of an univariate geometric Brownian motion has been developed in the unpublished work of Valkeila [129]. The basic idea of [46, 47] will be extended later for the two-dimensional dependent case in Section 3.4. Our chosen approach in the present section is based on an approximation

of the σ-algebra we condition on using finite observation samples, the simple prediction formula of Lemma 1.5.14 and classical results on conditional Gaussian distributions. We want to emphasize that our approach also covers the range $\kappa \in (-\frac{1}{2}, 0)$.

For notational convenience we fix for the rest of this section a Molchan-Golosov fBm $(B^\kappa(t))_{t\in[0,T]}$, $T > 0$, with $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Denote further $\mathcal{F}_s := \sigma\overline{\{B^\kappa(v), v \in [0,s]\}}$ for $0 \le s \le T$. The main theorem of this section follows.

Theorem 3.1.1. *Let $c \in \Lambda_T^\kappa$ and $0 \le s \le t \le T$. Then we have for $u \in \mathbb{R}$*

$$\begin{aligned} E\Big[e^{iu\int_0^t c(v)dB^\kappa(v)}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[\int_0^s c(v)dB^\kappa(v) + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\Big]\Big\} \\ &\quad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Big\}, \end{aligned}$$

i.e. $\int_0^t c(u)dB^\kappa(u)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E\Big[\int_0^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big] &= \int_0^s c(v)dB^\kappa(v) + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v) \\ Var\Big[\int_0^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2. \end{aligned}$$

Before we start with the proof of Theorem 3.1.1 we want to compare it to the classical Brownian case with its Markov property and independent increments.

Remark 3.1.2. (a) The variance formula above corresponds to

$$Var[X(t)|\mathcal{F}_s] = Var[X(t)] - Var[E[X(t)|\mathcal{F}_s]]$$

for $X(t) = \int_0^t c(v)dB^\kappa(v)$.
(b) Setting $c(\cdot) = \mathbf{1}_{[0,t]}(\cdot)$ we get by Theorem 3.1.1 for the conditional characteristic function of a fBm

$$\begin{aligned} E\Big[e^{iuB^\kappa(t)}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[B^\kappa(s) + \int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)\Big]\Big\} \\ &\quad \times \exp\Big\{-\frac{u^2}{2}\Big[\|\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Big\}, \quad u \in \mathbb{R}. \end{aligned}$$

If we compare this to the standard Brownian motion case, i.e. setting $\kappa = 0$, we get

$$E\Big[e^{iuB^0(t)}\Big|\mathcal{F}_s\Big] = \exp\Big\{iuB^0(s) - \frac{u^2}{2}\|\mathbf{1}_{[s,t]}(\cdot)\|_2^2\Big\}, \quad u \in \mathbb{R}.$$

It is not surprising that for $\kappa \neq 0$ the whole past path plays a role in the prediction. Theorem 3.1.1 and the equations above show that the conditional expectation changes by the term $\int_0^s \Psi^\kappa(s,t,v)dB^\kappa(v)$.

The proof of Theorem 3.1.1 follows.

Proof. There are different ways of proving Theorem 3.1.1, one possibility is to use Remark 3.1.2(a) for $X = \int_s^t c(v)dB^\kappa(v)$. However, having discrete observations in mind (as is the realistic statistical set-up), we base our proof on a discretization scheme for the past. Given the grid size and also κ we obtain then not only a limit result, but also an approximation based on the correct approximating matrices $(\Sigma_{22}^n)^{-1}\Sigma_{21}^n$ and $\Sigma_{12}^n(\Sigma_{22}^n)^{-1}\Sigma_{21}^n$. For this discrete approximation we shall need the well-known property of the multivariate normal distribution from Lemma 1.4.6.

Let $0 \le s \le t \le T$. To calculate the conditional characteristic function of $\int_0^t c(v)dB^\kappa(v)$ we invoke the fact that by Gaussianity and Lemma 1.4.6 the conditional random variable $\int_0^t c(v)dB^\kappa(v)|\mathcal{F}_s$ is again normally distributed. Since $\int_0^s c(v)dB^\kappa(v)$ is $\mathcal{F}_s$-measurable, it suffices to consider $\int_s^t c(v)dB^\kappa(v)|\mathcal{F}_s$.

As we know from Proposition 1.5.15,

$$E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right] = \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v).$$

Therefore we need only to calculate the conditional variance $Var[\int_s^t c(v)dB^\kappa(v)|\mathcal{F}_s]$.

Choose a sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ of $[0,s]$ such that for $n \in \mathbb{N}$ we have $\pi_n = (s_i^n)_{i=0,\dots,m_n}$ for $m_n \in \mathbb{N}$ with

$$0 = s_0^n < s_1^n < \cdots < s_{m_n}^n \le s \quad \text{and} \quad \sup_{i=1,\dots,m_n} |s_i^n - s_{i-1}^n| \to 0 \text{ as } n \to \infty.$$

Using this notation we know by Lemma 1.4.6 for $n \in \mathbb{N}$

$$\begin{aligned}
& E\left[\int_s^t c(v)dB^\kappa(v)\Big|(B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n))_{i=1,\dots,m_n}\right] \\
= \; & \Sigma_{12}^n(\Sigma_{22}^n)^{-1}\begin{pmatrix} \vdots \\ B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n) \\ \vdots \end{pmatrix}, \\
& Var\left[\int_s^t c(v)dB^\kappa(v)\Big|(B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n))_{i=1,\dots,m_n}\right] \\
= \; & \Sigma_{11}^n - \Sigma_{12}^n(\Sigma_{22}^n)^{-1}\Sigma_{21}^n, \qquad (3.1)
\end{aligned}$$

where $\Sigma_{11}^n = Var[\int_s^t c(v)dB^\kappa(v)]$,

$$\begin{aligned}
(\Sigma_{12}^n)^T &= \Sigma_{21}^n = \begin{pmatrix} \vdots \\ Cov\Big[\int_s^t c(v)dB^\kappa(v), B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)\Big] \\ \vdots \end{pmatrix} \in \mathbb{R}^{m_n} \\
\Sigma_{22}^n &= \Big(Cov\Big[B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n), B^\kappa(s_j^n) - B^\kappa(s_{j-1}^n)\Big]\Big)_{i,j=1,\dots,m_n} \in \mathbb{S}^{m_n \times m_n}.
\end{aligned}$$

It follows by Lemma 1.4.6 and p. 290 of Dudley [43] that a.s. and in $L^1(\Omega)$ as $n \to \infty$,

$$\begin{aligned}
E\left[\int_s^t c(v)dB^\kappa(v)\Big|(B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n))_{i=1,\dots,m_n}\right] &\to E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right], \\
Var\left[\int_s^t c(v)dB^\kappa(v)\Big|(B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n))_{i=1,\dots,m_n}\right] &\to Var\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right].
\end{aligned} \tag{3.2}$$

This implies by (3.1) and Proposition 1.5.15 that a.s. and in $L^1(\Omega)$ as $n \to \infty$,

$$\begin{aligned}
\Sigma_{12}^n(\Sigma_{22}^n)^{-1}\begin{pmatrix} \vdots \\ B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n) \\ \vdots \end{pmatrix} &= \sum_{i=1}^{m_n}(\Sigma_{12}^n(\Sigma_{22}^n)^{-1})_i[B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)] \\
&\to \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v).
\end{aligned} \tag{3.3}$$

Therefore, it follows pointwise and in $\|\cdot\|_{\kappa,T}$ as $n \to \infty$,

$$\sum_{i=1}^{m_n}(\Sigma_{12}^n(\Sigma_{22}^n)^{-1})_i \mathbf{1}_{[s_{i-1}^n, s_i^n]}(\cdot) \to \Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot). \tag{3.4}$$

With this result we can now calculate the conditional variance, since using the isometry (1.17)

$$\begin{aligned}
\Sigma_{12}^n(\Sigma_{22}^n)^{-1}\Sigma_{21}^n &= \Sigma_{12}^n(\Sigma_{22}^n)^{-1}\begin{pmatrix} \vdots \\ Cov\Big[\int_s^t c(v)dB^\kappa(v), B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)\Big] \\ \vdots \end{pmatrix} \\
&= \sum_{i=1}^{m_n}(\Sigma_{12}^n(\Sigma_{22}^n)^{-1})_i Cov\left[\int_s^t c(v)dB^\kappa(v), B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n)\right] \\
&= \sum_{i=1}^{m_n}(\Sigma_{12}^n(\Sigma_{22}^n)^{-1})_i < c(\cdot)\mathbf{1}_{[0,s]}(\cdot), \mathbf{1}_{[s_{i-1}^n, s_i^n]}(\cdot) >_{\kappa,T} \\
&= < c(\cdot)\mathbf{1}_{[0,s]}(\cdot), \sum_{i=1}^{m_n}(\Sigma_{12}^n(\Sigma_{22}^n)^{-1})_i \mathbf{1}_{[s_{i-1}^n, s_i^n]}(\cdot) >_{\kappa,T} \\
&\to < c(\cdot)\mathbf{1}_{[0,s]}(\cdot), \Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot) >_{\kappa,T} \qquad \text{as } n \to \infty
\end{aligned}$$

where we used in the last line the continuity of the scalar product.

It remains to observe that again by the isometry (1.17)

$$\begin{aligned}
& < c(\cdot)\mathbf{1}_{[0,s]}(\cdot), \Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot) >_{\kappa,T} \\
= \; & E\left[\int_s^t c(v)dB^\kappa(v)\int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\right] \\
= \; & E\left[\Big(\int_s^t c(v)dB^\kappa(v) - E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right]\Big)E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right]\right] \\
& +E\left[E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right]^2\right] \\
= \; & E\left[E\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right]^2\right] = \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2
\end{aligned}$$

by the projection property of the conditional expectation in $L^2(\Omega)$.

Finally we conclude that a.s.

$$\begin{aligned}
& Var\left[\int_s^t c(v)dB^\kappa(v)\Big|\mathcal{F}_s\right] \\
= \; & \lim_{n\to\infty} Var\left[\int_s^t c(v)dB^\kappa(v)\Big|(B^\kappa(s_i^n) - B^\kappa(s_{i-1}^n))_{i=1,\dots,m_n}\right] \\
= \; & \lim_{n\to\infty}(\Sigma_{11}^n - \Sigma_{21}^n(\Sigma_{22}^n)^{-1}\Sigma_{12}^n) \\
= \; & \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T} - < c(\cdot)\mathbf{1}_{[0,s]}(\cdot), \Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot) >_{\kappa,T} \\
= \; & \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T} - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2.
\end{aligned}$$

□

Our aim is now to use the results of Theorem 3.1.1 to derive the conditional distributions of more general fractional processes described by fractional Brownian sde's. Recall the results of Section 2.1:

Consider a general pathwise sde with fractional Brownian noise, i.e.

$$dZ(t) = \mu(Z(t))dt + \sigma(Z(t))dB^\kappa(t), \quad Z(0) \in \mathbb{R}, \quad t \in [0,T], \tag{3.5}$$

for $\kappa \in (0, \frac{1}{2})$ and a κ-proper triple (I, μ, σ). Section 2.1, especially Theorem 2.1.6, states that solutions for (3.5) are given by

$$Z(t) = f(X(t)) \tag{3.6}$$

$$dX(t) = -aX(t)dt + dB^\kappa(t), \quad X(0) = f^{-1}(Z(0)), \quad t \in [0,T], \tag{3.7}$$

for SST $f : \mathbb{R} \to \mathbb{R}$ and FC $a > 0$. In the light of the Ornstein-Uhlenbeck operator of Definition 2.1.5, the above restriction to $[0,T]$ is possible.

Motivated by this, we want to predict general OU-type processes driven by fBm in a next step. Therefore, we consider the sde with time dependent coefficient functions

$$dX(t) = (k(t) - a(t)X(t))dt + \sigma(t)dB^\kappa(t)\,, \quad X(0) \in \mathbb{R}, \quad t \in [0,T], \tag{3.8}$$

where integration is defined in Section 1.5. Here $k(\cdot), a(\cdot)$ are locally integrable and continuous on $\mathbb{R}_+$, $\sigma(\cdot) \neq 0$, continuous and $\sigma(\cdot) \in \Lambda_T^\kappa$. For $\kappa \in (-\frac{1}{2}, 0)$ assume further that $e^{-\int_\cdot^t a(w)dw}\sigma(\cdot) \in \Lambda_T^\kappa$ for $0 \leq t \leq T$. Then the unique solution to (3.8) is given by the process $X = (X(t))_{t\in[0,T]}$, defined for $t \in [0,T]$ by

$$X(t) = X(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du}k(s)ds + \int_0^t e^{-\int_s^t a(u)du}\sigma(s)dB^\kappa(s). \tag{3.9}$$

Because σ does not hit zero, we have the equality $\mathcal{F}_s = \overline{\sigma\{X(v), v \in [0,s]\}}$ for $0 \leq s \leq T$.

Theorem 3.1.3. *Let $0 \leq s \leq t \leq T$. Set $c(\cdot) = e^{-\int_\cdot^t a(w)dw}\sigma(\cdot)$ and recall Ψ_c^κ from* (1.20)*. Then we have for $u \in \mathbb{R}$*

$$\begin{aligned} & E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] \\ = \; & \exp\left\{iu\Big[X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\Big]\right\} \\ & \times \exp\left\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\right\}, \end{aligned}$$

i.e. $X(t)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E[X(t)|\mathcal{F}_s] &= X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v) \\ Var[X(t)|\mathcal{F}_s] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2. \end{aligned}$$

Proof. By (3.9) it follows that for $0 \leq s \leq t \leq T$

$$X(t) = X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv + \int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v). \tag{3.10}$$

Therefore $X(t)|\mathcal{F}_s$ is again Gaussian distributed. Since $X(s)$ is $\mathcal{F}_s$-measurable, a direct consequence is now that

$$\begin{aligned} E\Big[X(t)\Big|\mathcal{F}_s\Big] &= X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv \\ &+ E\Big[\int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big], \\ Var\Big[X(t)\Big|\mathcal{F}_s\Big] &= Var\Big[\int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big]. \end{aligned}$$

Invoking Theorem 3.1.1 with $c(\cdot) = e^{-\int_\cdot^t a(w)dw}\sigma(\cdot)$ concludes the proof. □

If we assume further that $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(\frac{1}{2} - \kappa)$, cf. Young [134] and Section 1.3, we can consider (3.8) as a pathwise sde and the fBm driven integral in (3.9) exists further as pathwise limit of Riemann-Stieltjes sums (Section 10 of Young [134]). An advantage of these stronger assumptions on $\sigma(\cdot)$ is that we are now able to invert the sde (3.8) (since a density formula like in Theorem 1.3.4 is needed for this step) and rewrite the prediction in terms of X:

Proposition 3.1.4. *In the situation of Theorem 3.1.3 assume that $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(\frac{1}{2} - \kappa)$. Let $0 \le s \le t \le T$. Set $c(\cdot) = e^{-\int_\cdot^t a(w)dw}\sigma(\cdot)$ and recall Ψ_c^κ from (1.20). Then we have for $u \in \mathbb{R}$*

$$\begin{aligned}
& E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] \\
= \; & \exp\Bigg\{iu\Big[X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv - \int_0^s \Psi_c^\kappa(s,t,v)\frac{k(v)}{\sigma(v)}dv \\
& \qquad + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v)\Big]\Bigg\} \\
& \times \exp\Bigg\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Bigg\},
\end{aligned}$$

i.e. $X(t)|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned}
E[X(t)|\mathcal{F}_s] &= X(s)e^{-\int_s^t a(v)dv} + \int_s^t e^{-\int_v^t a(w)dw}k(v)dv - \int_0^s \Psi_c^\kappa(s,t,v)\frac{k(v)}{\sigma(v)}dv \\
&\quad + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v) \\
Var[X(t)|\mathcal{F}_s] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2.
\end{aligned}$$

Proof. The main step of the proof of Proposition 3.1.4 is an application of a density formula for Riemann-Stieltjes integrals. By assumption on the coefficient functions, all appearing integrals in this proof can be considered in the pathwise Riemann-Stieltjes sense, cf. Young [134], Section 10.

Our goal is now to invert (3.8). By (3.9) we have for $0 \le s \le t \le T$

$$\int_s^t e^{-\int_v^t a(w)dw}\sigma(v)dB^\kappa(v) = X(t) - X(s)e^{-\int_v^t a(w)dw} - \int_s^t e^{-\int_v^t a(w)dw}k(v)dv$$

and, invoking the density formula of Theorem 1.3.4 we get for

$$\begin{aligned}
& B^\kappa(t) - B^\kappa(s) \\
= \; & \int_s^t \frac{e^{\int_v^t a(w)dw}}{\sigma(v)} d\left(-\int_v^t e^{-\int_z^t a(w)dw}\sigma(z)dB^\kappa(z)\right) \\
= \; & \int_s^t \frac{e^{\int_v^t a(w)dw}}{\sigma(v)} d\left(\int_v^t e^{-\int_z^t a(w)dw}k(z)dz + X(v)e^{-\int_v^t a(w)dw} - X(t)\right) \\
= \; & -\int_s^t \frac{k(v)}{\sigma(v)}dv + \int_s^t \frac{a(v)}{\sigma(v)}X(v)dv + \int_s^t \frac{1}{\sigma(v)}dX(v). \qquad (3.11)
\end{aligned}$$

It remains to plug this result into the formulas of Theorem 3.1.3 and the proof is finished.

□

Equation (3.6) shows that the fractional Ornstein-Uhlenbeck process with time-independent coefficient functions is important when considering general fractional sde's.

Corollary 3.1.5. *Consider in the sde* (3.8) *with* $k(\cdot) = 0$, $a(\cdot) = a > 0$ *and* $\sigma(\cdot) = 1$. *Then the solution* X *is given by*

$$X(t) = X(0)e^{-at} + \int_0^t e^{-a(t-s)}dB^\kappa(s)\,, \quad t \in [0,T]. \qquad (3.12)$$

Let $0 \le s \le t \le T$ *and set* $c(\cdot) = e^{-a(t-\cdot)}$. *For* $u \in \mathbb{R}$ *we have*

$$\begin{aligned}
& E\Big[e^{iuX(t)}\Big|\mathcal{F}_s\Big] \\
= \; & \exp\left\{iu\Big[X(s)e^{-a(t-s)} + a\int_0^s \Psi_c^\kappa(s,t,v)X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)dX(v)\Big]\right\} \\
& \times \exp\left\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\right\}, \qquad (3.13)
\end{aligned}$$

i.e. $X(t)|\mathcal{F}_s$ *is normally distributed with*

$$\begin{aligned}
E[X(t)|\mathcal{F}_s] &= X(s)e^{-a(t-s)} + a\int_0^s \Psi_c^\kappa(s,t,v)X(v)dv + \int_0^s \Psi_c^\kappa(s,t,v)dX(v) \\
Var[X(t)|\mathcal{F}_s] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2.
\end{aligned}$$

Proof. Set $k(\cdot) = 0$, $a(\cdot) = a > 0$ and $\sigma(\cdot) = 1$ in Proposition 3.1.4. The conditions on the p-variation are therefore satisfied. □

When calculating prices in a bond market the situation arises that not the short rate process r has to be predicted, but the integrated process. The next proposition will deal with this situation. For notational convenience we set

$$D(\cdot,t) = \int_\cdot^t e^{-\int_\cdot^v a(w)dw}dv, \quad t \in [0,T]. \qquad (3.14)$$

Proposition 3.1.6. *Denote by X the process given in* (3.9) *and let $0 \le s \le t \le T$. Set $c(\cdot) = D(\cdot,t)\sigma(\cdot)$ and recall Ψ_c^κ from* (1.20)*. For $\kappa \in (-\frac{1}{2}, 0)$ assume further that $D(\cdot,t)\sigma(\cdot) \in \Lambda_T^\kappa$. Then for $u \in \mathbb{R}$ we have*

$$\begin{aligned} E\Big[e^{iu\int_0^t X(v)dv}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[\int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\qquad + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v)\Big]\Big\} \\ &\quad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Big\}, \end{aligned}$$

i.e. $\int_0^t X(v)dv\Big|\mathcal{F}_s$ is normally distributed with

$$\begin{aligned} E\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= \int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\qquad + \int_0^s \Psi_c^\kappa(s,t,v)dB^\kappa(v) \\ Var\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= \|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2. \end{aligned}$$

If we assume further that $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(\frac{1}{2} - \kappa)$, then we have

$$\begin{aligned} E\Big[e^{iu\int_0^t X(v)dv}\Big|\mathcal{F}_s\Big] &= \exp\Big\{iu\Big[\int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\qquad - \int_0^s \Psi_c^\kappa(s,t,v)\frac{k(v)}{\sigma(v)}dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv \\ &\qquad + \int_0^s \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v)\Big]\Big\} \\ &\quad \times \exp\Big\{-\frac{u^2}{2}\Big[\|c(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|_{\kappa,T}^2 - \|\Psi_c^\kappa(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|_{\kappa,T}^2\Big]\Big\}. \end{aligned}$$

Proof. Let $0 \le s \le t \le T$. By Gaussianity we see again that $\int_0^t X(v)dv|\mathcal{F}_s$ is normally distributed and as before it remains to calculate its expectation and variance to achieve the conditional characteristic function. Since $\int_0^s X(v)dv$ is $\mathcal{F}_s$-measurable we just consider $\int_s^t X(v)dv|\mathcal{F}_s$. From (3.9) we obtain by (3.10) and Fubini's Theorem (Theorem 1 of Krvavich and Mishura [83])

$$\begin{aligned} &\int_s^t X(v)dv \\ &= \int_s^t \Big\{X(s)e^{-\int_s^v a(w)dw} + \int_s^v e^{-\int_z^v a(w)dw}k(z)dz + \int_s^v e^{-\int_z^v a(w)dw}\sigma(z)dB^\kappa(z)\Big\}dv \\ &= D(s,t)X(s) + \int_s^t D(v,t)k(v)dv + \int_s^t D(v,t)\sigma(v)dB^\kappa(v). \end{aligned}$$

It follows

$$\begin{aligned} E\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= \int_0^s X(v)dv + D(s,t)X(s) + \int_s^t D(v,t)k(v)dv \\ &\quad + E\Big[\int_s^t D(v,t)\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big], \\ Var\Big[\int_0^t X(v)dv\Big|\mathcal{F}_s\Big] &= Var\Big[\int_s^t D(v,t)\sigma(v)dB^\kappa(v)\Big|\mathcal{F}_s\Big]. \end{aligned}$$

Applying Theorem 3.1.1 with $c(\cdot) = D(\cdot,t)\sigma(\cdot)$ shows the first assertion. The second one follows by applying (3.11). □

The next theorem presents the conditional characteristic function of a process $Z = f \circ X$ as in (3.6). In general Z is no longer Gaussian. Note, however, that because of the assumptions on f, we have $\mathcal{F}_s = \sigma\overline{\{Z(v), v \in [0,s]\}}$ for $0 \le s \le T$.

Theorem 3.1.7. *Let the process Z be given by* (3.6) *with X as in* (3.7) *and* $0 \le s \le t \le T$. *Then we have for $u \in \mathbb{R}$*

$$E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] = \int_{\mathbb{R}} \Big(E\Big[e^{(i\xi+1)X(t)}\Big|\mathcal{F}_s\Big]\widehat{g_+}(\xi,u) + E\Big[e^{(i\xi-1)X(t)}\Big|\mathcal{F}_s\Big]\widehat{g_-}(\xi,u)\Big)\,d\xi$$

with $\widehat{g}_\pm(\xi,u) = (2\pi)^{-1}\int_{\mathbb{R}_\pm} e^{-(i\xi\pm1)x+iuf(x)}dx$ and $E[e^{(i\xi+1)X(t)}|\mathcal{F}_s]$ is given by the continuation of the characteristic function of $X(t)|\mathcal{F}_s$ to $\mathbb{C}$. This continuation exists due to the fact that $X(t)|\mathcal{F}_s$ is Gaussian.

Since the process Z as given in (3.6) does not have to be Gaussian any longer, there is no closed form for the prediction. However by Theorem 3.1.7 we can reduce this problem to an improper integral and the prediction of Ornstein Uhlenbeck type processes.

Proof. The proof uses Fourier techniques. Let $x \in \mathbb{R}$ and set $g(x,u) = \exp(iuf(x))$. First we decompose g into

$$\begin{aligned} g(x,u) &= e^x[e^{-x}g(x,u)\mathbf{1}_{[0,\infty)}(x)] + e^{-x}[e^x g(x,u)\mathbf{1}_{(-\infty,0)}(x)] \\ &=: e^x g_+(x,u) + e^{-x}g_-(x,u). \end{aligned}$$

Denote for fixed $u \in \mathbb{R}$ with $\widehat{g}_+(\cdot,u)$ and $\widehat{g}_-(\cdot,u)$ the Fourier transforms of $g_+(\cdot,u)$ and $g_-(\cdot,u)$ respectively. Using classical Fourier analysis we obtain for $x,\xi \in \mathbb{R}$

$$\begin{aligned} \widehat{g}_\pm(\xi,u) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}g_\pm(x,u)dx = \frac{1}{2\pi}\int_{\mathbb{R}_\pm} e^{-(i\xi\pm1)x+iuf(x)}dx, \\ g_\pm(x,u) &= \int_{\mathbb{R}} e^{i\xi x}\widehat{g}_\pm(\xi,u)d\xi, \end{aligned}$$

where we used the fact that $g_+(\cdot,u)$ and $g_-(\cdot,u)$ are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ since $g(\cdot,u)$ is bounded. It also follows that $\widehat{g}_+(\cdot,u)$ and $\widehat{g}_-(\cdot,u)$ are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Now we obtain

$$\begin{aligned} & E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] \\ = \; & E[g(X(t))|\mathcal{F}_s] = E[e^{X(t)}g_+(X(t),u)|\mathcal{F}_s] + E[e^{-X(t)}g_-(X(t),u)|\mathcal{F}_s] \\ = \; & E\Big[e^{X(t)}\int_{\mathbb{R}} e^{i\xi X(t)}\widehat{g}_+(\xi,u)d\xi\Big|\mathcal{F}_s\Big] + E\Big[e^{-X(t)}\int_{\mathbb{R}} e^{i\xi X(t)}\widehat{g}_-(\xi,u)d\xi\Big|\mathcal{F}_s\Big]. \end{aligned}$$

Since $E[e^{bX(t)}]<\infty$ for all $b\in\mathbb{C}$ we can interchange conditional expectation and integration and get

$$E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] = \int_{\mathbb{R}}\Big(E\Big[e^{(i\xi+1)X(t)}\Big|\mathcal{F}_s\Big]\widehat{g}_+(\xi,u) + E\Big[e^{(i\xi-1)X(t)}\Big|\mathcal{F}_s\Big]\widehat{g}_-(\xi,u)\Big)\,d\xi.$$

□

The following example considers the conditional characteristic function of a CIR type process.

Example 3.1.8. We consider for $\kappa\in(0,\frac{1}{2})$ a fractional CIR model given by the pathwise solution to the sde

$$dZ(t) = -\lambda Z(t)dt + \sigma\sqrt{|Z(t)|}dB^\kappa(t), \quad Z(0)\in\mathbb{R}, \quad t\in[0,T],$$

for some $\lambda,\sigma>0$. Then by Theorem 2.1.6 we know that a solution is given by

$$\begin{aligned} Z(t) &= f(X(t)) \\ dX(t) &= -\frac{\lambda}{2}X(t)dt + dB^\kappa(t)\,, \quad X(0)=f^{-1}(Z(0)), \quad t\in[0,T], \end{aligned}$$

where $f(x)=\operatorname{sign}(x)\frac{\sigma^2}{4}x^2$. We want to emphasize that this solution is in contrast to the classical Brownian case ($\kappa=0$) not unique; further details have been provided in Section 2.2.4. We considered there a similar case for fractional Lévy driven processes.

The following section covers the case of a multivariate fractional Brownian motion with independent entries. The proofs are straightforward and we will state them for the sake of completeness only.

3.2 d-dimensional fractional Brownian motion with independence

This short and straightforward section shall be dedicated to the case of multivariate fractional Brownian motion with independent entries. Nevertheless we shall state the result

and its short proof to compare the situation to Section 3.4 where we will introduce a certain dependence structure (cf. Elliott and van der Hoek [53]) into a two-dimensional fractional Brownian motion.

For $d \in \mathbb{N}$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})^d$ let $\mathbf{B}^\kappa(t) = (B_{(1)}^{\kappa(1)}(t), \ldots, B_{(d)}^{\kappa(d)}(t))^T$, $t \in [0,T]$, be a d-dimensional fractional Brownian motion. Furthermore the individual components $B_{(i)}^{\kappa(i)}$ for $i = 1 \ldots, d$ are assumed to be independent. We shall state the prediction theorem just in its simplest form without giving further thought to integration.

We fix for the rest of this section $\mathcal{F}_s := \sigma\overline{\{\mathbf{B}^\kappa(v), v \in [0,s]\}}$ for $0 \le s \le T$.

Theorem 3.2.1. *Let $0 \le s \le t \le T$. Then we have for $u \in \mathbb{R}^d$*

$$\begin{aligned} & E\Big[e^{i\sum_{j=1}^d u_j B_{(j)}^{\kappa(j)}(t)}\Big|\mathcal{F}_s\Big] \\ = \; & \exp\left\{ i\sum_{j=1}^d u_j \Big[B_{(j)}^{\kappa(j)}(t) + \int_0^s \Psi^{\kappa(j)}(s,t,v)dB_{(j)}^{\kappa(j)}(v)\Big]\right\} \\ & \times \exp\left\{ -\frac{1}{2}\sum_{j=1}^d u_j^2 \Big[\|\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa(j),T} - \|\Psi^{\kappa(j)}(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa(j),T}\Big]\right\}. \end{aligned}$$

Proof. Follows by independence and Theorem 3.1.1. □

Example 3.2.2. [2-dimensional case] If we set $d = 2$ in Theorem 3.2.1 above we get for $0 \le s \le t \le T$ and $u \in \mathbb{R}^2$

$$\begin{aligned} & E\Big[e^{i(u_1 B_{(1)}^{\kappa(1)}(t) + u_2 B_{(2)}^{\kappa(2)}(t))}\Big|\mathcal{F}_s\Big] \\ = \; & \exp\left\{ iu_1 \Big[B_{(1)}^{\kappa(1)}(t) + \int_0^s \Psi^{\kappa(1)}(s,t,v)dB_{(1)}^{\kappa(1)}(v)\Big]\right\} \\ & \times \exp\left\{ iu_2 \Big[B_{(2)}^{\kappa(2)}(t) + \int_0^s \Psi^{\kappa(2)}(s,t,v)dB_{(2)}^{\kappa(2)}(v)\Big]\right\} \\ & \times \exp\left\{ -\frac{u_1^2}{2} \Big[\|\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa(1),T} - \|\Psi^{\kappa(1)}(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa(1),T}\Big]\right\} \\ & \times \exp\left\{ -\frac{u_2^2}{2} \Big[\|\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa(2),T} - \|\Psi^{\kappa(2)}(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa(2),T}\Big]\right\}. \end{aligned}$$

3.3 Application: Fractional bond market

In this section we will consider an application of our results to bond markets. Recall that in many cases characteristic functions can be extended from arguments in $\mathbb{R}$ to $\mathbb{C}$.

3.3.1 Motivation

In a bond market driven by an adapted short rate process $r = (r(v))_{v\in[0,T]}$ the price of a non-defaultable zero coupon bond with maturity $T \geq 0$ at time $0 \leq s \leq t \leq T$ is given by the conditional expectation

$$B(s,t) = E^{\mathcal{Q}}\left[e^{-\int_s^t r(v)dv}\middle| r(v), v \in [0,s]\right] \tag{3.15}$$

under some risk-neutral measure $\mathcal{Q}$.

For a broad class of stochastic processes, in particular affine models (see e.g. Duffie [44] and Duffie, Filipovic and Schachermayer [45]), such predictions are easy to calculate and do only depend on the level of the process at time t due to their Markov property. However, staying in the bond framework above, Markov models may not be sufficient to catch the real market structure as was shown by the ongoing financial crisis. One reason behind this is that short rates, which are driven by macroeconomic variables like domestic gross products, supply and demand rates or volatilities, exhibit long range dependence, which cannot be captured by Markov models. Empirical evidence has been reported over the years and we refer to Henry and Zaffaroni [73] for a good overview on this research and further references. In particular, Backus and Zin [7] provide in their Section 4 evidence for long memory in the short rate process.

In this section we will derive a Vasicek model from the Heath-Jarrow-Morton (HJM) approach of Ohashi [100]. As in the classical setting of Heath et al. [72] we model the whole term structure under a measure $\mathcal{P}$ and show that with proportional transaction costs with proportionality factor $k > 0$ arbitrage can be ruled out. The existence of an average risk-neutral measure $\mathcal{Q}$ can be proven and we can formally calculate prices of defaultable bonds or more general contingent claims under this measure as suggested in Sottinen and Valkeila [125]. On the other hand it is - as said before - always possible to directly define prices via conditional expectations leading in general to an arbitrage-free model. We apply our formulas to calculate the price of zero coupon bonds.

The proportional transaction costs are crucial for this approach since the Markov setting of Duffie [44] and Duffie, Filipovic and Schachermayer [45] does not apply for fractional Brownian markets.

Ohashi's [100] work on a fractional HJM bond model with proportional transaction costs is based on an extension of the full support property of the logarithmic price processes in the set of continuous functions. This basic idea and its relevance to the absence of arbitrage was fully investigated by Guasoni, Rásonyi and Schachermayer [65]. Its remains

to observe that these properties are only sufficient for the market to be arbitrage-free. In Guasoni, Rásonyi and Schachermayer [66] a fundamental theorem with necessary and sufficient conditions for risk-neutral asset pricing under proportional transaction costs has been derived.

3.3.2 The fractional market model

The final time horizon of the market shall be $T^\star > 0$. We model on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ endowed with a filtration $(\mathcal{F}_t)_{0\le t\le T^\star}$, representing the complete market information and satisfying the usual conditions of completeness and right continuity. Assume further the existence of the *forward rate process*

$$f = (f(t,T))_{0\le t\le T\le T^\star}$$

on $(\Omega, \mathcal{F}, \mathcal{P})$ such that for each $0 \le T \le T^\star$ the stochastic process $(f(t,T))_{0\le t\le T}$ is adapted to $(\mathcal{F}_t)_{0\le t\le T}$. The stochastic process $r = (r(t))_{0\le t\le T^\star} := (f(t,t))_{0\le t\le T^\star}$ models the *short rate*.

3.3.2.1 Tradable bonds and numeraire

We specify the dynamics of f by the linear stochastic differential equation

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^d \int_0^t \sigma^{(i)}(s,T)dB_{(i)}^{\kappa(i)}(s), \quad 0 \le t \le T \le T^\star, \tag{3.16}$$

with a multivariate fBm given by $\mathbf{B}^\kappa(t) = (B_{(1)}^{\kappa(1)}(t), \dots, B_{(d)}^{\kappa(d)}(t))^T$, $t \in [0, T^\star]$, for $d \in \mathbb{N}$ and $\kappa \in (0, \frac{1}{2})^d$. The components $B_{(i)}^{\kappa(i)}$ for $i = 1 \dots, d$ are assumed to be independent. Further we will from now on impose the following assumptions:

(A1) The function $f : [0, T^\star] \to \mathbb{R}$, $T \mapsto f(0,T)$ is continuously differentiable.

(A2) The functions $\alpha, \sigma^{(i)} : [0, T^\star]^2 \to \mathbb{R}$ are continuous and bounded with $\sigma^{(i)} > 0$ on $[0, T^\star]^2$, $i \in \{1, \dots, d\}$.

Remark 3.3.1. Under the conditions (A1) and (A2) the technical integrability assumptions (2.6) – (2.9) of Ohashi [100] are satisfied.

We assume the existence of the following tradable bonds:

$$P(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big), \quad 0 \le t \le T \le T^\star.$$

Definition 3.3.2. *Define for* $0 \leq T \leq T^\star$ *the* relative bond price *of a* T*-maturity bond by*

$$Z_t(T) := P(t,T)/S_0(t), \quad 0 \leq t \leq T,$$

with

$$(S_0(t))_{0\leq t\leq T^\star} := \big(\exp\big(\int_0^t r(s)ds\big)\big)_{0\leq t\leq T^\star}$$

for the short rate

$$(r(t))_{0\leq t\leq T^\star} := (f(t,t))_{0\leq t\leq T^\star}.$$

We will take the process $(S_0(t))_{0\leq t\leq T^\star}$ as the *numeraire*.

Remark 3.3.3. We are aware that, as a Gaussian process, the forward rate f and the short rate r can also take negative values. However, it is always possible to shift and perhaps also scale the model such that the probability of becoming negative is arbitrarily small. On the other hand it is attractive and useful to have a benchmark model where quantities can be calculated explicitly.

3.3.2.2 Trading strategies and the wealth process

We will now describe which trading strategies are allowed. Therefore, we define the wealth process of a trading strategy by an integral over the whole relative bond price surface given by Z from Definition 3.3.2.

First we have to specify the idea above following Section 3 of Ohashi [100]: Let $\mathbb{B}([0,T^\star])$ denote the Borel sets of $[0,T^\star]$. Define admissible trading strategies in our market by the following procedure: Let $\mathcal{M}_{T^\star}$ be the space of all finite signed measures on $\mathbb{B}([0,T^\star])$ endowed with the total variation norm defined by

$$\|m\|_{TV} := \sup\{m(A)|A \in \mathbb{B}([0,T^\star])\} + |\inf\{m(A)|A \in \mathbb{B}([0,T^\star])\}|$$

for $m \in \mathcal{M}_{T^\star}$. Define further the total variation measure by

$$|m|(E) := \sup\{m(A)|A \in \mathbb{B}([0,T^\star]), A \subset E\} + |\inf\{m(A)|A \in \mathbb{B}([0,T^\star]), A \subset E\}|,$$

for $E \in \mathbb{B}([0,T^\star])$. Let φ be a measure-valued elementary process of the form

$$\varphi_t(\omega, A) = \sum_{i=0}^{N-1} \mathbf{1}_{F_i\times(t_i,t_{i+1}]}(\omega,t)m_i(A), \quad 0 \leq t \leq T^\star, \tag{3.17}$$

for $\omega \in \Omega$, $A \in \mathcal{F}$, $m_i \in \mathcal{M}_{T\star}$, $0 = t_0 < \cdots < t_i < \cdots < t_N \leq T^\star$ and $F_i \in \mathcal{F}_{t_i}$ for $i \in \{0, \ldots, N-1\}$ and $N \in \mathbb{N}$. The process φ represents an elementary trading strategy. Denote by $\mathcal{S}$ the set of all elementary processes of the form (3.17) endowed with the norm:

$$\|\varphi\|_{\mathcal{S}} := E\Big[\sup_{0 \leq t \leq T^\star} \|\varphi_t\|_{TV}^2\Big].$$

To define the wealth process we must specify integration with respect to φ and Z: For $\varphi \in \mathcal{S}$ define the random variable

$$\Big(\int_0^t \varphi_s dZ_s\Big)(\omega) := \sum_{i=0}^{N-1} \mathbf{1}_{F_i}(\omega)([Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t}] \bullet m_i)(\omega), \quad \omega \in \Omega$$

where we set

$$(X \bullet m)(\omega) := \int_0^{T^\star} X(s)(\omega) m(ds), \quad \omega \in \Omega,$$

for a stochastic process $X = (X(s))_{0 \leq s \leq T^\star}$ and $m \in \mathcal{M}_{T\star}$. From now on we will omit ω in the notation.

For $\varphi \in \mathcal{S}$ and proportional transactions costs with proportionality factor $k > 0$ define the wealth process $V^k(\varphi)$ via

$$V_t^k(\varphi) := \sum_{i=0}^{N-1} \mathbf{1}_{F_i}[Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t}] \bullet m_i - k \sum_{i=0}^{N-1} \mathbf{1}_{F_i} Z_{t_i \wedge t} \bullet |\varphi_{t_{i+1}\wedge t} - \varphi_{t_{i1}\wedge t}| - kZ_t \bullet |\varphi_t|,$$

for $0 \leq t \leq T^\star$.

Let $\bar{\mathcal{S}}$ be the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathcal{S}}$. Then equation (3.4) of Ohashi [100] shows that $\int_0^\cdot \varphi_s dZ_s$ and $V_\cdot^k(\varphi)$ can be defined for all $\varphi \in \bar{\mathcal{S}}$, as the next theorem states. The proof can be found in Ohashi [100], p.1559-1560.

Theorem 3.3.4. *For $\varphi \in \bar{\mathcal{S}}$ assume that the random variable*

$$\sup_{\pi} \sum_{t_i \in \pi} \|\varphi_{t_{i+1}} - \varphi_{t_i}\|_{TV} \tag{3.18}$$

is square-integrable, where the supremum is taken over all partitions π of $[0, T^\star]$. Then for each sequence φ^n of elementary processes with $\lim_{n\to\infty} \varphi^n = \varphi$ in $\bar{\mathcal{S}}$ we have that

$$\lim_{n\to\infty} E\Big[\sup_{0 \leq t \leq T^\star} |V_t^k(\varphi^n) - V_t^k(\varphi)|\Big] = 0, \quad k > 0,$$

where

$$V_t^k(\varphi) := \int_0^t \varphi_s dZ_s - k \int_0^t Z_s d|\varphi_s| - kZ_t \bullet |\varphi_t|, \quad 0 \leq t \leq T^\star.$$

Now we can define admissible trading strategies in our bond market.

Definition 3.3.5. *(a) For proportional transactions costs with proportionality factor $k > 0$ a trading strategy $\varphi \in \bar{\mathcal{S}}$ is called* admissible*, if it is adapted, the random variable* (3.18) *is square-integrable and if there exists $M > 0$ such that $V_t^k(\varphi) \geq -M$ a.s. for every $0 \leq t \leq T^\star$.*

(b) An admissible trading strategy $\varphi \in \bar{\mathcal{S}}$ is called an arbitrage opportunity with proportional transactions costs with proportionality factor $k > 0$ *if $V_{T^\star}^k(\varphi) \geq 0$ a.s. and $P(V_{T^\star}^k(\varphi) > 0) > 0$.*

*(c) The market is called k-*arbitrage free *with proportional transactions costs with proportionality factor $k > 0$, if for every admissible trading strategy $\varphi \in \bar{\mathcal{S}}$, $V_{T^\star}^k(\varphi) \geq 0$ a.s. implies $V_{T^\star}^k(\varphi) = 0$ a.s.*

3.3.2.3 No-arbitrage and average risk-neutral measure

Set $\phi_\kappa(x) := \kappa(2\kappa+1)|x|^{2\kappa-1}$ for $x \in \mathbb{R}, \kappa \in (0, \frac{1}{2})$ and define for $0 \leq s \leq T \leq T^\star$

$$\begin{aligned} \widetilde{\alpha}(s,T) &:= \sum_{i=1}^{d} \Big\{ \sigma^{(i)}(s,T) \int_0^s \int_0^{T-\theta} \sigma^{(i)}(\theta, \theta + x) \phi_{\kappa(i)}(s-\theta) dx d\theta \\ &+ \int_0^{T-s} \sigma^{(i)}(s, s+x) dx \int_0^s \sigma^{(i)}(\theta, T) \phi_{\kappa(i)}(s-\theta) d\theta \Big\}. \end{aligned} \tag{3.19}$$

Furthermore we impose from now on the following assumptions:

(A3) For all $i \in \{1, \ldots, d\}$, the function $(t,T) \mapsto \int_0^{T-t} \sigma^{(i)}(t, t+s) ds$ is λ-Hölder continuous on $0 \leq t \leq T \leq T^\star$ for all $1/2 < \lambda < 1$.

(A4) There exists an integrable function $\gamma : [0, T^\star] \to \mathbb{R}^d$ such that $< \sigma(t,T), \gamma(t) >= \widetilde{\alpha}(t,T) - \alpha(t,T)$ for $0 \leq t \leq T \leq T^\star$. Furthermore there is $\vartheta : [0, T^\star] \to \mathbb{R}$ square integrable satisfying for $0 \leq t \leq T^\star$

$$\int_0^t \gamma(s) ds = \Big(\frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \Big)^{\frac{1}{2}} \int_0^t [s^{-\kappa} I_{T-}^{\kappa}((\cdot)^\kappa \mathbf{1}_{[0,t)}(\cdot))(s)] \vartheta(s) ds. \tag{3.20}$$

Now we can state the main theorem of this section.

Theorem 3.3.6. *Let the proportionality factor $k > 0$. Then there exists a probability measure $\mathcal{Q} \sim \mathcal{P}$ (called average risk-neutral measure) such that for all $0 \leq t \leq T \leq T^\star$*

$$E_{\mathcal{Q}}[Z_t(T)] = P(0,T) \tag{3.21}$$

holds. The market is k-arbitrage free with proportional transactions costs with proportionality factor $k > 0$.

Proof. Since $\sigma(\cdot,\cdot) > 0$ is continuous and bounded by assumption (A2), it is also square-integrable on $[0, T^\star]^2$. Therefore, the conditions of Lemma 2.3 and Theorem 3.1 of Ohashi [100] are met. It follows that the market is k-arbitrage free with proportional transactions costs with proportionality factor $k > 0$ and a measure $\mathcal{Q}$, satisfying equation (3.21), exists. □

Motivated by equation (3.21) and using Remark 3.8 of Ohashi [100], we can price contingent claims under the measure $\mathcal{Q}$ in a formal way similar to Sottinen and Valkeila [125]: The price of an integrable, $\mathcal{F}_T$-measurable contingent claim X with maturity $T \in [0, T^\star]$ at time t is given by

$$E_{\mathcal{Q}}\Big[X \exp\Big(-\int_t^T r(s)ds\Big)\Big|\mathcal{F}_t\Big]. \tag{3.22}$$

Remark 3.3.7. As explained before Corollary 3.1 in Ohashi [100], there is a canonical choice for the measure change under which (3.21) holds. This leads to $\mathcal{Q}$ from Theorem 3.3.6.

3.3.2.4 Dynamics of the short rate under $\mathcal{Q}$

We need to be aware of the dynamics of the forward rate process under the measure $\mathcal{Q}$. Recall (3.19), then by Theorem 3.1 Ohashi [100] we have that the model (3.16) under $\mathcal{Q}$ has the form

$$f(t,T) = f(0,T) + \int_0^t \widetilde{\alpha}(s,T)ds + \sum_{i=0}^d \int_0^t \sigma^{(i)}(s,T)d\widetilde{B}_{(i)}^{\kappa(i)}(s), \quad 0 \le t \le T \le T^\star, \tag{3.23}$$

where $\widetilde{B}_{(i)}^{\kappa(i)}$, $i \in \{1,\dots,d\}$, are independent $\mathcal{Q}$-fBms.

Equation (3.22) shows that the payoff of a contingent claim must be discounted by the short rate r before taking the conditional expectation. In the following we are interested in models for which r is given by a fractional Vasicek model. To derive the Vasicek dynamics we have to impose an additional separability assumption on the volatility coefficient. This is similar to the situation in the classical Brownian HJM model of Heath et al. [72]. For details we refer to Section 5.3 of Brigo and Mercurio [24] and, in particular, Proposition 2.1 of Carverhill [30].

Assumption 3.3.8. *In addition to Assumptions (A1) - (A4) the volatility coefficients $\sigma^{(i)}(\cdot,\cdot)$ factorizes: $\sigma^{(i)}(t,T) = \xi^{(i)}(t)\nu(T)$, $0 \le t \le T \le T^\star$, where $\xi^{(i)}(\cdot)$ and $\nu(\cdot)$ are strictly positive and $\nu(\cdot)$ is differentiable, $i \in \{1,\dots,d\}$. Further $\xi^{(i)}(\cdot)$ is of bounded $p^{(i)}$-variation for some $0 < p^{(i)} < 1/(\frac{1}{2} - \kappa^{(i)})$, $i \in \{1,\dots,d\}$.*

Now we calculate the short rate for $0 \leq t \leq T^\star$ under $\mathcal{Q}$

$$\begin{aligned} r(t) &= f(t,t) = f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds + \sum_{i=0}^d \int_0^t \sigma^{(i)}(s,t)d\widetilde{B}_{(i)}^{\kappa(i)}(s) \\ &= f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds + \nu(t)\sum_{i=0}^d \int_0^t \xi^{(i)}(s)d\widetilde{B}_{(i)}^{\kappa(i)}(s). \end{aligned}$$

Furthermore for $0 \leq t \leq T^\star$

$$\begin{aligned} \int_0^t \widetilde{\alpha}(s,t)ds &= \sum_{i=0}^d \Big\{ \int_0^t \Big[\sigma^{(i)}(s,t)\int_0^s \int_0^{t-\theta} \sigma^{(i)}(\theta,\theta+x)\phi_{\kappa(i)}(s-\theta)dxd\theta \\ &\quad + \int_0^{t-s} \sigma^{(i)}(s,s+x)dx \int_0^s \sigma^{(i)}(\theta,t)\phi_{\kappa(i)}(s-\theta)d\theta\Big]ds\Big\} \\ &= \sum_{i=0}^d \Big\{ \int_0^t \Big[\sigma^{(i)}(s,t)\int_0^s \int_0^{t-\theta} \sigma^{(i)}(\theta,\theta+x)\phi_{\kappa(i)}(s-\theta)dxd\theta \\ &\quad + \int_0^{t-s} \sigma^{(i)}(s,s+x)dx \int_0^s \sigma^{(i)}(\theta,t)\phi_{\kappa(i)}(s-\theta)d\theta\Big]ds\Big\} \\ &= \nu(t)\sum_{i=0}^d \Big\{ \int_0^t \Big[\int_0^s \int_\theta^t \xi^{(i)}(s)\xi^{(i)}(\theta)\nu(x)\phi_{\kappa(i)}(s-\theta)dxd\theta \\ &\quad + \int_s^t \xi^{(i)}(s)\nu(x)dx \int_0^s \xi^{(i)}(\theta)\phi_{\kappa(i)}(s-\theta)d\theta\Big]ds\Big\} \\ &= \nu(t)\sum_{i=0}^d \Big\{\Big[\int_0^t \int_0^s \xi^{(i)}(s)\xi^{(i)}(\theta)\epsilon^{(i)}(\theta,t)\phi_{\kappa(i)}(s-\theta)d\theta ds \\ &\quad + \int_0^t \int_0^s \xi^{(i)}(s)\xi^{(i)}(\theta)\epsilon(s,t)\phi_{\kappa(i)}(s-\theta)d\theta ds\Big]\Big\}, \end{aligned}$$

where $\epsilon(s,t) := \int_s^t \nu(x)dx$, $i \in \{1,\ldots,d\}$.

The function $t \mapsto f(0,t)$ is by assumption differentiable. Further we have the following lemma.

Lemma 3.3.9. *The function* $[0,T^\star] \to \mathbb{R}, t \mapsto \int_0^t \widetilde{\alpha}(s,t)ds$ *is differentiable.*

Proof. Since ν is by Assumption 3.3.8 differentiable, we just need to show that for $i \in \{1,\ldots,d\}$ the functions

$$t \mapsto \int_0^t \int_0^s (\xi^{(i)}(s)\xi^{(i)}(\theta)\epsilon(\theta,t)\phi_{\kappa(i)}(s-\theta)d\theta ds \quad \textit{and} \tag{3.24}$$

$$t \mapsto \int_0^t \int_0^s (\xi^{(i)}(s)\xi^{(i)}(\theta)\epsilon(s,t)\phi_{\kappa(i)}(s-\theta)d\theta ds, \tag{3.25}$$

are differentiable. Fix $i \in \{1,\ldots,d\}$. We start by showing that the integrand function of (3.24) is differentiable in t. This follows by the classical rule for differentiation under the

integral sign since for all $0 \le t \le T^\star$

$$\begin{aligned} & \Big|\frac{\partial}{\partial t}\xi^{(i)}(s)\xi^{(i)}(\theta)\epsilon(\theta,t)\phi_{\kappa(i)}(s-\theta)\Big| = \xi^{(i)}(s)\xi^{(i)}(\theta)\phi_{\kappa(i)}(s-\theta)\nu(t) \\ \le\ & C\xi^{(i)}(s)\xi^{(i)}(\theta)\phi_{\kappa(i)}(s-\theta) \end{aligned}$$

for some constant $C > 0$ since ν is differentiable. Applying the Leipniz rule a second time shows that (3.24) is differentiable. Similar arguments work for (3.25). □

By Lemma 3.3.9 we conclude that $t \mapsto A(t) := f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds$ is differentiable. Under $\mathcal{Q}$ we have that for $0 \le t \le T^\star$

$$r(t) = A(t) + \nu(t)\sum_{i=0}^{d}\int_0^t \xi^{(i)}(s)d\widetilde{B}^{\kappa(i)}_{(i)}(s)$$

and therefore for $0 \le s \le t \le T^\star$ by using a pathwise product rule and density formula (like in Theorem 1.3.4)

$$\begin{aligned} & r(t) - r(s) \\ =\ & A(t) - A(s) + \sum_{i=0}^{d}\nu(t)\int_0^t \xi^{(i)}(u)d\widetilde{B}^{\kappa(i)}_{(i)}(u) - \nu(s)\sum_{i=0}^{d}\int_0^s \xi^{(i)}(u)d\widetilde{B}^{\kappa(i)}_{(i)}(u) \\ =\ & \int_s^t A'(u)du + \sum_{i=0}^{d}\Big\{\int_s^t\Big(\int_0^u \xi^{(i)}(v)d\widetilde{B}^{\kappa(i)}_{(i)}(v)\Big)d\nu(u) \\ & + \int_s^t \nu(u)d\Big(\int_0^u \xi^{(i)}(v)d\widetilde{B}^{\kappa(i)}_{(i)}(v)\Big)\Big\} \\ =\ & \int_s^t A'(u)du + \sum_{i=0}^{d}\Big\{\int_s^t \nu'(u)\Big(\int_0^u \xi^{(i)}(v)d\widetilde{B}^{\kappa(i)}_{(i)}(v)\Big)du \\ & + \int_s^t \nu(u)\xi^{(i)}(u)d\widetilde{B}^{\kappa(i)}_{(i)}(u)\Big\} \\ =\ & \int_s^t \Big[A'(u) + \nu'(u)\sum_{i=0}^{d}\Big(\int_0^u \xi^{(i)}(v)d\widetilde{B}^{\kappa(i)}_{(i)}(v)\Big)\Big]du \\ & + \sum_{i=0}^{d}\int_s^t \nu(u)\xi^{(i)}(u)d\widetilde{B}^{\kappa(i)}_{(i)}(u) \\ =\ & \int_s^t \Big[A'(u) + \nu'(u)\frac{r(u)-A(u)}{\nu(u)}\Big]du + \sum_{i=0}^{d}\int_s^t \nu(u)\xi^{(i)}(u)d\widetilde{B}^{\kappa(i)}(u) \\ =\ & \int_s^t \Big[k(u) - a(u)r(u)\Big]du + \sum_{i=0}^{d}\int_s^t \sigma^{(i)}(u)d\widetilde{B}^{\kappa(i)}(u), \end{aligned}$$

where

$$k(t) = A'(t) - \frac{\nu'(t)}{\nu(t)} A(t), \tag{3.26}$$

$$a(t) = -\frac{\nu'(t)}{\nu(t)}, \tag{3.27}$$

$$\sigma^{(i)}(t) = \sigma^{(i)}(t,t) = \xi^{(i)}(t)\nu(t), \quad i \in \{1,\dots,d\}, \quad t \in [0,T^\star]. \tag{3.28}$$

Therefore the short rate is described by a Vasicek dynamic under $\mathcal{Q}$.

From now on we want to model directly under the measure $\mathcal{Q}$. The only question which remains is that given $k(\cdot), a(\cdot), \sigma^{(i)}(\cdot)$, $i \in \{1,\dots,d\}$, continuous on $[0,T^\star]$, is it possible to find (not necessarily unique) $\xi(\cdot), \nu(\cdot), f(0,\cdot)$ such that (3.26) - (3.28) holds? The next lemma addresses this problem.

Lemma 3.3.10. *Given $k(\cdot), a(\cdot), \sigma^{(i)}(\cdot)$, $i \in \{1,\dots,d\}$, continuous on $[0,T^\star]$, take*

$$\nu(t) = \exp\Big(-\int_0^t a(s)ds\Big),$$

$$\xi^{(i)}(t) = \sigma^{(i)}(t)\exp\Big(\int_0^t a(s)ds\Big), \quad i \in \{1,\dots,d\},$$

and let $f(0,\cdot)$ be a solution of the first order linear differential equation

$$\frac{\partial}{\partial t} f(0,t) = -a(t)f(0,t) + \Big[k(t) - a(t)\int_0^t \widetilde{\alpha}(s,t)ds - \frac{\partial}{\partial t}\int_0^t \widetilde{\alpha}(s,t)ds\Big], \quad t \in [0,T^\star].$$

Then (3.26) - (3.28) *are satisfied.*

Proof. A solution to equation (3.27) is given by $\nu(t) = \exp(-\int_0^t a(s)ds)$, $t \in [0,T^\star]$ and therefore we get for $i \in \{1,\dots,d\}$

$$\xi^{(i)}(t)\nu(t) = \sigma^{(i)}(t)\exp\Big(\int_0^t a(s)ds\Big)\nu(t) = \sigma^{(i)}(t), \quad t \in [0,T^\star],$$

which shows (3.28). Remark that $\int_0^\cdot \widetilde{\alpha}(s,\cdot)ds$ is already fully specified by $\xi^{(i)}(\cdot)$, $t \in [0,T^\star]$, and $\nu(\cdot)$. If $f(0,\cdot)$ solves the first order linear differential equation in the assertion we get for $t \in [0,T^\star]$

$$\begin{aligned}
& \frac{\partial}{\partial t} f(0,t) = -a(t)f(0,t) + \Big[k(t) - a(t)\int_0^t \widetilde{\alpha}(s,t)ds - \frac{\partial}{\partial t}\int_0^t \widetilde{\alpha}(s,t)ds\Big] \\
\iff \quad & \frac{\partial}{\partial t}\Big(f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds\Big) = -a(t)\Big[f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds\Big] + k(t) \\
\iff \quad & A'(t) = \frac{\nu'(t)}{\nu(t)}A(t) + k(t)
\end{aligned}$$

which shows that (3.26) is satisfied. □

3.3.3 Modeling under $\mathcal{Q}$

Motivated by Section 3.3.2 we shall from now on for simplicity directly model under an (average) risk-neutral measure $\mathcal{Q}$ and consider prices as (discounted) conditional expectations. Remaining in this framework and given a maturity date $T^\star > 0$ we consider a multivariate fBm given by $\mathbf{B}^\kappa(t) = (B_{(1)}^{\kappa(1)}(t), \ldots, B_{(d)}^{\kappa(d)}(t))^T$, $t \in [0, T^\star]$, for $d \in \mathbb{N}$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})^d$. The components $B_{(i)}^{\kappa(i)}$ for $i = 1 \ldots, d$ are assumed to be independent. We remark that although empirical evidence shows long range dependence in short rates and Section 3.3.2 considered only this case, our calculations here also include $\kappa(i) \in (-\frac{1}{2}, 0)$.

Consider now for $\mathbf{X}(0) = (X^{(1)}(0), \ldots, X^{(n)}(0))^T \in \mathbb{R}^d$ a system of d fractional Vasicek sde's given for $i = 1, \ldots, d$ by

$$dX^{(i)}(t) = (k^{(i)}(t) - a(t)X^{(i)}(t))dt + \sigma^{(i)}(t)dB_{(i)}^{\kappa(i)}(t)\,, \quad t \in [0, T^\star]. \tag{3.29}$$

We assume that $k^{(i)}(\cdot), a(\cdot)$ are continuous on $[0, T^\star]$, $\sigma^{(i)}(\cdot) \neq 0$, continuous with $\sigma^{(i)}(\cdot) \in \Lambda_T^{\kappa(i)}$. If $\kappa(i) \in (-\frac{1}{2}, 0)$ assume further that $D^{(i)}(\cdot, t)\sigma^{(i)}(\cdot) \in \Lambda_{T^\star}^{\kappa(i)}$ for $0 \leq t \leq T^\star$ with D defined as in (3.14). Furthermore, let $\sigma^{(i)}(\cdot)$ and $1/\sigma^{(i)}(\cdot)$ are of bounded $p(i)$-variation for some $0 < p(i) < 1/(\frac{1}{2} - \kappa(i))$. Considering (3.8), the unique solution of (3.29) is given by $\mathbf{X}(t) = (X^{(1)}(t), \ldots, X^{(d)}(t))^T$, where $X^{(i)}$ is defined as in (3.9).

Now for $b \in (\mathbb{R}_+)^d$ fixed with $b \neq 0$, define for $t \in [0, T^\star]$,

$$r(t) = b^T \mathbf{X}(t). \tag{3.30}$$

Then it follows that $\mathcal{F}_s = \sigma\overline{\{r(v), v \in [0, s]\}}$ for $0 \leq s \leq T^\star$.

3.3.4 Zero coupon bonds

The price of a zero coupon bond for a short rate given in (3.30) is calculated in the next theorem.

Theorem 3.3.11. *Assume the situation above and let $0 \leq s \leq t \leq T^\star$. For $i = 1, \ldots, d$ set $c_i(\cdot) = D^{(i)}(\cdot, t)\sigma^{(i)}(\cdot)$ with D defined as in (3.14), and recall Ψ_c^κ from (1.20). For $\kappa \in (-\frac{1}{2}, 0)$ assume further that $D^{(i)}(\cdot, t)\sigma^{(i)}(\cdot) \in \Lambda_{T^\star}^{\kappa(i)}$. Then the price of a zero coupon*

bond $B(s,t)$ at time s with maturity t is given by

$$
\begin{aligned}
B(s,t) &= E\Big[e^{-\int_s^t r(v)dv}\Big|\mathcal{F}_s\Big] \\
&= \prod_{i=1}^{d}\exp\Big\{-b^{(i)}\Big[D^{(i)}(s,t)X^{(i)}(s)+\int_s^t D^{(i)}(v,t)k^{(i)}(v)dv \\
&\qquad -\int_0^s \Psi^{\kappa(i)}_{c^{(i)}}(s,t,v)\frac{k^{(i)}(v)}{\sigma^{(i)}(v)}dv \\
&\qquad +\int_0^s \Psi^{\kappa(i)}_{c^{(i)}}(s,t,v)\frac{a(v)}{\sigma^{(i)}(v)}X^{(i)}(v)dv \\
&\qquad +\int_0^s \Psi^{\kappa(i)}_{c^{(i)}}(s,t,v)\frac{1}{\sigma^{(i)}(v)}dX^{(i)}(v)\Big]\Big\} \\
&\quad \times\exp\Big\{\frac{(b^{(i)})^2}{2}\Big[\|c^{(i)}(\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2_{\kappa(i),T^\star}-\|\Psi^{\kappa(i)}_{c^{(i)}}(s,t,\cdot)\mathbf{1}_{[0,s]}(\cdot)\|^2_{\kappa(i),T^\star}\Big]\Big\}.
\end{aligned}
$$

Proof. We calculate

$$
\begin{aligned}
B(s,t) &= E\Big[e^{-\int_s^t r(v)dv}\Big|\mathcal{F}_s\Big] = E\Big[e^{-\int_s^t b^T\mathbf{X}(v)dv}\Big|\mathcal{F}_s\Big] \\
&= \prod_{i=1}^{d} E\Big[e^{\int_s^t b^{(i)}X^{(i)}(v)dv}\Big|\mathcal{F}_s\Big], \qquad (3.31)
\end{aligned}
$$

where we used the independence of the $X^{(i)}$ in the last equality. The result follows now by an application of Proposition 3.1.6. The extension of the conditional characteristic function to the whole of the complex plane $\mathbb{C}$ exists because of Gaussianity. □

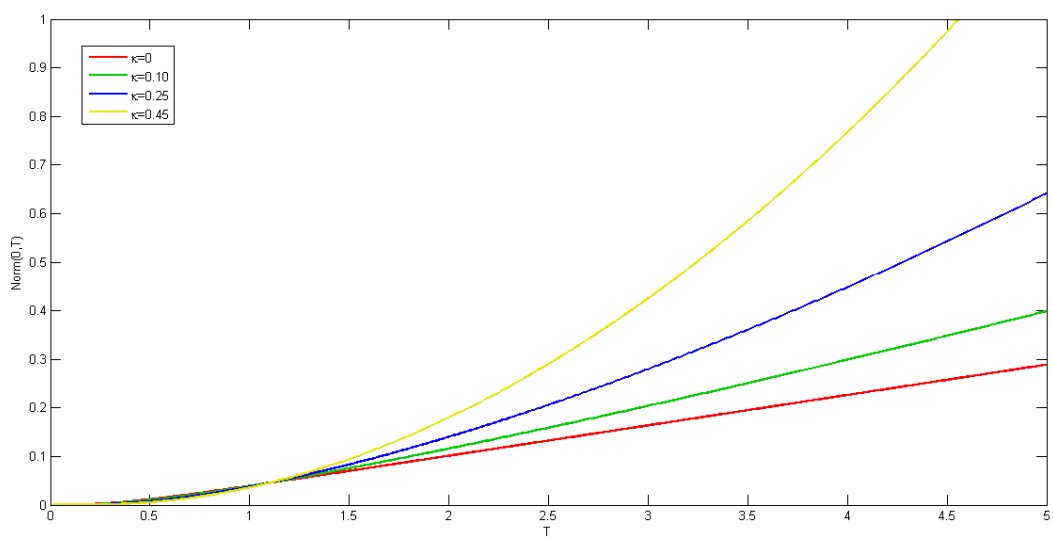

Figure 3.1: *Calculation of* $\|D(\cdot,T)\mathbf{1}_{[0,T]}(\cdot)\|^2_{\kappa,T^\star}$ *in the fractional one-factor model for varying* κ *and maturity* t*, using* $a=4$*. The case* $\kappa=0$ *has been calculated analytically.*

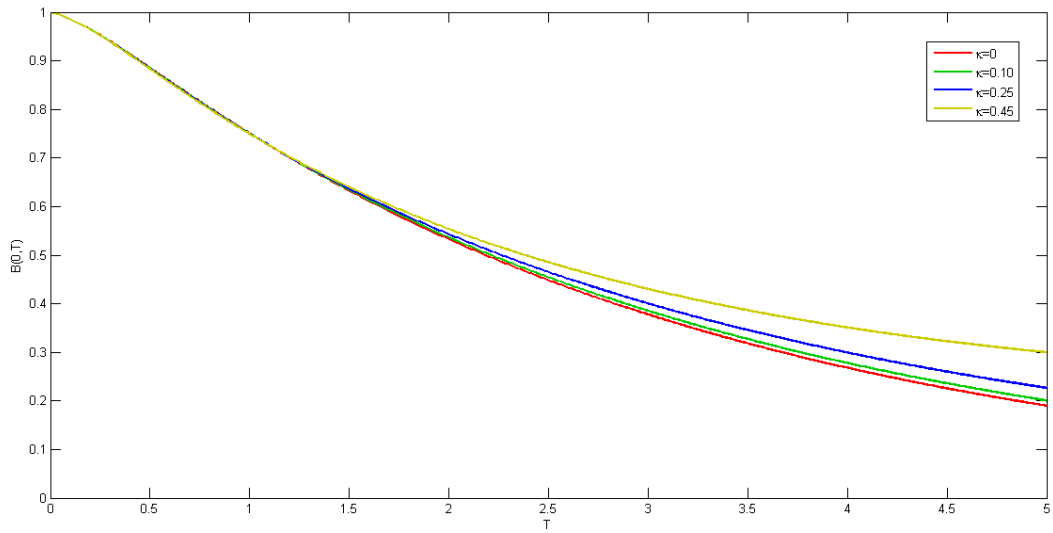

Figure 3.2: *Bond prices* $B(0,T)$ *in the fractional one-factor Vasicek model* (3.32) *for varying* $\kappa \geq 0$ *and maturity* T*, using constant coefficients* $a=4$*,* $k=1.5$*,* $\sigma=1$ *and* $X(0)=r(0)=0.1$*. Negative* κ *is not relevant as explained in the introduction to this section. Recall that* $\kappa=0$ *corresponds to the Brownian Vasicek model. Prices increase with* κ *as a consequence of long range dependence.*

Example 3.3.12. [Fractional one-factor model] We want to compare prices in our fractional model to the classical Brownian case, i.e. $\kappa=0$. For simplicity we assume constant coefficient functions in (3.29) and set $d=1$ with $b=1$. Today's prices of the zero coupon bonds are given by

$$B(0,t)=\exp\left\{-D(0,t)X(0)-k\int_0^t D(v,t)dv+\frac{\sigma^2}{2}\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T^\star}\right\}, \tag{3.32}$$

for $0\leq t\leq T^\star$.

Since negative κ is not relevant as explained in Section 3.3.1, we allow only for $\kappa\in[0,\frac{1}{2})$

in the following steps. Standard numerical methods may be instable here, because of the singularities in the norms in (3.32) whose exact values cannot be computed. Therefore we apply the following discretization scheme.

Remark 3.3.13 (Numerical procedure)**.** For $\kappa \in (0, \frac{1}{2})$ and $t \in [0, T^\star]$. We have

$$\begin{aligned} &\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T^\star} \\ = \; & \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)(\Gamma(\kappa))^2}\int_0^{T^\star} s^{-2\kappa}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds. \end{aligned} \tag{3.33}$$

In a first step we decompose the outer integral for $n \in \mathbb{N}$ and
$0 = s_0 \le s_1 \le \cdots \le s_n = t$

$$\begin{aligned} &\int_0^{T^\star} s^{-2\kappa}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds \\ = \; & \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} s^{-2\kappa}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds. \end{aligned}$$

For sufficiently small intervals $[s_i, s_{i+1}]$ we get a reasonable approximation by

$$\int_s^{T^\star}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s)^{1-\kappa}}dr \approx \int_{s_i}^{T^\star}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s_i)^{1-\kappa}}dr.$$

Now we take for $i = 0, \ldots, n-1$ a partition $s_i = u_0^i \le u_1^i \le \cdots \le u_{m_i}^i = s_{i+1}$ for some $m_i \in \mathbb{N}$

$$\begin{aligned} &\int_{s_i}^{T^\star}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s_i)^{1-\kappa}}dr = \sum_{j=0}^{m_i-1}\int_{u_j^i}^{u_{j+1}^i}\frac{r^\kappa D(r,t)\mathbf{1}_{[0,t]}(r)}{(r-s_i)^{1-\kappa}}dr \\ \approx \; & \frac{1}{\kappa}\sum_{j=0}^{m_i-1}[(u_{j+1}^i - s_i)^\kappa - (u_j^i - s_i)^\kappa]\frac{(u_j^i)^\kappa D(u_j^i,t) + (u_{j+1}^i)^\kappa D(u_{j+1}^i,t)}{2} \end{aligned}$$

Putting everything together and using $\Gamma(\kappa)\cdot\kappa = \Gamma(\kappa+1)$, we obtain

$$\begin{aligned} &\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T^\star} \\ \approx \; & \frac{\pi\kappa(2\kappa+1)}{\Gamma(2-2\kappa)\sin(\pi\kappa)(2\Gamma(\kappa+1))^2}\sum_{i=0}^{n-1}[s_{i+1}^{1-2\kappa} - s_i^{1-2\kappa}] \\ & \times\Big[\sum_{j=0}^{m_i-1}[(u_{j+1}^i - s_i)^\kappa - (u_j^i - s_i)^\kappa](u_j^i)^\kappa D(u_j^i,t) + (u_{j+1}^i)^\kappa D(u_{j+1}^i,t)\Big]^2. \end{aligned}$$

Choosing now $s_i = 0.01i$ for $i = 0, \ldots, 100t$, and $u_j^i = 0.01(i+j)$ for $j = 0, \ldots, 100t - i$,

we obtain

$$\begin{aligned}
&\|D(\cdot,t)\mathbf{1}_{[0,t]}(\cdot)\|^2_{\kappa,T^\star} \\
\approx\ & \frac{\pi\kappa(2\kappa+1)}{\Gamma(2-2\kappa)\sin(\pi\kappa)2\Gamma(\kappa+1)^2}0.01^{1+2\kappa}\sum_{i=0}^{100t-1}\Big([(i+1)^{1-2\kappa}-i^{1-2\kappa}] \\
&\times\Big(\sum_{j=0}^{100t-i-1}[(j+1)^\kappa-j^\kappa][(i+j)^\kappa D(0.01(i+j),t) \\
&+(i+j+1)^\kappa D(0.01(i+j+1),t)]\Big)^2\Big). \qquad (3.34)
\end{aligned}$$

Finally, examples of the norms and bond prices can be found in Figures 3.1 and 3.2.

3.4 Two-dimensional case with same driving factor

In this section we will consider the conditional characteristic function of a two-dimensional fractional Brownian motion. In contrast to the situation of Section 3.2 we shall use a Mandelbrot-Van Ness representation and introduce a certain dependence structure. However firstly, we want to provide a short motivation.

3.4.1 Motivation

The financial crisis of 2008/2009 showed that the counterparty risk can be significant when considering interest rate or credit derivatives. Therefore a natural extension of the fractional interest rate market of Section 3.3 would be the introduction of default possibilities. The simplest structure in such a market will be a defaultable zero coupon bond.

There are many examples, which consider the short (r) and default rate (λ) as functions of state vectors of Markov processes; see e.g. Duffie, Filipovic and Schachermayer [45] or Schönbucher [122], Chapter 7. Processes driven by Brownian motion are the most prominent ones. We will focus on the case where r and λ are given by Vasicek models, with possibly time-dependent coefficients, driven by fBms with fractional integration parameter κ strictly greater than zero. As explained in Section 3.3.1 this choice is motivated by the fact that macroeconomic variables like demand and supply, interest rates, or other economic activity measures often exhibit long range dependence (cf. Henry and Zaffaroni [73] for an overview).

3.4.2 Prediction results

The case of a one-dimensional fractional Brownian motion has already been covered in Section 3.1 and its multivariate extension with independent components followed directly (cf. Section 3.2). Of course we can describe the dynamics of short and default rate by such a two-dimensional Brownian motion with independent entries, however this assumption would be quite unrealistic for practical applications. Instead we introduce a certain dependence structure between two fractional Brownian motions as will be explained in a moment. A more general approach for Molchan-Golosov fractional Lévy processes will be covered in Chapter 4.

We introduce a bivariate fBm $(B^\kappa, \bar{B}^{\bar{\kappa}}) = (B^\kappa(t), \bar{B}^{\bar{\kappa}}(t))_{t\in\mathbb{R}}$ with $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$. The dependence structure between the fBm's will be modeled as in Elliott and van der Hoek [53] by assuming that both processes arise through an integral representation driven by the same two-sided Brownian motion $B = (B(t))_{t\in\mathbb{R}}$, which holds in $L^2(\Omega)$ and is stated in Proposition 1.5.8:

$$\begin{aligned} B^\kappa(t) &= \Gamma(\kappa+1)c_\kappa \int_{-\infty}^{\infty} \mathcal{I}_-^\kappa \mathbf{1}_{[0,t)}(s)dB(s), \quad c_\kappa := \frac{\sqrt{\Gamma(2\kappa+2)\sin((\kappa+1/2)\pi)}}{\Gamma(\kappa+1)}, \\ \bar{B}^{\bar{\kappa}}(t) &= \Gamma(\bar{\kappa}+1)c_{\bar{\kappa}} \int_{-\infty}^{\infty} \mathcal{I}_-^{\bar{\kappa}} \mathbf{1}_{[0,t)}(s)dB(s), \quad c_{\bar{\kappa}} := \frac{\sqrt{\Gamma(2\bar{\kappa}+2)\sin((\bar{\kappa}+1/2)\pi)}}{\Gamma(\bar{\kappa}+1)}, \end{aligned} \tag{3.35}$$

for $t \in \mathbb{R}$, with gamma function Γ and the classical Riemann-Liouville fractional integral from Definition 1.2.4.

Remark 3.4.1. The two fBms arising from the same Brownian motion have the economical interpretation that short rate and default rate are driven by the same market noise and macroeconomic factors. However, the influence of this noise may be different and depends on the long range dependence parameters as well as on the coefficient functions of the Langevin equations.
Also it is always possible to add several independent factors driven by independent Brownian motions. Using such a technique different dynamics for short and default rate can be constructed.

The dependence between B^κ and $\bar{B}^{\bar{\kappa}}$ is then given by the covariance function (see (2.17) of Elliott and van der Hoek [53]) for $0 \le s, t \le T^\star$ as

$$\begin{aligned} & Cov(B^\kappa(t), \bar{B}^{\bar{\kappa}}(s)) \\ = \; & \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)} [|t|^{\kappa+\bar{\kappa}+1} + |s|^{\kappa+\bar{\kappa}+1} + |t-s|^{\kappa+\bar{\kappa}+1}]. \end{aligned} \tag{3.36}$$

Remark 3.4.2. In Section 3.4 and 3.5 we will understand integration with respect to fBm in the $L^2(\Omega)$-sense of Pipiras and Taqqu [103]. However under certain additional assumptions also pathwise integrals will appear. When both types of integrals exists, they are the same in probability and therefore in distribution. For simplicity we shall furthermore restrict ourselves to integrands in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. It is however not difficult to weaken this assumption using the integration theory of Pipiras and Taqqu [103] (similar to Section 3.1).

Using (3.35) we can show the following proposition by approximation with step functions.

Proposition 3.4.3. *Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then*

$$\begin{aligned} & E\left[\int_{\mathbb{R}} f(s)dB^{\kappa}(s) \int_{\mathbb{R}} g(s)d\bar{B}^{\bar{\kappa}}(s)\right] \\ = \quad & \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)} \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{\kappa+\bar{\kappa}-1}dudv. \end{aligned}$$

The following proposition is a consequence of (3.13) of Pipiras and Taqqu [103] and equation (3.32) above.

Proposition 3.4.4. *Let $(B_t)_{t\in\mathbb{R}}$ be the two-sided Brownian motion of* (3.35) *and $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$. For every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the following integrals are equal in the $L^2(\Omega)$-sense:*

$$\int_{\mathbb{R}} f(s)dB^{\kappa}(s) \quad = \quad c_\kappa \Gamma(\kappa+1) \int_{\mathbb{R}} \mathcal{I}_-^{\kappa}(f(\cdot))(s)dB(s)$$

and

$$\int_{\mathbb{R}} f(s)d\bar{B}^{\bar{\kappa}}(s) \quad = \quad c_{\bar{\kappa}} \Gamma(\bar{\kappa}+1) \int_{\mathbb{R}} \mathcal{I}_-^{\kappa}(f(\cdot))(s)dB(s).$$

For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\kappa \in (0, \frac{1}{2})$ the following inner product is finite by Proposition 3.2 of Pipiras and Taqqu [103]:

$$\langle f, g\rangle_{\kappa,\infty} := \kappa(2\kappa+1) \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1}dudv.$$

We shall denote the induced norm by $\|\cdot\|_{\kappa,\infty}$. Define further $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$

$$\begin{aligned} & \langle f, g\rangle_{\kappa,\bar{\kappa},\infty} \\ := \quad & \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)} \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{\kappa+\bar{\kappa}-1}dudv \end{aligned}$$

and denote the induced norm by $\|\cdot\|_{\kappa,\bar{\kappa},\infty}$

As a first result we provide an extension of Proposition 3.4.4:

Lemma 3.4.5. *For $0 \le t \le T$ let $c \in L^2([t,T])$. Let B^κ and $\bar{B}^{\bar\kappa}$ be fBm's as in (3.35). Assume further that $\kappa \le \bar\kappa$. Then the equality of both integrals holds in the $L^2(\Omega)$-sense:*

$$\int_t^T c(v)d\bar{B}^{\bar\kappa}(v) = \frac{c_{\bar\kappa}\Gamma(\bar\kappa+1)}{c_\kappa\Gamma(\kappa+1)}\int_{\mathbb{R}} \mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[t,T]}(\cdot)c(\cdot))(v)dB^\kappa(v). \tag{3.37}$$

Proof. Set $a_\kappa := c_\kappa\Gamma(\kappa+1)$ and $a_{\bar\kappa} := c_{\bar\kappa}\Gamma(\bar\kappa+1)$. Then using repeatedly Proposition 3.4.4 we get

$$\begin{aligned}
\int_t^T c(v)d\bar{B}^{\bar\kappa}(v) &= \int_{\mathbb{R}} \mathbf{1}_{[t,T]}(v)c(v)d\bar{B}^{\bar\kappa}(v) \\
&= a_{\bar\kappa}\int_{\mathbb{R}} \mathcal{I}_-^{\bar\kappa}(\mathbf{1}_{[t,T]}(\cdot)c(\cdot))(v)dB(v) \\
&= \frac{a_{\bar\kappa}}{a_\kappa}\int_{\mathbb{R}} \mathcal{D}_-^\kappa\mathcal{I}_-^{\bar\kappa}(\mathbf{1}_{[t,T]}(\cdot)c(\cdot))(v)dB^\kappa(v) \\
&= \frac{a_{\bar\kappa}}{a_\kappa}\int_{\mathbb{R}} \mathcal{D}_-^\kappa\mathcal{I}_-^\kappa\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[t,T]}(\cdot)c(\cdot))(v)dB^\kappa(v) \\
&= \frac{a_{\bar\kappa}}{a_\kappa}\int_{\mathbb{R}} \mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[t,T]}(\cdot)c(\cdot))(v)dB^\kappa(v),
\end{aligned}$$

where we applied Theorem 6.1 of Samko, Kilbas and Marichev [117] in the last line. □

In Section 3.1 we calculated the conditional characteristic function by using an approximation of the conditional expectation by finite sample observations. Another way of approaching the prediction problem has been considered in Duncan [46] where the author used a special property of the Wick product to exchange the conditional expectation and the exponential function. However, Proposition 2 of that paper is not correct and an erratum is Duncan and Fink [47]. For the two-dimensional case considered in this section we shall apply a similar method as in Duncan and Fink [47]. The dependence structure of $(B^\kappa, \bar{B}^{\bar\kappa})$ will be used to apply the results of Elliott and van der Hoek [53].

First we recall some basic properties of the Wick product for fBm and refer to Biagini, Hu, Øksendal and Zhang [20], Section 3, Elliott and van der Hoek [53] or Duncan, Hu and Pasik-Duncan [48] for details and background.

There are various ways to introduce the Wick product and we will follow mainly Section 3.1 of Biagini et al. [20]. Let $\kappa \in (0, \frac{1}{2})$. First we consider for $c : \mathbb{R} \to \mathbb{R}$ with $\|c\|_{\kappa,\infty} < \infty$ exponentials of the form

$$\varepsilon(c) := \exp\left\{\int_{\mathbb{R}} c(s)dB^\kappa(s) - \frac{1}{2}\|c\|_{\kappa,\infty}\right\} \tag{3.38}$$

like in (3.7) of [20]. The set $\mathcal{E}$ of linear combinations of these exponentials is dense in $L^p(\Omega)$ for all $p \ge 1$.

Definition 3.4.6. *For $c, h : \mathbb{R} \to \mathbb{R}$ with $\|c\|_{\kappa,\infty}, \|h\|_{\kappa,\infty} < \infty$ the* Wick product *of the exponentials of c, h is defined as*

$$\varepsilon(c) \diamond \varepsilon(h) := \varepsilon(c + h). \tag{3.39}$$

By bilinearity the Wick product is defined on the whole of $\mathcal{E}$. A classical density argument (see Theorem 3.1 of [20]) extends this definition now to $L^p(\Omega)$ for all $p \geq 1$. The two main properties of the Wick product we need in this section are summarized in the next proposition.

Proposition 3.4.7. *Let $c : \mathbb{R} \to \mathbb{R}$ with $\|c\|_{\kappa,\infty} < \infty$.*
(1) Define the Wick exponential by $\exp^\diamond(\cdot) := \sum_{i=0}^\infty ((\cdot)^{\diamond i}/i!)$. Then

$$\exp^\diamond \left\{ \int_{\mathbb{R}} c(s) dB^\kappa(s) \right\} = \exp \left\{ \int_{\mathbb{R}} c(s) dB^\kappa(s) - \frac{1}{2} \|c\|_{\kappa,\infty} \right\} = \varepsilon(c). \tag{3.40}$$

(2) Define $\mathcal{G}_t := \sigma\{B^\kappa(s), s \in (a, t]\}$ for $-\infty \leq a < t < \infty$. Then

$$E\left[\exp^\diamond \left\{ \int_{\mathbb{R}} c(s) dB^\kappa(s) \right\} \middle| \mathcal{G}_t \right] = \exp^\diamond \left\{ E\left[\int_{\mathbb{R}} c(s) dB^\kappa(s) \middle| \mathcal{G}_t \right] \right\}$$

Proof. Part (1) is given by (3.25) of Biagini et al. [20] and part (2) is a consequence of (17) of Duncan [46] and the uniform convergence of the exponential Wick series. □

Remark 3.4.8. For the remainder of this section define

$$\mathcal{G}_t \quad := \quad \sigma\overline{\{(B_s^\kappa, \bar{B}_s^{\bar{\kappa}}), s \in [0, t]\}}, \quad t \geq 0.$$

Consider for $T > 0$ a bivariate fBm defined by fractional integration via a Molchan-Golosov kernel on a compact interval $(W^\kappa, \bar{W}^{\bar{\kappa}}) = (W_t^\kappa, \bar{W}_t^{\bar{\kappa}})_{t\in[0,T]}$ (for details cf. Molchan and Golosov [95], Kleptsyna, LeBreton and Roubaud [79] and Norros, Valkeila and Virtamo [99]) driven by the same Bm $W = (W_t)_{t\in[0,T]}$. A straightforward calculation shows that $(W_t^\kappa, \bar{W}_t^{\bar{\kappa}})_{t\in[0,T]}$ has the same second order structure as $(B_t^\kappa, \bar{B}_t^{\bar{\kappa}})_{t\in[0,T]}$. The equality of the finite dimensional distributions, i.e.

$$(W_t^\kappa, \bar{W}_t^{\bar{\kappa}})_{t\in[0,T]} \stackrel{d}{=} (B_t^\kappa, \bar{B}_t^{\bar{\kappa}})_{t\in[0,T]} \tag{3.41}$$

follows because both are Gaussian processes. Furthermore we have

$$(\mathcal{F}_t^{W^\kappa})_{t\in[0,T]} = (\mathcal{F}_t^{W})_{t\in[0,T]} = (\mathcal{F}_t^{\bar{W}^{\bar{\kappa}}})_{t\in[0,T]}$$

and therefore

$$E[f(W^\kappa(t))|\mathcal{F}_s^{W^\kappa}] = E[f(W^\kappa(t))|\mathcal{F}_s^{W^\kappa} \vee \mathcal{F}_s^{\bar{W}^{\bar{\kappa}}}]$$

for all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that the conditional expectation exists. From (3.41) follows also the equality of the conditional distributions and therefore by p. 290 of Dudley [43] that

$$E[f(B^\kappa(t))|\mathcal{F}_s^{B^\kappa}] = E[f(B^\kappa(t))|\mathcal{F}_s^{B^\kappa} \vee \mathcal{F}_s^{\bar{B}^{\bar{\kappa}}}].$$

Next we need an analog of the Theorem 3.1.1 for $(B^\kappa, \bar{B}^{\bar{\kappa}})$, where now in the exponential there is the sum of two integrals and the dependence between B^κ and $B^{\bar{\kappa}}$ matters. We shall proceed as follows. First we transform both integrals with respect to B^κ and $\bar{B}^\kappa$, respectively, into one integral with respect to B^κ and invoke afterwards Proposition 1.5.15. The proof will rely on the following technical lemma.

Lemma 3.4.9. *Given $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$ assume $\bar{\kappa} \geq \kappa$. Let $f : \mathbb{R} \to \mathbb{R}$ with norm $||f(\cdot)||_{\kappa,\infty} < \infty$ and $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then we have*

$$\left(\frac{a_{\bar{\kappa}}}{a_\kappa}\right)^2 \left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\|_{\kappa,\infty}^2 = \|g(\cdot)\|_{\bar{\kappa},\infty}^2, \tag{3.42}$$

$$\frac{a_{\bar{\kappa}}}{a_\kappa} \left\langle f(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\rangle_{\kappa,\infty} = \langle f(\cdot), g(\cdot)\rangle_{\kappa,\bar{\kappa},\infty}, \tag{3.43}$$

where $a_\kappa := c_\kappa \Gamma(\kappa + 1)$ and $a_{\bar{\kappa}} := c_{\bar{\kappa}} \Gamma(\bar{\kappa} + 1)$.

Proof. We know from Lemma 3.4.5 that in the $L^2(\Omega)$-sense

$$\int_{\mathbb{R}} g(v) d\bar{B}^{\bar{\kappa}}(v) = \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v) dB^\kappa(v)$$

and, therefore, variances are equal. Equation (3.42) follows. Furthermore, since equality in $L^2(\Omega)$ implies a.s. equality, we have with Lemma 3.4.5 again in the $L^2(\Omega)$-sense

$$\int_{\mathbb{R}} f(v) dB^\kappa(v) \int_{\mathbb{R}} g(v) d\bar{B}^{\bar{\kappa}}(v) = \int_{\mathbb{R}} f(v) dB^\kappa(v) \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v) dB^\kappa(v).$$

and, therefore, by Proposition 3.4.3

$$\begin{aligned}
\langle f(\cdot), g(\cdot)\rangle_{\kappa,\bar{\kappa},\infty} &= E\left[\int_{\mathbb{R}} f(v) dB^\kappa(v) \int_{\mathbb{R}} g(v) d\bar{B}^{\bar{\kappa}}(v)\right] \\
&= E\left[\int_{\mathbb{R}} f(v) dB^\kappa(v) \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v) dB^\kappa(v)\right] \\
&= \frac{a_{\bar{\kappa}}}{a_\kappa} \left\langle f(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\rangle_{\kappa,\infty}
\end{aligned}$$

□

Now we are ready to state and prove the result regarding the conditional distribution of $(B^\kappa, \bar{B}^{\bar{\kappa}})$. Since we want to use the following formula for pricing defaultable zero bonds in Section 3.5 we will directly state it in the form of the Laplace transform.

Theorem 3.4.10. *For $0 \le t < T$ let $c, \bar{c} \in L^2([t,T])$. Further let B^κ and $\bar{B}^{\bar{\kappa}}$ be fBm's as in* (3.35)*. Then*

$$\begin{aligned} & E\left[\exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\middle| \mathcal{G}_t\right] \\ = \; & e^{W(t,T)-V(t,T)} \exp\left\{\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\}. \end{aligned} \tag{3.44}$$

where

$$\begin{aligned} W(t,T) &= \frac{1}{2}\Big(\left\|\mathbf{1}_{[t,T]}(\cdot)c(\cdot)\right\|^2_{\kappa,\infty} + 2\left\langle \mathbf{1}_{[t,T]}(\cdot)c(\cdot), \mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty} \\ &\quad + \left\|\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot)\right\|^2_{\bar{\kappa},\infty}\Big), \\ V(t,T) &= \frac{1}{2}\Big(\left\|\mathbf{1}_{[0,t]}(\cdot)\Psi_c^\kappa(t,T,\cdot)\right\|^2_{\kappa,\infty} \\ &\quad + 2\left\langle \mathbf{1}_{[0,t]}(\cdot)\Psi_c^\kappa(t,T,\cdot), \mathbf{1}_{[0,t]}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty} \\ &\quad + \left\|\mathbf{1}_{[0,t]}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)\right\|^2_{\bar{\kappa},\infty}\Big). \end{aligned}$$

and $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ are as in (1.20) *and belong to $L^2([0,t])$ for all $0 \le t \le T$.*

Proof. To predict the exponential we transform it into a Wick exponential using Lemma 3.4.5 and then Proposition 3.4.7 as follows. W.l.o.g. assume that $\kappa \le \bar{\kappa}$. Define $a_\kappa = c_\kappa \Gamma(\kappa+1)$ and $a_{\bar{\kappa}} = c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)$. Then by Lemma 3.4.5 and Proposition 3.4.7

$$\begin{aligned} & \exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\} \\ = \; & \exp\left\{\int_{\mathbb{R}} \left(\mathbf{1}_{[t,T]}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\right\} \\ = \; & e^{W(t,T)} \exp^\diamond\left\{\int_{\mathbb{R}} \left(\mathbf{1}_{[t,T]}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\right\} \end{aligned}$$

and, as preliminary version,

$$\begin{aligned} W(t,T) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\mathbf{1}_{[t,T]}(u)c(u) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(u)\right) \\ &\quad \times \left(\mathbf{1}_{[t,T]}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(v)\right)|u-v|^{2\kappa-1}dudv \\ &= \frac{1}{2}\Bigg(\left\|\mathbf{1}_{[t,T]}(\cdot)c(\cdot)\right\|^2_{\kappa,\infty} + 2\frac{a_{\bar{\kappa}}}{a_\kappa}\left\langle \mathbf{1}_{[t,T]}(\cdot)c(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(\cdot)\right\rangle_{\kappa,\infty} \\ &\quad + \left(\frac{a_{\bar{\kappa}}}{a_\kappa}\right)^2 \left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(\cdot)\right\|^2_{\kappa,\infty}\Bigg) \end{aligned}$$

Next we take conditional expectation of the exponential integral which is nothing else than an $L^2(\Omega)$ projection. Therefore, Proposition 3.4.7 and Remark 3.4.8 apply, giving

$$\begin{aligned}
& E\left[\exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\middle|\,\mathcal{G}_t\right] \\
= \; & e^{W(t,T)}E\left[\exp^\diamond\left\{\int_{\mathbb{R}}\left(\mathbf{1}_{[t,T]}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\right\}\middle|\,\mathcal{G}_t\right] \\
= \; & e^{W(t,T)}\exp^\diamond\left\{E\left[\int_{\mathbb{R}}\left(\mathbf{1}_{[t,T]}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[t,T]}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\middle|\,\mathcal{G}_t\right]\right\}.
\end{aligned}$$

Now transform the integral in the conditional expectation back and apply the prediction formula from Proposition 1.5.15. Transforming the Wick exponential in a classical exponential yields the term

$$\begin{aligned}
& V(t,T) \\
= \; & \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\mathbf{1}_{[0,t]}(u)\Psi_c^\kappa(t,T,u) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[0,t]}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(u)\right) \\
& \times\left(\mathbf{1}_{[0,t]}(v)\Psi_c^\kappa(t,T,v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[0,t]}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(v)\right)|u-v|^{2\kappa-1}dudv \\
= \; & \frac{1}{2}\left(\left\|\mathbf{1}_{[0,t]}(\cdot)\Psi_c^\kappa(t,T,\cdot)\right\|_{\kappa,\infty}^2\right. \\
& +2\frac{a_{\bar{\kappa}}}{a_\kappa}\left\langle\mathbf{1}_{[0,t]}(\cdot)\Psi_c^\kappa(t,T,\cdot),\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[0,t]}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(\cdot)\right\rangle_{\kappa,\infty} \\
& \left.+\left(\frac{a_{\bar{\kappa}}}{a_\kappa}\right)^2\left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{[0,t]}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(\cdot)\right\|_{\kappa,\infty}^2\right).
\end{aligned}$$

Finally, we transform the indefinite integral $\mathcal{I}_-^{\bar{\kappa}-\kappa}$ within the conditional expectation back using Lemma 3.4.5. Combining these two steps yields (3.44). The final versions of $V(t,T)$ and $W(t,T)$ can be calculated by Lemma 3.4.9. □

3.5 Application: Defaultable bonds and credit derivatives

Using the results of Section 3.3 we will again directly model under an (average) risk-neutral measure $\mathcal{Q}$ from now on and consider prices as conditional expectations. Given $T^\star > 0$ and a complete probability space $(\Omega, \mathcal{F}, \mathcal{Q})$ we will for notational convenience work with the bivariate fBm $(B^\kappa, \bar{B}^{\bar{\kappa}})$ from Section 3.4 which shall be a fBm under $\mathcal{Q}$ and adapted to the filtration $(\mathcal{F}_t)_{0\le t\le T^\star}$ by assumption for the rest of the section.

Now we will consider defaultable bonds as specific contingent claims.

3.5.1 Defaultable claims

Let H be the *default indicator process* given by

$$H(t) = \mathbf{1}_{\{\tau \leq t\}}, \quad 0 \leq t \leq T^\star,$$

where τ is an $(\mathcal{F}_t)_{0\leq t\leq T^\star}$-stopping time, representing the default time of some firm or financial instrument. We denote by $(\mathcal{H}_t)_{0\leq t\leq T^\star}$ the filtration generated by H. We assume further that there exists a subfiltration $(\mathcal{G}_t)_{0\leq t\leq T^\star}$ of $(\mathcal{F}_t)_{0\leq t\leq T^\star}$ such that

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t, \quad 0 \leq t \leq T^\star,$$

Assumption 3.5.1 (Market structure; cf. Frey and Backhaus [62], Ass. 3.1)**.** *Remaining in the framework of the most reduced-form credit risk models in the literature we assume that there is a $(\mathcal{G}_t)_{0\leq t\leq T^\star}$-progressive stochastic process $\lambda = (\lambda_t)_{0\leq t\leq T^\star}$ modeling the intensity of H with the following properties (see also Corollary 5.1.5 of Bielecki and Rutkowski [21]): λ is positive, $\int_0^t \lambda(s)ds < \infty$ a.s. for all $0 \leq t \leq T^\star$ and it satisfies*

$$P(\tau > t \mid \mathcal{G}_t) = E[1 - H(t) \mid \mathcal{G}_t] = \exp\left\{-\int_0^t \lambda(s)ds\right\}. \tag{3.45}$$

Moreover, defining $\mathcal{G}_\infty := \bigvee_{0\leq t\leq T^\star} \mathcal{G}_t$, for all bounded $\mathcal{G}_\infty$-measurable random variables η, we have

$$E[\eta \mid \mathcal{F}_t] = E[\eta \mid \mathcal{G}_t], \quad 0 \leq t \leq T^\star. \tag{3.46}$$

We call λ the default rate.

Now we have to specify the joint dynamics of r and λ. Recall the bivariate fBm from Section 3.4. We model the short rate r and the default rate λ as pathwise solutions to Langevin equations on $[0, T^\star]$:

$$\begin{aligned} dr(t) &= (k(t) - a(t)r(t))dt + \sigma(t)dB^\kappa(t), \quad r(0) = r_0 \in \mathbb{R}, \\ d\lambda(t) &= (\bar{k}(t) - \bar{a}(t)\lambda(t))dt + \bar{\sigma}(t)d\bar{B}^{\bar{\kappa}}(t) \quad \lambda(0) = \lambda_0 \in \mathbb{R}, \end{aligned} \tag{3.47}$$

where $k(\cdot), \bar{k}(\cdot), a(\cdot), \bar{a}(\cdot)$ are continuous and locally integrable on $[0, T^\star]$. Further we assume that $\sigma(\cdot), \bar{\sigma}(\cdot) > 0$ are continuous and that $\sigma(\cdot), 1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(\frac{1}{2} - \kappa)$ and that $\bar{\sigma}(\cdot), 1/\bar{\sigma}(\cdot)$ are of bounded $\bar{p}$-variation for some $0 < \bar{p} < 1/(\frac{1}{2} - \bar{\kappa})$ on $[0, T^\star]$.

Although both fBms are driven by the same noise, its influence can vary through different coefficient functions of the Langevin equations.

Note that it is also possible to model different dynamics in r and λ by adding several independent factors driven by independent Brownian motions as explained in Remark 3.4.1.

Lemma 3.5.2. *Under the above conditions the pathwise solutions of the sde's* (3.47) *are given for* $0 \le t \le T \le T^\star$ *by*

$$r(T) = r(t)e^{-\int_t^T a(u)du} + \int_t^T e^{-\int_s^T a(u)du}k(s)ds + \int_t^T e^{-\int_s^T a(u)du}\sigma(s)dB^\kappa(s), \tag{3.48}$$

$$\lambda(T) = \lambda(t)e^{-\int_t^T \bar{a}(u)du} + \int_t^T e^{-\int_s^T \bar{a}(u)du}\bar{k}(s)ds + \int_t^T e^{-\int_s^T \bar{a}(u)du}\bar{\sigma}(s)d\bar{B}^{\bar{\kappa}}(s), \tag{3.49}$$

where the fBm integrals can be considered in the $L^2(\Omega)$*- or pathwise sense, cf. Young [134] and Section 1.3.*

Now the non-observable fBms can be replaced by observable processes given by solutions to (3.47).

Proposition 3.5.3. *Under the above conditions we have for* $0 \le t \le T \le T^\star$

$$\begin{aligned} dB^\kappa(t) &= \left(-\frac{k(t)}{\sigma(t)} + \frac{a(t)}{\sigma(t)}r(t)\right)dt + \frac{1}{\sigma(t)}dr(t) \quad \textit{and} \\ d\bar{B}^{\bar{\kappa}}(t) &= \left(-\frac{\bar{k}(t)}{\bar{\sigma}(t)} + \frac{\bar{a}(t)}{\bar{\sigma}(t)}\lambda(t)\right)dt + \frac{1}{\bar{\sigma}(t)}d\lambda(t). \end{aligned}$$

Proof. By Lemma 3.5.2 we have for $0 \le t \le T \le T^\star$

$$\int_t^T e^{-\int_s^T a(v)dv}\sigma(s)dB^\kappa(s) = r(T) - r(t)e^{-\int_t^T a(v)dv} - \int_t^T e^{-\int_s^T a(v)dv}k(s)ds$$

and, applying the density formula of Theorem 1.3.4 we get for $0 \le t \le T \le T^\star$

$$\begin{aligned} & B^\kappa(T) - B^\kappa(t) \\ = & \int_t^T \frac{e^{\int_u^T a(v)dv}}{\sigma(u)} d\left(-\int_u^T e^{-\int_s^T a(v)dv}\sigma(s)dB^\kappa(s)\right) \\ = & \int_t^T \frac{e^{\int_u^T a(v)dv}}{\sigma(u)} d\left(\int_u^T e^{-\int_s^T a(v)dv}k(s)ds + r(u)e^{-\int_u^T a(v)dv} - r(T)\right) \\ = & -\int_t^T \frac{k(u)}{\sigma(u)}du + \int_t^T \frac{a(u)}{\sigma(u)}r(u)du + \int_t^T \frac{1}{\sigma(u)}dr(u). \end{aligned}$$

The second equation can be obtained similarly. □

Corollary 3.5.4. *Let* $0 \le t \le T^\star$*. Then the sum* $r(t) + \lambda(t)$ *is normally distributed with*

mean zero and variance given by

$$\begin{aligned} & \kappa(2\kappa+1)\int_0^t\int_0^t e^{-\int_u^t a(w)dw-\int_v^t a(w)dw}\sigma(u)\sigma(v)|u-v|^{2\kappa-1}dudv \\ + \quad & 2\rho(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1)\int_0^t\int_0^t e^{-\int_u^t a(w)dw-\int_v^t \bar{a}(w)dw}\sigma(u)\bar{\sigma}(v)|u-v|^{\kappa+\bar{\kappa}-1}dudv \\ + \quad & \bar{\kappa}(2\bar{\kappa}+1)\int_0^t\int_0^t e^{-\int_u^t \bar{a}(w)dw-\int_v^t \bar{a}(w)dw}\bar{\sigma}(u)\bar{\sigma}(v)|u-v|^{2\bar{\kappa}-1}dudv, \end{aligned} \tag{3.50}$$

where the covariance of the two integrals is given by the second term of (3.50)*. Here*

$$\rho = \frac{c_\kappa c_{\bar{\kappa}}\Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)} \geq 0.$$

Remark 3.5.5. Corollary 3.5.4 implies that short rate and default rate are positively correlated, which makes sense economically. A high default rate indicates a higher probability of default before maturity. An investor will, therefore, request a compensation by a higher interest rate before taking this risk.

The information filtration given by the short rate and the default rate processes is now

$$\mathcal{G}_t \quad = \quad \sigma\{(r_s,\lambda_s), s\in[0,t]\} = \sigma\{(B_s^\kappa, \bar{B}_s^{\bar{\kappa}}), s\in[0,t]\}, \quad 0\leq t\leq T^\star.$$

Let $0\leq t\leq T\leq T^\star$ and X be $\mathcal{F}_T$-measurable. Since the contingent claim $\mathbf{1}_{\{\tau>T\}}$ is also $\mathcal{F}_T$-measurable we can apply equation (3.22) to get for the price of the defaultable, $\mathcal{F}_T$-measurable, integrable contingent claim $\mathbf{1}_{\{\tau>T\}}X$ at time t

$$E\Big[\mathbf{1}_{\{\tau>T\}}Xe^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big]. \tag{3.51}$$

Considering (3.51) and Lemma 13.2 of Filipovic [56] the price of a defaultable, $\mathcal{F}_T$-measurable, integrable contingent claim $\mathbf{1}_{\{\tau>T\}}X$ is for $0\leq t\leq T\leq T^\star$ given by

$$\overline{B}(t,T) \quad = \quad E\Big[\mathbf{1}_{\{\tau>T\}}Xe^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big] = \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}X\Big|\mathcal{G}_t\Big]. \tag{3.52}$$

Remark 3.5.6. Setting $X=1$ we get the situation of a defaultable zero coupon bond:

$$\overline{B}(t,T) \quad = \quad \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big].$$

3.5.2 Defaultable zero coupon bonds

Using the Remark 3.5.6 the pricing of a defaultable zero coupon bond boils down to the situation of a two-factor short rate model. However in contrast to Section 3.3.4 the two processes are dependent. Invoking Theorem 3.4.10 solves this issue and provides us with a analytic formula for the bond price manifesting a similar structure for the price as in the affine Markovian case.

Theorem 3.5.7. *Let $0 \le t < T \le T^\star$. Set $D(t,T) := \int_t^T e^{-\int_t^s a(u)du} ds$, $\bar{D}(t,T) := \int_t^T e^{-\int_t^s \bar{a}(u)du} ds$ and assume that $D(\cdot,T)\sigma(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot) \in L^2([t,T])$. Then*

$$\overline{B}(t,T) \;=\; \mathbf{1}_{\{\tau>t\}} e^{-A(t,T)-D(t,T)r(t)-\bar{D}(t,T)\lambda(t)}, \tag{3.53}$$

where

$$\begin{aligned} A(t,T) &= V(t,T) - W(t,T) + \int_t^T \big(D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v)\big)dv \\ &\quad + \int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v). \end{aligned}$$

Here $V(t,T)$, $W(t,T)$ are given in Theorem 3.4.10 and $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ are as in (1.20) with $c(\cdot) = D(\cdot,T)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,T)\bar{\sigma}(\cdot)$. Furthermore $\log(\overline{B}(t,T))$ is normally distributed with

$$\begin{aligned} E[\log(\overline{B}(t,T))] &= -D(t,T)e^{-\int_0^t a(u)du} r(0) - \bar{D}(t,T)e^{-\int_0^t \bar{a}(u)du}\lambda(0) \\ &\quad -D(t,T)\int_0^t e^{-\int_v^t a(u)du}k(v)dv - \bar{D}(t,T)\int_0^t e^{-\int_v^t \bar{a}(u)du}\bar{k}(v)dv \\ &\quad -\int_t^T (D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v))dv - V(t,T) + W(t,T)\,, \\ \mathrm{Var}(\log(\overline{B}(t,T)) &= \left\| \left(\Psi_c^\kappa(t,T,\cdot) + D(t,T)e^{-\int_\cdot^t a(u)du}\sigma(\cdot)\right)\mathbf{1}_{[0,t]}(\cdot)\right\|_{\kappa,\infty}^2 \\ &\quad +2\Big\langle \left(\Psi_c^\kappa(t,T,\cdot) + D(t,T)e^{-\int_\cdot^t a(u)du}\sigma(\cdot)\right)\mathbf{1}_{[0,t]}(\cdot), \\ &\quad \left(\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot) + \bar{D}(t,T)e^{-\int_\cdot^t \bar{a}(u)du}\bar{\sigma}(\cdot)\right)\mathbf{1}_{[0,t]}(\cdot)\Big\rangle_{\kappa,\bar{\kappa},\infty} \\ &\quad + \left\| \left(\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot) + \bar{D}(t,T)e^{-\int_\cdot^t \bar{a}(u)du}\bar{\sigma}(\cdot)\right)\mathbf{1}_{[0,t]}(\cdot)\right\|_{\bar{\kappa},\infty}^2. \end{aligned}$$

Proof. The case $t = 0$ is trivial by (3.50), so let $t > 0$. We obtain from Lemma 3.5.2 and Fubini's Theorem (Theorem 1 of Krvavich and Mishura [83])

$$\begin{aligned} &\int_t^T (r(s) + \lambda(s))ds \\ =\; &\int_t^T \Big[r(t)e^{-\int_t^s a(u)du} + \int_t^s e^{-\int_v^s a(u)du}k(v)dv + \int_t^s e^{-\int_v^s a(u)du}\sigma(v)dB^\kappa(v)\Big]ds \\ &+\int_t^T \Big[\lambda(t)e^{-\int_t^s \bar{a}(u)du} + \int_t^s e^{-\int_v^s \bar{a}(u)du}\bar{k}(v)dv + \int_t^s e^{-\int_v^s \bar{a}(u)du}\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\Big]ds \\ =\; &D(t,T)r(t) + \int_t^T D(v,T)k(v)dv + \int_t^T D(v,T)\sigma(v)dB^\kappa(v) \\ &+\bar{D}(t,T)\lambda(t) + \int_t^T \bar{D}(v,T)\bar{k}(v)dv + \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v). \end{aligned} \tag{3.54}$$

By Theorem 3.4.10 we have

$$\begin{aligned} & E\left[\exp\left\{\int_t^T D(v,T)\sigma(v)dB^\kappa(v) + \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Big|\mathcal{G}_t\right] \\ = \; & e^{W(t,T)-V(t,T)}\exp\left\{\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\}. \end{aligned}$$

Now we get for the price of the defaultable zero coupon bond by

$$\begin{aligned} \overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_0^T(r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\ &= \mathbf{1}_{\{\tau>t\}}e^{-D(t,T)r(t)-\int_t^T D(v,T)k(v)dv-\bar{D}(t,T)\lambda(t)-\int_t^T \bar{D}(v,T)\bar{k}(v)dv} \\ &\quad \times E\left[\exp\left\{-\int_t^T D(v,T)\sigma(v)dB^\kappa(v) - \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Big|\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}e^{-D(t,T)r(t)-\int_t^T D(v,T)k(v)dv-\bar{D}(t,T)\lambda(t)-\int_t^T \bar{D}(v,T)\bar{k}(v)dv} \\ &\quad \times e^{W(t,T)-V(t,T)}\exp\left\{-\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) - \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\} \\ &= \mathbf{1}_{\{\tau>t\}}e^{-A(t,T)-D(t,T)r(t)-\bar{D}(t,T)\lambda(t)} \end{aligned}$$

with $A(t,T)$, c and $\bar{c}$ as given in the assertion. The formulas for the expectation and variance of $\log(\overline{B}(t,T))$ can be obtained by simple calculations. □

Remark 3.5.8. If we compare (3.53) with Proposition 7.2 of Schönbucher [122], we realize that in the case $t = 0$ the zero coupon bond prices differ only by a deterministic factor. However, if we calculate the price at time $t > 0$, the whole paths of the fractional Brownian motions up to time t enter because of the dependent increments. Those integrals do not appear in a Markovian model.

By Proposition 3.5.3 we rewrite the bond price in terms of r and λ.

Corollary 3.5.9. *In the situation of Theorem 3.5.7 we have for* $0 \le t < T \le T^\star$

$$\begin{aligned} \overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}}\exp\Big\{-\widetilde{A}(t,T) - D(t,T)r(t) - \bar{D}(t,T)\lambda(t)\Big\} \\ &\quad \times\exp\Big\{-\int_0^t(\Psi_c^\kappa(t,T,v)\frac{a(t)}{\sigma(t)}r(t) + \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)\frac{\bar{a}(t)}{\bar{\sigma}(v)}\lambda(t))dv\Big\} \\ &\quad \times\exp\Big\{-\int_0^t \Psi_c^\kappa(t,T,v)\frac{1}{\sigma(v)}dr(v) - \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)\frac{1}{\bar{\sigma}(v)}d\lambda(v)\Big\} \end{aligned}$$

where $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ *are as in* (1.20) *with* $c(\cdot) = D(\cdot,T)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,T)\bar{\sigma}(\cdot)$ *and*

$$\widetilde{A}(t,T) = V(t,T) - W(t,T) + \int_t^T \big(D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v)\big)dv$$

with $W(t,T)$ *and* $V(t,T)$ *as in Theorem 3.4.10.*

3.5.3 Option pricing

Now we explain how derivatives prices can be calculated. First we aim for a European call price with a defaultable zero coupon bond as underlying. Today's price can be found similar to the classical Brownian case and a closed formula is obtained. For more general options and times, Fourier techniques can be applied and we show, how to do this.

In Theorem 3.5.11 below we will price a European call option invoking a change of numéraire. Therefore, we need a Girsanov theorem. For the elementary case, where we the drift of a fBm is changed by a deterministic factor, the measure change has been derived in Norros, Valkeila and Virtamo [99], Theorem 4.1, using pathwise integration. In our case we need some result for the other direction. We need to know the distribution of a fBm after a given measure change. Theorem 3.3 of Duncan et al. [48] considers a general situation, which we can use. Moreover, their result also covers the result of Norros, Valkeila and Virtamo [99].

Proposition 3.5.10. *Let $0 \leq t < T < S \leq T^\star$. Consider a European call at strike $K > 0$ and maturity T based on a defaultable zero coupon bond maturing at time S as underlying given by the contingent claim*

$$\mathbf{1}_{\{\tau > T\}}(\overline{B}(T,S) - K)_+.$$

At time t the price $\mathcal{V}(t,T,S)$ is given by

$$\begin{aligned} \mathcal{V}(t,T,S) &= \mathbf{1}_{\{\tau > t\}} E\left[e^{-\int_t^T (r(s)+\lambda(s))ds} (\overline{B}(T,S) - K)_+ \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau > t\}} \overline{B}(t,T) E^T \left[(\overline{B}(T,S) - K)_+ \middle| \mathcal{G}_t \right], \end{aligned} \tag{3.55}$$

where E^T is the expectation with respect to the T-forward measure defined by the Radon-Nikodym derivative

$$\frac{d\mathcal{Q}^T}{d\mathcal{Q}} = \exp\left\{ -\int_0^T (r(s) + \lambda(s))ds \right\} e^{-\overline{B}(0,T)}. \tag{3.56}$$

Proof. The first equality follows by (3.52). As in the classical Brownian motion case we calculate the European call price by means of a T-forward measure (using the expressions defined in Theorem 3.5.7)

$$\begin{aligned} \frac{d\mathcal{Q}^T}{d\mathcal{Q}} &= \exp\left\{ -\int_0^T (r(s) + \lambda(s))ds \right\} e^{-\overline{B}(0,T)} \\ &= \exp\left\{ -\int_0^T D(v,T)\sigma(v)dB^\kappa(v) - \int_0^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v) - W(0,T) \right\} \end{aligned}$$

Using Bayes' theorem for conditional expectations we obtain (3.55). □

Denote by N the standard normal distribution function.

Theorem 3.5.11. *Let $0 < T < S \leq T^\star$. At time 0 the price $\mathcal{V}(0,T,S)$ of a European call at strike $K > 0$ and maturity T based on a defaultable zero coupon bond maturing at time S as underlying is given by*

$$\begin{aligned}\mathcal{V}(0,T,S) &= \overline{B}(0,T)\\ &\times \left\{ e^{\frac{\Sigma(0,T,S)^2}{2} - A(0,T,S)} N\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} + \Sigma(0,T,S)\right)\right.\\ &\left. -KN\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)}\right)\right\}\end{aligned}$$

with

$$\begin{aligned}A(0,T,S) &= V(T,S) - W(T,S) + \int_T^S \big(D(v,S)k(v) + \bar{D}(v,S)\bar{k}(v)\big)dv\\ &+D(T,S)\left(r(0)e^{-\int_0^T a(u)du} + \int_0^T e^{-\int_v^T a(u)du}k(v)dv\right)\\ &+\bar{D}(T,S)\left(\lambda(0)e^{-\int_0^T \bar{a}(u)du} + \int_0^T e^{-\int_v^T \bar{a}(u)du}\bar{k}(v)dv\right)\\ &-\left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\infty}\\ &-\left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty}\\ &-\left\langle \bar{\Phi}(\cdot)\mathbf{1}_{[0,T]}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty}\\ &-\left\langle \bar{\Phi}(\cdot)\mathbf{1}_{[0,T]}(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\bar{\kappa},\infty}\end{aligned}$$

where $V(T,S), W(T,S)$ are as in Theorem 3.4.10. Furthermore,

$$\begin{aligned}\Sigma(0,T,S)^2 &= Var\left(-\int_0^T \Phi(v)dB^\kappa(v) - \int_0^T \bar{\Phi}(v)d\bar{B}^{\bar{\kappa}}(v)\right)\\ &= \left\|\mathbf{1}_{[0,T]}(\cdot)\Phi(\cdot)\right\|^2_{\kappa,\infty} + 2\left\langle \mathbf{1}_{[0,T]}(\cdot)\Phi(\cdot), \mathbf{1}_{[0,T]}(\cdot)\bar{\Phi}(\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty}\\ &\quad + \left\|\mathbf{1}_{[0,T]}(\cdot)\bar{\Phi}(\cdot)\right\|^2_{\bar{\kappa},\infty}. \qquad (3.57)\end{aligned}$$

Here we have set

$$\begin{aligned}\Phi(\cdot) &:= \Psi^\kappa_c(S,T,\cdot) + D(T,S)e^{-\int_\cdot^S a(u)du}\sigma(\cdot) \quad \textit{and}\\ \bar{\Phi}(\cdot) &:= \Psi^{\bar{\kappa}}_{\bar{c}}(S,T,\cdot) + \bar{D}(T,S)e^{-\int_\cdot^S \bar{a}(u)du}\bar{\sigma}(\cdot), \qquad (3.58)\end{aligned}$$

where $\Psi^\kappa_c(S,T,\cdot)$, $\Psi^{\bar{\kappa}}_{\bar{c}}(S,T,\cdot)$ are as in (1.20) with $c(\cdot) = D(\cdot,S)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,S)\bar{\sigma}(\cdot)$.

Proof. W.l.o.g. assume $\bar{\kappa} \geq \kappa$. Recall $\overline{B}(S,T)$ from Theorem 3.5.7. We replace $r(S)$ and $\lambda(S)$ as in the proof of Theorem 3.5.7 by the solutions to the sde's given in Lemma 3.5.2.

Then we collect those terms, which are deterministic and those, which are not. This yields the following definition of a function F on the paths of the fBm B^κ as

$$F(B^\kappa) := \left(\exp\left\{-\overline{A}(0,T,S) - \int_{\mathbb{R}} \left(\Phi(v)\mathbf{1}_{[0,T]}(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[0,T]}(\cdot)\bar\Phi(\cdot))(v)\right) dB^\kappa(v)\right\} - K\right)_+$$

with Φ and $\bar\Phi$ as in (3.58) and

$$\begin{aligned}\overline{A}(0,T,S) &= V(T,S) - W(T,S) + \int_T^S \big(D(v,S)k(v) + \bar D(v,S)\bar k(v)\big)dv \\ &\quad + D(T,S)\left(r(0)e^{-\int_0^T a(u)du} + \int_0^T e^{-\int_v^T a(u)du}k(v)dv\right) \\ &\quad + \bar D(T,S)\left(\lambda(0)e^{-\int_0^T \bar a(u)du} + \int_0^T e^{-\int_v^T \bar a(u)du}\bar k(v)dv\right)\end{aligned}$$

where $V(T,S), W(T,S)$ are as in Theorem 3.4.10. Starting with (3.55) from Proposition 3.5.10 we obtain

$$\begin{aligned}\mathcal{V}(0,T,S) &= \overline{B}(0,T)E^T[(\overline{B}(T,S)-K)_+] \\ &= \overline{B}(0,T)E^T\left[F(B^\kappa)\right] \\ &= \overline{B}(0,T)E\left[F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dv\right)ds\right]\end{aligned}$$

with

$$\Upsilon(v) := -\left(D(v,T)\sigma(v)\mathbf{1}_{[0,T]}(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[0,T]}(\cdot)\bar D(\cdot,T)\bar\sigma(\cdot))(v)\right).$$

For the last equality we applied Theorem 3.3 of Duncan et al. [48] to calculate the expectation under the T-forward measure $\mathcal{Q}^T$. (In fact, we have to extend their result to Wick exponentials defined on the whole of $\mathbb{R}$ as in (3.38).) We further calculate

$$\begin{aligned}&F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dvds\right) \\ =\ &\left(\exp\left\{-\overline{A}(0,T,S) - \int_{\mathbb{R}}\left(\Phi(v)\mathbf{1}_{[0,T]}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[0,T]}(\cdot)\bar\Phi(\cdot))(v)\right)dB^\kappa(v)\right.\right. \\ &\left.\left. - \left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[0,T]}(\cdot)\bar\Phi(\cdot))(\cdot), \Upsilon(\cdot)\right\rangle_{\kappa,\infty}\right\} - K\right)_+.\end{aligned}$$

With Lemma 3.4.9 we can show that

$$\begin{aligned}
& -\left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{[0,T]}(\cdot)\bar\Phi(\cdot))(\cdot), \Upsilon(\cdot)\right\rangle_{\kappa,\infty} \\
= \; & \left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\infty} \\
& + \left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\bar\kappa,\infty} \\
& + \left\langle \bar\Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\bar\kappa,\infty} \\
& + \left\langle \bar\Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\bar\kappa,\infty}.
\end{aligned}$$

Collecting all terms and transforming the integral back we finally arrive at

$$\begin{aligned}
& F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\left(\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dv\right)ds\right) \\
= \; & \left(\exp\left\{-A(0,T,S) - \int_0^T \Phi(v)dB^\kappa(v) - \int_0^T \bar\Phi(v)d\bar B^{\bar\kappa}(v)\right\} - K\right)_+,
\end{aligned}$$

where

$$\begin{aligned}
& A(0,T,S) \\
:= \; & \overline{A}(0,T,S) - \left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\infty} \\
& - \left\langle \Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\bar\kappa,\infty} \\
& - \left\langle \bar\Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\kappa,\bar\kappa,\infty} \\
& - \left\langle \bar\Phi(\cdot)\mathbf{1}_{[0,T]}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{[0,T]}(\cdot)\right\rangle_{\bar\kappa,\infty}.
\end{aligned}$$

Finally, we can calculate the expectation in the pricing formula. This works now exactly as in the case of the classical Black-Scholes setting, since the appearing integrals are Gaussian. This results in

$$\begin{aligned}
\mathcal{V}(0,T,S) \; &= \; \overline{B}(0,T) \\
& \times E\left[\left(\exp\left\{-A(0,T,S) - \int_0^T \Phi(v)dB^\kappa(v) - \int_0^T \bar\Phi(v)d\bar B^{\bar\kappa}(v)\right\} - K\right)_+\right] \\
= \; & \overline{B}(0,T) \\
& \times e^{-A(0,T,S)}E\left[\left(\exp\left\{-\int_0^T \Phi(v)dB^\kappa(v) - \int_0^T \bar\Phi(v)d\bar B^{\bar\kappa}(v)\right\} - e^{A(0,T,S)}K\right)_+\right] \\
= \; & \overline{B}(0,T) \times \left\{e^{\frac{\Sigma(0,T,S)^2}{2} - A(0,T,S)}N\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} + \Sigma(0,T,S)\right)\right. \\
& \left. - KN\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)}\right)\right\}
\end{aligned}$$

where $\Sigma(0,T,S)^2$ is defined in (3.57). The expression for the variance can be deduced by calculating the characteristic function analogously to the moment generating function

in Theorem 3.4.10, then apply Lemma 3.4.9 to rewrite the appearing norms and scalar products. □

Remark 3.5.12. We want to compare the price of Theorem 3.5.11 to the European call price in a classical Brownian Vasicek model. For simplicity we choose a model with constant coefficient functions. Given two dependent standard Brownian motions $B, \bar{B}$ with correlation $\rho > 0$, we model the short and hazard rate by the sde's

$$\begin{aligned} dr(t) &= (k - ar(t))dt + \sigma dB(t), \quad r(0) = r_0 \in \mathbb{R}, \\ d\lambda(t) &= (\bar{k} - \bar{a}\lambda(t))dt + \bar{\sigma} d\bar{B}(t), \quad \lambda(0) = \lambda_0 \in \mathbb{R}, \end{aligned}$$

where we will assume that $\sigma, \bar{\sigma} > 0$. We know by Proposition 5.3 of Schönbucher [122] that this model eventually boils down to a two-factor short rate model. Using for example Theorem 4.2.1 of Brigo and Mercurio [24], today's price of the defaultable zero coupon bond is given by

$$\begin{aligned} \overline{B}(0,T) &= \exp\left\{ -A(0,T) - \frac{k}{a}\left[T - \frac{e^{-aT}-1}{a}\right] - \frac{\bar{k}}{\bar{a}}\left[T - \frac{e^{-\bar{a}T}-1}{\bar{a}}\right] \right. \\ &\quad \left. - \frac{1-e^{-aT}}{a} r_0 - \frac{1-e^{-\bar{a}T}}{\bar{a}\lambda_0} \right\} \end{aligned}$$

with

$$\begin{aligned} A(0,T) &= -\frac{1}{2}\left(\frac{\sigma^2}{a^2}\left[T + \frac{2}{a}e^{-aT} - \frac{1}{2a}e^{-2aT} - \frac{3}{2a}\right] \right. \\ &\quad + \frac{\bar{\sigma}^2}{\bar{a}^2}\left[T + \frac{2}{\bar{a}}e^{-\bar{a}T} - \frac{1}{2\bar{a}}e^{-2\bar{a}T} - \frac{3}{2\bar{a}}\right] \\ &\quad \left. + 2\rho\frac{\sigma\bar{\sigma}}{a\bar{a}}\left[T + \frac{e^{-aT}-1}{a} + \frac{e^{-\bar{a}T}-1}{\bar{a}} - \frac{e^{-(a+\bar{a})T}-1}{a+\bar{a}}\right]\right). \end{aligned}$$

Let $0 \le T \le S \le T^\star$. Applying Theorem 4.2.2 of Brigo and Mercurio [24] we get for the price $\mathcal{V}(0,T,S)$ of a call option with maturity T and strike K, written on a defaultable zero coupon bond maturing at time S:

$$\begin{aligned} &\mathcal{V}(0,T,S) \\ &= \overline{B}(0,S)N\left(\frac{\log\left(\frac{\overline{B}(0,S)}{K\overline{B}(0,T)}\right)}{\Sigma(0,T,S)} + \frac{1}{2}\Sigma(0,T,S)\right) \\ &\quad - \overline{B}(0,T)KN\left(\frac{\log\left(\frac{\overline{B}(0,S)}{K\overline{B}(0,T)}\right)}{\Sigma(0,T,S)} - \frac{1}{2}\Sigma(0,T,S)\right), \end{aligned}$$

where

$$\begin{aligned}\Sigma^2(0,T,S) &= \frac{\sigma^2}{2a^3}\Big(1-e^{-a(S-T)}\Big)^2\Big(1-e^{-2a(T-t)}\Big)\\ &\quad+\frac{\bar\sigma^2}{2\bar a^3}\Big(1-e^{-\bar a(S-T)}\Big)^2\Big(1-e^{-2\bar a(T-t)}\Big)\\ &\quad+2\rho\frac{\sigma\bar\sigma}{a\bar a(a+\bar a)}\Big(1-e^{-a(S-T)}\Big)\Big(1-e^{-\bar a(S-T)}\Big)^2\Big(1-e^{-(a+\bar a)(T-t)}\Big).\end{aligned}$$

Note now that the main structure of bond and call prices is the same in both models, especially today's bond prices differ only by a deterministic multiplicative factor; however, if we look further "into the future" the path of the fBm does matter, which results in a more complex option price.

We want to emphasize that we have in the situation of Theorem 3.5.11

$$\overline{B}(0,T)e^{\frac{\Sigma(0,T,S)^2}{2}-A(0,T,S)} \neq \overline{B}(0,S)$$

and, therefore, cannot get exactly the same structure as in the Brownian case.

Numerical evaluations of the formulas in the fractional case are significantly more complicated than in the classical Brownian model. Especially calculating the norms $\|\cdot\|_{\kappa,\infty}$ is challenging due to the singularity of the weight function $(x,y)\mapsto|x-y|^{2\kappa-1}$ on the diagonal.

The following pricing method allows for more general payoff functions, but it is less explicit. Note that it also includes the European call price calculated explicitly in Theorem 3.5.11.

Theorem 3.5.13. *Let $0\le t<T\le T^\star$. Denote by X an $\mathcal{F}_T$-measurable payoff of the form*

$$X=\mathbf{1}_{\{\tau>T\}}f\left(\int_0^T\phi(s)dB^\kappa(s)+\int_0^T\bar\phi(s)d\bar B^{\bar\kappa}(s)\right)$$

for some $f:\mathbb{R}\to\mathbb{R}$ and $\phi,\bar\phi\in L^2([0,T])$. Assume further that there exist $b>0$ and $z\in\mathbb{R}$ such that $f_+^{b,z}(\cdot):=e^{-b\cdot}f(\cdot)\mathbf{1}_{[z,\infty)}(\cdot)$, $f_-^{b,z}(\cdot):=e^{b\cdot}f(\cdot)\mathbf{1}_{(-\infty,z)}(\cdot)$ and their Fourier transforms $\widehat{f}_+^{b,z}(\cdot)$, $\widehat{f}_-^{b,z}(\cdot)$ are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Define for $\xi\in\mathbb{R}$ and $\star\in\{+,-\}$

$$\Phi^{\xi,\star}(\cdot):=D(\cdot,T)\sigma(\cdot)-(i\xi\star b)\phi(\cdot),\quad \bar\Phi^{\xi,\star}(\cdot):=\bar D(\cdot,T)\bar\sigma(\cdot)-(i\xi\star b)\bar\phi(\cdot).\tag{3.59}$$

Then the price of X at time t is given by

$$\begin{aligned}
&\mathcal{V}(t,T) = \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T (r(s)+\lambda(s))ds} X \middle| \mathcal{G}_t\right] \\
= \;& \mathbf{1}_{\{\tau>t\}} \exp\left\{-\int_t^T D(s,T)k(s)ds - \int_t^T \bar{D}(s,T)\bar{k}(s)ds - D(t,T)r(t) - \bar{D}(t,T)\lambda(t)\right\} \\
&\times \int_{\mathbb{R}} \left[\exp\left\{V^\xi(t,T) - W^\xi(t,T) - \int_0^t \Psi^\kappa_{c^+_\xi}(t,T,v)dB^\kappa(v) - \int_0^t \Psi^{\bar{\kappa}}_{\bar{c}^+_\xi}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\} \widehat{f}^{b,z}_+(\xi)\right. \\
&\left. + \exp\left\{V^\xi(t,T) - W^\xi(t,T) - \int_0^t \Psi^\kappa_{c^-_\xi}(t,T,v)dB^\kappa(v) - \int_0^t \Psi^{\bar{\kappa}}_{\bar{c}^-_\xi}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\} \widehat{f}^{b,z}_-(\xi)\right] d\xi
\end{aligned} \tag{3.60}$$

where $c^\star_\xi(\cdot) = \Phi^{\xi,\star}(\cdot)$ and $\bar{c}^\star_\xi(\cdot) = \bar{\Phi}^{\xi,\star}(\cdot)$,

$$\begin{aligned}
W^{\xi,\star}(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{[t,T]}(\cdot)\Phi^{\xi,\star}(\cdot)\right\|^2_{\kappa,\infty} + 2\left\langle \mathbf{1}_{[t,T]}(\cdot)\Phi^{\xi,\star}(\cdot), \mathbf{1}_{[t,T]}(\cdot)\bar{\Phi}^{\xi,\star}(\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty}\right. \\
&\quad \left. + \left\|\mathbf{1}_{[t,T]}(\cdot)\bar{\Phi}^{\xi,\star}(\cdot)\right\|^2_{\bar{\kappa},\infty}\right), \\
V^{\xi,\star}(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{[0,t]}(\cdot)\Psi^\kappa_{c^\star_\xi}(t,T,\cdot)\right\|^2_{\kappa,\infty} + 2\left\langle \mathbf{1}_{[0,t]}(\cdot)\Psi^\kappa_{c^\star_\xi}(t,T,\cdot), \mathbf{1}_{[0,t]}(\cdot)\Psi^{\bar{\kappa}}_{\bar{c}^\star_\xi}(t,T,\cdot)\right\rangle_{\kappa,\bar{\kappa},\infty}\right. \\
&\quad \left. + \left\|\mathbf{1}_{[0,t]}(\cdot)\Psi^{\bar{\kappa}}_{\bar{c}^\star_\xi}(t,T,\cdot)\right\|^2_{\bar{\kappa},\infty}\right),
\end{aligned} \tag{3.61}$$

with $\widehat{f}^{b,z}_+$ and $\widehat{f}^{b,z}_-$ the Fourier transforms of $f^{b,z}_+$ and $f^{b,z}_-$ respectively.

Proof. Applying - as in the theorem before - equation (3.52) we obtain the first equality in (3.60). For some $a < 0$ and $z \in \mathbb{R}$ we have

$$\begin{aligned}
f(x) &= e^{bx}[e^{-bx}f(x)\mathbf{1}_{[z,\infty)}(x)] + e^{-bx}[e^{bx}f(x)\mathbf{1}_{(-\infty,z)}(x)] \\
&=: e^{bx}f^{b,z}_+(x) + e^{-bx}f^{b,z}_-(x).
\end{aligned} \tag{3.62}$$

Denote by $\widehat{f}^{b,z}_+$ and $\widehat{f}^{b,z}_-$ the Fourier transforms of $f^{b,z}_+$ and $f^{b,z}_-$ respectively. Using classical Fourier analysis we obtain for $\xi, x \in \mathbb{R}$ and $\star \in \{+,-\}$

$$\widehat{f}^{b,z}_\star(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} f^{b,z}_\star(x)dx, \quad f^{b,z}_\star(x) = \int_{\mathbb{R}} e^{i\xi x}\widehat{f}^{b,z}_\star(\xi)d\xi,$$

where we used the fact that $f^{b,z}_+$ and $f^{b,z}_-$ are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Set

$$J(t,T) := \int_t^T \phi(s)dB^\kappa(s) + \int_t^T \bar{\phi}(s)d\bar{B}^{\bar{\kappa}}(s).$$

We get by the definition and (3.62)

$$\begin{aligned}
X &= f\left(J(0,T)\right) \\
&= e^{bJ(0,T)}f^{b,z}_+\left(J(0,T)\right) + e^{-bJ(0,T)}f^{b,z}_-\left(J(0,T)\right) \\
&= \int_{\mathbb{R}} \left(e^{(i\xi+b)J(0,T)}\widehat{f}^{b,z}_+(\xi) + e^{(i\xi-b)J(0,T)}\widehat{f}^{b,z}_-(\xi)\right) d\xi
\end{aligned}$$

Since by Gaussianity $E[e^{bJ(0,T)}] < \infty$ for all $b \in \mathbb{R}$, we can interchange expectation and integration as follows using (3.54)

$$\begin{aligned}
& \mathcal{V}(t,T) \\
= \; & \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T (r(s)+\lambda(s))ds} X \middle| \mathcal{G}_t \right] \\
= \; & \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T (r(s)+\lambda(s))ds} \int_{\mathbb{R}} \left[e^{(i\xi+b)J(0,T)} \widehat{f}_+^{b,z}(\xi) + e^{(i\xi-b)J(0,T)} \widehat{f}_-^{b,z}(\xi) \right] d\xi \middle| \mathcal{G}_t \right] \\
= \; & \mathbf{1}_{\{\tau>t\}} e^{C(t,T)+D(t,T)r(t)+\bar{D}(t,T)\lambda(t)} \\
& \times \int_{\mathbb{R}} \left[E\left[e^{G(t,T)+(i\xi+b)J(0,T)} \middle| \mathcal{G}_t \right] \widehat{f}_+^{b,z}(\xi) + E\left[e^{G(t,T)+(i\xi-b)J(0,T)} \middle| \mathcal{G}_t \right] \widehat{f}_-^{b,z}(\xi) \right] d\xi
\end{aligned}$$

with

$$C(t,T) := -\int_t^T D(v,T)k(v)dv - \int_t^T \bar{D}(v,T)\bar{k}(v)dv$$

and

$$G(t,T) := -\int_t^T D(s,T)\sigma(s)dB^{\kappa}(s) - \int_t^T \bar{D}(s,T)\bar{\sigma}(s)d\bar{B}^{\bar{\kappa}}(s).$$

The case $t = 0$ is again simple because we just need to calculate the expectations. Let further be $t > 0$. Prediction works now the same way as in Theorem 3.4.10 and we obtain for $\star \in \{+,-\}$ with $\Phi^{\xi,\star}$ and $\bar{\Phi}^{\xi,\star}$ as in (3.59):

$$\begin{aligned}
& E\left[e^{G(t,T)+(i\xi\star b)J(0,T)} \middle| \mathcal{G}_t \right] = e^{(i\xi\star b)J(0,t)} E\left[e^{G(t,T)+(i\xi\star b)J(t,T)} \middle| \mathcal{G}_t \right] \\
= \; & e^{(i\xi\star b)J(0,t)} E\left[e^{-\int_t^T (D(s,T)\sigma(s)-(i\xi\star b)\phi(s))dB^{\kappa}(s) - \int_t^T (\bar{D}(s,T)\bar{\sigma}(s)-(i\xi\star b)\bar{\phi}(s))d\bar{B}^{\bar{\kappa}}(s)} \middle| \mathcal{G}_t \right] \\
= \; & e^{(i\xi\star b)J(0,t)} E\left[e^{-\int_t^T \Phi^{\xi,\star}(s)dB^{\kappa}(s) - \int_t^T \bar{\Phi}^{\xi,\star}(s)d\bar{B}^{\bar{\kappa}}(s)} \middle| \mathcal{G}_t \right] \\
= \; & e^{(i\xi\star b)J(0,t)+V^{\xi,\star}(t,T)-W^{\xi,\star}(t,T)} \exp\left\{ -\int_0^t \Psi^{\kappa}_{c^{\star}_{\xi}}(t,T,v)dB^{\kappa}(v) - \int_0^t \Psi^{\bar{\kappa}}_{\bar{c}^{\star}_{\xi}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v) \right\}
\end{aligned}$$

where $c^{\star}_{\xi}(\cdot) = \Phi^{\xi,\star}(\cdot)$ and $\bar{c}^{\star}_{\xi}(\cdot) = \bar{\Phi}^{\xi,\star}(\cdot)$ and $W^{\xi,\star}(t,T)$, $V^{\xi,\star}(t,T)$ are as in (3.61). $\square$

Chapter 4

Molchan-Golosov fractional Lévy processes

In this chapter we introduce the class of (multivariate) Molchan-Golosov fractional Lévy processes (MG-fLps), including fractional Brownian motion and fractional subordinators (as defined in Bender and Marquardt [16]) by a Molchan-Golosov transformation (cf. Molchan and Golosov [95]) based on general finite second moment Lévy processes. This idea has been proposed by Tikanmäki and Mishura [127], who however considered only the univariate case and used centered, i.e. zero-mean, driving Lévy processes without Brownian parts, basically excluding fBm and fractional subordinators, cf. Definition 1.6.6.

We calculate the conditional characteristic functions of MG-fLp-related processes using fractional calculus and results on infinitely divisible distributions. Important examples like fractional Lévy Ornstein-Uhlenbeck processes or Cox-Ingersoll-Ross processes are considered.

As a first application we propose a credit market similar to Section 3.5. However the Vasicek sde's describing the dynamics of the short and default rate will be driven by dependent MG-fLps which will also rule out the problem of potential negative paths. The model will still allow for fairly explicit calculations of zero coupon bond prices.

In a second application we use our prediction results to calculate the prices of European call options in the framework of Bender and Marquardt [16] who introduced a Black-Scholes stock market time changed by convoluted Lévy processes.

4.1 Multivariate Molchan-Golosov fractional Lévy processes

As already mentioned in Remark 1.6.9, we will generalize the concept introduced in the work of Tikanmäki and Mishura [127], cf. Definition 1.6.6, to the multivariate case. Therefore, our processes will also be defined by a compactly supported Molchan-Golosov transformation (cf. Molchan and Golosov [95]). However, in contrast to [127] we will also allow for Brownian components which can result in fractional Brownian motions, and non-centered driving Lévy processes, possibly leading to fractional subordinators as introduced by Bender and Marquardt [16].

4.1.1 Definition

Recall from Section 1.4 that we will only consider Lévy processes with existing second moments. The very general definition of MG-fLps follows.

Definition 4.1.1. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, we define the kernel function $z^d : [0,T] \times [0,T] \to \mathbb{R}^{n\times n}$ by the diagonal matrix*

$$z^d(t,s) := \mathbf{1}_{\{s\le t\}} \begin{pmatrix} c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)}\mathbf{1}_{[0,t)}(\cdot))(s) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)}\mathbf{1}_{[0,t)}(\cdot))(s) \end{pmatrix}$$

where for $1 \le j \le n$

$$c_{d(j)} = \Big(\frac{(2d(j)+1)\Gamma(d(j)+1)\Gamma(1-d(j))}{\Gamma(1-2d(j))}\Big)^{\frac{1}{2}}.$$

Then a Molchan-Golosov fractional Lévy process (MG-fLp)

$$\boldsymbol{L}^d = (\boldsymbol{L}^d(t))_{t\in[0,T]} = (L^{d(1)}(t), \dots, L^{d(n)}(t))^\top_{t\in[0,T]}$$

is defined by

$$\boldsymbol{L}^d(t) = \int_0^t z^d(t,s)d\boldsymbol{L}(s), \quad t \in [0,T]. \tag{4.1}$$

Remark 4.1.2. The integral in (4.1) can be considered in the $L^2(\Omega)$- or in the pathwise Riemann-Stieltjes-sense. The first assertion is clear by Rajput and Rosinski [109] and the fact that the kernel is square-integrable. The second one follows because as a finite second moment Lévy process, $\mathbf{L}$ is of bounded p-variation for all $p > 2$ (cf. Monroe [96], Theorem 2, based on the Blumenthal-Getoor-index introduced in Blumenthal and

Getoor [23]) and $z^d(t,s)$ is on $(0,t)$ componentwise of bounded variation in the variable s, cf. Young [134].

Remark 4.1.3. Clearly $\mathbf{L}^d$ is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Further we want to remark that this is not the case for MVN-fLps as in Definition 1.6.1. However in contrast to those processes, MG-fLps do not have stationary increments in general, cf. Proposition 3.11 of Tikanmäki and Mishura [127].

Using the Lévy-Itô decomposition we obtain

$$\begin{aligned}\mathbf{L}(t) &= E[\mathbf{L}(t)] + (\mathbf{L}(t) - E[\mathbf{L}(t)]) \\ &\stackrel{d}{=} E[\mathbf{L}(1)]\cdot t + \mathbf{B}(t) + \mathbf{S}(t), \quad t\in[0,T],\end{aligned}$$

where $\mathbf{B}$ is a n-dimensional Brownian motion with $\mathbf{B}(1)\sim N(0,\Sigma)$, $\Sigma\in\mathbb{S}^{n\times n}$, and $\mathbf{S}$ is a zero-mean Lévy process without Brownian part. Furthermore $\mathbf{B}$ and $\mathbf{S}$ are independent. This leads to the MG-fLp decomposition

$$\mathbf{L}^d(t) \stackrel{d}{=} \int_0^t z^d(t,s)ds\cdot E[\mathbf{L}(1)] + \mathbf{B}^d(t) + \mathbf{S}^d(t), \quad t\in[0,T], \tag{4.2}$$

where the first integral is understood componentwise. $\mathbf{B}^d$ and $\mathbf{S}^d$ are defined as in (4.1), i.e. are a multivariate fBm and a zero-mean MG-fLp.

Example 4.1.4. Let $n=1$ in Definition 4.1.1.

(i) Choosing as driving Lévy process a standard Brownian motion, we get a classical fBm on $[0,T]$, cf. Proposition 1.5.10 or Samorodnitsky and Taqqu [119].

(ii) Taking a strictly increasing subordinator as driving Lévy process leads to a fractional subordinator in the sense of Example 1 of Bender and Marquardt [16]. In particular, the resulting MG-fLp is a.s. increasing.

The next results follow by standard properties of Lévy processes, see e.g. Rajput and Rosinski [109], Marcus and Rosinski [91] or Sato [120]. A brief look at the autocovariance of a MG-fLp leads to the following proposition.

Proposition 4.1.5. *For $s,t\in[0,T]$ we have for the mean-value and autocovariance function*

(i) $E[\boldsymbol{L}^d(t)] = \int_0^t z^d(t,s)ds\cdot E[\boldsymbol{L}(1)]$.

(ii) $Cov[\boldsymbol{L}^d(t), \boldsymbol{L}^d(s)]$
$= \frac{1}{2}\left(c_{d(i),d(j)} Cov[L^i(1), L^j(1)](t^{d(i)+d(j)+1} + s^{d(i)+d(j)+1} - |t-s|^{d(i)+d(j)+1})\right)_{1\le i,j\le n}$
where

$$c_{d(i),d(j)} = \frac{\sqrt{\Gamma(2d(i)+2)\sin(\pi(d(i)+\frac{1}{2}))}\sqrt{\Gamma(2d(j)+2)\sin(\pi(d(j)+\frac{1}{2}))}}{\Gamma(d(i)+d(j)+2)\sin(\pi(d(i)+d(j)+1)/2)}.$$

Proof. The first part is clear using the decomposition (4.2). The second one follows by

$$\begin{aligned} Cov[\mathbf{L}^d(t), \mathbf{L}^d(s)] &= \int_0^T z^d(t,u) Cov[\mathbf{L}(t), \mathbf{L}(s)](z^d(s,u))^\top du \\ &= \left(Cov[L^i(t), L^j(s)] \int_0^T z^d_{ii}(t,u) z^d_{jj}(s,u) du\right)_{1\le i,j\le n}. \end{aligned}$$

The calculation of the last integrals is similar to (2.17) of Elliott and van der Hoek [53]. □

Remark 4.1.6. Proposition 4.1.5 (*ii*) shows that MG-fLps have the same second-order structure as fBm (up to a constant) and therefore it is clear that for all $1 \le i \le n$ the increments of the univariate process $L^{d(i)}$ exhibit a similar memory structure as in the case of fBm.

The next theorem states the characteristic function of a MG-fLp which determines its distribution. The proof uses Theorem 1.4.5 and the fact that $z^d(t,s)$ is Hermitian and symmetric for $s, t \in [0,T]$.

Theorem 4.1.7. *For each fixed $t \in [0,T]$ the random vector $\boldsymbol{L}^d(t)$ is infinitely divisible and its characteristic function is for $u \in \mathbb{R}^n$ given by*

$$\begin{aligned} & E[\exp\{i\langle u, \boldsymbol{L}^d(t)\rangle\}] = \exp\Big\{\int_0^T \psi(z^d(t,s)u) ds\Big\} \\ = \ & \exp\Big\{i\int_0^T \langle z^d(t,s)\gamma, u\rangle ds - \frac{1}{2}\int_0^T u^\top z^d(t,s)\Sigma z^d(t,s) u ds \\ & + \int_0^T \int_{\mathbb{R}^n} \Big(e^{i\langle u, z^d(t,s)x\rangle} - 1 - i\langle u, z^d(t,s)x\rangle \mathbf{1}_{\{\|z^d(t,s)x\|<1\}}\Big)\nu(dx) ds\Big\} \end{aligned}$$

The next lemma summarizes main properties of MG-fLps and is the multivariate extension of Proposition 3.7 of Tikanmäki and Mishura [127].

Lemma 4.1.8. *We have for $d = (d(1), \ldots, d(n))^\top$:*

(i) A MG-fLp without Gaussian component has a.s. continuous paths if and only if $d \in (0, \frac{1}{2})^n$.

(ii) A MG-fLp without Gaussian component has a.s. Hölder continuous paths of any order $\alpha < \min[d]$ if and only if $d \in (0, \frac{1}{2})^n$.

(iii) If $d(i) \in (-\frac{1}{2}, 0)$ for some $1 \leq i \leq n$, then the MG-fLp has discontinuous and unbounded sample paths with positive probability.

4.1.2 Integration

Before coming to the prediction of MG-fLps we need to define integration. This will be done by the usual $L^2(\Omega)$-approach, as e.g. in Pipiras and Taqqu [103, 104], Marquardt [92] or Tikanmäki and Mishura [127].

For $d = (d(1), \ldots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, define

$$\Lambda_T^d := \Big\{ f : [0,T] \to \mathbb{R}^{n\times n} \Big| f_{ij} \in \Lambda_T^{d(j)}, 1 \leq i,j \leq n \Big\},$$

where f_{ij} denotes the ij-th component of f.

Consider simple functions of the form

$$f(\cdot) = \sum_{k=1}^m a_k \mathbf{1}_{[t_k, t_{k+1})}(\cdot),$$

where $m \in \mathbb{N}$, $m \geq 1$, $0 \leq t_1 \leq \cdots \leq t_m \leq T$ and $a_k \in \mathbb{R}^{n\times n}$ for $1 \leq k \leq m$. Then we define

$$\begin{aligned} \int_0^T f(s) d\mathbf{L}^d(s) &:= \sum_{k=1}^m a_k (\mathbf{L}^d(t_{k+1}) - \mathbf{L}^d(t_k)) \\ &= \sum_{k=1}^m \begin{pmatrix} \sum_{j=1}^n (a_k)_{1j} (L^{d(j)}(t_{k+1}) - L^{d(j)}(t_k)) \\ \vdots \\ \sum_{j=1}^n (a_k)_{nj} (L^{d(j)}(t_{k+1}) - L^{d(j)}(t_k)) \end{pmatrix}. \end{aligned}$$

A simple calculation leads to

$$\int_0^T f(s) d\mathbf{L}^d(s) = \int_0^T z^d(f,s) d\mathbf{L}(s),$$

where

$$z^d(f,s) := \begin{pmatrix} c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)} f_{11}(\cdot)(\cdot))(s) & \ldots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)} f_{1n}(\cdot)(\cdot))(s) \\ \vdots & \ddots & \vdots \\ c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)} f_{n1}(\cdot)(\cdot))(s) & \ldots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)} f_{nn}(\cdot)(\cdot))(s) \end{pmatrix}.$$

Now we obtain from the definition of Λ_T^d and Theorem 1.4.5:

Theorem 4.1.9. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$. Then the integral $\int_0^T f(s) d\boldsymbol{L}^d(s)$ exists as a (componentwise) $L^2(\Omega)$-limit of approximating step functions in Λ_T^d (also componentwise). Furthermore, we have the identity*

$$\int_0^T f(s) d\boldsymbol{L}^d(s) = \int_0^T z^d(f,s) d\boldsymbol{L}(s),$$

which holds (componentwise) in $L^2(\Omega)$.

Further we find some distributional results on MG-fLp driven integrals.

Theorem 4.1.10. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$. Then we have for all $u \in \mathbb{R}^n$*

$$\begin{aligned} & E\Big[\exp\Big\{i\Big\langle u, \int_0^T f(s) d\boldsymbol{L}^d(s)\Big\rangle\Big\}\Big] = \exp\Big\{\int_0^T \psi(z^d(f,s)^\top u) ds\Big\} \\ = \; & \exp\Big\{i\int_0^T \langle z^d(f,s)\gamma, u\rangle ds - \frac{1}{2}\int_0^T u^\top z^d(f,s)\Sigma z^d(f,s)^\top u ds \\ & + \int_0^T \int_{\mathbb{R}^n} \Big(e^{i\langle u, z^d(f,s)x\rangle} - 1 - i\langle u, z^d(f,s)x\rangle \mathbf{1}_{\{\|z^d(f,s)x\|<1\}}\Big)\nu(dx) ds\Big\}. \end{aligned}$$

Proof. We calculate

$$\begin{aligned} & E\Big[\exp\Big\{i\Big\langle u, \int_0^T f(s) d\mathbf{L}^d(s)\Big\rangle\Big\}\Big] = E\Big[\exp\Big\{i\sum_{k=1}^n u_k\Big(\int_0^T f(s) d\mathbf{L}^d(s)\Big)_k\Big\}\Big] \\ = \; & E\Big[\exp\Big\{i\sum_{l=1}^n \int_0^T \Big(\sum_{l=1}^n u_k f^{kl}(s)\Big) dL^{d(l)}(s)\Big\}\Big] \\ = \; & E\Big[\exp\Big\{i\sum_{l=1}^n \int_0^T \Big(\sum_{l=1}^n u_k f^{kl}(s)\Big) dL^{d(l)}(s)\Big\}\Big] \\ = \; & E\Big[\exp\Big\{i\sum_{l=1}^n \int_0^T c_{d(l)} s^{-d(l)} I_{T-}^{d(l)}\Big((\cdot)^{d(l)} \sum_{l=1}^n u_k f^{kl}(\cdot)\Big)(s) dL^l(s)\Big\}\Big]. \end{aligned}$$

Using Theorem 1.4.5 we obtain the assertion. □

Remark 4.1.11. When $d \in (0, \frac{1}{2})^n$, it is also possible to define pathwise integration with respect to MG-fLps using Hölder continuity like in Buchmann and Klüppelberg [28] or a p-variation approach like in Section 1.6.1.

4.2 Prediction results

In this section we will state and prove our main theorems about the conditional characteristic functions of MG-fLp driven integrals and related processes. Recall that by Remark 4.1.3 a MG-fLp is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the corresponding Lévy process.

4.2.1 Prediction of integrals

First we need a technical lemma, which will be crucial to derive the prediction formula. Define for $d \in (-\frac{1}{2}, \frac{1}{2})$ and suitable $f : [0,T] \to \mathbb{R}^{n\times n}$ the deconvolution operator

$$z_\star^d(f,s) := \begin{pmatrix} c_{-d(1)}^{-1} s^{d(1)} I_{T-}^{d(1)}((\cdot)^{-d(1)} f_{11}(\cdot)(\cdot))(s) & \dots & c_{-d(n)}^{-1} s^{d(n)} I_{T-}^{d(n)}((\cdot)^{-d(n)} f_{1n}(\cdot)(\cdot))(s) \\ \vdots & \ddots & \vdots \\ c_{-d(1)}^{-1} s^{d(1)} I_{T-}^{d(1)}((\cdot)^{-d(1)} f_{n1}(\cdot)(\cdot))(s) & \dots & c_{-d(n)}^{-1} s^{d(n)} I_{T-}^{d(n)}((\cdot)^{-d(n)} f_{nn}(\cdot)(\cdot))(s) \end{pmatrix}.$$

Lemma 4.2.1. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$. Then the following assertions hold for all $0 \le s \le t \le T$:*

(i) For all $1 \le i, j \le n$ we have that $z_{ij}^d(f\mathbf{1}_{[s,t]}, \cdot) \in L^2([0,T]) \subset L^1([0,T])$.

(ii) The function

$$[0,s] \to \mathbb{R}^{n\times n}, \quad v \mapsto z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(f\mathbf{1}_{[s,t]}, \cdot), v)$$

exists componentwise and belongs to $\widetilde{\Lambda}_s^d$.

(iii) For all $v \in [0,T]$,

$$z^d(z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(f\mathbf{1}_{[s,t]}, \cdot), \cdot), v) = \mathbf{1}_{[0,s]}(v) z^d(f\mathbf{1}_{[s,t]}, v).$$

Proof. Since $f \in \Lambda_T^d$ it follows that $f\mathbf{1}_{[s,t]} \in \Lambda_T^d$. By definition of Λ_T^d we therefore get that $z_{ij}^d(f\mathbf{1}_{[s,t]}, \cdot) \in L^2([0,T]) \subset L^1([0,T])$, which leads to (i). The existence of the function in assertion (ii) can be obtained by using a similar approximation argument as in the proof of Lemma 1 of Duncan [46]. The second statement in (ii) and (iii) follows by definition of $\widetilde{\Lambda}_s^d$ and by applying Theorem 2.5 of Samko, Kilbas and Marichev [117] componentwise. □

Theorem 4.2.2. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$ and*

$$\mathcal{F}_t = \overline{\sigma\Big\{ \int_0^s f(v) d\mathbf{L}^d(v), s \in [0,t] \Big\}}, \quad t \in [0,T].$$

Then we have for all $0 \le s \le t \le T$ and $u \in \mathbb{R}^n$

$$\begin{aligned} & E\Big[\exp\Big\{ i\Big\langle u, \int_0^t f(v) d\boldsymbol{L}^d(v) \Big\rangle \Big| \mathcal{F}_s \Big\} \Big] \\ = \; & \exp\Big\{ i\Big\langle u, \int_0^s f(v) d\boldsymbol{L}^d(v) + \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(f\mathbf{1}_{[s,t]}, \cdot), v) d\boldsymbol{L}^d(v) \Big\rangle \Big\} \\ & \times \Big\{ \int_s^t \psi(z^d(f\mathbf{1}_{[s,t]}, v)^\top u) dv \Big\}. \end{aligned}$$

Proof. Since the random variable $\int_0^t f(v)d\mathbf{L}^d(v)$ is $\mathcal{F}_s$-measurable, it is enough to consider the conditional characteristic function of $\int_s^t f(v)d\mathbf{L}^d(v)$. Applying Theorem 4.1.9 we switch from the MG-fLp to the corresponding Lévy process and obtain

$$\begin{aligned}\int_s^t f(v)d\mathbf{L}^d(v) &= \int_0^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\\ &= \int_0^s z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)+\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v).\end{aligned}$$

The first summand on the righthand side is again $\mathcal{F}_s$-measurable and therefore we obtain

$$\begin{aligned}&E\Big[\exp\Big\{i\Big\langle u,\int_0^t f(v)d\mathbf{L}^d(v)\Big\rangle\Big|\mathcal{F}_s\Big\}\Big]\\ =\ &\exp\Big\{i\Big\langle u,\int_0^s f(v)d\mathbf{L}^d(v)+\int_0^s z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big\}\\ &\times E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big|\mathcal{F}_s\Big\}\Big].\end{aligned}$$

However due to independent increments of Lévy processes we have

$$E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big|\mathcal{F}_s\Big\}\Big]=E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big\}\Big]$$

and applying Theorem 1.4.5 leads to

$$E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big\}\Big]=\exp\Big\{\int_s^t \psi(z^d(f\mathbf{1}_{[s,t]},v)^\top u)dv\Big\},$$

where we have used the fact that $z^d(f\mathbf{1}_{[s,t]},v)=0$ if $v\in[t,T]$. Since we want the prediction formula in terms of the MG-fLp and not the driving Lévy process, we invoke Lemma 4.2.1 (ii) and apply again Theorem 4.1.9 to obtain

$$\int_0^s z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)=\int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(f\mathbf{1}_{[s,t]},\cdot),v)d\mathbf{L}^d(v).$$

Putting everything together we get the assertion. □

Remark 4.2.3. Every $f\in\Lambda_T^d$, with $f_{ij}(u)\neq 0$ for all $u\in[0,T]$ and $1\le i,j\le n$, satisfies the conditions of Theorem 4.2.2.

Example 4.2.4. [Univariate fBm] Choose in Theorem 4.2.2 $n=1$, $f=\mathbf{1}_{[0,t)}$, $0\le t\le T$, and take as driving Lévy process a standard Brownian motion, i.e. $\mathbf{L}=B$. Then $\mathbf{L}^d=B^d$ is an univariate fBm. Using Theorem 3.1.1, the conditional characteristic function is for $0\le s\le t\le T$ given by

$$\begin{aligned}&E[\exp\{iuB^d(t)|\mathcal{F}_s\}]\\ =\ &\exp\Big\{iu\big[B^d(s)+\int_0^s\Psi^d(s,t,v)dB^d(v)\big]-\frac{u^2}{2}\big[\|\mathbf{1}_{[s,t]}\|_{d,T}^2-\|\Psi^d(s,t,\cdot)\mathbf{1}_{[0,s]}\|_{d,T}^2\big]\Big\},\end{aligned}$$

where

$$\Psi^d(s,t,v) = v^{-d}(I_{s-}^{-d}(I_{t-}^{d}(\cdot)^d \mathbf{1}_{[s,t]}(\cdot)))(v), \quad v \in (0,s),$$

and $\|\cdot\|_{d,T}^2$ as defined in Section 1.5. This matches the result of Theorem 4.2.2 since $I_{t-}^{-d}\mathbf{1}_{[0,s]}(\cdot)f(\cdot) = I_{s-}^{-d}f(\cdot)$ in the situation above. Furthermore we have

$$\|\mathbf{1}_{[s,t]}\|_{d,T}^2 - \|\Psi^d(s,t,\cdot)\mathbf{1}_{[0,s]}\|_{d,T}^2 = \|z^d(\mathbf{1}_{[s,t]},\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2.$$

Example 4.2.5. [Univariate Gamma MG-fLp] In Theorem 4.2.2 choose $n = 1$, $f = \mathbf{1}_{[0,t)}$, $0 \le t \le T$, and take as driving Lévy process a univariate Gamma process $G = (G(t))_{t\in[0,T]}$. Its distribution is then characterized by

$$\psi(u) = -\gamma \log\big(1 - \frac{u}{\lambda}\big)\mathbf{1}_{[0,\lambda)}(u),$$

with $\gamma, \lambda > 0$. By Theorem 4.2.2 we obtain for the Gamma MG-fLp $G^d = (G^d(t))_{t\in[0,T]}$ and $0 \le s \le t \le T$

$$\begin{aligned} & E[\exp\{iuG^d(t)|\mathcal{F}_s\}] \\ = \; & \exp\Big\{iu\big[G^d(s) + \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(\mathbf{1}_{[s,t]},\cdot),v)dG^d(v)\big]\Big\} \\ & \times \exp\Big\{\gamma\log(\lambda)[t-s] - \gamma\int_s^t \log(\lambda - z^d(\mathbf{1}_{[s,t]},v)u)dv\Big\}. \end{aligned}$$

Example 4.2.6. [Bivariate Poisson MG-fLp] Choose in Theorem 4.2.2 $n = 2$, $f = \mathbf{1}_{[0,t)}$, $0 \le t \le T$, and take as driving Lévy process a bivariate Poisson process, i.e. here take independent Poisson processes Z_i on $[0,T]$ with intensities $\eta_i \ge 0$ for $i = 1,2,3$ and define

$$\mathbf{L} := (Z_1 + Z_2, Z_2 + Z_3)^\top.$$

The distribution of this bivariate Lévy process is then characterized by

$$\begin{aligned} \psi(u) &= \int_{\mathbb{R}^2} (\exp\{i\langle u,x\rangle - 1 - i\langle u,x\rangle \mathbf{1}_{\{\|x\|<1\}})\nu(dx) \\ &= \int_{\mathbb{R}^2} (\exp\{i\langle u,x\rangle\} - 1)\nu(dx), \quad u \in \mathbb{R}^2, \end{aligned}$$

with $\nu(dx) = \eta_1\delta_{\{1\}\times\{0\}}(dx) + \eta_2\delta_{\{1\}\times\{1\}}(dx) + \eta_3\delta_{\{0\}\times\{1\}}(dx)$. Theorem 4.2.2 leads now

for $0 \leq s \leq t \leq T$ to

$$\begin{aligned}
& E[\exp\{i\langle u, \mathbf{L}^d(t)\rangle | \mathcal{F}_s\}] \\
= \; & \exp\Big\{ i\Big\langle u, \mathbf{L}^d(s) + \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(\mathbf{1}_{[s,t]}, \cdot), v) d\mathbf{L}^d(v) \Big\rangle \Big\} \\
& \times \exp\Big\{ \eta_1 \int_s^t \big(\exp\big(i \sum_{j=1}^2 z_{1j}^d(\mathbf{1}_{[s,t)}, v) u_j \big) - 1 \big) dv \Big\} \\
& \times \exp\Big\{ \eta_2 \int_s^t \big(\exp\big(\sum_{k=1}^2 \sum_{j=1}^2 z_{kj}^d(\mathbf{1}_{[s,t)}, v) u_j \big) - 1 \big) dv \Big\} \\
& \times \exp\Big\{ \eta_3 \int_s^t \big(\exp\big(i \sum_{j=1}^2 z_{2j}^d(\mathbf{1}_{[s,t)}, v) u_j \big) - 1 \big) dv \Big\}
\end{aligned}$$

4.2.2 Ornstein-Uhlenbeck type processes

In a next step we will consider MG-fLp-driven Ornstein-Uhlenbeck processes, starting with the definition. Similar processes were considered by Marquardt [92], Klüppelberg and Matsui [82] and in Fink and Klüppelberg [59], cf. Section 2.2. However, in contrast to our work, they define their underlying fractional Lévy processes by an integral representation over the whole real line as in Definition 1.6.1. Postponing the usual question of existence and uniqueness until Proposition 4.2.8 we define:

Definition 4.2.7. *For $d = (d(1), \ldots, d(n))^\top \in (0, \frac{1}{2})^n$, take $\sigma \in \Lambda_T^d$, $k : [0,T] \to \mathbb{R}^n$ and $a : [0,T] \to \mathbb{R}^{n \times n}$, k, a componentwise locally integrable such that $e^{-\int_\cdot^t a(v)dv}\sigma(\cdot) \in \Lambda_T^d$ for all $t \in [0,T]$. Then the (unique) solution to the sde*

$$d\mathfrak{L}^d(t) = (k(t) - a(t)\mathfrak{L}^d(t))dt + \sigma(t)d\mathbf{L}^d(t), \quad t \in [0,T], \quad \mathfrak{L}^d(0) \in \mathbb{R}^n.$$

is called a fractional Lévy Ornstein-Uhlenbeck process.

The next proposition ensures the existence of a solution. Its uniqueness follows by a simple application of Gronwall's Lemma (e.g. Theorem 3.1 of Ikeda and Watanabe [77]) similar to the classical Brownian case.

Proposition 4.2.8. *In the situation of Definition 4.2.7 assume further that σ_{ij} is of bounded $p(j)$-variation for some $0 < p(j) < 1/(1-d(j))$ (cf. Young [134] and Section 1.3) for all $1 \leq i, j \leq n$. Then we have for $t \in [0,T]$*

$$\mathfrak{L}^d(t) = e^{-\int_0^t a(s)ds}\mathfrak{L}^d(0) + \int_0^t e^{-\int_s^t a(v)dv} k(s)ds + \int_0^t e^{-\int_s^t a(v)dv}\sigma(s)d\mathbf{L}^d(s),$$

where the matrix exponential is defined as usual.

Proof. Define $\mathfrak{L}^d$ as above and calculate for $0 \le s \le t \le T$

$$
\begin{aligned}
& -\int_s^t a(z)\mathfrak{L}^d(z)dz \\
= & -\int_s^t a(z)e^{-\int_0^z a(v)dv}dz\mathfrak{L}^d(0) - \int_s^t a(z)\int_0^z e^{-\int_w^z a(v)dv}k(w)dwdz \\
& -\int_s^t a(z)\int_0^z e^{-\int_w^z a(v)dv}\sigma(w)d\mathbf{L}^d(w)dz \\
= & \left[e^{-\int_0^t a(v)dv} - e^{-\int_0^s a(v)dv}\right]\mathfrak{L}^d(0) \\
& -\int_s^t a(z)\int_0^s e^{-\int_w^z a(v)dv}k(w)dwdz - \int_s^t a(z)\int_s^z e^{-\int_w^z a(v)dv}k(w)dwdz \\
& -\int_s^t a(z)\int_0^s e^{-\int_w^z a(v)dv}\sigma(w)d\mathbf{L}^d(w)dz - \int_s^t a(z)\int_s^z e^{-\int_w^z a(v)dv}\sigma(w)d\mathbf{L}^d(w)dz \\
= & \left[e^{-\int_0^t a(v)dv} - e^{-\int_0^s a(v)dv}\right]\mathfrak{L}^d(0) \\
& -\int_0^s\int_s^t a(z)e^{-\int_w^z a(v)dv}k(w)dzdw - \int_s^t\int_w^t a(z)e^{-\int_w^z a(v)dv}k(w)dzdw \\
& -\int_0^s\int_s^t a(z)e^{-\int_w^z a(v)dv}\sigma(w)dzd\mathbf{L}^d(w) - \int_s^t\int_w^t a(z)e^{-\int_w^z a(v)dv}\sigma(w)dzd\mathbf{L}^d(w) \\
= & \left[e^{-\int_0^t a(v)dv} - e^{-\int_0^s a(v)dv}\right]\mathfrak{L}^d(0) \\
& +\int_0^s \left[e^{-\int_w^t a(v)dv} - e^{-\int_w^s a(v)dv}\right]k(w)dw + \int_s^t \left[e^{-\int_w^t a(v)dv} - I\right]k(w)dw \\
& +\int_0^s \left[e^{-\int_w^t a(v)dv} + e^{-\int_w^s a(v)dv}\right]\sigma(w)d\mathbf{L}^d(w) + \int_s^t \left[e^{-\int_w^t a(v)dv} - I\right]\sigma(w)d\mathbf{L}^d(w) \\
= & \mathfrak{L}^d(t) - \mathfrak{L}^d(s) - \int_s^t \sigma(w)d\mathbf{L}^d(w) - \int_s^t k(w)dw
\end{aligned}
$$

where I is the unit matrix of size n. □

Remark 4.2.9. Using potential Hölder continuity of the MG-fLp paths (Lemma 4.1.8 (ii)), we can also define integration via a pathwise approach, cf. Young [134]. If the pathwise and the L^2-integral both exist, they have to be equal. The next lemma is based on this fact.

Lemma 4.2.10. *In addition to the assumptions of Definition 4.2.7, let the matrix $\sigma(t)$ be non-singular for every $0 \le t \le T$ and $d = (d(1), \dots, d(n))^\top \in (0, \frac{1}{2})^n$. Furthermore assume σ_{ij} and $(\sigma)^{-1}_{ij}$ are of bounded $p(j)$-variation for some $0 < p(j) < 1/(1-d(j))$ (cf. Young [134] and Section 1.3) for all $1 \le i, j \le n$. Then we have*

$$
d\mathbf{L}^d(t) = \Big(-\sigma(t)^{-1}k(t) + \sigma(t)^{-1}a(t)\mathfrak{L}^d(t)\Big)\,dt + \sigma(t)^{-1}d\mathfrak{L}^d(t), \quad 0 \le t \le T.
$$

Proof. The proof is analogous to the proof of Proposition 4.2.8 and the univariate inversion part of Proposition 3.1.4. □

The prediction result follows. The proof is a combination of Theorem 4.2.2 and Lemma 4.2.10.

Theorem 4.2.11. *For $d = (d(1), \dots, d(n))^\top \in (0, \frac{1}{2})^n$ take $\sigma \in \Lambda_T^d$, $k : [0,T] \to \mathbb{R}^n$ and $a : [0,T] \to \mathbb{R}^{n\times n}$, k, a componentwise locally integrable such that $e^{-\int_\cdot^t a(v)dv}\sigma(\cdot) \in \Lambda_T^d$ for all $t \in [0,T]$. Let further $\sigma(t)$ be non-singular for every $0 \le t \le T$. Then we have for $0 \le s \le t \le T$ and $u \in \mathbb{R}^n$*

$$
\begin{aligned}
& E[\exp\{i\langle u, \mathfrak{L}^d(t)\rangle | \mathcal{F}_s\}] \\
= \; & \exp\Big\{ i\Big\langle u, e^{-\int_s^t a(v)dv}\mathfrak{L}^d(s) + \int_s^t e^{-\int_w^t a(v)dv} k(w)dw \Big\rangle \Big\} \\
& \times \exp\Big\{ i\Big\langle u, \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(h\mathbf{1}_{[s,t]}, \cdot), v) d\boldsymbol{L}^d(v) \Big\rangle + \int_s^t \psi(z^d(h\mathbf{1}_{[s,t]}, v)^\top u)dv \Big\} \\
= \; & \exp\Big\{ i\Big\langle u, e^{-\int_s^t a(v)dv}\mathfrak{L}^d(s) + \int_s^t e^{-\int_w^t a(v)dv} k(w)dw \Big\rangle \Big\} \\
& \times \exp\Big\{ -i\Big\langle u, \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(h\mathbf{1}_{[s,t]}, \cdot), v)\sigma(v)^{-1}k(v)dv \Big\rangle \Big\} \\
& \times \exp\Big\{ \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(h\mathbf{1}_{[s,t]}, \cdot), v)\sigma(v)^{-1}a(v)\mathfrak{L}^d(v)dv \Big\} \\
& \times \exp\Big\{ \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(h\mathbf{1}_{[s,t]}, \cdot), v)\sigma(v)d\mathfrak{L}^d(v) + \int_s^t \psi(z^d(h\mathbf{1}_{[s,t]}, v)^\top u)dv \Big\},
\end{aligned}
\tag{4.3}
$$

where $h(\cdot) = e^{-\int_\cdot^t a(v)dv}\sigma(\cdot)$.

Remark 4.2.12. [General sde's] Using the MG-fLp decomposition (4.2) and Proposition 3.9 of Tikanmäki and Mishura [127] we see that the i-th component of a MG-fLp is of zero-quadratic variation if $d(i) \in (0, \frac{1}{2})$. Therefore for $d = (d(1), \dots, d(n))^\top \in (0, \frac{1}{2})^n$, general sde's driven by MG-fLps can be considered using for example the theory of Zähle [136], Section 5. However as already mentioned in the introduction of Section 2.2.2, this does not cover CIR type processes. Using the zero-quadratic variation property and the integration concept of Russo and Vallois [115], a similar theory as in [59] can be proven (at least up to stationary solutions, where we have to face Remark 4.1.3). Therefore it will be interesting to calculate the prediction of transforms of MG-fLp driven integrals, which can be achieved by Fourier methods (cf. Theorem 3.1.7 for the univariate fBm case). An example is the next theorem.

Theorem 4.2.13. *For $d \in (0, \frac{1}{2})$ and $a > 0$ let $X = (X(t))_{t\in[0,T]}$ be the solution to the sde*

$$
dX(t) = -\frac{a}{2}X(t)dt + dL^d(t), \quad t \in [0,T], \quad X(0) \in \mathbb{R}.
$$

Assume further that $E[\exp\{X(t)\}] < \infty$ *for* $t \in [0,T]$. *Take* $f(x) = \operatorname{sign}(x)x^2\frac{\sigma^2}{4}$ *for* $x \in \mathbb{R}$ *and* $\sigma > 0$. *Define the process* $Z = (Z(t))_{t\in[0,T]}$ *by* $Z(t) = f(X(t))$. *Then for* $0 \le s \le t \le T$

$$Z(t) - Z(s) = -a\int_s^t Z(v)dv + \sigma\int_s^t \sqrt{|Z(v)|}d^{\mathrm{RV}}L^d(v)$$

holds with $Z(0) = f(X(0))$. *Here the integral* $\int_s^t \sqrt{|Z(v)|}d^{\mathrm{RV}}L^d(v)$ *is the forward integral of Definition 1 of Russo and Vallois [115]. Furthermore for* $u \in \mathbb{R}$ *we have*

$$E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] = \int_{\mathbb{R}}\Big(E\Big[e^{(i\xi+1)X(t)}\Big|\mathcal{F}_s\Big]g_+(\xi,u) + E\Big[e^{(i\xi-1)X(t)}\Big|\mathcal{F}_s\Big]g_-(\xi,u)\Big)\,d\xi$$

with $g_\star(\xi,u) = (2\pi)^{-1}\int_{\mathbb{R}_\star} e^{-(i\xi\star 1)x+iuf(x)}dx$, $\star \in \{+,-\}$, *where* $E[e^{(i\xi+1)X(t)}|\mathcal{F}_s]$ *is given by the analytic extension of* (4.3) *to* $\mathbb{C}$.

Proof. Observe that for $g \in \Lambda_T^d$ with bounded variation the equality

$$\int_0^T g(v)dL^d(v) = \int_0^T g(v)d^{RV}L^d(v)$$

holds in the $L^2(\Omega)$-sense by Proposition 1 of [115]. Therefore the sde's

$$\begin{aligned} dX(t) &= -\frac{a}{2}X(t)dt + dL^d(t), \quad t \in [0,T], \quad X(0) \in \mathbb{R}, \\ d^{RV}X(t) &= -\frac{a}{2}X(t)dt + d^{RV}L^d(v), \quad t \in [0,T], \quad X(0) \in \mathbb{R}. \end{aligned}$$

lead to the same process. Now we calculate for $0 \le s \le t \le T$ using an Itô formula (Proposition 12 of [115]),

$$\begin{aligned} Z(t) - Z(s) &= \int_s^t f'(X(v))d^{RV}X(v) \\ &= -\frac{a}{2}\int_s^t f'(X(v))X(v)dv + \int_s^t f'(X(v))d^{RV}L^d(v) \\ &= -\frac{a}{2}\int_s^t f'(f^{-1}(Z(v)))f^{-1}(Z(v))dv + \int_s^t f'(f^{-1}(Z(v)))d^{RV}L^d(v), \end{aligned}$$

where $-\frac{a}{2}f'(f^{-1}(x))f^{-1}(x) = -ax$ and $f'(f^{-1}(x)) = \sigma\sqrt{|x|}$ for $x \in \mathbb{R}$ by Proposition 5.7 of Buchmann and Klüppelberg [28].

The prediction part follows similar to the proof of Theorem 3.1.7 and Example 3.1.8 using $E[\exp\{X(t)\}] < \infty$. □

4.3 Application: Interest rates with credit risk

Similar to Section 3.5 we shall work in the framework of the most reduced-form credit risk models in the literature. Given a finite time horizon $T^\star > 0$, a credit market shall be described by the bivariate process $(r, H) = (r(t), H(t))_{0\le t\le T^\star}$ on a given probability space $(\Omega, \mathcal{F}, \mathcal{Q})$ endowed with the filtration $(\mathcal{F}_t)_{0\le t\le T^\star}$, which represents the market information and satisfies the usual conditions of completeness and right continuity. The process r models the *short rate* and H the *default indicator*, i.e.

$$H(t) = \mathbf{1}_{\{\tau\le t\}}, \quad 0 \le t \le T^\star,$$

for a given $(\mathcal{F}_t)_{0\le t\le T^\star}$-stopping time τ, the *default time*. Denote by $(\mathcal{H}_t)_{0\le t\le T^\star}$ the filtration generated by H.

We will extend Assumption 3.5.1 such that it fits to the situation of this section:

Assumption 4.3.1 (Market structure; cf. Frey and Backhaus [62], Ass. 3.1)**.**

(i) We assume that there is a subfiltration $(\mathcal{G}_t)_{0\le t\le T^\star}$ of $(\mathcal{F}_t)_{0\le t\le T^\star}$ with

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t\,, \quad 0 \le t \le T^\star,$$

r is $(\mathcal{G}_t)_{0\le t\le T^\star}$-progressive, and that there exists a positive $(\mathcal{G}_t)_{0\le t\le T^\star}$-progressive process $\lambda = (\lambda_t)_{0\le t\le T^\star}$, called the default rate*, describing the intensity of H (cf. Corollary 5.1.5 of Bielecki and Rutkowski [21]) with $\int_0^t \lambda(s)ds < \infty$ a.s. for all $0 \le t \le T^\star$. Furthermore assume that*

$$P(\tau > t \mid \mathcal{G}_t) = E[1 - H(t) \mid \mathcal{G}_t] = \exp\left\{-\int_0^t \lambda(s)ds\right\}. \tag{4.4}$$

Setting $\mathcal{G}_\infty := \bigvee_{0\le t\le T^\star} \mathcal{G}_t$, assume that for all bounded $\mathcal{G}_\infty$-measurable random variables η,

$$E[\eta \mid \mathcal{F}_t] = E[\eta \mid \mathcal{G}_t] \tag{4.5}$$

holds.

(ii) $\mathcal{Q}$ is a risk-neutral pricing measure, such that the price of any $\mathcal{F}_T$-measurable claim $X \in L^1(\Omega)$ with maturity $0 \le T \le T^\star$ at time $0 \le t \le T$ is given by $\mathcal{V}(t, T) = E[X \mid \mathcal{F}_t]$ for $0 \le t \le T$.

In the framework above, the default history $(\mathcal{F}_t)_{0\le t\le T^\star}$ is the investor information at time t, meaning that the investor knows the short rate r, the default rate λ and the default

indicator process H at time t. Using Lemma 13.2 of Filipovic [56] we see that the price of a defaultable zero coupon bond is for $0 \le t \le T \le T^\star$ given by

$$\overline{B}(t,T) \;=\; E\Big[\mathbf{1}_{\{\tau>T\}}e^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big] = \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big].$$

Therefore it is sufficient to specify the dynamics of the bivariate process (r,λ), forgetting H. We propose a fractional Vasicek model:

Assumption 4.3.2 (Fractional Vasicek model)**.** *For $d = (d(1),\dots,d(n))^\top \in (0,\frac{1}{2})^n$, $n \in \mathbb{N}$, take $\sigma \in \Lambda^d_{T^\star}$, $k : [0,T^\star] \to \mathbb{R}^n$ and $a : [0,T^\star] \to \mathbb{R}^{n\times n}$, k, a componentwise locally integrable such that $e^{-\int_\cdot^t a(v)dv}\sigma(\cdot) \in \Lambda^d_{T^\star}$ for all $t \in [0,T^\star]$. Consider the corresponding Ornstein-Uhlenbeck sde*

$$d\mathfrak{L}^d(t) = (k(t) - a(t)\mathfrak{L}^d(t))dt + \sigma(t)d\mathbf{L}^d(t), \quad t \in [0,T^\star], \quad \mathfrak{L}^d(0) \in \mathbb{R}^{n\times n}.$$

Assume further as in Lemma 4.2.10 that $\sigma(t)$ is non-singular for every $0 \le t \le T^\star$ and σ_{ij} and $(\sigma)^{-1}_{ij}$ are of bounded $p(j)$-variation for some $0 < p(j) < 1/(1-d(j))$ (cf. Young [134] and Section 1.3) for all $1 \le i,j \le n$. Then define for fixed weights $\theta, \phi \in (\mathbb{R}_+)^n$, where $\theta \neq 0$,

$$r(t) = \langle \theta, \mathfrak{L}^d(t)\rangle \quad \text{and} \quad \lambda(t) = \langle \phi, \mathfrak{L}^d(t)\rangle, \quad t \in [0,T^\star]. \tag{4.6}$$

Therefore we have $\mathcal{G}_t = \sigma\overline{\{\mathbf{L}^d(s), s \in [0,t]\}}$ for all $t \in [0,T^\star]$.

The following theorem considers the price of a defaultable zero coupon bond in the fractional Vasicek credit market 4.3.2.

Theorem 4.3.3. *Let $0 \le t \le T \le T^\star$. In the model of Assumption 4.3.2, set $D(t,T) := \int_t^T e^{-\int_t^s a(v)dv}ds$ where the integral is taken componentwise. Assume further that $D(\cdot,T)\sigma(\cdot) \in \Lambda^d_T$ and*

$$E\Big[\exp\Big\{-\Big\langle \theta+\phi, \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big\rangle\Big\}\Big] < \infty. \tag{4.7}$$

Then we have

$$\begin{aligned}
\overline{B}(t,T) \;&=\; \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\
&=\; \mathbf{1}_{\{\tau>t\}}\exp\Big\{-\Big\langle \theta+\phi, D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(v)dv\Big\rangle\Big\} \\
&\quad\times \exp\Big\{-\Big\langle \theta+\phi, \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]}z^d(h\mathbf{1}_{[t,T]},\cdot),v)d\boldsymbol{L}^d(v)\Big\rangle\Big\} \\
&\quad\times \exp\Big\{\int_t^T \psi(z^d(h\mathbf{1}_{[t,T]},v)^\top i(\theta+\phi))dv\Big\},
\end{aligned}$$

with $h(\cdot) = D(\cdot,T)\sigma(\cdot)$.

Proof. By Assumption 4.3.2 the stochastic integrals also exist in the pathwise sense. By the proof of Proposition 4.2.8 and Fubini's Theorem (pathwise) we get

$$\begin{aligned}
&\int_t^T (r(s)+\lambda(s))ds = (\theta+\phi)^\top \int_t^T \mathfrak{L}^d(s)ds \\
= \;& (\theta+\phi)^\top \int_t^T \Big[e^{-\int_t^s a(v)dv}\mathfrak{L}^d(t) + \int_t^s e^{-\int_w^s a(v)dv} k(w)dw \\
&+ \int_t^s e^{-\int_w^s a(v)dv}\sigma(w)d\mathbf{L}^d(w)\Big]ds \\
= \;& (\theta+\phi)^\top \Big[D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(v)dv + \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big].
\end{aligned}$$

We calculate

$$\begin{aligned}
&E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\
= \;& \exp\Big\{ -\Big\langle \theta+\phi, \Big[D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(w)dw\Big\rangle\Big\} \\
&\times E\Big[\exp\Big\{ -\Big\langle \theta+\phi, \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big\rangle\Big\}\Big|\mathcal{G}_t\Big].
\end{aligned}$$

By assumption, the conditional expectation is a.s. smaller than infinity and therefore we can invoke the prediction result of Theorem 4.2.2 (by extending it to $u \in \mathbb{C}$) to achieve

$$\begin{aligned}
&E\Big[\exp\Big\{ -\Big\langle \theta+\phi, \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big\rangle\Big\}\Big|\mathcal{G}_t\Big] \\
= \;& \exp\Big\{ -\Big\langle \theta+\phi, \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]}z^d(h\mathbf{1}_{[t,T]},\cdot),v)d\mathbf{L}^d(v)\Big\rangle\Big\} \\
&\times \exp\Big\{ \int_t^T \psi(z^d(h\mathbf{1}_{[t,T]},v)^\top i(\theta+\phi))dv\Big\},
\end{aligned}$$

with $h(\cdot) = D(\cdot,T)\sigma(\cdot)$. Putting everything together we obtain the assertion. □

Remark 4.3.4. Condition (4.7) is met if we assume the components of $\mathbf{L}^d$ to be fractional subordinators and the function $\sigma(\cdot)$ to be componentwise positive. These assumptions are economically justified since interest rates should be positive in most cases. We refer to Theorem 3.3 of Rajput and Rosinski [109] for more general conditions.

Remark 4.3.5. The Gaussian Vasicek model was already considered in Section 3.3 and 3.5 using different, partly more direct approaches. As explained on many occasions, it has the serious drawback that processes like the short rate could be negative. However in practice the Gaussian models are fast to implement, very tractable and, of course, it is always possible to scale and shift a Gaussian Vasicek model such that the probability

of becoming negative is arbitrarily small. However with the theory of this section we can propose models with non-negative short and default rates where bond prices still are comparably easy to calculate.

The remainder of the section is dedicated to the consideration of a specific example using the bivariate Poisson MG-fLp of Example 4.2.6. The components of this process are fractional subordinators in the sense of Remark 4.1.4 and have therefore the advantage of delivering non-negative short and hazard rates.

Example 4.3.6. [Fractional Poisson market] Assume $d(1), d(2) > 0$. Take the bivariate Poisson MG-fLp of Example 4.2.6 as driving process in the fractional market 4.3.2. Further, for simplicity, take $k(\cdot) = (k_1, k_2)^\top$, with $k_1, k_2 > 0$,

$$a(\cdot) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \sigma(\cdot) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

for $a_1, a_2, \sigma_1, \sigma_2 > 0$ and $\theta = (1, 0)^\top$, $\phi = (0, 1)^\top$. Therefore

$$\begin{aligned} dr(t) &= (k_1 - a_1 r(t))dt + \sigma_1 dL^{d(1)}(t), \quad r(0) = r_0 \in \mathbb{R}, \\ d\lambda(t) &= (k_2 - a_2 \lambda(t))dt + \sigma_2 dL^{d(2)}(t), \quad \lambda(0) = \lambda_0 \in \mathbb{R}, \end{aligned}$$

where $L^{d(1)}$ and $L^{d(2)}$ are dependent.

We apply now Theorem 4.3.3. Condition (4.7) is met since the process in the exponential is non-positive. Further we have

$$\begin{aligned} D(t,T) &= \int_t^T e^{-\int_t^s a(v)dv} ds = \begin{pmatrix} \int_t^T e^{-a_1(s-t)} ds & 0 \\ 0 & \int_t^T e^{-a_2(s-t)} ds \end{pmatrix} \\ &=: \begin{pmatrix} D_1(t,T) & 0 \\ 0 & D_2(t,T) \end{pmatrix}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & -\Big\langle \theta + \phi, D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(v)dv \Big\rangle \\ = \; & -D_1(t,T)r(t) - D_2(t,T)\lambda(t) - k_1 \int_t^T D_1(v,T)dv - k_2 \int_t^T D_2(v,T)dv \end{aligned}$$

and

$$
\begin{aligned}
& -\Big\langle \theta+\phi, \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(h(\cdot)\mathbf{1}_{[t,T]},\cdot),v)d\mathbf{L}^d(v)\Big\rangle \\
= \; & -\sigma_1 \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_1(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)dL^{d(1)}(v) \\
& -\sigma_2 \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_2(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)dL^{d(2)}(v) \\
= \; & -\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_1(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_1 dv + a_1 r(v)dv + dr(v)) \\
& -\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_2(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_2 dv + a_2 \lambda(v)dv + d\lambda(v))
\end{aligned}
$$

where we used Lemma 4.2.10 in the last line. Applying the characterization of the Lévy measure from Example 4.2.6, we calculate

$$
\begin{aligned}
& \psi(z^d(h\mathbf{1}_{[t,T]},v)^\top i(\theta+\phi)) \\
= \; & \psi\left(\begin{pmatrix} \sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) & 0 \\ 0 & \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v) \end{pmatrix} i(\theta+\phi)\right) \\
= \; & \psi\left(\begin{pmatrix} i\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) \\ i\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v) \end{pmatrix}\right) \\
= \; & \eta_1(\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1) \\
& +\eta_2(\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) - \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1) \\
& +\eta_3(\exp(-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)
\end{aligned}
$$

Putting everything together and applying Theorem 4.3.3 leads for all $0 \le t \le T \le T^\star$ to

the bond prices

$$
\begin{aligned}
&\overline{B}(t,T)\\
=\;& \mathbf{1}_{\{\tau>t\}} E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big]\\
=\;& \mathbf{1}_{\{\tau>t\}} \exp\Big\{ -D_1(t,T)r(t) - D_2(t,T)\lambda(t) - k_1\int_t^T D_1(v,T)dv - k_2\int_t^T D_2(v,T)dv\Big\}\\
&\times \exp\Big\{ -\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_1(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_1dv + a_1 r(v)dv + dr(v))\Big\}\\
&\times \exp\Big\{ -\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_2(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_2dv + a_2 \lambda(v)dv + d\lambda(v))\Big\}\\
&\times \exp\Big\{\eta_1\int_t^T (\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)dv\Big\}\\
&\times \exp\Big\{\eta_2\int_t^T (\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) - \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)dv\Big\}\\
&\times \exp\Big\{\eta_3\int_t^T (\exp(-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)dv\Big\}.
\end{aligned}
$$

Remark 4.3.7. If we want to choose $d = (0,0)^\top$ in the context of Example 4.3.6, short and hazard rate are driven by Lévy processes which are a subclass of affine Markov processes. Therefore the bond prices can be calculated to

$$
\begin{aligned}
&\overline{B}(t,T)\\
=\;& \mathbf{1}_{\{\tau>t\}} \exp\Big\{ -D_1(0,T)r(0) - D_2(0,T)\lambda(0) - k_1\int_0^T D_1(v,T)dv - k_2\int_0^T D_2(v,T)dv\Big\}\\
&\times \exp\Big\{\eta_1\int_t^T (\exp(-\sigma_1 D_1(v,T)) - 1)dv + \eta_3\int_t^T (\exp(-\sigma_2 D_2(v,T)) - 1)dv\Big\}\\
&\times \exp\Big\{\eta_2\int_t^T (\exp(-\sigma_1 D_1(v,T) - \sigma_2 D_2(v,T)) - 1)dv\Big\},
\end{aligned}
$$

which represents the affine structure, see e.g. Duffie [44] and Duffie, Filipovic and Schachermayer [45]. However, if $d \neq (0,0)^\top$, the past paths of short and default rate matter and will enter the prices.

To compare prices we consider the case $t = 0$.

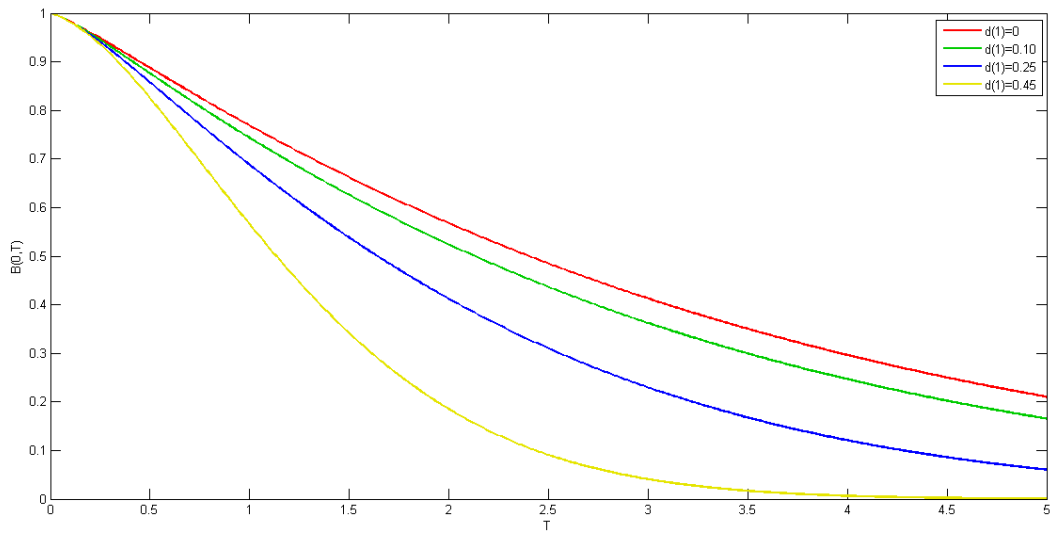

Figure 4.1: *Bond prices $B(0,t)$ in the fractional Poisson market 4.3.6 for varying $d(1)$ and maturity t, using $(\eta_1,\eta_2,\eta_3)^\top = (2,1,2)^\top$, $(r(0),\lambda(0))^\top = (0.1,0.05)^\top$, $k_1 = 0.5$, $k_2 = 1$, $a_1 = 4$, $a_2 = 8$, $\sigma_1 = 2$, $\sigma_2 = 1$ and $d(2) = 0.25$. Recall that $(d(1),d(2))^\top = (0,0)^\top$ corresponds to the Lévy Vasicek model of Remark 4.3.7. Prices decrease with $d(1)$ as a consequence of the long range dependence, which is very surprising, cf. Remark 4.3.9.*

Example 4.3.8. In the situation of Example 4.3.6 we have

$$\begin{aligned}
&\overline{B}(0,T)\\
=\;& \exp\Big\{-D_1(0,T)r(0) - D_2(0,T)\lambda(0) - k_1\int_0^T D_1(v,T)dv - k_2\int_0^T D_2(v,T)dv\Big\}\\
&\times\exp\Big\{\eta_1\int_0^T(\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]},v))-1)dv\Big\}\\
&\times\exp\Big\{\eta_2\int_0^T(\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]},v)-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[0,T]},v))-1)dv\Big\}\\
&\times\exp\Big\{\eta_3\int_0^T(\exp(-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[0,T]},v))-1)dv\Big\}.
\end{aligned}$$

Because of the singularities in the operator z^d classical numerical methods have to be used with care. We will choose a similar discretization scheme, as in the fBm case, cf. Remark 3.3.13. This will be explained for the first occurring fractional integration:

For $d(1) \in (0, \frac{1}{2})$ and $0 \leq t \leq T \leq T^\star$ we have

$$\begin{aligned} &\int_0^T \exp(-\sigma_1 z^{d(1)}(D_1(\cdot, T)\mathbf{1}_{[0,T]}, v))dv \\ = &\int_0^T \exp\Big(- \sigma_1 c_{d(1)} v^{-d(1)} \int_v^{T^\star} \frac{r^{d(1)} D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr\Big) dv. \end{aligned}$$

First we decompose the outer integral for $m \in \mathbb{N}$ and $0 = v_0 \leq v_1 \leq \cdots \leq v_m = T$

$$= \sum_{i=0}^{n-1} \int_{v_i}^{v_{i+1}} \exp\Big(- \sigma_1 c_{d(1)} v^{-d(1)} \int_v^T \frac{r^{d(1)} D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr\Big) dv.$$

Now by Remark 3.3.13 we have for sufficiently small intervals $[v_i, v_{i+1}]$, subpartitions $v_i = w_0^i \leq w_1^i \leq \cdots \leq w_{m_i}^i = v_{i+1}$ for some $m_i \in \mathbb{N}$, $i = 0, \ldots, m-1$, and $v \in [v_i, v_{i+1}]$

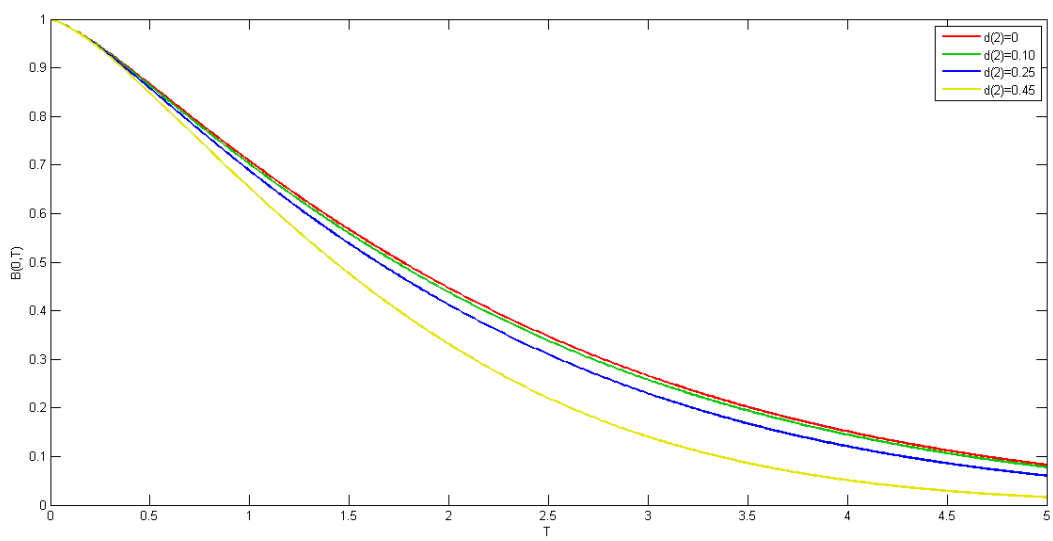

Figure 4.2: *Bond prices $B(0,t)$ in the fractional Poisson market 4.3.6 for varying $d(2)$ and maturity t, using $(\eta_1, \eta_2, \eta_3)^\top = (2,1,2)^\top$, $(r(0), \lambda(0))^\top = (0.1, 0.05)^\top$, $k_1 = 0.5$, $k_2 = 1$, $a_1 = 4$, $a_2 = 8$, $\sigma_1 = 2$, $\sigma_2 = 1$ and $d(1) = 0.25$. Recall that $(d(1), d(2))^\top = (0,0)^\top$ corresponds to the Lévy Vasicek model of Remark 4.3.7. Prices decrease with $d(2)$ as a consequence of the long range dependence, cf. Remark 4.3.9.*

$$\begin{aligned}
&\int_v^T \frac{r^{d(1)} D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr \\
\approx\ & \frac{1}{2d(1)} \sum_{j=0}^{m_i-1} [(w_{j+1}^i - v_i)^{d(1)} - (w_j^i - v_i)^{d(1)}][(w_j^i)^{d(1)} D(w_j^i, T) + (w_{j+1}^i)^{d(1)} D(w_{j+1}^i, T)] \\
=:\ & A(v_i)
\end{aligned}$$

Putting everything together we get

$$\begin{aligned}
&\int_0^T \exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]}, v))dv \\
=\ & \sum_{i=0}^{n-1} \int_{v_i}^{v_{i+1}} \exp\Big(- \sigma_1 c_{d(1)} v^{-d(1)} \int_v^T \frac{r^{d(1)} D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr\Big) dv \\
\approx\ & \sum_{i=0}^{n-1} \int_{v_i}^{v_{i+1}} \exp\Big(- \sigma_1 c_{d(1)} A(v_i) v^{-d(1)}\Big) dv \\
\approx\ & \sum_{i=0}^{n-1} [v_{i+1} - v_i] \exp\Big(- \sigma_1 c_{d(1)} A(v_i)[v_{i+1} - v_i]/2\Big).
\end{aligned}$$

Choosing now $v_i = 0.01i$, $i = 0, \ldots, 100t$, and $w_j^i = 0.01(i+j)$, $j = 0, \ldots, 100t - i$, we obtain

$$A(v_i) = \sum_{j=0}^{100t-i-1} [(j+1)^\kappa - j^\kappa][(i+j)^\kappa D(0.01(i+j), t) + (i+j+1)^\kappa D(0.01(i+j+1), t)]$$

and

$$\int_0^T \exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]}, v))dv \approx 0.01 \sum_{i=0}^{n-1} \exp\Big(- 0.005\sigma_1 c_{d(1)} A(v_i)\Big). \tag{4.8}$$

Remark 4.3.9. At first sight it is surprising that in the case of a fractional Poisson market prices decrease with $d(1)$ and $d(2)$, since in the Gaussian case the contrary is the case (cf. Section 3.3 and further considerations in Chapter 6). The reason behind this is the following: in a fractional Poisson market, short and default rate increase with the shocks of the driving Poisson subordinators and decrease between these exponentially. An increase in $d(1)$ and $d(2)$ means an increase in the (positive) correlation between these shocks. Therefore short and default rate are more likely to go up, which leads the bond price to drop.

In the Gaussian case, the driving processes are not increasing and an increase in $d(1)$ and $d(2)$ does no longer affect short and default rate as above, since there is now also a (positive) correlation between decreases of the driving processes.

4.4 Application: Fractional volatility in a Black Scholes market

In this section we want to show another possible application of our prediction result from Theorem 4.2.2. While we concentrated mainly on interest rate and credit markets in the previous sections, we will now focus on classical stock markets. It is our aim to price classical vanilla call options in a Black-Scholes market with fractional volatility.

In the following we will use the setup of Bender and Marquardt [16] who introduced a financial market model driven by a standard Brownian motion, time changed by a univariate convoluted Lévy process. For the later, [16] allows for many different kernel functions including the fractional subordinators of Example 4.1.4, (ii). However we will directly state the model of [16] using such fractional subordinators. By Theorem 4.2.2 we are able to calculate the whole price process of a European call.

4.4.1 The market model

For a time horizon $T^\star > 0$ assume a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T^\star}, \mathcal{P})$ where the filtration satisfies the usual conditions of completeness and right continuity. Let $W = (W(t))_{0\le t\le T^\star}$ be a standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{0\le t\le T^\star}$. Furthermore let $L^d = (L^d(t))_{0\le t\le T^\star}$, $d \in (0, \frac{1}{2})$, be a fractional subordinator in the sense of Example 4.1.4, (ii), independent of W and also adapted $(\mathcal{F}_t)_{0\le t\le T^\star}$. There exist two tradable assets: for $r \ge 0$, the process

$$B = (B(t))_{0\le t\le T^\star} = (\exp(rt))_{0\le t\le T^\star}$$

describes the risk-free money account while the price process of the only stock in the market is given by

$$S = (S(t))_{0\le t\le T^\star} = \Big(S(0)\exp\Big\{rt + \Big(\mu - \frac{\sigma^2}{2}\Big)L^d(t) + \sigma W(L^d(t))\Big\}\Big)_{0\le t\le T^\star}$$

for some $S(0) > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$.

In a first step we can calculate the conditional characteristic function of the logarithmic stock price using Theorem 4.2.2.

Theorem 4.4.1. *We have for all $0 \le s \le t \le T^\star$ and $u \in \mathbb{R}$*

$$\begin{aligned}
& E\Big[\exp\Big\{iu\log(S(t))\Big\}\Big|\mathcal{F}_s\Big] \\
= \;& \exp\{iu(\log(S(0)) + rt)\} \\
& \times \exp\Big\{\Big(iu\Big(\mu - \frac{\sigma^2}{2}\Big) - u^2\frac{\sigma^2}{2}\Big)L^d(s)\Big\} \\
& \times \exp\Big\{\Big(iu\Big(\mu - \frac{\sigma^2}{2}\Big) - u^2\frac{\sigma^2}{2}\Big)\int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(\mathbf{1}_{[s,t]},\cdot),v)dL^d(v)\Big\} \\
& \times \exp\Big\{\int_s^t \psi\Big(z^d(\mathbf{1}_{[s,t]},v)\Big(u\Big(\mu - \frac{\sigma^2}{2}\Big) + iu^2\frac{\sigma^2}{2}\Big)\Big)dv\Big\}.
\end{aligned}$$

Proof. Let $0 \le s \le t \le T^\star$. We calculate using the independence of W and L^d

$$\begin{aligned}
& E\Big[\exp\Big\{iu\log(S(t))\Big\}\Big|\mathcal{F}_s\Big] \\
= \;& \times E\Big[\exp\Big\{iu\Big(\log(S(0)) + rt + \Big(\mu - \frac{\sigma^2}{2}\Big)L^d(t) + \sigma W(L^d(t))\Big)\Big\}\Big|\mathcal{F}_s\Big] \\
= \;& \exp\{iu(\log(S(0)) + rt)\} \times E\Big[\exp\Big\{iu\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)L^d(t) + \sigma W(L^d(t))\Big)\Big\}\Big|\mathcal{F}_s\Big] \\
= \;& \exp\{iu(\log(S(0)) + rt)\} \times E\Big[E\Big[\exp\Big\{iu\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)L^d(t) + \sigma W(L^d(t))\Big)\Big\}\Big|\mathcal{F}^{L^d}_{T^\star}\Big]\Big|\mathcal{F}_s\Big] \\
= \;& \exp\{iu(\log(S(0)) + rt)\} \\
& \times E\Big[\exp\Big\{iu\Big(\mu - \frac{\sigma^2}{2}\Big)L^d(t)\Big\}E\Big[\exp\Big\{iu\sigma W(L^d(t))\Big\}\Big|\mathcal{F}^{L^d}_{T^\star}\Big]\Big|\mathcal{F}_s\Big] \\
= \;& \exp\{iu(\log(S(0)) + rt)\} \times E\Big[\exp\Big\{\Big(iu\Big(\mu - \frac{\sigma^2}{2}\Big) - u^2\frac{\sigma^2}{2}\Big)L^d(t)\Big\}\Big|\mathcal{F}_s\Big].
\end{aligned}$$

Since L^d is strictly increasing, its conditional Laplace transform $E[\exp\{-wL^d(t)\}|\mathcal{F}_s]$ exists for $w \in \mathbb{R}^+$. Therefore we can invoke Theorem 4.2.2 for $f = \mathbf{1}_{[0,t]}$ and

$$u^\star := \Big(u\Big(\mu - \frac{\sigma^2}{2}\Big) + iu^2\frac{\sigma^2}{2}\Big)$$

by extending it $\mathbb{C}^-$. Then we obtain

$$\begin{aligned}
& E\Big[\exp\Big\{\Big(iu\Big(\mu - \frac{\sigma^2}{2}\Big) - u^2\frac{\sigma^2}{2}\Big)L^d(t)\Big\}\Big|\mathcal{F}_s\Big] \\
= \;& E\Big[\exp\Big\{iu^\star L^d(t)\Big\}\Big|\mathcal{F}_s\Big] \\
= \;& \exp\Big\{iu^\star L^d(s) + iu^\star\int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(\mathbf{1}_{[s,t]},\cdot),v)dL^d(v) + \int_s^t \psi(z^d(\mathbf{1}_{[s,t]},v)^\top u^\star)dv\Big\} \\
= \;& \exp\Big\{\Big(iu\Big(\mu - \frac{\sigma^2}{2}\Big) - u^2\frac{\sigma^2}{2}\Big)L^d(s)\Big\} \\
& \times \exp\Big\{\Big(iu\Big(\mu - \frac{\sigma^2}{2}\Big) - u^2\frac{\sigma^2}{2}\Big)\int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(\mathbf{1}_{[s,t]},\cdot),v)dL^d(v)\Big\} \\
& \times \exp\Big\{\int_s^t \psi\Big(z^d(\mathbf{1}_{[s,t]},v)\Big(u\Big(\mu - \frac{\sigma^2}{2}\Big) + iu^2\frac{\sigma^2}{2}\Big)\Big)dv\Big\},
\end{aligned}$$

which concludes the proof. □

Remark 4.4.2. *Setting $s = 0$ in Theorem 4.4.1 we obtain*

$$\begin{aligned} & E\Big[\exp\Big\{iu\log(S(t))\Big\}\Big] \\ = \; & \exp\Big\{iu(\log(S(0)) + rt) + \int_0^t \psi\Big(z^d(\mathbf{1}_{[0,t]}, v)\Big(u\Big(\mu - \frac{\sigma^2}{2}\Big) + iu^2\frac{\sigma^2}{2}\Big)\Big)dv\Big\}. \end{aligned}$$

which equals the result of Theorem 2 of Bender and Marquardt [16] for $k(t,s) := z^d(\mathbf{1}_{[0,t]}, s)$.

4.4.2 Absence of arbitrage and option pricing

In a next step Bender and Marquardt [16] showed in their Section 4 that the model above is arbitrage-free but incomplete by explicitly constructing equivalent martingale measures. The authors further showed that under such risk-neutral dynamics the stock price equals

$$S = (S(t))_{0\le t\le T^\star} = \Big(S(0)\exp\Big\{rt + \sigma\widetilde{W}(L^d(t)) - \frac{1}{2}\sigma^2 L^d(t)\Big\}\Big)_{0\le t\le T^\star},$$

where $\widetilde{W} = (\widetilde{W}(t))_{0\le t\le T^\star}$ is a standard Brownian motion independent of L^d. We will from now on work under such a measure without further comment. Remark that the occurring option prices are not the only who lead to an arbitrage-free market due to incompleteness.

By conditioning on $\mathcal{F}^{L^d}_{T^\star}$ we have:

Theorem 4.4.3 (Version of Theorem 4 of Bender and Marquardt [16])**.** *Let $X = (S(T^\star) - K)_+$ be a European call option with strike $K > 0$ and maturity $T^\star$. Then, for the initial fair price $C_0(K)$ of X, we have*

$$\begin{aligned} C_0(K) \;=\; & \int_0^\infty \Big[S(0)N\Big(\frac{\log(S(0)/K) + \sigma^2 t/2}{\sigma\sqrt{t}}\Big) \\ & -Ke^{-rT^\star}N\Big(\frac{\log(S(0)/K) - \sigma^2 t/2}{\sigma\sqrt{t}}\Big)\Big]F_{L^d(T^\star)}(dt), \end{aligned}$$

where N is the distribution function of the standard normal distribution and $F_{L^d(T^\star)}$ is the distribution function of $L^d(T^\star)$.

Using the result from Theorem 4.4.1 we can use Fourier pricing similar to Theorem 3.5.13 to calculate the whole price process $C(K) = (C_t(K))_{0\le t\le T^\star}$ of the European call X:

Theorem 4.4.4. *Let $X = (S(T^\star) - K)_+$ be a European call option with strike $K > 0$ and maturity $T^\star$. Assume that there exists $a > 1$ such that*

$$E\Big[e^{a\log(S(T^\star))}\Big] < \infty.$$

Then, for the fair price process $C(K)$ of X, we have

$$\begin{aligned}
& C_t(K) \\
= \; & \int_{\mathbb{R}} E\Big[e^{(a+i\xi)\log(S(T^\star))}\Big|\mathcal{F}_t\Big]\widehat{f}_a(\xi)d\xi \\
= \; & e^{a(\log(S(0))+rt)}\exp\Big\{\frac{\sigma^2}{2}(a^2-a)\Big(L^d(t)+\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]}z^d(\mathbf{1}_{[0,T^\star]},\cdot),v)dL^d(v)\Big)\Big\} \\
& \times\int_{\mathbb{R}}\Big(\exp\Big\{i\xi(\log(S(0))+rt)\Big\} \\
& \exp\Big\{\frac{\sigma^2}{2}(2ia\xi-i\xi-\xi^2)\Big(L^d(t)+\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]}z^d(\mathbf{1}_{[0,T^\star]},\cdot),v)dL^d(v)\Big)\Big\} \\
& \times\exp\Big\{\int_t^{T^\star}\psi\Big(\frac{\sigma^2 z^d(\mathbf{1}_{[0,T^\star]},v)}{2}\Big((ia-\xi)+(2a\xi+i(\xi^2-a^2))\Big)\Big)dv\Big\}\widehat{f}_a(\xi)\Big)d\xi
\end{aligned}$$

for $0\le t\le T^\star$ with

$$\widehat{f}_a(\xi)=\frac{e^{-(a-1+i\xi)\log(K)}}{(a+i\xi)(a-1+i\xi)},\quad \xi\in\mathbb{R}.$$

Proof. Define for $a>1$ the function

$$f_a(x):=e^{-ax}(e^x-K)_+,\quad x\in\mathbb{R}.$$

Then $f_a\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. By Example 5.6, (B), of Biagini, Fuschini and Klüppelberg [19] we get for the Fourier transform $\widehat{f}_a$

$$\widehat{f}_a(\xi)=\int_{\log(K)}^\infty e^{-i\xi x}f_a(x)dx=\int_{\log(K)}^\infty e^{-x(a+i\xi)}(e^x-K)dx=\frac{e^{-(a-1+i\xi)\log(K)}}{(a+i\xi)(a-1+i\xi)},\quad \xi\in\mathbb{R}.$$

Further Example 5.6, (B), of [19] shows that $\widehat{f}_a\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Therefore we have by classical Fourier analysis

$$f_a(x)=\int_{\mathbb{R}}e^{i\xi x}\widehat{f}_a(\xi)d\xi,\quad x\in\mathbb{R}.$$

Using this we obtain for $0\le t\le T^\star$

$$\begin{aligned}
C_t(K) &= E[(S(T^\star)-K)_+|\mathcal{F}_t]=E\Big[e^{a\log(S(T^\star))}f_a(\log(S(T^\star))\Big|\mathcal{F}_t\Big] \\
&= E\Big[e^{a\log(S(T^\star))}f_a(\log(S(T^\star))\Big|\mathcal{F}_t\Big] \\
&= E\Big[e^{a\log(S(T^\star))}\int_{\mathbb{R}}e^{i\xi\log(S(T^\star))}\widehat{f}_a(\xi)d\xi\Big|\mathcal{F}_t\Big] \\
&= E\Big[e^{a\log(S(T^\star))}\int_{\mathbb{R}}e^{i\xi\log(S(T^\star))}\widehat{f}_a(\xi)d\xi\Big|\mathcal{F}_t\Big] \\
&= \int_{\mathbb{R}}E\Big[e^{(a+i\xi)\log(S(T^\star))}\Big|\mathcal{F}_t\Big]\widehat{f}_a(\xi)d\xi
\end{aligned}$$

where we used the fact that $E\Big[e^{a\log(S(T^\star))}\Big] < \infty$. Therefore we can also extend Theorem 4.4.1 to get

$$\begin{aligned}
& E\Big[e^{(a+i\xi)\log(S(T^\star))}\Big|\mathcal{F}_t\Big] \\
= \;& \exp\{(a+i\xi)(\log(S(0))+rT^\star))\} \\
& \times \exp\Big\{\frac{\sigma^2}{2}(a^2+2ia\xi-a-i\xi-\xi^2)\Big(L^d(t)+\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]}z^d(\mathbf{1}_{[0,T^\star]},\cdot),v)dL^d(v)\Big)\Big\} \\
& \times \exp\Big\{\int_t^{T^\star}\psi\Big(\frac{\sigma^2 z^d(\mathbf{1}_{[0,T^\star]},v)}{2}\Big((ia-\xi)+(2a\xi+i(\xi^2-a^2))\Big)\Big)dv\Big\}.
\end{aligned}$$

Putting everything together concludes the proof.

□

Remark 4.4.5. *If we chose $t=0$ in Theorem 4.4.4 we get*

$$\begin{aligned}
C_0(K) \;=\; & S(0)^a\int_{\mathbb{R}}\Big(\exp\Big\{i\xi(\log(S(0)))\Big\} \\
& \times\exp\Big\{\int_0^{T^\star}\psi\Big(\frac{\sigma^2 z^d(\mathbf{1}_{[0,T^\star]},v)}{2}\Big((ia-\xi)+(2a\xi+i(\xi^2-a^2))\Big)\Big)dv\Big\}\widehat{f}_a(\xi)\Big)d\xi
\end{aligned}$$

Chapter 5

Application: Interest rate models and parameter sensitivity

In Sections 3.3, 3.5 and 4.3 we motivated and introduced two different interest rate models driven by fractional Vasicek dynamics. In particular the approach in Section 3.3 was based on a fractional Brownian motion. In the classical work of Vasicek [130] the drawback of getting a negative short rate r with positive probability was justified by the significant advantage of having a market model that provides easily tractable bond prices with analytical formulas. As mentioned in the introduction of Section 3.3 one can always shift and scale the model to make the probability of a negative short rate as small as possible.

The fractional Brownian Vasicek model of Section 3.3 (which includes the classical Vasicek model by setting $\kappa = 0$) has the same drawback of a potential negative short rate and still provides an analytical pricing formula for zero coupon bonds (cf. Theorem 3.3.11). However in comparison to [130] the numerics have become more difficult since the appearing norms cannot be calculated analytically due to fractional integration, cf. Remark 3.3.13. Still, it is the natural extension of the classical model and allows long range dependence in the increments of the short rate.

The fractional Lévy models of Section 4.3 addressed the above mentioned drawback of the Gaussian setting from Section 3.3: When using fractional subordinators as driving processes it is ensured that the short rate r cannot become negative. Also the model still allows for fairly explicit calculations of zero coupon bond prices as can be seen by Theorem 4.3.3. For simplicity we shall set $\phi = 0$ in (4.6) for the following considerations. As a consequence, no default is possible and we can compare the fractional Lévy models directly to the fractional Vasicek model from 3.3.

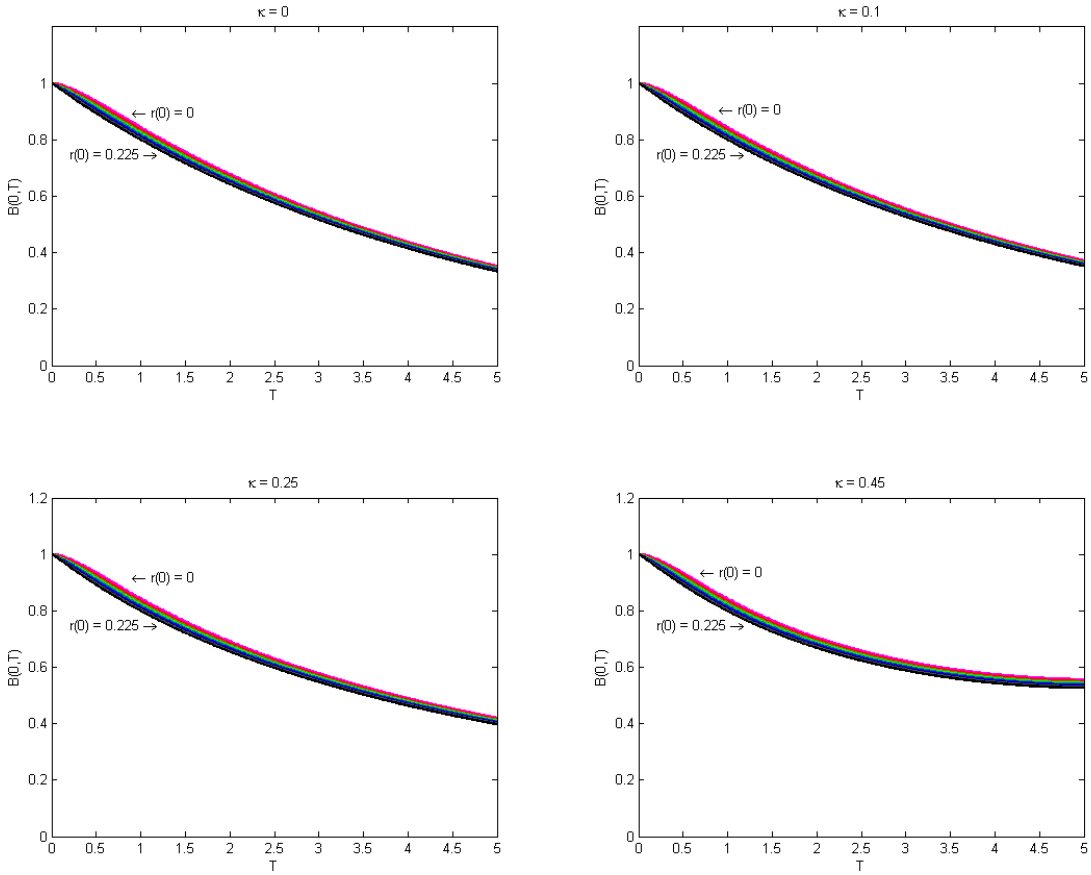

Figure 5.1: *Bond prices* $B^{fBm}(0,T)$ *in the fractional Brownian model for varying* $r(0)$*, maturity* T *and fractional parameter* κ*, using constant coefficients* $a = 4$*,* $k = 1$ *and* $\sigma = 1$*.* $\kappa = 0$ *corresponds to the classical Brownian Vasicek model. In particular* $r(0)$ *increases by steps of size* 0.025 *from* 0 *to* 0.225*.*

The section will be dedicated to a detailed analysis of the bond price dynamics in these models since in practical considerations parameter sensitivities play an important role. For example in the classical Black-Scholes model of Black and Scholes [22] these sensitives are captured by the so-called *greeks* which are basically the derivatives with respect to the individual parameters. Market participants can apply the greeks to carry out a ceteris paribus analysis and approximate how the prices would change under certain assumptions.

In the Black-Scholes model the greeks also play an important role when building a hedging strategy. Of course these considerations are only valid if the model assumptions are fulfilled. Therefore such derivatives should be used with care. Now we want to compare the following fractional short rate models (cf. Theorem 3.3.11 and Theorem 4.3.3):

Model type	**Zero bond price for $\mathbf{t = 0}$ and $\mathbf{0 \leq T \leq T^\star}$**	**Parameters**
Fractional Brownian	$\exp\big\{ - D(0,T)r(0) - k\int_0^T D(v,T)dv\big\}$ $\times \exp\big\{\frac{\sigma^2}{2}\|D(\cdot,T)\mathbf{1}_{[0,T]}(\cdot)\|^2_{\kappa,T^\star}\big\}$	$r(0), k, \sigma \geq 0$ $a > 0, \kappa \in [0, \frac{1}{2})$
Fractional Poisson	$\exp\big\{ - D(0,T)r(0) - k\int_0^T D(v,T)dv\big\}$ $\times \exp\big\{\eta\int_0^T \exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]}, v))dv\big\}$	$r(0), k, \sigma \geq 0$ $a, \eta > 0, d \in [0, \frac{1}{2})$

In the following we will execute a numerical study to analyze the impact of parameter changes on the bond price and calculate the derivatives with respect to the individual parameters. Recall that in the case of time independent coefficient functions as above we have for $0 < T < T^\star$

$$D(t,T) = \int_t^T e^{-\int_t^v a(w)dw}dv = \frac{1-e^{-a(T-t)}}{a}, \quad t \in [0,T].$$

In particular we have

$$D(0,T) = \frac{1-e^{-aT}}{a}, \quad t \in [0,T].$$

We will carry out this study by letting one parameter vary while the rest will be fixed for the time being. Reference parameters shall be

$$\begin{aligned} r(0) &= 0.1 \\ a &= 4 \\ k &= 1 \\ \sigma &= 1 \end{aligned}$$

For the fractional Poisson model the reference intensity shall be $\eta = 1$. As in Sections 3.3 and 4.3 the fractional parameters κ and d will take the values $0, 0.1, 0.25$ and 0.45.

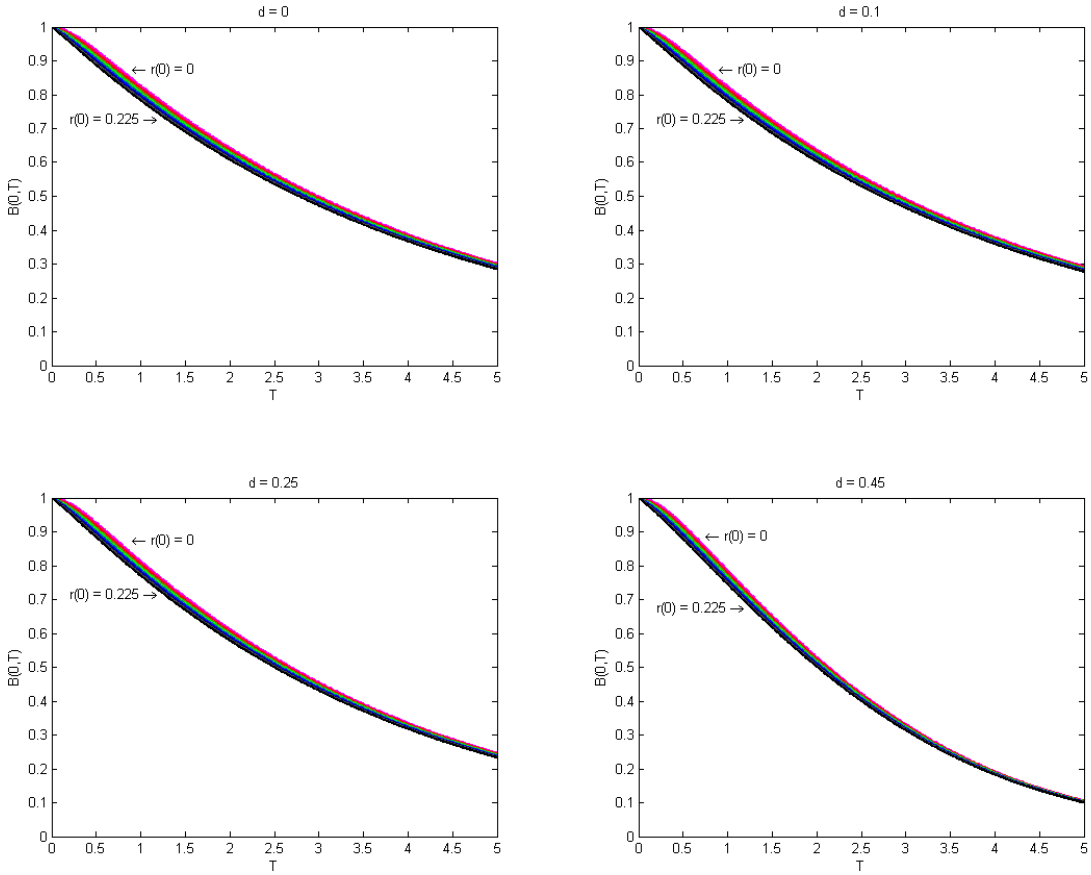

Figure 5.2: *Bond prices* $B^{Poi}(0,T)$ *in the fractional Poisson model for varying* $r(0)$*, maturity* T *and fractional parameter* d*, using constant coefficients* $a = 4$*,* $k = 1$*,* $\sigma = 1$ *and* $\eta = 1$*. In particular* $r(0)$ *increases by steps of size* 0.025 *from* 0 *to* 0.225*.*

Denote for the rest of this section the zero coupon bond price in the fractional Brownian model by $B^{fBm}(0,T)$ and in the fractional Poisson model by $B^{Poi}(0,T)$, $0 \leq T \leq T^{\star}$. Considering the bond price as a function with respect to a certain parameter will be denoted by subscription, e.g. $B^{fBm}_{r(0)}(0,T)$.

5.1 Parameter sensitivity with respect to $r(0)$

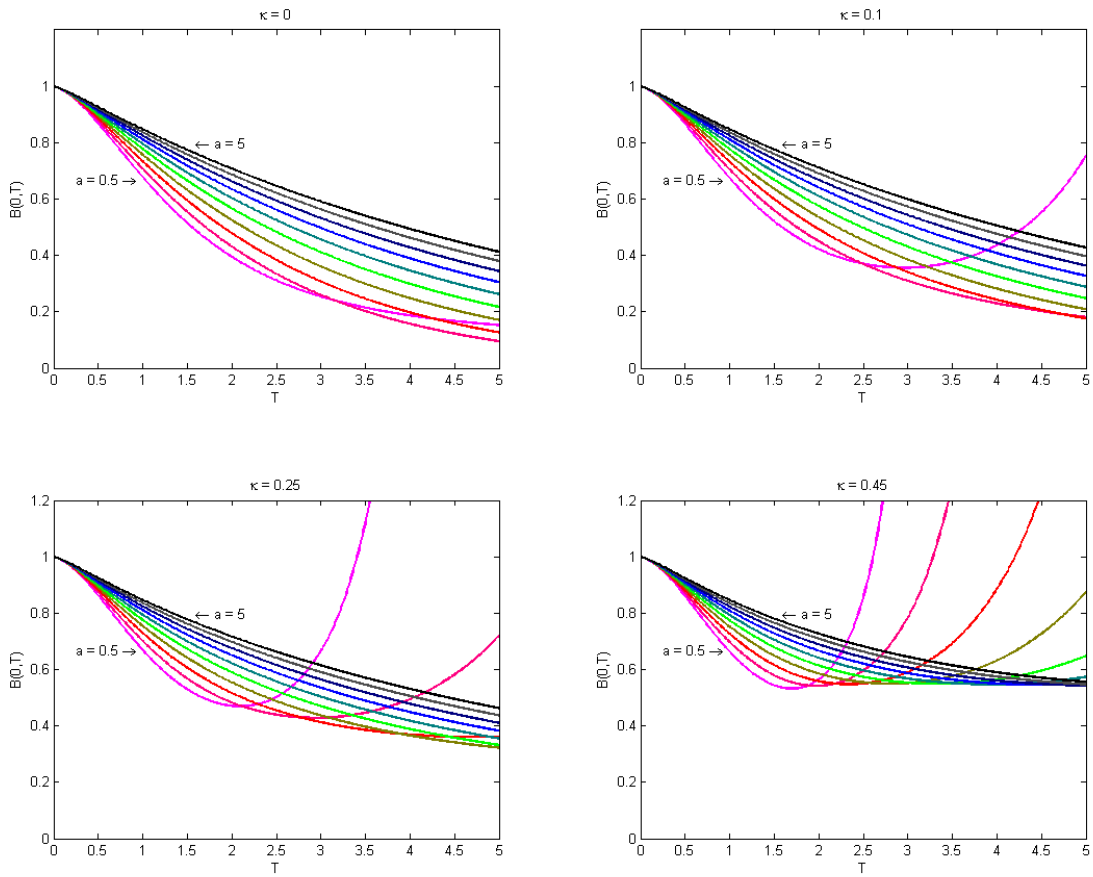

Figure 5.3: *Bond prices $B^{fBm}(0,T)$ in the fractional Brownian model for varying a, maturity T and fractional parameter κ, using constant coefficients $r(0) = 0.1$, $k = 1$ and $\sigma = 1$. $\kappa = 0$ corresponds to the classical Brownian Vasicek model. In particular a increases by steps of size 0.5 from 0.5 to 5.*

At $T = 0$ all bond prices are equal to 1 but afterwards a higher start interest rate $r(0)$ also leads to a lower bond price as can be seen in Figure 5.1 and Figure 5.2. For higher maturities T this difference becomes smaller since the weighting factor $D(0,T)$ in the pricing formula is bounded by a^{-1}. For all $\kappa, d \in [0, \frac{1}{2})$ the absolute influence of $r(0)$

on the bond price is equal while the relative influence varies. The derivative with respect to $r(0)$ exists and is given by

$$\begin{aligned}\frac{\partial B^{fBm}_{r(0)}(0,T)}{\partial r(0)} &= -D(0,T)\cdot B^{fBm}_{r(0)}(0,T),\\ \frac{\partial B^{Poi}_{r(0)}(0,T)}{\partial r(0)} &= -D(0,T)\cdot B^{Poi}_{r(0)}(0,T).\end{aligned}$$

5.2 Parameter sensitivity with respect to a

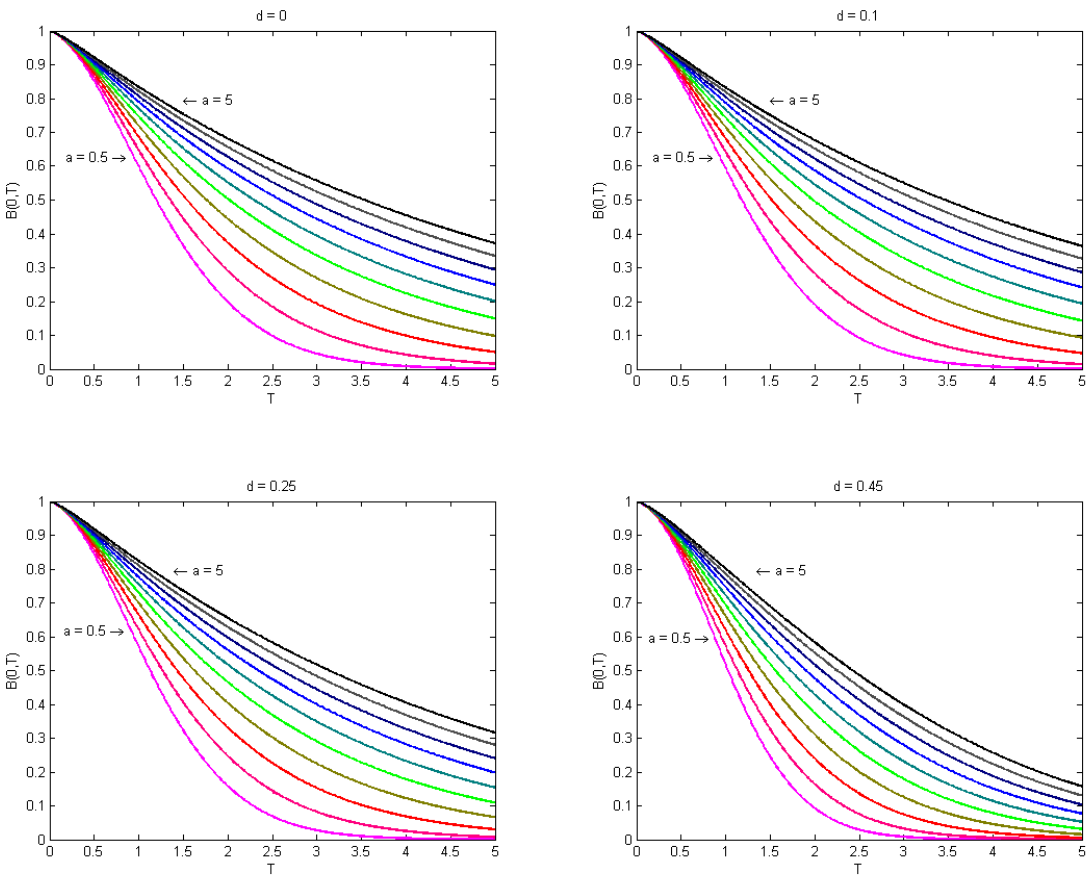

Figure 5.4: *Bond prices $B^{Poi}(0,T)$ in the fractional Poisson model for varying a, maturity T and fractional parameter d, using constant coefficients $r(0) = 0.1$, $k = 1$, $\sigma = 1$ and $\eta = 1$. In particular a increases by steps of size 0.5 from 0.5 to 5.*

As can be seen in Figure 5.3, the influence of the parameter a is more difficult now. The reason behind this is the fact that $D(\cdot, T)$ (and therefore a) is also involved via the fractional integration in the bond price formulas. However it can be seen from these formulas that the impact of the parameter a is still mostly monotone. If the probability of getting a negative short rate r is small enough, a higher value of a leads to a higher bond price. This can be explained by the following: the parameter a manages the speed of mean reversion of the short rate. With high probability any short time divergence from the mean will lead to a higher short rate (since negative values occur only with small probability) and therefore to a lower bond price. High values of a leads to a stronger mean-reversion and therefore the impact of such a potential divergence will be small and vice versa.

The suddenly increasing prices are explained by the positive probability of getting a negative short rate. The variance of r increases with the fractional parameter and therefore the probability of negative values increases. As a consequence bond prices tend to get higher for longer maturities. Combined with a weaker mean reversion (i.e. small values of a) this effect is even stronger. Since in the fractional Poisson model the short rate is always positive the influence of a is more straightforward as Figure 5.4 shows: a higher value of a leads ceteris paribus to a higher bond price. The calculation of the derivative with respect to a is more complicated and follows:

$$\begin{aligned}
&\frac{\partial B_a^{fBm}(0,T)}{\partial a} \\
= \quad & \left(-\frac{\partial D(0,T)}{\partial a} r(0) - k \frac{\partial \int_0^T D(v,T)dv}{\partial a} + \frac{\sigma^2}{2} \frac{\partial}{\partial a} \|D(\cdot,T)\mathbf{1}_{[0,T]}\|^2_{\kappa,T^\star} \right) B_a^{fBm}(0,T)
\end{aligned}$$

where we calculate using (3.33) and the classical rule for differentiation under the integral

sign

$$
\begin{aligned}
& \frac{\partial}{\partial a}\|D(\cdot,T)\mathbf{1}_{[0,T]}\|^2_{\kappa,T^\star} \\
= \quad & \frac{\partial}{\partial a}\frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)(\Gamma(\kappa))^2}\int_0^{T^\star} s^{-2\kappa}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds \\
= \quad & \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)(\Gamma(\kappa))^2}\int_0^{T^\star} s^{-2\kappa}\frac{\partial}{\partial a}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-s)^{1-\kappa}}dr\Big)^2 ds \\
= \quad & \frac{2\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)(\Gamma(\kappa))^2}\int_0^{T^\star} s^{-2\kappa}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-s)^{1-\kappa}}dr\Big) \\
& \times\frac{\partial}{\partial a}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-s)^{1-\kappa}}dr\Big)ds \\
= \quad & \frac{2\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)(\Gamma(\kappa))^2}\int_0^{T^\star} s^{-2\kappa}\Big(\int_s^{T^\star}\frac{r^\kappa D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-s)^{1-\kappa}}dr\Big) \\
& \times\Big(\int_s^{T^\star}\frac{r^\kappa \frac{\partial}{\partial a}D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-s)^{1-\kappa}}dr\Big)ds.
\end{aligned}
$$

We finally see that

$$
\frac{\partial}{\partial a}\|D(\cdot,T)\mathbf{1}_{[0,T]}\|^2_{\kappa,T^\star} = 2\langle D(\cdot,T),\frac{\partial}{\partial a}D(\cdot,T)\rangle_{\kappa,T^\star}.
$$

In the fractional Poisson case we have

$$
\begin{aligned}
& \frac{\partial B_a^{Poi}(0,T)}{\partial a} \\
= \quad & \left(-\frac{\partial D(0,T)}{\partial a}r(0) - k\frac{\partial\int_0^T D(v,T)dv}{\partial a} + \eta\frac{\partial}{\partial a}\int_0^T \exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v))dv\right) \\
& \times B_a^{fBm}(0,T)
\end{aligned}
$$

where we obtain again by interchanging differentiation and integration

$$
\begin{aligned}
& \frac{\partial}{\partial a}\int_0^T \exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v))dv \\
= \quad & \int_0^T \frac{\partial}{\partial a}\exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v))dv \\
= \quad & -\sigma\int_0^T \frac{\partial}{\partial a}z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v)\exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v))dv.
\end{aligned}
$$

Therefore we have in total

$$
\begin{aligned}
& \frac{\partial B_a^{fBm}(0,T)}{\partial a} \\
= \quad & \left(-\frac{\partial D(0,T)}{\partial a}r(0) - k\int_0^T \frac{\partial}{\partial a}D(v,T)dv + \sigma^2\langle D(\cdot,T),\frac{\partial}{\partial a}D(\cdot,T)\rangle_{\kappa,T^\star}\right) B_a^{fBm}(0,T)
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{\partial B_a^{Poi}(0,T)}{\partial a} \\
= &\Bigg(-\frac{\partial D(0,T)}{\partial a} r(0) - k\int_0^T \frac{\partial}{\partial a} D(v,T)dv \\
&-\eta\sigma \int_0^T \frac{\partial}{\partial a} z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v)\exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v))dv\Bigg) B_a^{Poi}(0,T).
\end{aligned}$$

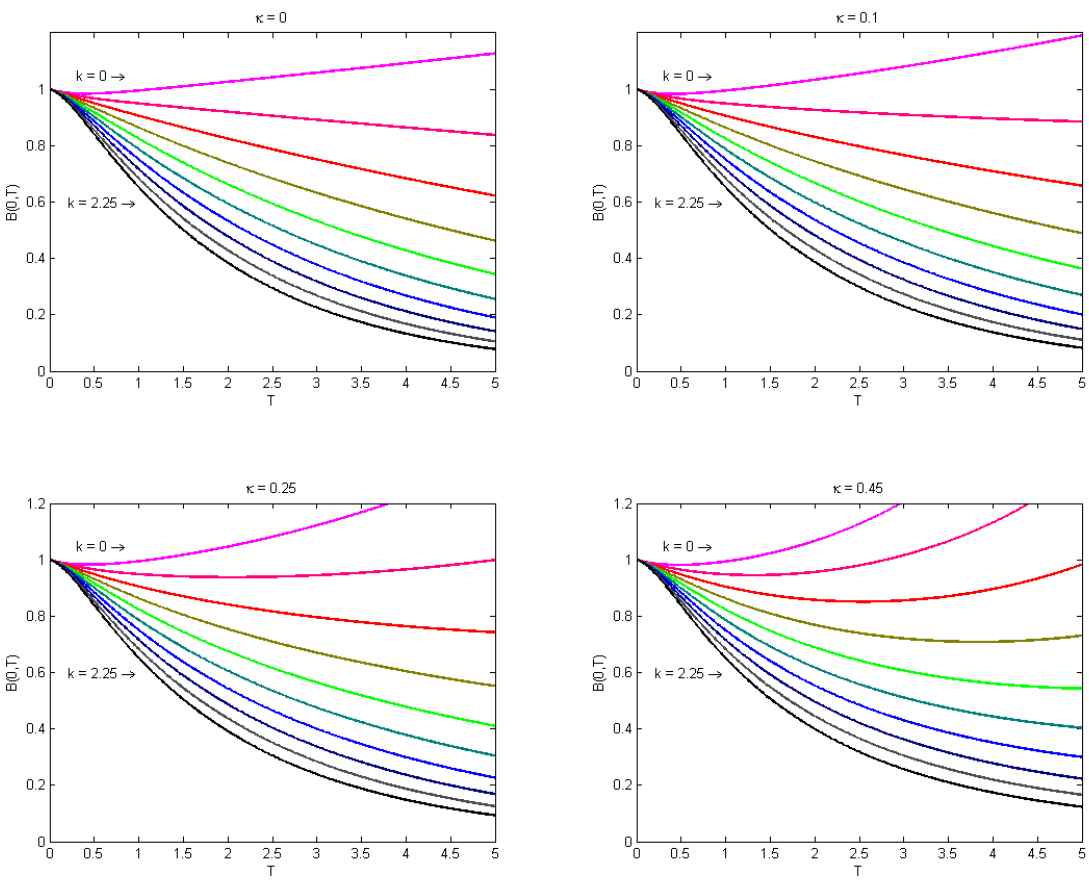

Figure 5.5: *Bond prices $B^{fBm}(0,T)$ in the fractional Brownian model for varying k, maturity T and fractional parameter κ, using constant coefficients $r(0) = 0.1$, $a = 4$ and $\sigma = 1$. $\kappa = 0$ corresponds to the classical Brownian Vasicek model. In particular k increases by steps of size 0.25 from 0 to 2.25.*

5.3 Parameter sensitivity with respect to k

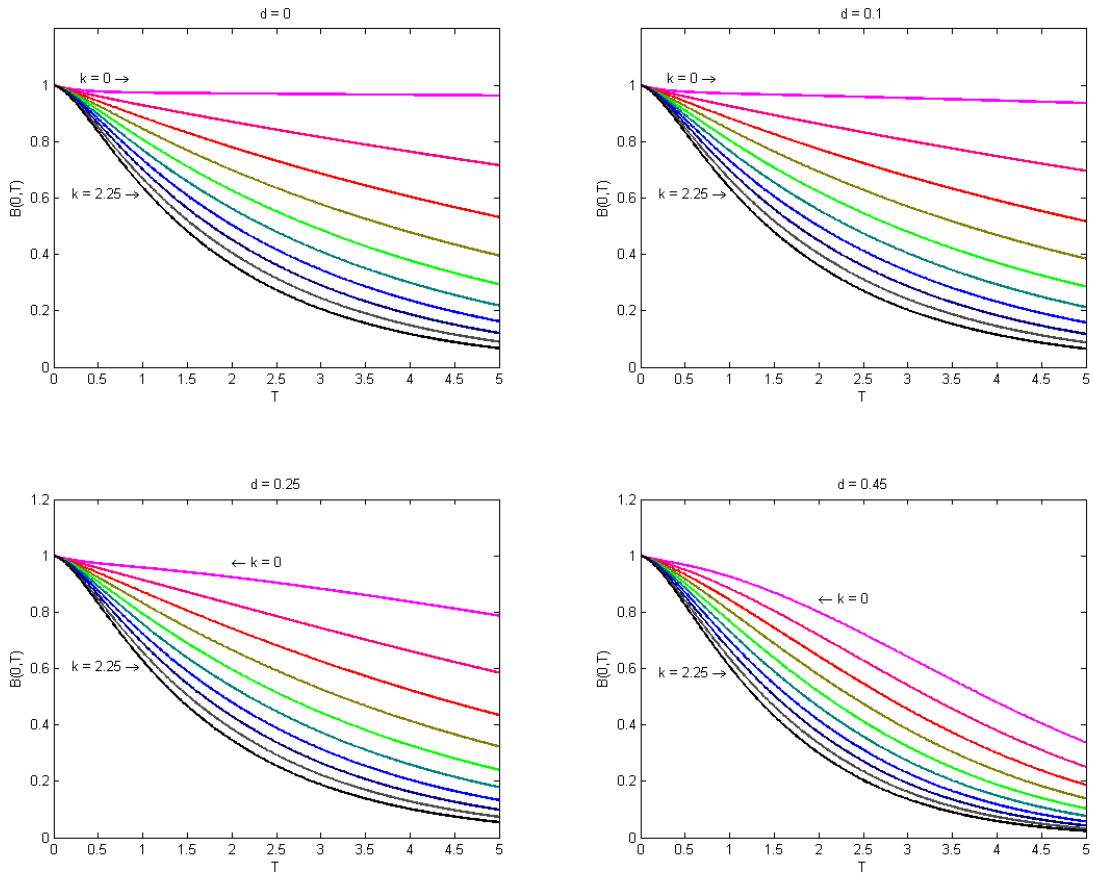

Figure 5.6: *Bond prices* $B^{Poi}(0,T)$ *in the fractional Poisson model for varying* k*, maturity* T *and fractional parameter* d*, using constant coefficients* $r(0) = 0.1$*,* $a = 4$*,* $\sigma = 1$ *and* $\eta = 1$*. In particular* k *increases by steps of size* 0.25 *from* 0 *to* 2.25*.*

The impact of the parameter k is again a bit more straightforward. Figure 5.5 shows that a higher value of k results mostly in a lower bond price since for fixed a, the parameter k controls the long term mean of the process r. For longer maturities the influence of k becomes stronger. A special case are again the suddenly increasing bond prices. As before they can be explained by the probability of negative values of the short rate. Lower values

of k increase this probability. Of course this is not an issue in the fractional Poisson model, cf. Figure 5.6. The derivative with respect to k exists and is given by

$$\begin{aligned}\frac{\partial B_k^{fBm}(0,T)}{\partial k} &= -\int_0^T D(v,T)dv \cdot B_k^{fBm}(0,T),\\ \frac{\partial B_k^{Poi}(0,T)}{\partial k} &= -\int_0^T D(v,T)dv \cdot B_k^{Poi}(0,T).\end{aligned}$$

5.4 Parameter sensitivity with respect to σ

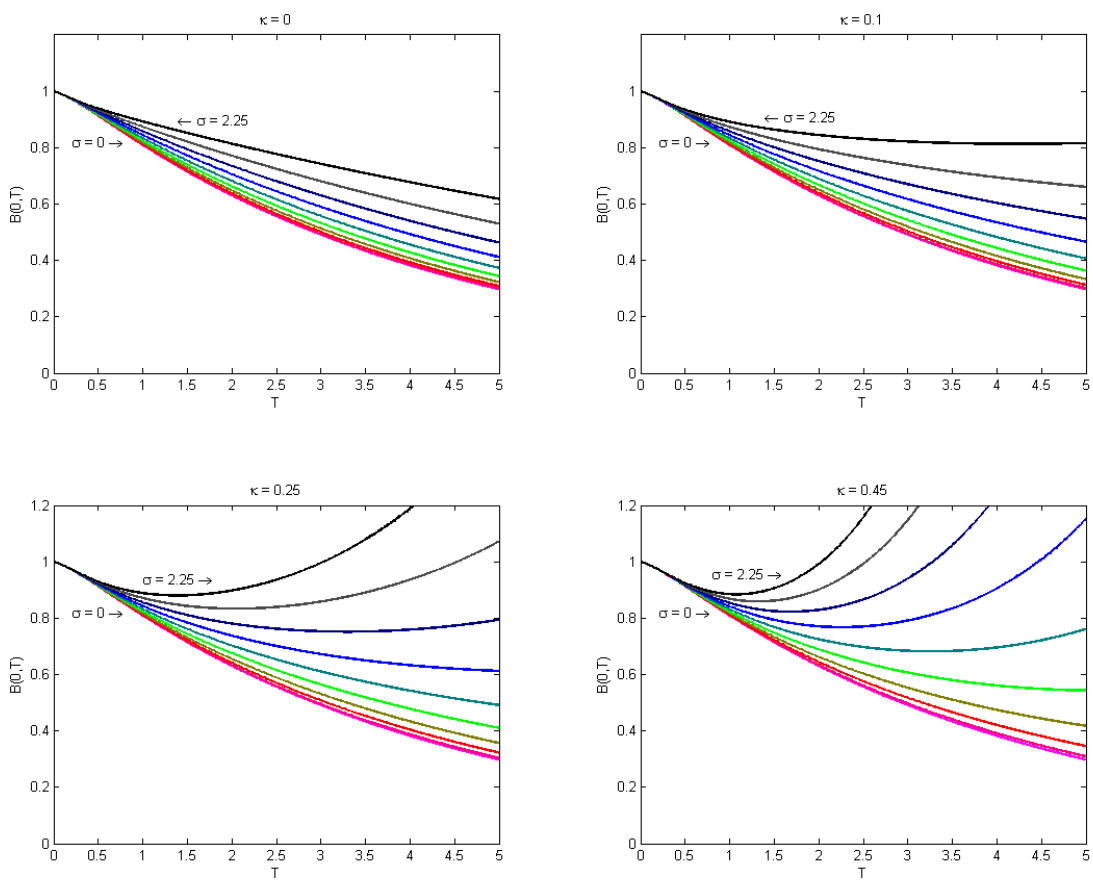

Figure 5.7: *Bond prices $B^{fBm}(0,T)$ in the fractional Brownian model for varying σ, maturity T and fractional parameter κ, using constant coefficients $r(0) = 0.1$, $a = 4$ and $k = 1$. $\kappa = 0$ corresponds to the classical Brownian Vasicek model. In particular σ increases by steps of size* 0.25 *from* 0 *to* 2.25.

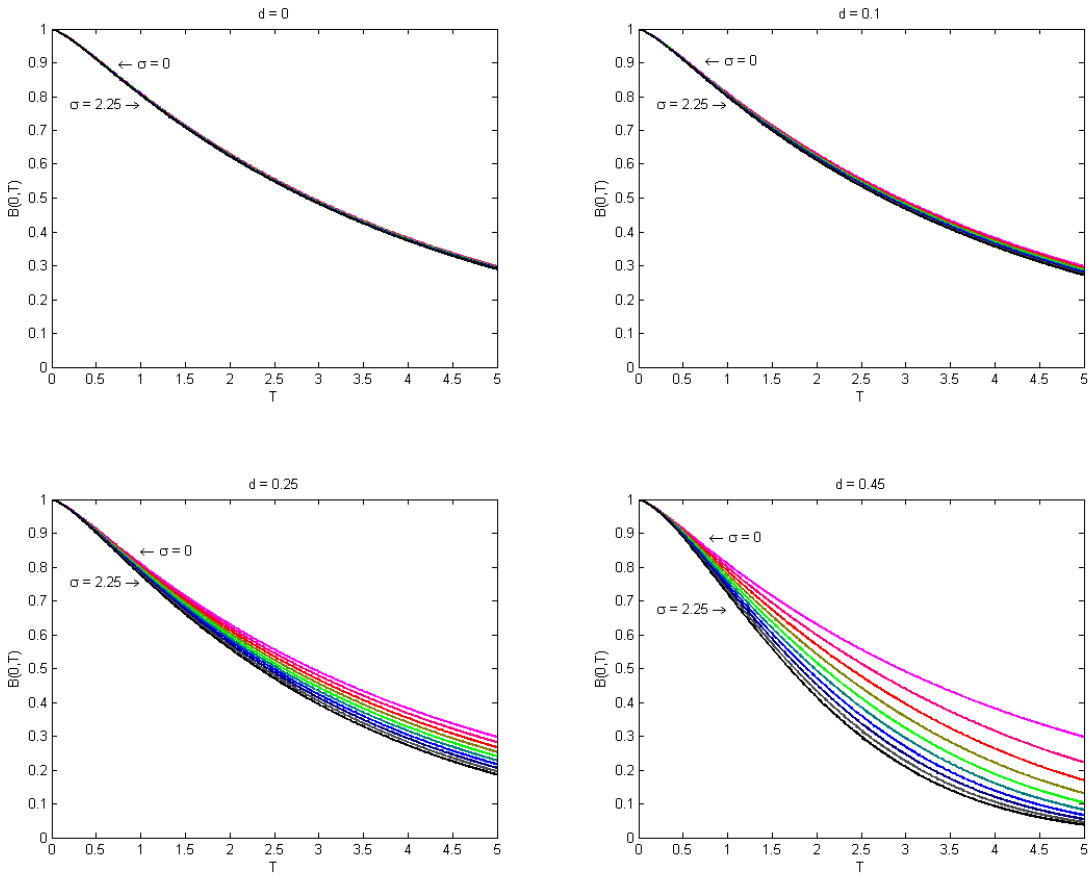

Figure 5.8: *Bond prices* $B^{Poi}(0,T)$ *in the fractional Poisson model for varying* σ*, maturity* T *and fractional parameter* d*, using constant coefficients* $r(0) = 0.1$*,* $a = 4$*,* $k = 1$ *and* $\eta = 1$*. In particular* σ *increases by steps of size* 0.25 *from* 0 *to* 2.25*.*

The parameter σ has a positive impact on the bond price in the fractional Brownian model, cf. Figure 5.7. If σ equals zero the short rate r is deterministic and cannot become negative. Bond prices decrease with longer maturity. However if σ is positive and takes high values, the probability of r getting negative gets higher, which means that bond prices will be higher, too. In the fractional Poisson model, cf. Figure 5.8, the influence of

the parameter σ is very different. Since the short rate is non-negative the impact of σ is asymmetric. A higher value will increase the probability of seeing higher values of r which leads to a lower bond price. The derivative with respect to σ is given by

$$\begin{aligned}
\frac{\partial B_\sigma^{fBm}(0,T)}{\partial \sigma} &= \sigma \|D(\cdot,T)\mathbf{1}_{[0,T]}(\cdot)\|^2_{\kappa,T^\star} \cdot B_\sigma^{fBm}(0,T), \\
\frac{\partial B_\sigma^{Poi}(0,T)}{\partial \sigma} &= -\eta \int_0^T z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v)) \exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v)dv \cdot B_\sigma^{Poi}(0,T),
\end{aligned}$$

where we used the classical rule for differentiation under the integral sign in the fractional Poisson case.

5.5 Parameter sensitivity with respect to η

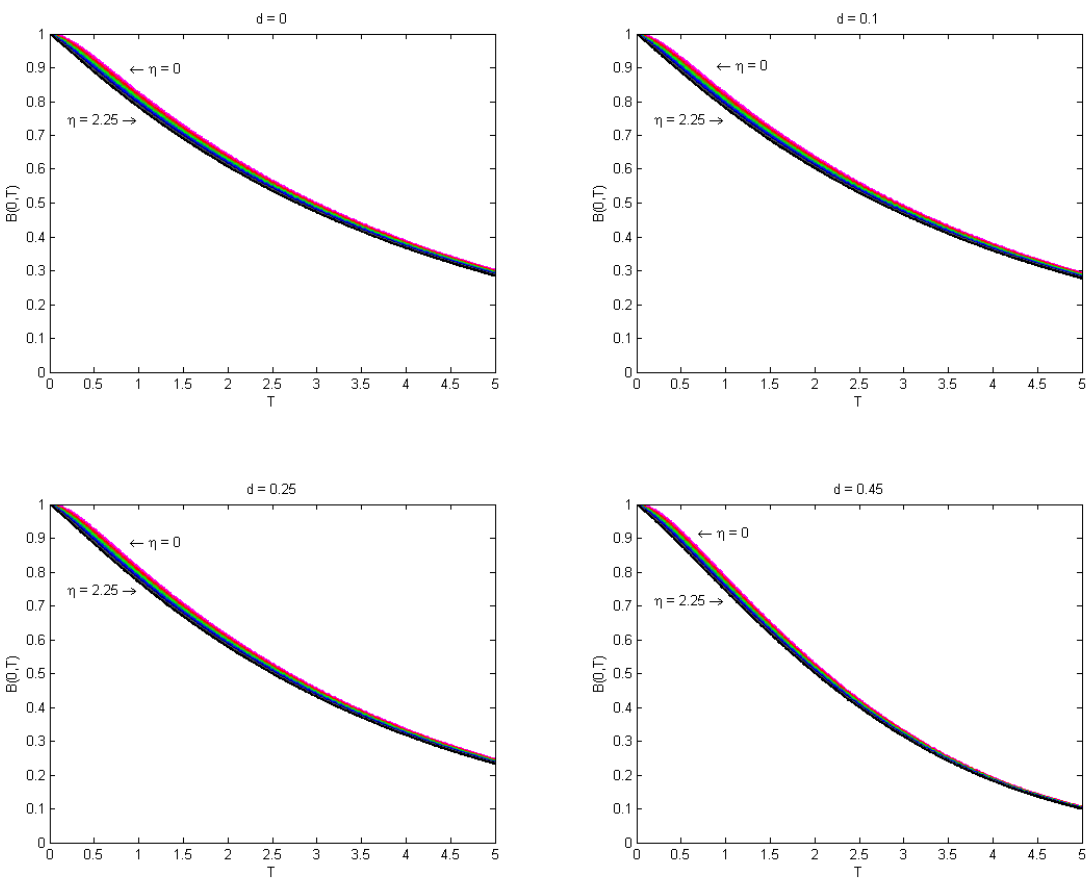

Figure 5.9: *Bond prices $B^{Poi}(0,T)$ in the fractional Poisson model for varying η, maturity T and fractional parameter d, using constant coefficients $r(0) = 0.1$, $a = 4$ and $k = 1$. In*

particular η *increases by steps of size* 0.25 *from* 0 *to* 2.25.

As can be seen in Figure 5.9 the impact of the Poisson parameter η is always monotone: higher values of η lead to lower bond prices. The reason behind this is that with increasing η the fractional Poisson subordinator driving the short rate process will have a higher upward drift. The derivative with respect to η is given by

$$\frac{\partial B_{\eta}^{Poi}(0,T)}{\partial \eta} = \int_0^T \exp(-\sigma z^d(D(\cdot,T)\mathbf{1}_{[0,T]},v)dv \cdot B_{\eta}^{Poi}(0,T).$$

Part II

High tick data modeling by discrete valued processes

Chapter 6

Discrete-valued Lévy models

In this chapter we develop a new class of continuous time models to describe integer based prices, which are integer processes when normalized by the tick size. This research is carried out to give us models to deal with situations where the discrete tick size plays an important empirical role and is particularly helpful in dealing with low latency financial data. Our models will allow for volatility clustering and statistical leverage, extending the Lévy model of Barndorff-Nielsen, Pollard and Shephard [10] which delivers independent and stationary returns.

Recently low latency data have become available for research. These data from specialist data providers are recorded very close to the data exchange itself and are therefore of the highest available quality. Typically low latency data are added to the data providers database less than 1 millisecond after they leave the exchange.

There has been considerable interest in using high frequency financial data to aid decision making. Recent reviews are given by Russell and Engle [114] and Bauwens and Hautsch [13]. Leading applied reasons include:

(i) Building models to design efficient trading methods with low transaction costs. These methods are typically implemented electronically and are called "automated trading". An interesting recent example being Avellaneda and Stoikov [6]. Such methods often study the relative utility of market and limit orders, see for example Lehmann [84] for a theoretical discussion.

(ii) Harnessing the data to better estimate medium term financial volatility or dependence e.g. by Andersen, Bollerslev, Diebold and Labys [3], Barndorff-Nielsen and Shephard [12], Barndorff-Nielsen, Hansen, Lunde and Shephard [9] and Mykland and Zhang [98].

(iii) Studies of the relationships between the many quantities of economic interest. For example relationships between trade volumes and price changes have been studied by Potters and Bouchaud [106] and Lo and Wang [85] amongst many and between order flow and tick price changes by Weber and Rosenow [133] and others.

Our starting point is the recent integer-valued Lévy process based model of Barndorff-Nielsen et al. [10]. They modeled prices as

$$P_t = P_0 + L_t, \quad L_t = L_t^+ - L_t^-, \quad t \geq 0,$$

where $L^\pm = (L_t^\pm)_{t\geq 0}$ are independent integer-valued subordinators and $P_0 \in \mathbb{R}^+$.

To improve the readability of this chapter (which includes many time-changed stochastic processes) we shall - in contrast to the first part of the thesis - denote a stochastic process using the time as a subscript (i.e. $L_t^\pm$) instead of $L^\pm(t)$.

Recall that subordinators are non-negative Lévy processes and therefore have non-negative, independent and stationary increments. The simplest example of this, corresponding to one-tick markets, is where $L^\pm$ are Poisson processes. Then Barndorff-Nielsen et al. [10] called L a Skellam Lévy process.

Related integer-valued econometric models include those discussed by, for example, Hausman, Lo and MacKinlay [71], Rydberg and Shephard [116], Russel and Engle [113], Hasbrouck [69], Phillips and Yu [102] and Hansen and Horel [67]. Delattre and Jacod [37] considered diffusion processes with round-off errors and their stochastic properties while Rosenbaum [112] used their results to propose a model for asset prices with focus on estimating integrated volatility. Statistical approaches, working directly with the discrete-valued observations, can be found in e.g. by Müller and Czado [97] or Haug and Czado [70].

The Barndorff-Nielsen et al. [10] model structure has many interesting advantages, as it is a continuous time model which obeys the tick structure. The model has the disadvantage that it delivers integer-valued returns which are independent and stationary, the latter features are shared with Brownian motion. Clearly this is unsatisfactory for financial data. However, in the same way that Brownian motion can be time-changed to deliver Gaussian stochastic volatility, e.g. the reviews by, for example, Shephard [124] and Ghysels, Harvey and Renault [63], we can adjust integer-valued Lévy processes to make them more realistic.

Clearly the direct application of non-Gaussian stochastic volatility approaches (e.g. Renault and Werker [111]), where we would define

$$P_t = P_0 + \int_0^t \sigma_s \mathrm{d}L_s, \quad t \geq 0,$$

where $\sigma = (\sigma_t)_{t\geq 0}$ is a non-negative stochastic process, is unsatisfactory for low latency data for the resulting process is not integer-valued in general. Likewise

$$P_t = P_0 + L \circ T_t, \quad t \geq 0,$$

where $T = (T_t)_{t\geq 0}$ is a time-change (i.e. a process with non-negative increments) independent of L is also unsatisfactory. It will be integer-valued but it does not allow for statistical leverage effects since the time change will affect upticks and downticks likewise. Recall the statistical leverage effect is where there is a negative correlation between the stock price changes and future volatility, meaning that stock price falls are associated with future high levels of volatility. We develop here a new approach to deliver tick structure, volatility clustering and statistical leverage. Our model differs from the previous work of Carr and Wu [29] and the more general discussion given by Veraart and Veraart [132].

Although the models developed in this chapter deal with most of the features we see in low latency data, this approach is not complete for it ignores some microstructure effects which deliver autocorrelation in returns. This will inevitably limit its detailed application until this extension is fully developed.

Firstly, we will introduce our model in Section 6.1 and give a detailed analysis of its basic properties in Section 6.2. Various econometric methods for inference are outlined in Section 6.3. An application is provided in Section 6.4.

6.1 Stochastic volatility

Now we want to extend the integer-valued Lévy processes from Barndorff-Nielsen et al. [10] to allow for volatility clustering and statistical leverage. Throughout this chapter we will always assume a given Lévy subordinator $L^+ = (L_t^+)_{t\geq 0}$ on a complete filtered probability space, satisfying the usual hypotheses of right-continuity and completeness.

Using its characteristic triple, L^+ can also be described in terms of its cumulant function (instead of its characteristic function), i.e.

$$\mathrm{C}\{\theta \ddagger L_t^+\} := \log E[\exp\{iuL_t^+\}] = t\psi^+(u), \quad t \geq 0, \quad u \in \mathbb{R}.$$

with

$$\psi^+(u) = iu\gamma^+ + \int_{\mathbb{R}^+} \left\{\exp(iux) - 1 - iux\mathbf{1}_{\{|x|<1\}}\right\} \nu^+(\mathrm{d}x), \quad u \in \mathbb{R}.$$

Here we have $\gamma^+ \in \mathbb{R}^+$ and the Lévy measure ν^+ satisfies (cf. Section 1.4)

$$\begin{aligned} \nu^+(\{0\}) &= 0, \quad \nu^+((-\infty, 0]) = 0, \\ \int_{\mathbb{R}^+} (x \wedge 1)\nu^+(\mathrm{d}x) &< \infty. \end{aligned}$$

We shall also assume that the appearing moments of L^+ always exist. Mostly it will be enough assume the existence of fourth moments, i.e.

$$\int_{\mathbb{R}^+} x^4 \nu^+(\mathrm{d}x) < \infty,$$

however, when considering cumulant functions of the price process P, we will need the existence of exponential moments of L^+.

The same structure will also be assumed to apply to the subordinator L^- which shall be independent of L^+.

6.1.1 Model specification

Recall the tick prices $P = (P_t)_{t \geq 0}$ from Barndorff-Nielsen et al. [10] as

$$P_t - P_0 = L_t^+ - L_t^-, \quad t \geq 0,$$

where L^+ and L^- are independent *integer-valued* subordinators. This means that P is integer valued if P_0 is and therefore is suitable as a tick price model. As both L^+ and L^- are finite activity processes, so P must be as well.

In this chapter we will study the following model in detail:

$$P_t - P_0 = R_t^+ - R_t^-, \quad t \geq 0,$$

where

$$R_t^+ = L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} = L^+_{T_t + \alpha\psi t E(Z_1)}, \quad t \geq 0,$$

and

$$R_t^- = L^- \circ (T_t + \psi Z_t) = L^-_{T_t + \psi Z_t}, \quad t \geq 0.$$

Here L_t^+ and L_t^- are independent integer-valued subordinators, independent of the process $Z = (Z_t)_{t \geq 0}$. It is important to note that it is not possible to write the process P as being an univariate Lévy process which is time-changed by a single process.

In this model $T = (T_t)_{t \geq 0}$ and Z are particular types of time-changes, so are non-decreasing stochastic processes. Later we will think of $\psi \geq 0$ as a statistical leverage

parameter, while $\alpha \geq 0$ controls the drift. In this chapter we will often impose the constraint that

$$\alpha = \frac{E(L_1^-)}{E(L_1^+)},$$

which means that the component

$$L^+_{\alpha\psi t E(Z_1)} - L^-_{\psi Z_t}, \quad t \geq 0,$$

is a martingale. The literature on time-changes in the context of finance is reviewed by, for example, Veraart and Winkel [131].

Remark 6.1.1. Carr and Wu [29] considered time changed Lévy processes in general. However after calculating the cumulant functions they focussed in their work mainly on time changes by Lévy processes or "instantaneous rate of activities", i.e. continuous and differentiable processes. This does not match our model, since the process $T_\cdot + \psi Z_\cdot$ is neither in general a Lévy process nor differentiable. Veraart and Veraart [132] have a more general discussion of the topic.

Remark 6.1.2. As $L^\pm$ are of finite activity we can write them as compound Poisson processes

$$L_t^\pm = \sum_{j=1}^{N_t^\pm} C_j^\pm, \quad t \geq 0,$$

where $C_j^\pm \in \{1, 2, \ldots\}$ are stochastically independent and equally distributed. Furthermore $N^\pm = (N_t^\pm)_{t\geq 0}$ are Poisson processes. Consequently

$$R_t^+ = \sum_{j=1}^{N^+_{T_t+\alpha\psi t E(Z_1)}} C_j^+, \quad R_t^- = \sum_{j=1}^{N^-_{T_t+\psi Z_t}} C_j^-, \quad t \geq 0,$$

which shows the intensity of the upticks and downticks are changed by the time-changes, but not the distribution of the jumps which does not change over time. Of course this means that the increments of $R^\pm$ are dependent (unlike $L^\pm$), due to the common measures in the two time changes T and Z.

6.1.2 Linear subordinator

For simplicity of analysis, the non-decreasing time-change T will nearly always be assumed to be linear in a *strictly increasing* Lévy subordinator Z, i.e.

$$T_t = a(t) + \int_0^t k(t,s)\mathrm{d}Z_s \quad t \geq 0.$$

Throughout we will impose the following assumptions. The non-stochastic kernel

$$k : (\mathbb{R}_+)^2 \to \mathbb{R}_+$$

satisfies (cf. Definition 1 of Bender and Marquardt [16]):

(i) For fixed $t \geq 0$, the mapping $s \mapsto k(t,s)$ is integrable.

(ii) For fixed $s \geq 0$, the mapping $t \mapsto k(t,s)$ is continuous and increasing. There exists an $\varepsilon > 0$ such that $t \mapsto k(t,s)$ is strictly increasing on $[s, s+\varepsilon]$.

(iii) For $s > t \geq 0$, $k(t,s) = 0$.

Moreover we will assume that the non-stochastic function

$$a : \mathbb{R}_+ \to \mathbb{R}_+$$

is increasing. Then it follows by Proposition 1 of [16] that T has a.s. increasing and continuous paths. . Clearly non-linear time-changes could also be used but we will not explore that here.

Remark 6.1.3. This definition of the time-change also includes fractional subordinators in the sense of Bender and Marquardt [16], cf. Definition 4.1.1 and Example 4.1.4.

Remark 6.1.4. Straightforwardly the time-changes T and Z are related with the features

$$\begin{aligned} E(T_t) &= a_t + E(Z_1)\int_0^t k(t,s)\mathrm{d}s, \quad Var(T_t) = E(Z_1^2)\int_0^t k(t,s)^2\mathrm{d}s, \\ Cov\,(T_t, Z_t) &= Cov\left(\int_0^t k(t,s)\mathrm{d}Z_s, Z_t\right) = E(Z_1^2)\int_0^t k(t,s)\mathrm{d}s, \end{aligned}$$

for $t \geq 0$. This implies T and Z can only be positively correlated.

Example 6.1.5. An example of this is the continuous non-Gaussian OU process driven time-change

$$T_t = \int_0^t \tau_s \mathrm{d}s, \quad \mathrm{d}\tau_t = -\lambda\tau_t\mathrm{d}t + \mathrm{d}Z_t, \quad \lambda > 0, \quad t \geq 0,$$

where for $t \geq 0$

$$\tau_t = e^{-\lambda t}\tau_0 + \int_0^t e^{-\lambda(t-s)}\mathrm{d}Z_s, \quad T_t = \lambda^{-1}(1-e^{-\lambda t})\tau_0 + \lambda^{-1}\int_0^t \left\{1 - e^{-\lambda(t-s)}\right\}\mathrm{d}Z_s,$$

which was used in the context of Gaussian stochastic volatility by Barndorff-Nielsen and Shephard [11]. Hence for $0 \leq s \leq t$

$$\begin{aligned} k(t,s) &= \lambda^{-1}\Big\{1 - e^{-\lambda(t-s)}\Big\}, \\ \int_0^t k(t,s)ds &= \lambda^{-1}\Big\{t + \lambda^{-1} - \lambda{-}1e^{-\lambda t}\Big\}, \\ \int_0^t k(t,s)^2 ds &= \lambda^{-2}\Big\{t + 2\lambda^{-1} - 2\lambda{-}1e^{-\lambda t} - (2\lambda)^{-1} + (2\lambda)^{-1}e^{-2\lambda t}\Big\}. \end{aligned}$$

For small $t \downarrow 0$ we have

$$\int_0^t k(t,s)ds \simeq 2t\lambda^{-1}, \quad \int_0^t k(t,s)^2 ds \simeq 2t\lambda^{-2},$$

while for large t we obtain

$$\frac{1}{\lambda}\int_0^t k(t,s)ds \to \lambda^{-1}, \quad \frac{1}{\lambda}\int_0^t k(t,s)^2 ds \to \lambda^{-2}.$$

Extensions to ARMA type processes include Brockwell [25], Brockwell and Marquardt [27] and Todorov and Tauchen [128]. Long-memory versions can be found in Barndorff-Nielsen [8], Marquardt [92] and Fink and Klüppelberg [59]. Figure 1 shows three simulated sample paths.

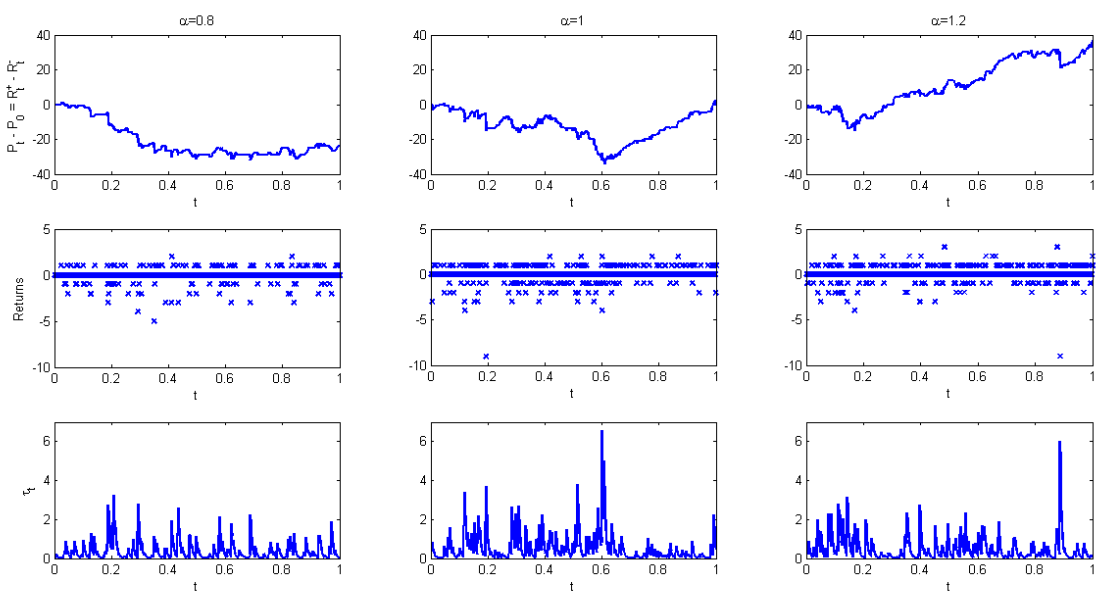

Figure 6.1: *Sample paths of the process P using for τ an OU process with $\lambda = 2$ driven by a gamma subordinator Z with expectation 1 and variance 2, Poisson processes L^+ and L^- with common intensity 1 and $\psi = 1$.*

6.2 Properties of the model

In a next step will consider the higher order properties of our model, showing that it leads to volatility clustering and statistical leverage.

6.2.1 Second moments

From now on we shall denote for $i \in \mathbb{N}$ the i-th cumulant (i.e. the i-th differential of the cumulant function at 0) of $L^{\pm}$ by $\kappa_i^{\pm}$ and that of Z by η_i.

Remember the following well-known property. For a Lévy process $X = (X_t)_{t\geq 0}$ with existing second moments and $0 \leq s \leq t$ we have

$$E\left(X_t X_s\right) = Var\left(X_s\right) + \mathrm{E}\left(X_t\right) E\left(X_s\right) = Var\left(X_1\right) s + E\left(X_1\right)^2 ts. \tag{6.1}$$

In a first step we consider returns in the model. Define for $0 \leq s \leq t$

$$r_{s,t} = r_{s,t}^{+} - r_{s,t}^{-}, \quad r_{s,t}^{\pm} = R_t^{\pm} - R_s^{\pm}.$$

Then

$$r_{s,t}^{+} = L^{+} \circ \left\{T_t + \alpha\psi t E\left(Z_1\right)\right\} - L^{+} \circ \left\{T_s + \alpha\psi s E\left(Z_1\right)\right\}$$

and

$$r_{s,t}^{-} = L^{-} \circ \left(T_t + \psi Z_t\right) - L^{-} \circ \left(T_s + \psi Z_s\right).$$

Proposition 6.2.1 and Proposition 6.2.2 give the mean and covariance of the returns, respectively. Set $\mathcal{F}_{\infty}^{Z} = \overline{\sigma\{Z_t, t \in [0,\infty)\}}$.

Proposition 6.2.1. *For $\alpha = \kappa_1^{-}/\kappa_1^{+}$ we have for $0 \leq s \leq t$*

$$E(r_{s,t}) = (\kappa_1^{+} - \kappa_1^{-})E(T_t - T_s)$$

and in general

$$E(r_{s,t}^{+}) = \kappa_1^{+}\{E(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\}, \quad E(r_{s,t}^{-}) = \kappa_1^{-}\{E(T_t - T_s) + \psi(t-s)E(Z_1)\}.$$

Proof. Consider

$$\begin{aligned} E(r^+_{s,t}) &= EE(L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} \,|\mathcal{F}^Z_\infty) - EE(L^+ \circ \{T_s + \alpha\psi s E(Z_1)\} \,|\mathcal{F}^Z_\infty) \\ &= E(T_t + \alpha\psi t E(Z_1))\kappa^+_1 - E(T_s + \alpha\psi t E(Z_1))\kappa^+_1 \\ &= \kappa^+_1\{E(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\}. \end{aligned}$$

The calculations for $E(r^-_{s,t})$ and $E(r_{s,t})$ work similar. □

Proposition 6.2.2. *In general we have for* $0 \le s \le t$

$$\begin{aligned} Var(r_{s,t}) &= (\kappa^+_1 - \kappa^-_1)^2 Var(T_t - T_s) + (\kappa^+_2 + \kappa^-_2)\Big(E(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\Big) \\ &\quad +2\psi((\kappa^-_1)^2 - \kappa^+_1\kappa^-_1)Cov(T_t - T_s, Z_t - Z_s) + \psi^2(\kappa^-_1)^2 Var(Z_t - Z_s) \end{aligned}$$

where

$$\begin{aligned} Var(r^+_{s,t}) &= (\kappa^+_1)^2 Var(T_t - T_s) + \kappa^+_2\Big(E(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\Big), \\ Var(r^-_{s,t}) &= (\kappa^-_1)^2\Big(Var(T_t - T_s) + \psi^2 Var(Z_t - Z_s) + 2\psi Cov(T_t - T_s, Z_t - Z_s)\Big) \\ &\quad +\kappa^-_2\Big(E(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\Big), \\ Cov(r^+_{s,t}, r^-_{s,t}) &= \kappa^+_1\kappa^-_1 Cov(T_t - T_s, T_t - T_s + \psi(Z_t - Z_s)). \end{aligned}$$

Furthermore for $0 \le u \le v \le s \le t$

$$\begin{aligned} & Cov(r_{s,t}, r_{u,v}) \\ = \; & Cov(r^+_{s,t}, r^+_{u,v}) - Cov(r^+_{s,t}, r^-_{u,v}) - Cov(r^-_{s,t}, r^+_{u,v}) + Cov(r^-_{s,t}, r^-_{u,v}) \\ = \; & (\kappa^+_1 - \kappa^-_1)^2 Cov(T_t - T_s, T_v - T_u) + \psi\Big((\kappa^-_1)^2 - \kappa^+_1\kappa^-_1\Big)Cov(T_t - T_s, Z_v - Z_u), \end{aligned}$$

with

$$\begin{aligned} Cov(r^+_{s,t}, r^+_{u,v}) &= (\kappa^+_1)^2 Cov(T_t - T_s, T_v - T_u), \\ Cov(r^+_{s,t}, r^-_{u,v}) &= \kappa^+_1\kappa^-_1 Cov(T_t - T_s, T_v - T_u + \psi(Z_v - Z_u)), \\ Cov(r^-_{s,t}, r^+_{u,v}) &= \kappa^+_1\kappa^-_1 Cov(T_t - T_s, T_v - T_u), \\ Cov(r^-_{s,t}, r^-_{u,v}) &= (\kappa^-_1)^2 Cov(T_t - T_s, T_v - T_u + \psi(Z_v - Z_u)). \end{aligned}$$

Proof. Let $0 \le u \le v \le s \le t$. We will provide the calculation for $Cov(r^+_{s,t}, r^+_{u,v})$. The rest follows by similar arguments. Consider

$$Cov(r^+_{s,t}, r^+_{u,v}) = E(r^+_{s,t} r^+_{u,v}) - \mathrm{E}(r^+_{s,t})E(r^+_{u,v}).$$

We have

$$\begin{aligned} r^+_{s,t} r^+_{u,v} &= L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} L^+ \circ \{T_v + \alpha\psi v E(Z_1)\} \\ &\quad - L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} L^+ \circ \{T_u + \alpha\psi u E(Z_1)\} \\ &\quad - L^+ \circ \{T_s + \alpha\psi s E(Z_1)\} L^+ \circ \{T_v + \alpha\psi v E(Z_1)\} \\ &\quad + L^+ \circ \{T_s + \alpha\psi s E(Z_1)\} L^+ \circ \{T_u + \alpha\psi u E(Z_1)\}. \end{aligned}$$

Therefore it is sufficient to calculate $E(L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} L^+ \circ \{T_v + \alpha\psi v E(Z_1)\})$. Applying (6.1) we obtain

$$\begin{aligned} & E(L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} L^+ \circ \{T_v + \alpha\psi v E(Z_1)\}) \\ = \; & EE(L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} L^+ \circ \{T_v + \alpha\psi v E(Z_1)\} | \mathcal{F}^Z_\infty) \\ = \; & E\Big(Var(L^+ \circ \{T_v + \alpha\psi v E(Z_1)\}) + E(L^+ \circ \{T_t + \alpha\psi t E(Z_1)\} | \mathcal{F}^Z_\infty) \\ & \times E(L^+ \circ \{T_v + \alpha\psi v E(Z_1)\} | \mathcal{F}^Z_\infty)\Big) \\ = \; & E\Big(\{T_v + \alpha\psi v E(Z_1)\} \kappa^+_2 + \{T_t + \alpha\psi t E(Z_1)\} \{T_v + \alpha\psi v E(Z_1)\} (\kappa^+_1)^2 \Big) \\ = \; & E(T_v + \alpha\psi v E(Z_1))\kappa^+_2 + E(\{T_t + \alpha\psi t E(Z_1)\} \{T_v + \alpha\psi v E(Z_1)\})(\kappa^+_1)^2. \end{aligned}$$

Using this result repeatedly we get

$$\begin{aligned} E(r^+_{s,t} r^+_{u,v}) &= E(T_v + \alpha\psi v E(Z_1))\kappa^+_2 + E(\{T_t + \alpha\psi t E(Z_1)\} \{T_v + \alpha\psi v E(Z_1)\})(\kappa^+_1)^2 \\ &\quad - E(T_u + \alpha\psi u E(Z_1))\kappa^+_2 - E(\{T_t + \alpha\psi t E(Z_1)\} \{T_u + \alpha\psi u E(Z_1)\})(\kappa^+_1)^2 \\ &\quad - E(T_v + \alpha\psi v E(Z_1))\kappa^+_2 - E(\{T_s + \alpha\psi s E(Z_1)\} \{T_v + \alpha\psi v E(Z_1)\})(\kappa^+_1)^2 \\ &\quad + E(T_u + \alpha\psi u E(Z_1))\kappa^+_2 + E(\{T_s + \alpha\psi s E(Z_1)\} \{T_v + \alpha\psi v E(Z_1)\})(\kappa^+_1)^2 \\ &= (\kappa^+_1)^2 E\Big(\{T_t - T_s - \alpha\psi(t-s)E(Z_1)\}\{T_v - T_u - \alpha\psi(v-u)E(Z_1)\}\Big). \end{aligned}$$

On the other hand we have by Proposition 6.2.1

$$E(r^+_{s,t})E(r^+_{u,v}) = (\kappa^+_1)^2 E\Big(\{T_t - T_s - \alpha\psi(t-s)E(Z_1)\}\Big) E\Big(\{T_v - T_u - \alpha\psi(v-u)E(Z_1)\}\Big).$$

Putting everything together we obtain the assertion for $Cov(r^+_{s,t}, r^+_{u,v})$. For the other statements of Proposition 6.2.2 we use

$$\begin{aligned} Cov((Z_t - Z_s), T_v - T_u) &= 0, \quad \text{and} \\ Cov((Z_t - Z_s), (Z_v - Z_u)) &= 0. \end{aligned}$$

□

Hence a sufficient condition for returns to be zero mean weak white noise is $\kappa^+_1 = \kappa^-_1$ and $\alpha = 1$.

6.2.2 Statistical leverage

For the statistical leverage effect we would like to have a negative correlation between the stock price changes and future volatility, meaning that stock price falls are associated with future high levels of volatility.

The following theorem provides a general expression for this correlation.

Theorem 6.2.3. *Now we have for* $0 \leq u \leq v \leq s \leq t$

$$\begin{aligned}
& Cov\left(r_{s,t}^2, r_{u,v}\right) \\
= \; & (\kappa_1^+ - \kappa_1^-)^3 Cov\left\{(T_t - T_s)^2, T_v - T_u\right\} \\
& + \Big(\kappa_1^+\kappa_2^+ - \kappa_1^-\kappa_2^+ + \kappa_1^+\kappa_2^- - \kappa_1^-\kappa_2^-\Big) Cov\left(T_t - T_s, T_v - T_u\right) \\
& + \psi\Big(-(\kappa_1^-)^3 + 2(\kappa_1^-)^2\kappa_1^+ - (\kappa_1^+)^2\kappa_1^-\Big) Cov\left\{(T_t - T_s)^2, Z_v - Z_u\right\} \\
& + \psi\Big(-2(\kappa_1^-)^3 + 4(\kappa_1^-)^2\kappa_1^+ - 2(\kappa_1^+)^2\kappa_1^-\Big) Cov\left\{(T_t - T_s)(Z_t - Z_s), T_v - T_u\right\} \\
& + \psi^2\Big(-2(\kappa_1^-)^3 + 2(\kappa_1^-)^2\kappa_1^+\Big) Cov\left\{(T_t - T_s)(Z_t - Z_s), Z_v - Z_u\right\} \\
& + 2\alpha\psi\Big((\kappa_1^+)^3 + (\kappa_1^-)^2\kappa_1^+ - 2(\kappa_1^+)^2\kappa_1^-\Big)[t-s]E(Z_1) Cov\left(T_t - T_s, T_v - T_u\right) \\
& + \alpha(\psi^2)\Big(2(\kappa_1^-)^2\kappa_1^+ - 2(\kappa_1^+)^2\kappa_1^-\Big)[t-s]E(Z_1) Cov\left(T_t - T_s, Z_v - Z_u\right) \\
& + \psi\left(-\kappa_1^-\kappa_2^- - \kappa_1^-\kappa_2^+\right) Cov\left(T_t - T_s, Z_v - Z_u\right).
\end{aligned}$$

Before we start with the proof, we want to clarify the consequence of Theorem 6.2.3 by considering Example 6.2.4, which gives a clear result in terms of statistical leverage.

Example 6.2.4. Let the expectations of L^+ and L^- be equal, i.e. $\kappa_1^+ = \kappa_1^-$. Then we have for $0 \leq u \leq v \leq s \leq t$

$$Cov\left(r_{s,t}^2, r_{u,v}\right) = -2\kappa_1^+(\kappa_2^+ + \kappa_2^-)\psi Cov\left(T_t - T_s, Z_v - Z_u\right).$$

Therefore we have if $Cov\left(T_t - T_s, Z_v - Z_u\right) > 0$,

$$Cov\left(r_{s,t}^2, r_{u,v}\right) < 0,$$

if $\psi > 0$ and

$$Cov\left(r_{s,t}^2, r_{u,v}\right) = 0,$$

if $\psi = 0$.

For higher moments equation (6.1) gets much more complicated and the cases of third and fourth moments are considered below.

Theorem 6.2.5. *Let X be a Lévy process with existing third moments. Denote by κ_i its cumulants. Then we have for $0 \leq u \leq v \leq s \leq t$*

$$E[X_t X_s X_v] = \kappa_1^3 tsv + \kappa_1\kappa_2(tv + 2sv) + \kappa_3 v.$$

If even fourth moments exists, we have

$$\begin{aligned} E[X_t X_s X_v X_u] &= \kappa_1^4 tsvu + \kappa_4 u + \kappa_1\kappa_3\Big(tu + su + 2vu\Big) + \kappa_2^2\Big(sv + 3vu - v^2\Big) \\ &+ \kappa_1^2\kappa_2\Big(5svu + tsu + 2tvu - s^2u + s^2v - 2sv^2 - v^2u + v3\Big). \end{aligned}$$

Proof. This is a lengthy calculation and we will only give a sketch. It is well-known that for $t \geq 0$, $E[X_t^3]$ is a polynomial of order 3 in t. Furthermore for $v, s \geq 0$ we have the recursion formula

$$E[X_{v+s}^3] = \sum_{k=0}^{3} \begin{pmatrix} n \\ k \end{pmatrix} E[X_v^k] E[X_s^{n-k}].$$

Using this relation we deduce the coefficients of $E[X_t^3]$. Now using independent increments we can rewrite $E[X_t X_s X_v]$ in terms of the moments of X up to order 3 at times t, v, s. Putting this together with the closed form of $E[X_\cdot^n]$, $n \leq 3$, delivers the assertion. The proof regarding the fourth moments works similar. □

Theorem 6.2.6. *Let X and Y be independent Lévy processes with existing third moments. Denote by κ_i and η_i their cumulants, $i = 1, 2$. Further let A and B be increasing processes independent of X and Y with finite second moments and*

$$\overline{\sigma\{A_t, t \in [0, \infty)\}} = \overline{\sigma\{B_t, t \in [0, \infty)\}}.$$

Then we have for $0 \leq v \leq s \leq t$

$$\begin{aligned} (i) \quad Cov[(X_{A_t} - X_{A_s})^2, X_{A_v} - X_{A_u}] &= (\kappa_1)^3 Cov[(A_t - A_s)^2, A_v - A_u] \\ &+ \kappa_1\kappa_2 Cov[A_t - A_s, A_v - A_u], \end{aligned}$$

$$\begin{aligned} (ii) \quad Cov[(X_{A_t} - X_{A_s})^2, Y_{B_v} - Y_{B_u}] &= \kappa_1^2\eta_1 Cov[(A_t - A_s)^2, B_v - B_u] \\ &+ \kappa_2\eta_1 Cov[A_t - A_s, B_v - B_u], \end{aligned}$$

$$\begin{aligned} (iii) \quad & Cov[(X_{A_t} - X_{A_s})(Y_{B_t} - Y_{B_s}), Y_{B_v} - Y_{B_u}] \\ & = (\kappa_1)^3 Cov[(A_t - A_s)(B_t - B_s), B_v - B_u]. \end{aligned}$$

Proof. Set $\mathcal{F}_\infty^A = \overline{\sigma\{A_t, t \in [0,\infty)\}}$. Consider (i) first and calculate using Theorem 6.2.5 for $0 \le v \le s \le t$,

$$\begin{aligned}
& Cov[X_{A_t}X_{A_s}, X_{A_v}] = E[X_{A_t}X_{A_s}Y_{B_v}] - E[X_{A_t}X_{A_s}]E[Y_{B_v}] \\
= \;& EE[X_{A_t}X_{A_s}Y_{B_v}|\mathcal{F}_\infty^A] - \mathrm{E}E[X_{A_t}X_{A_s}|\mathcal{F}_\infty^A]E[Y_{B_v}] \\
= \;& E\Big[(\kappa_1)^3 A_tA_sA_v + \kappa_1\kappa_2(A_tA_v + 2A_sA_v) + \kappa_3 A_v\Big] \\
& -\kappa_1\kappa_2 E[A_s]E[A_v] - (\kappa_1)^3 E[A_tA_s]E[A_v] \\
= \;& (\kappa_1)^3 Cov[A_tA_s, A_v] + \kappa_1\kappa_2\Big(Cov[A_s, A_v] + E[A_tA_v] + E[A_sA_v]\Big) + \kappa_3 E[A_v]
\end{aligned}$$

and therefore

$$Cov[(X_{A_t})^2, X_{A_v}] = (\kappa_1)^3 Cov[A_t^2, A_v] + \kappa_1\kappa_2\Big(Cov[A_t, A_v] + 2E[A_tA_v]\Big) + \kappa_3 E[A_v].$$

Using this we obtain

$$\begin{aligned}
& Cov[(X_{A_t} - X_{A_s})^2, X_{A_v} - X_{A_u}] \\
= \;& (\kappa_1)^3 Cov[A_t^2, A_v] + \kappa_1\kappa_2\Big(Cov[A_t, A_v] + 2E[A_tA_v]\Big) + \kappa_3 E[A_v] \\
+ \;& (\kappa_1)^3 Cov[A_s^2, A_v] + \kappa_1\kappa_2\Big(Cov[A_s, A_v] + 2E[A_sA_v]\Big) + \kappa_3 E[A_v] \\
- \;& (\kappa_1)^3 Cov[A_t^2, A_u] - \kappa_1\kappa_2\Big(Cov[A_t, A_u] + 2E[A_tA_u]\Big) - \kappa_3 E[A_u] \\
- \;& (\kappa_1)^3 Cov[A_s^2, A_u] - \kappa_1\kappa_2\Big(Cov[A_s, A_u] + 2E[A_sA_u]\Big) - \kappa_3 E[A_u] \\
- \;& 2(\kappa_1)^3 Cov[A_tA_s, A_v] - 2\kappa_1\kappa_2\Big(Cov[A_s, A_v] + E[A_tA_v] + E[A_sA_v]\Big) - 2\kappa_3 E[A_v] \\
+ \;& 2(\kappa_1)^3 Cov[A_tA_s, A_u] + 2\kappa_1\kappa_2\Big(Cov[A_s, A_u] + E[A_tA_u] + E[A_sA_u]\Big) + 2\kappa_3 E[A_u] \\
= \;& (\kappa_1)^3 Cov[(A_t - A_s)^2, A_v - A_u] + \kappa_1\kappa_2 \mathrm{Cov}[A_t - A_s, A_v - A_u].
\end{aligned}$$

Now let us have a look at (ii). Invoking (6.1) for $0 \le v \le s \le t$, supplies us with

$$\begin{aligned}
& Cov[X_{A_t}X_{A_s}, Y_{B_v}] = E[X_{A_t}X_{A_s}Y_{B_v}] - E[X_{A_t}X_{A_s}]E[Y_{B_v}] \\
= \;& EE[X_{A_t}X_{A_s}Y_{B_v}|\mathcal{F}_\infty^A] - \mathrm{E}E[X_{A_t}X_{A_s}|\mathcal{F}_\infty^A]E[Y_{B_v}] \\
= \;& E\Big[E[X_{A_t}X_{A_s}|\mathcal{F}_\infty^A]E[Y_{B_v}|\mathcal{F}_\infty^A]\Big] - EE[X_{A_t}X_{A_s}|\mathcal{F}_\infty^A]E[Y_{B_v}] \\
= \;& E\Big[(\kappa_2 A_s + \kappa_1^2 A_tA_s)E[Y_1]E[B_v]\Big] - E\Big[\kappa_2 A_s + \kappa_1^2 A_tA_s\Big]E[Y_1]E[B_v] \\
= \;& \kappa_1^2\eta_1 Cov[A_tA_s, B_v] + \kappa_2\eta_1 Cov[A_s, B_v]
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
& Cov[(X_{A_t} - X_{A_s})^2, Y_{B_v} - Y_{B_u}] \\
= \;& \kappa_1^2\eta_1 \mathrm{Cov}[(A_t - A_s)^2, B_v - B_u] + \kappa_2\eta_1 Cov[A_t - A_s, B_v - B_u].
\end{aligned}$$

Assertion (iii) follows by similar arguments. □

Proof. (of Theorem 6.2.3) Again we will have to work through a lengthy calculation. We start by decomposing the covariance:

$$\begin{aligned} Cov[r_{s,t}^2, r_{u,v}] &= Cov[(r_{s,t}^+)^2, r_{u,v}^+] + \mathrm{Cov}[(r_{s,t}^-)^2, r_{u,v}^+] \\ &\quad -Cov[(r_{s,t}^+)^2, r_{u,v}^-] - Cov[(r_{s,t}^-)^2, r_{u,v}^-] \\ &\quad -2Cov[r_{s,t}^+ r_{s,t}^-, r_{u,v}^+] + 2Cov[r_{s,t}^+ r_{s,t}^-, r_{u,v}^-]. \end{aligned}$$

In the following we will calculate each of these 6 terms, starting with the first. Using Theorem 6.2.6 (i) we get

$$\begin{aligned} Cov[(r_{s,t}^+)^2, r_{u,v}^+] &= (\kappa_1^+)^3 Cov[(T_t - T_s)^2, T_v - T_u] \\ &+ (2(\kappa_1^+)^3 \alpha\psi[t-s]E(Z_1) + b^{L^+}) Cov[T_t - T_s, T_v - T_u], \\ Cov[(r_{s,t}^-)^2, r_{u,v}^-] &= (\kappa_1^-)^3 Cov[(T_t - T_s + \psi(Z_t - Z_s))^2, T_v - T_u + \psi(Z_v - Z_u)] \\ &+ b^{L^-} Cov[T_t - T_s + \psi(Z_t - Z_s), T_v - T_u + \psi(Z_v - Z_u)]. \end{aligned}$$

Theorem 6.2.6 (ii) supplies us with

$$\begin{aligned} Cov[(r_{s,t}^+)^2, r_{u,v}^-] &= (\kappa_1^+)^2 \kappa_1^- Cov[(T_t - T_s + \alpha\psi[t-s]E(Z_1))^2, T_v - T_u + \psi(Z_v - Z_u)] \\ &\quad + \kappa_2^+ \kappa_1^- Cov[T_t - T_s + \alpha\psi[t-s]E(Z_1), T_v - T_u + \psi(Z_v - Z_u)], \\ Cov[(r_{s,t}^-)^2, r_{u,v}^+] &= (\kappa_1^-)^2 \kappa_1^+ Cov[(T_t - T_s + \psi(Z_t - Z_s))^2, T_v - T_u + \alpha\psi[v-u]E(Z_1)] \\ &\quad + \kappa_2^- \kappa_1^+ Cov[T_t - T_s + \psi(Z_t - Z_s), T_v - T_u + \alpha\psi[v-u]E(Z_1)]. \end{aligned}$$

And at last with Theorem 6.2.6 (iii) we obtain

$$\begin{aligned} Cov[r_{s,t}^+ r_{s,t}^-, r_{u,v}^+] &= (\kappa_1^+)^2 \kappa_1^- \mathrm{Cov}[(T_t - T_s + \psi(Z_t - Z_s))(T_t - T_s + \alpha\psi[t-s]E(Z_1)), \\ &\quad T_v - T_u + \alpha\psi[v-u]E(Z_1)], \\ Cov[r_{s,t}^+ r_{s,t}^-, r_{u,v}^-] &= (\kappa_1^-)^2 \kappa_1^+ \mathrm{Cov}[(T_t - T_s + \alpha\psi[t-s]E(Z_1))(T_t - T_s + \psi(Z_t - Z_s)), \\ &\quad T_v - T_u + \psi(Z_v - Z_u)]. \end{aligned}$$

Now we put everything together and write the covariances out. Reordering the terms leads to the assertion. □

Hence the model is able to deliver statistical leverage.

6.2.3 Volatility clustering

The following theorem gives the cumbersome general autocovariance function for squared returns. A simpler special case will be given in a moment.

Theorem 6.2.7. *We have for* $0 \leq u \leq v \leq s \leq t$

$$
\begin{aligned}
& Cov\left(r_{s,t}^2, r_{u,v}^2\right) \\
= \; & (\kappa_1^+ - \kappa_1^-)^4 Cov\left\{(T_t - T_s)^2, (T_v - T_u)^2\right\} \\
& +(\kappa_1^+ - \kappa_1^-)^2\Big(\kappa_2^+ + \kappa_2^- + 2((\kappa_1^+)^2 - \kappa_1^+\kappa_1^-)\alpha\psi(v-u)E(Z_1)\Big) \\
& \quad \times Cov\left\{(T_t - T_s)^2, T_v - T_u\right\} \\
& +(\kappa_1^+ - \kappa_1^-)^2\Big(\kappa_2^-\psi - 2\kappa_1^+\kappa_1^-\alpha\psi(v-u)E(Z_1)\Big)Cov\left\{(T_t - T_s)^2, Z_v - Z_u\right\} \\
& +(\kappa_1^+ - \kappa_1^-)^2(\kappa_1^-)^2\psi^2 Cov\left\{(T_t - T_s)^2, (Z_v - Z_u)^2\right\} \\
& +(\kappa_1^+ - \kappa_1^-)^2 2\psi\Big((\kappa_1^-)^2 - \kappa_1^+\kappa_1^-\Big)Cov\left\{(T_t - T_s)^2, (T_v - T_u)(Z_v - Z_u)\right\} \\
& +\Big(\kappa_2^+ + \kappa_2^- + 2((\kappa_1^+)^2 - \kappa_1^+\kappa_1^-)\alpha\psi(v-u)E(Z_1)\Big) \\
& \times\Big\{(\kappa_1^+ - \kappa_1^-)^2 Cov\left\{T_t - T_s, (T_v - T_u)^2\right\} \\
& \quad +\Big(\kappa_2^+ + \kappa_2^- + 2((\kappa_1^+)^2 - \kappa_1^+\kappa_1^-)\alpha\psi(v-u)E(Z_1)\Big)Cov\left\{T_t - T_s, T_v - T_u\right\} \\
& \quad +\Big(\kappa_2^-\psi - 2\kappa_1^+\kappa_1^-\alpha\psi(v-u)E(Z_1)\Big)Cov\left\{T_t - T_s, Z_v - Z_u\right\} \\
& \quad +(\kappa_1^-)^2\psi^2 Cov\left\{T_t - T_s, (Z_v - Z_u)^2\right\} \\
& \quad +2\psi\Big((\kappa_1^-)^2 - \kappa_1^+\kappa_1^-\Big)Cov\left\{T_t - T_s, (T_v - T_u)(Z_v - Z_u)\right\}\Big\} \\
& +2\kappa_1^-(\kappa_1^- - \kappa_1^+)(\kappa_1^+ - \kappa_1^-)^2\psi Cov\left\{(T_t - T_s)(Z_t - Z_s), (T_v - T_u)^2\right\} \\
& +2\kappa_1^-(\kappa_1^- - \kappa_1^+)\psi\Big(\kappa_2^+ + \kappa_2^- + 2((\kappa_1^+)^2 - \kappa_1^+\kappa_1^-)\alpha\psi(v-u)E(Z_1)\Big) \\
& \quad \times Cov\left\{(T_t - T_s)(Z_t - Z_s), T_v - T_u\right\} \\
& +2\kappa_1^-(\kappa_1^- - \kappa_1^+)\psi\Big(\kappa_2^-\psi - 2\kappa_1^+\kappa_1^-\alpha\psi(v-u)E(Z_1)\Big)Cov\left\{(T_t - T_s)(Z_t - Z_s), Z_v - Z_u\right\} \\
& +2\kappa_1^-(\kappa_1^- - \kappa_1^+)(\kappa_1^-)^2\psi^3 Cov\left\{(T_t - T_s)(Z_t - Z_s), (Z_v - Z_u)^2\right\} \\
& +4(\kappa_1^-)^2(\kappa_1^- - \kappa_1^+)^2\psi^2 Cov\left\{(T_t - T_s)(Z_t - Z_s), (T_v - T_u)(Z_v - Z_u)\right\}.
\end{aligned}
$$

Proof. The lengthy calculation works similar to the one of Theorem 6.2.3 and we will omit it. □

To illustrate the last theorem, we look at the following example where the first cumulants of $L^{\pm}$ are assumed to be equal.

Example 6.2.8. Let the expectations of L^+ and L^- be equal, i.e. $\kappa_1^+ = \kappa_1^-$. We have for $0 \leq u \leq v \leq s \leq t$

$$\begin{aligned} & Cov\left(r_{s,t}^2, r_{u,v}^2\right) \\ = \; & (\kappa_2^+ + \kappa_2^-)^2 Cov\left\{T_t - T_s, T_v - T_u\right\} \\ & +(\kappa_2^+ + \kappa_2^-)\psi\Big(\kappa_2^- - 2(\kappa_1^+)^2\alpha(v-u)E(Z_1)\Big)Cov\left\{T_t - T_s, Z_v - Z_u\right\} \\ & +(\kappa_2^+ + \kappa_2^-)(\kappa_1^-)^2\psi^2 Cov\left\{T_t - T_s, (Z_v - Z_u)^2\right\}. \end{aligned}$$

6.2.4 Cumulant function

From now on, for ease of exposition, we will set $P_0 = 0$. We will consider the cumulants of

$$P_t = L^+_{T_t + \alpha\psi t E(Z_1)} - L^-_{T_t + \psi Z_t}, \quad t \geq 0.$$

Given Z, both terms in P are independent by definition. Now we have for $t \geq 0$

$$\begin{aligned} \mathrm{C}\left\{\theta \ddagger P_t | T_t, Z_t\right\} &= \log\left[E\exp\left\{i\theta P_t\right\}|T_t, Z_t\right] \\ &= T_t\Big(\mathrm{C}\left\{\theta \ddagger L_1\right\} + \mathrm{C}\left\{-\theta \ddagger L_1^-\right\}\Big) + \alpha\psi t E\left(Z_1\right)\mathrm{C}\left\{\theta \ddagger L_1^+\right\} \\ &\quad +\psi Z_t \mathrm{C}\left\{-\theta \ddagger L_1^-\right\}. \end{aligned}$$

This conditional form is attractive as it is linear in T_t and Z_t. This means that for $t \geq 0$

$$\begin{aligned} \mathrm{C}\left\{\theta \ddagger P_t\right\} &= \alpha\psi t E\left(Z_1\right)\mathrm{C}\left\{\theta \ddagger L_1^+\right\} \\ &\quad +\mathrm{K}\left\{\mathrm{C}\left\{\theta \ddagger L_1\right\} + \mathrm{C}\left\{-\theta \ddagger L_1^-\right\}, \psi\mathrm{C}\left\{-\theta \ddagger L_1^-\right\} \ddagger T_t, Z_t\right\} \end{aligned}$$

where

$$\mathrm{K}\left\{a, b \ddagger T_t, Z_t\right\} = \log\left[E\left\{\exp\left(aT_t + bZ_t\right)\right\}\right], \quad t \geq 0,$$

the is joint cumulant function of T_t and Z_t evaluated at the complex a and b.

6.2.5 A small tick size limit

An important model for price processes used in mathematical finance and financial econometrics is the stochastic volatility model presented in Barndorff-Nielsen and Shephard [11]. Hence it is attractive to think through how our new models relate to these established ones.

The basis of our analysis will be the small jump approximations of Lévy processes. These are discussed e.g. by Asmussen and Rosinski [5] and Cont and Tankov [34], Chapter 6.3.

In order to derive a valid approximation we think of the process over a finite time interval and allow the tick size to go to zero at the same time as the intensity of the process goes to infinity. Hence this is an active trading approximation.

This type of asymptotic device is necessary for integer-valued processes, for the usual small jump threshold approximation is not appropriate here due to the discrete nature. We will work with the conditional compensated versions of the time-changed Lévy processes

$$\begin{aligned}\overline{L}^{+}_{T_t+\alpha\psi tE(Z_1)} &= (\kappa_2^+)^{-1/2}\Big(L^+_{T_t+\alpha\psi tE(Z_1)} - \kappa_1^+(T_t+\alpha\psi tE(Z_1))\Big),\\ \overline{L}^{-}_{T_t+\psi Z_t} &= (\kappa_2^-)^{-1/2}\Big(L^-_{T_t+\psi Z_t} - \kappa_1^-(T_t+\psi Z_t)\Big), \quad t\geq 0.\end{aligned}$$

Then denote for $t \geq 0$

$$a^{1/2}\left\{R_{a^{-1}t} - E\left(R_{a^{-1}t}\right)\right\} := a^{1/2}\overline{L}^{+}\circ\left[a^{-1}\left\{T_t+\alpha\psi tE\left(Z_1\right)\right\}\right] - a^{1/2}\overline{L}^{-}\circ\left[a^{-1}\left\{T_t+\psi Z_t\right\}\right].$$

Our main result is as follows. For $d > 0$ denote by $D[0,d]$ the space of right continuous functions on $[0,d]$ with left limits equipped with the uniform metric, cf. Pollard [105]. Define limits as follows: for a sequence of stochastic processes $(X^t)_{t\geq 0}$ and Y both a.s. in $D[0,d]$ we write that $X^t \overset{D}{\to} Y$ as $t\to\infty$ if for every measurable, bounded and continuous (with respect to the uniform metric) function $f : D[0,d]\to\mathbb{R}$

$$E[f(X^t)] \to E[f(Y)], \quad t\to\infty.$$

Theorem 6.2.9. *Fix a time horizon $d > 0$. Then*

$$a^{1/2}\left\{R_{a^{-1}\bullet} - E\left(R_{a^{-1}\bullet}\right)\right\} \overset{D}{\to} W^+_{T_\bullet+\alpha\psi\bullet E(Z_1)} - W^-_{T_\bullet+\psi Z_\bullet}, \quad a\downarrow 0,$$

as a process, where W^+ and W^- are independent Brownian motions, independent of the process Z.

Proof. In a first step let $S : \mathbb{R}^+ \to \mathbb{R}^+$ be a deterministic increasing right-continuous time change with left limits and L an integer-valued zero-mean Lévy process with jumps bounded by $C > 0$. Remark that we have for the scaled process $\Delta[aC^{-1}L_{a^{-1}\bullet}] \leq a$. Therefore the jumps of $aC^{-1}L_{a^{-1}\bullet}$ are bounded by a. Further we have the limes condition $\lim_{a\to 0}\sigma(a)/a = \infty$, where $\sigma(a) = \sqrt{Var(aC^{-1}L_{a^{-1}})}$ and therefore are in the context of Theorem 2.1 of Asmussen and Rosinski [5]. We can deduce that (at time 1) for $\theta\in\mathbb{R}$

$$C\{\theta \ddagger \sigma(a)^{-1}a^n L_{a^{-1}}\} \to -\frac{1}{2}\theta^2, \quad a\to 0.$$

Considering now the time changed process $aC^{-1}L_{a^{-1}S_\cdot}$, we get for $t\geq 0$ and $\theta\in\mathbb{R}$

$$C\{\theta \ddagger \sigma(a)^{-1}aC^{-1}L_{a^{-1}S_t}\} = S_t C\{\theta \ddagger \sigma(a)^{-1}aC^{-1}L_{a^{-1}}\} \to -\frac{1}{2}\theta^2 S_t, \quad a\to 0,$$

concluding that $\sigma(a)^{-1}aC^{-1}L_{a^{-1}S_t} \to W_{S_t}$ in distribution as $a \to 0$, where W is a standard Brownian motion. Since the increments of the time changed process $aC^{-1}L_{a^{-1}S_\bullet}$ are still independent, we get that its finite dimensional distributions convergence to those of $W_{S_\bullet}$. By Theorem V.19 of Pollard [105] follows now that $aC^{-1}L_{a^{-1}S_\bullet} \xrightarrow{D} W_{S_\bullet}$ as $a \to 0$.

Now back to the assertion. Define $\mathcal{F}_R^Z := \sigma\overline{\{Z_s, s \in [0,R]\}}$ and $\mathcal{F}_R^{L^+} := \sigma\overline{\{L_s^+, s \in [0,R]\}}$. Consider a measurable bounded function $f : D[0,R] \to \mathbb{R}$. Then by repeatedly applying the theorem of dominated convergence and the result from above we get

$$\begin{aligned}
& E[f(a^{1/2}\{R_{a^{-1}\bullet} - E(R_{a^{-1}\bullet})\})] \\
= \;& EE[f(a^{1/2}\overline{L}_{a^{-1}\{T_\bullet+\alpha\psi\bullet E(Z_1)\}} - a^{1/2}\overline{L}_{a^{-1}\{T_\bullet-\psi Z_\bullet\}})|\mathcal{F}_R^Z, \mathcal{F}_R^{L^+}] \\
\to \;& EE[f(a^{1/2}\overline{L}_{a^{-1}\{T_\bullet+\alpha\psi\bullet E(Z_1)\}} - W^-_{T_\bullet+\psi Z_\bullet})|\mathcal{F}_R^Z, \mathcal{F}_R^{L^+}] \\
= \;& EE[f(a^{1/2}\overline{L}_{a^{-1}\{T_\bullet+\alpha\psi\bullet E(Z_1)\}} - W^-_{T_\bullet+\psi Z_\bullet})|\mathcal{F}_R^Z] \\
\to \;& EE[f(W^+_{T_\bullet+\alpha\psi\bullet E(Z_1)} - W^-_{T_\bullet+\psi Z_\bullet})|\mathcal{F}_\infty^Z] \\
= \;& E[f(W^+_{T_\bullet+\alpha\psi\bullet E(Z_1)} - W^-_{T_\bullet+\psi Z_\bullet})], \quad a \to 0.
\end{aligned}$$

□

Remark 6.2.10. If $\psi = 0$, $\kappa_1^+ = \kappa_1^-$ and T is of the form $T_t = \int_0^t \tau_u^2 \mathrm{d}u$, then

$$a^{1/2}R_{a^{-1}\bullet} \xrightarrow{D} \int_0^\bullet \sqrt{2}\tau_u \mathrm{d}W_u, \quad a \downarrow 0,$$

where W is another Brownian motion independent of W^+ and W^-.

Example 6.2.11. Here we build a non-stochastic T to mimic the diurnal feature of high frequency data, where financial activity tends to peak at the start and end of the day. The following is an example of this, for a single day where $t \in [0,1]$:

$$\begin{aligned}
T_t &= \int_0^t (1-2u)^2 \,\mathrm{d}u, \quad t \le 1/2, \\
T_t &= \int_{1/2}^t 8\,(u-1/2)^2 \,\mathrm{d}u, \quad t \in (1/2, 1]\,.
\end{aligned}$$

Let L^+ and L^- be independent Poisson processes with the same intensities $\lambda = 1$ and take $\psi = 0$, i.e. no leverage. Then $E\left(R_{a^{-1}t}\right) = 0$ and we plot in Figure 6.2

$$a^{1/2}R_{a^{-1}\bullet}$$

for a variety of values of a as $a \downarrow 0$. Hence for small a the path looks like a time changed Brownian motion.

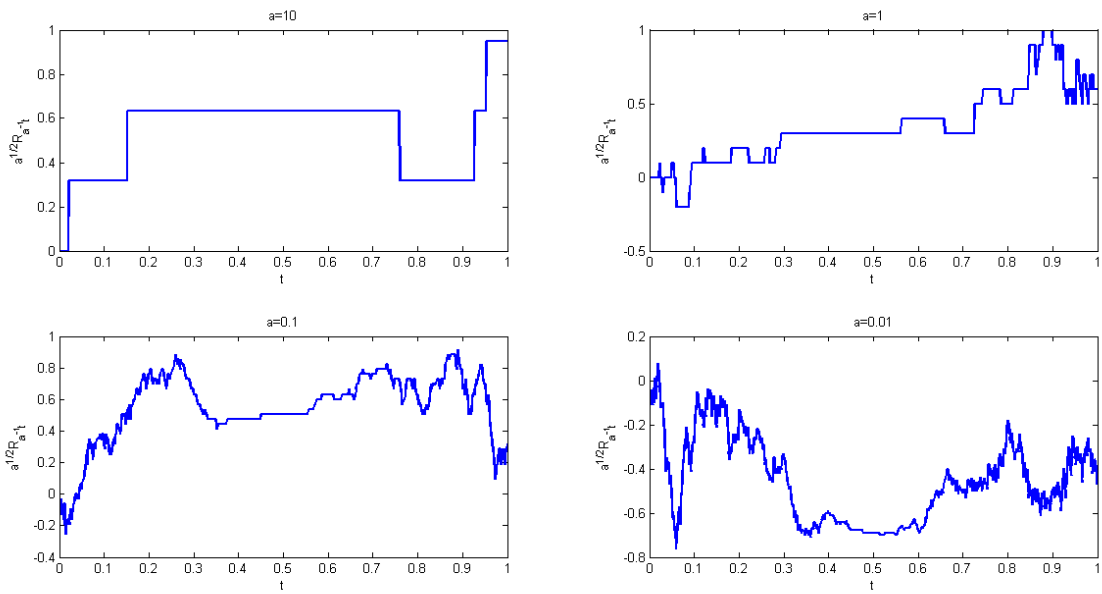

Figure 6.2: *Convergence of the process* $a^{1/2}R_{a^{-1}\bullet}$ *from Example 6.2.11.*

6.2.6 No-arbitrage and incompleteness

Denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by P. An important observation is, that the process P is a semimartingale which can be validated by the following decomposition

$$P_t = L^+_{T_t+\alpha\psi t E(Z_1)} - L^-_{T_t+\psi Z_t} = M_t + A_t$$

where

$$\begin{aligned} M_t &= L^+_{T_t+\alpha\psi t E(Z_1)} - L^-_{T_t+\psi Z_t} + \psi\left\{\kappa_1^- Z_t - \kappa_1^+ t\alpha E(Z_1)\right\} - \left(\kappa_1^+ - \kappa_1^-\right) T_t, \\ A_t &= -\psi\left\{\kappa_1^- Z_t - \kappa_1^+ t\alpha E(Z_1)\right\} + \left(\kappa_1^+ - \kappa_1^-\right) T_t. \end{aligned}$$

The process M is a (local) martingale and since Z is a subordinator (i.e. a Lévy process with upward only jumps) and T is a non-decreasing time-change, A is of locally bounded variation. Thus P is a semimartingale. This has an important consequence when considering possible arbitrage in the market. Theorem 1 of Delbaen and Schachermayer [40] states that in this case, the concept of 'no free lunch without vanishing risk' (NFLVR) is

equivalent to the existence of an equivalent martingale measure (EMM). Therefore we are interested in actually constructing such an EMM.

If we assume that $\alpha = \kappa_1^- / \kappa_1^+$ we have

$$\begin{aligned} M_t &= L^+_{T_t + \alpha\psi t E(Z_1)} - L^-_{T_t + \psi Z_t} + \psi\kappa_1^- \left\{ Z_t - tE(Z_1) \right\} - \left(\kappa_1^+ - \kappa_1^- \right) T_t, \\ A_t &= -\psi\kappa_1^- \left\{ Z_t - tE(Z_1) \right\} + \left(\kappa_1^+ - \kappa_1^- \right) T_t. \end{aligned}$$

Hence A_t is a (local) martingale if $\kappa_1^+ = \kappa_1^-$.

In their work on the Lévy special case, Barndorff-Nielsen et al. [10] obtained an EMM by simply changing the probability masses in the integer-valued Lévy processes so that $\kappa_1^+ = \kappa_1^-$ under the EMM. Exactly the same approach solves this problem here, whatever the form of T and Z. This is rather simple compared to the usual results on EMM obtained for Gaussian stochastic volatility models.

Markets driven by Lévy processes are usually not complete and therefore there is more than one EMM. This is also the case in our model, even in the simple Lévy case discussed by Barndorff-Nielsen et al. [10].

6.3 Econometric inference

Now we will outline various possible methods of estimating our model class.

Given the sample path of the price process we can separately observe up and down ticks. This means we have to make inference on two conditionally independent processes

$$R_t^+ = L^+_{T_t + \alpha\psi t E(Z_1)} =: L^+_{T_t^+} \quad \text{and} \quad R_t^- = L^-_{T_t + \psi Z_t} =: L^-_{T_t^-}, \tag{6.2}$$

although they are linked by a common T_t.

Inference can be made through

- the moments of the R_t^+, R_t^- processes or their increments,
- the likelihood of the R_t^+, R_t^- sample paths.

6.3.1 Identification

All integer-valued subordinators are simply compound Poisson processes, which have Poisson process arrivals and i.i.d. jump sizes of 1 or more. This is discussed extensively in Barndorff-Nielsen et al. [10]. When we time-change them, all we do is to make the arrival process time-varying. Hence we can assume, without loss of generality that $L^\pm$ have unit

arrival rates. In econometrics this is called an identification constraint. In turn this means we can estimate $\kappa_j^{\pm}$ simply off the distribution of price up or down moves.

6.3.2 Moment based inference

Recall from Section 6.2 that if $0 \leq s \leq t$ and $\alpha = \kappa_1^- / \kappa_1^+$

$$E(r_{s,t}) = (\kappa_1^+ - \kappa_1^-) E(T_t - T_s)$$

which allows us to identify $E(T_t - T_s)$ if $\kappa_1^+ \neq \kappa_1^-$. Further we know that if $0 \leq u \leq v \leq s \leq t$ we have for $r_{s,t}^+ = R_t^+ - R_s^+$

$$\begin{aligned} E(r_{s,t}^+) &= \kappa_1^+ \{E(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\}, \\ Cov(r_{s,t}^+, r_{u,v}^+) &= (\kappa_1^+)^2 Cov(T_t - T_s, T_v - T_u), \\ Var(r_{s,t}^+) - \frac{\kappa_2^+}{\kappa_1^+} E(r_{s,t}^+) &= \left(\kappa_1^+\right)^2 Var(T_t - T_s). \end{aligned}$$

This means we can non-parametrically identify ψ, $Var(T_t - T_s)$ and $Cov(T_t - T_s, Z_t - Z_s)$. Similar results can be collected for $r_{s,t}^- = R_t^- - R_s^-$, with, in particular, for $0 \leq u \leq v \leq s \leq t$

$$\begin{aligned} E(r_{s,t}^-) &= \kappa_1^- \{E(T_t - T_s) + \psi(t-s)E(Z_1)\}, \\ Cov(r_{s,t}^-, r_{u,v}^-) &= (\kappa_1^-)^2 Cov(T_t - T_s, T_v - T_u) + \psi(\kappa_1^-)^2 Cov(T_t - T_s, Z_v - Z_u) \\ Var(r_{s,t}^-) - \frac{\kappa_2^-}{\kappa_1^-} E(r_{s,t}^-) &= \left(\kappa_1^-\right)^2 Var\left\{T_t - T_s + \psi(Z_t - Z_s)\right\}. \end{aligned}$$

Example 6.3.1. If $L^{\pm}$ are standard Poisson processes, which corresponds to a one-tick market, then $\kappa_1^{\pm} = \kappa_2^{\pm} = 1$ and $\alpha = 1$. This implies for $0 \leq u \leq v \leq s \leq t$

$$\begin{aligned} E(r_{s,t}^+) &= E(T_t - T_s) + \psi(t-s)E(Z_1), \\ Cov(r_{s,t}^+, r_{u,v}^+) &= Cov(T_t - T_s, T_v - T_u), \\ Var(r_{s,t}^+) - E(r_{s,t}^+) &= Var(T_t - T_s), \\ Cov(r_{s,t}^-, r_{u,v}^-) - Cov(r_{s,t}^+, r_{u,v}^+) &= \psi Cov(T_t - T_s, Z_v - Z_u). \end{aligned}$$

Hence we can non-parametrically estimate the variogram of the increments of T.

6.3.3 Intensity function inference

Recalling Remark 6.1.2 we know that as L^+ and L^- are integer-valued Lévy processes, they are also compound Poisson processes

$$L_t^{\pm} = \sum_{j=1}^{N_t^{\pm}} C_j^{\pm}, \quad t \geq 0,$$

where $N^{\pm}$ are independent Poisson processes recording the number of times the processes move and $C_j^{\pm} \in \{1, 2, \ldots\}$ are the sizes of the jumps. Under this model the jump sizes are independent of the times of jumps and the T and Z processes. Hence inference about T and Z is really inference from a bivariate Cox process

$$N^+ \circ \{T_t + \alpha\psi t E(Z_1)\} = N^+_{T_t + \alpha\psi t E(Z_1)} = N^+_{T_t^+}, \quad t \geq 0,$$

and

$$N^- \circ \{T_t + \psi Z_t\} = N^-_{T_t + \psi Z_t} = N^-_{T_t^-}, \quad t \geq 0.$$

There is substantial literature about this topic, ranging over martingale inference methods in Andersen, Borgan, Gill and Keiding [2], nonparametric methods in Diggle [41] or Ramlau-Hansen [110], Markov Chain Monte Carlo methods in Adams, Murray and MacKay [1] and particle filters in Fearnhead, Papaspiliopoulos and Roberts [55] and Papaspiliopoulos, Belmonte and Pitt [101].

6.3.3.1 Linear filtering using the counting process

Define the increments of the counting processes by

$$n_{s,t}^{\pm} = N^{\pm}_{T_t^{\pm}} - N^{\pm}_{T_s^{\pm}}, \quad 0 \leq s \leq t,$$

Using the Poisson setting of Section 3 of Durbin and Koopman [49] we can directly estimate the path of the time changes $T^{\pm}$ by approximating their distribution by a Gaussian density. Another possibility is to linearize the problem. Consider the compensated processes

$$\begin{aligned} u_{s,t}^+ &= n_{s,t}^+ - \{(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\}, \\ u_{s,t}^- &= n_{s,t}^- - \{(T_t - T_s) + \psi(Z_t - Z_s)\}, \quad 0 \leq s \leq t. \end{aligned}$$

Then $u_{s,t}^{\pm}$ are uncorrelated zero mean weak white noise.

One approach to analysing the model is to use a bivariate linear representation of the measurements

$$\begin{aligned} n^+_{s,t} &= \{(T_t - T_s) + \alpha\psi(t-s)E(Z_1)\} + u^+_{s,t}, \\ n^-_{s,t} &= \{(T_t - T_s) + \psi(Z_t - Z_s)\} + u^-_{s,t}. \end{aligned}$$

This can be combined with a linear representation of the dynamics of the joint T and Z processes as in the following example.

Example 6.3.2. Suppose

$$T_t = \int_0^t \tau_s ds, \quad \mathrm{d}\tau_t = -\lambda\tau_t \mathrm{d}t + \mathrm{d}Z_t, \quad \lambda > 0.$$

Then

$$T_t = \lambda^{-1}(Z_t - \tau_t + \tau_0)$$

so the state variables are Markovian $(Z_t, \tau_t, \tau_0)'$. Now

$$\begin{aligned} \tau_t &= e^{-\lambda t}\tau_0 + \int_0^t e^{-\lambda(t-s)}\mathrm{d}Z_s = e^{-\lambda t}\tau_0 + \frac{1}{\lambda}\Big(1 - e^{-\lambda t}\Big)E(Z_1) + \int_0^t e^{-\lambda(t-s)}\mathrm{d}\overline{Z}_s, \\ Z_t &= tE(Z_1) + \int_0^t \mathrm{d}\overline{Z}_s, \end{aligned}$$

where $\overline{Z}_t := Z_t - tE(Z_1)$. Then

$$\begin{aligned} E(\tau_t|\tau_0) &= e^{-\lambda t}\tau_0 + \frac{1}{\lambda}\Big(1 - e^{-\lambda t}\Big)E(Z_1), \\ Var\Big(\begin{array}{c} \tau_t \\ Z_t \end{array}|\tau_0\Big) &= Var(Z_1)\begin{pmatrix} \frac{1}{2\lambda}\big(1 - e^{-2\lambda t}\big) & \frac{1}{\lambda}\big(1 - e^{-\lambda t}\big) \\ \frac{1}{\lambda}\big(1 - e^{-\lambda t}\big) & t \end{pmatrix}. \end{aligned}$$

Hence we can use the Kalman filter (e.g. Harvey [68] and Durbin and Koopman [49]) to provide a computationally simple way to compute a best linear forecast, filter and smoothed estimators of $(T_t - T_s, Z_t - Z_s)'$ using the time series of $(n^+_{s,t}, n^-_{s,t})$.

6.3.3.2 Approximation by piecewise linear time changes

The drawback of 6.3.3.1 is the fact that such a linear model usually has to be approximated by a Gaussian setting, cf. Durbin and Koopman [49], Section 3. Another possibility of estimating the counting process is by approximating the time changes. Since $N^\pm$ is independent of $T^\pm$ we can work with a given but unknown realized path of $T^\pm$. For simplicity let $\kappa^\pm_1 = 1$, i.e. the counting processes have unit intensities.

Split the observation period into equally spaced time brackets

$$[0], (0, \delta], \quad (i\delta, (i+1)\delta], \quad \ldots, \quad ((M-1)\delta, M\delta]$$

with $\delta > 0$ and $M \in \mathbb{N}$. We assume the time changes $T^{\pm}$ to be piecewise linear in $((i\delta, (i+1)\delta]$, $i = 0, 1, \ldots, M-1$, i.e.

$$T_t^{\pm} = \frac{1}{\delta}\Big[(t - i\delta)T_{(i+1)\delta}^{\pm} + ((i+1)\delta - t)T_{i\delta}^{\pm}\Big], \quad t \in (i\delta, (i+1)\delta], \quad i = 0, 1, \ldots, M-1,$$

for some $(T_{i\delta}^{\pm})_{i=0,1,\ldots,M}$ where we set $T_0^{\pm} = 0$. Now we have, given the paths of $T^{\pm}$,

$$N_{T_t^{\pm}}^{\pm} - N_{T_s^{\pm}}^{\pm} \stackrel{d}{=} N_{\delta^{-1}(t-s)(T_{(i+1)\delta}^{\pm} - T_{i\delta}^{\pm})}^{\pm}, \quad t, s \in (i\delta, (i+1)\delta], \quad s \leq t, \quad i = 0, 1, \ldots, M-1,$$

For $i = 0, 1, \ldots, M-1$ an unbiased estimator for $T_{(i+1)\delta}^{\pm} - T_{i\delta}^{\pm}$ is given by

$$\widehat{T}_{(i+1)\delta}^{\pm} - \widehat{T}_{i\delta}^{\pm} = \frac{1}{j-1}\sum_{h=1}^{j-1} \frac{N_{T_{t_{h+1}}^{\pm}}^{\pm} - N_{T_{t_h}^{\pm}}^{\pm}}{(t_{h+1} - t_h)/\delta},$$

with observations $(N_{T_{t_h}^{\pm}}^{\pm})_{h=1,\ldots,j}$ in time period $(i\delta, (i+1)\delta]$, since

$$E\Big(\widehat{T}_{(i+1)\delta}^{\pm} - \widehat{T}_{i\delta}^{\pm}\Big|T^{\pm}\Big) = \frac{1}{j-1}\sum_{h=1}^{j-1} \frac{E\Big(N_{T_{t_{h+1}}^{\pm}}^{\pm} - N_{T_{t_h}^{\pm}}^{\pm}\Big|T^{\pm}\Big)}{(t_{h+1} - t_h)/\delta} = T_{(i+1)\delta}^{\pm} - T_{i\delta}^{\pm}.$$

Now a parametric model can be applied to estimate the process (T^+, T^-, Z) from $(\widehat{T}^+, \widehat{T}^-)$.

A big advantage of this approach is that $T^{\pm}$ can be estimated separately by splitting the price process into up and down ticks. An application is provided in Section 6.4.3.4.

6.4 Application

We will follow Barndorff-Nielsen et al. [10] in studying tick price processes in low latency data from futures exchanges. Futures exchanges trade many assets ranging from equity indices to interest rate products and commodities. Liquidity on the electronic marketplace in many of these futures contracts is good and the exchanges well established. They are able to provide low latency data feeds recording every price and new order update seen on the matching engines order book.

6.4.1 Dataset

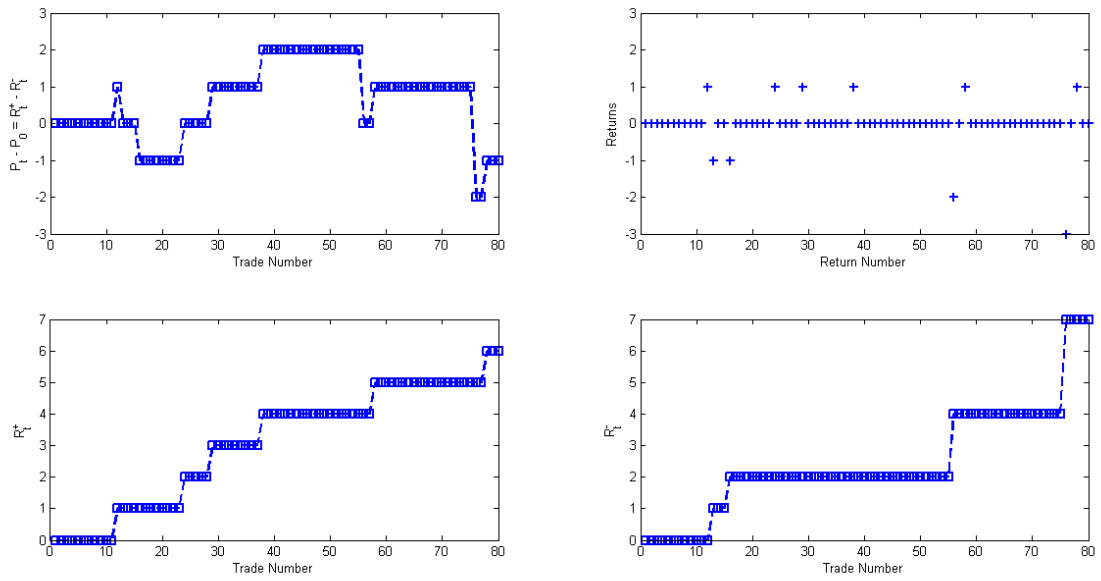

Figure 6.3: *Euro-Dollar IMM FX futures contract on 11th December 2009. Top left: ask price for the first 80 trades of the day. Top right: returns from ask price. Bottom left:* R^+*, positive component of the ask. Bottom right:* R^-*, negative component of the ask. They are all integers.*

We study, in particular, futures data for the Euro-Dollar IMM FX futures contract on 11th December 2009. These markets are sufficiently different to demonstrate a range of tick price behaviours. These data was provided to us by QuantHouse (`www.quanthouse.com`) from data feeds at the Chicago Mercantile Exchange (CME) which is one of the largest Futures exchanges.

Type of cumulant	Up ticks	Down ticks
Expectation	0.0543	0.0560
Variance	0.0557	0.0702

Figure 6.4: *Euro-Dollar IMM FX futures contract on 11th December 2009. Expectation and variance of the up and down ticks.*

6.4.2 Trades and prices

In a first step we will just consider moments when trades occur. The times at which there are trades will be written as

$$\tau_i, \quad i = 1, 2, \ldots, N.$$

The justification being that when trades occur there is agreement by at least two market participants about the market price and so we have more confidence in its accuracy. Throughout we will model one side of the spread, focusing on the ask. We then scale the prices by the tick size, so the resulting series is integer-valued. An alternative integer-valued process is the pure mid-price suggested by Barndorff-Nielsen et al. [10] but it is not used here.

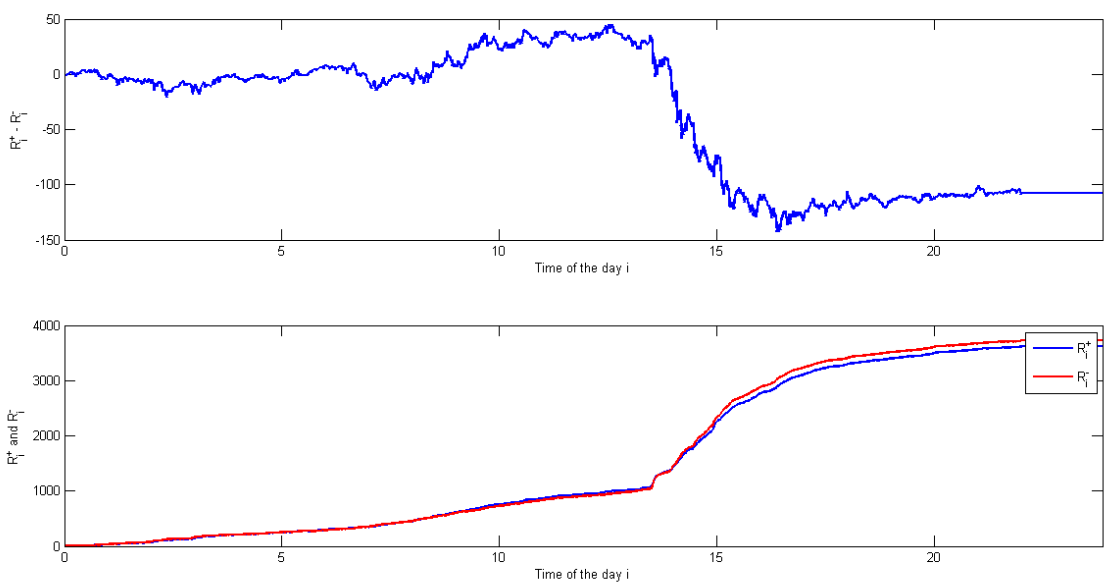

Figure 6.5: *Euro-Dollar IMM FX futures contract on 11th December 2009. Price process* $R_i^+ - R_i^-$*, up and down tick processes* R_i^+ *and* R_i^- *in 30 seconds intervals.*

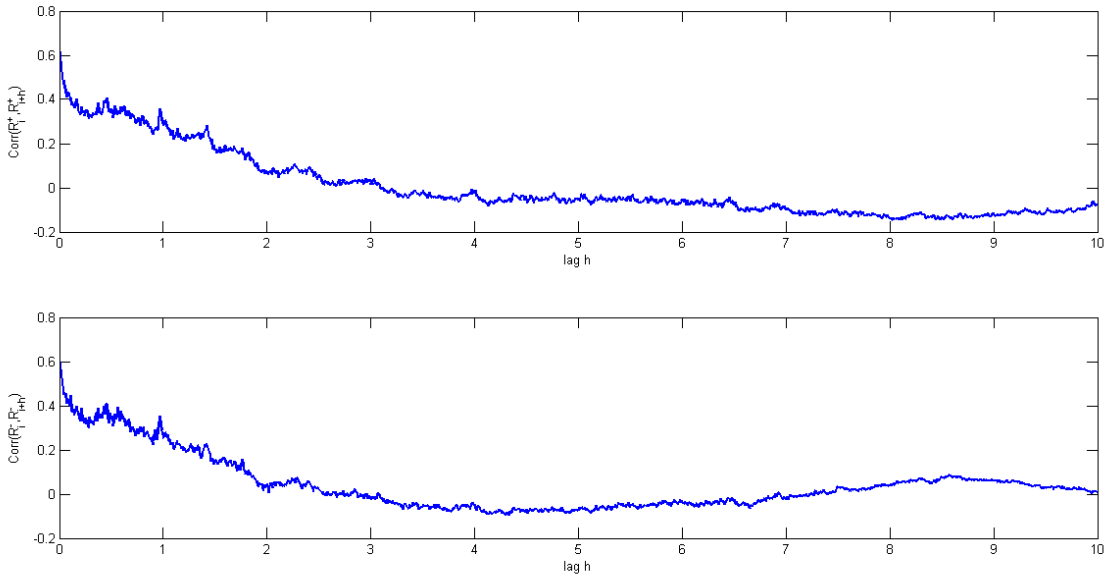

Figure 6.6: *Euro-Dollar IMM FX futures contract on 11th December 2009. Autocorrelation function* $\mathrm{Corr}(\overline{R}_i^{\pm}, \overline{R}_{i+h}^{\pm})$ *for different lags h up to 10 hours.*

Figure 6.3 shows this for the Euro-Dollar IMM FX futures contract during 11th December 2009, which had 66481 trades on the ask that day. For this contract the tick size is 0.0001 of a unit, i.e. prices move from, for example, 1.4624 to 1.4623 U.S. Dollar to the Euro. We plotted the ask at the times of the first 80 trades. The corresponding returns are also given in the figure. It shows integer returns, with most being -1, 0 and 1. However, there is also a move of 2 ticks. Looking at the whole day we even see a move of 6 ticks.

6.4.3 Summary statistics

6.4.3.1 Distributions of the jumps

Barndorff-Nielsen et al. [10] study the distribution of positive and negative jumps, which is given in Figure 8 of their paper. Our first interest here is in studying the size of up and down jumps. The results are given in Figure 6.4, for 11th December 2009. As can be seen there is a slightly downside trend in the price process on that day. Remark that only the sizes of the jumps at trade times have been considered here - the actual time at which this

jumps have occurred has not played a role yet.

This will change now as we apply the procedure provided in Section 6.3.3.2.

6.4.3.2 Avoiding microstructure effects

Since there are many ticks of just size 1 we shall from now on assume $L^{\pm} = N^{\pm}$ to be Poisson processes. Our focus will be on the time series

$$R_i^{\pm} = N^{\pm}_{T^{\pm}_{(i+1)\delta}} - N^{\pm}_{T^{\pm}_{i\delta}}, \quad i = 0, 1, \ldots, M-1,$$

where $\delta > 0$ represents a time interval of $\delta = 30$, in order to avoid the worst of the unmodeled market microstructure effects, and $M = 2880$ to reflect the whole day. We will from now on work with the assumption of Section 6.3.3.2, i.e. we will assume $T^{\pm}$ to be piecewise linear in $(i\delta, (i+1)\delta]$, $i = 0, 1, \ldots, M-1$. Figure 6.5 shows the process $R^{\pm}$ and the whole price process for the 11th December 2009 and indicates that there are strong diurnal features. Remark that there is no more activity at the end of the observation day.

6.4.3.3 Autocorrelation in the jumps

To consider autocorrelation we must first deal with the diurnal features. Therefore we calculate the average number of up- and downmovements in each 30 second time bracket which we denote by μ_i, $i = 1, 2 \ldots, M-1$. Then define

$$\overline{R}_i^{\pm} := R_i^{\pm} - \mu_i, \quad i = 1, 2 \ldots, M-1.$$

and consider for lag $h \in \{1, \ldots M-1\}$

$$Corr(\overline{R}_i^{\pm}, \overline{R}_{i+h}^{\pm}), \quad i = 1, 2 \ldots, M-h-1.$$

The results for different h can be found in Figure 6.6. The slowly decaying functions suggest that there is in fact strong autocorrelation in the returns.

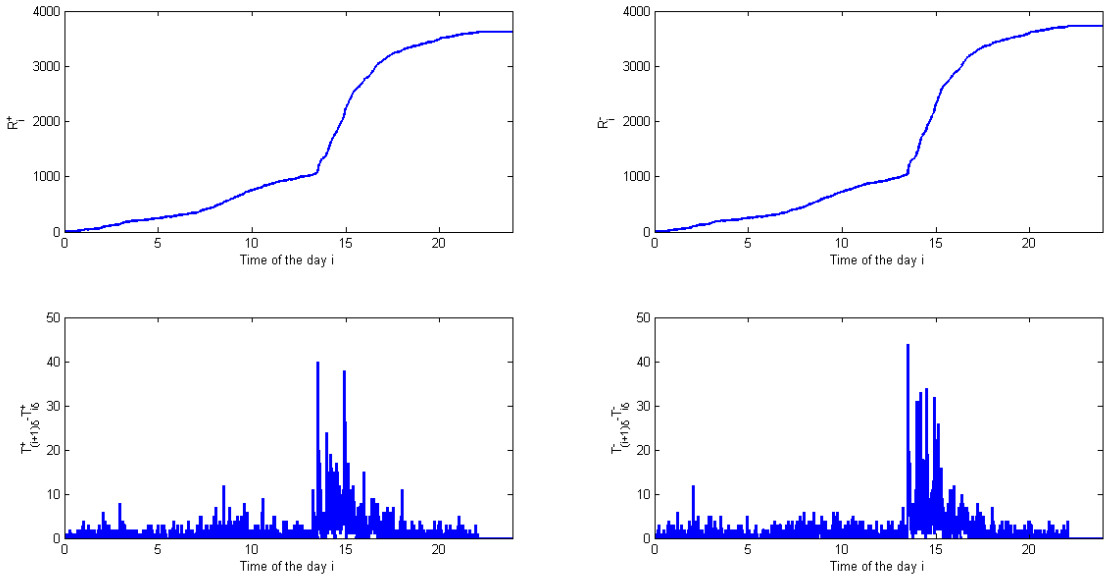

Figure 6.7: *Euro-Dollar IMM FX futures contract on 11th December 2009. Up and down tick processes* R_i^+ *and* R_i^- *in 30 seconds intervals. Returns of estimated time changes* $\widehat{T}^{\pm}$.

6.4.3.4 Estimating the time changes

Since we assumed $L^{\pm}$ to be independent Poisson processes with unit intensity we can directly apply the method from Section 6.3.3.2 for the up and down tick processes. The estimated time changes $T^{\pm}$ and their returns can be found in Figure 6.7.

Chapter 7

Conclusion

In the first part of this thesis we considered conditional distributions of fractional processes in increasing generality. Starting with a known prediction formula for the conditional expectation of a univariate fBm we derived its conditional characteristic function and transferred this result to related processes like fractional Vasicek or CIR models. Afterwards we introduced a bivariate fBm by assuming a certain dependence structure between two univariate fBms (cf. Elliott and van der Hoek [53]) and used the Wick product to obtain similar results for this case. Finally we introduced multivariate fractional Lévy processes by Molchan-Golosov kernels and investigated again conditional distributions using deconvolution. This general definition included fractional subordinators and multivariate fBm.

Motivated by the empirical studies of Henry and Zaffaroni [73] and Backus and Zin [7] which suggest the presence of long range dependence in macroeconomic variables like short and default rates, we provided several fractional interest rate and credit models as applications. Starting point was the fractional Brownian HJM approach of Ohashi [100] which we used to derive an arbitrage-free setting for a fractional Vasicek model. Zero coupon bonds - the building blocks of interest rate markets - were priced using the earlier results on conditional distributions. Afterwards we introduced credit risk in this model and calculated the prices of defaultable zero coupon bonds and general credit derivatives. In a next step we generalized the fBms to MG-fLps to overcome the issue of potential negative paths of short and default rate and compared zero coupon bond prices and parameter sensitivities between those two model types.

Credit default swaps are of a special type of credit derivatives and their pricing is of crucial importance especially for practitioners in mathematical finance. Their valuation in the above fractional model is still part of ongoing research.

The second part of this thesis focussed on extending the discrete-valued Lévy model of Barndorff-Nielsen et al. [10] introduced to describe low latency financial data. One of the major drawbacks of this setting is that it generates independent and stationary returns which is often not supported by empirical observations. Therefore we introduced stochastic volatility by two time changes affecting up and down ticks separately. We analysed various properties of this model and showed that it also captures statistical leverage, volatility clustering and diurnal features which are often observed in financial data. Even if the price process is no longer a Lévy process in general, it is still a semimartingale and the known theorems of no-arbitrage-pricing are valid. The classical Ornstein-Uhlenbeck stochastic volatility model of Barndorff-Nielsen and Shephard [11] was obtained by a limit process. Finally we outlined several estimation techniques and provided an example using the Euro-Dollar IMM FX futures.

However the proposed model leaves market microstructure effects still unmodelled which provides a good starting point for further research.

Bibliography

[1] R. P. Adams, I. Murray, and D. J. C. MacKay. Tractable nonparametric Bayesian inference in Poisson processes with Gaussian process intensities. In *Proceedings of the 26th International Conference on Machine Learning, Montreal, Canada*. 2009.

[2] P. K. Andersen, O. Borgan., R. D. Gill, and N. Keiding. *Statistical Models based on Counting Processes*. Springer, New York, 1992.

[3] T. G. Andersen, T. Bollerslev, F. X. Diebold, and P. Labys. The distribution of exchange rate volatility. *Journal of the American Statistical Association*, 96:42–55, 2001. Correction published in 2003, volume 98, page 501.

[4] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2004.

[5] S. Asmussen and J. Rosinski. Approximation of small jumps of Lévy processes with a view towards simulation. *J. Appl. Probab.*, 38:482–493, 2001.

[6] M. Avellaneda and S. Stoikov. High frequency trading in a limit order book. *Quantitative Finance*, 8:217–224, 2008.

[7] D. K. Backus and S. E. Zin. Long memory inflation uncertainty: evidence from the term structure of interest rates. *Journal of Money, Credit and Banking*, 25:681–700, 1993.

[8] O. E. Barndorff-Nielsen. Superposition of Ornstein-Uhlenbeck type processes. *Theory of Probability and its Applications*, 46:175–194, 2001.

[9] O. E. Barndorff-Nielsen, P. R. Hansen, A. Lunde, and N. Shephard. Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise. *Econometrica*, 76:1481–1536, 2008.

[10] O. E. Barndorff-Nielsen, D. Pollard, and N. Shephard. Integer-valued Lévy processes and low latency financial econometrics. Submitted for publication, 2010.

[11] O. E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics (with discussion). *Journal of the Royal Statistical Society, Series B*, 63:167–241, 2001.

[12] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society, Series B*, 64:253–280, 2002.

[13] L. Bauwens and N. Hautsch. Modelling financial high frequency data using point processes. In T. G. Andersen, R. A. Davis, J. P. Kreiss, and T. Mikosch, editors, *Handbook of Financial Time Series*, pages 953–979. Springer, Heidelberg, 2009.

[14] C. Bender and R. J. Elliott. On the Clark-Ocone Theorem for fractional Brownian motion with Hurst parameter bigger than a half. *Stochastics Stochastics Rep.*, 75(6):391–905, 2003.

[15] C. Bender, A. Lindner, and M. Schicks. Finite variation of fractional Lévy processes. *J. Theor. Probab.*, 2011. online first.

[16] C. Bender and T. Marquardt. Integrating volatility clustering into exponential Lévy models. *J. Appl. Probab.*, 46(3):609–628, 2009.

[17] C. Bender, T. Sottinen, and E. Valkeila. Arbitrage with fractional Brownian motion? *Theory Stoch. Process.*, 13(1-2):23–34, 2007.

[18] F. Biagini, H. Fink, and C. Klüppelberg. A fractional credit model with long range dependent default rate. Submitted for publication. Available at http://www-m4.ma.tum.de/forschung/preprints-veroeffentlichungen, 2010.

[19] F. Biagini, S. Fuschini, and C. Klüppelberg. Credit contagion in a long range dependent macroeconomic factor model. In G. Di Nunno and B. Øksendal, editors, *Advanced Mathematical Methods in Finance*, pages 105–132. Springer, Heidelberg, 2011.

[20] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Springer, London, 2008.

[21] T. R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging.* Springer, Berlin, 2002.

[22] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Pol. Econ.*, 87:637–659, 1973.

[23] R. M. Blumenthal and R. K. Getoor. Sample functions of stochastic processes with stationary independent increments. *J. Math. and Mech.*, 10:493–516, 1961.

[24] D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice.* Springer, Berlin, 2001.

[25] P. J. Brockwell. Lévy driven CARMA processes. *Annals of the Institute of Statistical Mathematics*, 52:113–124, 2001.

[26] P. J. Brockwell. Representations of continuous-time ARMA processes. *J. Appl. Probab.*, 41A:357–382, 2004.

[27] P. J. Brockwell and T. Marquardt. Lévy-driven and fractionally integrated ARMA processes with continuous time parameter. *Statistica Sinica*, 15:477–494, 2005.

[28] B. Buchmann and C. Klüppelberg. Fractional integral equations and state space transforms. *Bernoulli*, 12(3):431–456, 2006.

[29] P. Carr and L. Wu. Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 71:113–141, 2004.

[30] A. Carverhill. When is the short rate Markovian? *Mathematical Finance*, 4:305–312, 1994.

[31] P. Cheridito. Arbitrage in fractional Brownian motion models. *Finance and Stochastics*, 7(4):533–553, 2003.

[32] P. Cheridito, H. Kawaguchi, and M. Maejima. Fractional Ornstein-Uhlenbeck processes. *Electronic Journal of Probability*, 8:1–14, 2003.

[33] P. K. Clark. A subordinated stochastic process model with fixed variance for speculative prices. *Econometrica*, 41:135–156, 1973.

[34] R. Cont and P. Tankov. *Financial Modelling with Jump Processes.* Chapman and Hall, London, 2004.

[35] J. C. Cox, J. E. Ingersoll, and S. A. Ross. A theory of term structure of interest rates. *Econometrica*, 53:385–407, 1985.

[36] L. Decreusefond and A. S. Üstünel. Stochastic analysis of the fractional Brownian motion. *Potential Analysis*, 10:177–214, 1999.

[37] S. Delattre and J. Jacod. A central limit theorem for normalized functions of the increments of a diffusion process in the presence of round off errors. *Bernoulli*, 3:1–28, 1997.

[38] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Annalen*, 312:215–250, 1989.

[39] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Annalen*, 312:463–520, 1994.

[40] F. Delbaen and W. Schachermeyer. *The Mathematics of Arbitrage*. Springer, Heidelberg, 2006.

[41] P. Diggle. A kernel method for smoothing point process data. *Applied Statistics*, 34:138–147, 1985.

[42] H. Doss. Liens entre équations différentielles stochastiques et ordinaires. *Ann. Inst. Henri Poincaré*, 13:99–125, 1977.

[43] R. M. Dudley. *Real Analysis and Probability*. Wadsworth and Brooks/Cole Advanced Books and Software, 1989; republished, Cambridge University Press, Cambridge, 2006.

[44] D. Duffie. Credit risk modeling with affine processes. Stanford University and Scuola Normale Superiore, Pisa, 2004.

[45] D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. *Ann. Appl. Prob.*, 13:984–1053, 2003.

[46] T. E. Duncan. Prediction for some processes related to a fractional Brownian motion. *Statistics & Probability Letters*, 76:128–134, 2006.

[47] T. E. Duncan and H. Fink. Corrigendum to "Prediction for some processes related to a fractional Brownian motion". *Statistics & Probability Letters*, 81(8):1336–1337, 2011.

[48] T. E. Duncan, Y. Hu, and B. Pasik-Duncan. Stochastic calculus for fractional Brownian motion. *SIAM J. Control and Optimization*, 38(2):582–612, 2000.

[49] J. Durbin and S. J. Koopman. A simple and efficient simulation smoother for state space time series analysis. *Biometrika*, 89:603–616, 2002.

[50] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1:281–299, 1995.

[51] E. Eberlein, U. Keller, and K. Prause. New insights into smile, mispricing and value at risk: the hyperbolic model. *Journal of Business*, 71:371–405, 1998.

[52] E. Eberlein and S. Raible. Term structure models driven by general Lévy processes. *Mathematical Finance*, 9:31–53, 1999.

[53] R. J. Elliott and J. van der Hoek. A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13:301–330, 2003.

[54] R. F. Engle. Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50:987–1007, 1982.

[55] P. Fearnhead, O. Papaspiliopoulos, and G. O. Roberts. Particle filters for partially-observed diffusions. *Journal of the Royal Statistical Society, Series B*, 70:755–777, 2008.

[56] D. Filipovic. *Term-Structure Models*. Springer, New York, 2003.

[57] H. Fink. Fractional Lévy Ornstein-Uhlenbeck Processes. Bachelor's thesis, Technische Universität München, 2008.

[58] H. Fink. Conditional characteristic functions of Molchan-Golosov fractional Lévy processes with application to credit risk. Submitted for publication. Available at http://www-m4.ma.tum.de/forschung/preprints-veroeffentlichungen, 2012.

[59] H. Fink and C. Klüppelberg. Fractional Lévy driven Ornstein-Uhlenbeck processes and stochastic differential equations. *Bernoulli*, 17(1):484–506, 2011.

[60] H. Fink, C. Klüppelberg, and M. Zähle. Conditional characteristic functions of fractional Brownian motion and related processes. Accepted by *J. Appl. Probab.* Available at http://www-m4.ma.tum.de/forschung/preprints-veroeffentlichungen, 2012.

[61] H. Fink and N. Shephard. Integer-valued volatility clustering and statistical leverage for low latency financial data. Preprint, 2011.

[62] R. Frey and J. Backhaus. Pricing and hedging of portfolio credit derivatives with interacting default intensities. *International Journal of Theoretical and Applied Finance*, 11(6):611–634, 2008.

[63] E. Ghysels, A. C. Harvey, and E. Renault. Stochastic volatility. In C. R. Rao and G. S. Maddala, editors, *Statistical Methods in Finance*, pages 119–191. North-Holland, Amsterdam, 1996.

[64] G. Gripenberg and I. Norros. On the prediction of fractional Brownian motion. *J. Appl. Prob.*, 33:400–410, 1996.

[65] P. Guasoni, M. Rásonyi, and W. Schachermeyer. Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Prob.*, 18:491–520, 2008.

[66] P. Guasoni, M. Rásonyi, and W. Schachermeyer. The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Annals of Finance*, 6(2):157–191, 2010.

[67] P. R. Hansen and G. Horel. Quadratic variation by Markov chains. Department of Economics, Stanford University, Preprint, 2009.

[68] A. C. Harvey. *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge University Press, Cambridge, 1989.

[69] J. Hasbrouck. The dynamics of discrete bid and ask quotes. *Journal of Finance*, 54:2109–2142, 1999.

[70] S. Haug and C. Czado. Mixed effect models for absolute log-returns of ultra high frequency data. *Applied Stochastic Models in Business and Industry*, 22(3):243–267, 2006.

[71] J. Hausman, A. W. Lo, and A. C. MacKinlay. An ordered probit analysis of transaction stock prices. *Journal of Financial Economics*, 31:319–30, 1992.

[72] D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica*, 60:77–105, 1992.

[73] M. Henry and P. Zaffaroni. The long-range dependence paradigm for macroeconomics and finance. In P. Doukhan, G. Oppenheim, and M. Taqqu, editors, *Long-Range Dependence*. Birkhäuser, Boston, 2003.

[74] J. Hull and A. White. Pricing interest rate derivative securities. *Review of Financial Studies*, 3:573–592, 1990.

[75] H. Hurst. Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers*, 116:770–808, 1951.

[76] H. Hurst. Methods of using long-term storage in reservoirs. *Proceedings of the Institution of Civil Engineers*, Part I:519–577, 1955.

[77] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes.* North-Holland Publ. Co., Amsterdam, 2nd edition, 1989.

[78] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, New York, 1988.

[79] M. L. Kleptsyna, A. LeBreton, and M. C. Roubaud. Rudiments of stochastic fractional calculus and statistical applications. 1999b. Preprint.

[80] C. Klüppelberg, A. Lindner, and R. Maller. Continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour. *J. Appl. Probab.*, 41:601–622, 2004.

[81] C. Klüppelberg, A. Lindner, and R. Maller. A continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. In Y. Kabanov, R. Lipster, and J. Stoyanov, editors, *From Stochastic Analysis to Mathematical Finance, Festschrift for A. Shiryaev*, pages 393–419. Springer, Berlin, 2006.

[82] C. Klüppelberg and M. Matsui. Generalized fractional Lévy processes with fractional Brownian limit and applications to stochastic volatility models. *Submitted for publication*, 2010.

[83] V. Krvavich and S. Mishura. Differentiability of fractional integrals whose kernels contain fractional Brownian motions. *Ukrainian Mathematical Journal*, 53:35–47, 2001.

[84] B. Lehmann. Arbitrage-free limit order books and the pricing of order flow risk. Working Paper No. 13848, 2008.

[85] A. W. Lo and J. Wang. Stock market trading volume. In Y. Ait-Sahalia and L. P. Hansen, editors, *Handbook of Financial Econometrics: volume 2 — applications*, pages 241–342. 2010.

[86] T. Lyons. Differential equations driven by rough signals (I): an extension of an equality of L. C. Young. *Math. Research Letters*, 1:451–464, 1994.

[87] B. B. Mandelbrot. The variation of certain speculative prices. *Journal of Business*, 36:394–419, 1963.

[88] B. B. Mandelbrot. Une classe de processus stochastiques homothetiques a soi; application a loi climatologique de H. E. Hurst. *Comptes Rendus Academic Sciences Paris*, 240:3274–3277, 1965.

[89] B. B. Mandelbrot and J. W. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437, 1968.

[90] B. B. Mandelbrot and J. R. Wallis. Noah, Joseph and operational hydrology. *Water Resources Research*, 4:909–918, 1968.

[91] M. B. Marcus and J. Rosinski. Continuity and boundedness of infinitely divisible processes: a Poisson point process approach. *J. Theor. Probab.*, 4:109–160, 2005.

[92] T. Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1009–1126, 2006.

[93] R. C. Merton. Rational theory of option pricing. *Bell Journal of Economics and Management Science*, 4:141–83, 1973.

[94] T. Mikosch and R. Norvaiša. Stochastic integral equation without probability. *Bernoulli*, 6:401–434, 2000.

[95] G. Molchan and J. Golosov. Gaussian stationary processes with asymptotic power spectrum. *Soviet Math. Dokl.*, 10:134–137, 1969.

[96] I. Monroe. On the γ-variation of processes with stationary independent increments. *Ann. Math. Statist.*, 43:1213–1220, 1972.

[97] G. Müller and C. Czado. An autoregressive ordered probit model with application to high-frequency finance. *Journal of Computational and Graphical Statistics*, 14(2):320–338, 2005.

[98] P. A. Mykland and L. Zhang. The econometrics of high frequency data. In M. Kessler, A. Lindner, and M. Sørensen, editors, *Statistical Methods for Stochastic Differential Equations*. Chapman & Hall/CRC Press, 2010. Forthcoming.

[99] I. Norros, E. Valkeila, and J. Virtamo. An elementary approach to a Girsanov formula and other analytical results for fractional Brownian motion. *Bernoulli*, 5(4):571–587, 1999.

[100] A. Ohashi. Fractional term structure models: no arbitrage and consistency. *Ann. Appl. Prob.*, 19(4):1533–1580, 2009.

[101] O. Papaspiliopoulos, M. A. G. Belmonte, and M. K. Pitt. Particle filters for state and parameter estimation of cox process. Department of Economics, University of Warwick, Preprint, 2009.

[102] P. C. B. Phillips and J. Yu. Information loss in volatility measurement with flat price trading. Cowles Foundation for Research in Economics, Yale University, Preprint, 2008.

[103] V. Pipiras and M. S. Taqqu. Integration questions related to fractional Brownian motion. *Prob. Theory Rel. Fields*, 118:251–291, 2000.

[104] V. Pipiras and M. S. Taqqu. Are classes of deterministic integrals for fractional Brownian motion on an interval complete? *Bernoulli*, 7:878–897, 2001.

[105] D. Pollard. *Convergence of Stochastic Processes.* Springer, New York, 1984.

[106] M. Potters and J.-P. Bouchaud. More statistical properites of order books and price impact. *Physics A: Stat. Mech. Appl.*, 324:133–140, 2003.

[107] S. J. Press. A compound events model for security prices. *Journal of Business*, 40:317–335, 1967.

[108] P. E. Protter. *Stochastic Integration and Differential Equations.* Springer, Berlin, 2nd edition, 2004.

[109] B. S. Rajput and J. Rosinski. Spectral representations of infinitely divisible processes. *J. Theor. Probab.*, 82(3):453–487, 1989.

[110] H. Ramlau-Hansen. Smoothing counting process intensities by means of kernel functions. *Annals of Statistics*, 11:453–466, 1983.

[111] E. Renault and B. Werker. Stochastic volatility models with transaction time risk. Department of Economics, University of North Carolina, Preprint, 2009.

[112] M. Rosenbaum. Integrated volatility and round-off error. *Bernoulli*, 15, 2009.

[113] J. R. Russell and R. F. Engle. A discrete-state continuous-time model of financial transaction prices and times. *Journal of Business and Economic Statistics*, 23:166–180, 2006.

[114] J. R. Russell and R. F. Engle. Analysis of high-frequency data. In Y. Ait-Sahalia and L. P. Hansen, editors, *Handbook of Financial Econometrics: volume 1 — tools and techniques*, pages 383–426. 2010.

[115] F. Russo and P. Vallois. Elements of stochastic calculus via regularisation. *Lecture Notes in Math.*, 1899:147–186, 2007.

[116] T. H. Rydberg and N. Shephard. Dynamics of trade-by-trade price movements: decomposition and models. *Journal of Financial Econometrics*, 1:2–25, 2003.

[117] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach Science Publishers, Yverdon, 1993.

[118] G. Samorodnitsky. Long Range Dependence. *Foundations and Trends ® in Stochastic Systems*, 1(3):163–257, 2006.

[119] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes.* Chapman & Hall, New York, 1994.

[120] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[121] K.-I. Sato. Additive processes and stochastic integrals. *Illinois Journal of Mathematics*, 50:825–851, 2006.

[122] P. J. Schönbucher. *Credit Derivatives Pricing Models.* Wiley, Chichester, 2003.

[123] N. Shephard. Statistical aspects of ARCH and stochastic volatility. In D. R. Cox, D. V. Hinkley, and O. E. Barndorff-Nielsen, editors, *Time Series Models in Econometrics, Finance and Other Fields*, pages 1–67. Chapman & Hall, London, 1996.

[124] N. Shephard, editor. *Stochastic Volatility: Selected Readings*. Oxford University Press, Oxford, 2005.

[125] T. Sottinen and E. Valkeila. Fractional Brownian motion as a model in finance. Department of Mathematics, Preprint 302, 2001.

[126] H. Sussman. On the gap between deterministic and stochastic ordinary differential equations. *Ann. Probab.*, 6:19–41, 1978.

[127] H. J. Tikanmäki and Y. Mishura. Fractional Lévy processes as a result of compact interval integral transformation. *Stochastic Analysis and Applications*, 29:1081–1101, 2011.

[128] V. Todorov and G. Tauchen. Simulation methods for Lévy-driven CARMA stochastic volatility models. *Journal of Business and Economic Statistics*, 24:450–469, 2006.

[129] E. Valkeila. On some properties of geoometric fractional Brownian motion. Preprint, 1999.

[130] O. Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5:177–188, 1977.

[131] A. E. D. Veraart and M. Winkel. Time change. In R. Cont, editor, *Encyclopedia of Quantitative Finance*. Wiley, 2010.

[132] A. E. D. Veraat and L. A. M. Veraat. Stochastic volatility and stochastic leverage. *Annals of Finance*, 2010. Forthcoming.

[133] P. Weber and B. Rosenow. Order book approach to price impact. *Quantitative Finance*, 5:357–364, 2005.

[134] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67:251–282, 1936.

[135] M. Zähle. Integration with respect to fractal functions and stochastic calculus I. *Prob. Theory Rel. Fields*, 111:333–374, 1998.

[136] M. Zähle. Integration with respect to fractal functions and stochastic calculus II. *Math. Nachr.*, 225:145–183, 2001.