PIDES FOR PRICING EUROPEAN OPTIONS IN LÉVY MODELS – A FOURIER APPROACH

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Abstract.

Our aim is to establish a precise link between prices of European options in Lévy models and PIDEs. We follow a Fourier transform based approach and outline a structural affinity of PIDE and Fourier methods in this context. Our analysis provides a framework that is extendible to more complex problems, such as to PIDEs for pricing barrier options.

Since the payoff functions of a wide range of options such as calls or puts, written as functions on the log-price of the underlying, are not in $L^2(\mathbb{R})$, exponentially weighted Sobolev–Slobodeckii spaces are studied. It turns out that the exponential weight corresponds to a shift of the symbol of the Lévy process in the complex plane. We derive natural assumptions on the symbol and its analytic extension, under which the associated evolution problem has a unique weak solution in an exponentially weighted Sobolev–Slobodeckii space.

To provide the characterization of option prices as weak solutions of PIDEs in weighted Sobolev–Slobodeckii spaces, a Feynman–Kac formula for weak solutions of PIDEs is proved. Furthermore, an explicit solution in terms of the Fourier transform is derived.

1. Introduction

The Feynman–Kac formula builds a bridge between conditional expectations and solutions of Partial Integro Differential Equations (PIDEs). In recent years this has lead to a remarkable development of algorithms to price options in Lévy models by solving PIDEs based on the finite element method. For the development of wavelet-Galerkin methods for pricing European and American options, see Matache et al. (2004), Matache et al. (2005b), Matache et al. (2005a), a multivariate extension is provided in Reich et al. (2010), and Winter (2009). We refer to Hilber et al. (2009) for an overview of different numerical methods, including finite element methods, for option pricing in Lévy models.

Our central concern of this article is to establish a precise link between conditional expectations that represent European option prices and weak solutions of PIDEs. The payoff functions of options such as calls, puts or up and out digitals are, written as functions of the logarithmic returns of the underlying, not square integrable. To enforce sufficient integrability, we have to work with exponentially weighted spaces.

Let us point out that our analysis of PIDEs for European options in Lévy models is not motivated by a numerical method for solving PIDEs compatible with Fourier transform based methods. Instead our intention is to provide a framework that is extendible to more complex

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situations, such as the pricing of barrier and American options; that is to problems for which solving a PIDE can actually be more efficient than following a Fourier approach. The results of the present article are used in Glau (2010) in the proof of a Feynman–Kac formula which allows to characterize the distribution of the supremum of a Lévy process or more generally the prices of barrier options. We would like to point out that our analysis of PIDEs in exponentially weighted spaces can also be useful for numerical methods. This may be surprising since numerical procedures for solving PIDEs typically start with the restriction to a finite domain. In this situation, exponential weighting is unnecessary. There are also recent results on algorithms to solve PDEs numerically on unbounded domains without restricting the domain, see Kestler and Urban (2010). The finite element methods developed therein might also be useful for solving PIDEs related to option prices in Lévy models.

The papers mentioned above by Schwab et. al. concentrate on the numerics of PIDEs. Their starting point already assumes the characterization of prices as weak solutions of PIDEs. Feynman–Kac formulas for PIDEs related to Lévy models have been studied in Cont and Voltchkova (2005). These authors characterize the prices of European and barrier options as viscosity solutions of PIDEs. Let us mention that they assume polynomial boundedness of the payoff functions, written as functions of the logarithm of the stock price, which is not satisfied for the payoff function of a European call. They additionally assume Lipschitz-continuity of the payoff written as a function of the stock price itself. This is not the case for a digital up and out barrier option.

In this article, we restrict ourselves to the case of European options. On the other side our framework is designed to capture all standard payoff functions.

We follow a Fourier transform based approach by understanding the PIDE as a pseudo differential equation and show that exponentially weighting the underlying Sobolev-Slobodeckii space corresponds to shifting the symbol in the complex plane. This fact is used to derive conditions on the symbol, under which the related parabolic equation has a unique solution in an exponentially weighted Sobolev-Slobodeckii space. These techniques are also to be used in the more complex situation of pricing e.g. barrier or lookback options.

We further derive an explicit solution of PIDEs associated to European options. This arises in a natural way following the Fourier approach to PIDEs. Even more, it allows to reveal a structural affinity between PIDE and Fourier methods. We emphasize that this is true in the context of European option prices, but there is no comparable analogy in the case of path-dependent options like barrier or American options.

The basic notation and results concerning an analytic extension of the symbol resp. on the characteristic function of time-inhomogeneous Lévy processes in the complex plane, are provided in Section 2.

In Section 3, exponentially weighted Sobolev-Slobodeckii spaces are introduced via Fourier transforms, which allows to conclude that the dual space of $H^s_\eta(\mathbb{R}^d)$ and the space $H^{-s}_\eta(\mathbb{R}^d)$ are isomorphic. This fact is used in Section 4, where we derive under additional assumptions on the symbol, that the infinitesimal generator of a time-inhomogeneous Lévy process represents a family of continuous linear mappings from a weighted Sobolev space into its dual space. The proof is based on the observation that exponential weighting of a function corresponds to a shift of its Fourier transform in the complex plane. The same is true for the operator and the symbol, which leads to a polynomial growth condition for the symbol on a certain complex domain. The proof provided in Section 4 relies on a version of Cauchy's theorem, that is given in the appendix.

In section 4, we additionally translate the Gårding condition for the bilinear form to a condition on the symbol, and from the ellipticity of the operator, we obtain under suitable conditions on the symbol, that the related evolution problem has a unique solution in a certain weighted Sobolev–Slobodeckii space.

For the homogeneous evolution problem, i.e. when the right hand side of the equation equals 0, in Section 6 the equation is transformed by a Fourier transform into an ordinary differential equation. The resulting equation has an explicit solution. This allows for example for an elementary proof of the smoothness of the solution of the homogeneous Cauchy problem. In particular, one can derive a Feynman–Kac formula for the solution.

2. Time-inhomogeneous Lévy processes, infinitesimal generator and symbol

This section provides the basic notation and preliminary results on the symbol of time-inhomogeneous Lévy processes. Lévy processes are adapted stochastic processes with càdlàg paths with stationary and independent increments. The wider class of time-inhomogeneous Lévy processes, also called PIIAC (process with independent increments and absolutely continuous characteristics), consists of those adapted stochastic processes with càdlàg paths, that have independent increments, compare Eberlein et al. (2005). This class of processes is closely related to the class of additive processes, in particular, every time-inhomogeneous Lévy process is an additive process, see Sato (1999) and Kluge (2005, Lemma 1.3).

An introduction to Lévy processes is provided in Sato (1999), Bertoin (1996), Kyprianou (2006), and Applebaum (2009). A detailed introduction to time-inhomogeneous Lévy processes is given in Kluge (2005).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis. The distribution of an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process L is determined by the characteristic functions of the distributions of L_t for $t \geq 0$,

$$E\,\mathrm{e}^{i\langle \xi, L_t\rangle} = \mathrm{e}^{\int_0^t \theta_s(i\xi)\,\mathrm{d}s}\,, \tag{1}$$

where the cumulant function θ_s for any fixed $s \geq 0$ equals

$$\theta_s(i\xi) = -\frac{1}{2}\langle \xi, \sigma_s \xi\rangle + i\langle \xi, b_s\rangle + \int_{\mathbb{R}^d} \Big(\mathrm{e}^{i\langle \xi, y\rangle} - 1 - i\langle \xi, h(y)\rangle\Big)\, F_s(\mathrm{d}y) \tag{2}$$

for a truncation function $h : \mathbb{R}^d \to \mathbb{R}$. A bounded measurable function $h : \mathbb{R}^d \to \mathbb{R}$ with compact support is called a *truncation function*, if $h(x) = x$ in a neighbourhood of 0.

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. Furthermore, for every $s > 0$, σ_s is a symmetric, positive semi-definite $d \times d$-matrix, $b_s \in \mathbb{R}^d$, and F_s is a Lévy measure, i.e. a Borel measure on $\mathbb{R}^d$ with $\int (|x|^2 \wedge 1) F_s(\mathrm{d}x) < \infty$. The maps $s \mapsto \sigma_s$, $s \mapsto b_s$ and $s \mapsto \int (|x|^2 \wedge 1) F_s(\mathrm{d}x)$ are Borel-measurable with

$$\int_0^T \Big(|b_s| + \|\sigma_s\|_{\mathcal{M}(d\times d)} + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x)\Big)\,\mathrm{d}s < \infty$$

for every $T > 0$, where $\|\cdot\|_{\mathcal{M}(d\times d)}$ is a norm on the vector space formed by the $d\times d$-matrices. For Lévy processes L, the identity (1) is the Lévy-Khintchine formula, and in this case the quantities b, σ, F and θ do not depend on s.

The canonical representation of the process is, according to Jacod and Shiryaev (1987, Theorem II.2.34), given by

$$L = \int_0^{\cdot} b_s \,\mathrm{d}s + L^c + h * (\mu - \nu) + \big(x - h(x)\big) * \mu\,,$$

where L^c denotes the continuous martingale part of L and μ is the random measure of jumps of the process L. The continuous martingale part L^c can be written in the form $L^c = \int_0^{\cdot} \sigma_s^{1/2} \,\mathrm{d}W_s$ with a standard Brownian motion W with values in $\mathbb{R}^d$, see Karatzas and Shreve (1991, Theorem 3.4.2). Choosing the truncation function $h(x) := x\mathbb{1}_{\{|x|\leq 1\}}(x)$, one obtains the more explicit representation

$$L = \int_0^{\cdot} b_s \,\mathrm{d}s + \int_0^{\cdot} \sigma_s^{1/2} \,\mathrm{d}W_s + h * (\mu - \nu) + \sum_{s\leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s|>1\}}\,.$$

In case L is a special semimartingale, we can chose h to be the identity, $h(x) = x$, which leads to the more convenient representation

$$L = \int_0^{\cdot} b_s \,\mathrm{d}s + \int_0^{\cdot} \sigma_s^{1/2} \,\mathrm{d}W_s + x * (\mu - \nu)$$

with different coefficients b_s, see Jacod and Shiryaev (1987, Corollary II.2.38).

Of special interest in the next sections is the infinitesimal generator $\mathcal{G}_s$ of time-inhomogeneous Lévy processes L, that is

$$\begin{aligned}\mathcal{G}_s\varphi(x) \;=\;& \frac{1}{2}\sum_{j,k=1}^{d} \sigma_s^{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{d} b_s^j \frac{\partial\varphi}{\partial x_j}(x) \\ &+ \int_{\mathbb{R}^d} \Big(\varphi(x+y) - \varphi(x) - \sum_{j=1}^{d} \frac{\partial\varphi}{\partial x_j}(x)(h(y))_j\Big) F_s(\mathrm{d}y)\end{aligned} \tag{3}$$

for $\varphi \in C_0^2(\mathbb{R}^d, \mathbb{R})$, compare e.g. Dynkin (1965).

We define $\mathcal{A}_t := -\mathcal{G}_t$ for every $t \geq 0$. It turns out that $\mathcal{A}_t$ can be written in the form

$$\mathcal{A}_t u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-i\langle \xi, x\rangle}\, A_t(\xi)\mathcal{F}(u)(\xi)\,\mathrm{d}\xi \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^d, \mathbb{R})\,.$$

For short we write

$$\mathcal{A}_t u = \mathcal{F}^{-1}\big(A_t\mathcal{F}(u)\big) \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^d, \mathbb{R})\,, \tag{4}$$

where

$$\begin{aligned} A_t(\xi) :=& \frac{1}{2}\langle \xi, \sigma_t\xi\rangle + i\langle \xi, b_t\rangle - \int \Big(\mathrm{e}^{-i\langle \xi, y\rangle} - 1 + i\langle \xi, h(y)\rangle\Big)\, F_t(\mathrm{d}y) \\ =& -\theta_t(-i\xi) \qquad\qquad (\xi \in \mathbb{R}^d)\,, \end{aligned} \tag{5}$$

$\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. It is standard to show that

$$\big|A_t(\xi)\big| \leq C\big(1 + |\xi|\big)^2 \qquad \text{for all } \xi \in \mathbb{R}^d\,. \tag{6}$$

As a consequence, the Fourier inversion $\mathcal{F}^{-1}$ in (4) is well defined. (4) shows that $\mathcal{A}_t$ is a so-called pseudo differential operator (PDO) with symbol A_t. In analogy to the symbol of a Lévy process, compare Jacob (2001), we call the family $(A_t)_{t\in[0,T]}$ the symbol of the time-inhomogeneous Lévy process.

We outline in the following remarks and lemmas some properties of the symbol of time-inhomogeneous Lévy processes, where we focus on its analytic extension which allows to interpret the operator $\mathcal{A}$ as a continuous linear operator between exponentially weighted Sobolev–Slobodeckii spaces (see Section 4). In the sequel we restrict ourselves to a finite time horizon $[0,T]$ and we will concentrate on an analytic extension of the symbol to the domain

$$U_{-\eta} = U_{-\eta^1} \times \cdots \times U_{-\eta^d}\,, \tag{7}$$

which is defined for $\eta = (\eta^1, \dots, \eta^d) \in \mathbb{R}^d$ by the strips $U_{-\eta^j} := \mathbb{R} - i\,\mathrm{sgn}(\eta^j)[0, |\eta^j|)$ in the complex plane for $\eta^j \neq 0$. For $\eta^j = 0$ we define $U_{-\eta^j} = U_0 := \mathbb{R}$. We denote by R_η the d-dimensional cuboid $R_\eta = \mathrm{sgn}(\eta^1)[0, |\eta^1|] \times \cdots \times \mathrm{sgn}(\eta^d)[0, |\eta^d|]$.

The following lemma generalises Lemma 25.17 (ii) and (iii) in Sato (1999) from Lévy processes to time-inhomogeneous Lévy processes. In particular, we show that an analytic extension of the symbol to the domain $U_{-\eta}$ exists, if a related exponential moment condition is satisfied. For z, $w \in \mathbb{C}^d$ we define

$$\langle z, w\rangle := \sum_{j=1}^{d} z_j w_j\,.$$

Note that this is *not* the scalar product in $\mathbb{C}^d$.

Lemma 2.1. *Let $\eta \in \mathbb{R}^d$.*

(a) *$E\,\mathrm{e}^{\langle \eta, L_t\rangle} < \infty$ for every $0 \le t \le T$ if and only if*

$$\int_0^T \int_{|x|>1} \mathrm{e}^{\langle \eta, x\rangle}\, F_s(\mathrm{d}x)\,\mathrm{d}s < \infty\,.$$

(b) *If $E\,\mathrm{e}^{\langle \eta, L_t\rangle} < \infty$ for every $0 \le t \le T$, then*

$$E\,\mathrm{e}^{\langle i\xi+\eta, L_t\rangle} = \mathrm{e}^{\int_0^t \theta_s(i\xi+\eta)\,\mathrm{d}s} = \mathrm{e}^{-\int_0^t A_s(-\xi+i\eta)\,\mathrm{d}s}$$

for every $t \in [0,T]$ and $\xi \in \mathbb{R}^d$.

(c) *If $E\,\mathrm{e}^{\langle \eta', L_t\rangle} < \infty$ for every $0 \le t \le T$ and every $\eta' \in R_\eta$, then the maps $z \mapsto A_s(-z)$ for every $s \ge 0$ as well as*

$$z \mapsto E\,\mathrm{e}^{i\langle z, L_t\rangle} = \mathrm{e}^{\int_0^t \theta_s(iz)\,\mathrm{d}s} = \mathrm{e}^{-\int_0^t A_s(-z)\,\mathrm{d}s}$$

have a continuous extension to the domain $\overline{U_{-\eta}}$ which is analytic in $\overset{\circ}{U}_{-\eta}$.

Proof. Parts (a) and (b) are straightforward extensions of Theorem 25.17 in Sato (1999) as shown more explicitly in Lemma 6 and formula (14) in Eberlein and Kluge (2006).

In order to prove (c), let γ_j be an arbitrary (compact) triangle, that lies inside the strip $U_{\eta^j} = \mathbb{R} + i\,\mathrm{sgn}(\eta^j)[0, |\eta^j|)$, and let $w = (w_1, \dots, w_d) \in \mathbb{C}^d$ for fixed $w_k \in U_{\eta^k}$ for every $k \in \{1, \dots, d\} \setminus \{j\}$.

We shall derive

$$\int\limits_{\partial\gamma_j} A_t(w_1,\ldots,w_{j-1},w_j,w_{j+1},\ldots,w_d)\,\mathrm{d}w_j = 0\,.$$

Then, the analyticity of the mapping $w_j \mapsto A_t(w_1,\ldots,w_{j-1},w_j,w_{j+1},\ldots,w_d)$ in the interior of U_{η^j} follows from the theorem of Morera. Since this is true for every coordinate $j\in\{1,\ldots,d\}$, the analyticity of the map $w\mapsto A_t(w)$ in $\overset{\circ}{U}_\eta$ follows.

Consider the symbol as given in (5). The mapping $w_j \mapsto \frac{1}{2}\langle w,\Sigma_t w\rangle + i\langle w,b_t\rangle$ is analytic in $\mathbb{C}$. The same is true for the mapping $w_j \mapsto \big(\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\big)$. An application of the theorem of Fubini and the lemma of Goursat yields

$$\begin{aligned}
&\int\limits_{\partial\gamma_j} A_t(w_1,\ldots,w_{j-1},w_j,w_{j+1},\ldots,w_d)\,\mathrm{d}w_j\\
&\quad= \int\limits_{\partial\gamma_j}\Big(\frac{1}{2}\langle w,\Sigma_t w\rangle + i\langle w,b_t\rangle\Big)\,\mathrm{d}w_j - \int\limits_{\partial\gamma_j}\int\limits_{\mathbb{R}^d}\Big(\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big)\,F_t(\mathrm{d}y)\,\mathrm{d}w_j\\
&\quad= -\int\limits_{\mathbb{R}^d}\int\limits_{\partial\gamma_j}\Big(\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big)\,\mathrm{d}w_j\,F_t(\mathrm{d}y)\\
&\quad= 0\,.
\end{aligned}$$

To justify the use of Fubini's theorem, we derive an upper bound of $\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big|$ in $L^1(F_t(\mathrm{d}y),\mathbb{R}^d)$ which does not depend on w_j.

For $|y|\le 1$ we have

$$\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big| \le \frac{1}{2}|\langle w,y\rangle|^2 \le \frac{1}{2}|w|^2|y|^2 \le \frac{1}{2}\Big(\max_{w_j\in\partial\gamma_j}|w|^2\Big)|y|^2\,.$$

For $|y|>1$, we get when chosen $h(y) = y\mathbb{1}_{|y|\le 1}$

$$\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big| \;=\; \Big|\mathrm{e}^{-i\langle w,y\rangle} - 1\Big| \le \mathrm{e}^{\langle \Re(-iw),y\rangle} + 1 = \mathrm{e}^{\langle \Im(w),y\rangle} + 1\,,$$

where we denote by $\Re(w)$ resp. $\Im(w)$ the vector of the real parts (resp. the imaginary parts) of the components of the vector $w\in\mathbb{C}^d$. By assumption $\Im(w)$ belongs to $R_{-\eta}$. This is also the case for

$$v^1 := \big(\Im(w_1),\ldots,\Im(w_{j-1}),\max_{w_j\in\partial\gamma_j}\Im(w_j),\Im(w_{j+1}),\ldots,\Im(w_d)\big)$$

and for

$$v^2 := \big(\Im(w_1),\ldots,\Im(w_{j-1}),\min_{w_j\in\partial\gamma_j}\Im(w_j),\Im(w_{j+1}),\ldots,\Im(w_d)\big)$$

It follows that

$$\int\limits_{|y|>1}\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big|F_t(\mathrm{d}y) \le \int\limits_{|y|>1}\Big(\mathrm{e}^{\langle v^1,y\rangle} + \mathrm{e}^{\langle v^2,y\rangle} + 1\Big)\,F_t(\mathrm{d}y) < \infty\,.$$

For every fixed choice of complex numbers $w_k\in\overline{U_{\eta^k}}$ for $k\in\{1,\ldots,d\}\setminus\{j\}$ and every fixed $y\in\mathbb{R}^d$, the mapping $w_j\mapsto \mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle$ is continuous. Furthermore, the following estimate,

$$\left|\mathrm{e}^{-i\langle w,y\rangle}-1+i\langle w,h(y)\rangle\right| \leq \frac{1}{2}\Big(\max_{w_j\in\partial\gamma_j}|w|^2\Big)|y|^2\mathbb{1}_{\{|y|\leq 1\}}(y) + \left(\mathrm{e}^{\langle v^1,y\rangle}+\mathrm{e}^{\langle v^2,y\rangle}+1\right)\mathbb{1}_{\{|y|>1\}}(y) \tag{8}$$

which is an upper bound in $L^1(F_t(\mathrm{d}y),\mathbb{R}^d)$, from where the continuity of the mapping $w\mapsto A_t(w)$ on $\overline{U_\eta}$ follows. □

The next remark collects further elementary properties of $\big(A_t(\cdot-i\eta)\big)_{t\in[0,T]}$, where $\big(A_t\big)_{t\in[0,T]}$ is the symbol of a time-inhomogeneous Lévy process.

Lemma 2.2. *Let L be a PIIAC with characteristic triplet $(b_t,\sigma_t,F_t)_{t\in[0,T]}$, with symbol $(A_t)_{t\in[0,T]}$. If*

$$\int_0^T\int_{|x|>1}\mathrm{e}^{-\langle\eta',x\rangle}F_t(\mathrm{d}x)<\infty \qquad \textit{for all } \eta'\in R_\eta\,,$$

then the following holds.

(a) *For every $\eta'\in R_\eta$ we have*

$$A_t(\xi-i\eta')=A_t(-i\eta')+A_t^{L^{-\eta'}}(\xi)\,,$$

where $A^{L^{-\eta'}}$ is the symbol of a time-inhomogeneous Lévy process $L^{-\eta'}$ which is given by the characteristic triplet $(b_t^{-\eta'},\sigma_t,F_t^{-\eta'})_{t\geq 0}$ with

$$b_t^{-\eta'} = b_t-\sigma_t\eta'+\int\left(\mathrm{e}^{-\langle\eta',x\rangle}-1\right)h(x)F_t(\mathrm{d}x) \quad \textit{and}$$
$$F_t^{-\eta'}(\mathrm{d}x) = \mathrm{e}^{-\langle\eta',x\rangle}F_t(\mathrm{d}x)\,.$$

(b) *For every $\eta'\in R_\eta$ there is equivalence between*

$$A_t(-i\eta')=0 \qquad \textit{for all } t\in[0,T]\big)$$

and the martingale property of the process $\left(\mathrm{e}^{-\langle\eta',L_t\rangle}\right)_{t\geq 0}$.

(c) *For every $\xi\in\mathbb{R}^d$ we have*

$$\Re\big(A_t(\xi-i\eta')\big)=A_t(-i\eta')+\frac{1}{2}\langle\xi,\sigma_t\xi\rangle-\int\big(\cos(\langle\xi,x\rangle)-1\big)F_t^{-\eta'}(\mathrm{d}x)$$
$$\geq A_t(-i\eta')\,.$$

Proof. The derivation of the decomposition (a) follows in a straightforward way, compare Glau (2010, Satz II.15), as does (c).
To show (b), we notice that

$$E\,\mathrm{e}^{-\langle\eta',L_t\rangle}=\mathrm{e}^{\int_0^t\theta_s(-\eta')\,\mathrm{d}s}=\mathrm{e}^{-\int_0^t A_s(-i\eta')\,\mathrm{d}s}$$

and for $s\leq t$ the equality

$$E\left(\mathrm{e}^{-\langle\eta',L_t\rangle}\,\middle|\,\mathcal{F}_s\right)=\mathrm{e}^{-\langle\eta',L_s\rangle}\,E\,\mathrm{e}^{-\langle\eta',L_t-L_s\rangle}=\mathrm{e}^{-\langle\eta',L_s\rangle}\,\mathrm{e}^{-\int_s^t A_u(-i\eta')\,\mathrm{d}u}$$

follows. Hence $\mathrm{e}^{-\langle\eta',L\rangle}$ is a martingale, if and only if $A_t(-i\eta')=0$ holds for every $t\in[0,T]$. □

3. Exponentially weighted Sobolev–Slobodeckii spaces

We consider so-called weighted Sobolev–Slobodeckii spaces with weight functions of the form $x \mapsto \mathrm{e}^{\langle \eta, x\rangle}$ with a vector $\eta \in \mathbb{R}^d$. We study only Sobolev–Slobodeckii spaces with exponential weight functions and define these spaces analogously to the definition of Sobolev–Slobodeckii spaces based on Fourier transformed functions in Wloka (1987). The main reason besides the benefits of an appropriate access via the symbol is the result which will be given in theorem 3.4, that the dual space $\big(H^s_\eta(\mathbb{R}^d)\big)'$ of $H^s_\eta(\mathbb{R}^d)$ is isomorphic isometric to the space $H^{-s}_\eta(\mathbb{R}^d)$. This property of the Sobolev space is necessary for the interpretation in section 5 of the PDOs $\mathcal{A}_t$ associated with the symbols A_t as linear operators from the Hilbert space $H^s_\eta(\mathbb{R}^d)$ to its dual space $\big(H^s_\eta(\mathbb{R}^d)\big)'$.

We denote by $L^2_\eta(\mathbb{R}^d)$ the Hilbert space of complex-valued square integrable functions

$$L^2_\eta(\mathbb{R}^d) := \big\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \,\big|\, x \mapsto u(x)\,\mathrm{e}^{\langle \eta, x\rangle} \in L^2(\mathbb{R}^d) \big\} \tag{9}$$

with scalar product

$$\langle u, v\rangle_{L^2_\eta} := \int_{\mathbb{R}^d} u(x)\overline{v(x)}\,\mathrm{e}^{2\langle \eta, x\rangle}\,\mathrm{d}x \quad \text{for all } u,\, v \in L^2_\eta(\mathbb{R}^d)\,. \tag{10}$$

The crucial step for a definition of the Sobolev spaces via Fourier transforms is based on Parseval's identity,

$$\langle u, v\rangle_{L^2} = \int_{\mathbb{R}^d} u(x)\overline{v(x)}\,\mathrm{d}x = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{u}(\xi)\overline{\hat{v}(\xi)}\,\mathrm{d}\xi\,. \tag{11}$$

In order to derive the analogous identity for functions in the space $L^2_\eta(\mathbb{R}^d)$, we denote

$$\begin{aligned} u_\eta(x) \quad &:= \quad u(x)\,\mathrm{e}^{\langle \eta, x\rangle} \\ \hat{u}(\xi - i\eta) \quad &:= \quad \int \mathrm{e}^{i\langle \xi, x\rangle}\,u(x)\,\mathrm{e}^{\langle \eta, x\rangle}\,\mathrm{d}x = \mathcal{F}(u_\eta)(\xi) \end{aligned} \tag{12}$$

for functions $u : \mathbb{R}^d \to \mathbb{C}$ with $\int |u(x)|\,\mathrm{e}^{\langle \eta, x\rangle}\,\mathrm{d}x < \infty$. Let us further notice the equality

$$\langle u, v\rangle_{L^2_\eta} = \frac{1}{(2\pi)^d}\int \hat{u}(\xi - i\eta)\overline{\hat{v}(\xi - i\eta)}\,\mathrm{d}\xi\,, \tag{13}$$

for functions $u\,,\, v \in L^2_\eta(\mathbb{R}^d)$. This leads to the following equivalent definition of the space $L^2_\eta(\mathbb{R}^d)$.

Remark 3.1. The space $L^2_\eta(\mathbb{R}^d)$ is isomorphic isometric to the space

$$\big\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \,\big|\, \mathcal{F}(u_\eta) \in L^2(\mathbb{R}^d) \big\}\,.$$

Furthermore the space $\big(L^2_\eta(\mathbb{R}^d), \langle \cdot, \cdot\rangle_{L^2_\eta}\big)$ is a separable Hilbert space. The space $C^\infty_0(\mathbb{R}^d, \mathbb{C})$ of infinitely differentiable complex functions with compact support is a dense subspace.

For a consistent definition of the Sobolev–Slobodeckii spaces with exponential weights, we first define the analogue of the Schwartz space $S(\mathbb{R}^d)$ and of the generalized functions.

Definition 3.2 (Exponentially weighted Schwartz space)**.** *For $\eta \in \mathbb{R}^d$ let*

$$S_\eta(\mathbb{R}^d) := \big\{ u \in C^\infty(\mathbb{R}^d, \mathbb{C}) \big|\, \|u\|_{m,\eta} < \infty \big\}$$

with

$$\|\varphi\|_{m,\eta} := \|\varphi\, \mathrm{e}^{\langle \eta, \cdot \rangle}\|_m$$

and we denote by $S'_\eta(\mathbb{R}^d)$ the dual space of $S_\eta(\mathbb{R}^d)$.

For every $m \in \mathbb{N}_0$ the norms $\|\cdot\|_m$ are defined as usual by

$$\|\varphi\|_m := \sup_{|p| \le m} \sup_{x \in \mathbb{R}^d} \big(1 + |x|^2\big)^m \, |D^p \varphi(x)| \,,$$

compare e.g. (Rudin, 1973, section 7.3). In the following remark, we define a Fourier transform $\mathcal{F}_\eta$ for functions in the weighted Schwartz space which is the analogue of the Fourier transform $\mathcal{F}$ on the Schwartz space.

Remark 3.3. [Fourier transform with weights] The mapping

$$\mathcal{F}_\eta(\varphi) := \mathrm{e}^{-\langle \eta, \cdot \rangle} \, \mathcal{F}\big(\varphi\, \mathrm{e}^{\langle \eta, \cdot \rangle}\big) \qquad (\varphi \in S_\eta(\mathbb{R}^d) \quad \text{resp.} \quad \varphi \in L^2_\eta(\mathbb{R}^d))$$

and

$$\mathcal{F}_\eta^{-1}(\varphi) := \mathrm{e}^{-\langle \eta, \cdot \rangle} \, \mathcal{F}^{-1}\big(\varphi\, \mathrm{e}^{\langle \eta, \cdot \rangle}\big) \qquad (\varphi \in S_\eta(\mathbb{R}^d) \quad \text{resp.} \quad \varphi \in L^2_\eta(\mathbb{R}^d))$$

is a continuous bijection, $\mathcal{F}_\eta : S_\eta(\mathbb{R}^d) \to S_\eta(\mathbb{R}^d)$ resp. $\mathcal{F}_\eta : L^2_\eta(\mathbb{R}^d) \to L^2_\eta(\mathbb{R}^d)$.

It follows similarly that the transformation $\mathcal{F}_\eta : S'_\eta(\mathbb{R}^d) \to S'_\eta(\mathbb{R}^d)$, defined by the Parseval identity

$$\mathcal{F}_\eta(u)(\varphi) := (2\pi)^d u\big(\mathcal{F}_\eta^{-1}(\varphi)\big) \quad \big(u \in S'_\eta(\mathbb{R}^d),\, \varphi \in S_\eta(\mathbb{R}^d)\big)$$

resp.

$$u(\varphi) = \frac{1}{(2\pi)^d} \mathcal{F}_\eta(u)\big(\mathcal{F}_\eta(\varphi)\big) \quad \big(u \in S'_\eta(\mathbb{R}^d),\, \varphi \in S_\eta(\mathbb{R}^d)\big) \tag{14}$$

is continuous and bijective. The weighted Sobolev–Slobodeckii spaces for $s \in \mathbb{R}$ and $\eta \in \mathbb{R}^d$ are defined via

$$H^s_\eta(\mathbb{R}^d) := \big\{ u \in S'_\eta(\mathbb{R}^d) \,\big|\, \|\, \mathrm{e}^{\langle \eta, \cdot \rangle} \, \mathcal{F}_\eta(u)\|_{\widehat{H}^s} < \infty \big\}$$

with the scalar product

$$\langle u, v \rangle_{H^s_\eta} := \langle \mathcal{F}_\eta(u), \mathcal{F}_\eta(v) \rangle_{\widehat{H}^s_\eta} := \langle \mathrm{e}^{\langle \eta, \cdot \rangle} \, \mathcal{F}_\eta(u), \mathrm{e}^{\langle \eta, \cdot \rangle} \, \mathcal{F}_\eta(v) \rangle_{\widehat{H}^s} \tag{15}$$

where

$$\langle \varphi, \psi \rangle_{\widehat{H}^s} := \int \varphi(\xi) \overline{\psi(\xi)} \, \big(1 + |\xi|\big)^{2s} \, \mathrm{d}\xi \,. \tag{16}$$

For the scalar product of the weighted space, this entails

$$\langle u, v \rangle_{H^s_\eta} = \int \mathcal{F}_\eta(u)(\xi) \overline{\mathcal{F}_\eta(v)(\xi)} \, \big(1 + |\xi|\big)^{2s} \, \mathrm{e}^{2\langle \eta, \xi \rangle} \, \mathrm{d}\xi \,. \tag{17}$$

The space of Fourier transforms of functions in $H^s_\eta(\mathbb{R}^d)$ is given by

$$\widehat{H}^s_\eta(\mathbb{R}^d) := \big\{ \mathcal{F}_\eta(u) \,\big|\, u \in H^s_\eta(\mathbb{R}^d) \big\}$$

with scalar product $\langle \cdot, \cdot \rangle_{\widehat{H}^s_\eta}$. Inserting the notation $u_\eta = u\, \mathrm{e}^{\langle \eta, \cdot \rangle}$ and $v_\eta = v\, \mathrm{e}^{\langle \eta, \cdot \rangle}$ yields

$$\langle u, v \rangle_{H^s_\eta} = \big\langle \, \mathrm{e}^{\langle \eta, \cdot \rangle} \, \mathcal{F}_\eta(u), \mathrm{e}^{\langle \eta, \cdot \rangle} \, \mathcal{F}_\eta(v) \big\rangle_{\widehat{H}^s} = \big\langle \mathcal{F}(u_\eta), \mathcal{F}(v_\eta) \big\rangle_{\widehat{H}^s} = \langle u_\eta, v_\eta \rangle_{H^s} \,. \tag{18}$$

In particular, we have $\|u\|_{H^s_\eta} = \|u_\eta\|_{H^s}$ and $H^s_0(\mathbb{R}^d) = H^s(\mathbb{R}^d)$.

Theorem 3.4. *The dual space $\big(H^s_\eta(\mathbb{R}^d)\big)'$ of $H^s_\eta(\mathbb{R}^d)$ is isomorphic isometric to $H^{-s}_\eta(\mathbb{R}^d)$.*

Proof. We argue similar to Eskin (1981, S. 62–63). Let $l \in \big(H^s_\eta(\mathbb{R}^d)\big)'$. From the representation theorem of Riesz, we conclude the unique existence of a function $v \in H^s_\eta(\mathbb{R}^d)$ with $\|v\|_{H^s_\eta} = \|l\|_{(H^s_\eta(\mathbb{R}^d))'}$, such that by equation (18) and (15)

$$l(\varphi) \quad = \quad \langle v, \varphi\rangle_{H^s_\eta} = \int (1+|\xi|)^{2s}\,\hat{v}(\xi - i\eta)\overline{\hat{\varphi}(\xi - i\eta)}\,\mathrm{d}\xi$$

for every $\varphi \in H^s_\eta(\mathbb{R}^d)$. If we then define $w := \mathrm{e}^{-\langle \eta,\cdot\rangle}\,\mathcal{F}^{-1}\big(\,(1+|\cdot|)^{2s}\,\hat{v}(\cdot - i\eta)\big)$, we obtain

$$\begin{aligned}
\|w\|^2_{H^{-s}_\eta} &= \int\limits_{\mathbb{R}^d} \big(1+|\xi|\big)^{-2s}\Big|\mathcal{F}\big(\,\mathrm{e}^{\langle\eta,\cdot\rangle}\,w\big)(\xi)\Big|^2\,\mathrm{d}\xi \\
&= \int\limits_{\mathbb{R}^d} \big(1+|\xi|\big)^{2s}\big|\hat{v}(\xi - i\eta)\big|^2\,\mathrm{d}\xi \\
&= \|v\|^2_{H^s_\eta}
\end{aligned}$$

and hence $w \in H^{-s}_\eta(\mathbb{R}^d)$ with $\|w\|_{H^{-s}_\eta} = \|v\|_{H^s_\eta} = \|l\|_{(H^s_\eta(\mathbb{R}^d))'}$. Hence $l \mapsto w$ defines an isometry from $\big(H^s_\eta(\mathbb{R}^d)\big)'$ to the space $H^{-s}_\eta(\mathbb{R}^d)$. Since the Riesz mapping $l \mapsto v$ and the mapping defined by $v \mapsto w := \mathrm{e}^{-\langle\eta,\cdot\rangle}\,\mathcal{F}^{-1}\Big(\big(1+|\cdot - i\eta|^2\big)^s\,\hat{v}(\cdot - i\eta)\Big)$ are both bijective maps, their composition defines the desired isomorphism. □

4. Symbol and Pseudo-Differential operator

Let $\mathcal{A}$ be a PDO with symbol A as in (5), i.e.

$$\mathcal{A}u = \mathcal{F}^{-1}\big(A\mathcal{F}(u)\big) \qquad \text{for all } u \in S(\mathbb{R}^d)$$

with $A : \mathbb{R}^d \to \mathbb{C}$ measurable satisfying

$$\big|A(\xi)\big| \le c\big(1+|\xi|\big)^\alpha \qquad \text{for all } \xi \in \mathbb{R}^d$$

for an $\alpha \in \mathbb{R}$ and a constant $c \ge 0$. The latter estimate guarantees that the Fourier inversion operator $\mathcal{F}^{-1}$ is well defined. In (Eskin, 1981, Lemma 4.4) it is shown that

$$\|\mathcal{A}u\|_{H^{s-\alpha}} \le c\|u\|_{H^s} \qquad \text{for all } u \in S(\mathbb{R}^d)\,.$$

As a consequence, $\mathcal{A}$ has a unique extension to a continuous linear operator $\mathcal{A} : H^s(\mathbb{R}^d) \to H^{s-\alpha}(\mathbb{R}^d)$.

In this section we derive conditions on the symbol, that allow the interpretation of $\mathcal{A}$ as a continuous linear operator

$$\mathcal{A} : H^s_\eta(\mathbb{R}^d) \to H^{s-\alpha}_\eta(\mathbb{R}^d)\,.$$

Let $U_{-\eta}$ be given as in (7). We denote by $S_\alpha(-\eta)$ the set of symbols A that have a continuous extension

$$A : \overline{U_{-\eta}} \to \mathbb{C}$$

that is analytic in the interior of $U_{-\eta}$, and further satisfies the *continuity condition*

$$\big|A(z)\big| \le C_\eta\big(1+|z|\big)^\alpha \quad \big(\text{for all } z \in U_{-\eta}\big)\,. \tag{19}$$

Note that as a consequence of the identity theorem for holomorphic functions, the extension is unique on $U_{-\eta}$. By continuity, the extension is unique on the closure $\overline{U_{-\eta}}$.

Let us observe that by definition of the Fourier transforms $\mathcal{F}_\eta$, $\mathcal{F}_\eta^{-1}$ and the estimate (19), it is obvious that

$$u \mapsto \mathcal{F}_\eta^{-1}\big(A(\cdot - i\eta)\mathcal{F}_\eta(u)\big)$$

is a linear continuous mapping from $H^s_\eta(\mathbb{R}^d)$ to $H^{s-\alpha}_\eta(\mathbb{R}^d)$. We prove the following consistency result.

Theorem 4.1. *Let $\mathcal{A}$ be a PDO with symbol $A \in S_\alpha(-\eta)$ for an index $\alpha \in \mathbb{R}$ and a weight index $\eta \in \mathbb{R}^d$. Then*

$$\mathcal{A}\,u = \mathcal{F}^{-1}\big(A\mathcal{F}(u)\big) = \mathcal{F}_\eta^{-1}\Big(A(\cdot - i\eta)\mathcal{F}_\eta(u)\Big) \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$$

and there exists a constant $c(\eta) > 0$ with

$$\big\|\mathcal{A}\,u\big\|_{H^{s-\alpha}_\eta(\mathbb{R}^d)} \le c(\eta)\|u\|_{H^s_\eta(\mathbb{R}^d)} \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^d, \mathbb{C})\,.$$

Moreover, the operator $\mathcal{A}$ can be extended to a linear continuous operator $\mathcal{A} : H^s_\eta(\mathbb{R}^d) \to H^{s-\alpha}_\eta(\mathbb{R}^d)$ in a unique way.

Proof. For $u \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ we have

$$\mathcal{A}\,u(x) \;=\; \mathcal{F}^{-1}\big(A\mathcal{F}(u)\big)(x) = \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \mathrm{e}^{-i\langle \xi, x\rangle}\, A(\xi)\mathcal{F}(u)(\xi)\,\mathrm{d}\xi \qquad \text{for all } x \in \mathbb{R}^d\,.$$

The map $\xi \mapsto \mathrm{e}^{-i\langle \xi, x\rangle}\, A(\xi)\mathcal{F}(u)(\xi)$ is continuous on $\overline{U_{-\eta}}$ and holomorphic in the interior $\mathring{U}_{-\eta}$. The continuity of A on $\overline{U_{-\eta}}$ entails

$$|A(z)| \le C_\eta\big(1 + |z|\big)^\alpha \qquad \text{for all } z \in \overline{U_{-\eta}}\,. \tag{20}$$

Furthermore, for $\eta' := (\eta'_1, \ldots, \eta'_d)$ with $\eta'_j \in \operatorname{sgn}(\eta^j)[0, |\eta^j|]$, we obtain

$$\begin{aligned}\big|\,\mathrm{e}^{-i\langle \xi - i\eta', x\rangle}\, A(\xi - i\eta')\hat{u}(\xi - i\eta')\big| \;&=\; \mathrm{e}^{-\langle \eta', x\rangle}\,\big|A(\xi - i\eta')\hat{u}(\xi - i\eta)\big| \\ &\le\; \mathrm{e}^{-\langle \eta', x\rangle} C_\eta\big(1 + |\xi - i\eta'|\big)^\alpha C_N \frac{\mathrm{e}^{R|\eta'|}}{\big(1 + |\xi - i\eta'|\big)^N}\end{aligned}$$

with a constant C_N for arbitrary $N \in \mathbb{N}_0$, if the support of the function u is inside of the open ball with radius $R \in \mathbb{R}_+$, $tr(u) \subset B_R(0)$. This is a direct consequence of $A \in S_\alpha(-\eta)$ and the Paley-Wiener-Schwartz theorem, compare Jacob (2001, Theorem 3.4.6).

Now define $f(\xi - i\eta') := \mathrm{e}^{-i\langle \xi - i\eta', x\rangle}\, A(\xi - i\eta')\hat{u}(\xi - i\eta')$ for $x \in \mathbb{R}^d$ fixed, then for any $N > \alpha + d + 1$ this yields in particular

$$\big|f(\xi - i\eta')\big| \le C_\eta C_N\,\mathrm{e}^{-\langle \eta', x\rangle}\,\mathrm{e}^{R|\eta'|}\,\frac{1}{(1 + |\xi - i\eta'|)^{N-\alpha}} \le C(x, \eta, N, R)\big(1 + |\xi|\big)^{-(d+1)} \tag{21}$$

with a constant $C(x, \eta, N, R)$ independent of ξ and η' for all $(\xi - i\eta') \in U_{-\eta}$ and

$$f(\xi - i\eta') \to 0 \quad \text{for } |\xi| \to \infty\,,$$

which shows assumption (i) of lemma A.2. The integrability assumption (ii) in the same lemma is obviously satisfied, hence we can apply the version of Cauchy's theorem, that we provide in lemma A.2, from where we obtain

$$\begin{aligned}\mathcal{A}\,u(x) &= \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi,x\rangle}\,A(\xi)\hat{u}(\xi)\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi-i\eta,x\rangle}\,A(\xi-i\eta)\hat{u}(\xi-i\eta)\,\mathrm{d}\xi\,.\end{aligned}$$

We then insert the definition of $\mathcal{F}_\eta$, compare equation (12) and remark 3.3 to obtain

$$\begin{aligned}\mathcal{A}\,u(x) &= \mathrm{e}^{-\langle\eta,x\rangle}\,\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi,x\rangle}\,A(\xi-i\eta)\,\mathrm{e}^{\langle\eta,\xi\rangle}\,\mathrm{e}^{-\langle\eta,\xi\rangle}\,\hat{u}(\xi-i\eta)\,\mathrm{d}\xi\\ &= \mathrm{e}^{-\langle\eta,x\rangle}\,\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi,x+i\eta\rangle}\,A(\xi-i\eta)\mathcal{F}_\eta(u)(\xi)\,\mathrm{d}\xi\,. \qquad (22)\end{aligned}$$

Furthermore we show that the mapping $\xi\mapsto A(\xi-i\eta)\mathcal{F}_\eta(u)(\xi)$ belongs to $L^1_\eta(\mathbb{R}^d)$ where the latter space is defined in an analogous way to (9). Using the definition of $\mathcal{F}_\eta$ and (21) we get

$$\begin{aligned}\big\|A(\cdot-i\eta)\mathcal{F}_\eta(u)(\cdot)\big\|_{L^1_\eta(\mathbb{R}^d)} &= \int\limits_{\mathbb{R}^d} |A(\xi-i\eta)\mathcal{F}_\eta(u)(\xi)|\,\mathrm{e}^{\langle\eta,\xi\rangle}\,\mathrm{d}\xi\\ &= \int\limits_{\mathbb{R}^d} \big|A(\xi-i\eta)\big|\big|\hat{u}(\xi-i\eta)\big|\,\mathrm{d}\xi\\ &\le C(x,\eta,N,R)\int\limits_{\mathbb{R}^d}\big(1+|\xi|\big)^{-(d+1)}\,\mathrm{d}\xi\\ &< \infty\,.\end{aligned}$$

It follows from the last line in (22) that

$$\mathcal{A}\,u(x) = \mathcal{F}_\eta^{-1}\Big(A(\cdot-i\eta)\mathcal{F}_\eta(u)\Big)(x)\,.$$

In order to prove the continuity property, we choose $u\in C_0^\infty(\mathbb{R}^d,\mathbb{C})$, and inserting (17) we estimate

$$\begin{aligned}\big\|\mathcal{A}\,u\big\|^2_{H^{s-\alpha}_\eta(\mathbb{R}^d)} &= \int\limits_{\mathbb{R}^d}\big|\mathcal{F}_\eta\big(\mathcal{A}\,u\big)\big|^2\big(1+|\xi|\big)^{2(s-\alpha)}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi\\ &= \int\limits_{\mathbb{R}^d}\big|A(\xi-i\eta)\big|^2\big|\mathcal{F}_\eta(u)(\xi)\big|^2\big(1+|\xi|\big)^{2(s-\alpha)}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi\\ &\le C_\eta^2(1+|\eta|)^{2\alpha}\int\limits_{\mathbb{R}^d}\big|\mathcal{F}_\eta(u)(\xi)\big|^2\big(1+|\xi|\big)^{2s}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi\\ &= c(\eta)\|u\|^2_{H^s_\eta(\mathbb{R}^d)}\,.\end{aligned}$$

By density, the operator has a unique continuous extension $\mathcal{A}: H^s_\eta(\mathbb{R}^d)\to H^{s-\alpha}_\eta(\mathbb{R}^d)$. □

5. Parabolic Equation

In Glau (2011) a Sobolev index is introduced, and it is shown that the evolution problem associated with a Lévy process with Sobolev index α has a unique weak solution in the Sobolev–Slobodeckii space $H^{\alpha/2}$. In this section, we generalize the results obtained in Glau (2011) to the case of weighted Sobolev–Slobodeckii spaces, and we examine $\mathbb{R}^d$-valued time-inhomogeneous Lévy processes instead of genuine Lévy processes.

Let L be an $\mathbb{R}^d$-valued PIIAC with local characteristics $\big(b_s, \sigma_s, F_s\big)$ for $s \geq 0$. Let us consider the following assumptions on the symbol A of the process as defined in (5).

(A1) *Assume*

$$\int\limits_0^T \int\limits_{|x|>1} \mathrm{e}^{-\langle \eta', x\rangle}\, F_s(\mathrm{d}x)\,\mathrm{d}s < \infty \qquad \forall \eta' \in R_\eta\,.$$

(A2) *There exists a constant $C_1 > 0$ with*

$$\big|A_t(z)\big| \leq C_1\big(1+|z|\big)^{\alpha}$$

for all $z \in U_{-\eta}$ and for all $t \in [0,T]$. *(Continuity condition)*

(A3) *There exist constants $C_2 > 0$ and $C_3 \geq 0$, such that for a certain $0 \leq \beta < \alpha$*

$$\Re\big(A_t(z)\big) \geq C_2\big(1+|z|\big)^{\alpha} - C_3\big(1+|z|\big)^{\beta}$$

for all $z \in U_{-\eta}$ and for all $t \in [0,T]$. *(Gårding condition)*

Let us make the following remarks.

Remark 5.1. (i) For Lévy processes with Brownian part the conditions (A2) and (A3) are valid for $\alpha = 2$ and those $\eta \in \mathbb{R}$ that satisfy assumption (A1). In particular the Brownian motion (with drift) satisfies the assumptions for every $\eta \in \mathbb{R}$, compare section 7, where also further examples are studied.

(ii) Conditions (A1)–(A3) are for example satisfied for CGMY-processes with parameters C, G, $M > 0$ and $Y \in [1, 2)$ with $\alpha = Y$ and $\eta \in (-M, G)$.

Remark 5.2. Conditions (A2) and (A3) are not necessary assumptions of theorem 5.3 about the existence and uniqueness of the weak solution of the corresponding PIDE in a weighted Sobolev-Slobodeckii space. We choose this set of assumptions, since usually the symbol is well known for real arguments and it is hence convenient to extend the polynomial growth conditions to the complex domain $U_{-\eta}$. Moreover by theorem 4.1, this approach allows us to work in a unique framework for the PDO $\mathcal{A}$ associated with exponentially weighted Sobolev-Slobodeckii spaces $H^s_\eta(\mathbb{R}^d)$ with different weights η.

However, it is instead also possible to assume the growth conditions (A2) and (A3) only for a fixed imaginary part of the arguments, i.e. for the function $x \mapsto A_t(x - i\eta)$. Instead of (A1) one would then assume $\int_0^T \int_{|x|>1} \mathrm{e}^{-\langle \eta, x\rangle}\, F_s(\mathrm{d}x)\,\mathrm{d}s < \infty$.

Under assumptions (A1) and (A2), we conclude from theorem 4.1, that for every fixed $t \in [0,T]$ the operator $\mathcal{A}_t|_{C_0^\infty(\mathbb{R}^d,\mathbb{C})} : C_0^\infty(\mathbb{R}^d,\mathbb{C}) \to C^\infty(\mathbb{R}^d,\mathbb{C})$ associated with the symbol A_t has a unique linear and continuous extension

$$\mathcal{A}_t : H^{\alpha/2}_\eta(\mathbb{R}^d) \to H^{-\alpha/2}_\eta(\mathbb{R}^d)$$

with $\mathcal{A}_t u = \mathcal{F}_\eta^{-1}\big(A_t(\cdot - i\eta)\mathcal{F}_\eta(u)\big)$ for all $u \in H_\eta^{\alpha/2}(\mathbb{R}^d)$. Since the Hilbert spaces $H_\eta^{-\alpha/2}(\mathbb{R}^d)$ and $\big(H_\eta^{\alpha/2}(\mathbb{R}^d)\big)'$ are isomorphic, the operators $\mathcal{A}_t$ can be identified with continuous linear operators

$$\mathcal{A}_t : H_\eta^{\alpha/2}(\mathbb{R}^d) \to \Big(H_\eta^{\alpha/2}(\mathbb{R}^d)\Big)' .$$

Let us further define the family of bilinear forms a_t by

$$a_t(\varphi, \psi) := \big(\mathcal{A}_t \varphi\big)(\psi) \quad \text{for } \varphi,\, \psi \in H_\eta^{\alpha/2}(\mathbb{R}^d) \tag{23}$$

for every $t \in [0,T]$. Inserting Parseval's equality (11), we obtain for every φ, $\psi \in H_\eta^{\alpha/2}(\mathbb{R}^d)$ the equality

$$\begin{aligned} a_t(\varphi,\psi) &= \frac{1}{(2\pi)^d}\langle \mathcal{F}_\eta(\mathcal{A}_t\varphi), \mathcal{F}_\eta(\psi)\rangle_{L^2_\eta(\mathbb{R}^d)} \\ &= \frac{1}{(2\pi)^d}\int A_t(\xi - i\eta)\mathcal{F}_\eta(\varphi)(\xi)\,\overline{\mathcal{F}_\eta(\psi)(\xi)}\,\mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int A_t(\xi - i\eta)\hat{\varphi}(\xi - i\eta)\overline{\hat{\psi}(\xi - i\eta)}\,\mathrm{d}\xi \end{aligned}$$

for each $t \in [0,T]$.

Theorem 5.3. *Let L be an $\mathbb{R}^d$-valued PIIAC with local characteristics $\big(b_s, \sigma_s, F_s\big)$ for $s \geq 0$ and symbol $A = (A_t)_{t\in[0,T]}$ and associated pseudo differential operators $(\mathcal{A}_t)_{t\in[0,T]}$. If the assumptions (A1)–(A3) are satisfied, the parabolic equation*

$$\begin{aligned} \partial_t u + \mathcal{A}_t u &= \quad f \\ u(0) &= \quad g\,, \end{aligned} \tag{24}$$

with real-valued $f \in L^2\big(0,T; H_\eta^{-\alpha/2}(\mathbb{R}^d)\big)$ and real-valued initial condition $g \in L^2_\eta(\mathbb{R}^d)$ has a unique weak solution $u \in W^1\big(0,T; H_\eta^{\alpha/2}(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$, and the estimate

$$\|u\|_{W^1(0,T;H_\eta^{\alpha/2}(\mathbb{R}^d),L^2_\eta(\mathbb{R}^d))} \leq C(T)\big(\|f\|_{L^2(0,T;H_\eta^{-\alpha/2}(\mathbb{R}^d))} + \|g\|_{L^2_\eta(\mathbb{R}^d)}\big)$$

with a constant $C(T) > 0$, only depending on T, is satisfied.

The space $W^1\big(0,T; H_\eta^{\alpha/2}(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ consists of those functions $u \in L^2\big(0,T; H_\eta^{\alpha/2}(\mathbb{R}^d)\big)$ that have a derivative with respect to time $\partial_t u$ in a distributional sense that belongs to the space $L^2\big(0,T; \big(H_\eta^{\alpha/2}(\mathbb{R}^d)\big)'\big)$. For a Hilbert space H, the space $L^2\big(0,T;H\big)$ denotes the space of functions $u : [0,T] \to H$, that are weakly measurable and that satisfy $\int_0^T \|u(t)\|_H^2\,\mathrm{d}t < \infty$. For the definition of weak measurability and for a detailed introduction of the space $W^1\big(0,T; H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d)\big)$ that relies on the Bochner integral, we refer to the book of Wloka (1987).

Proof. To apply the classical result on existence and uniqueness of solutions of linear parabolic equations in Hilbert spaces, see e.g. Wloka (1982, Satz 25.5, S. 381), it is at this point sufficient to verify the Gårding inequality of the bilinear form $a : [0,T] \times H_\eta^{\alpha/2}(\mathbb{R}^d) \times H_\eta^{\alpha/2}(\mathbb{R}^d) \to \mathbb{R}$ uniformly in $t \in [0,T]$. For $\varphi \in H_\eta^{\alpha/2}(\mathbb{R}^d)$, we conclude

$$\Re\big(a_t(\varphi,\varphi)\big) = \frac{1}{(2\pi)^d}\int \Re\big(A_t(\xi - i\eta)\big)\big|\mathcal{F}_\eta(\varphi)(\xi)\big|^2\,\mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi\,.$$

From the Gårding condition (A3) and an elementary calculation the Gårding inequality

$$\Re\big(a_t(\varphi,\varphi)\big) \geq C_2 \|\varphi\|^2_{H^{\alpha/2}_\eta(\mathbb{R}^d)} - C_3' \|\varphi\|^2_{L^2_\eta(\mathbb{R}^d)} .$$

with constants $C_2 > 0$ and $C_3' \geq 0$ follows uniformly in $t \in [0,T]$. □

6. Explicit solution of the Fourier transformed equation and a Feynman–Kac formula

In Theorem 5.3, we showed the existence of a unique solution of the parabolic equation (24) under (A1)–(A3). We will now look for a more explicit form of this solution in the homogeneous case $f \equiv 0$. Since we are moreover interested in a stochastic representation of the solution that usually corresponds to an evolution problem with given terminal condition, we replace the operator $\mathcal{A}_t$ in equation (24) with $\mathcal{A}_{T-t}$. Thus we consider

$$\begin{aligned} \partial_t u + \mathcal{A}_{T-t} u = &\quad 0 \\ u(0) = &\quad g \,, \end{aligned} \tag{25}$$

with real-valued initial condition $g \in L^2_\eta(\mathbb{R}^d)$. It will turn out that the weak solution has an explicit Fourier transform. Furthermore it is smooth.

In order to derive the Fourier representation, let us notice that a function u that belongs to the space $W^1\big(0,T; H^\alpha_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ is a solution of the linear parabolic equation (25), if and only if

$$\mathcal{F}_\eta(\partial_t u) + \mathcal{F}_\eta(\mathcal{A}_{T-t} u) = 0 \quad \text{in } L^2\big(0,T; \widehat{H}^{-\alpha}(\mathbb{R}^d)\big) \tag{26}$$

and

$$\mathcal{F}_\eta\big(L^2_\eta\text{-}\lim_{t\downarrow 0} u(t)\big) = \mathcal{F}_\eta(g) \,. \tag{27}$$

It is a consequence of the continuity of the Fourier transform $\mathcal{F}_\eta$ with respect to the L^2_η-norm, that equation (27) is equivalent to $L^2_\eta - \lim_{t\downarrow 0} \mathcal{F}_\eta(u(t)) = \mathcal{F}_\eta(g)$. Furthermore the equality $\mathcal{F}_\eta\big(\partial_t u\big) = \partial_t \mathcal{F}_\eta(u)$ can be derived inserting the definition of the Bochner integral: For every $\psi \in C_0^\infty((0,T))$ the following chain of equalities for elements in the Hilbert space $\big(H^\alpha_\eta(\mathbb{R}^d)\big)'$ holds,

$$\begin{aligned} \int_0^T \mathcal{F}_\eta\big(\partial_s u(s)\big)\big(\psi(s)\big)\,\mathrm{d}s &= \mathcal{F}_\eta\left(\int_0^T \big(\partial_s u(s)\big)\big(\psi(s)\big)\,\mathrm{d}s\right) = -\mathcal{F}_\eta\left(\int_0^T \big(u(s)\big)\big(\partial_s \psi(s)\big)\,\mathrm{d}s\right) \\ &= -\int_0^T \mathcal{F}_\eta\big(u(s)\big)\big(\partial_s\psi(s)\big)\,\mathrm{d}s = \int_0^T \big(\partial_s \mathcal{F}_\eta\big(u(s)\big)\big)\big(\psi(s)\big)\,\mathrm{d}s \,. \end{aligned}$$

From Theorem 4.1 we conclude

$$\mathcal{F}_\eta(\mathcal{A}_t v) = A_t(\cdot - i\eta)\mathcal{F}_\eta(v) \quad \text{for all } v \in H^s_\eta(\mathbb{R}^d) \,.$$

Altogether, we have $u \in W^1\big(0,T; H^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ is a solution of equation (25), iff $\mathcal{F}_\eta(u)$ belongs to the space $W^1\big(0,T; \widehat{H}^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$, and if $\mathcal{F}_\eta(u)$ solves the ordinary differential equation (ODE)

$$\begin{aligned} \partial_t \mathcal{F}_\eta(u) + A_{T-t}(\cdot - i\eta)\mathcal{F}_\eta(u) = 0 \\ \mathcal{F}_\eta(u)(t=0) = \mathcal{F}_\eta(g) \,. \end{aligned}$$

Theorem 6.1. *Assume (A1)–(A3). The function* $u \in W^1\big(0,T; H_\eta^{\alpha/2}(\mathbb{R}^d), L_\eta^2(\mathbb{R}^d)\big)$ *is a weak solution of equation* (25)*, iff* $\mathcal{F}_\eta(u) \in W^1\big(0,T; \widehat{H}_\eta^{\alpha/2}(\mathbb{R}^d), L_\eta^2(\mathbb{R}^d)\big)$ *and the Fourier transform* $\mathcal{F}_\eta(u)$ *solves the ODE*

$$\begin{aligned} \partial_t \mathcal{F}_\eta\big(u(t)\big)(\xi) + A_{T-t}(\xi - i\eta)\mathcal{F}_\eta(u(t))(\xi) &= 0 \qquad \text{in } (0,T) \text{ for a.e. } \xi \in \mathbb{R}^d \\ \mathcal{F}_\eta\big(u(t=0)\big) &= \mathcal{F}_\eta(g)\,. \end{aligned} \tag{28}$$

The solution of (28) *is given by*

$$\mathcal{F}_\eta\big(u(t)\big)(\xi)) = \mathcal{F}_\eta(g)(\xi)\, \mathrm{e}^{-\int_{T-t}^T A_s(\xi - i\eta)\,\mathrm{d}s} \tag{29}$$

and hence

$$u(t,x) = \frac{\mathrm{e}^{-\langle \eta, x\rangle}}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-i\langle \xi, x + i\eta\rangle}\, \mathcal{F}_\eta(g)(\xi)\, \mathrm{e}^{-\int_{T-t}^T A_s(\xi - i\eta)\,\mathrm{d}s}\, \mathrm{d}\xi \tag{30}$$

is the weak solution of equation (25)*. If furthermore the mapping* $t \mapsto A_t(\xi - i\eta)$ *is continuous for every fixed* $\xi \in \mathbb{R}^d$*, then we have* $u \in C^1\big((0,T); H_\eta^m(\mathbb{R}^d)\big)$ *for every* $m \in \mathbb{N}$ *and hence* $u \in C^1\left((0,T), C^m(\mathbb{R}^d)\right)$ *for every* $m \geq 0$ *is the pointwise solution of the equation* (25)*.*

As a direct consequence, under the additional assumption that $g_\eta \in L^1$, we obtain the stochastic representation

$$u(T-t,x) = E\big(g(L_{T-t}^t + x)\big) \tag{31}$$

with $L_{T-t}^t := L_T - L_t$. We use this notation since the process $L^u := \big(L_{u+s} - L_u\big)_{s\geq 0}$ is a PIIAC as well, and the local characteristics of L^u, $(b_s^{L^u}, \sigma_s^{L^u}, F_s^{L^u})$, with respect to the truncation function h are given by $\big(b_{u+s}, \sigma_{u+s}, F_{u+s}\big)$.

To show equation (31), we fix t and T and we set

$$U(x) := E\big(g(L_{T-t}^t + x)\big) = \mathrm{e}^{-\langle \eta, x\rangle}\, E\Big(\mathrm{e}^{-\langle \eta, L_{T-t}^t\rangle}\, g_\eta(L_{T-t}^t + x)\Big)\,,$$

then a short calculation based on Fubini's theorem provides

$$\mathcal{F}_\eta(U)(\xi) = \mathrm{e}^{-\langle \eta, \xi\rangle}\, \mathcal{F}(U_\eta)(\xi) = \mathcal{F}_\eta(g) E\Big(\mathrm{e}^{i\langle L_T - L_t, i\eta - \xi\rangle}\Big) = \mathcal{F}_\eta(g)\, \mathrm{e}^{-\int_t^T A_s(\xi - i\eta)\,\mathrm{d}s}\,.$$

Let us notice that the real-valued initial function g results in a real-valued solution u of the parabolic equation. This stems from the fact that $\mathcal{A}_t\varphi$ is real-valued, if $\varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$. Let us mention that this property of a PDO $\mathcal{A}$ can be translated to symmetry properties of the corresponding symbol, compare e.g. p. 206 in Glau (2010).

Remark 6.2. Theorem 6.1 illuminates the parallelism between Fourier and PIDE methods for option pricing: The PIDEs for European options are interpreted as a pseudo differential equation, then the Fourier transform is applied which results in an ordinary differential equation that can be solved explicitly. The solution leads to equation (30) which coincides with the famous convolution formula for option prices, derived independently in Carr and Madan (1999) and Raible (2000). See Eberlein et al. (2010) for a derivation of the formula under conditions similar to (A1)–(A3).

Proof of theorem 6.1: Our previous arguments show the equivalence of equations (25) and (28). Equations (29) and (30) are immediate consequences. Hence we are left to show that the function u defined by equation (29) resp. (30) satisfies $u \in C^1\left((0,T), H_\eta^m(\mathbb{R}^d)\right)$ and $u \in C^1\left((0,T), C^m(\mathbb{R}^d)\right)$ for every $m \geq 0$.

An elementary calculation provides that the Gårding inequality yields

$$\Re\big(A_s(\xi - i\eta)\big) \geq C_1|\xi|^\alpha - C_2 \qquad \big(s \in (0,T),\, \xi \in \mathbb{R}^d\big)\,.$$

with a strictly positive constant C_1 and $C_2 \geq 0$. Whence the inequality

$$\mathrm{e}^{-\int_s^t \Re(A_u(\xi - i\eta))\,\mathrm{d}u} \leq c_2\, \mathrm{e}^{-(t-s)C_1|\xi|^\alpha} \tag{32}$$

with a positive constant c_2 independent of $s \in [0,T]$.

We derive successively for every $t \in (0,T)$ and for every $m \geq 0$

(i) $u(t) \in H^m_\eta(\mathbb{R}^d)$,
(ii) $\lim_{s\to t} \|u(t) - u(s)\|_{H^m_\eta(\mathbb{R}^d)} = 0$,
(iii) $\partial_t u(t) = \mathcal{F}^{-1}_\eta\Big(A_{T-t}(\cdot - i\eta)\mathcal{F}_\eta\big(u(t)\big)\Big) \in H^m_\eta(\mathbb{R}^d)$ and
(iv) $\lim_{s\to t} \|\partial_t u(t) - \partial_s u(s)\|_{H^m_\eta(\mathbb{R}^d)} = 0$

hence $u \in C^1\left((0,T), H^m_\eta(\mathbb{R}^d)\right)$ for every $m \geq 0$. In view of the smoothness of the weight function, we conclude from the Sobolev embedding theorem, compare e.g. Wloka (1982), that the function u also belongs to the space $C^1\big((0,T), C^m(\mathbb{R}^d)\big)$ for every $m \geq 0$.

Let us first estimate the norm of u,

$$\begin{aligned}
\|u(t)\|^2_{H^m_\eta(\mathbb{R}^d)} &= \int |\mathcal{F}_\eta(u(t))(\xi)|^2 (1+|\xi|)^{2m}\, \mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi \\
&= \int |\mathcal{F}_\eta(g)(\xi)|^2 \big|\,\mathrm{e}^{-2\int_{T-t}^T A_s(\xi - i\eta)\,\mathrm{d}s}\,\big| (1+|\xi|)^{2m}\, \mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi \\
&= \int |\mathcal{F}_\eta(g)(\xi)|^2\, \mathrm{e}^{-2\int_{T-t}^T \Re(A_s(\xi - i\eta)\,\mathrm{d}s} (1+|\xi|)^{2m}\, \mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi \\
&\leq c_2 \int |\mathcal{F}_\eta(g)(\xi)|^2\, \mathrm{e}^{2\langle \eta,\xi\rangle}\, \mathrm{e}^{-2tC_1|\xi|^\alpha} (1+|\xi|)^{2m}\,\mathrm{d}\xi \\
&< \infty
\end{aligned}$$

for every $t > 0$ and every $m \geq 0$.

In order to derive (ii) we conclude

$$\begin{aligned}
&\|u(t) - u(s)\|^2_{H^m_\eta(\mathbb{R}^d)} \\
&\quad = \Big\| \mathcal{F}_\eta(g)\, \mathrm{e}^{-\int_{T-s}^T A_u(\cdot - i\eta)\,\mathrm{d}u}\, \big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\cdot - i\eta)\,\mathrm{d}u} - 1\big| \Big\|^2_{\widehat{H}^m_\eta(\mathbb{R}^d)} \\
&\quad = \int_{\mathbb{R}^d} \big|\mathcal{F}(g_\eta)(\xi)\big|^2\, \mathrm{e}^{-2\int_{T-s}^T \Re(A_u(\xi - i\eta))\,\mathrm{d}u} \Big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\xi - i\eta)\,\mathrm{d}u} - 1\Big|^2 \big(1+|\xi|\big)^{2m}\,\mathrm{d}\xi \\
&\quad \to 0 \quad (s \to t)\,,
\end{aligned}$$

which follows by dominated convergence if $t > \epsilon > 0$ or $m = 0$, since $\big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\xi - i\eta)\,\mathrm{d}u} - 1\big| \to 0$ for $s \to t$ and

$$\sup_{\xi \in \mathbb{R}^d} \big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\xi - i\eta)\,\mathrm{d}u} - 1\big| \leq const\,.$$

In order to derive the explicit expression for the Fourier transform of the time derivative of u given in (iii) we consider

$$\begin{aligned}&\left\|\frac{u(t)-u(s)}{t-s}-\mathcal{F}_\eta^{-1}\big(A_{T-t}(\cdot-i\eta)\mathcal{F}_\eta(u(t))\big)\right\|_{H_\eta^m(\mathbb{R}^d)} \qquad (33)\\ &=\left\|\mathcal{F}_\eta(g)\left(\frac{\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}-\mathrm{e}^{-\int_{T-s}^T A_u(\cdot-i\eta)\,\mathrm{d}u}}{t-s}-A_{T-t}(\cdot-i\eta)\,\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}\right)\right\|_{\widehat{H}_\eta^m(\mathbb{R}^d)}.\end{aligned}$$

From the continuity of $t\mapsto A_t(\xi-i\eta)$ for every fixed $\xi\in\mathbb{R}^d$ we get

$$\frac{\mathrm{e}^{-\int_{T-t}^T A_u(\xi-i\eta)\,\mathrm{d}u}-\mathrm{e}^{-\int_{T-s}^T A_u(\xi-i\eta)\,\mathrm{d}u}}{t-s}\to A_{T-t}(\xi-i\eta)\,\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}\qquad\text{for } s\to t$$

for every fixed $\xi\in\mathbb{R}^d$. From inequality (32) and assumption (A2) it follows

$$\left|A_{T-t}(\cdot-i\eta)\,\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}\right|\le C_2\big(1+|\xi|\big)^\alpha\,\mathrm{e}^{-tC_1|\xi|^\alpha} \qquad (34)$$

with a positive constant C_2. Because of the continuity of $t\mapsto A_t(\xi-i\eta)$ for every fixed $\xi\in\mathbb{R}^d$ the mean-value theorem moreover yields together with inequality (32) and assumption (A2)

$$\left|\frac{\mathrm{e}^{-\int_{T-t}^T A_u(\xi-i\eta)\,\mathrm{d}u}-\mathrm{e}^{-\int_{T-s}^T A_u(\xi-i\eta)\,\mathrm{d}u}}{t-s}\right|\le C_3\big(1+|\xi|\big)^\alpha\,\mathrm{e}^{-(t\wedge s)C_1|\xi|^\alpha} \qquad (35)$$

with a constant $C_3>0$. Hence by dominated convergence we get that the term (33) vanishes for $s\to t$ for any $t\in(0,T)$, which shows $\partial_t u(t)=\mathcal{F}_\eta^{-1}\big(A_{T-t}(\cdot-i\eta)\mathcal{F}_\eta(u(t))\big)\in H_\eta^m(\mathbb{R}^d)$ for every $m\ge 0$.

The continuity of the time derivative as a function $[\epsilon,T]\to\widehat{H}_\eta^m(\mathbb{R}^d)$ i.e. assertion (iv) follows in a similar way. □

7. Examples

We provide examples of classes of time-inhomogeneous Lévy processes that satisfy the assumptions (A1)–(A3). Let us first consider Lévy processes.

Example 7.1 (σ positive definite)**.** For Lévy processes with a Brownian part and a positive definite covariance matrix σ, the assumptions (A2) and (A3) with $\alpha=2$ are satisfied for every choice of $\eta\in\mathbb{R}$ such that assumption (A1) is satisfied. In particular, for $\alpha=2$ the assumptions are satisfied for the Brownian motion with or without drift for every $\eta\in\mathbb{R}$.

In order to derive the Gårding condition, we conclude by lemma 2.2 (c)

$$\Re\big(A(\xi-i\eta')\big)=A(-i\eta')+\frac{1}{2}\langle\xi,\sigma\xi\rangle-\int\big(\cos(\langle\xi,x\rangle)-1\big)F^{-\eta'}(\mathrm{d}x)\,.$$

Since the integrand is nonpositive and σ is positive definite we get

$$\Re\big(A(\xi-i\eta')\big)\ge A(-i\eta')+\frac{1}{2}\underline{\sigma}|\xi|^2\,,$$

where $\underline{\sigma}$ denotes the smallest eigenvalue of the matrix σ. The Gårding condition follows, since $|A(-i\eta')|$ is bounded for all $\eta'\in R_\eta$ by some constant only depending on η which can be shown similar to inequality (8) by summing up over all possible combinations of signs. In a similar way, the continuity condition can be derived.

Example 7.2 (GH-processes)**.** GH-processes with parameters λ, α', β and δ satisfy the assumptions (A1)–(A3) with index $\alpha = 1$, if the GH-parameters α' and β satisfy

$$\beta - \alpha' < \eta < \beta + \alpha' .$$

Let us briefly derive that statement. It is shown in Raible (2000, Appendix A.1) that the characteristic function of the GH-distribution has an analytic extension for $z \in \mathbb{C}$ to the domain $-\alpha' < \beta - \Im(z) < \alpha'$. In particular this entails $E\,\mathrm{e}^{-\eta L_t} < \infty$ and hence $\int_{|x|>1} \mathrm{e}^{-\eta x} F(\mathrm{d}x) < \infty$ for $-\alpha' < \beta - \eta < \alpha'$. From lemma 2.2 we obtain the following representation of the symbol A of the GH-process

$$A(\xi - i\eta) = A(-i\eta) + ib^{-\eta}\xi + \int \left(\mathrm{e}^{-i\xi x} - 1 - i\xi x\right) \mathrm{e}^{-\eta x} F^{GH}(\mathrm{d}x) ,$$

where $b^{-\eta} = \mu + \int (\mathrm{e}^{-\eta x} - 1)\, x F_t^{GH}(\mathrm{d}x)$ and $(\mu, 0, F^{GH})$ are the local characteristics of the GH-process with respect to the truncation function $h(x) = x$.

Moreover, the Lévy measure F^{GH} has a Lebesgue density f^{GH}, whose behaviour around the origin is explored in Raible (2000). The asymptotic behaviour around the origin remains unaffected when multiplying with the term $\mathrm{e}^{-\eta \cdot}$. Therefore the statement can be proven as in the case $\eta = 0$ which is treated in Glau (2011).

CGMY processes can be discussed along the same lines.

7.1. **Examples of time-inhomogeneous Lévy processes.** For time-inhomogeneous Lévy processes we make the following assumptions on the local characteristics $(b_t, \sigma_t, F_t)_{t \geq 0}$.

Assumption 7.3.

$$\sup_{s \in [0,T]} \left\{ |b_s| + \|\sigma_s\|_{\mathcal{M}(d \times d)} + \int \left(|x|^2 \wedge 1\right) F_s(\mathrm{d}x) \right\} < \infty .$$

Time-inhomogeneous Lévy processes with local characteristics $(b_t, \sigma_t, F_t)_{t \in [0,T]}$, that have a Brownian part with $(\sigma_t)_{t \in [0,T]}$ being uniformly positive definite and that satisfy an appropriate exponential moment condition, satisfy assumptions (A1)–(A3):

Example 7.4. Let L be a PIIAC with symbol $(A_t)_{t \in [0,T]}$, PDO $(\mathcal{A}_t)_{t \in [0,T]}$ and characteristic triplet $(b_t, \sigma_t, F_t)_{t \in [0,T]}$. If Assumption 7.3 is satisfied and

$$\sup_{t \in [0,T]} \int_{|x|>1} \mathrm{e}^{\langle \eta', x \rangle} F_t(\mathrm{d}x) < \infty \qquad \left(\eta' \in U_{-\eta}\right) ,$$

which is a stronger condition than assumption (A1), and if furthermore the family of matrices $(\sigma_t)_{t \geq 0}$ is uniformly positive definite in the following sense,

$$\inf_{t \in [0,T]} \|\sigma_t\|_{\mathcal{M}(d \times d)} \geq \underline{\sigma} > 0 ,$$

then A satisfies the continuity and Gårding condition (A2) and (A3) with index $\alpha = 2$.

In order to show this, we conclude from lemma 2.2

$$\left|A_t(\xi - i\eta)\right| \leq |A_t(-i\eta)| + \left|\langle b_t^{-\eta}, \xi \rangle\right| + \frac{1}{2}\left|\langle \xi, \sigma_t \xi \rangle\right| + \left| \int \left(\mathrm{e}^{-i\langle \xi, x \rangle} - 1 + i\langle \xi, h(x) \rangle\right) F_t^{-\eta}(\mathrm{d}x) \right|$$

with $b^{-\eta}$ and $F^{-\eta}$ as in the lemma. Inserting assumption (7.3) yields

$$\begin{aligned}|A_t(-i\eta)| &\leq \sup_{t\in[0,T]}\left\{|b_t||\eta| + \frac{1}{2}\|\sigma_t\|^2_{\mathcal{M}(d\times d)}|\eta|^2 + \left|\int\left(\mathrm{e}^{-\langle\eta,x\rangle} - 1 + \langle\eta, h(x)\rangle\right)F_t(\mathrm{d}x)\right|\right\}\\ &\leq c_1(\eta)\,,\end{aligned}$$

and together with

$$|b_t^{-\eta}| \leq |b_t| + \left|\sigma_t\eta'\right| + \int\left|\mathrm{e}^{-\langle\eta,x\rangle} - 1\right|h(x)F_t(\mathrm{d}x) \leq c_2(\eta)$$

and

$$\left|\langle b_t^{-\eta},\xi\rangle\right| + \frac{1}{2}\left|\langle\xi,\sigma_t\xi\rangle\right| \leq \sup_{t\in[0,T]}|b_t^{-\eta}||\xi| + \sup_{t\in[0,T]}\|\sigma_t\|^2_{\mathcal{M}(d\times d)}|\xi|^2$$

we get the continuity condition

$$\begin{aligned}\left|A_t(\xi - i\eta)\right| &\leq |A_t(-i\eta)| + \left|\langle b_t^{-\eta},\xi\rangle\right| + \frac{1}{2}\left|\langle\xi,\sigma_t\xi\rangle\right| + \left|\int\left(\mathrm{e}^{-i\langle\xi,x\rangle} - 1 + i\langle\xi, h(x)\rangle\right)F_t^{-\eta}(\mathrm{d}x)\right|\\ &\leq c(\eta)\big(1 + |\xi| + |\xi|^2\big)\,.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\Re\big(A_t(\xi - i\eta)\big) &= \Re\big(A_t(-i\eta)\big) + \frac{1}{2}\langle\xi,\sigma_t\xi\rangle - \int\left(\cos\left(\langle\xi,x\rangle\right) - 1\rangle\right)F_t^{-\eta}(\mathrm{d}x)\\ &\geq \min_{t\in[0,T]}\|\sigma_t\|^2_{\mathcal{M}(d\times d)}|\xi|^2 - \sup_{t\in[0,T]}\left|A_t(-i\eta)\right|\,,\end{aligned}$$

whence the Gårding condition.

Appendix A. A multivariate version of Cauchy's theorem

We start with a formulation of Cauchy's theorem for rectangles. This formulation is based on the usual version of Cauchy's theorem for rectangles, that is e.g. provided in Jänich (1996). From standard arguments we obtain the following lemma.

Lemma A.1. (A) *Let* $R_1 := [a_1, a_2] \times i[b_1, b_2]$ *with* $-\infty < a_1 < a_2 < \infty$ *and* $-\infty < b_1 < b_2 < \infty$ *and let* $\gamma := \partial R_1$.

If f *is holomorphic in the interior* $\overset{\circ}{R}_1$ *of the rectangle* R_1 *and continuous on* $\overline{R_1} = R_1$, *then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0\,.$$

(B) *Let* $R := (-\infty,\infty) \times i[b_1, b_2]$ *with* $-\infty < b_1 < b_2 < \infty$ *and let* f *be holomorphic in the interior* $\overset{\circ}{R}$ *of* R *and continuous on* $\overline{R} = R$. *Furthermore assume* $f(\cdot + ib_1)$, $f(\cdot + ib_2) \in L^1(\mathbb{R})$. *For every* $y \in [b_1, b_2]$ *we assume*

$$\left|f(K + iy)\right| \to 0 \quad \text{for } K \in \mathbb{R},\ |K| \to \infty$$

and that there exists an upper bound $h \in L^1\big((b_1, b_2)\big)$ *such that*

$$\left|f(K + iy)\right| + \left|f(-K + iy)\right| \leq h(y) \quad \text{for all } y \in [b_1, b_2] \text{ uniformly in } K.$$

Then

$$\int_{-\infty}^{\infty} f(x+ib_1)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x+ib_2)\,\mathrm{d}x\,.$$

We prove a generalization of the version of Cauchy's theorem provided in lemma A.1 for the multivariate case.

Lemma A.2. *Let $R_j := (-\infty,\infty)\times i[b_j,\beta_j]$ with $-\infty < b_j < \beta_j < \infty$ for $j=1,\dots,d$ and $Q_d = R_1\times\dots R_d$. Let $f: Q_d\to\mathbb{C}$ be holomorphic in the interior $\mathring{Q}_d$ of Q_d, and continuous on $\overline{Q_d} = Q_d$. Further, we assume the following integrability and convergence properties.*

(i) *Assume*

$$f(z)\to 0 \quad \textit{for } |\Re(z)|\to\infty \textit{ and } \Im(z)\in[b_1,\beta_1]\times\dots\times[b_d,\beta_d]$$

with $z=(z_1,\dots,z_d)$.

(ii) *For $z_j = x_j+iy_j$ with $x_j\in\mathbb{R}$ and $y_j\in[b_j,\beta_j]$ for $j=1,\dots,d$ we assume*

$$\big|f(z_1,\dots,z_d)\big| \le h(x_1,\dots,x_d) \quad \textit{uniformly for } y\in[b_1,\beta_1]\times\dots\times[b_d,\beta_d]$$

with a function $h\in L^1(\mathbb{R}^d)$. For $d=1$ let $c>0$ be a constant such that

$$\big|h(x)\big|\le c \quad \textit{for all } x\in\mathbb{R}\,.$$

In the case $d>1$ we additionally assume for every $j\in\{1,\dots,d\}$ the existence of a function $h_j\in L^1(\mathbb{R}^{d-1})$ such that

$$\big|h(x_1,\dots,x_d)\big| \le h_j(x_1,\dots x_{j-1},x_{j+1},\dots,x_d)$$

for all $(x_1,\dots,x_d)\in\mathbb{R}^d$.

Then the following is true

$$\begin{aligned}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} & f(x_1+ib_1,\dots,x_d+ib_d)\,\mathrm{d}x_1\dots\mathrm{d}x_d \\ &= \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} f(x_1+iy_1,\dots,x_d+iy_d)\,\mathrm{d}x_1\dots\mathrm{d}x_d\,.\end{aligned}$$

for every $y\in[b_1,\beta_1]\times\dots\times[b_d,\beta_d]$.

Proof. We verify the assertion by induction over the dimension d. For the base case $d=1$ the assertion follows from part (B) of lemma A.1.

Assume the assertion is true for the dimensions $1,\dots,d$. We observe that the integrability condition (ii) assures that the integral

$$I := \int_{-\infty}^{\infty}\mathrm{d}x_{d+1}\int_{-\infty}^{\infty}\mathrm{d}x_d\dots\int_{-\infty}^{\infty}\mathrm{d}x_1\, f(x_1+ib_1,\dots,x_{d+1}+ib_{d+1}) \tag{36}$$

is well defined and

$$I = \lim_{K\to\infty}\int_{-K}^{K}\mathrm{d}x_{d+1}\int_{-\infty}^{\infty}\mathrm{d}x_d\dots\int_{-\infty}^{\infty}\mathrm{d}x_1\, f(x_1+ib_1,\dots,x_{d+1}+ib_{d+1})\,.$$

For $y \in [b_1, \beta_1] \times \ldots \times [b_d, \beta_d]$ and $z \in R_{d+1}$ we let

$$g^y(z) := \int_{-\infty}^{\infty} \mathrm{d}x_d \ldots \int_{-\infty}^{\infty} \mathrm{d}x_1\, f(x_1 + iy_1, \ldots, x_d + iy_d, z)\,.$$

In this notation the integral in (36) reads

$$I = \int_{-\infty}^{\infty} g^{(b_1,\ldots,b_d)}(x + ib_{d+1})\,\mathrm{d}x_{d+1}\,.$$

We prove successively the following assertions.

(a) For every $y \in \overline{Q_d} = Q_d$ the mapping $z \mapsto g^y(z)$ is continuous in R_{d+1}.

(b) Let $b = (b_1, \ldots, b_d)$. For every $y_{d+1} \in [b_{d+1}, \beta_{d+1}]$ and every $x \in \mathbb{R}$ we have

$$g^b(x + iy_{d+1}) = g^y(x + iy_{d+1}) \qquad \text{for all } y \in \overline{Q_d} = Q_d\,.$$

(c) For every fixed $y \in \mathring{Q}_d$ the mapping $z \mapsto g^y(z)$ is holomorphic in the interior $\mathring{R}_{d+1} = (-\infty, \infty) \times (b_{d+1}, \beta_{d+1})$ of R_{d+1}.

(d) For every $y \in \mathring{Q}_d$

$$g^y(z) \to 0 \quad \text{for } \big|\Re(z)\big| \to 0 \text{ with } \Im(z) \in [b_{d+1}, \beta_{d+1}]$$

and there exists a constant $c > 0$ such that $\big|g^y(z)\big| \le c$ for all $y \in \mathring{Q}$.

(e) For every fixed $y \in \mathring{Q}_d$

$$\int_{-\infty}^{\infty} g^y(x + ib_{d+1})\,\mathrm{d}x = \int_{-\infty}^{\infty} g^y(x + iy')\,\mathrm{d}x \qquad \text{for all } y' \in [b_{d+1}, \beta_{d+1}]\,.$$

When assertions (a)–(e) are shown, they allow for the following chain of equalities

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} g^{(b_1,\ldots,b_d)}(x + ib_{d+1})\,\mathrm{d}x \\
&\overset{\text{(b)}}{=} \int_{-\infty}^{\infty} g^{(y'_1,\ldots,y'_d)}(x + ib_{d+1})\,\mathrm{d}x \qquad \big(y'_j \in (b_j, \beta_j)\,, j = 1, \ldots, d\big) \\
&\overset{\text{(e)}}{=} \int_{-\infty}^{\infty} g^{(y'_1,\ldots,y'_d)}(x + iy_{d+1})\,\mathrm{d}x \qquad \big(y_{d+1} \in [b_{d+1}, \beta_{d+1}]\big) \\
&\overset{\text{(b)}}{=} \int_{-\infty}^{\infty} g^{(y_1,\ldots,y_d)}(x + iy_{d+1})\,\mathrm{d}x \qquad \big(y_j \in [b_j, \beta_j]\,, j = 1, \ldots, d+1\big) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_{d+1} \ldots \int_{-\infty}^{\infty} \mathrm{d}x_1\, f(x_1 + iy_1, \ldots, x_{d+1} + iy_{d+1})
\end{aligned}$$

for every $(y_1, \ldots, y_d, y_{d+1}) \in [b_1, \beta_1] \times \ldots \times [b_d, \beta_d] \times [b_{d+1}, \beta_{d+1}]$, whence the lemma is proved.

Assertion (a) is a direct consequence of the continuity of the function f in $\overline{Q_{d+1}} = Q_{d+1}$ and the estimate

$$\big|f(z_1,\dots,z_d,z) - f(z_1,\dots,z_d,z')\big| \le 2h_{d+1}\big(\Re(z_1),\dots,\Re(z_d)\big)$$

for an integrable function $h_{d+1} \in L^1(\mathbb{R}^d)$, which exists by assumption (ii).

Assertion (b) does not follow directly from the induction hypothesis, since the mapping $y \mapsto g^y(x+ib_{d+1})$ is not holomorphic in general for $y \in \overset{\circ}{Q}_d$, as for $z \in \overset{\circ}{Q}_d$ and $x \in \mathbb{R}$ the points $(z_1,\dots,z_d,x+ib_{d+1})$ do not lie in the interior of Q_{d+1} where f is known to be holomorphic.

Note that for every $0 < \epsilon' < \beta_{d+1} - b_{d+1}$ the mapping

$$(z_1,\dots,z_d) \mapsto f(z_1,\dots,z_d,x+i(b_{d+1}+\epsilon'))$$

is holomorphic in the interior of Q_d and continuous on Q_d. The convergence and integrability assumptions (i) and (ii) are obviously satisfied for the function $(z_1,\dots,z_d) \mapsto f(z_1,\dots,z_d,x+i(b_{d+1}+\epsilon'))$ and the induction hypothesis yields

$$g^b(z) = g^y(z) \quad \text{for all } z \in \overset{\circ}{R}_{d+1}\,.$$

The continuity of the map $z \mapsto g^y(z)$ for $y \in Q^d$ and $z \in R_{d+1}$, provided in assertion (a) shows

$$g^b(z) = g^y(z) \quad \text{for all } z \in R_{d+1}\,.$$

Proof of assertion (c): To verify the analyticity of the map $z \mapsto g^y(z)$ in the interior of R_{d+1}, let $\triangle$ be an arbitrary triangle (non-degenerate), that lies completely in R_{d+1}. Let us further denote by γ a curve surrounding the triangle $\triangle$. For $y \in \overset{\circ}{R}_d$ we conclude with the help of the theorem of Fubini,

$$\begin{aligned}\int_\gamma g^y(z)\,\mathrm{d}z &= \int_\gamma \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x_d \dots \int_{-\infty}^{\infty} \mathrm{d}x_1\, f(x_1+iy_1,\dots,x_d+iy_d,z) \\ &= \int_{-\infty}^{\infty} \mathrm{d}x_d \dots \int_{-\infty}^{\infty} \mathrm{d}x_1 \left(\int_\gamma f(x_1+iy_1,\dots,x_d+iy_d,z)\,\mathrm{d}z \right).\end{aligned}$$

Fubini's theorem is justified here, since we have $\big|f(x_1+iy_1,\dots,x_d+iy_d,z)\big| \le h_{d+1}(x_1,\dots,x_d)$ with $h_{d+1} \in L^1(\mathbb{R}^d)$ by assumption and the curve γ is of finite length. Further, as (y,z) lies inside of $\overset{\circ}{Q}_{d+1}$ for $z \in \gamma$, the mapping

$$z \mapsto f(x_1+iy_1,\dots,x_d+iy_d,z)$$

is holomorphic in $\overset{\circ}{R}_{d+1} \supset \triangle$. An application of the lemma of Goursat yields

$$\int_\gamma f(x_1+iy_1,\dots,x_d+iy_d,z)\,\mathrm{d}z = 0$$

for every $(x_1+iy_1,\dots,x_d+iy_d) \in \overset{\circ}{Q}_d$. Hence we get

$$\int_\gamma g^y(z)\,\mathrm{d}z = 0\,.$$

In view of the continuity assertion (a), the theorem of Morera yields that for $y \in \mathring{Q}_d$ the map $z \mapsto g^y(z)$ is holomorphic in $\mathring{R}_{d+1}$.

Proof of assertion (d): From assumption (i) and (ii) of the theorem, we know that

$$\big|f(x_1 + iy_1, \dots, x_d + iy_d, z)\big| \to 0$$

for $\big|\Re(z)\big| \to \infty$ and $\Im(z) \in [b_{d+1}, \beta_{d+1}]$ and

$$\big|f(x_1 + iy_1, \dots, x_d + iy_d, z)\big| \le h_{d+1}(x_1, \dots, x_d)$$

with $h_{d+1} \in L^1(\mathbb{R}^d)$. As a direct consequence, we obtain

$$\big|g^y(z)\big| \le \int_{-\infty}^{\infty} \mathrm{d}x_d \dots \int_{-\infty}^{\infty} \mathrm{d}x_1 \big|f(x_1 + iy_1, \dots, x_d + iy_d, z)\big| \le const. < \infty$$

uniformly for all $z \in \mathbb{C}$ with $\Im(z) \in [b_{d+1}, \beta_{d+1}]$, and via dominated convergence, we obtain furthermore

$$g^y(z) = \int_{-\infty}^{\infty} \mathrm{d}x_d \dots \int_{-\infty}^{\infty} \mathrm{d}x_1\, f(x_1 + iy_1, \dots, x_d + iy_d, z) \to 0$$

for $\big|\Re(z)\big| \to \infty$ and $\Im(z) \in [b_{d+1}, \beta_{d+1}]$, which proves assertion (d).

Assertion (e) is a direct consequence of the assertions (a), (c), (d) and lemma A.2. □

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