Technische Universität München
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Fractional Lévy Ornstein-Uhlenbeck Processes

Bachelor's Thesis
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Abstract

This thesis introduces an Ornstein-Uhlenbeck model by a stochastic integral representation where the driving stochastic process is a fractional Lévy process (FLP). Since FLPs are in general not semimartingales, pathwise Riemann-Stieltjes integration can be applied. This works quite well for differentiable integrands such as in the Ornstein-Uhlenbeck case. To achieve the convergence of improper integrals the long time behavior of FLPs is derived. This is sufficient to define the fractional Lévy Ornstein-Uhlenbeck process (FLOUP) pathwise as an improper Riemann-Stieltjes integral.
As one would expect there is also a close relation to stochastic differential equations: We show that the FLOUP is the unique stationary solution of the corresponding Langevin equation. Furthermore we calculate the autocovariance function of a FLOUP and prove that its increments exhibit long range dependence.
So far only integration of deterministic differentiable integrands has been considered. However when one wants to look at more general stochastic differential equations it is clear that pathwise Riemann-Stieltjes integration becomes more difficult. We therefore invoke a generalization of the concept of bounded variation and restrict ourselves to FLPs whose sample paths are in that class. It is then possible to prove a chain rule and density formula as known from classical Riemann-Stieltjes calculus. Finally we look at stochastic differential equations driven by FLPs and present conditions on the coefficient functions such that solutions can be constructed from the corresponding FLOUP. In fact stationary solutions are obtained by monotone transformation of the FLOUP.

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Contents

1 Notation

Symbols

$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^0$, $\overline{\mathbb{R}}$	$(-\infty,\infty)$, $[0,\infty)$, $\mathbb{R}\setminus\{0\}$, $\mathbb{R}\cup\{\infty\}$
$\mathbb{N}$	$\{1,2,\ldots\}$
$\mathcal{B}(A)$	Borel σ-algebra over $A\subset\mathbb{R}$
$a\wedge b$, $a\vee b$	minimum, maximum of $a,b\in\mathbb{R}$
a_+, a_-	$0\vee a$, $0\vee -a$
$\Delta f(t)$	$f(t)-f(t-)$
f'	first derivative of f
$f(t)\sim g(t)$	$f(t)/g(t)\to 1$ as $t\to\infty$
log, exp	natural logarithm, exponential function
ζ, Γ	zeta function, gamma function
P, E, Var, Cov	probability, expectation, variance, covariance
1_A	indicator function of set A
$\lvert x\rvert$	absolute value of $x\in\mathbb{R}$
$\mathcal{C}^0(M)$	continuous functions on $M\subset\mathbb{R}$
$\mathcal{C}^k(M)$	k-times continuously differentiable functions on $M\subset\mathbb{R}$
$\mathbf{Lip}(M)$	$\{f:\mathbb{R}\to\mathbb{R}: f\rvert K$ Lipschitz for all compact $K\subset M\}$, $M\subset\mathbb{R}$
L^0	measurable functions
L^p	space of all p-integrable functions
$\mathcal{L}_C(M)$	$\{f:M\to\overline{\mathbb{R}}:\int_K \lvert f(t)\rvert dt<\infty$ for all compact $K\subset M\}$
$\mathcal{AC}(M)$	$\{f:M\to\mathbb{R}:\exists g\in\mathcal{L}_C(M)$ with $f(y)=f(x)+\int_x^y g(z)dz$ for all $[x,y]\subset M\}$
$v_p(f,[a,b])$	p-variation of f
$\mathcal{W}_p([a,b])$	$\{f:[a,b]\to\mathbb{R}:v_p(f,[a,b])<\infty\}$
$\mathfrak{W}_p(\mathbb{R})$	$\{f:\mathbb{R}\to\mathbb{R}:\forall\,[s,t]\subseteq\mathbb{R} f\in\mathcal{W}_p([s,t])\}$
$\mathbf{im}(g)$	image of g
$Z(f)$	zero points of f
H	Hurst index
d	fractional integration parameter
$(B_t)_{t\in\mathbb{R}}$	two-sided Brownian motion
$(B_t^H)_{t\in\mathbb{R}}$	fractional Brownian motion
$(L_t)_{t\in\mathbb{R}}$	two-sided Lévy process
$(L_t^d)_{t\in\mathbb{R}}$	fractional Lévy process
$(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$	fractional Lévy Ornstein-Uhlenbeck process

Abbreviations

a.e.	almost everywhere
a.s.	almost sure
FBM	fractional Brownian motion
FLOUP	fractional Lévy Ornstein-Uhlenbeck process
FLP	fractional Lévy process
i.i.d.	independent identically distributed
SDE	stochastic differential equation

2 Introduction

Continuous time modeling has become a very important part of modern finance and various nobel prices have been awarded to financial mathematicians like Robert Merton and Myron Scholes (1997) or Robert Engle (2003). For example widely spread and used in praxis is the so called Black-Scholes model which has been presented in Black and Scholes (1973) and Merton (1979): One describes the price of an asset $(S_t)_{t\geq 0}$ by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad S_0 \in \mathbb{R}, \sigma > 0,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion. The unique continuous solution given by Itô integration is

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}, \quad t \in \mathbb{R}.$$

From this representation follows that in the Black-Scholes model the log-price is normally distributed. The parameter $\mu - \frac{\sigma^2}{2}$ can be seen as a deterministic time drift and σ as the volatility. It is widely known that many assumptions of this model are not matched in praxis. In particular the normal distribution and the continuity of the sample paths. These problems can be avoided by using a more general Lévy process instead of Brownian motion as the driving process of the SDE above.

However in the time of the subprime crisis and its consequences one can see that there may be other very important mismatches in the Black-Scholes model. The volatility σ for instance is taken to be constant - an assumption which does not hold in praxis: The volatilities of most assets have increased drastically during the financial crisis. Such features lead to so-called stochastic volatility models. For example Barndorff-Nielsen and Shephard (2001b) proposed to model the volatility as an Ornstein-Uhlenbeck process and suggested

$$\begin{aligned} dY_t &= \mu + \beta\sigma_t^2 dt + \sigma_t dB_t + \rho d(L_{\lambda t} - E[L_{\lambda t}]), \quad Y_0 \in \mathbb{R}, \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dL_{\lambda t}, \quad \sigma_0^2, \lambda > 0 \end{aligned}$$

where $(L_t)_{t\geq 0}$ is a Lévy process. Note that the stochastic process $(Y_t)_{t\geq 0}$ models here directly the log-price of the asset. It can be shown that under this assumptions we get

$$\sigma_t^2 = e^{-\lambda t}\left\{\sigma_0^2 + \int_0^t e^{\lambda s} dL_{\lambda s})\right\}, \quad t \geq 0,$$

and the autocovariance function of $(\sigma_t^2)_{t\geq 0}$ decreases exponentially. However statistical analysis suggests that often there is in fact a strong dependence structure in the sense that the autocovariance function decreases much slower. To overcome this one can built models driven by processes whose increments exhibit that kind of long memory and this leads to fractional Brownian motion (FBM) or fractional Lévy processes (FLP). FBM driven

Ornstein-Uhlenbeck models have been considered for example in Mikosch and Norvaiša [8] or Buchmann and Klüppelberg [2].
This thesis will take a closer look at so-called fractional Lévy Ornstein-Uhlenbeck processes (FLOUPs) where the driving processes will be FLPs and it is organized as follows.
In Chapter 3 we will recall useful facts about Lévy processes, classical Riemann-Stieltjes integration and fractional Brownian motion. Chapter 4 introduces the p-variation of a real-valued function and states a Theorem from Young [13] regarding the existence of the Riemann-Stieltjes integral when integrand and integrator may be of unbounded variation, but of bounded p-variation and q-variation respectively, for certain $p, q > 0$. Furthermore we will state and prove a chain rule and density formula for those integrals. Chapter 5 will introduce FLPs $(L_t^d)_{t\in\mathbb{R}}$ and state important properties which we will need later. Here the structure will mainly follow Marquardt [5]. Also we will state and prove a Theorem about the long time behavior of the sample paths of a FLP. In Chapter 6 a FLOUP $(\mathcal{L}_t^d)_{t\in\mathbb{R}}$ will be introduced as the pathwise improper Riemann-Stieltjes integral

$$\mathcal{L}_t^{d,\lambda} = \int_{-\infty}^{t} e^{-\lambda(t-u)} dL_u^d, \quad t \in \mathbb{R},$$

and we will show that it is the unique stationary pathwise solution of the SDE

$$d\mathcal{L}_t^{d,\lambda} = -\lambda \mathcal{L}_t^{d,\lambda} dt + dL_t^d, \quad t \in \mathbb{R}.$$

Moreover we will calculate its autocovariance function and show that the increments of a FLOUP exhibit long range dependence. In Chapter 7 we will combine our results about Riemann-Stieltjes integrals and FLOUPs to consider more general fractional integral equations

$$dX_t = \mu(X_t)dt + \sigma(X_t)dL_t^d, \quad t \in \mathbb{R}$$

and impose assumptions on the coefficient functions μ and σ under which solutions can be constructed from the corresponding FLOUP. Chapter 8 mainly extends Buchmann and Klüppelberg [2] and will state structural properties of the coefficient functions we used before. Chapter 9 considers concrete examples of such fractional integral equations. In Chapter 10 we will present some simulations using a (a, b)-gamma process as an underlying Lévy process.

3 Preliminaries

In this Chapter we will briefly recall important facts about Lévy processes, FBM and classical Riemann-Stieltjes integration. We will omit proofs completely and refer to the known standard literature like Applebaum [1], Protter [9] or Sato [11].

Throughout the whole thesis we will always assume a given complete probability space $(\Omega, \mathcal{F}, P)$.

3.1 Lévy Processes

This Section is dedicated to useful facts about Lévy processes and we will first state the Definition:

3.1.1 Definition[Lévy process]

A stochastic process $(L_t)_{t\geq 0}$ is called a *Lévy process* if the following conditions are satisfied:

(i) $L_0 = 0$ a.s.

(ii) $L_t - L_s$ and $L_u - L_v$ are independent for $(t-s) \cap (u-v) = \emptyset$;

(iii) $L_t - L_s \overset{d}{=} L_u - L_v$ for $s \leq t$ and $v \leq u$ with $t - s = u - v$;

(iv) $(L_t)_{t\geq 0}$ is stochastically continuous, i.e. for all $\varepsilon > 0$ and all $s > 0$

$$\lim_{t\to s} P(|L_t - L_s| > \varepsilon) = 0.$$

It can be shown that every Lévy-process has a unique càdlàg modification, i.e. almost all sample paths have left limits and are right-continuous. From now on we will always consider this modification when speaking of a Lévy process.
A concept with a very close relation to Lévy processes is the infinitely divisibility. We will now give its definition:

3.1.2 Definition[Infinitely divisibility]

A random variable X is said to be *infinitely divisible* if for all $n \in \mathbb{N}$ exist i.i.d. random variables $Y_1^n, \ldots, Y_n^n$ such that

$$X \overset{d}{=} \sum_{i=1}^{n} Y_i^n$$

For any $A \in \mathcal{B}(\mathbb{R}^0)$ one can define a stochastic process by

$$N_t^A = \sum_{0<s\leq t} 1_A(\Delta L_s), \quad t \in \mathbb{R}.$$

For $0 \notin \overline{A}$ it can be shown that $(N_t^A)_{t\geq 0}$ is a Poisson process with intensity $\nu(A) = E[N_1^A] < \infty$. Moreover we have:

3.1.3 Theorem

Let $(L_t)_{t\geq 0}$ be a Lévy process and $0 \notin \overline{A}$. The set function $A \mapsto N_t^A(\omega)$ defines for every fixed $(t, \omega) \in \mathbb{R}^+ \times \Omega$ a σ-finite measure on $\mathbb{R}^0$. The set function $A \mapsto \nu(A)$ also defines a σ-finite measure on $\mathbb{R}^0$. ν is called the *Lévy measure* of $(L_t)_{t\geq 0}$.

Having this construction in mind one can show the famous Lévy -Itô Decomposition Theorem which we will state now.

3.1.4 Theorem[Lévy -Itô Decomposition]

Let $(L_t)_{t\geq 0}$ be a Lévy process. Then we have for all $t \geq 0$

$$L_t = \alpha t + B_t + \int_{\{|x|\leq 1\}} x(N_t(\cdot, dx) - t\nu(dx)) + \int_{\{|x|>1\}} xN_t(\cdot, dx) \tag{3.1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion, $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$ and $\alpha = E\left[L_1 - \int_{\{|x|>1\}} xN_t(\cdot, dx)\right]$.

With this decomposition one can now show:

3.1.5 Theorem[Lévy-Khintchine Formula]

Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy measure ν. Then for every $t \geq 0$ the distribution of L_t is infinitely divisible and we have

$$E[e^{iuL_t}] = \exp\{t\psi(u)\}$$

where $\psi(u)$ is given by

$$\psi(u) = i\alpha u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx), \quad u \in \mathbb{R}. \tag{3.2}$$

Moreover given α, σ^2, ν the corresponding Lévy process is unique in distribution.

3.1.6 Remark

(i) When constructing FLPs we will only consider Lévy processes without Brownian part, i.e. setting $\sigma = 0$ in the Lévy-Khintchine formula. We will assume further that

$$\int_{|x|>1} |x|^2 \nu(dx) < \infty \tag{3.3}$$

which is equivalent for $(L_t)_{t\geq 0}$ to have finite mean and variance. In that case we can suppose $E[L(1)] = 0$ which means $\alpha = -\int_{\{|x|>1\}} x\nu(dx)$ in (3.2) and finally we get

$$\psi(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx)), \quad u \in \mathbb{R}. \tag{3.4}$$

(ii) When speaking of a two-sided Lévy process we mean the following: Taking two independent copies $(L_t^1)_{t\geq 0}$ and $(L_t^2)_{t\geq 0}$ of a Lévy process we define a two-sided Lévy process by

$$L_t := \begin{cases} L_t^1 & t \geq 0 \\ -L_{-t-}^2 & t < 0 \end{cases}, \quad t \in \mathbb{R}. \tag{3.5}$$

3.2 Fractional Brownian Motion

We will now briefly review definition and some basic facts of FBM. For a more detailed analysis we refer to Samorodnitsky and Taqqu [10]. We start by the Definition:

3.2.1 Definition[Fractional Brownian Motion]

Let $H \in (0,1)$. The Gaussian process $(B_t^H)_{t\in\mathbb{R}}$ is called a *fractional Brownian motion with Hurst index H* if the following conditions are satisfied:

(i) $B_0^H = 0$ a.s.,

(ii) $E[B_t^H] = 0$,

(iii) $E[B_t^H B_s^H] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]$ for all $t, s \geq 0$.

The next Lemma states some important properties of FBM.

3.2.2 Lemma

Let $H \in (0,1)$ and $(B_t^H)_{t\in\mathbb{R}}$ a FBM. Then we have:

(i) $(B_t^H)_{t\in\mathbb{R}}$ is *self-similar*, i.e. for all $c > 0$

$$\left\{B_{ct}^H\right\}_{t\in\mathbb{R}} \stackrel{d}{=} c^H \left\{B_t^H\right\}.$$

(ii) For every $0 < \alpha < H$ there exists a modification of $(B_t^H)_{t\in\mathbb{R}}$ whose sample paths are a.s. locally Hölder continuous. Moreover $(B_t^H)_{t\in\mathbb{R}}$ has a.s. continuous sample paths.

(iii) $(B_t^H)_{t\in\mathbb{R}}$ has stationary increments.

FBM is a classical example of processes with long memory. There are various definitions for this property and as already mentioned in the introduction we will define it by the rate of decrease of the autocovariance function:

3.2.3 Definition[Longe Range Dependence]

Let $(X_t)_{t\in\mathbb{R}}$ be a stationary process and let $\gamma_X := \mathrm{Cov}(X_{t+h}, X_t)$ its autocovariance function. We say that $(X_t)_{t\in\mathbb{R}}$ is a stationary process with *longe range dependence* if $d \in (0, \frac{1}{2})$ and $c_\gamma > 0$ exists such that

$$\lim_{h\to\infty} \frac{\gamma_X(h)}{h^{2d-1}} = c_\gamma. \tag{3.6}$$

Consider the covariance between two increments of a FBM. Then we get by stationarity

$$\delta(n) := \mathrm{Cov}(B_k^H - B_{k-1}^H, B_{k+n}^H - B_{k+n-1}^H) = \frac{1}{2}[|n+1|^{2H} - |n-1|^{2H} - 2|n|^{2H}], \quad n, k \in \mathbb{N}.$$

In particular one can prove:

3.2.4 Lemma

(i) If $H \in (\frac{1}{2}, 1)$, $\delta(n)$ is positive and $\sum_{n=1}^{\infty} |\delta(n)| = \infty$. Moreover one has

$$\delta(n) \sim Cn^{2H-2}, \quad \text{as } n \to \infty \text{ for some } C > 0. \tag{3.7}$$

(ii) If $H = \frac{1}{2}$, $\delta(n) = 0$.

(iii) If $H \in (0, \frac{1}{2})$, $\delta(n)$ is negative and $\sum_{n=1}^{\infty} |\delta(n)| < \infty$.

3.2.5 Corollary

For $H \in (\frac{1}{2}, 1)$ the increments of FBM exhibt long range dependence.

Proof: This follows directly from (3.7) with $d := H - \frac{1}{2}$. □

3.3 Classical Riemann-Stieltes Integration

In this section we give a short review of classical Riemann-Stieltjes calculus.

3.3.1 Definition

Let $[a, b]$ be a compact interval.

(i) A *subdivision* κ of $[a, b]$ is a finite sequence $(x_i)_{i=0,\ldots,n}$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

(ii) An *intermediate subdivision* ρ of a subdivision κ is a finite sequence $(y_i)_{i=1,\ldots,n}$ such that for all $i \in \{1, \ldots, n\}$ $x_{i-1} \leq y_i \leq x_i$ holds.

(iii) For a subdivision κ we further define $\mathbf{mesh}(\kappa) := \sup_{i=1,\ldots,n} |x_i - x_{i-1}|$.

As known from standard analysis courses we define the Riemann-Stieltjes Integral:

3.3.2 Definition

Let f, g be real-valued functions on the compact interval $[a, b]$. We define a *Riemann-Stieltjes Sum* by

$$S(f, g, \kappa, \rho) := \sum_{i=1}^{n} f(y_i)[g(x_i) - g(x_{i-1})] \tag{3.8}$$

where κ is a subdivision of $[a, b]$ and ρ an intermediate subdivision of κ. If there is an $I \in \mathbb{R}$ such that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, g, \kappa, \rho) - I| < \varepsilon$$

for all subdivisions κ with $\mathbf{mesh}(\kappa) < \delta$ and for all intermediate subdivisions ρ of κ, then I is unique and will be denoted by

$$\int_a^b f dg. \tag{3.9}$$

This number is also called the *Riemann-Stieltjes Integral of f with respect to g* and f is called *Riemann-Stieltjes integrable with respect to g*.

3.3.3 Remark

(i) If g is of finite variation then the Riemann-Stieltjes integral exists for all continuous f.

(ii) With the Banach-Steinhaus Theorem one can show the converse:
If g is a right continuous function an $[a, b]$ and the Riemann-Stieltjes sums converge for every continuous f then g is of finite variation.

4 Extended Riemann-Stieltes Integration

As we will see later FLPs are like FBM in general not semimartingales. Consequently we cannot use the methods of semimartingale integration which can be found in Protter [9] for instance. However with the following techniques regarding Riemann-Stieltjes integration we can use pathwise integrals for a brood class of FLPs. In the following section we will state the needed Theorems of existence of such integrals and prove a chain rule and density formula. Our approach follows mainly Mikosch and Norvaiša [8], however as we are content with continuous integrators and integrands, proofs will become much more easier.

4.1 Functions of bounded p-variation

4.1.1 Definition

Let f be a real-valued function on the compact interval $[a, b]$.

(i) We define for $0 < p < \infty$ the *p-variation* of f as

$$v_p(f, [a, b]) := \sup_{\kappa} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p \tag{4.1}$$

where the supremum is taken over all subdivisions κ of $[a, b]$.

(i) If $v_p(f, [a, b]) < \infty$ then we will say that f is of *bounded p-variation on* $[a, b]$.

(iii) We further define

$$\begin{aligned} \mathcal{W}_p([a, b]) &:= \{f : [a, b] \to \mathbb{R} : v_p(f, [a, b]) < \infty\} \quad \text{and} \\ \mathfrak{W}_p(\mathbb{R}) &:= \{f : \mathbb{R} \to \mathbb{R} : \forall\, [s, t] \subseteq \mathbb{R} \quad f \in \mathcal{W}_p([s, t])\}. \end{aligned}$$

4.1.2 Remark

Obviously we have for $0 < q \leq p < \infty$ the inclusion $\mathcal{W}_q([a, b]) \subseteq \mathcal{W}_p([a, b])$.

4.1.3 Proposition

For $p > 1$ $\mathcal{W}_p([a, b])$ is a $\mathbb{R}$-algebra by pointwise addition and multiplication. Furthermore $(v_p(\cdot, [a, b]))^{\frac{1}{p}}$ is a seminorm an $\mathcal{W}_p([a, b])$.

Proof: Let $f, g \in \mathcal{W}_p([a,b])$ and $\lambda \in \mathbb{R}$. We will first show that $(v_p(\cdot,[a,b]))^{\frac{1}{p}}$ has the properties of a seminorm. By Minkowski's inequality we have

$$(v_p(f+g,[a,b]))^{\frac{1}{p}} \leq (v_p(f,[a,b]))^{\frac{1}{p}} + (v_p(g,[a,b]))^{\frac{1}{p}}.$$

Obviously

$$(v_p(\lambda \cdot f,[a,b]))^{\frac{1}{p}} = |\lambda|(v_p(f,[a,b]))^{\frac{1}{p}}$$

also holds and for $f=0$ we clearly have

$$(v_p(f,[a,b]))^{\frac{1}{p}} = 0.$$

Thus we have already shown that $\mathcal{W}_p([a,b])$ is a $\mathbb{R}$-vector space. For multiplication we calculate

$$\begin{aligned} v_p(f \cdot g,[a,b]) &= \sup_\kappa \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|^p \\ &= \sup_\kappa \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_i) + f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})|^p \\ &\leq \sup_\kappa \sum_{i=1}^n (|g(x_i)||f(x_i) - f(x_{i-1})| + |f(x_{i-1})||g(x_i) - g(x_{i-1})|)^p \\ &\leq \sup_\kappa \sum_{i=1}^n \left(||g||_{[a,b]}|f(x_i) - f(x_{i-1})| + ||f||_{[a,b]}|g(x_i) - g(x_{i-1})|\right)^p. \end{aligned} \quad (4.2)$$

We know that for $p > 1$ and $x, y \in \mathbb{R}$

$$|x+y|^p \leq 2^p(|x|^p + |y|^p) \quad (4.3)$$

and thus we get

$$\begin{aligned} &\leq 2^p \sup_\kappa \sum_{i=1}^n \left(||g||^p_{[a,b]}|f(x_i) - f(x_{i-1})|^p + ||f||^p_{[a,b]}|g(x_i) - g(x_{i-1})|^p\right) \\ &\leq 2^p \left(||g||^p_{[a,b]} v_p(f,[a,b]) + ||f||^p_{[a,b]} v_p(g,[a,b])\right). \end{aligned} \quad (4.4)$$

According to this inequality $f \cdot g \in \mathcal{W}_p([a,b])$ follows. □

We already know that the Riemann-Stieltjes integrals exists if f and g are continuous and g is of finite variation (which means of bounded 1-variation). However one can prove much more general results. We will consequently make use of the following Theorem which was proven by Young [13].

4.1.4 Theorem [Young [13], Section 10]

Let $[a,b]$ be a compact interval and f,g be continuous on $[a,b]$ where $g \in \mathcal{W}_p([a,b])$ and $f \in \mathcal{W}_q([a,b])$ for some $p,q > 0$ with $p^{-1} + q^{-1} > 1$. Then (3.9) exists and moreover we have for any $y \in [a,b]$

$$\left| \int_a^b f dg - f(y)[g(b) - g(a)] \right| \leq \left\{ 1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right) \right\} (v_q(f,[a,b]))^{\frac{1}{q}} (v_p(g,[a,b]))^{\frac{1}{p}} . \quad (4.5)$$

4.2 A Chain Rule

As promised we will now prove a chain rule sufficiant for our further needs in the next Chapters.

4.2.1 Theorem

Let $[a, b]$ be a compact interval and g be continuous on $[a, b]$ where $g \in \mathcal{W}_p([a, b])$ for some $p \in (0, 2)$. Furthermore let $F \in \mathcal{C}^1(\mathbb{R})$ with $F' \in \mathbf{Lip}(\mathbb{R})$. Then the Riemann-Stieltjes integral $\int_a^b (F' \circ g) dg$ exists and we have

$$(F \circ g)(b) - (F \circ g)(a) = \int_a^b (F' \circ g) dg. \tag{4.6}$$

Proof: We will first have a look at the existence of $\int_a^b (F' \circ g) dg$. Since g is continuous on $[a, b]$ it is surely bounded. So we find a constant $M > 0$ such that $|g(x)| \leq M$ for all $x \in [a, b]$. Because $[-M, M]$ is clearly compact we find another constant $K > 0$ such that $|F'(y) - F'(z)| \leq K|y - z|$ for all $y, z \in [-M, M]$. Here we used the known fact that $F' \in \mathbf{Lip}(\mathbb{R})$. It follows now that

$$\begin{aligned} v_p(F' \circ g, [a, b]) &= \sup_\kappa \sum_{i=1}^n |(F' \circ g)(x_i) - (F' \circ g)(x_{i-1})|^p \\ &\leq K \sup_\kappa \sum_{i=1}^n |g(x_i) - g(x_{i-1})|^p < \infty \end{aligned}$$

where we used $\mathbf{im}(g) \in [-M, M]$. From Theorem 4.1.4 we get the existence of $\int_a^b (F' \circ g) dg$ because for $p \in (0, 2)$ we cleary have $p^{-1} + p^{-1} > 1$. For every subdivsion κ of $[a, b]$ we have with the Mean Value Theorem

$$\begin{aligned} (F \circ g)(b) - (F \circ g)(a) &= \sum_{i=1}^n [(F \circ g)(x_i) - (F \circ g)(x_{i-1})] \\ &= \sum_{i=1}^n F'(y_i)[g(x_i) - g(x_{i-1})], \end{aligned} \tag{4.7}$$

for some $x_{i-1} \leq y_i \leq x_i$. The right-hand side of (4.7) converges by Theorem 4.1.4 to $\int_a^b (F' \circ g) dg$ and the proof is complete. □

4.2.2 Remark

For continuous g of finite variation one can easily show that the assumption $F' \in \mathbf{Lip}(\mathbb{R})$ is not needed. This follows mainly from the proof above where the existence of $\int_a^b (F' \circ g) dg$

is now a consequence of classical Riemann-Stieltjes calculus because $F' \circ g$ is clearly continuous.

As an easy consequence of our chain rule we get a Theorem about a product formula:

4.2.3 Theorem

Let $[a,b]$ be a compact interval and f, g be continuous on $[a,b]$ where $f, g \in \mathcal{W}_p([a,b])$ for some $p \in (0,2)$. Then we have

$$f(b)g(b) - f(a)g(a) = \int_a^b f dg + \int_a^b g df. \tag{4.8}$$

Proof: We define $\phi(x) := x^2$ and see that $\phi \in \mathcal{C}^1(\mathbb{R})$ and $\phi' \in \mathbf{Lip}(\mathbb{R})$ therefore with Theorem 4.1.3 the conditions of Theorem 4.2.1 are fullfilled for f, g and $f+g$. We obtain then by polarization

$$\begin{aligned}
& f(b)g(b) - f(a)g(a) \\
= \; & \frac{1}{2}\left[(f+g)^2(b) - (f+g)^2(a) - [f^2(b) - f^2(a)] - [g^2(b) - g^2(a)]\right] \\
= \; & \frac{1}{2}\left[(\phi \circ (f+g))(b) - (\phi \circ (f+g))(a) - [(\phi \circ f)(b) - (\phi \circ f)(a)] - [(\phi \circ g)(b) - (\phi \circ g)(a)]\right] \\
= \; & \frac{1}{2}\left[2\int_a^b (f+g)d(f+g) - 2\int_a^b f df - 2\int_a^b g dg]\right] \\
= \; & \int_a^b f dg + \int_a^b g df,
\end{aligned}$$

because the Riemann-Stieltjes integral is additive with respect to the integrators if the Riemann-Stieltjes integrals exists separately which is here the case. □

4.3 Density Formula

The next Theorem we are going to prove will provide us with a much needed density formula. For that however we will need the following Lemma:

4.3.1 Lemma

Let $[a,b]$ be a compact interval and f,g be continuous on $[a,b]$ where $g \in \mathcal{W}_p([a,b])$ and $f \in \mathcal{W}_q([a,b])$ where $q>0$ and $p>1$ with $p^{-1}+q^{-1}>1$. For all $x \in [a,b]$ we define $\phi(x) := \int_a^x f dg$. Then $\phi \in \mathcal{W}_p([a,b])$. Moreover we have

$$v_p(\phi,[a,b]) \le 2^p \left(\left\{ 1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}^p v_q(f,[a,b])^{\frac{p}{q}} + ||f||^p_{[a,b]} \right) v_p(g,[a,b]). \tag{4.9}$$

Proof: Using again (4.3) we compute for $z_i \in [x_{i-1},x_i] \subset [a,b]$

$$\begin{aligned}
v_p(\phi,[a,b]) &= \sup_\kappa \sum_{i=1}^n \left| \int_a^{x_i} f dg - \int_a^{x_{i-1}} f dg \right|^p = \sup_\kappa \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f dg \right|^p \\
&= \sup_\kappa \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f dg - f(z_i)[g(x_i)-g(x_{i-1})] + f(z_i)[g(x_i)-g(x_{i-1})] \right|^p \\
&\le \sup_\kappa \sum_{i=1}^n \left(\left| \int_{x_{i-1}}^{x_i} f dg - f(z_i)[g(x_i)-g(x_{i-1})] \right| + |f(z_i)[g(x_i)-g(x_{i-1})]| \right)^p \\
&= \sup_\kappa \sum_{i=1}^n \left(\left| \int_{x_{i-1}}^{x_i} (f-f(z_i)) dg \right| + |f(z_i)|\,|g(x_i)-g(x_{i-1})| \right)^p \\
&\le 2^p \sup_\kappa \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} (f-f(z_i)) dg \right|^p + |f(z_i)|^p\,|g(x_i)-g(x_{i-1})|^p .
\end{aligned}$$

Recalling (4.5) we get

$$\begin{aligned}
&\le 2^p \sup_\kappa \left(\sum_{i=1}^n \left\{ 1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}^p v_q(f,[x_{i-1},x_i])^{\frac{p}{q}} v_p(g,[x_{i-1},x_i]) \right. \\
&\qquad \left. + \sum_{i=1}^n ||f||^p_{[a,b]}\,|g(x_i)-g(x_{i-1})|^p \right) \\
&\le 2^p \sup_\kappa \left(\left\{ 1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}^p v_q(f,[a,b])^{\frac{p}{q}} \sum_{i=1}^n v_p(g,[x_{i-1},x_i]) \right. \\
&\qquad \left. + ||f||^p_{[a,b]} \sum_{i=1}^n |g(x_i)-g(x_{i-1})|^p \right).
\end{aligned}$$

Finally with the inequality $\sum_{i=1}^n v_p(g,[x_{i-1},x_i]) \leq v_p(g,[a,b])$ we arrive at

$$\begin{aligned} &\leq 2^p \sup_\kappa \left(\left\{ 1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}^p v_q(f,[a,b])^{\frac{p}{q}} v_p(g,[a,b]) \right. \\ &\quad \left. + ||f||_{[a,b]}^p \sum_{i=1}^n |g(x_i)-g(x_{i-1})|^p \right) \\ &= 2^p \left(\left\{ 1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right\}^p v_q(f,[a,b])^{\frac{p}{q}} v_p(g,[a,b]) + ||f||_{[a,b]}^p v_p(g,[a,b]) \right), \end{aligned}$$

which proves the assertion. □

Now we are able to state and prove a density formula:

4.3.2 Theorem

Let $[a,b]$ be a compact interval and f,g,h be continuous on $[a,b]$ where $f,h \in \mathcal{W}_q([a,b])$ and $g \in \mathcal{W}_p([a,b])$ for some $q>0$ and $p>1$ with $p^{-1}+q^{-1}>1$. For all $x \in [a,b]$ we define $\phi(x) := \int_a^x hdg$. Then we have

$$\int_a^b f d\phi = \int_a^b fhdg. \tag{4.10}$$

Proof: The integrals in (4.10) exists because by Lemma 4.3.1 $\phi \in \mathcal{W}_p([a,b])$ and by Proposition 4.1.3 $fh \in \mathcal{W}_q([a,b])$. First we will look at the corresponding Riemann-Stieltjes sums of (4.10) for some subdivision κ and intermediate subdivision ρ. For $f_{\kappa,\rho} := \sum_{i=1}^n f(y_i)1_{(x_{i-1},x_i]}$ we have

$$\begin{aligned} \int_a^b f_{\kappa,\rho} d\phi &= \sum_{i=1}^n f(y_i)[\phi(x_i)-\phi(x_{i-1})] \\ &= \sum_{i=1}^n f(y_i) \int_{x_{i-1}}^{x_i} hdg \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(y_i)hdg \\ &= \int_a^b f_{\kappa,\rho} hdg. \end{aligned} \tag{4.11}$$

For easier notation we will take a to be contained in $(x_0,x_1]$ from now on. By our main Theorem 4.1.4 we know that

$$\int_a^b f_{\kappa,\rho} d\phi = \sum_{i=1}^n f(y_i)[\phi(x_i)-\phi(x_{i-1})] \overset{\mathbf{mesh}(\kappa)\to 0}{\longrightarrow} \int_a^b f d\phi$$

and for $(fh)_{\kappa,\rho} := \sum_{i=1}^n f(y_i)h(y_i)1_{(x_{i-1},x_i]}$,

$$\int_a^b (fh)_{\kappa,\rho} dg = \sum_{i=1}^n f(y_i)h(y_i)[g(x_i) - g(x_{i-1})] \overset{\mathbf{mesh}(\kappa)\to 0}{\longrightarrow} \int_a^b fhdg.$$

To show now (4.10) it is sufficient to prove

$$\left| \int_a^b f_{\kappa,\rho} hdg - \int_a^b (fh)_{\kappa,\rho} dg \right| \overset{\mathbf{mesh}(\kappa)\to 0}{\longrightarrow} 0. \tag{4.12}$$

Since h is continuous for given $\delta > 0$ we can choose $\varepsilon > 0$ such that for all subdivisions κ and corresponding intermediate subdivisions ρ with $\mathbf{mesh}(\kappa) < \varepsilon$ we get $|h(x)-h(y)| < \delta$ for all $x, y \in (x_{i-1}, x_i]$, $i \in \{1, \dots, n\}$. Under this assumption we will now look at the difference of the Riemann-Stieltjes sums of (4.12) given some subdivision $\pi = (z_j)_{j=0,\dots,m}$ and an intermediate subdivision $o = (z_j)_{j=1,\dots,m}$ of $[a, b]$:

$$\begin{aligned} & \left| \sum_{i=1}^m (f_{\kappa,\rho} h)(w_j))[g(z_j) - g(z_{j-1})] - \sum_{i=1}^m ((fh)_{\kappa,\rho})(w_j)[g(z_j) - g(z_{j-1})] \right| \\ = & \left| \sum_{i=1}^m [(f_{\kappa,\rho} h)(w_j)) - ((fh)_{\kappa,\rho})(w_j)][g(z_j) - g(z_{j-1})] \right|. \end{aligned}$$

Setting $p^* := \frac{p}{p-1}$ and using Hölder's inequality we get

$$\begin{aligned} \le & \left(\sum_{i=1}^m |(f_{\kappa,\rho} h)(w_j)) - ((fh)_{\kappa,\rho})(w_j)|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^m |g(z_j) - g(z_{j-1})|^p \right)^{\frac{1}{p}} \\ \le & ||f||_{[a,b]} \left(\sum_{i=1}^m |h(w_j) - h(y_{i_j})|^{p^*} \right)^{\frac{1}{p^*}} (v_p(g, [a, b]))^{\frac{1}{p}}, \end{aligned}$$

with i_j such that $w_j \in (x_{i_j - 1}, x_{i_j}]$. With $y_{i_j} \in (x_{i_j-1}, x_{i_j}]$ we have

$$\begin{aligned} = & ||f||_{[a,b]} \left(\sum_{i=1}^m |h(w_j) - h(y_{i_j})|^q |h(w_j) - h(y_{i_j})|^{p^*-q} \right)^{\frac{1}{p^*}} (v_p(g, [a, b]))^{\frac{1}{p}} \\ < & \delta^{1-\frac{q}{p^*}} ||f||_{[a,b]} \left(\sum_{i=1}^m |h(w_j) - h(y_{i_j})|^q \right)^{\frac{1}{p^*}} (v_p(g, [a, b]))^{\frac{1}{p}} \\ \le & \delta^{1-\frac{q}{p^*}} ||f||_{[a,b]} (v_q(h, [a, b]))^{\frac{1}{p^*}} (v_p(g, [a, b]))^{\frac{1}{p}} \overset{\mathbf{mesh}(\kappa)\to 0}{\longrightarrow} 0. \end{aligned}$$

Hence we have shown (4.12) and therefore proven the Theorem. □

5 Fractional Lévy Processes

In this Chapter we will introduce fractional Lévy processes (FLP) as a natural generalization of the integral representation of fractional Brownian Motion. We will further derive second order and sample path properties and also state and prove a Theorem about long time behavior of FLPs. The last Section will concern integration methods, where we will make use of our preceding investigations of extended Riemann-Stieltjes integrals. In our approach we will follow mainly Marquardt [5] and proofs are in most cases omitted.

5.1 Construction and Integral-Representations

For notational convenience we will work with the fractional integration parameter $d \in (-\frac{1}{2}, \frac{1}{2})$ instead of the Hurst index H, where $d = H - \frac{1}{2}$. Because we are only interested in long memory behavior we will restrict ourselves to $d \in (0, \frac{1}{2})$. We will furthermore only consider FLPs with existing second moments because we defined long range dependence via the autocovariance function. Following Mandelbrot and Hudson [4] for FBM we choose like Marquardt [5] the Definition:

5.1.1 Definition [Marquardt [5], Definition 2.1]

Let $(L_t)_{t\in\mathbb{R}}$ be a zero-mean two-sided L^2-Lévy process, i.e.

$$E[L(1)^2] < \infty, \tag{5.1}$$

without Brownian component. We define a stochastic process $(L_t^d)_{t\in\mathbb{R}}$ by

$$L_t^d := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(ds), \quad t \in \mathbb{R}, \tag{5.2}$$

where $d \in (0, \frac{1}{2})$ is the fractional integration parameter. We call $(L_t^d)_{t\in\mathbb{R}}$ *fractional Lévy process (FLP)* and $(L_t)_{t\in\mathbb{R}}$ the *driving Lévy process of the FLP.*

5.1.2 Remark

As Marquardt [5] we will only consider driving Lévy processes without Brownian component. A Lévy process can by Lévy-Itô Decomposition represented as a Brownian motion and a jump part which are independent. However the first one gives rise to a FBM which has been studied already for a long time; for instance refer to Samorodnitsky and Taqqu [10].

The next Theorem specifies the integration method and insures the existence of the indefinite integral in (5.2):

5.1.3 Theorem [Marquardt [5], Theorem 2.7]

Let $(L_t)_{t\in\mathbb{R}}$ be a zero-mean two-sided L^2-Lévy process without Brownian component. For $t, s \in \mathbb{R}$ define $f_t(s) := \frac{1}{\Gamma(d+1)}(t-s)_+^d - (-s)_+^d$. Then the integrals

$$L_t^d = \int_{\mathbb{R}} f_t(s) L(ds), \quad t \in \mathbb{R},$$

exists as limits in probability of step functions approximating f_t. The finite dimensional distributions of L^d have the characteristic functions

$$E\left[\exp\left\{\sum_{j=1}^m iu_j L_{t_j}^d\right\}\right] = \exp\left\{\int_{\mathbb{R}} \psi_L\left(\sum_{j=1}^m u_j f_{t_j}(s)\right) ds\right\}, \quad u_1, \dots, u_m \in \mathbb{R}, \tag{5.3}$$

for $-\infty < t_1 < \ldots < t_m < \infty$ and $\psi_L(u) := \int_{\mathbb{R}} (e^{iux} - 1 - iux)\,\nu(dx)$ where ν is the Lévy measure of $(L_t)_{t\in\mathbb{R}}$.

5.1.4 Remark

(i) From general theory of infinite divisible distributions (see Sato [11]) we know that a Lévy process satisfies (5.1) if and only if

$$\int_{|x|>1} |x|^2 \nu(dx) < \infty \tag{5.4}$$

(ii) It can be shown that (5.2) is well-defined for all $d \in (0, \frac{1}{2})$ if and only if (5.4) holds.

(iii) From (5.3) can be seen that for fixed $t \in \mathbb{R}$ the generating triple of the infinite divisible random variable L_t^d is given by $(\alpha_L^t, 0, \nu_L^t)$ with

$$\begin{aligned} \alpha_L^t &= -\int_{\mathbb{R}}\int_{\mathbb{R}} f_t(s) x 1_{|f_t(s)x|>1} \nu(dx) ds \\ \nu_L^t(B) &= \int_{\mathbb{R}}\int_{\mathbb{R}} 1_B(f_t(s)x)\nu(dx)ds, \quad B \in \mathcal{B}(\mathbb{R}^0). \end{aligned}$$

The next Theorem shows that FLPs can also be defined in a pathwise sense through an improper Riemann integral. This representation is very important to prove long time behavior as we will see later.

5.1.5 Theorem [Marquardt [6], Theorem 2.12]

Let $(L_t)_{t\in\mathbb{R}}$ a zero-mean two-sided L^2-Lévy process without Brownian component. Then $(L_t^d)_{t\in\mathbb{R}}$ has a modification which equals the following improper Riemann integral

$$\frac{1}{\Gamma(d)}\int_{\mathbb{R}}[(t-s)_+^{d-1}-(-s)_+^{d-1}]L(s)ds, \quad t\in\mathbb{R}. \tag{5.5}$$

Furthermore (5.5) is continuous in t.

5.1.6 Remark

(i) From now on when speaking of a FLP we will always consider the modification of Theorem 5.1.5.

(ii) Given a FLP $(L_t^d)_{t\in\mathbb{R}}$ we will always assume that there is also a driving zero-mean two-sided L^2-Lévy process $(L_t)_{t\in\mathbb{R}}$ without Brownian component given, without explicitly stating it.

5.2 Second Order and Sample Path Properties

In this Section we will derive the second order structure and calculate the autocovariance function of FLPs. Moreover we will state a Theorem which ensures Hölder continuity similar to the case of FBM. However only Hölder exponents up to the fractional integration parameter are allowed. This is different to FBM where these exponents may go up to the Hurst index. This fact has important consequences for defining pathwise integrals regarding FLPs as we will see in the next Section.

5.2.1 Theorem [Marquardt [5], Theorem 2.15]

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP with $d \in (0, \frac{1}{2})$. Then we have for $t, s \in \mathbb{R}$

$$\mathrm{Cov}(L_t^d, L_s^d) = \frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\left[|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}\right] \tag{5.6}$$

5.2.2 Theorem [Marquardt [5], Theorem 2.16]

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP with $d \in (0, \frac{1}{2})$. Let further $h > 0$ and consider $t, s \in \mathbb{R}$ with $s + h \leq t$ and $t - s = nh$ for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \delta_d(n) &= \mathrm{Cov}(L_{t+h}^d - L_t^d, L_{s+h}^d - L_s^d) \\ &= \frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} h^{2d+1} n^{2d-1} + O(n^{2d-2}), \quad \text{as } n \to \infty. \end{aligned} \tag{5.7}$$

5.2.3 Corollary [Marquardt [5], Corollary 2.17]

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP with $d \in (0, \frac{1}{2})$. Let further $h > 0$ and consider $t, s \in \mathbb{R}$ with $s + h \leq t$ and $t - s = nh$ for some $n \in \mathbb{N}$. Then we have

$$\delta_d(n) \overset{n\to\infty}{\to} 0$$

Moreover we have $\delta_d(n) > 0$ and

$$\sum_{n=1}^{\infty} \delta_d(n) = \infty.$$

At last (5.7) shows that the increments of a FLP exhibit long range dependence.

The next Theorem provides some other properties needed later:

5.2.4 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP with $d \in (0, \frac{1}{2})$. Then the following assertions holds:

(i) For every $\beta < d$ there exists a modification of $(L_t^d)_{t\in\mathbb{R}}$ such that its sample paths are a.s. locally Hölder continuous of order β.

(ii) For every $\beta > d$ there exists a modification of $(L_t^d)_{t\in\mathbb{R}}$ such that its sample paths are not locally Hölder continuous of order β on a non-zero set $A \subseteq \Omega$.

(iii) $(L_t^d)_{t\in\mathbb{R}}$ has stationary increments and is symmetric, i.e. $\left\{L_{-t}^d\right\}_{t\in\mathbb{R}} =^d \left\{-L_t^d\right\}_{t\in\mathbb{R}}$.

(iv) $(L_t^d)_{t\in\mathbb{R}}$ cannot be selfsimilar.

As already mentioned FLPs are not semimartingales in general. A bit more precise is the next Theorem:

5.2.5 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP with $d \in (0, \frac{1}{2})$. Let ν be the Lévy measure of the driving Lévy process. Then we have:

(i) If $\nu(\mathbb{R}) < \infty$, then $(L_t^d)_{t\in\mathbb{R}}$ is of finite total variation thus it is a semimartingale.

(ii) If we have

$$\int_{|x|\geq\varepsilon} |x|\nu(dx) \geq C\varepsilon^{-\alpha}$$

for some $\alpha \geq 1$, $C > 0$ and $\varepsilon > 0$, then $(L_t^d)_{t\in\mathbb{R}}$ is not a semimartingale.

Now we will state and prove a Theorem about long time behavior of FLPs which will be needed in the next Chapters. A similar Theorem considering $t \to \infty$ can be found in [7].

5.2.6 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP with $d \in (0, \frac{1}{2})$ and $\alpha > d + \frac{1}{2}$. Then we have

$$\lim_{t\longrightarrow-\infty} \frac{|L_t^d|}{|t|^\alpha} = 0 \quad \text{a.s.} \tag{5.8}$$

Proof: Without loss of generality we can $t < 0$ assume. A first look provides us with the following inequality

$$\begin{aligned}\frac{1}{|t|^\alpha}|L_t^d| &= \frac{1}{|t|^\alpha}\frac{1}{\Gamma(d)}\left|\int_{-\infty}^t [(t-s)_+^{d-1} - (-s)_+^{d-1}]L(s)ds\right| \\ &\leq \frac{1}{|t|^\alpha}\frac{1}{\Gamma(d)}\int_{-\infty}^t \left|[(t-s)_+^{d-1} - (-s)_+^{d-1}]\right| |L(s)|\, ds \\ &= \frac{1}{|t|^\alpha}\frac{1}{\Gamma(d)}\int_{-\infty}^t [(-s)^{d-1} - (t-s)^{d-1}]\, |L(s)|\, ds.\end{aligned}$$

Therefore it suffices to show

$$\lim_{t\to-\infty} \frac{1}{|t|^\alpha}\int_{-\infty}^t [(-s)^{d-1} - (t-s)^{d-1}]\, |L(s)|\, ds = 0 \quad \text{a.s.}$$

By the law of iterated logarithm for Lévy processes from Sato [11], Proposition 48.9, we find a random variable T such that a.s. for all $s < T$

$$|L(s)| \leq M(2|s|\log\log|s|)^{\frac{1}{2}}. \tag{5.9}$$

Because we will show the result pathwise we can without loss of generality assume $t < T$. Therefore we get by (5.9)

$$\begin{aligned}&\frac{1}{|t|^\alpha}\int_{-\infty}^t [(-s)^{d-1} - (t-s)^{d-1}]|L(s)|ds \\ \leq\ &\frac{M}{|t|^\alpha}\int_{-\infty}^t [(-s)^{d-1} - (t-s)^{d-1}](2|s|\log\log|s|)^{\frac{1}{2}}ds \\ =\ &\frac{M}{|t|^\alpha}\int_{-\infty}^{-|t|} [(-s)^{d-1} - (-|t|-s)^{d-1}](2|s|\log\log|s|)^{\frac{1}{2}}ds.\end{aligned}$$

Setting $e^{-1}|t|u = s$ we obtain by change of variable

$$= \frac{M|t|}{e|t|^\alpha}\int_{-\infty}^{-e} [(-e^{-1}|t|u)^{d-1} - (-|t| - e^{-1}|t|u)^{d-1}](2e^{-1}|t||u|\log\log(e^{-1}|t||u|))^{\frac{1}{2}}du. \tag{5.10}$$

Now we will use that for large $|t|$ and for $|u| \geq e$

$$\begin{aligned}&|t||u|\log\log(e^{-1}|t||u|) \\ =\ &|t||u|\log(\log(e^{-1}|t|) + \log|u|) \\ \leq\ &|t||u|\log\left(\log(e^{-1}|t|)\left|1 + \frac{\log|u|}{\log e^{-1}|t|}\right|\right) \\ =\ &|t||u|\log\log(e^{-1}|t|) + |t||u|\log\left(\left|1 + \frac{\log|u|}{\log(e^{-1}|t|)}\right|\right) \\ \leq\ &|t||u|\log\log|t| + |t||u|\log(1 + \log|u|).\end{aligned} \tag{5.11}$$

Combining (5.10), (5.11) with $|a+b|^{\frac{1}{2}} \leq |a|^{\frac{1}{2}} + |b|^{\frac{1}{2}}$ for $a, b \in \mathbb{R}$ we get

$$\begin{aligned}
&\leq \frac{M(2e^{-1}|t|\log\log|t|)^{\frac{1}{2}}}{e|t|^{\alpha-d}} \int_{-\infty}^{-e} [(-e^{-1}u)^{d-1} - (-1-e^{-1}u)^{d-1}]|u|^{\frac{1}{2}} du \\
&\quad + \frac{M(2e^{-1}|t|)^{\frac{1}{2}}}{e|t|^{\alpha-d}} \int_{-\infty}^{-e} [(-e^{-1}u)^{d-1} - (-1-e^{-1}u)^{d-1}](|u|\log(1+\log|u|))^{\frac{1}{2}} du \\
&= \frac{M(2e^{-1}\log\log|t|)^{\frac{1}{2}}}{e|t|^{\alpha-(d+\frac{1}{2})}} \int_{e}^{\infty} [(e^{-1}u)^{d-1} - (-1+e^{-1}u)^{d-1}]u^{\frac{1}{2}} du \\
&\quad + \frac{M(2e^{-1})^{\frac{1}{2}}}{e|t|^{\alpha-(d+\frac{1}{2})}} \int_{e}^{\infty} [(e^{-1}u)^{d-1} - (-1+e^{-1}u)^{d-1}](u\log(1+\log u))^{\frac{1}{2}} du. \qquad (5.12)
\end{aligned}$$

By a binomial extension we get

$$(e^{-1}u-1)^{d-1} = (e^{-1}u)^{d-1} - (d-1)(e^{-1}u)^{d-2} + O(u^{d-3})$$

and therefore

$$\left[(e^{-1}u)^{d-1} - (-1+e^{-1}u)^{d-1}\right](u\log(1+\log|u|))^{\frac{1}{2}} \sim (d-1)(e^{-1})^{d-2}u^{d-\frac{3}{2}}(\log\log(u))^{\frac{1}{2}} \qquad (5.13)$$

which ensures the existence of the last integrals in (5.12). Letting $t \to -\infty$ we obtain the assertion. □

5.3 Integration with respect to Fractional Lévy Processes

In this Section we will define integrals with respect to FLPs. As we showed before FLPs are in general not semimartingales, thus the known theory for this kind of processes (refer to Protter [9]) will not work. Hence our aim is to definde integration in a pathwise Riemann-Stieltjes sense and there are many possibilities to do this. For instance Zähle [14] introduced the spaces

$$\begin{aligned}\mathcal{C}^{\beta-}(\mathbb{R}) &:= \{f:\mathbb{R}\to\mathbb{R}:\ \forall K\subset\mathbb{R}\ f|K \text{ is Hölder continuous of all orders } \beta<d\}\\ \mathcal{C}^{\beta+}(\mathbb{R}) &:= \{f:\mathbb{R}\to\mathbb{R}:\ \forall K\subset\mathbb{R}\ f|K \text{ is Hölder continuous of some order } \beta(K)>d\}\end{aligned}$$

and proved that the Riemann-Stieltjes integral $\int_a^b f dg$ exists, if $f\in C^{(1-\beta)+}(\mathbb{R})$ and $g\in C^{\beta-}(\mathbb{R})$. Having Theorem 5.2.4 in mind we see that this would be a possibility do define integrals with respect to FLPs. In fact Buchmann and Klüppelberg [2] used this approach to consider SDEs driven by FBM. However problems arise very fast if we want to prove a chain rule for such integrals because this is only possible if $\beta\in(\frac{1}{2},1)$. Now speaking in terms of the Hurst index the sample paths of FBM lie in $C^{H-}(\mathbb{R})$ where these of FLPs are only element of $C^{H-\frac{1}{2}}(\mathbb{R})$. Thus that approach will not be very useful to our aims for further Chapters. Another way of ensuring the convergence of the Riemann-Stieltjes sums is what we have done in Chapter 4. There we proved a chain rule and a density formula if the integrand is of bounded p-variation for $p\in(1,2)$. In the following Section we will only consider integration with respect to FLPs of bounded p-variation for $p\in(0,2)$.

5.3.1 Definition

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded 1-variation, i.e. $L^d\in\mathfrak{W}_1(\mathbb{R})$ a.s., $d\in(0,\frac{1}{2})$. We define for every $X\in\mathcal{C}^0(\mathbb{R})$ a.s. the integral

$$\int_a^b X dL^d, \quad -\infty\le a\le b\le\infty, \tag{5.14}$$

pathwise in the classical Riemann-Stieltjes sense.

5.3.2 Remark

Obviously by 4.1.2 we have for a FLP of bounded p-variation with $p<1$ that it is of finite variation.

5.3.3 Definition

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded p-variation, i.e. $L^d\in\mathfrak{W}_p(\mathbb{R})$ a.s., $d\in(0,\frac{1}{2})$ and $p\in(1,2)$. Then we define for every continuous $X\in\mathfrak{W}_q(\mathbb{R})$ a.s. with $p^{-1}+q^{-1}>1$ the integral

$$\int_a^b X dL^d, \quad -\infty \le a \le b \le \infty, \tag{5.15}$$

pathwise in the extended Riemann-Stieltjes sense of Theorem 4.1.4.

5.3.4 Remark

From now on we will understand integrals with respect to FLPs always in the pathwise sense of the above definitions.

5.3.5 Example

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded p-variation, $d \in (0, \frac{1}{2})$ and $p \in (0, 2)$. Then we have for $s \le t$

$$\int_s^t L_u^d dL_u^d = \frac{1}{2}\left[(L_t^d)^2 - (L_s^d)^2\right].$$

This resembles the fact that L^d is of finite p-variation for some $p < 2$.
Proof: Assume $p \in (1, 2)$. We define $\phi(x) := x^2$. Surely the assumptions of Theorem 4.2.1 are fullfilled. We get then

$$(L_t^d)^2 - (L_s^d)^2 = \phi(L_t^d) - \phi(L_s^d) = \int_s^t \phi'(L_u^d) dL_u^d = 2\int_s^t L_u^d dL_u^d$$

If $p \in (0, 1]$ then the result follows with classical Riemann-Stieltjes calculus. □

6 Fractional Lévy Ornstein-Uhlenbeck Processes

In this Chapter we will introduce fractional Lévy Ornstein-Uhlenbeck processes (FLOUPs) as improper Riemann-Stieltjes integrals and prove that they are stationary solutions of Langevin equations where the driving processes are the corresponding FLP, i.e.

$$d\mathcal{L}_t^{d,\lambda} = -\lambda \mathcal{L}_t^{d,\lambda} dt + dL_t^d. \tag{6.1}$$

We will further consider the second order structure and prove that the increments exhibit long range dependence. For FBM similar Theorems can be found in Cheridito [3].

6.1 Existence of the Pathwise Improper Riemann-Stieltjes Integral

Recalling some basic facts about Ornstein-Uhlenbeck processes we want to define the FLOUP like

$$\int_{-\infty}^{t} e^{-\lambda(t-s)} dL_s^d.$$

Our chosen approach will be the pathwise Riemann-Stieltjes sense and it will now be shown that with Theorem 5.2.6 we are already able to prove the existence of the above integral. No condition about bounded p-variation of the driving FLP is required. That follows from the fact that the integrand is continuously differentiable as can be seen in the proof of the next Theorem.

6.1.1 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$, $-\infty \le a < \infty$, and $\lambda > 0$. Then for almost all $\omega \in \Omega$

$$\int_a^t e^{\lambda s} dL_s^d(\omega), \quad t > a, \tag{6.2}$$

exists as a Riemann-Stieltjes integral and is equal to

$$e^{\lambda t} L_t^d(\omega) - e^{\lambda a} L_a^d(\omega) - \lambda \int_a^t L_s^d(\omega) e^{\lambda s} ds. \tag{6.3}$$

Furthermore the function $t \mapsto \int_a^t e^{\lambda s} dL_s^d(\omega), t > a$ is continuous.

Proof: From Theorem 5.2.6 we know that there null set $N \subset \Omega$ such that for $\omega \in \Omega \setminus N$ and all $\alpha > d + \frac{1}{2}$ we have

$$\lim_{t\to-\infty} \frac{L_t^d(\omega)}{|t|^\alpha} = 0 \tag{6.4}$$

and hence for all $\omega \in \Omega \backslash N$ and $t > a$, $\int_a^t L_u^d(\omega) e^{\lambda u} du$ exists as a Riemann-Stieltjes integral. For a compact interval $[a, t]$ this is clear. Now consider $a = -\infty$. It is clearly enough to show that $\int_{-\infty}^T L_u^d(\omega) e^{\lambda u} du$ exists for $T < -1$. This follows by the inequality

$$\left| \int_R^T L_u^d(\omega) e^{\lambda u} du \right| \leq \int_R^T \left| L_u^d(\omega) \right| e^{\lambda u} du = \int_R^T \underbrace{\frac{|L_u^d(\omega)|}{|u|^\alpha}}_{\leq C} e^{\lambda u} |u|^\alpha du \leq C \int_R^T e^{\lambda u} |u|^\alpha du,$$

for some $C > 0$, where the integral on the right hand side exists for $R \to -\infty$. Similar one shows

$$\lim_{a \to -\infty} e^{\lambda a} L_a^d(\omega) = 0. \tag{6.5}$$

Now by Wheeden and Zygmund [12], Theorem 2.21, follows that (6.2) also exists as a Riemann-Stieltjes integral and is equal to (6.3). Because of (6.3) is continuous in t for $t > a$ we are finished. □

From now on we will work outside the null set N from the proof above.

6.1.2 Definition

Let $(L_t^d)_{t \in \mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$ and $\lambda > 0$. Then we define a stochastic process $(\mathcal{L}_t^{d,\lambda})_{t \in \mathbb{R}}$ by

$$\mathcal{L}_t^{d,\lambda} := \int_{-\infty}^t e^{-\lambda(t-s)} dL_s^d, \quad t \in \mathbb{R}. \tag{6.6}$$

and will call it *Fractional Lévy Ornstein-Uhlenbeck Process (FLOUP)*.

The next Lemma states a very useful condition of FLOUPs.

6.1.3 Lemma

The process $(\mathcal{L}_t^{d,\lambda})_{t \in \mathbb{R}}$ is stationary, i.e. for all $t_1 < t_2 < \cdots < t_m$, $m \in \mathbb{N}$, $h \in \mathbb{R}^+$

$$(\mathcal{L}_{t_1}^{d,\lambda}, \mathcal{L}_{t_2}^{d,\lambda}, \ldots, \mathcal{L}_{t_n}^{d,\lambda}) \stackrel{d}{=} (\mathcal{L}_{t_1+h}^{d,\lambda}, \mathcal{L}_{t_2+h}^{d,\lambda}, \ldots, \mathcal{L}_{t_n+h}^{d,\lambda}).$$

Proof: For $u_1, \ldots, u_n$ and $-\infty < t_1 < \cdots < t_n$, $n \in \mathbb{R}$, we get by the stationary increments of $(L_t^d)_{t \in \mathbb{R}}$

$$\sum_{i=1}^n u_i \mathcal{L}_{t_i+h}^{d,\lambda} = \sum_{i=1}^n u_i \int_{-\infty}^{t_i+h} e^{-\lambda(t_i+h-s)} dL_s^d \stackrel{d}{=} \sum_{i=1}^n u_i \int_{-\infty}^{t_i} e^{-\lambda(t_i-s)} dL_s^d = \sum_{i=1}^n u_i \mathcal{L}_{t_i}^{d,\lambda}.$$

Hence the characteristic function of the left and right hand side of (6.7) coincide and we are finished. □

6.2 Langevin Equation

In this Section we will consider a FLP-driven Langevin equation and prove that it has a unique stationary solution. This solution will be the corresponding FLOUP as defined in the previous Section. However we first need some facts about ordinary differential equations.

6.2.1 Theorem

For $c, \xi \in \mathbb{R}$, $\lambda, \sigma > 0$ and $f \in C^0(\mathbb{R})$ a continuous function $y : \mathbb{R} \to \mathbb{R}$ solves the integral equation

$$y(t) = \xi - \lambda \int_c^t y(s)ds + \sigma(f(t) - f(c)), \quad , t \geq c, y(c) = \xi \tag{6.7}$$

if and only if the function $z(t) := \int_c^t y(s)ds$, $t \geq c$, solves the linear differential equation

$$z'(t) = \xi - \lambda z(t) + \sigma(f(t) - f(c)), \quad t \geq c, z(c) = 0. \tag{6.8}$$

Proof: Let y be a continuous solution of (6.7). For z as defined as above we get with the fundamental theorem of calculus and (6.7)

$$z'(t) = y(t) = \xi - \lambda \int_c^t y(s)ds + \sigma(f(t) - f(c)) = \xi - \lambda z(t) + \sigma(f(t) - f(c))$$

and $z(c) = 0$. On the other hand if z solves (6.8) we have

$$y(t) = z'(t) = \xi - \lambda z(t) + \sigma(f(t) - f(c)) = \xi - \lambda \int_c^t y(s)ds + \sigma(f(t) - f(c)),$$

with $y(c) = z'(c) = \xi$. □

6.2.2 Proposition

For $c, \xi \in \mathbb{R}$, $\lambda, \sigma > 0$ and $f \in C^0(\mathbb{R})$, the unique continuous solution of (6.8) is given by

$$z(t) = e^{-\lambda t} \int_c^t e^{\lambda u}(\xi + \sigma(f(u) - f(c)))du, \quad t \geq c. \tag{6.9}$$

Proof: The differential equation (6.8) satisfies obviously a Lipschitz condition, hence has a unique continuous solution. Furthermore we have for z defined as above

$$\begin{aligned}
z'(t) &= -\lambda e^{-\lambda t}\int_c^t e^{\lambda u}(\xi+\sigma(f(t)-f(c)))du + e^{-\lambda t}e^{\lambda t}(\xi+\sigma(f(t)-f(c))) \\
&= -\lambda z(t)+\xi+\sigma(f(t)-f(c)).
\end{aligned}$$

□

Now we will have a first look at pathwise SDEs.

6.2.3 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $\xi \in L^0$, $d \in (0, \frac{1}{2})$, $c \in \mathbb{R}$ and $\lambda, \sigma > 0$. For almost all $\omega \in \Omega$ the unique continuous solution of

$$y(t) = \xi(\omega) - \lambda\int_c^t y(s)ds + \sigma(L_t^d(\omega) - L_c^d(\omega)), \quad t \geq c, \tag{6.10}$$

is given by

$$y(t) = e^{-\lambda t}\left\{e^{\lambda c}\xi(\omega) + \sigma\int_c^t e^{\lambda u}dL_u^d(\omega)\right\}, \quad t \geq c. \tag{6.11}$$

Proof: Let $\omega \in \Omega - N$ where N is defined as in 6.1.1. According to Theorem 6.2.1 we solve the corresponding linear differential equation

$$z'(t) = -\lambda z(t) + \xi(\omega) + \sigma(L_t^d(\omega) - L_c^d(\omega)), \quad t \geq c, z(c) = 0,$$

with $z(t) := \int_c^t y(s)ds$, $t \geq c$. Proposition 6.2.2 states that

$$z(t) = e^{-\lambda t}\int_c^t e^{\lambda u}(\xi(\omega) + \sigma(L_u^d(\omega) - L_c^d(\omega)))du, \quad t \geq c,$$

is the unique continuous solution of this differential equation. To obtain $y(t)$ we differentiate z

$$\begin{aligned}
y(t) &= \frac{\partial}{\partial t}\left\{e^{-\lambda t}\int_c^t e^{\lambda u}(\xi(\omega) + \sigma(L_u^d(\omega) - L_c^d(\omega)))du\right\} \\
&= -\lambda e^{-\lambda t}\int_c^t e^{\lambda u}(\xi(\omega) + \sigma(L_u^d(\omega) - L_c^d(\omega)))du \\
&+ e^{-\lambda t}e^{\lambda t}(\xi(\omega) + \sigma(L_t^d(\omega) - L_c^d(\omega))) \\
&= -\lambda e^{-\lambda t}\int_c^t e^{\lambda u}(\xi(\omega) + \sigma(L_u^d(\omega) - L_c^d(\omega)))du \\
&+ \xi(\omega) + \sigma(L_t^d(\omega) - L_c^d(\omega)),
\end{aligned} \tag{6.12}$$

where the first term in the right hand side equals

$$
\begin{aligned}
& -\lambda e^{-\lambda t}\int_c^t e^{\lambda u}(\xi(\omega)+\sigma(L_u^d(\omega)-L_c^d(\omega)))du \\
= \; & -\lambda e^{-\lambda t}\left\{\int_c^t e^{\lambda u}\xi(\omega)du+\int_c^t e^{\lambda u}\sigma(L_u^d(\omega)-L_c^d(\omega))du\right\} \\
= \; & -\lambda e^{-\lambda t}\left\{\left[\frac{e^{\lambda u}}{\lambda}\right]_c^t\xi(\omega)+\int_c^t e^{\lambda u}\sigma L_u^d(\omega)du-\left[\frac{e^{\lambda u}}{\lambda}\right]_c^t L_c^d(\omega)\right\} \\
= \; & -\xi(\omega)+e^{\lambda(c-t)}\xi(\omega)-\lambda e^{-\lambda t}\int_c^t e^{\lambda u}\sigma L_u^d(\omega)du+\sigma L_c^d(\omega)-e^{\lambda(c-t)}\sigma L_c^d(\omega). \quad (6.13)
\end{aligned}
$$

The term with the integral on the right hand side of (6.13) equals by Theorem 6.1.1

$$
\begin{aligned}
& -e^{-\lambda t}\left\{e^{\lambda t}\sigma L_t^d(\omega)-e^{\lambda c}\sigma L_c^d(\omega)-\sigma\int_c^t e^{\lambda u}dL_u^d(\omega)\right\} \\
= \; & -\sigma L_t^d(\omega)+e^{\lambda(c-t)}\sigma L_c^d(\omega)+\sigma e^{-\lambda t}\int_c^t e^{\lambda u}dL_u^d(\omega). \quad (6.14)
\end{aligned}
$$

Putting (6.12), (6.13) and (6.14) together we finally obtain

$$
y(t)=e^{-\lambda t}\left\{e^{\lambda c}\xi(\omega)+\sigma\int_c^t e^{\lambda u}dL_u^d(\omega)\right\}, \quad t\geq c.
$$

□

Now we are able to state and prove a connection between the FLP-driven Langevin equations (6.1) and FLOUPs. Recall that we work outside the null set N. Furthermore equations and statements will from now on be understood in the almost surely sense if not said otherwise.

6.2.4 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d\in(0,\frac{1}{2})$ and $\lambda>0$. Then the unique stationary pathwise solution of

$$
\mathcal{L}_t^{d,\lambda}-\mathcal{L}_s^{d,\lambda}=-\lambda\int_s^t\mathcal{L}_u^{d,\lambda}du+L_t^d-L_s^d, \quad s\leq t, \quad (6.15)
$$

is given a.s. by the corresponding FLOUP

$$
\mathcal{L}_t^{d,\lambda}=\int_{-\infty}^t e^{-\lambda(t-u)}dL_u^d, \quad t\in\mathbb{R}.
$$

Proof: From Theorem 6.1.1 we know that $\int_{-\infty}^{t} e^{-\lambda(t-u)} dL_u^d$ exists for all $t \in \mathbb{R}$ almost surely as a Riemann-Stieltjes integral. We fix $s \in \mathbb{R}$ and consider the pathwise SDE

$$\mathcal{L}_t^{d,\lambda} = \xi_s - \lambda \int_s^t \mathcal{L}_u^{d,\lambda} du + L_t^d - L_s^d, \quad s \le t, \tag{6.16}$$

where $\xi_s := \int_{-\infty}^{s} e^{-\lambda(s-u)} dL_u^d$. Obviously we have $\xi \in L^0$. Setting $\sigma = 1$ Theorem 6.2.3 ensures that

$$\mathcal{L}_t^{d,\lambda} = e^{-\lambda t} \left\{ e^{\lambda s} \int_{-\infty}^{s} e^{-\lambda(s-u)} dL_u^d + \int_s^t e^{\lambda u} dL_u^d \right\} = \int_{-\infty}^{t} e^{-\lambda(t-u)} dL_u^d, \quad t \in \mathbb{R},$$

is the unique pathwise solution of (6.16) and therefore by Lemma 6.1.3 a stationary solution of (6.15).
On the other hand let $(X_t)_{t\in\mathbb{R}}$ be a stationary solution of (6.15). We will show that then $(X_t)_{t\in\mathbb{R}} = (\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ holds for almost all $\omega \in \Omega$. Set $A := \left\{ \omega \in \Omega : (X_t(\omega))_{t\in\mathbb{R}} \neq (\mathcal{L}_t^{d,\lambda}(\omega))_{t\in\mathbb{R}} \right\}$ and assume $P(A) > 0$. For $\omega \in A$ fix $t \in \mathbb{R}$ with $X_t(\omega) \neq \mathcal{L}_t^{d,\lambda}(\omega)$. Then we have for $\omega \in A$ and $s \le t$ by Theorem 6.2.3

$$\begin{aligned} 0 \neq \left| X_t - \mathcal{L}_t^{d,\lambda} \right| &= \left| e^{-\lambda t} \left\{ e^{\lambda s} X_s + \int_s^t e^{\lambda u} dL_u^d \right\} - \int_{-\infty}^{t} e^{-\lambda(t-v)} dL_v^d \right| \\ &= \left| e^{-\lambda(t-s)} X_s + \int_s^t e^{-\lambda(t-u)} dL_u^d - \int_{-\infty}^{t} e^{-\lambda(t-v)} dL_v^d \right| \\ &= \left| e^{-\lambda(t-s)} X_s - \int_{-\infty}^{s} e^{-\lambda(t-u)} dL_u^d \right| \\ &= \left| e^{-\lambda t} \right| \left| e^{\lambda s} X_s - \int_{-\infty}^{s} e^{\lambda u} dL_u^d \right| \\ &= \underbrace{\left| e^{-\lambda t} \right|}_{\le C} \underbrace{\left| e^{\lambda s} \right|}_{\to 0} \left| X_s - \mathcal{L}_s^{d,\lambda} \right| \qquad \text{for} \quad s \to -\infty \end{aligned}$$

where we surpressed the chosen ω for simplicity. Hence $\left| X_s(\omega) - \mathcal{L}_s^{d,\lambda}(\omega) \right| \to \infty$ for $s \to -\infty$. Therefore on A we have $\left| X_t - \mathcal{L}_t^{d,\lambda} \right| \to \infty$ for $t \to -\infty$. For a given $K > 0$ we define ω-wise the random number $T : A \longrightarrow \mathbb{R}$ with $|X_t - \mathcal{L}_t^{d,\lambda}| \ge \frac{K}{P(A)}$ for $t \le T$ on A. Hence,

$$\begin{aligned} E|X_t - \mathcal{L}_t^{d,\lambda}| &\ge E\left\{ |X_t - \mathcal{L}_t^{d,\lambda}| 1_{\{t \le T\}} 1_A \right\} + E\left\{ |X_t - \mathcal{L}_t^{d,\lambda}| 1_{\{t > T\}} 1_A \right\} \\ &\ge \frac{K}{P(A)} P(\{t \le T\} \cap A). \end{aligned} \tag{6.17}$$

Further we know that $\{t \leq T\} \cap A \subseteq \{s \leq T\} \cap A$ for $s \leq t$. Choosing a sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers with $\lim_{n\to\infty} t_n = -\infty$ we get by continuity of P

$$\lim_{n\to\infty} P(\{t_n \leq T\} \cap A) = P\left(\bigcup_{n\in\mathbb{N}} \{t_n \leq T\} \cap A\right) = P(A) \tag{6.18}$$

Putting this together with (6.17) we arrive at

$$\lim_{n\to\infty} E|X_{t_n} - \mathcal{L}_{t_n}^{d,\lambda}| \geq \lim_{n\to\infty} \frac{K}{P(A)} P(\{t_n \leq T\} \cap A) = K \tag{6.19}$$

Hence $\lim_{n\to\infty} E|X_{t_n} - \mathcal{L}_{t_n}^{d,\lambda}| = \infty$. However we have now

$$E|X_{t_n}| = E\left|X_{t_n} - \mathcal{L}_{t_n}^{d,\lambda} - (-\mathcal{L}_{t_n}^{d,\lambda})\right| \geq E|X_{t_n} - \mathcal{L}_{t_n}^{d,\lambda}| - \underbrace{E|\mathcal{L}_{t_n}^{d,\lambda}|}_{\text{constant}}$$

Thus $\lim_{n\to\infty} E|X_{t_n}| = \infty$ and by stationary $E|X_t| = \infty$ for all $t \in \mathbb{R}$. However we also have for fixed $s \leq t$

$$\lim_{t\to\infty} (X_t - \mathcal{L}_t^{d,\lambda}) = \lim_{t\to\infty} e^{-\lambda t} \left\{ e^{\lambda s} X_s - \int_{-\infty}^{s} e^{\lambda u} dL_u^d \right\} = 0 \quad \text{a.s.} \tag{6.20}$$

Hence by stationary $X_t \stackrel{d}{=} \mathcal{L}_t^{d,\lambda}$ but $E\left|\mathcal{L}_t^{d,\lambda}\right| < \infty$ which is a contradiction and thus we get $P(A) = 0$.

□

6.2.5 Remark

We will sometimes also write $d\mathcal{L}_t^{d,\lambda} = -\lambda \mathcal{L}_t^{d,\lambda} dt + dL_t^d$ instead of (6.15).

For later purposes we will now introduce the Ornstein-Uhlenbeck Operator.

6.2.6 Definition

Let $L^d = (L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ the corresponding FLOUP. We define the so called *Ornstein-Uhlenbeck Operator* by

$$\begin{aligned} \mathfrak{L}^\lambda((L^d, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathcal{C}^0(\mathbb{R}) \\ (\tau, z) &\longmapsto \mathcal{L}_t^{d,\lambda} - e^{-\lambda(t-\tau)} \mathcal{L}_\tau^{d,\lambda} + e^{-\lambda(t-\tau)} z. \end{aligned} \tag{6.21}$$

The next Theorem provides us with a connection to stochastic differential equations:

6.2.7 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ the corresponding FLOUP. For a continuous process $(l_t)_{t\in\mathbb{R}}$ the identity $l_t = \mathfrak{L}_t^\lambda(L^d, \tau, l_\tau)$ holds for all $\tau, t \in \mathbb{R}$ if and only if

$$l_t - l_s = -\lambda \int_s^t l_u du + L_t^d - L_s^d, \quad s \le t. \tag{6.22}$$

Proof: First we assume that $l_t = \mathfrak{L}_t^\lambda(L^d, \tau, l_\tau)$ holds for all $\tau, t \in \mathbb{R}$. Then we can easily compute for every fixed $\tau \in \mathbb{R}$ with (6.21)

$$\begin{aligned} l_t - l_s &= \mathfrak{L}_t^\lambda(L^d, \tau, l_\tau) - \mathfrak{L}_s^\lambda(L^d, \tau, l_\tau) \\ &= \mathcal{L}_t^d - e^{-\lambda(t-\tau)}\mathcal{L}_\tau^d + e^{-\lambda(t-\tau)} l_\tau - \mathcal{L}_s^d - e^{-\lambda(s-\tau)}\mathcal{L}_\tau^d + e^{-\lambda(s-\tau)} l_\tau \\ &= \mathcal{L}_t^d - \mathcal{L}_s^d - \mathcal{L}_\tau^d(e^{-\lambda(t-\tau)} - e^{-\lambda(s-\tau)}) + l_\tau(e^{-\lambda(t-\tau)} - e^{-\lambda(s-\tau)}) \\ &= -\lambda \int_s^t \mathcal{L}_u^{d,\gamma} du + L_t^d - L_s^d - \mathcal{L}_\tau^d \left(\int_s^t -\lambda e^{-\lambda(u-\tau)} du \right) + l_\tau \left(\int_s^t -\lambda e^{-\lambda(u-\tau)} du \right) \\ &= -\lambda \int_s^t \mathcal{L}_u^d - e^{-\lambda(u-\tau)}\mathcal{L}_\tau^d + e^{-\lambda(u-\tau)} l_\tau du + L_t^d - L_s^d \\ &= -\lambda \int_s^t l_u du + L_t^d - L_s^d, \quad s \le t. \end{aligned}$$

On the other hand if we have a solution l of (6.22), then for every fixed $\tau \in \mathbb{R}$ and all $t \ge \tau$ the process l solves also the SDE

$$l_t - l_\tau = -\lambda \int_\tau^t l_u du + L_t^d - L_\tau^d, \quad \tau \le t. \tag{6.23}$$

With Theorem 6.2.3 we conclude that for all $t \ge \tau$

$$\begin{aligned} l_t &= e^{-\lambda t} \left\{ e^{\lambda\tau} l_\tau + \int_\tau^t e^{\lambda u} dL_u^d \right\} \\ &= e^{-\lambda(t-\tau)} l_\tau + \int_\tau^t e^{-\lambda(t-u)} dL_u^d + \int_{-\infty}^\tau e^{-\lambda(t-u)} dL_u^d - \int_{-\infty}^\tau e^{-\lambda(t-u)} dL_u^d \\ &= e^{-\lambda(t-\tau)} l_\tau + \mathcal{L}_t^d - e^{-\lambda(t-\tau)} \int_{-\infty}^\tau e^{-\lambda(\tau-u)} dL_u^d \\ &= e^{-\lambda(t-\tau)} l_\tau + \mathcal{L}_t^d - e^{-\lambda(t-\tau)} \mathcal{L}_\tau^d. \end{aligned}$$

For $t \le \tau$ we get the same result by interchanging t and τ in the above calculation and multiplying each side with (-1). $\square$

6.3 Second Order Properties

We will now have a closer look at the second order structure of FLOUPs and show that the increments exhibit long range dependence.

6.3.1 Lemma

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $-\infty \leq a < b \leq c < f < \infty$. Then we have

$$E\left\{\int_a^b e^{\lambda t} dL_t^d \int_c^f e^{\lambda s} dL_s^d\right\} = \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \int_a^b e^{\lambda t}\left(\int_c^f e^{\lambda s}(s-t)^{2d-1} ds\right) dt \tag{6.24}$$

Proof: We first prove the result for $b = 0 = c$. There we get by using partial integration

$$\begin{aligned}
& E\left\{\int_a^0 e^{\lambda t} dL_t^d \int_0^f e^{\lambda s} dL_s^d\right\} \\
= \quad & E\left\{\left(-e^{\lambda a} L_a^d - \lambda \int_a^0 e^{\lambda t} L_t^d dt\right)\left(e^{\lambda e} L_e^d - \lambda \int_0^f e^{\lambda s} L_s^d ds\right)\right\} \\
= \quad & E\left\{-e^{\lambda a} e^{\lambda e} L_a^d L_e^d + \lambda e^{\lambda a} L_a^d \int_0^f e^{\lambda s} L_s^d ds - \lambda e^{\lambda e} L_e^d \int_a^0 e^{\lambda t} L_t^d dt\right. \\
& \left. +\lambda^2 \int_a^0 e^{\lambda t} L_t^d dt \int_0^f e^{\lambda s} L_s^d ds\right\}.
\end{aligned}$$

Using now Fubini and Theorem 5.2.1

$$\begin{aligned}
= \quad & \left\{-e^{\lambda a} e^{\lambda e}[(-a)^{2d+1} + e^{2d+1} - (e-a)^{2d+1}]\right. \\
& +\lambda e^{\lambda a} \int_0^f e^{\lambda s}[(-a)^{2d+1} + s^{2d+1} - (s-a)^{2d+1}] ds \\
& -\lambda e^{\lambda e} \int_a^0 e^{\lambda t}[(-t)^{2d+1} + e^{2d+1} - (e-t)^{2d+1}] dt \\
& \left. +\lambda^2 \int_0^f e^{\lambda s}\left(\int_a^0 e^{\lambda t}[(-t)^{2d+1} + s^{2d+1} - (s-t)^{2d+1}] dt\right) ds\right\} \cdot \frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}.
\end{aligned} \tag{6.25}$$

Again we apply partial integration with respect to t and get

$$\begin{aligned}
&-\lambda e^{\lambda f}\int_a^0 e^{\lambda t}[(-t)^{2d+1}+f^{2d+1}-(f-t)^{2d+1}]dt \\
= \;& -\lambda e^{\lambda f}\left\{\left[-\frac{1}{\lambda}e^{\lambda t}[(-t)^{2d+1}+f^{2d+1}-(f-t)^{2d+1}]\right]_a^0\right. \\
&\left.-\frac{1}{\lambda}\int_a^0 e^{\lambda t}[-(2d+1)(-a)^{2d}+(2d+1)(f-a)^{2d}]dt\right\} \\
= \;& e^{\lambda f}\left\{e^{-\lambda a}[(-a)^{2d+1}+f^{2d+1}-(f-a)^{2d+1}]\right. \\
&\left.-(2d+1)\int_a^0 e^{\lambda t}[(-t)^{2d}+(f-t)^{2d}]dt\right\}. \qquad (6.26)
\end{aligned}$$

With partial integration with respect to s we get by the same arguments

$$\begin{aligned}
&\lambda^2\int_0^f e^{\lambda s}\left(\int_a^0 e^{\lambda t}[(-t)^{2d+1}+s^{2d+1}-(s-t)^{2d+1}]dt\right)ds \\
= \;& -\lambda e^{\lambda a}\int_0^f e^{\lambda s}[(-a)^{2d+1}+s^{2d+1}-(s-a)^{2d+1}]ds \\
&+\lambda(2d+1)\int_0^f e^{\lambda s}\left(\int_a^0 e^{\lambda t}[(-t)^{2d}+(s-t)^{2d}]dt\right)ds. \qquad (6.27)
\end{aligned}$$

Putting (6.25), (6.26) and (6.27) together we finally get

$$\begin{aligned}
&E\left\{\int_a^0 e^{\lambda t}dL_t^d\int_0^f e^{\lambda s}dL_s^d\right\} \\
= \;& (2d+1)\frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\left\{-e^{\lambda f}\int_a^0 e^{\lambda t}[(-t)^{2d}+(f-t)^{2d}]dt\right. \\
&\left.+\lambda\int_0^f e^{\lambda s}\left(\int_a^0 e^{\lambda t}[(-t)^{2d}+(s-t)^{2d}]dt\right)ds\right\},
\end{aligned}$$

which simplifies again by partial integraton with respect to s to

$$\begin{aligned}
&E\left\{\int_a^0 e^{\lambda t}dL_t^d\int_0^f e^{\lambda s}dL_s^d\right\} \\
= \;& \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\int_a^0 e^{\lambda t}\left(\int_0^f e^{\lambda s}(s-t)^{2d-1}ds\right)dt.
\end{aligned}$$

The next case we will have a look at is $b = 0 < c$. From our previous calculations we get

$$
\begin{aligned}
& E\left\{\int_a^0 e^{\lambda t} dL_t^d \int_c^f e^{\lambda s} dL_s^d\right\} \\
= \; & E\left\{\int_a^0 e^{\lambda t} dL_t^d \int_0^f e^{\lambda s} dL_s^d - \int_a^0 e^{\lambda t} dL_t^d \int_0^c e^{\lambda s} dL_s^d\right\} \\
= \; & \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\left[\int_a^0 e^{\lambda t}\left(\int_0^f e^{\lambda s}(s-t)^{2d-1} ds\right) dt\right. \\
& \left. - \int_a^0 e^{\lambda t}\left(\int_0^c e^{\lambda s}(s-t)^{2d-1} ds\right) dt\right] \\
= \; & \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\int_a^0 e^{\lambda t}\left(\int_c^f e^{\lambda s}(s-t)^{2d-1} ds\right) dt.
\end{aligned}
$$

Finally for general $-\infty \leq a < b \leq c < f < \infty$ we recall that $(L_t^d)t \in \mathbb{R}$ has stationary increments and hence the new process $\widetilde{L}_t^d := L_{t+b}^d - L_b^d$, $t \in \mathbb{R}$, has the same distribution as $(L_t^d)_{t\in\mathbb{R}}$. In the end we get

$$
\begin{aligned}
& E\left\{\int_a^b e^{\lambda t} dL_t^d \int_c^f e^{\lambda s} dL_s^d\right\} \\
= \; & E\left\{\int_{a-b}^0 e^{\lambda(x+b)} d\widetilde{L}_x^d \int_{c-b}^{f-b} e^{\lambda(y+b)} d\widetilde{L}_y^d\right\} \\
= \; & \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\int_{a-b}^0 e^{\lambda(x+b)}\left(\int_{c-b}^{f-b} e^{\lambda(y+b)}(s-t)^{2d-1} dy\right) dx \\
= \; & \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))}\int_a^b e^{\lambda t}\left(\int_c^f e^{\lambda s}(s-t)^{2d-1} ds\right) dt.
\end{aligned}
$$

□

The next Lemma can be also found in standard analysis literature.

6.3.2 Lemma

Let $\alpha < 0$ and $N \in \mathbb{N}_0$, then we have

$$
e^x \int_x^\infty e^{-y} y^\alpha dy = x^\alpha + \sum_{n=1}^N \left(\prod_{k=0}^{n-1}(\alpha - k)\right) x^{\alpha-n} + O(x^{\alpha-N-1}), \quad \text{as } x \to \infty, \tag{6.28}
$$

and

$$
e^{-x} \int_1^x e^{y} y^\alpha dy = x^\alpha + \sum_{n=1}^N (-1)^n \left(\prod_{k=0}^{n-1}(\alpha - k)\right) x^{\alpha-n} + O(x^{\alpha-N-1}), \quad \text{as } x \to \infty \tag{6.29}
$$

Proof: By partial integration we get for general $\beta < 0$

$$\begin{aligned} e^x \int_x^\infty e^{-y} y^\beta dy &= e^x \left(e^{-x} x^\beta + \beta \int_x^\infty e^{-y} y^{\beta-1} dy \right) \quad \text{and} \\ e^{-x} \int_1^x e^{y} y^\beta dy &= e^x \left(e^{x} x^\beta - e - \beta \int_1^x e^{y} y^{\beta-1} dy \right). \end{aligned}$$

Applying this iteratively we get

$$\begin{aligned} e^x \int_x^\infty e^{-y} y^\alpha dy &= e^x \left(e^{-x} x^\alpha + \alpha \int_x^\infty e^{-y} y^{\alpha-1} dy \right) \\ &= x^\alpha + \alpha e^x \int_x^\infty e^{-y} y^{\alpha-1} dy \\ &= x^\alpha + \alpha e^x \left(e^{-x} x^{\alpha-1} + (\alpha-1) \int_x^\infty e^{-y} y^{\alpha-2} dy \right) \\ &= x^\alpha + \sum_{n=1}^{N} \left(\prod_{k=0}^{n-1} (\alpha-k) \right) x^{\alpha-n} + e^x \prod_{k=0}^{N} (\alpha-k) \int_x^\infty e^{-y} y^{\alpha-N-1} dy. \end{aligned}$$

For the last term we have the following inequality

$$e^x \prod_{k=0}^{N} (\alpha-k) \int_x^\infty e^{-y} y^{\alpha-N-1} dy \le e^x \prod_{k=0}^{N} (\alpha-k) \int_x^\infty e^{-y} x^{\alpha-N-1} dy = \prod_{k=0}^{N} (\alpha-k) x^{\beta-N-1},$$

which proves the first part of the lemma. For the second one we use similar arguments to get

$$\begin{aligned} e^{-x} \int_1^x e^{y} y^\alpha dy &= x^\alpha + \sum_{n=1}^{N} (-1)^n \left(\prod_{k=0}^{n-1} (\alpha-k) \right) x^{\alpha-n} - e^{-x} e \left(1 + \sum_{n=1}^{N} (-1)^n \prod_{k=0}^{n-1} (\alpha-k) \right) \\ &\quad - e^{-x} (-1)^N \prod_{k=0}^{N} (\alpha-k) \int_1^x e^{y} y^{\alpha-N-1} dy \end{aligned}$$

and we observe

$$\begin{aligned} e^{-x} \int_1^x e^{y} y^{\alpha-N-1} dy &\le e^{-x} \left(\int_1^{\frac{x}{2}} e^y dy + \int_{\frac{x}{2}}^x e^y \left(\frac{x}{2} \right)^{\alpha-N-1} dy \right) \\ &\le e^{-x} \left(e^{\frac{x}{2}} + e^x \left(\frac{x}{2} \right)^{\alpha-N-1} \right) \\ &= e^{-\frac{x}{2}} + \left(\frac{x}{2} \right)^{\alpha-N-1} \end{aligned}$$

which proves the second assertion. □

6.3.3 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ a FLOUP. Then for $N \in \mathbb{N}_0$ and for fixed $t \in \mathbb{R}$ we have

$$\begin{aligned}\mathrm{Cov}(\mathcal{L}_t^{d,\lambda}, \mathcal{L}_{t+s}^{d,\lambda}) &= \frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \sum_{n=1}^{N} \left(\prod_{k=0}^{2n-1} (2d+1-k) \right) \lambda^{-2n} s^{2d+1-2n} + O(s^{2d-2N-1}) \\ &\quad \text{as } s \to \infty. \end{aligned} \tag{6.30}$$

Proof: By the stationarity of $\mathcal{L}_t^{d,\lambda}$ and the Lemma 6.3.1 we have

$$\begin{aligned}
\mathrm{Cov}(\mathcal{L}_t^{d,\lambda}, \mathcal{L}_{t+s}^{d,\lambda}) &= \mathrm{Cov}(\mathcal{L}_0^{d,\lambda}, \mathcal{L}_s^{d,\lambda}) = E\left\{ \int_{-\infty}^{0} e^{\lambda u} dL_u^d \int_{-\infty}^{s} e^{-\lambda(s-v)} dL_v^d \right\} \\
&= e^{-\lambda s} E\left\{ \int_{-\infty}^{0} e^{\lambda u} dL_u^d \int_{-\infty}^{s} e^{\lambda v} dL_v^d \right\} \\
&= e^{-\lambda s} E\left\{ \int_{-\infty}^{0} e^{\lambda u} dL_u^d \left(\int_{-\infty}^{\frac{1}{\lambda}} e^{\lambda v} dL_v^d + \int_{\frac{1}{\lambda}}^{s} e^{\lambda v} dL_v^d \right) \right\} \\
&= e^{-\lambda s} E\left\{ \int_{-\infty}^{0} e^{\lambda u} dL_u^d \int_{-\infty}^{\frac{1}{\lambda}} e^{\lambda v} dL_v^d \right\} \\
&\quad + e^{-\lambda s} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \int_{-\infty}^{0} e^{\lambda u} \left(\int_{\frac{1}{\lambda}}^{s} e^{\lambda v} (v-u)^{2d-1} dv \right) du.
\end{aligned}$$

By setting $x = \lambda u$ and $y = \lambda v$ we get

$$\begin{aligned}
&= \frac{e^{-\lambda s}}{\lambda^2} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \int_{-\infty}^{0} e^{x} \left(\int_{1}^{\lambda s} e^{y} \left(\frac{y}{\lambda} - \frac{x}{\lambda} \right)^{2d-1} dy \right) dx \\
&\quad + O(e^{-\lambda s}) \\
&= \frac{e^{-\lambda s}}{\lambda^{2d+1}} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \int_{-\infty}^{0} \left(\int_{1}^{\lambda s} e^{y+x} (y-x)^{2d-1} dy \right) dx \\
&\quad + O(e^{-\lambda s}).
\end{aligned}$$

Changing variables again by $w = y - x$ and $z = y + x$ brings us to

$$\begin{aligned}
&= \frac{e^{-\lambda s}}{\lambda^{2d+1}} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \int_{-\infty}^{0} \left(\int_{1}^{\lambda s} e^z w^{2d-1} dy \right) dx \\
&\quad + O(e^{-\lambda s}) \\
&= \frac{e^{-\lambda s}}{2\lambda^{2d+1}} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \left\{ \int_{1}^{\lambda s} w^{2d-1} \left(\int_{2-w}^{w} e^z dz \right) dw \right. \\
&\quad \left. + \int_{\lambda s}^{\infty} w^{2d-1} \left(\int_{2-y}^{2\lambda s - y} e^z dz \right) dw \right\} + O(e^{-\lambda s}) \\
&= \frac{e^{-\lambda s}}{2\lambda^{2d+1}} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \left\{ \int_{1}^{\lambda s} w^{2d-1} \left(\int_{2-w}^{w} e^z dz \right) dw \right. \\
&\quad \left. + \int_{\lambda s}^{\infty} w^{2d-1} \left(\int_{2-y}^{2\lambda s - y} e^z dz \right) dw \right\} + O(e^{-\lambda s}).
\end{aligned}$$

Solving the integrals leads us to

$$\begin{aligned}
&= \frac{e^{-\lambda s}}{2\lambda^{2d+1}} \frac{d(2d+1)E[L(1)]^2}{\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \left\{ \int_{1}^{\lambda s} e^w w^{2d-1} dw - \int_{1}^{\lambda s} e^{2-w} w^{2d-1} dw \right. \\
&\quad \left. + \int_{\lambda s}^{\infty} e^{2\lambda s - w} w^{2d-1} dw - \int_{\lambda s}^{\infty} e^{2-w} w^{2d-1} dw \right\} + O(e^{-\lambda s}) \\
&= \lambda^{-(2d+1)} \frac{d(2d+1)E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \left\{ e^{-\lambda s} \int_{1}^{\lambda s} e^w w^{2d-1} dw + e^{-\lambda s} \int_{\lambda s}^{\infty} e^{2\lambda s - w} w^{2d-1} dw \right. \\
&\quad \left. - e^{-\lambda s} \int_{1}^{\infty} e^{2-w} w^{2d-1} dw \right\} + O(e^{-\lambda s}).
\end{aligned}$$

The last term we can again draw into $O(e^{-\lambda s})$ and we arrive at

$$= \lambda^{-(2d+1)} \frac{d(2d+1)E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \left\{ e^{-\lambda s} \int_{1}^{\lambda s} e^w w^{2d-1} dw + e^{\lambda s} \int_{\lambda s}^{\infty} e^{-w} w^{2d-1} dw \right\} + O(e^{-\lambda s}).$$

Using Lemma (6.3.2) we find that for $s \to \infty$

$$e^{-\lambda s} \int_{1}^{\lambda s} e^w w^{2d-1} dw = (\lambda s)^{2d-1} + \sum_{n=1}^{2N-1} (-1)^n \left(\prod_{k=0}^{n-1} (2d-1-k) \right) (\lambda s)^{2d-1-n} + O((\lambda s)^{2d-2N-1}) \tag{6.31}$$

and

$$e^{\lambda s} \int_{\lambda s}^{\infty} e^{-w} w^{2d-1} dw = (\lambda s)^{2d-1} + \sum_{n=1}^{2N-1} \left(\prod_{k=0}^{n-1} (2d-1-k) \right) (\lambda s)^{2d-1-n} + O((\lambda s)^{2d-2N-1}). \tag{6.32}$$

Adding (6.31) and (6.32) we obtain

$$\begin{aligned} 2(\lambda s)^{2d-1} + & \sum_{n=1}^{2N-1} (-1)^n \left(\prod_{k=0}^{n-1}(2d-1-k)\right)(\lambda s)^{2d-1-n} + O((\lambda s)^{2d-2N-1}) \\ + & \sum_{n=1}^{2N-1} \left(\prod_{k=0}^{n-1}(2d-1-k)\right)(\lambda s)^{2d-1-n} + O((\lambda s)^{2d-2N-1}) \\ = 2(\lambda s)^{2d-1} + & \sum_{n=1}^{N-1} \left(\prod_{k=0}^{2n-1}(2d-1-k)\right) 2(\lambda s)^{2d-1-2n} + O((\lambda s)^{2d-2N-1}) \\ = 2(\lambda s)^{2d-1} + & \sum_{n=2}^{N} \left(\prod_{k=0}^{2n-3}(2d-1-k)\right) 2(\lambda s)^{2d-1-2(n-1)} + O((\lambda s)^{2d-2N-1}) \\ = 2(\lambda s)^{2d-1} + & \sum_{n=2}^{N} \left(\prod_{k=2}^{2n-1}(2d+1-k)\right) 2(\lambda s)^{2d+1-2n} + O((\lambda s)^{2d-2N-1}). \quad (6.33) \end{aligned}$$

Putting (6.31) and (6.33) together we finally get for $s \to \infty$

$$= \frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \sum_{n=1}^{N} \left(\prod_{k=0}^{2n-1}(2d+1-k)\right) \lambda^{-2n} s^{2d+1-2n} + O(s^{2d-2N-1}).$$

□

6.3.4 Corollary

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP, $d \in (0, \frac{1}{2})$, $\lambda > 0$ and $(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ a FLOUP. Let further $h > 0$ and consider $t, s \in \mathbb{R}$ with $s + h \leq t$ and $t - s = nh$ for a $n \in \mathbb{N}$. Then we have

$$\delta_d(n) := \mathrm{Cov}(\mathcal{L}_{t+h}^{d,\lambda} - \mathcal{L}_t^{d,\lambda}, \mathcal{L}_{s+h}^{d,\lambda} - \mathcal{L}_s^{d,\lambda}) \overset{n\to\infty}{\longrightarrow} 0 \qquad (6.34)$$

Moreover we have $\delta_d(n) > 0$ and

$$\sum_{n=1}^{\infty} \delta_d(n) = \infty \qquad (6.35)$$

At last the increments of FLOUPs exhibit long range dependence.

Proof: This is clear because the increments have a similar autocovariance function as the FLOUPs. □

7 Fractional Integral Equations

In this Chapter we want to consider stochastic differential equations or integral equations where the driving process is a FLP. We will follow here closely Buchmann and Klüppelberg [2] which states this theory already for FBM.

7.1 State Space Transforms and Proper Triples

We are going to recall important definitions from Buchmann and Klüppelberg [2] first, mainly the concept of proper tripels.

7.1.1 Definition

A function $f : \mathbb{R} \to \mathbb{R}$ which is continuous and strictly increasing is called a *state space transform (SST)* and the open intervall $f(\mathbb{R})$ is called the *state space*.

7.1.2 Definition [Buchmann and Klüppelberg [2], Definition 3.2]

A triple (I, μ, σ) is called *proper* if and only if it satisfies the following properties:

(P1) $I = (a, b) \subseteq \mathbb{R}$ is an open interval where $-\infty \leq a < b \leq \infty$ and $\mu, \sigma \in \mathcal{C}^0(I)$.

(P2) There exists $\psi \in \mathcal{AC}(I)$ strictly decreasing with $\psi = \mu/\sigma$ on $I \setminus Z(\sigma)$ and

$$\lim_{x \nearrow b} \psi(x) = -\lim_{x \searrow a} \psi(x) = -\infty.$$

(P3) There exists $\lambda > 0$ such that $\sigma\psi' \equiv \lambda$ holds on I Lebesgue-a.e.

7.1.3 Remark [Buchmann and Klüppelberg [2], Remark 3.2]

$\psi : I \to \psi(I) = \mathbb{R}$ is by (P2) strictly decreasing and a.e. differentiable on I with $\psi' \leq 0$. Now (P3) implies that $Z(\sigma)$ and $Z(\psi')$ have Lebesgue measure zero. Also we have that σ is non-negative and $1/\sigma \in \mathcal{L}_C(I)$; $I \setminus Z(\sigma)$ is dense and open in I by (P1). Therefore the equality $\mu = \sigma\psi$ extends to I. It follows that ψ and λ are uniquely determined by μ and σ.

7.1.4 Definition

A proper triple (I, μ, σ) is called *strongly proper* if and only if the inverse function $\psi^{-1} : \mathbb{R} \to \psi^{-1}(\mathbb{R}) = I$ is differentiable and $(\psi^{-1})' \in \mathbf{Lip}(\mathbb{R})$.

7.1.5 Definition [Buchmann and Klüppelberg [2], Definition 3.3]

Let (I, μ, σ) be a proper triple.

(i) The interval I is called the *state space*.

(ii) The unique $\lambda > 0$ in (P3) is called the *friction coefficient (FC)* for (I, μ, σ).

(iii) The unique SST $f : \mathbb{R} \to I = f(\mathbb{R})$, $f(x) := \psi^{-1}(-\lambda x)$, is called the *SST* for (I, μ, σ).

7.1.6 Definition [Buchmann and Klüppelberg [2], Definition 3.4]

Let $(a, b) \subset \mathbb{R}$ and $h \in \mathcal{C}^0((a, b))$. $\xi \in (a, b)$ is called a *centre* for h if $h(x) \geq 0$ for $x \in (a, \xi)$, $h(y) \leq 0$ for $y \in (\xi, b)$ and $Z(h)$ has Lebesgue measure zero.

The following Lemmata concern properties of proper triples (I, μ, σ). The proofs are omitted and can be found in Buchmann and Klüppelberg [2].

7.1.7 Lemma [Buchmann and Klüppelberg [2], Lemma 3.1]

Let (I, μ, σ) be a proper triple. Then there exists a unique centre ξ for μ which satisfies the following properties:

$$Z(\psi) = \{\xi\}, \quad f(0) = \xi, \quad Z(\mu) = Z(\sigma) \cup \{\xi\}. \tag{7.1}$$

7.1.8 Lemma [Buchmann and Klüppelberg [2], Lemma 3.2]

Let (I, μ, σ) be a proper triple with the corresponding SST f. Furthermore let ξ be the centre for μ. Then the following properties are satisfied:

(i) $f \in \mathcal{C}^1(\mathbb{R})$ with $f' = \sigma \circ f$ and $f(0) = \xi$. Also $f^{-1} \in \mathcal{C}^1(I \setminus Z(\sigma))$ with $(f^{-1})'(x) = 1/\sigma(x)$ for all $x \in I \setminus Z(\sigma)$.

(ii) If $g \in \mathcal{C}^1(\mathbb{R})$ is a SST with state space I such that $g' = \sigma \circ g$ and $g(0) = \xi$, then

$$f(x) \leq g(x), \quad \text{for all } x \leq 0 \quad \text{and} \quad f(x) \geq g(x), \quad \text{for all } x \geq 0.$$

Furthermore $f = g$ if and only if $g^{-1} \in \mathcal{AC}(I)$.

7.1.9 Lemma

Let (I, μ, σ) be a proper triple with the corresponding SST f. Then we have

(i) (I, μ, σ) is strongly proper if and only if $f \in \mathcal{C}^1(\mathbb{R})$ and $f \in \mathbf{Lip}(\mathbb{R})$.

(ii) If $\sigma \in \mathbf{Lip}(I)$, then $f' \in \mathbf{Lip}(\mathbb{R})$ and (I, μ, σ) is strongly proper.

(iii) If (I, μ, σ) is strongly proper, then $(f^{-1})', \sigma \in \mathbf{Lip}(I \setminus Z(\sigma))$.

(iv) If $Z(\sigma) = \emptyset$, then (I, μ, σ) is strongly proper if and only if $\sigma \in \mathbf{Lip}(I)$.

Proof: The function $x \mapsto -\lambda x$ is a diffeomorphism and therefore $f \in \mathbf{Lip}(\mathbb{R})$ if and only if $\psi^{-1} \in \mathbf{Lip}(\mathbb{R})$. Hence (i) is immediate. For (ii) we recall from the preceding Lemma that $f' = \sigma \circ f$. Because f is differentiable, it is locally Lipschitz and the composition of two locally Lipschitz functions is again locally Lipschitz. Furthermore $f' = \sigma \circ f$ implies $f(Z(f')) = Z(\sigma)$ and $\sigma = f' \circ f^{-1}$ holds on I. To show (iii) one uses the following fact: If $h \in \mathcal{C}^1(\mathbb{R})$, $h' \in \mathbf{Lip}(\mathbb{R})$ with h strictly increasing, then we have $h^{-1} \in \mathcal{C}^1(\mathbf{im}(h) \setminus h(Z(h')))$ and $(h^{-1})' \in \mathbf{Lip}(\mathbf{im}(h) \setminus h(Z(h')))$. Using this on f we get with $f(\mathbb{R}) = I$ the assertion. (iv) follows from (ii) and (iii). $\square$

7.2 Pathwise Integral Equations - Finite Variation Case

Before looking at the more complex case of finite p-variation we want to consider integral equations where the driving FLP is of finite variation. Solving those equations is rather simple and gives a good overview over the methods we will also use in the more complex case later. Throughout these Section we will only consider FLP of finite variation. Calculations and Theorems are to be understood in the almost surely sense.

7.2.1 Definition

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of finite variation, $d \in (0, \frac{1}{2})$. Suppose that $I \subseteq \mathbb{R}$ is a non-empty interval and $\mu, \sigma \in \mathcal{C}^0(I)$. We refer to a stochastic process $(X_t)_{t\in\mathbb{R}}$ as a *pathwise solution* of

$$dX_t = \mu(X_t)dt + \sigma(X_t)dL_t^d, \tag{7.2}$$

if we have a.s. the following: $X \in \mathcal{C}^0(\mathbb{R})$ and $\mathbf{im}(X) \in I$ such that for $s \le t$

(S) the integral equation

$$X_t - X_s = \int_s^t \mu(X_u)du + \int_s^t \sigma(X_u)dL_u^d$$

holds.

The space of all solutions of (7.2) is denoted by $\mathcal{S}(I, \mu, \sigma, L^d)$.

7.2.2 Construction

We consider an integral equation as given in Definition 7.2.1. If we assume that the triple (I, μ, σ) is proper with SST f and FC λ we define the following function

$$\begin{aligned} X^{f,\lambda}(L^d, \cdot, \cdot) : \mathbb{R} \times I &\longrightarrow \mathcal{C}^0(\mathbb{R}) \\ (\tau, z) &\longmapsto f(\mathfrak{L}_t^\lambda(L^d, \tau, f^{-1}(z))) \end{aligned} \tag{7.3}$$

with $\mathfrak{L}_t^\lambda$ as in Definition 6.2.6. We remark that by Definition

$$X^{f,\lambda}(L^d, \tau, f(\mathcal{L}_\tau^{d,\lambda})) = f(\mathcal{L}^{d,\lambda}). \tag{7.4}$$

7.2.3 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of finite variation, $d \in (0, \frac{1}{2})$. If (I, μ, σ) is proper with SST f and FC $\lambda > 0$, then we have

$$\{X^{f,\lambda}(L^d, \tau, z) : \tau \in \mathbb{R}, z \in I\} \subseteq \mathcal{S}(I, \mu, \sigma, L^d). \tag{7.5}$$

Proof: First fix $\tau \in \mathbb{R}$ and $z \in I$. Now define the following processes

$$l_t := \mathfrak{L}_t^\lambda(L^d, \tau, f^{-1}(z)) \qquad t \in \mathbb{R}$$

$$Y_t := X_t^{f,\lambda}(L^d, z) \qquad t \in \mathbb{R}$$

We shall show that $Y \in \mathcal{S}(I, \mu, \sigma, (L_t^d)_{t\in\mathbb{R}})$. Obviously Y tales values in I. We also know from Lemma 7.1.8 that $f \in \mathcal{C}^1(\mathbb{R})$, therefore $Y = f \circ l \in \mathcal{C}^0(\mathbb{R})$ and l is of finite variation. With the change of variables formula for Riemann-Stieljes integrals we get

$$Y_t - Y_s = f(l_t) - f(l_s) = \int_s^t f'(l_u) dl_u, \quad s \le t. \tag{7.6}$$

As l is the solution of (6.22), we get

$$l_u = l_s - \lambda \int_s^u l_v dv + L_u^d - L_s^d, \quad s \le u. \tag{7.7}$$

The Riemann-Stieltjes integral is additive with respect to a sum of integrators of the Riemann-Stieltes integrals exist separately for each integrator. This is here the case because every term on the right-hand side has finite variation as a function of u and $f'(l_u)$ is continuous. Thus (7.6) and (7.7) imply

$$Y_t - Y_s = \int_s^t f'(l_u) d\left(-\lambda \int_s^u l_v dv\right) + \int_s^t f'(l_u) dL_u^d \qquad s \le t$$

Furthermore $-\lambda \int_s^u l_v dv$ is differentiable and $f'(l_u)l_u$ is continuous as a function of u. Hence we obtain by the density formula for Riemann Stieltjes integrals

$$Y_t - Y_s = -\lambda \int_s^t f'(l_u) l_u du + \int_s^t f'(l_u) dL_u^d, \quad s \le t.$$

Finally we know from Lemma 7.1.8(i) $f' = \sigma \circ f$, hence $\sigma \circ Y = f' \circ l \in \mathcal{C}^0(\mathbb{R})$. Furthermore by 7.1.5(iii) and Remark 7.1.3 we observe $\sigma f^{-1} = -\sigma\psi/\lambda = -\mu/\lambda$. At last we see that

$$\begin{aligned} Y_t - Y_s &= -\lambda \int_s^t f'(l_u) l_u du + \int_s^t f'(l_u) dL_u^d \\ &= -\lambda \int_s^t \underbrace{\sigma(f(l_u)) l_u}_{=\sigma(Y_u) f^{-1}(Y_u) = -\mu(Y_u)/\lambda} du + \int_s^t \sigma(f(l_u)) dL_u^d \\ &= \int_s^t \mu(Y_u) du + \int_s^t \sigma(Y_u) dL_u^d, \quad s \le t. \end{aligned}$$

Finally we have $Y \in \mathcal{S}(I, \mu, \sigma, L^d)$. $\square$

7.2.4 Theorem

Let $(L^d_t)_{t\in\mathbb{R}}$ be a FLP of finite variation, $d \in (0, \frac{1}{2})$, (I, μ, σ) be proper with SST f and FC $\lambda > 0$. Furthermore be $Z(\sigma) = \emptyset$. Then we have:

$$\{X^{f,\lambda}(L^d, \tau, z) : \tau \in \mathbb{R}, z \in I\} = \mathcal{S}(I, \mu, \sigma, L^d). \tag{7.8}$$

Proof: Because of $Z(\sigma) = \emptyset$, we know by Lemma 7.1.8 that $f \in \mathcal{C}^1(\mathbb{R})$. Furthermore we have $(f^{-1})'(x) = 1/\sigma(x)$ for all $x \in I$. Be $X \in \mathcal{S}(I, \mu, \sigma, L^d)$. From Definition 7.2.1 we know that $X \in \mathcal{C}^0(\mathbb{R})$ and therefore $(1/\sigma) \circ X$ is also continuous. By change of variable we get

$$f^{-1}(X_t) - f^{-1}(X_s) = \int_s^t f^{-1}(X_u) dX_u = \int_s^t \frac{1}{\sigma(X_u)} dX_u, \quad s \le t. \tag{7.9}$$

Also we know that X satisfies

$$X_u = X_s + \int_s^u \mu(X_v) dv + \int_s^u \sigma(X_v) dL^d_v. \tag{7.10}$$

Every term on the right-hand side of (7.10) is of finite variation as a function of u and so we get by putting (7.9) and (7.10) together and the density formula for Riemann-Stieltjes integrals

$$\begin{aligned}
f^{-1}(X_t) - f^{-1}(X_s) &= \int_s^t \frac{1}{\sigma(X_u)} d\left(X_s + \int_s^u \mu(X_v) dv + \int_s^u \sigma(X_v) dL^d_v\right) \\
&= \int_s^t \frac{1}{\sigma(X_u)} d\left(\int_s^u \mu(X_v) dv\right) + \int_s^t \frac{1}{\sigma(X_u)} d\left(\int_s^u \sigma(X_v) dL^d_v\right) \\
&= \int_s^t \frac{\mu(X_u)}{\sigma(X_u)} du + \int_s^t dL^d_u \\
&= \int_s^t \frac{\mu(X_u)}{\sigma(X_u)} du + L^d_t - L^d_s, \quad s \le t.
\end{aligned}$$

At last we know from (I, μ, σ) beeing proper that $\psi(x) = \frac{\mu(x)}{\sigma(x)}$ and $\psi(x) = -\lambda f^{-1}(x)$ for all $x \in I$ from the definition of corresponding the SST. Hence we arrive at

$$\begin{aligned}
f^{-1}(X_t) - f^{-1}(X_s) &= \int_s^t \frac{\mu(X_u)}{\sigma(X_u)} du + L^d_t - L^d_s \\
&= \int_s^t \psi(X_u) du + L^d_t - L^d_s \\
&= -\lambda \int_s^t f^{-1}(X_u) du + L^d_t - L^d_s, \quad s \le t.
\end{aligned}$$

Hence $f^{-1}(X)$ is a solution of (6.22). Fixing a $\tau \in \mathbb{R}$ we know from theorem 6.2.7 that $f^{-1}(X) = \mathfrak{L}^{\lambda}(L^d, \tau, f^{-1}(X_\tau))$ and finally $X = X^{f,\lambda}(L^d, \tau, X_\tau)$. □

If we are interested in particular in stationary solutions of our integral equation, then Theorem 7.2.4 seems not to be enough. A closer look at the construction and the proof of the Theorem however leads us to the following result.

7.2.5 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of finite variation, $d \in (0, \frac{1}{2})$. Furthermore let (I, μ, σ) be proper with SST f and FC $\lambda > 0$. Set $X = f(\mathcal{L}^{d,\lambda})$, where $(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ is a FLOUP. Then we have:

(i) X is a stationary pathwise solution of the stochastic integral equation

$$X_t - X_s = \int_s^t \mu(X_u)du + \int_s^t \sigma(X_u)dL_u^d. \tag{7.11}$$

(ii) If $Z(\sigma) = \emptyset$, then X is the unique stationary pathwise solution of (7.11).

Proof: (i) From Theorem 7.2.3 we know that $X = X^{f,\lambda}(L^d, 0, f(\mathcal{L}_0^{d,\lambda})) = f(\mathcal{L}^{d,\lambda})$ is a pathwise solution of (7.11). Furthermore from Theorem 6.1.3 we know that X is as a strictly montone transformation of the stationary FLOUP again stationary. To show (ii) we invoke Theorem 7.2.4: Given a pathwise solution of (7.11) the Theorem supplies us with a random variable W such that $Y_t = f(e^{-\lambda t}W + \mathcal{L}_t^{d,\lambda})$. If we want Y to be strictly stationary we must have $W = 0$ a.s. and get $Y = X$. □

7.3 Pathwise Integral Equations - Bounded p-Variation Case

After getting a good insight in the case of finite variation we will now generalize our results to the more complex problem when our driving FLP is not of finite variation but of bounded p-variation for some $p \in (1, 2)$. This is a much needed assumption because for a $p \geq 2$ we would loose our chain rule which played an important role in proving the correctness of constructed solutions. Again Theorems and calculations shall be understood in the almost surely sense.

7.3.1 Definition

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded p-variation, $p \in (1, 2)$ and $d \in (0, \frac{1}{2})$. Suppose that $I \subseteq \mathbb{R}$ is a non-empty interval and $\mu, \sigma \in \mathcal{C}^0(I)$. We refer to a stochastic process $(X_t)_{t\in\mathbb{R}}$ as a *pathwise solution* of

$$dX_t = \mu(X_t)dt + \sigma(X_t)dL_t^d \tag{7.12}$$

if a.s. the following conditions are satisfied: $X \in \mathfrak{W}_p$ and $\mathbf{im}(X) \in I$ such that for $s \leq t$,

(S1) $\sigma \circ X$ is a.s. Riemann-Stieltjes integrabel with respect to $(L_t^d)_{t\in\mathbb{R}}$ on $[s, t]$;

(S2) the integral equation

$$X_t - X_s = \int_s^t \mu(X_u)du + \int_s^t \sigma(X_u)dL_u^d$$

holds.

The space of all solutions of (7.12) is denoted by $\mathcal{S}(I, \mu, \sigma, L^d)$.

7.3.2 Construction

For a strongly proper tripel (I, μ, σ) with SST f and FC λ we define as in the finite variation case

$$\begin{aligned} X^{f,\lambda}(L^d, \cdot, \cdot) : \mathbb{R} \times I &\longrightarrow \mathcal{C}^0(\mathbb{R}) \\ (\tau, z) &\longmapsto f(\mathfrak{L}_t^\lambda(L^d, \tau, f^{-1}(z))) \end{aligned} \tag{7.13}$$

with $\mathfrak{L}_t^\lambda$ as in Definition 6.2.6.

7.3.3 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded p-variation, $p \in (1, 2)$ and $d \in (0, \frac{1}{2})$. Then the following assertions hold: If (I, μ, σ) is p-proper with SST f and FC $\lambda > 0$, then we have

$$\{X^{f,\lambda}(L^d, \tau, z) : \tau \in \mathbb{R}, z \in I\} \subseteq \mathcal{S}(I, \mu, \sigma, L^d). \tag{7.14}$$

Proof: First fix $\tau \in \mathbb{R}$ and $z \in I$ and define the following processes

$$l_t := \mathfrak{L}_t^\lambda(L^d, \tau, f^{-1}(z)), \quad t \in \mathbb{R}, \tag{7.15}$$

and

$$Y_t := X_t^{f,\lambda}(L^d, \tau, z), \quad t \in \mathbb{R}. \tag{7.16}$$

Again we will show that $Y \in \mathcal{S}(I, \mu, \sigma, (L_t^d)_{t\in\mathbb{R}})$. Obviously Y tales values in I. Hence $f \in \mathcal{C}^1(\mathbb{R})$ and l is of bounded p-variation we know that $Y = f \circ l \in \mathfrak{W}_p(\mathbb{R})$. With Theorem 4.2.1 we get

$$Y_t - Y_s = f(l_t) - f(l_s) = \int_s^t f'(l_u) dl_u, \quad s \le t. \tag{7.17}$$

Hence l solves (6.22) we know

$$l_u = l_s - \lambda \int_s^u l_v dv + L_u^d - L_s^d, \quad s \le u. \tag{7.18}$$

The Riemann-Stieltjes integral is additive with respect to a sum of integrators if the Riemann-Stieltjes integrals exist separately for each integrator. That is true in our case because l_s and L_s^d as functions of u are constant and therefore of finite variation. Furthermore $-\lambda \int_s^u l_v dv$ is also of finite variation and L_u^d is of bounded p-variation. Hence $f'(l_u)$ is continuous and also of bounded p-variation (7.17) and (7.18) imply

$$Y_t - Y_s = \int_s^t f'(l_u) d\left(-\lambda \int_s^u l_v dv\right) + \int_s^t f'(l_u) dL_u^d, \quad s \le t.$$

Furthermore $-\lambda \int_s^u l_v dv$ is differentiable and $f'(l_u) l_u$ is continuous as a function of u and thus we get by the density formula for Riemann-Stieltjes integrals

$$Y_t - Y_s = -\lambda \int_s^t f'(l_u) l_u du + \int_s^t f'(l_u) dL_u^d, \quad s \le t.$$

From Lemma 7.1.8(i) we obtain $f' = \sigma \circ f$, hence $\sigma \circ Y = f' \circ l \in \mathfrak{W}_p(\mathbb{R})$. By 7.1.5(iii) and Remark 7.1.3 we see that $\sigma f^{-1} = -\sigma\psi/\lambda = -\mu/\lambda$. Finally this leads us to

$$\begin{aligned} Y_t - Y_s &= -\lambda \int_s^t f'(l_u) l_u du + \int_s^t f'(l_u) dL_u^d \\ &= -\lambda \int_s^t \underbrace{\sigma(f(l_u)) l_u}_{=\sigma(Y_u) f^{-1}(Y_u) = -\mu(Y_u)/\lambda} du + \int_s^t \sigma(f(l_u)) dL_u^d \\ &= \int_s^t \mu(Y_u) du + \int_s^t \sigma(Y_u) dL_u^d, \quad s \le t. \end{aligned} \tag{7.19}$$

At last we have $Y \in \mathcal{S}(I, \mu, \sigma, L^d)$. □

7.3.4 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded p-variation, $p \in (1,2)$ and $d \in (0, \frac{1}{2})$, (I, μ, σ) be strongly proper with SST f and FC $\lambda > 0$. Furthermore be $Z(\sigma) = \emptyset$. Then we have:

$$\{X^{f,\lambda}(L^d, \tau, z) : \tau \in \mathbb{R}, z \in I\} = \mathcal{S}(I, \mu, \sigma, L^d). \tag{7.20}$$

Proof: From $Z(\sigma) = \emptyset$, we know by Lemma 7.1.8 that $f \in \mathcal{C}^1(\mathbb{R})$ and $(f^{-1})'(x) = 1/\sigma(x)$ for all $x \in I$. Be $X \in \mathcal{S}(I, \mu, \sigma, L^d)$. From Definition 7.3.1 we see that $X \in \mathfrak{W}_p(\mathbb{R})$ and by $(f^{-1})' \in \mathbf{Lip}(I)$ we get by Theorem 4.2.1

$$f^{-1}(X_t) - f^{-1}(X_s) = \int_s^t f^{-1}(X_u) dX_u = \int_s^t \frac{1}{\sigma(X_u)} dX_u, \quad s \le t. \tag{7.21}$$

From $X \in \mathcal{S}(I, \mu, \sigma, (L_t^d)_{t\in\mathbb{R}})$ we know

$$X_u = X_s + \int_s^u \mu(X_v) dv + \int_s^u \sigma(X_v) dL_v^d. \tag{7.22}$$

Now X_s and $\int_s^u \mu(X_v)dv$ are of finite variation and by Lemma 4.3.1 $\int_s^u \sigma(X_v)dL_v^d$ is of bounded p-variation as functions in u. By putting (7.21) and (7.22) together and using the density formula of Theorem 4.3.2 we get

$$\begin{aligned} f^{-1}(X_t) - f^{-1}(X_s) &= \int_s^t \frac{1}{\sigma(X_u)} d\left(X_s + \int_s^u \mu(X_v)dv + \int_s^u \sigma(X_v)dL_v^d\right) \\ &= \int_s^t \frac{1}{\sigma(X_u)} d\left(\int_s^u \mu(X_v)dv\right) + \int_s^t \frac{1}{\sigma(X_u)} d\left(\int_s^u \sigma(X_v)dL_v^d\right) \\ &= \int_s^t \frac{\mu(X_u)}{\sigma(X_u)} du + \int_s^t dL_u^d \\ &= \int_s^t \frac{\mu(X_u)}{\sigma(X_u)} du + L_t^d - L_s^d, \quad s \le t. \end{aligned}$$

Because (I, μ, σ) is proper $\psi(x) = \frac{\mu(x)}{\sigma(x)}$ and $\psi(x) = -\lambda f^{-1}(x)$ hold for all $x \in I$. Thus we have

$$\begin{aligned} f^{-1}(X_t) - f^{-1}(X_s) &= \int_s^t \frac{\mu(X_u)}{\sigma(X_u)} du + L_t^d - L_s^d \\ &= \int_s^t \psi(X_u) du + L_t^d - L_s^d \\ &= -\lambda \int_s^t f^{-1}(X_u) du + L_t^d - L_s^d, \quad s \le t. \end{aligned}$$

Hence $f^{-1}(X)$ is a solution of (6.22). Fixing $\tau \in \mathbb{R}$ we see by Theorem 6.2.7 that $f^{-1}(X) = \mathfrak{L}^\lambda(L^d, \tau, f^{-1}(X_\tau))$ and at last $X = X^{f,\lambda}(L^d, \tau, X_\tau)$. □

Again we are interested in finding stationary solutions:

7.3.5 Theorem

Let $(L_t^d)_{t\in\mathbb{R}}$ be a FLP of bounded p-variation, $p \in (1,2)$ and $d \in (0, \frac{1}{2})$. Furthermore let (I, μ, σ) be strongly proper with SST f and FC $\lambda > 0$. Set $X = f(\mathcal{L}^{d,\lambda})$, where $(\mathcal{L}_t^{d,\lambda})_{t\in\mathbb{R}}$ is a FLOUP. Then we have:

(i) X is a stationary pathwise solution of the stochastic integral equation

$$X_t - X_s = \int_s^t \mu(X_u)du + \int_s^t \sigma(X_u)dL_u^d. \tag{7.23}$$

(ii) If $Z(\sigma) = \emptyset$, then X is the unique stationary pathwise solution of (7.23).

Proof: Refer to the proof of 7.2.5. □

8 Structural Properties of Proper Triples

In these Section we want to analyse structural properties of proper triples (I, μ, σ). We will do this by first giving us some interval I and $\sigma \in \mathcal{C}^0(I)$ and trying to find a function $\mu \in \mathcal{C}^0(I)$ such that we get a proper tripel, then we will do this the other way around. The results will then be used in Chapter 9 to solve concrete integral equations. Our analysis follows mainly Section 4 on Buchmann and Klüppelberg [2] and the proofs are omitted.

8.1 Construction of Proper Tripels when σ is given

We start with the easy case where we will assume that we have a given interval $I \subseteq \mathbb{R}$ which can also contain $\pm\infty$ and a non-negative function $\sigma \in \mathcal{C}^0(I)$. The question we will try to answer is: Can we find a $\mu \in \mathcal{C}^0(I)$ such that (I, μ, σ) is proper? To formulate this more mathematically we define

$$\mathcal{K}_\sigma^I := \left\{(\lambda, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : (I, \mu, \sigma) \quad \text{is proper with FC } \lambda\right\} \tag{8.1}$$

The following Lemmata state some properties of this construct.

8.1.1 Lemma [Buchmann and Klüppelberg [2], Proposition 4.1 (i)]

Let $I = (a, b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $\sigma \in \mathcal{C}^0(I)$. The the following assertions are equivalent:

(i) $\mathcal{K}_\sigma^I \neq \emptyset$,

(ii) $1/\sigma \in \mathcal{L}_C(I)$ and for all $x \in I$

$$\int_a^x \frac{dz}{\sigma(z)} = \int_x^b \frac{dz}{\sigma(z)} = \infty. \tag{8.2}$$

8.1.2 Lemma [Buchmann and Klüppelberg [2], Proposition 4.1 (ii)]

Let $I = (a, b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $\sigma \in \mathcal{C}^0(I)$. If $\mathcal{K}_\sigma^I \neq \emptyset$, then $\mathcal{K}_\sigma^I$ is a cone,

$$\mathcal{K}_\sigma^I = \left\{(\lambda, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : \mu(x) = -\lambda\sigma(x) \int_\xi^x \frac{dz}{\sigma(z)}, \xi \in I\right\} \tag{8.3}$$

and λ, ξ are uniquely determined by σ and μ.

8.2 Construction of Proper Tripels when μ is given

In the more complex case we will assume a given interval I as before and a continuous function μ in I. How we could find a continuous non-negative σ such that (I, μ, σ) is proper? We recall the Definition 7.1.2 of a proper triple, especially (P2) and (P3). Combining these two conditions we find a differential equation for the function $\psi = \mu/\sigma$:

$$\psi' = -\lambda \frac{1}{\sigma} = -\lambda \frac{\mu}{\sigma \cdot \mu} = -\lambda \frac{\psi}{\mu}. \tag{8.4}$$

It is clear that every solution of (8.4) delivers us a potential σ by setting $\sigma := \mu/\psi$. However the problem that arises is that there is always a $\xi \in I$ such that $\mu(\xi) = 0$, namely the center of μ. According to that it is not trivial that a solution of the above differential equations provides us with a continuous σ. A more technical analysis of the problem leads us to the following Lemma.

8.2.1 Lemma [Buchmann and Klüppelberg [2], Lemma 4.2]

Let $I = (a, b) \subseteq \mathbb{R}$, $-\infty \le a < b \le \infty$ and $\mu \in \mathcal{C}^0(I)$. Furthermore $\xi \in I$ be a centre of μ and $1/\mu \in \mathcal{L}_C(I)$. Take $\lambda > 0$, $x \in (a, \xi)$ and $y \in (\xi, b)$. Then the following function is well-defined

$$\begin{aligned} \psi_{x,y,\lambda} : I \setminus \{\xi\} &\longrightarrow \mathbb{R} \\ w &\longmapsto \begin{cases} \exp\left(-\lambda \int_x^w \frac{dz}{\mu(z)}\right) & w \in (a, \xi) \\ -\exp\left(-\lambda \int_y^w \frac{dz}{\mu(z)}\right) & w \in (\xi, b) \end{cases} . \end{aligned} \tag{8.5}$$

Furthermore $\psi_{x,y,\lambda}$ is the unique absolut continuous solution of (8.4) on $I \setminus \{\xi\}$ with $\psi_{x,y,\lambda}(x) = -\psi_{x,y,\lambda}(y) = 1$ and if $\psi_{x,y,\lambda}$ extends continuously on I, then $\psi_{x,y,\lambda} \in \mathcal{AC}(I)$ and $\psi_{x,y,\lambda}$ is strictly increasing on I.

In difference to the case where σ is given, we cannot choose a FC λ freely here. In order to that we have to look at the following set first:

$$\Lambda_\mu^I := \left\{\lambda \in \mathbb{R}^+ : \exists \sigma \in \mathcal{C}^0(I) \text{ with } (I, \mu, \sigma) \text{ is proper with FC } \lambda\right\}. \tag{8.6}$$

If there is a $\lambda \in \Lambda_\mu^I$ we will have a look at the set

$$\mathcal{H}_{\mu,\lambda}^I = \left\{\sigma \in \mathcal{C}^0(I) : (I, \mu, \sigma) \text{ is proper with FC } \lambda\right\}. \tag{8.7}$$

As in the former case we get a similar result about the structure of $\mathcal{H}_{\mu,\lambda}^I$:

8.2.2 Lemma [Buchmann and Klüppelberg [2], Proposition 4.3]

Let $I = (a, b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$, $\mu \in \mathcal{C}^0(I)$ and $\lambda \in \Lambda_\mu^I$. Then $\mathcal{H}_{\mu,\lambda}^I$ is a cone.

Two answer the main question of this section we obviously have to analysis both sets seperately.

8.2.3 Lemma [Buchmann and Klüppelberg [2], Proposition 4.4]

Let $I = (a, b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $\mu \in \mathcal{C}^0(I)$. Then the following assertions are equivalent:

(i) $\Lambda_\mu^I \neq \emptyset$.

(ii) There is a centre for μ $\xi \in I$ such that

 (a) $1/\mu \in \mathcal{L}_C(I \setminus \{\xi\})$,

 (b) for all $a < x < \xi < y < b$ we have

$$\int_a^x \frac{dz}{\mu(z)} = \int_y^b \frac{dz}{|\mu(z)|} = \int_x^\xi \frac{dz}{\mu(z)} = \int_\xi^y \frac{dz}{|\mu(z)|} = \infty, \tag{8.8}$$

 (c) the following set is not empty:

$$\Theta_\mu^I := \left\{ \lambda \in \mathbb{R}^+ : \exists x \in (a, \xi) \wedge y \in (\xi, b) \text{ with } \lim_{w \nearrow \xi} \frac{\mu(w)}{\psi_{x,y,\lambda}(w)} = \lim_{w \searrow \xi} \frac{\mu(w)}{\psi_{x,y,\lambda}(w)} = 0 \right\}. \tag{8.9}$$

The next lemma will teach us a relation between the two sets Λ_μ^I and Θ_μ^I and a first look into the structure of the cone $\mathcal{H}_{\mu,\lambda}^I$. Therefore we define

$$\lambda_\mu^I := \sup \Lambda_\mu^I.$$

8.2.4 Lemma [Buchmann and Klüppelberg [2], Proposition 4.5]

Let $I = (a, b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $\mu \in \mathcal{C}^0(I)$. Furthermore let $\Lambda_\mu^I \neq \emptyset$ and $\xi \in I$ be the centre of μ. Then we have:

(i) $(0, \lambda_\mu^I) \subseteq \Theta_\mu^I \subseteq \Lambda_\mu^I$,

(ii) If $\lambda \in \Theta_\mu^I$ then the following holds

$$\mathcal{H}_{\mu,\lambda}^I = \left\{ \sigma \in \mathcal{C}^0(I) : \exists x \in (a,\xi) \wedge y \in (\xi,b) \text{ with } \sigma(w) = \left\{ \begin{array}{ll} \frac{\mu(w)}{\psi_{x,y,\lambda}(w)} & w \in I \setminus \{\xi\} \\ 0 & w = \xi \end{array} \right\} \right. . \tag{8.10}$$

The leaves us only with the case $\lambda_\mu^I \in \Lambda_\mu^I$ but $\lambda_\mu^I \notin \Theta_\mu^I$, which we will consider in the next two Lemmata.

8.2.5 Lemma [Buchmann and Klüppelberg [2], Proposition 4.6 (i)]

Let $I = (a,b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $\mu \in \mathcal{C}^0(I)$. Let further $\xi \in I$ be the centre of μ, $\Lambda_\mu^I \neq \emptyset$ bounded and $\lambda_\mu^I \notin \Theta_\mu^I$. Then there is equivalent:

(i) $\lambda_\mu^I \in \Lambda_\mu^I$.

(ii) ξ is an isolated point of $Z(\mu)$ and for all $\bar{x} \in (a,\xi)$ and $\bar{y} \in (\xi,b)$ we have

$$\lim_{w \nearrow \xi} \lim_{x \nearrow \xi} \frac{\mu(w)}{\mu(x)} \psi_{w,\bar{y},\lambda_\mu^I} = \lim_{w \searrow \xi} \lim_{x \searrow \xi} \frac{\mu(w)}{\mu(x)} \psi_{\bar{x},w,\lambda_\mu^I} = 1. \tag{8.11}$$

8.2.6 Lemma [Buchmann and Klüppelberg [2], Proposition 4.6 (ii)]

Let $I = (a,b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$ and $\mu \in \mathcal{C}^0(I)$. Let further $\xi \in I$ be the centre of μ, $\Lambda_\mu^I \neq \emptyset$ bounded and $\lambda_\mu^I \notin \Theta_\mu^I$. If $\lambda_\mu^I \in \Lambda_\mu^I$, then for $\bar{x} \in (a,\xi)$ and $\bar{y} \in (\xi,b)$ the function

$$\sigma_{\lambda_\mu^I}(w) = \left\{ \begin{array}{ll} \lim_{x \nearrow \xi} \frac{\mu(w)}{\mu(x)} \psi_{w,\bar{y},\lambda_\mu^I} & w \in (a,\xi) \\ 1 & w = \xi \\ \lim_{x \searrow \xi} \frac{\mu(w)}{\mu(x)} |\psi_{\bar{x},w,\lambda_\mu^I}| & w \in (\xi,b) \end{array} \right. \tag{8.12}$$

is well-defined and continuous on I. The representation of $\sigma_{\lambda_\mu^I}$ does not depend on the choice of $\bar{x}$ and $\bar{y}$ and we have

$$\mathcal{H}_{\mu,\lambda}^I = \left\{ c\sigma_{\lambda_\mu^I} : c \in \mathbb{R}^+ \right\}. \tag{8.13}$$

9 Examples

In our examples we want to consider like Buchmann and Klüppelberg [2] two main cases: An affine drift term and power volatility function. Most of the Theorems from Section 5 of Buchmann and Klüppelberg [2] hold true here because they consider only proper triple and we will like before omit the proofs. Changes have been necessary when strongly proper triples are concerned.

9.1 Power Volatility

In difference to Buchmann and Klüppelberg [2] we want to begin with power volatility functions. Our motivation is to solve equations like

$$dr_t = -\gamma r_t dt + \sqrt{|r_t|}\sigma dL_t^d, \tag{9.1}$$

namely a simplified Cox-Ingersoll-Ross model with non-time-dependend coefficiants and zero mean. So we will from now give us some function $\sigma : \mathbb{R} \to [0, \infty)$ with $\sigma(x) := \sigma_0 |x|^\delta$ for $\sigma_0 > 0$ and $\delta \in \mathbb{R}$. We will understand $\sigma(0) = \infty$ if $\delta < 0$. Applying our Theorems from the last section we get:

9.1.1 Lemma [Buchmann and Klüppelberg [2], Proposition 5.4]

Given $\sigma : \mathbb{R} \to [0, \infty)$ by $\sigma(x) := \sigma_0 |x|^\delta$ for some $\sigma_0 > 0$ and $\delta \in \mathbb{R}$. Then there exist $I \subseteq \mathbb{R}$ and $\mu \in \mathcal{C}^0(I)$ such that (I, μ, σ) is proper if and only if $\delta \in [0, 1]$.

Now we will consider the the cases $\delta = 0$, $\delta = 1$ and $\delta \in (0, 1)$ separately. Lets assume $\delta = 0$ first. Then for a triple (I, μ, σ) to be proper means that $\sigma\psi' \equiv -\lambda$ with $\psi = \mu/\sigma$. Obviously this is only the case when μ is affine and we will consider this case in the next Section. Now let us have a look at:

9.1.2 Theorem

Given $\sigma : \mathbb{R} \to [0, \infty)$ by $\sigma(x) := \sigma_0 |x|$ for some $\sigma_0 > 0$. Then the following assertions hold:

(i) $\mathcal{K}_\sigma^I \neq \emptyset$ if and only if $I = (0, \infty)$ or $I = (-\infty, 0)$.

(ii) If $\mathcal{K}_\sigma^I \neq \emptyset$ then (I, μ, σ) is strongly proper for all $(\lambda, \mu) \in \mathcal{K}_\sigma^I$ and we have

$$\begin{aligned} \mathcal{K}_\sigma^{(0,\infty)} &= \left\{(|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : \mu(x) = \alpha x + \beta x \log x, \alpha \in \mathbb{R}, \beta < 0, x \in (0, \infty)\right\} \quad \text{and} \\ \mathcal{K}_\sigma^{(-\infty,0)} &= \left\{(|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : \mu(x) = \alpha x + \beta x \log |x|, \alpha \in \mathbb{R}, \beta < 0, x \in (-\infty, 0)\right\} \end{aligned} \tag{9.2}$$

(iii) If $\mu(x) = \alpha x + \beta x \log x$ with $(|\beta|, \mu) \in \mathcal{K}_\sigma^{(0,\infty)}$ the we have for the SST f and the centre ξ:

$$f(x) = \exp\left(\sigma_0 x - \frac{\alpha}{\beta}\right), \quad \xi = \exp\left(-\frac{\alpha}{\beta}\right). \tag{9.3}$$

Proof: From Lemma 8.1.1 we know that $\mathcal{K}_\sigma^I \neq \emptyset$ if and only if (8.2) holds for the chosen I. This is clearly only the case if $I = (0, \infty)$ or $I = (-\infty, 0)$. For (ii) we invoke Lemma 8.1.2 and calculate for $I = (-\infty, 0)$ and some $\lambda > 0$, $\xi < 0$, $x \in I$

$$\begin{aligned} \mu(x) &= -\lambda\sigma(x) \int_\xi^x \frac{dz}{\sigma(z)} = -\lambda\sigma_0|x| \int_\xi^x \frac{dz}{\sigma_0|z|} = -\lambda x \int_\xi^x \frac{dz}{z} \\ &= -\lambda x(\log|x| - \log|\xi|) = \lambda \log|\xi| x - \lambda x \log|x|. \end{aligned} \tag{9.4}$$

Hence $\phi : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R} \times (-\infty, 0)$, $\phi(\xi, \lambda) := (\lambda \log|\xi|, -\lambda)$ is a bijection we get our stated result. The same arguments are used for $\mathcal{K}_\sigma^{(0,\infty)}$. To show that all occurring triples are strongly proper we calculate the SST and see that it is differentiable and its derivation lies in $\mathbf{Lip}(\mathbb{R})$. This calculations we will do in order to show (iii): First we find that for $x \in (-\infty, 0)$

$$\psi(x) = \mu(x)/\sigma(x) = \frac{\alpha x + \beta x \log|x|}{\sigma_0|x|} = -\frac{\alpha}{\sigma_0} - \frac{\beta \log(-x)}{\sigma_0} \tag{9.5}$$

and we get

$$\psi^{-1}(x) = -\exp\left(-\frac{\alpha + \sigma_0 x}{\beta}\right). \tag{9.6}$$

Now we calculate $f(x) = \psi^{-1}(-|\beta|x)$ and obtain

$$f(x) = -\exp\left(-\sigma_0 x - \frac{\alpha}{\beta}\right). \tag{9.7}$$

Obviously $f \in \mathcal{C}^\infty(\mathbb{R})$ and thus the occuring triples are strongly proper. At last we see for $x \in (-\infty, 0)$ $\mu(x) = 0$ if and only if $x = -\exp\left(-\frac{\alpha}{\beta}\right)$ and using Lemma 7.1.7 we get the desired result. Similar arguments hold true for the case $I = (0, \infty)$. $\square$

The last case is then where δ is allowed to be in the open interval $(0, 1)$:

9.1.3 Theorem

Given $\sigma : \mathbb{R} \to [0, \infty)$ by $\sigma(x) := \sigma_0|x|^\delta$ for some $\sigma_0 > 0$, $\delta \in (0, 1)$. Then we have:

(i) $\mathcal{K}_\sigma^I \neq \emptyset$ if and only if $I = \mathbb{R}$ and we have

$$\mathcal{K}_\sigma^{\mathbb{R}} = \left\{((1-\delta)|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}^0(I) : \mu(x) = \alpha|x|^\delta + \beta x, \alpha \in \mathbb{R}, \beta < 0, x \in \mathbb{R}\right\}. \tag{9.8}$$

(ii) If $(\lambda,\mu)\in\mathcal{K}_\sigma^{\mathbb{R}}$ then (I,μ,σ) is strongly proper if and only if $\delta\in[\frac{1}{2},1)$.

(iii) For $(\lambda,\mu)\in\mathcal{K}_\sigma^{\mathbb{R}}$ with $\mu(x)=\alpha|x|^\delta+\beta x$ then we have the following for the SST f and the centre ξ:

$$f(x)=\operatorname{sign}\left((1-\delta)\sigma_0 x-\frac{\alpha}{\beta}\right)\left|(1-\delta)\sigma_0 x-\frac{\alpha}{\beta}\right|^{\frac{1}{1-\delta}},\quad \xi=\operatorname{sign}(\alpha)\left|\frac{\alpha}{\beta}\right|^{\frac{1}{1-\delta}}. \tag{9.9}$$

Proof: Using Lemma 8.1.1 and calculating the integrals of (8.2) we see instantaneously that (i) is true. Also we observe that

$$\begin{aligned}\int_\xi^x\frac{dz}{|z|^\delta} &= \left\{\begin{array}{ll}\int_\xi^x\frac{dz}{z^\delta}=\frac{x^{1-\delta}-\xi^{1-\delta}}{1-\delta} & \xi,x\geq 0\\ \int_\xi^x\frac{dz}{(-z)^\delta}=\frac{(-\xi)^{1-\delta}-(-x)^{1-\delta}}{1-\delta} & \xi,x\leq 0\\ \int_\xi^0\frac{dz}{(-z)^\delta}+\int_0^x\frac{dz}{z^\delta}=\frac{(-\xi)^{1-\delta}+x^{1-\delta}}{1-\delta} & \xi\leq 0,x\geq 0\\ -\left(\int_x^0\frac{dz}{(-z)^\delta}+\int_0^\xi\frac{dz}{z^\delta}\right)=\frac{-\xi^{1-\delta}-(-x)^{1-\delta}}{1-\delta} & \xi\geq 0,x\leq 0\end{array}\right\}\\ &= \frac{\operatorname{sign}(x)|x|^{1-\delta}-\operatorname{sign}(\xi)|\xi|^{1-\delta}}{1-\delta}.\end{aligned} \tag{9.10}$$

Now we get by (9.10) and Lemma 8.1.2

$$\mu(x)=-\lambda\sigma(x)\int_\xi^x\frac{dz}{\sigma(z)}=-\lambda|x|^\delta\int_\xi^x\frac{dz}{|z|^\delta}=\frac{\lambda}{1-\delta}\operatorname{sign}(\xi)|\xi|^{1-\delta}|x|^\delta-\frac{\lambda}{1-\delta}x.$$

By bijection arguments as in the preceding proof we get the stated representation of $\mathcal{K}_\sigma^{\mathbb{R}}$. For (iii) we calculate

$$\psi(x)=\frac{\alpha}{\beta}+\operatorname{sign}(x)|x|^{1-\delta}\frac{\beta}{\sigma_0}\Longrightarrow\psi^{-1}(x)=\operatorname{sign}\left(\frac{\sigma_0 x-\alpha}{\beta}\right)\left|\frac{\sigma_0 x-\alpha}{\beta}\right|^{\frac{1}{1-\delta}}.$$

Thus we get for friction coefficient $(1-\delta)|\beta|$

$$f(x)=\operatorname{sign}\left((1-\delta)\sigma_0 x-\frac{\alpha}{\beta}\right)\left|(1-\delta)\sigma_0 x-\frac{\alpha}{\beta}\right|^{\frac{1}{1-\delta}}. \tag{9.11}$$

At last we can see that $\psi(x)=0$ if and only if $x=\operatorname{sign}(\alpha)\left|\frac{\alpha}{\beta}\right|^{\frac{1}{1-\delta}}$. It is clear from (9.11) that f is differentiable if and only if $\delta\in[\frac{1}{2},1)$. For such δ we calculate the derivation and obtain

$$f'(x)=\left\{\begin{array}{ll}\sigma_0((1-\delta)\sigma_0 x-\frac{\alpha}{\beta})^{\frac{\delta}{1-\delta}} & x\geq\frac{\alpha}{\sigma_0(1-\delta)\beta}\\ \sigma_0(\frac{\alpha}{\beta}-(1-\delta)\sigma_0 x)^{\frac{\delta}{1-\delta}} & x\leq\frac{\alpha}{\sigma_0(1-\delta)\beta}\end{array}\right..$$

Hence the function $h(x):=|x|^\gamma$ for $\gamma\geq 1$ is clearly Lipschitz the proof is finished. □

9.1.4 Example

Now we can consider for some $\sigma > 0$, $\gamma > 0$ and a FLP $(L_t^d)_{t\in\mathbb{R}}$ of bounded p-variation, $d \in (0, \frac{1}{2})$, $p \in (0, 2)$ the pathwise SDE

$$dr_t = -\gamma r_t dt + \sqrt{|r_t|}\sigma dL_t^d. \tag{9.12}$$

Therefore we define $\widetilde{\sigma}(x) := \sigma|x|^{\frac{1}{2}}$ and choose $\widetilde{\mu}(x) = -\gamma x$ and take $I = \mathbb{R}$. With Theorem 9.1.3 we see that $(I, \widetilde{\mu}, \widetilde{\sigma})$ is strongly proper with SST $f(x) = \operatorname{sign}\left(\frac{1}{2}\sigma x\right)\left|\frac{1}{2}\sigma x\right|^2$ and b y Theorem 7.2.5 or 7.3.5 a stationary solution of (9.12) is given by $(f(\mathcal{L}_t^{d,\lambda}))_{t\in\mathbb{R}}$ with $\lambda = \frac{\gamma}{2}$.
□

9.2 Affine Drift

As already announced we want to consider models with an affine drift term $\mu : \mathbb{R} \to \mathbb{R}$, $\mu(x) := \alpha + \beta x$, for some $\alpha, \beta \in \mathbb{R}$. Our first Lemma limits the choices of the coefficient, the proof is omitted and can be found in Buchmann and Klüppelberg [2]:

9.2.1 Lemma [Buchmann and Klüppelberg [2], Proposition 5.1]

Given $\mu : \mathbb{R} \to \mathbb{R}$, $\mu(x) := \alpha + \beta x$, for some $\alpha, \beta \in \mathbb{R}$. Then there exists an interval $I \subseteq \mathbb{R}$ and $\sigma \in \mathcal{C}^0(I)$ such that (I, μ, σ) is proper if and only if $\beta < 0$. In that situation we have

$$I = \mathbb{R}, \quad \Lambda_\mu^I = (0, |\beta|], \quad \Theta_\mu^I = (0, |\beta|) \quad \xi = -\frac{\alpha}{\beta}, \tag{9.13}$$

where ξ is the centre of ψ.

Remembering our results from Section 8 we have to consider the cases where the friction coefficient is an element of $\Theta_\mu^I = (0, |\beta|)$ or $|\beta|$. We start with the last case which is pretty simple:

9.2.2 Theorem

Given $\mu : \mathbb{R} \to \mathbb{R}$, $\mu(x) := \alpha + \beta x$, for some $\alpha \in \mathbb{R}$ and $\beta < 0$. Then we have

$$\mathcal{H}_{\mu,|\beta|}^{\mathbb{R}} = \left\{ \sigma \in C^0(\mathbb{R}) : \sigma(x) \equiv \sigma_0, x \in I, \sigma_0 > 0 \right\}. \tag{9.14}$$

For the SST f we get $f(x) = \sigma_0 - \frac{\alpha}{\beta}$ and every $\sigma \in \mathcal{H}_{\mu,|\beta|}^{\mathbb{R}}$ leads to a strongly proper triple.

Proof: The statements for $\mathcal{H}_{\mu,|\beta|}^{\mathbb{R}}$ and f are very easy to calculate if one uses Lemma 8.2.6 and (8.12). Cleary $f \in C^\infty(\mathbb{R})$ thus every occurring triple is strongly proper. □

Now to the other case:

9.2.3 Theorem

Given $\mu : \mathbb{R} \to \mathbb{R}$, $\mu(x) := \alpha + \beta x$, for some $\alpha \in \mathbb{R}$ and $\beta < 0$. Take $\delta \in (0, 1)$, then $(1 - \delta)|\beta| \in \Theta_\mu^{\mathbb{R}}$ and we have:

(i) $\sigma \in \mathcal{H}_{\mu,|\beta|}^{\mathbb{R}} \iff$ there exist constants $\sigma_1, \sigma_2 > 0$ such that for $\xi = -\frac{\alpha}{\beta}$

$$\sigma(x) = \begin{cases} \sigma_1 \left|\alpha + \beta x\right|^\delta & x \leq \xi \\ \sigma_2 \left|\alpha + \beta x\right|^\delta & x \geq \xi \end{cases}. \tag{9.15}$$

(ii) For σ as in (i) the SST f is given by

$$f(x) = \begin{cases} -f_1\,|x|^{\frac{1}{1-\delta}} + \xi & x \leq 0 \\ f_2\,|x|^{\frac{1}{1-\delta}} + \xi & x \geq 0 \end{cases}, \tag{9.16}$$

for $f_i := |\beta|^{\frac{\delta}{1-\delta}} \sigma_i^{\frac{1}{1-\delta}} (1-\delta)^{\frac{1}{1-\delta}}$.

(iii) For $\sigma \in \mathcal{H}^{\mathbb{R}}_{\mu,|\beta|}$ $(\mathbb{R}, \mu, \sigma)$ is strongly proper if and only if $\delta \in [\frac{1}{2}, 1)$.

Proof: We use Lemma 8.2.1 and Lemma 8.2.4 to calculate the formula to show (i). Setting $\lambda := (1-\delta)|\beta|$ we obtain first for some $\overline{x} \in (-\infty, \xi)$ and $\overline{y} \in (\xi, \infty)$

$$\exp\left(-\lambda \int_{\overline{x}}^{x} \frac{dz}{\mu(z)}\right) = \exp\left(-\frac{\lambda}{\beta} \int_{\overline{x}}^{x} \frac{\beta dz}{\alpha + \beta z}\right)$$
$$= \exp\left(-\frac{\lambda}{\beta}[\log(\alpha + \beta z)]_{\overline{x}}^{x}\right) = (\alpha + \beta x)^{-\frac{\lambda}{\beta}} (\alpha + \beta \overline{x})^{\frac{\lambda}{\beta}}, \quad x < \xi \tag{9.17}$$

Similar we get

$$-\exp\left(-\lambda \int_{\overline{y}}^{x} \frac{dz}{\mu(z)}\right) = |\alpha + \beta x|^{-\frac{\lambda}{\beta}} |\alpha + \beta \overline{y}|^{\frac{\lambda}{\beta}}, \quad x > \xi \tag{9.18}$$

Putting (9.17), (9.18) and (8.10) we obtain

$$\sigma_{\overline{x},\overline{y},\lambda} = \begin{cases} |\alpha + \beta x|^{1+\frac{\lambda}{\beta}} |\alpha + \beta \overline{x}|^{-\frac{\lambda}{\beta}} & x < \xi \\ 0 & x = \xi \\ |\alpha + \beta x|^{1+\frac{\lambda}{\beta}} |\alpha + \beta \overline{y}|^{-\frac{\lambda}{\beta}} & x > \xi \end{cases}.$$

A bijection argument with

$$\begin{aligned} (-\infty, \xi) &\longrightarrow \mathbb{R}^+ \\ \overline{x} &\longmapsto \sigma_1 = |\alpha + \beta \overline{x}|^{-\frac{\lambda}{\beta}}, \end{aligned}$$

$$\begin{aligned} (\xi, \infty) &\longrightarrow \mathbb{R}^+ \\ \overline{y} &\longmapsto \sigma_1 = |\alpha + \beta \overline{y}|^{-\frac{\lambda}{\beta}}, \end{aligned}$$

$$\begin{aligned} (0, |\beta|) &\longrightarrow (0,1) \\ \lambda &\longmapsto \delta = 1 + \frac{\lambda}{\beta}, \end{aligned}$$

proves the assertion. The formula for the SST f can be obtained by easy calculations. (iii) follows by the same arguments as in the proof of Theorem 9.1.3. □

10 Simulations

The last Chapter will be dedicated to a few simulations of FLPs, FLOUPs and solutions of more general fractional integral equations like we considered in this thesis. Let $a, b > 0$. We start with our driving process which shall be a (a, b)-gamma process, i.e. a Lévy process $(\widetilde{L}_t^{a,b})_{t\geq 0}$ where the density of $\widetilde{L}_t$ is for every $t \in \mathbb{R}$ given by

$$f_{\widetilde{L}_t}(x) = \frac{b^{-at}}{\Gamma(at)} x^{at-1} e^{-\frac{x}{b}}, \quad x \geq 0$$

A gamma process has finite second moments; in particular we have

$$E[\widetilde{L}_t^{a,b}] = tab, \quad \mathrm{Var}(\widetilde{L}_t^{a,b}) = tab^2.$$

Because we are only interested in zero-mean driving processes we have to subtract its expectation. This leads us to a gamma process with negative drift $(L_t^{a,b})_{t\geq 0}$, i.e. for every $t \geq 0$ we define $L_t^{a,b} := \widetilde{L}_t^{a,b} - tab$. If we simulate now two independent paths of this process we can put them together by (6.2.1) to obtain a two sided Lévy process $(L_t^{a,b})_{t\in\mathbb{R}}$:

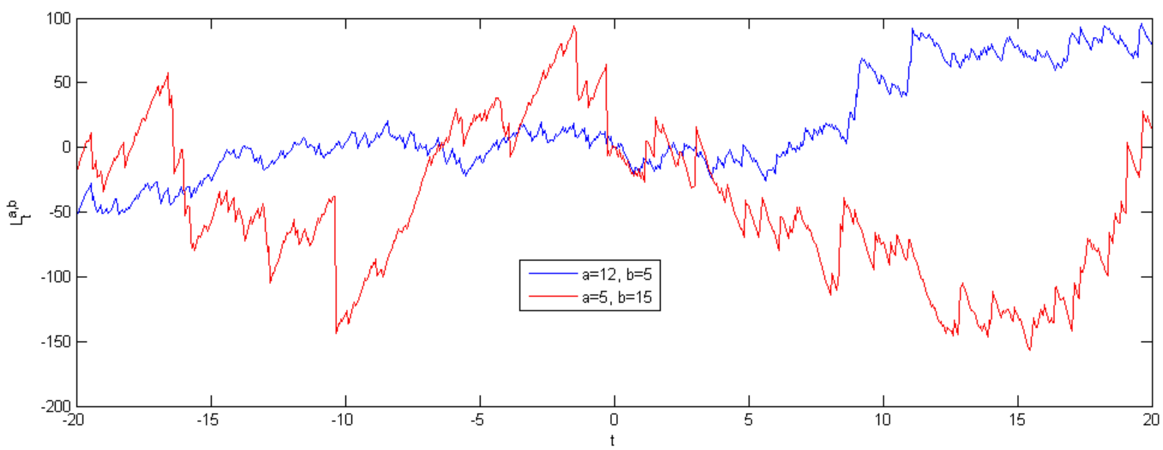

Figure 1: The sample paths of a gamma process for varying parameters.

We used the same random numbers for both sample paths and from now on we will use these computed paths to obtain the next processes. By using an approximation by

Riemann-Stieltjes sums

$$L_t^d \approx \frac{1}{\Gamma(d+1)} \left\{ \sum_{k=-n^2}^{0} \left[\left(t - \frac{k}{n}\right)^d - \left(-\frac{k}{n}\right)^d \right] \left(L^{a,b}_{\frac{k+1}{n}} - L^{a,b}_{\frac{k}{n}} \right) + \sum_{k=1}^{[nt]} \left(t - \frac{k}{n}\right)^d \left(L^{a,b}_{\frac{k+1}{n}} - L^{a,b}_{\frac{k}{n}} \right) \right\}, \quad t \in \mathbb{R}$$

(for a more detailed analysis refer to Marquardt [5]) for FLPs we construct sample paths of a (a, b)-gamma-driven FLP:

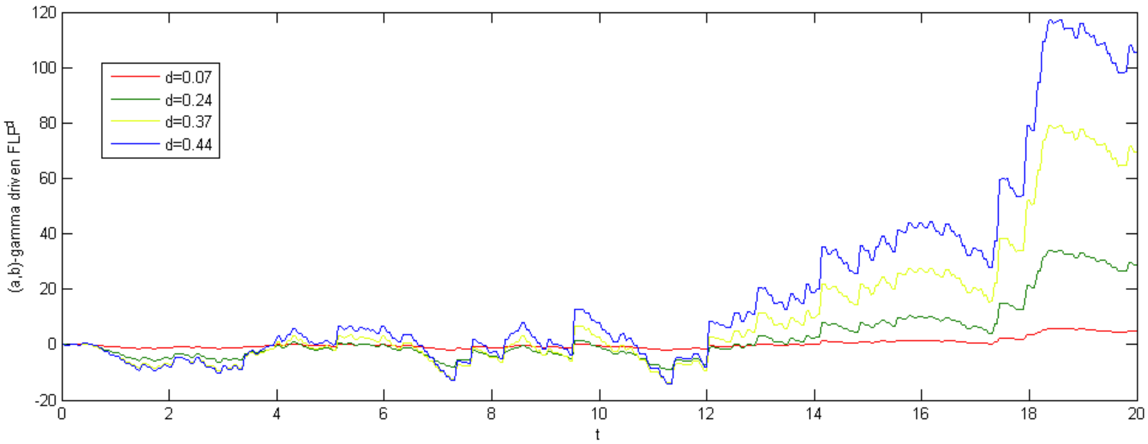

Figure 2: The sample paths of a FLP driven by a (a, b)-gamma process for varying d and fixed $a = 12$, $b = 5$.

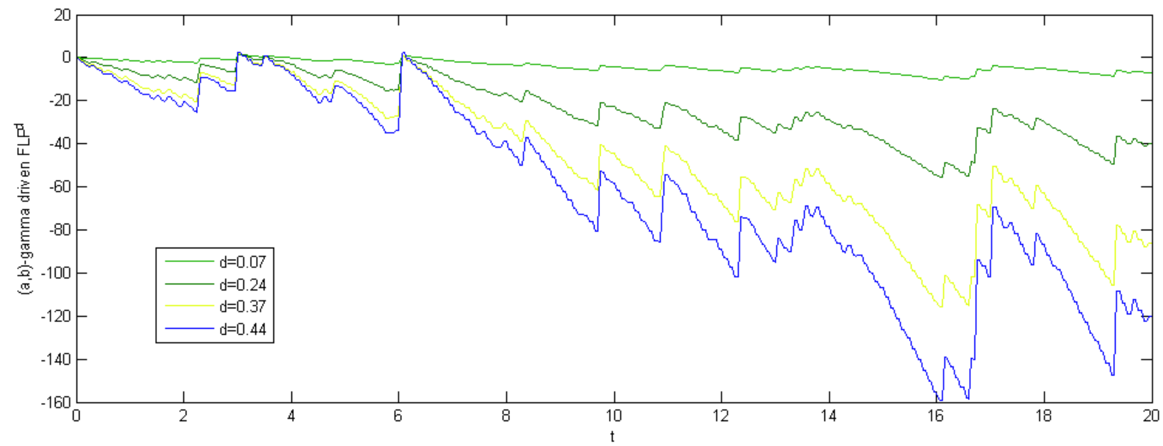

Figure 3: The sample paths of a FLP driven by a (a, b)-gamma process for varying d and fixed a=5, b=15.

Using now a version of the explicit Euler method to get paths of the solution of the Langevin equation, i.e. sample paths of FLOUPs driven by (a, b)-gamma FLPs.

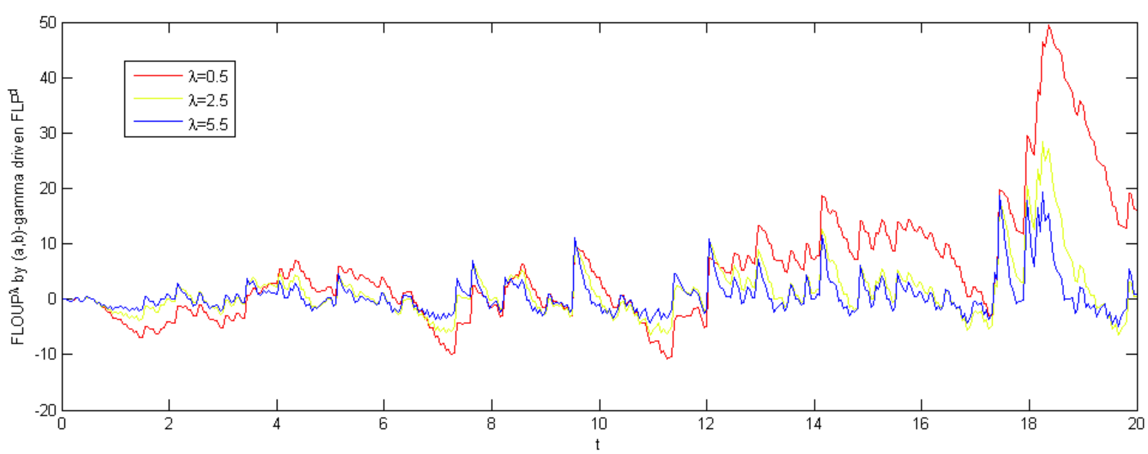

Figure 4: The sample paths of a FLOUP corrpesonding to a (a, b)-gamma-driven FLP for varying λ and fixed $a = 12$, $b = 5$, $d = 0.37$.

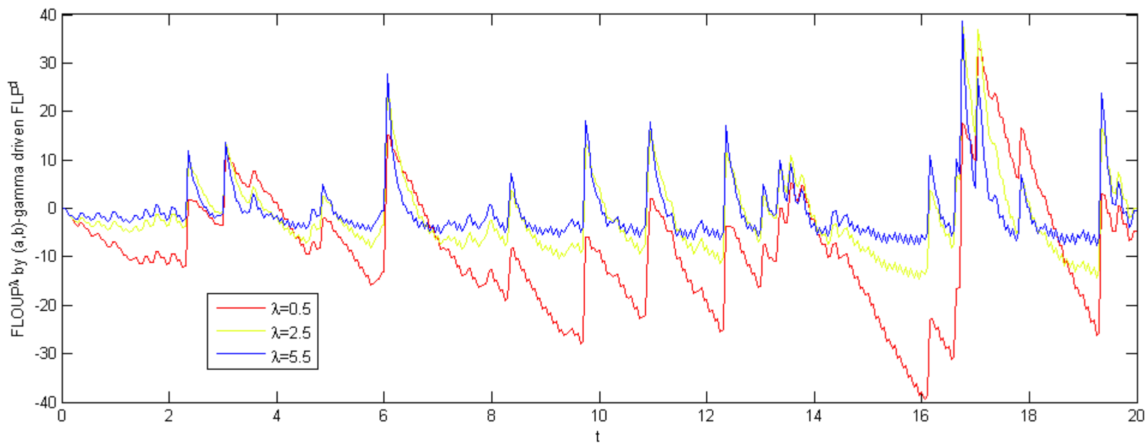

Figure 5: The sample paths of a FLOUP corrpesonding to a (a, b)-gamma-driven FLP for varying λ and fixed $a = 5$, $b = 15$, $d = 0.37$.

Returning to example 9.1.4 we are now interested in a solution of the pathwise SDE

$$dr_t = -\gamma r_t dt + \sqrt{|r_t|}\sigma dL_t^d,$$

namely a simplified Cox-Ingersoll-Ross model with non-time-dependend coefficiants and zero mean. We already know that a stationary solution is given by $(X_t)_{t\in\mathbb{R}}$ with

$$X_t = \text{sign}\left(\frac{1}{2}\sigma\mathcal{L}_t^{d,\frac{\gamma}{2}}\right)\left|\frac{1}{2}\sigma\mathcal{L}_t^{d,\frac{\gamma}{2}}\right|^2, \quad t \in \mathbb{R}. \tag{10.1}$$

where $(\mathcal{L}_t^{d,\frac{\gamma}{2}})_{t\in\mathbb{R}}$ is the corresponding FLOUP. If we use our (a, b)-gamma FLP as driving process of the SDE above we can easily compute sample paths of $(X_t)_{t\in\mathbb{R}}$ bearing in mind that $2\lambda = \gamma$:

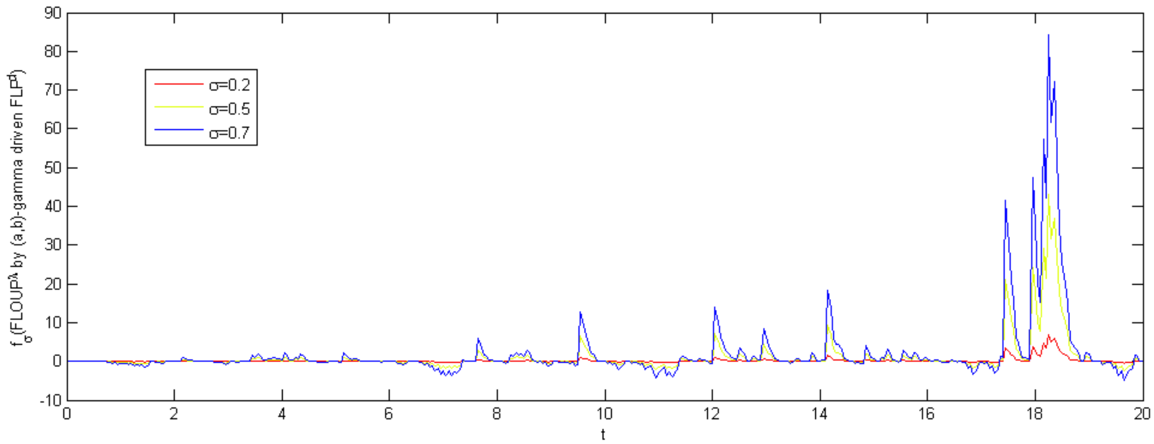

Figure 6: The sample paths of a solution of the Cox-Ingersoll-Ross model for varying σ and fixed $a = 12$, $b = 5$, $d = 0.37$, $\lambda = 3$.

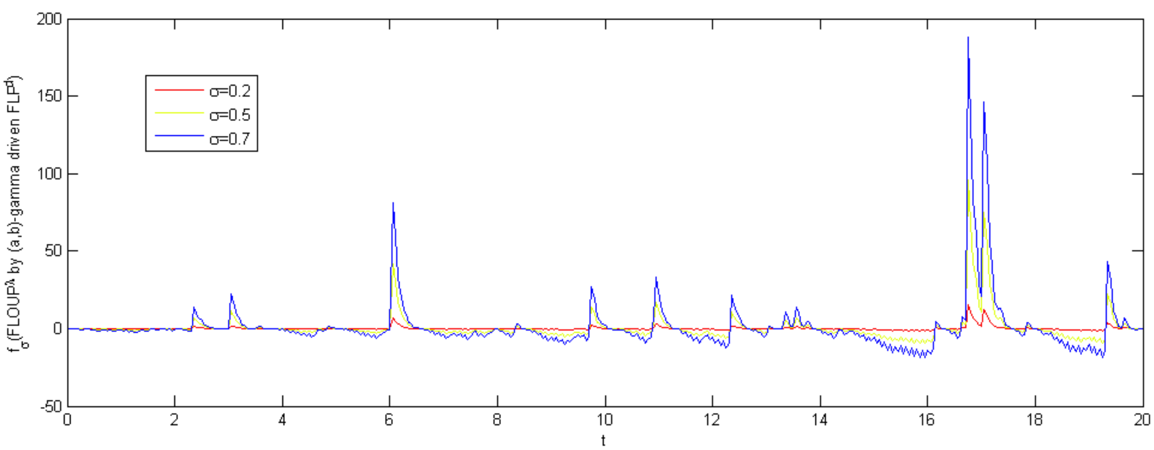

Figure 7: The sample paths of a solution of the Cox-Ingersoll-Ross model for varying σ and fixed $a = 5$, $b = 12$, $d = 0.37$, $\lambda = 3$.

At last we compare two of those paths which came from different parameter a and b:

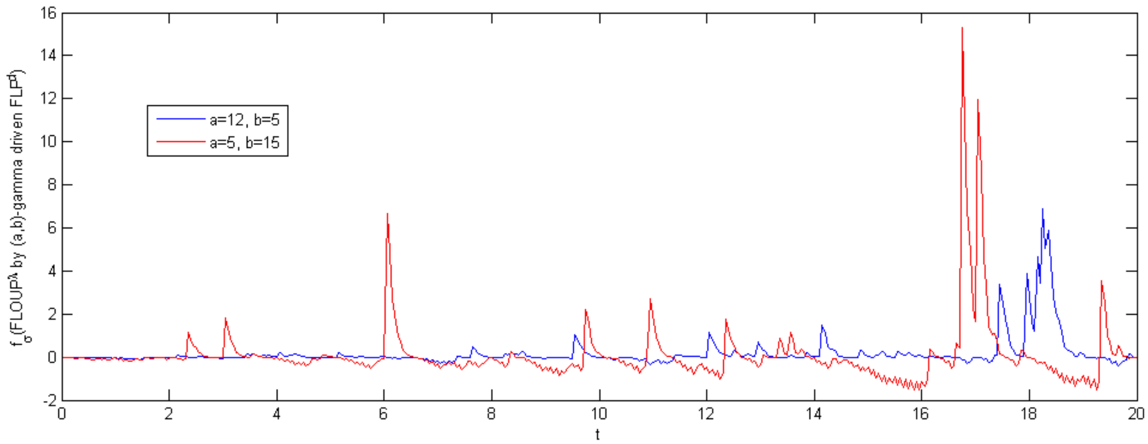

Figure 8: The sample paths of a solution of the Cox-Ingersoll-Ross model for varying a, b and fixed $d = 0.37$, $\lambda = 3$, $\sigma = 0.2$.

As we can observe the processes also take negative values.

11 Outlook

At the end we want to give some thoughts about possible topics and questions for future research. On the one hand we have in the last Chapters restricted ourselves to FLPs of bounded p-variation for some $p \in (0, 2)$. It is only natural to ask if this may be dropped when considering general fractional integral equations. As already stated before this is not possible if one wants to integrate pathwise in the Riemann-Stieltjes sense and still have some kind of a chain rule and density formula. However it might be possible to prove the theory concerning proper triples and solutions of corresponding fractional integral equations if one uses another way of integration. For instance Marquardt [5] suggested an approach by a Wick-Itô integral with respect to FLPs in terms of the S-transform.

Another interesting question is to find general results about the p-variation of FLPs. There is in fact a conjecture that the p-variation of a FLP can be calculated by the p-variation of the underlying Lévy process. However nothing has been proven yet.

At last when considering the results of Buchmann and Klüppelberg [2] one can find there concrete calculations of the stationary density of solutions to some fractional integral equations in the case of FBM as driving process. The question arises if one can prove similar results for at least the characteristic functions in the FLP case. The last question is also topic of our current research and hopefully there will be a satisfying answer soon.

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