Pareto Lévy measures and multivariate regular variation

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Abstract

We consider regular variation of a Lévy process $\boldsymbol{X} := (\boldsymbol{X}_t)_{t\geq 0}$ in $\mathbb{R}^d$ with Lévy measure Π in consideration of dependence between the jumps. By transforming the one-dimensional marginal Lévy measures to those of a standard 1-stable Lévy process, we decouple the marginal Lévy measures from the dependence structure. The dependence between the jump components is modeled by a so-called *Pareto Lévy measure*, which was introduced in [18] for spectrally positive Lévy processes in $\mathbb{R}^d$. In contrast to the use of a Lévy copula, also after standardisation of the marginal Lévy measures we stay within the class of Lévy measures. We give conditions on the one-dimensional marginal Lévy measures and the Pareto Lévy measure such that $\boldsymbol{X}$ is α-stable or multivariate regularly varying, respectively. Finally, we present graphical tools to visualize the dependence structure in terms of the spectral density and the tail integral for homogeneous and non-homogeneous Pareto Lévy measures.

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1 Introduction

In a series of papers Hult and Lindskog [11, 12, 13, 14] have defined and investigated regular variation of measures and additive processes, which apply in particular to Lévy measures and Lévy processes. Their concept of regular variation of a stochastic process with càdlàg sample paths is for a Lévy process $\boldsymbol{X}$ equivalent to regular variation of the random vector $\boldsymbol{X}_1$ and its Lévy measure; cf. [14], Lemma 2.1. Similar concepts have been used to study the extremal behaviour of stochastic processes in De Haan and Lin [7] and in Giné, Hahn and Vatan [9].

Since regular variation of a random vector $\boldsymbol{X}_1$ is well understood (cf. Resnick [20, 21]), it seems that all such results can be translated to the corresponding Lévy measure. Of course, this is in principle true, but we argue that new insight in the dynamic of the Lévy process $\boldsymbol{X}$ can be gained by investigating regular variation of the Lévy measure itself. This is particularly true when our main emphasis is on dependence modeling between the jump components of a multivariate Lévy process. From the Lévy-Khinchine representation in (2.1) below it is clear that the Lévy measure determines dependence in the jump part of $\boldsymbol{X}$ for every arbitrary time interval.

With a view towards the overwhelming success story of the distributional copula, Tankov and collaborators (cf. [24, 25, 17, 6]) have introduced the so-called Lévy copula by standardizing the marginal Lévy measure to Lebesgue measure. Contrary to the distributional copula model, which is always a distribution function, the Lévy copula defines a measure which is not a Lévy measure.

It has been suggested in Definition 2.2 of [1] that a simple inversion transformation of the one-dimensional marginal Lebesgue measures leads to a Lévy measure with standard 1-stable margins. For multivariate distributions [18] suggested the transformation of marginals not to uniform but to standard Pareto distributions. This has the advantage that the well-developed theory of multivariate regular variation can be applied. In this paper functional limit theorems for partial sums and partial maxima to stable Lévy processes are proved. The corresponding Lévy measures reflect the transformation of marginals by a representation in terms of marginal standard 1-stable Lévy measures and a *Pareto Lévy copula* as denoted in [18].

Both papers [1, 18] restrict their investigations to spectrally positive Lévy processes, i.e. Lévy measures concentrated on the positive cone $[0,\infty)^d$. Moreover, their respective focus is on applications and not so much on the basic understanding of the marginal transformation. In the present paper we start with a Lévy process in $\mathbb{R}^d$ with positive or negative jumps possible in every component and explore the transformation of the marginal Lévy measures to Lévy measures of standard 1-stable Lévy processes.

Our paper is organised as follows. In Section 2 we recall basic knowledge about multivariate regular variation of Lévy measures and formulate Sklar's Theorem for Pareto Lévy measures. The main results about regular variation of a Lévy measure and its Pareto Lévy measure we prove in Section 3. In Section 4 we calculate the limit measure of a bivariate regular varying Lévy measure for homogeneous Pareto Lévy measures and consider also a non-homogenous example outside regular variation. For $d = 2$ we also show graphical representation of the dependence structure in Section 5. Some technical proofs are postponed to the Appendix.

2 Preliminaries

We assume that all random elements considered are defined on a common probability space $(\Omega, \mathcal{F}, P)$. For a topological space $\mathbb{T}$ its Borel-σ-algebra is denoted by $\mathcal{B}(\mathbb{T})$. For $B \in \mathcal{B}(\mathbb{T})$, we denote by B° and by $\overline{B}$ the interior and the closure of B, respectively, and $\partial B = \overline{B} \setminus B^\circ$ is the boundary of B.

Regular variation of Lévy measures is formulated in terms of vague convergence of Radon measures on $\mathbb{E} := \overline{\mathbb{R}}^d \setminus \{\mathbf{0}\} := [-\infty, \infty]^d \setminus \{\mathbf{0}\}$, where $\mathbf{0} := (0, \dots, 0)$ is the zero in $\mathbb{R}^d$, and we also denote $\boldsymbol{\infty} := (\infty, \dots, \infty)$. $\mathbb{E}$ is equipped with the usual topology such that $\mathcal{B}(\mathbb{E}) = \mathcal{B}(\overline{\mathbb{R}}^d) \cap \mathbb{E}$, and the Borel sets of $\mathbb{R}^d$ bounded away from $\mathbf{0}$ are relatively compact in $\mathbb{E}$.

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ we write $\boldsymbol{a} < \boldsymbol{b}$, if this holds componentwise. For $a, b \in \mathbb{R}$ we write $a \vee b := \max\{a, b\}$. For a set I we define $|I|$ as its cardinality.

Let $\boldsymbol{X} := (\boldsymbol{X}_t)_{t \geq 0}$ be a Lévy process in $\mathbb{R}^d$ with characteristic triple $(\boldsymbol{\gamma}, A, \Pi)$, where $\boldsymbol{\gamma} \in \mathbb{R}^d$, A is a symmetric non-negative definite $d \times d$ matrix, and the *Lévy measure* Π is a measure on $\mathbb{R}^d$ satisfying $\Pi(\{\mathbf{0}\}) = 0$ and $\int_{\mathbb{R}^d} \min\{1, |\boldsymbol{x}|^2\} \Pi(\mathrm{d}\boldsymbol{x}) < \infty$, where $|\cdot|$ denotes any arbitrary norm in $\mathbb{R}^d$. The Lévy-Khintchine representation gives for every $t \geq 0$,

$$\mathbb{E}\left[e^{i(\boldsymbol{z}, \boldsymbol{X}_t)}\right] = e^{-t\Psi(\boldsymbol{z})}, \quad \boldsymbol{z} \in \mathbb{R}^d,$$

where

$$\Psi(\boldsymbol{z}) = i(\boldsymbol{\gamma}, \boldsymbol{z}) + \frac{1}{2}\boldsymbol{z}^\top A \boldsymbol{z} + \int_{\mathbb{R}^d} \left(1 - e^{i(\boldsymbol{z}, \boldsymbol{x})} + i(\boldsymbol{z}, \boldsymbol{x}) 1_{\{|\boldsymbol{x}| \leq 1\}}\right) \Pi(\mathrm{d}\boldsymbol{x}) \tag{2.1}$$

and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^d$. We consider a Lévy measure Π on $\mathbb{E}$ by setting $\Pi(B) := \Pi(B \cap \mathbb{R}^d)$ for $B \in \mathcal{B}(\mathbb{E})$. Further, we assume throughout that the sample paths of $\boldsymbol{X}$ are almost surely (a.s.) right-continuous and have left limits. For details and more background on Lévy processes we refer to [23].

2.1 Multivariate regular variation of Lévy measures

The notion of multivariate regular variation of a random vector has been in the focus of multivariate extreme value theory for years; cf. [20, 21]. Of course, a reformulation of the definition in terms of a multivariate probability measure is obvious. Since Lévy measures may, however, be infinite measures, they require an extension to such a setting, which has been investigated in all generality in [13].

Definition 2.1. [[13], Section 3]
A Lévy measure Π *on* $\mathbb{E}$ *is called* regularly varying *if one of the following equivalent definitions (1) or (2) holds.*
(1) There exists a norming sequence $\{c_n\}_{n\in\mathbb{N}}$ *of positive numbers with* $c_n \uparrow \infty$ *as* $n \to \infty$ *and a non-zero Radon measure* μ *on* $\mathcal{B}(\mathbb{E})$ *with* $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ *such that*

$$n\Pi(c_n \cdot) \xrightarrow{v} \mu(\cdot) \quad \text{as } n \to \infty\,, \tag{2.2}$$

where $\xrightarrow{v}$ *denotes vague convergence on* $\mathcal{B}(\mathbb{E})$*. Then necessarily the limit measure* μ *is homogeneous of some degree* $\alpha > 0$ *(called the* index of regular variation*), i.e.* $\mu(t\,\cdot) = t^{-\alpha}\mu(\cdot)$ *for all* $t > 0$*.*
For Π *regularly varying with index* α*, norming sequence* c_n *and limit measure* μ *we shall write* $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$*.*
(2) There exists a finite non-zero measure $\mu_{\mathbb{S}}$ *on* $\mathcal{B}(\mathbb{S})$*, where* $\mathbb{S} := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| = 1\}$ *denotes the unit sphere with respect to a norm* $|\cdot|$ *on* $\mathbb{R}^d$*, such that for all* $u > 0$

$$\frac{\Pi(\{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| > tu, \boldsymbol{x}/|\boldsymbol{x}| \in \cdot\})}{\Pi(\{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| > t\})} \xrightarrow{w} u^{-\alpha}\mu_{\mathbb{S}}(\cdot) \quad \text{as } t \to \infty\,, \tag{2.3}$$

where $\xrightarrow{w}$ *denotes weak convergence on* $\mathcal{B}(\mathbb{S})$*. We call* $\mu_{\mathbb{S}}$ *the* spectral measure *of* Π*.*

We did not specify the norm $|\cdot|$, since regular variation does not depend on the choice of the norm; see [10], Lemma 2.1.

2.2 The Pareto Lévy measure

We use the same notation as in [17]. Define $\mathrm{sgn}(x) := 1_{\{x\geq 0\}} - 1_{\{x<0\}}$ and

$$\mathcal{I}(x) := \begin{cases} (x, \infty), & x \geq 0, \\ (-\infty, x], & x < 0\,. \end{cases} \tag{2.4}$$

Definition 2.2. [Tail integral of a Lévy measure]
Let $\boldsymbol{X}$ *be a Lévy process in* $\mathbb{R}^d$ *with Lévy measure* Π*. The* tail integral *of* $\boldsymbol{X}$ *is the function* $\overline{\Pi} : (\overline{\mathbb{R}} \setminus \{0\})^d \to \mathbb{R}$ *defined as*

$$\overline{\Pi}(x_1, \ldots, x_d) := \Pi\left(\prod_{i=1}^d \mathcal{I}(x_i)\right) \prod_{j=1}^d \mathrm{sgn}(x_j)\,.$$

By definition (2.4) all tail integrals are right-continuous functions on $(\mathbb{R}\setminus\{0\})^d$. The tail integral does not determine the Lévy measure uniquely, because it does not specify its mass on $\mathbb{R}^d\setminus(\mathbb{R}\setminus\{0\})^d$.

Definition 2.3. [Margins of a Lévy process/Lévy measure/tail integral]
Let $\boldsymbol{X}=(X^1,\dots,X^d)=(X^1_t,\dots,X^d_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Π and $I\subseteq\{1,\dots,d\}$ a non-empty index set. We define the following quantities:
(1) The I-margin *of $\boldsymbol{X}$ is the Lévy process $X^I:=(X^i)_{i\in I}$.*
(2) Π_I denotes the Lévy measure of X^I and is the I-marginal Lévy measure. *It is given by*

$$\Pi_I(A)=\Pi\big(\{\boldsymbol{x}\in\mathbb{R}^d:(x_i)_{i\in I}\in A\}\big)\,,\quad A\in\mathcal{B}(\mathbb{R}^{|I|}\setminus\{\mathbf{0}\})\,.$$

(3) The I-marginal tail integral *of $\boldsymbol{X}$ is given by $\overline{\Pi}_I:(\overline{\mathbb{R}}\setminus\{0\})^{|I|}\to\mathbb{R}$ with*

$$\overline{\Pi}_I((x_i)_{i\in I})=\Pi_I\Big(\prod_{i\in I}\mathcal{I}(x_i)\Big)\prod_{i\in I}^{d}\operatorname{sgn}(x_i)\,.$$

To simplify notation, we denote one-dimensional margins by X^i, Π_i and $\overline{\Pi}_i$.

By [17], Lemma 3.5, the set of all marginal tail integrals $\{\overline{\Pi}_I: I\subseteq\{1,\dots,d\}\}$ determines the Lévy measure Π uniquely and vice versa.

The following lemma is well-known. Indeed, [21] introduces multivariate regular variation in two versions: firstly, he defines *standard regular variation with uniform normalization*, see p. 173, and, secondly, *non-standard regular variation with componentwise normalization*, see p. 204. For the sake of selfcontainedness of our paper, we prove the following result in the Appendix. Theorem 3.1 below can be viewed as a converse of this lemma.

Lemma 2.4. *Let Π be a d-dimensional Lévy measure. If $\Pi\in\mathrm{RV}(\alpha,c_n,\mu)$, then for $x>0$ and all $i=1,\dots,d$,*

$$n\overline{\Pi}_i(c_nx)\to\overline{\mu}_i(1)x^{-\alpha}\quad\text{and}\quad n\overline{\Pi}_i(-c_nx)\to\overline{\mu}_i(-1)x^{-\alpha}\quad\text{as } n\to\infty\,, \tag{2.5}$$

where $\mu_i(B):=\mu(\{\boldsymbol{x}\in\mathbb{E}:x_i\in B\})$ for $B\in\mathcal{B}(\overline{\mathbb{R}}\setminus\{0\})$ and $\overline{\mu}_i(1),|\overline{\mu}_i(-1)|\in[0,\infty)$. Further, there exists an index $i_*\in\{1,\dots,d\}$ such that $\overline{\mu}_{i_*}(1)-\overline{\mu}_{i_*}(-1)>0$ and $\Pi_{i_*}\in\mathrm{RV}(\alpha,c_n,\mu_{i_*})$.

Now we present our reference Lévy measure, which will appear for every d-dimensional Lévy measure after transformation of the marginal Lévy measures. It has been proposed in [18].

Definition 2.5. [Pareto Lévy measure, Pareto Lévy copula]
Let Γ be a d-dimensional Lévy measure with one-dimensional marginals $\Gamma_i(\mathrm{d}x_i) = |x_i|^{-2}\,\mathrm{d}x_i$ on $\mathbb{R}\setminus\{0\}$. Then we call Γ the Pareto Lévy measure (PLM) *and its tail integral $\overline{\Gamma}$ is called* Pareto Lévy copula (PLC).

The margins Γ_i are Lévy measures of 1-stable Lévy processes, but Γ is in general not the Lévy measure of a 1-stable Lévy process.

The following result has been proved for Lévy copulas in [17], Theorem 3.6.

Theorem 2.6. [Sklar's Theorem for Pareto Lévy measures]
(1) Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Π. Let $\emptyset \neq I \subseteq \{1,\dots,d\}$ be an arbitrary index set. Then there exists a PLM Γ such that

$$\overline{\Pi}_I\left((x_i)_{i\in I}\right) = \overline{\Gamma}_I\left(\left(\frac{1}{\overline{\Pi}_i(x_i)}\right)_{i\in I}\right), \quad (x_i)_{i\in I} \in (\overline{\mathbb{R}}\setminus\{0\})^{|I|}\,. \tag{2.6}$$

The PLM Γ is unique on $\prod_{i=1}^d \overline{\mathrm{Ran}(1/\overline{\Pi}_i)}$.
(2) Let Γ be a d-dimensional PLM and $\overline{\Pi}_i$ for $i=1,\dots,d$ one-dimensional tail integrals of arbitrary Lévy processes. Then there exists a Lévy process $\boldsymbol{X}$ in $\mathbb{R}^d$, whose components have tail integrals $\overline{\Pi}_1,\dots,\overline{\Pi}_d$ and whose marginal Lévy measures satisfy (2.6) for every non-empty $I\subseteq\{1,\dots,d\}$ and every $(x_i)_{i\in I}\in(\overline{\mathbb{R}}\setminus\{0\})^{|I|}$. The Lévy measure Π of $\boldsymbol{X}$ is uniquely determined by Γ and $\overline{\Pi}_1,\dots,\overline{\Pi}_d$.

Proof. (1) Recall the following tools from [17], Theorem 3.6. For $x\in(-\infty,\infty]$ and $i=1,\dots,d$ we define

$$\dot{\overline{\Pi}}_i(x) := \begin{cases} \overline{\Pi}_i(x) & \text{for } x\neq 0\,, \\ \infty & \text{for } x = 0 \end{cases}$$

and

$$\Delta\overline{\Pi}_i(x) := \begin{cases} \lim_{\xi\uparrow x}\overline{\Pi}_i(\xi) - \overline{\Pi}_i(x) = \Pi_i(\{x\}) & \text{for } x\neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Since Π_i may have atoms or may be finite, we construct an atomless infinite measure m on $\mathcal{B}((-\infty,\infty]^d\setminus\{\mathbf{0}\}\times[0,1]^d\times\mathbb{R})$. Denote by λ the Lebesgue measure on $\mathbb{R}$ and by $\delta_{\boldsymbol{x}}$ the Dirac measure in $\boldsymbol{x}$. Then we take the product measure

$$\begin{aligned} m \;&:=\; \Pi\otimes\lambda|_{[0,1]^d}\otimes\delta_0 \\ &\quad + \sum_{i=1}^d \delta_{(\underbrace{0,\dots,0}_{i-1},\infty,0,\dots,0)}\otimes\delta_{(\underbrace{0,\dots,0}_{d})}\otimes\lambda|_{\left((-\infty,-\Pi_i(-\infty,0))\cup(\Pi_i(0,\infty),\infty)\right)}\,. \end{aligned} \tag{2.7}$$

For $B \in \mathcal{B}(\mathbb{R}^d \setminus \{\mathbf{0}\})$ we define

$$\Gamma(B) := m\Bigg(\Big\{(x_1,\ldots,x_d,y_1,\ldots,y_d,z) \in ((-\infty,\infty]^d \setminus \{\mathbf{0}\}) \times [0,1]^d \times \mathbb{R} : \tag{2.8}$$
$$\left(\frac{1}{\dot{\overline{\Pi}}_i(x_i) + y_i \Delta\overline{\Pi}_i(x_i) + z}\right)_{i=1,\ldots,d} \in B\Big\}\Bigg),$$

where we set $1/\infty := 0$ and $1/0 := \infty$. Note that we use $1/(\dot{\overline{\Pi}}_i(x_i + y_i\Delta\overline{\Pi}_i(x_i) + z)$ in (2.8) instead of $\dot{\overline{\Pi}}_i(x_i + y_i\Delta\overline{\Pi}_i(x_i) + z$ as in [17], equation (3.5) (corresponding to componentwise inversion: $x \mapsto 1/x$). By the proof of Theorem 3.6, [17], it follows $\overline{\Gamma}_i(x) = x^{-1}$ for $x \neq 0$ and (1) is proved.
(2) Define the set of marginal tail integrals of Π by equation (2.6). Then the Lévy measures Π_i are the one-dimensional margins of Π and, therefore, Π is a Lévy measure. With [17], Lemma 3.5, Π is uniquely defined by its marginal tail integrals. $\square$

Remark 2.7. [Relation between Lévy copula and Pareto Lévy measure]
Every Lévy copula $\widehat{C}$ defines uniquely a PLM Γ, given for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ such that $\mathbf{0} \notin [\boldsymbol{x}, \boldsymbol{y})$ by

$$\Gamma([\boldsymbol{x}, \boldsymbol{y})) = \sum_{\boldsymbol{u} \in \{1/y_1, 1/x_1\} \times \cdots \times \{1/y_d, 1/x_d\}} (-1)^{N(\boldsymbol{u})} \widehat{C}(\boldsymbol{u}), \tag{2.9}$$

where $\boldsymbol{u} = (u_1, \ldots, u_d) \in (-\infty, \infty]^d$, $N(\boldsymbol{u}) := \#\{k : u_k = 1/y_k\}$ and $1/0 := \infty$. Furthermore, for the PLC $\overline{\Gamma}$ we have for $\boldsymbol{x} = (x_1, \ldots, x_d) \in (\overline{\mathbb{R}} \setminus \{0\})^d$

$$\overline{\Gamma}(x_1, \ldots, x_d) = \widehat{C}\left(\frac{1}{x_1}, \ldots, \frac{1}{x_d}\right). \tag{2.10}$$

Consequently, for a PLM Γ with corresponding Lévy copula $\widehat{C}$ the following assertions are equivalent:
(1) Γ is the Lévy measure of a 1-stable Lévy process.
(2) Γ is homogeneous of degree 1; i.e. $\Gamma(tA) = t^{-1}\Gamma(A)$ for all $t > 0$ and $A \in \mathcal{B}(\mathbb{E})$.
(3) $\widehat{C}$ satisfies for all $t > 0$

$$\widehat{C}(tx_1, \ldots, tx_d) = t\widehat{C}(x_1, \ldots, x_d), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

With equation (2.9) we can reformulate Theorem 4.6 of [17] for PLMs .

Theorem 2.8. *Let Π be a Lévy measure with one-dimensional margins $\Pi_i, i = 1, \ldots, d$, and $\alpha \in (0, 2)$. Π is homogenous of degree α if and only if all Π_i are homogenous of degree α and if it has a PLM Γ that is homogeneous of degree 1.*

The advantage of working with a Pareto Lévy measure instead of a Lévy copula should be clear: the Pareto Lévy measure is always a Lévy measure. The corresponding Lévy process extends the class of stable processes in a natural way; see Sklar's Theorem 2.6.

In our next results we formulate the relationship between a Lévy measure Π and its (by Sklar's Theorem) transformed PLM Γ for sets in the generating semi-algebra of rectangular sets. For one-dimensional tail integrals we define (recall the possible singularity in 0)

$$\overline{\Pi}_i(x+) := \lim_{\beta\downarrow x}\overline{\Pi}_i(\beta) \quad \text{and} \quad \overline{\Pi}_i(x-) := \lim_{\beta\uparrow x}\overline{\Pi}_i(\beta) \quad \text{for } x \in \mathbb{R}. \tag{2.11}$$

The tail integral $\overline{\Pi}_i$ is continuous on $\mathbb{R}\setminus\{0\}$ if and only if Π_i has no atoms on $\mathbb{R}\setminus\{0\}$.

Since Lévy measures can be finite or infinite in $\mathbf{0}$, the Lévy measure on the hyperplanes through the axes needs special attention. Furthermore, since Pareto Lévy copulas are defined quadrantwise, special care has to be taken for sets which are not concentrated in one single quadrant. The following result presents the Lévy measure Π of arbitrary rectangles in terms of the PLM Γ and the one-dimensional marginal tail integrals $\overline{\Pi}_i, i = 1,\dots,d$. The proof is given in the Appendix.

Proposition 2.9. *Let Γ be a PLM and Π_i for $i = 1,\dots,d$ one-dimensional Lévy measures. Let Π be the Lévy measure defined by equation (2.6). With $\overline{\Pi}_i(0) = \overline{\Pi}_i(0+)$ the following assertions hold.*

(1) For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$ with $\mathbf{0}\notin\prod_{i=1}^d(a_i,b_i]$,

$$\Pi\left(\prod_{i=1}^d (a_i,b_i]\right) = \Gamma\left(\prod_{i=1}^d\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right). \tag{2.12}$$

(2) Let $\emptyset \neq K \subset \{1,\dots,d\}$. Define

$$A_i := \begin{cases} \left[\frac{1}{\overline{\Pi}_i(0-)},\frac{1}{\overline{\Pi}_i(0+)}\right], & \text{if } \overline{\Pi}_i(0-)<0, \overline{\Pi}_i(0+)>0\,,\\ \left[0,\frac{1}{\overline{\Pi}_i(0+)}\right], & \text{if } \overline{\Pi}_i(0-)=0, \overline{\Pi}_i(0+)>0\,,\\ \left[\frac{1}{\overline{\Pi}_i(0-)},0\right], & \text{if } \overline{\Pi}_i(0-)<0, \overline{\Pi}_i(0+)>0\,. \end{cases} \tag{2.13}$$

For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$ with $\mathbf{0}\notin\prod_{i\in K}\{0\}\times\prod_{i\notin K}(a_i,b_i]$,

$$\Pi\left(\prod_{i\in K}\{0\}\times\prod_{i\notin K}(a_i,b_i]\right) = \Gamma\left(\prod_{i\in K}A_i\times\prod_{i\notin K}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right). \tag{2.14}$$

The following result presents the PLM defined in (2.8) of arbitrary rectangles in terms of the Lévy measure Π and its one-dimensional marginal tail integrals $\overline{\Pi}_i$ for $i = 1,\dots,d$. It directly follows from the construction (2.7) and (2.8).

Proposition 2.10. *Let* Π *be a Lévy measure with one-dimensional margins* Π_i *for* $i = 1,\dots,d$. *For the PLM* Γ *defined in (2.8) the following holds.*
(1) Define

$$\mathcal{D}_i := \mathcal{I}\left(\frac{1}{\overline{\Pi}_i(0-)}\right) \cup \mathcal{I}\left(\frac{1}{\overline{\Pi}_i(0+)}\right) \cup \{0\}\,. \tag{2.15}$$

For $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^d$ *with* $(\boldsymbol{a},\boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$

$$\begin{aligned}\Gamma((\boldsymbol{a},\boldsymbol{b}]) \;=\; \Pi\otimes\lambda|_{[0,1]^d}\Bigg(\Bigg\{&(x_1,\dots,x_d,y_1,\dots,y_d)\in(\mathbb{R}^d\setminus\{\mathbf{0}\})\times[0,1]^d : \\ &\frac{1}{\dot{\overline{\Pi}}_i(x_i)+y_i\Delta\overline{\Pi}_i(x_i)}\in(a_i,b_i] \text{ for } i=1,\dots,d\Bigg\}\Bigg)\end{aligned} \tag{2.16}$$

(2) For $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^d$ *with* $(\boldsymbol{a},\boldsymbol{b}] \subset \mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$

$$\Gamma((\boldsymbol{a},\boldsymbol{b}]) = \sum_{i=1}^d \underbrace{\delta_0\otimes\cdots\otimes\delta_0}_{i-1}\otimes\Gamma_i\otimes\underbrace{\delta_0\otimes\cdots\otimes\delta_0}_{d-i}((\boldsymbol{a},\boldsymbol{b}])\,, \tag{2.17}$$

where $\Gamma_i(\mathrm{d}x_i) = |x_i|^{-2}\mathrm{d}x_i$ *for* $x_i \in \mathbb{R}\setminus\{0\}$.

Remark 2.11. For a multivariate compound Poisson process the sets $\mathcal{D}_i$ are equal to

$$\mathcal{D}_i = (-\infty, -1/\lambda_i^-] \cup (1/\lambda_i^+, \infty) \cup \{0\}$$

for $i = 1,\dots,d$ where $\lambda_i^-, \lambda_i^+ > 0$ are the intensities of the positive and negative Poisson processes, respectively. The above construction of Proposition 2.10(b) ensures that the resulting PLM has indeed 1-stable margins.

Example 2.12. [Independence PLM]
The jumps of a Lévy process are independent, if the Lévy measure is supported by the coordinate axes $\{x\boldsymbol{e}_i : x \in \mathbb{R}, i = 1,\dots,d\}$, see [23], E 12.10, where the $\boldsymbol{e}_i$ denote the unit vectors in $\mathbb{R}^d$. So the independence PLM is given by

$$\Gamma_\perp(A) = \sum_{i=1}^d \Gamma_i(A_i) \quad \text{for } A \in \mathcal{B}(\mathbb{R}^d\setminus\{\mathbf{0}\}),$$

where $A_i := \{x_i \in \mathbb{R} : (0,\dots,0,x_i,0,\dots,0) \in A\}$ and Γ_i denote the Lévy measure of a one-dimensional standard 1-stable Lévy process. By [17], Lemma 3.5, Γ is characterized by the family of marginal tail integrals $(\overline{\Gamma}_{\perp,I})_{I\subseteq\{1,\dots,d\}}$, given for $(x_1,\dots,x_{|I|}) \in (\overline{\mathbb{R}}\setminus\{0\})^{|I|}$ by

$$\overline{\Gamma}_{\perp,I}(x_1,\dots,x_{|I|}) = \begin{cases} 0\,, & \text{if } |I|>1\,,\\ x^{-1}\,, & \text{if } |I|=1\,.\end{cases}$$

Example 2.13. [Complete positive dependence PLM]
The jumps of a Lévy process are completely dependent or comonotonic, if there exists a strictly ordered set $S \subset K := \{x \in \mathbb{R}^d : \operatorname{sgn}(x_1) = \cdots = \operatorname{sgn}(x_d)\}$ such that for almost all sample paths $\Delta \boldsymbol{X}_t \in S$ for $t > 0$, see [17], Definition 4.2. In this case, all components jump a.s. together and, therefore, the PLM of complete positive dependence is concentrated on $(\mathbb{R} \setminus \{0\})^d$. So Γ is characterized by the corresponding PLC, given for $(x_1 \ldots, x_d) \in (\overline{\mathbb{R}} \setminus \{0\})^d$ by

$$\overline{\Gamma}_{||}(x_1, \ldots, x_d) = \frac{1}{|x_1| \vee \cdots \vee |x_d|} 1_K((x_1, \ldots, x_d)) \prod_{j=1}^{d} \operatorname{sgn}(x_j)$$

and is supported by $\{\boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\} : x_1 = \cdots = x_d\}$.

Example 2.14. [Archimedean PLM]
Analogously to the Archimedean copula construction (cf. [19], Section 4), *Archimedean Pareto Lévy measures* can be defined by constructing their Pareto Lévy copula on $(\mathbb{R} \setminus \{0\})^d$ and setting $\Gamma(\mathbb{R}^d \setminus (\mathbb{R} \setminus \{0\})^d) = 0$.

Let $\varphi : [-1, 1] \to [-\infty, \infty]$ be a strictly increasing continuous function with $\varphi(1) = \infty$, $\varphi(0) = 0$ and $\varphi(-1) = -\infty$, having derivatives of order up to d on $(-1, 0)$ and $(0, 1)$, satisfying for all $k = 1, \ldots, d$,

$$\frac{\partial^k \varphi(u)}{\partial u^k} \geq 0\,, \quad u \in (0, 1)\,, \quad \text{and} \quad (-1)^k \frac{\partial^k \varphi(u)}{\partial u^k} \leq 0\,, \quad u \in (-1, 0)\,.$$

Set $\tilde{\varphi}(u) := 2^{d-2} \left(\varphi(u) - \varphi(-u)\right)$ for $u \in [-1, 1]$. Then

$$\overline{\Gamma}(x_1, \ldots, x_d) = \varphi\left(\prod_{i=1}^{d} \tilde{\varphi}^{-1}\left(\frac{1}{x_i}\right)\right)\,, \quad (x_1 \ldots, x_d) \in (\overline{\mathbb{R}} \setminus \{0\})^d\,,$$

is a Pareto Lévy copula, see [17], Theorem 6.1.

If we construct a Lévy measure Π by margins Π_i for $i = 1, \ldots, d$ and an Archimedean PLM, Π may have mass on $\mathbb{R}^d \setminus (\mathbb{R} \setminus \{0\})^d$, although Γ has not. In equation (2.14) we see that $\Pi(\mathbb{R}^d \setminus (\mathbb{R} \setminus \{0\})^d) > 0$ if and only if $\overline{\Pi}_i(0+) < \infty$ or $\overline{\Pi}_i(0-) > -\infty$ for at least one i.

Example 2.15. [Clayton PLM]
Setting in Example 2.14

$$\varphi(x) = (-\log|x|)^{-1/\theta} \left(\eta 1_{\{x>0\}} - (1-\eta) 1_{\{x<0\}}\right)\,, \quad \theta > 0, \eta \in (0, 1)\,,$$

corresponds to the *Clayton Pareto Lévy measure* and the *Clayton Pareto Lévy copula* is for $\theta > 0, \eta \in [0, 1]$ given for $(x_1, \ldots, x_d) \in (\overline{\mathbb{R}} \setminus \{0\})^d$ by

$$\overline{\Gamma}_{\eta,\theta}(x_1, \ldots, x_d) = 2^{2-d} \left(\sum_{i=1}^{d} |x_i|^{\theta}\right)^{-1/\theta} \left(\eta 1_{\{x_1 \cdots x_d > 0\}} - (1-\eta) 1_{\{x_1 \cdots x_d < 0\}}\right)\,.$$

For $d = 2$ this reduces to

$$\overline{\Gamma}_{\eta,\theta}(x_1, x_2) = \left(|x_1|^\theta + |x_2|^\theta\right)^{-1/\theta} \left(\eta 1_{\{x_1 x_2 > 0\}} - (1-\eta) 1_{\{x_1 x_2 < 0\}}\right) , \tag{2.18}$$

which was frequently used, e.g. in [4, 5, 8]. Obviously, a Clayton PLM is homogeneous of degree 1.

Example 2.16. [Non-homogeneous PLM]
Setting in Example 2.14

$$\varphi(x) = \zeta \frac{|x|}{1-|x|} \left(\eta 1_{\{x>0\}} - (1-\eta) 1_{\{x<0\}}\right) , \quad \zeta > 0, \eta \in (0,1) ,$$

yields the PLC for $\zeta > 0, \eta \in [0,1]$ given for $(x_1, \dots, x_d) \in (\overline{\mathbb{R}} \setminus \{0\})^d$ as

$$\overline{\Gamma}_{\eta,\zeta}(x_1, \dots, x_d) = \frac{\zeta \prod_{i=1}^d |1/x_i|}{\prod_{i=1}^d (|1/x_i| + \zeta) - \prod_{i=1}^d |1/x_i|} \left(\eta 1_{\{x_1 \cdots x_d > 0\}} - (1-\eta) 1_{\{x_1 \cdots x_d < 0\}}\right) .$$

For $d = 2$ the PLC $\overline{\Gamma}_{\eta,\zeta}$ reduces to

$$\overline{\Gamma}_{\eta,\zeta}(x_1, x_2) = \frac{1}{|x_1| + |x_2| + \zeta |x_1 x_2|} \left(\eta 1_{\{x_1 x_2 > 0\}} - (1-\eta) 1_{\{x_1 x_2 < 0\}}\right) ,$$

which was treated in [8], Example 2.8(d). Obviously, this PLM has homogeneous one-dimensional margins, but is not homogeneous of degree 1.

3 Main Results

It is well-known and we proved it in Lemma 2.4 that multivariate regular variation of Π implies regular variation of at least one of the one-dimensional marginal Lévy measures Π_i. To prove the converse, we assume w.l.o.g. that $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$. We also assume that the following tail balance conditions hold for $x > 0$ and for all $i = 1, \dots, d$

$$\lim_{n\to\infty} n \overline{\Pi}_i(c_n x) = p_i^+ x^{-\alpha} \quad \text{and} \quad \lim_{n\to\infty} -n \overline{\Pi}_i(-c_n x) = p_i^- x^{-\alpha} , \tag{3.1}$$

where $p_i^+, p_i^- \in [0, \infty)$. For $x \in \overline{\mathbb{R}}$ we define

$$p_i^{\mathrm{sgn}(x)}(x) := \begin{cases} p_i^+ , & \text{if } x \geq 0, \\ p_i^- , & \text{if } x < 0. \end{cases}$$

The following result is an analogon of [18], Theorem 3.1, for distributional copulas.

Theorem 3.1. *Let Γ be a PLM and Π_i for $i = 1, \dots, d$ one-dimensional Lévy measures. Let Π be the d-dimensional Lévy measure defined in (2.12). Suppose that $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$ and that the tail balance conditions (3.1) for the margins hold. Furthermore, suppose that $\Gamma \in \mathrm{RV}(1, n, \nu)$. Then $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$, where for $\boldsymbol{a} = (a_i)_{i=1,\dots,d}, \boldsymbol{b} = (b_i)_{i=1,\dots,d} \in \mathbb{R}^d$ and for $i = 1, \dots, d$,*

$$\widetilde{a_i} := \begin{cases} 0\,, & \text{if } a_i = 0\,, \\ \mathrm{sgn}(a_i)(p_i^{\mathrm{sgn}(a_i)})^{-1}|a_i|^\alpha\,, & \text{if } a_i \neq 0, p_i^{\mathrm{sgn}(a_i)} > 0\,, \\ \infty\,, & \text{if } a_i > 0, p_i^{\mathrm{sgn}(a_i)} = 0\,, \\ -\infty\,, & \text{if } a_i < 0, p_i^{\mathrm{sgn}(a_i)} = 0\,, \end{cases} \tag{3.2}$$

and $\widetilde{b_i}$ is defined analogously. Furthermore, we have

$$\mu((\boldsymbol{a}, \boldsymbol{b}]) = \nu\left(\prod_{i=1}^{d}(\widetilde{a}_i, \widetilde{b}_i]\right)\,. \tag{3.3}$$

Proof. First we show that $\{n\Pi(c_n\cdot)\}_{n\in\mathbb{N}}$ is relatively compact in the vague topology. Since Π is a Lévy measure, for the ball $B_{\boldsymbol{0},r} = \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x} - \boldsymbol{0}| < r\}$ we get

$$\sup_{n\in\mathbb{N}} n\Pi\left(c_n(\mathbb{R}^d \setminus B_{\boldsymbol{0},r})\right) < \infty \quad \text{for all } r > 0$$

and by [16], Theorem 15.7.5, the sequence $\{n\Pi(c_n\cdot)\}_{n\in\mathbb{N}}$ is relatively compact. So there are subsequential vague limits and by [13], Theorem 2.8, we have to show convergence for sets in a determining class. The sets $\{(\boldsymbol{a}, \boldsymbol{b}] : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}, \boldsymbol{a} \leq \boldsymbol{b}\}$ are an $\cap$-stable generator of $\mathcal{B}(\mathbb{E})$, but since $\Pi(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ it is sufficient to investigate convergence on the sets $\{(\boldsymbol{a}, \boldsymbol{b}] : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}, \boldsymbol{a} \leq \boldsymbol{b}\}$. Consequently, we have to show that $n\Pi(c_n(\boldsymbol{a}, \boldsymbol{b}]) \to \mu((\boldsymbol{a}, \boldsymbol{b}])$ as $n \to \infty$ for all sets $(\boldsymbol{a}, \boldsymbol{b}]$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$, $\boldsymbol{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and

$$\mu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = \mu\left(\bigcup_{k=1}^{d}\prod_{i<k}[a_i, b_i] \times \{a_k, b_k\} \times \prod_{i>k}[a_i, b_i]\right) = 0\,,$$

where μ is a non-zero Radon measure with $\mu(\overline{\mathbb{R}}^d \setminus \{\boldsymbol{0}\}) = 0$ and homogeneous of degree α.

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}$ and the weight constants p_i given in (3.1) we define index sets

$$\begin{aligned} &K_1 := \{i : a_i b_i \neq 0, p_i^{\mathrm{sgn}(a_i)} p_i^{\mathrm{sgn}(b_i)} > 0\}\,, && K_2 := \{i : a_i b_i \neq 0, p_i^{\mathrm{sgn}(a_i)} > 0, p_i^{\mathrm{sgn}(b_i)} = 0\}\,, \\ &K_3 := \{i : a_i b_i \neq 0, p_i^{\mathrm{sgn}(a_i)} = 0, p_i^{\mathrm{sgn}(b_i)} > 0\}\,, && K_4 := \{i : a_i b_i > 0, p_i^{\mathrm{sgn}(a_i)} = p_i^{\mathrm{sgn}(b_i)} = 0\}\,, \\ &K_5 := \{i : a_i < 0 < b_i, p_i^{\mathrm{sgn}(a_i)} = p_i^{\mathrm{sgn}(b_i)} = 0\}\,, && K_6 := \{i : a_i = 0, p_i^{\mathrm{sgn}(b_i)} > 0\}\,, \\ &K_7 := \{i : a_i = 0, p_i^{\mathrm{sgn}(b_i)} = 0\}\,, && K_8 := \{i : b_i = 0, p_i^{\mathrm{sgn}(a_i)} > 0\}\,, \\ &K_9 := \{i : b_i = 0, p_i^{\mathrm{sgn}(a_i)} = 0\}\,. && \end{aligned} \tag{3.4}$$

Moreover, we set for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}$ with $\mathbf{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$

$$
\begin{aligned}
&\mu((\boldsymbol{a}, \boldsymbol{b}]) \qquad (3.5)\\
&:= \nu\Bigg(\prod_{i \in K_1} \left(\frac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}} |a_i|^\alpha, \frac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}} |b_i|^\alpha \right] \times \prod_{i \in K_2} \left(\frac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}} |a_i|^\alpha, \infty \right) \\
&\quad \times \prod_{i \in K_3} \left(-\infty, \frac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}} |b_i|^\alpha \right] \times \prod_{i \in K_4} \emptyset \times \prod_{i \in K_5} (-\infty, \infty) \times \prod_{i \in K_6} \left(0, \frac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}} |b_i|^\alpha \right] \\
&\quad \times \prod_{i \in K_7} (0, \infty) \times \prod_{i \in K_8} \left(\frac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}} |a_i|^\alpha, 0 \right] \times \prod_{i \in K_9} (-\infty, 0] \Bigg).
\end{aligned}
$$

Consider sets $(\boldsymbol{a}, \boldsymbol{b}]$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$, $\mathbf{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and $\mu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$. From relation (2.12) we obtain

$$
\begin{aligned}
&n\Pi(c_n(\boldsymbol{a}, \boldsymbol{b}]) \qquad (3.6)\\
&= n\Gamma\Bigg(n \prod_{i \in K_1, K_2, K_3, K_4, K_5} \left(\frac{1}{n\overline{\Pi}_i(c_n a_i)}, \frac{1}{n\overline{\Pi}_i(c_n b_i)} \right] \times \prod_{i \in K_6, K_7} \left(\frac{1}{n\overline{\Pi}_i(0+)}, \frac{1}{n\overline{\Pi}_i(c_n b_i)} \right] \\
&\quad \times \prod_{i \in K_8, K_9} \left(\frac{1}{n\overline{\Pi}_i(c_n a_i)}, \frac{1}{n\overline{\Pi}_i(0+)} \right] \Bigg).
\end{aligned}
$$

From the definition of the p_i in (3.1) we conclude for $n \to \infty$ that

$$
\begin{aligned}
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to \left(\tfrac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}} |a_i|^\alpha, \tfrac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}} |b_i|^\alpha \right] =: B_1 && \text{for } i \in K_1, \\
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to \left(\tfrac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}} |a_i|^\alpha, \infty \right) := B_2 && \text{for } i \in K_2, \\
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to \left(-\infty, \tfrac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}} |b_i|^\alpha \right] =: B_3 && \text{for } i \in K_3, \\
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to \emptyset =: B_4 && \text{for } i \in K_4, \\
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to (-\infty, \infty) =: B_5 && \text{for } i \in K_5, \\
\left(\tfrac{1}{n\overline{\Pi}_i(0+)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to \left(0, \tfrac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}} |b_i|^\alpha \right] =: B_6 && \text{for } i \in K_6, \\
\left(\tfrac{1}{n\overline{\Pi}_i(0+)}, \tfrac{1}{n\overline{\Pi}_i(c_n b_i)} \right] &\to (0, \infty) =: B_7 && \text{for } i \in K_7, \\
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(0+)} \right] &\to \left(\tfrac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}} |a_i|^\alpha, 0 \right] =: B_8 && \text{for } i \in K_8, \\
\left(\tfrac{1}{n\overline{\Pi}_i(c_n a_i)}, \tfrac{1}{n\overline{\Pi}_i(0+)} \right] &\to (-\infty, 0] =: B_9 && \text{for } i \in K_9.
\end{aligned}
$$

Furthermore,

$$
\mathbf{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]} \quad \Rightarrow \quad \mathbf{0} \notin \overline{\prod_{i=1}^d \left(\frac{1}{n\overline{\Pi}_i(c_n a_i)}, \frac{1}{n\overline{\Pi}_i(c_n b_i)} \right]} \quad \Rightarrow \quad \mathbf{0} \notin \prod_{i=1}^9 \overline{B}_i
$$

and

$$\mu(\partial(\boldsymbol{a},\boldsymbol{b}]) = 0 \quad \Rightarrow \quad \nu\left(\partial \prod_{i=1}^{d}\left(\frac{1}{n\overline{\Pi}_i(c_n a_i)}, \frac{1}{\overline{\Pi}_i(c_n b_i)}\right]\right) = 0 \quad \Rightarrow \nu\left(\partial \prod_{i=1}^{9} B_i\right) = 0\,.$$

Since $n\Gamma(n\cdot) \xrightarrow{v} \nu(\cdot)$ as $n \to \infty$ and ν has no atoms on the considered sets applying Propositions A.1 and A.2 yields that expression (3.6) converges to μ and relation (3.3) holds.
The properties of μ can easily be seen. μ is an α-homogeneous Radon measure on $\mathcal{B}(\mathbb{E})$ with $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ since ν is a 1-homogeneous Lévy measure. Moreover, μ is a non-zero Radon measure because the one-dimensional margin μ_1 is a non-zero measure. □

Corollary 3.2. *Assume the situation of Theorem 3.1. Then*

$$\lim_{t\to\infty} \frac{\overline{\Pi}(t,\dots,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(1,\dots,1)}{\overline{\mu}_1(1)} = \frac{1}{p_1^+}\overline{\nu}\left(\frac{1}{p_1^+},\dots,\frac{1}{p_d^+}\right), \tag{3.7}$$

$$\lim_{t\to-\infty} \frac{\overline{\Pi}(t,\dots,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(-1,\dots,-1)}{\overline{\mu}_1(-1)} = -\frac{1}{p_1^-}\overline{\nu}\left(-\frac{1}{p_1^-},\dots,-\frac{1}{p_d^-}\right). \tag{3.8}$$

Proof. As a consequence of (3.3) we have

$$\lim_{t\to\infty} \frac{\overline{\Pi}(t,\dots,t)}{\overline{\Pi}_1(t)} = \lim_{n\to\infty} \frac{\overline{\Pi}(c_n,\dots,c_n)}{\overline{\Pi}_1(c_n)} = \frac{\overline{\mu}(1,\dots,1)}{\overline{\mu}_1(1)} = \frac{1}{p_1^+}\overline{\nu}\left(\frac{1}{p_1^+},\dots,\frac{1}{p_d^+}\right).$$

□

Note that both limits are independent of α, i.e. they are defined by the dependence structure given by the PLM and the weight constants of the marginal tail integrals. The following notion is well-known for bivariate probability measures and has attracted much attention when modelling joint extremes; see e.g. [15].

Definition 3.3 (Tail dependence coefficient)**.**
(1) Let Π *be a Lévy measure with PLM* Γ*. We define the* upper and lower tail dependence coefficient *of* Γ *as*

$$\Lambda_U := \lim_{t\to\infty} t\overline{\Gamma}(t,\dots,t) \quad \textit{and} \quad \Lambda_L := \lim_{t\to-\infty} |t\overline{\Gamma}(t,\dots,t)|.$$

If $\Lambda_U > 0$ *we call* Γ upper tail dependent*, and if* $\Lambda_L > 0$ *the PLM* Γ *is called* lower integral dependent*.*
(2) If $\Gamma_U > 0$ *(*$\Gamma_L > 0$*) and the conditions (3.1) hold with* $p_i^+ > 0$ *(*$p_i^- > 0$*) for all* $i = 1,\dots,d$*, then we call* Π *upper (lower) tail dependent.*

Note that Λ_U and Λ_L always exist, since by the standardized one-dimensional margins we have

$$|t\overline{\Gamma}(t,\dots,t)| \le 1 \quad \text{for } t \neq 0.$$

Remark 3.4. If $\Gamma \in \mathrm{RV}(1, n, \nu)$, then

$$\Lambda_U = \overline{\nu}(1,\dots,1) \quad \text{and} \quad \Lambda_L = |\overline{\nu}(-1,\dots,-1)|.$$

The following result is a converse of Theorem 3.1 and extends Proposition 2.4.

Theorem 3.5. *Let Π be a d-dimensional Lévy measure with one-dimensional margins Π_i for $i = 1,\dots,d$, and Γ the PLM given in (2.8). Suppose that $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$. Then the tail balance conditions* (3.1) *hold and $\Gamma \in \mathrm{RV}(1, n, \nu)$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$ with $\mathcal{D}_i$ defined in (2.15), the relation between μ and ν is given as*

$$\nu((\boldsymbol{a}, \boldsymbol{b}]) = \mu\left(\prod_{i=1}^d (\hat{a}_i, \hat{b}_i]\right), \tag{3.9}$$

where for $i = 1,\dots,d$

$$\hat{a}_i := \begin{cases} 0\,, & \text{if } p_i^{\mathrm{sgn}(a_i)} = 0, \\ \mathrm{sgn}(a_i)\left(p_i^{\mathrm{sgn}(a_i)} a_i\right)^{1/\alpha}, & \text{if } p_i^{\mathrm{sgn}(a_i)} > 0\,, \\ \infty\,, & \text{if } a_i > 0, p_i^{\mathrm{sgn}(a_i)} = 0\,, \\ -\infty\,, & \text{if } a_i < 0, p_i^{\mathrm{sgn}(a_i)} = 0\,, \end{cases} \tag{3.10}$$

and the $\hat{b}_i$ are defined in the same way. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$ it holds $\nu((\boldsymbol{a}, \boldsymbol{b}]) = \Gamma((\boldsymbol{a}, \boldsymbol{b}])$.

Proof. By Lemma 2.4, the tail balance conditions (3.1) hold with $p_i^+ := \mu_i(1)$ and $p_i^- := -\overline{\mu}_i(-1)$ and there exists at least one index i_* such that $p_{i_*}^+ + p_{i_*}^- > 0$.
Analogously to the proof of Theorem 3.1 we have to show that $n\Gamma(n(\boldsymbol{a}, \boldsymbol{b}]) \to \nu((\boldsymbol{a}, \boldsymbol{b}])$ as $n \to \infty$ for all sets $(\boldsymbol{a}, \boldsymbol{b}]$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$, $\boldsymbol{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and $\nu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$, where ν is a non-zero 1-homogeneous Radon measure with $\nu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$.
Recall the definition of the sets $\mathcal{D}_i$ in (2.15). By relation (2.17) we have that Γ is 1-homogeneous on $\mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$ and so we define $\nu((\boldsymbol{a}, \boldsymbol{b}]) := \Gamma((\boldsymbol{a}, \boldsymbol{b}])$ for sets $(\boldsymbol{a}, \boldsymbol{b}] \subset (\mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i)$. Further, we define ν on $\mathcal{B}(\mathbb{E})$ by $\nu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) := 0$ and for $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$ we set with Definition (3.2) for $\widetilde{x}_i$

$$\nu((\boldsymbol{a}, \boldsymbol{b}]) := \mu(\{(x_1,\dots,x_d) \in \mathbb{R}^d \setminus \{\boldsymbol{0}\} : \ \widetilde{x}_i \in (a_i, b_i] \text{ for } i = 1,\dots,d\}).$$

ν is a non-zero 1-homogeneous Radon measure since μ is an α-homogeneous Lévy measure and Γ is 1-homogeneous on $R^d \setminus \prod_{i=1}^d \mathcal{D}_i$. Moreover, ν is a non-zero measure because μ_{i_*}

is a non-zero measure and $p_{i_*}^+ + p_{i_*}^- > 0$.
Suppose that $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ with $\boldsymbol{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and $\nu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$.
With relation (2.16) we obtain for $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$ that

$$n\Gamma(n(\boldsymbol{a}, \boldsymbol{b}]) = n\Pi \otimes \lambda|_{[0,1]^d}\Bigg(\Bigg\{(c_n x_1, \ldots, c_n x_d, y_1, \ldots, y_d) \in (\mathbb{R}^d \setminus \{\boldsymbol{0}\}) \times [0,1]^d : \frac{1}{n\dot{\overline{\Pi}}_i(c_n x_i) + n y_i \Delta\overline{\Pi}_i(c_n x_i)} \in (a_i, b_i] \text{ for } i = 1, \ldots, d\Bigg\}\Bigg). \tag{3.11}$$

With $\lim_{n\to\infty} n\Delta\overline{\Pi}_i(c_n x_i) = 0$ it holds

$$\lim_{n\to\infty} \frac{1}{n\dot{\overline{\Pi}}_i(c_n x_i) + y_i \Delta\overline{\Pi}_i(c_n x_i)} = \begin{cases} 0\,, & \text{if } x_i = 0 \\ \operatorname{sgn}(x_i)(p_i^{\operatorname{sgn}(x_i)})^{-1}|x_i|^\alpha\,, & \text{if } x_i \neq 0, p_i^{\operatorname{sgn}(x_i)} > 0\,, \\ \infty\,, & \text{if } x_i > 0, p_i^{\operatorname{sgn}(x_i)} = 0\,, \\ -\infty\,, & \text{if } x_i < 0, p_i^{\operatorname{sgn}(x_i)} = 0\,. \end{cases}$$

We see that $\nu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$ holds if and only if $\mu(\partial(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}]) = 0$ and $\boldsymbol{0} \notin (\boldsymbol{a}, \boldsymbol{b}]$ implies $\boldsymbol{0} \notin (\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}]$. So with Propositions A.1 and A.3 it results for (3.11) as $n \to \infty$ that

$$\lim_{n\to\infty} n\Gamma(n(\boldsymbol{a}, \boldsymbol{b}]) = \mu \otimes \lambda|_{[0,1]^d}(\{(x_1, \ldots, x_d, y_1, \ldots, y_d) \in (\mathbb{R}^d \setminus \{\boldsymbol{0}\}) \times [0,1]^d : \widetilde{x}_i \in (a_i, b_i] \text{ for } i = 1, \ldots, d\}) = \nu((\boldsymbol{a}, \boldsymbol{b}])\,,$$

and (3.9) follows. □

4 Examples

To simplify notation, we only consider the case $d = 2$. Moreover, we assume that we are in the framework of Theorem 3.1 with $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$, $p_1^+, p_1^- > 0$ and $p_2^+, p_2^- \geq 0$, i.e. $\mu_i(x) = \operatorname{sgn}(x) p_i^{\operatorname{sgn}(x)}|x|^{-\alpha}$ for $x \neq 0$.

Example 4.1. [Independence PLM, continuation of Example 2.12]
Since $\Gamma_\perp$ is homogeneous of degree 1, by Theorem 3.1 we get $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$ and with

$$\Gamma_\perp(\mathrm{d}x_1, \mathrm{d}x_2) = \delta_0(\mathrm{d}x_1)|x_2|^{-2}\mathrm{d}x_2 + \delta_0(\mathrm{d}x_2)|x_1|^{-2}\mathrm{d}x_1\,,\ (x_1, x_2) \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\},$$

the limit measure μ is supported on the axes. Hence, μ is given by

$$\mu(\mathrm{d}x_1, \mathrm{d}x_2) = \delta_0(\mathrm{d}x_1) p_2^{\operatorname{sgn}(x_2)} \frac{\alpha}{|x_2|^{\alpha+1}}\,\mathrm{d}x_2 + \delta_0(\mathrm{d}x_2) p_1^{\operatorname{sgn}(x_1)} \frac{\alpha}{|x_1|^{\alpha+1}}\,\mathrm{d}x_1\,,\ (x_1, x_2) \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}.$$

Then $\Lambda_U = \Lambda_L = 0$ and the limits in (3.7) and (3.8) are equal to 0.

Example 4.2. [Complete positive dependence PLM, continuation of Example 2.13]
The complete positive dependence PLM $\Gamma_\parallel$ is homogeneous of degree 1 and with relation (3.3) we obtain for the limit measure μ that

$$\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) = \left(p_1^{\mathrm{sgn}(x_1)}|x_1|^{-\alpha} \wedge p_2^{\mathrm{sgn}(x_2)}|x_2|^{-\alpha}\right) 1_K((x_1,x_2)),(x_1,x_2)\in(\mathbb{R}\setminus\{0\})^2.$$

For $x_1\in\mathbb{R}\setminus\{0\}$ we get

$$\begin{aligned}\mu(\mathcal{I}(x_1)\times\{0\}) &= \mu_1(\mathcal{I}(x_1)) - \lim_{x_2\uparrow 0}\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) - \lim_{x_2\downarrow 0}\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2))\\ &= \begin{cases} p_1^{\mathrm{sgn}(x_1)}|x_1|^{-\alpha}, & \text{if } p_2^{\mathrm{sgn}(x_1)}=0,\\ 0, & \text{if } p_2^{\mathrm{sgn}(x_1)}>0.\end{cases}\end{aligned}$$

Analogously for $x_2\in\mathbb{R}\setminus\{0\}$ with $p_1^+>0, p_1^->0$ we get $\mu(\{0\}\times\mathcal{I}(x_2))=0$. Since $\Gamma_\parallel$ is supported by $\{(x_1,x_2)\in(\mathbb{R}\setminus\{0\})^2 : x_1=x_2\}$, μ is supported by $\{(x_1,x_2)\in(\mathbb{R}\setminus\{0\})^2 : x_2=(p_2^{\mathrm{sgn}(x_2)}/p_1^{\mathrm{sgn}(x_1)})^{1/\alpha}x_1\}$. Finally, the limit measure μ results in

$$\mu(\mathrm{d}x_1,\mathrm{d}x_2) = p_1^{\mathrm{sgn}(x_1)}\frac{\alpha}{|x_1|^{\alpha+1}}1_{\{x_2=(p_2^{\mathrm{sgn}(x_2)}/p_1^{\mathrm{sgn}(x_1)})^{1/\alpha}x_1\}}\mathrm{d}x_1, x_1\in\mathbb{R}\setminus\{0\}.$$

Then $\Lambda_U=\Lambda_L=1$ and the limits in (3.7) and (3.8) are given as

$$\lim_{t\to\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = \frac{p_1^+\wedge p_2^+}{p_1^+} \quad\text{and}\quad \lim_{t\to-\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = -\frac{p_1^-\wedge p_2^-}{p_1^-}.$$

Example 4.3. [Clayton PLM, continuation of Example 2.15]
The Clayton PLM $\Gamma_{\eta,\theta}$ is homogeneous of degree 1 and we have $\Gamma_{\eta,\theta}(\mathbb{R}^2\setminus(\mathbb{R}\setminus\{0\})^2)=0$. For $x_1\in\mathbb{R}\setminus\{0\}$ we get

$$\begin{aligned}\mu(\mathcal{I}(x_1)\times\{0\}) &= \lim_{\epsilon\uparrow 0}\mu(\mathcal{I}(x_1)\times(\epsilon,0]) \qquad (4.1)\\ &= \begin{cases} 0, & \text{if } p_2^->0,\\ \lim_{\epsilon\uparrow 0}\Gamma_{\eta,\theta}\left(\mathcal{I}\left(\frac{\mathrm{sgn}(x_1)}{p_1^{\mathrm{sgn}(x_1)}}|x_1|^\alpha\right)\times\mathcal{I}(\epsilon)\right), & \text{if } p_2^-=0,\end{cases}\\ &= \begin{cases} 0, & \text{if } p_2^->0,\\ p_1^{\mathrm{sgn}(x_1)}|x_1|^{-\alpha}\left(\eta 1_{\{x_1<0\}}+(1-\eta)1_{\{x_1>0\}}\right), & \text{if } p_2^-=0,\end{cases}\end{aligned}$$

and for $x_2\in\mathbb{R}\setminus\{0\}$ we obtain

$$\begin{aligned}\mu(\{0\}\times\mathcal{I}(x_2)) &= \lim_{\epsilon\uparrow 0}\mu((\epsilon,0]\times\mathcal{I}(x_2)) = \lim_{\epsilon\uparrow 0}\Gamma\left(\left(\frac{-1}{p_1^-}|\epsilon|^\alpha,0\right]\times\mathcal{I}(\widetilde{x}_2)\right)\\ &= \Gamma_{\eta,\theta}(\{0\}\times\mathcal{I}(\widetilde{x}_2))=0,\end{aligned}$$

Let $x_1,x_2\in\mathbb{R}\setminus\{0\}$. If $p_2^{\mathrm{sgn}(x_2)}>0$, then

$$\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) = \left((p_1^{\mathrm{sgn}(x_1)})^{-\theta}|x_1|^{\alpha\theta}+(p_2^{\mathrm{sgn}(x_2)})^{-\theta}|x_2|^{\alpha\theta}\right)^{-1/\theta}\left(\eta 1_{\{x_1x_2>0\}}+(1-\eta)1_{\{x_1x_2<0\}}\right).$$

If $p_i^{\mathrm{sgn}(x_2)} = 0$, then

$$\mu(\mathcal{I}(x_1) \times \mathcal{I}(x_2)) \;=\; \Gamma_{\eta,\theta}\left(\mathcal{I}\left(\frac{\mathrm{sgn}(x_1)}{p_1^{\mathrm{sgn}(x_1)}}|x_1|^{\alpha}\right) \times \emptyset\right) = 0\,.$$

Moreover, the limits in (3.7) and (3.8) are given by

$$\lim_{t\to\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(1,1)}{\overline{\mu}(1)} = \begin{cases} \frac{\eta\left((p_1^+)^{-\theta}+(p_2^+)^{-\theta}\right)^{-1/\theta}}{p_1^+}\,, & \text{if } p_2^+ > 0\,,\\ 0\,, & \text{if } p_2^+ = 0\,,\end{cases}$$

and

$$\lim_{t\to-\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(-1,-1)}{\overline{\mu}(-1)} = \begin{cases} \frac{-\eta\left((p_1^-)^{-\theta}+(p_2^-)^{-\theta}\right)^{-1/\theta}}{p_1^-}\,, & \text{if } p_2^- > 0\,,\\ 0\,, & \text{if } p_2^- = 0\,,\end{cases}$$

and $\Lambda_U = \Lambda_L = \eta 2^{-1/\theta}$. So if $\eta > 0$ and $p_2^+ > 0$ ($p_2^- > 0$), then we always have upper (lower) tail dependence.

Example 4.4. [Non-homogeneous PLM, continuation of 2.16]
$\Gamma_{\eta,\zeta}$ is concentrated on $(\mathbb{R}\setminus\{0\})^2$ and we get

$$n\Gamma_{\eta,\zeta}(n(\mathcal{I}(x_1)\times\mathcal{I}(x_2))) = \frac{1}{|x_1|+|x_2|+n|x_1x_2|} \to 0 \quad \text{as } n\to\infty$$

for all $x_1, x_2 \in \mathbb{R}\setminus\{0\}$. Therefore, Γ is not only non-homogeneous, but also not regularly varying. Consequently, by Theorem 3.1, Π defined by (2.6) is not multivariate regularly varying. Moreover, $\Lambda_U = \Lambda_L = 0$, so there is no tail dependence in this model.

It has been shown in [8] that for $\eta = 1$ the Lévy measure of the sum of the two components of the bivariate Lévy process can, for certain marginal models, be calculated explicitly. This is also true for the Pareto Lévy measure, which is not bivariate regularly varying. Surprisingly, it turns out that the Lévy measure of the sum

$$\Gamma_+(\cdot) := \Gamma_{1,\zeta}(\{(x_1,x_2)\in\mathbb{R}^2\setminus\{\mathbf{0}\} : x_1+x_2\in\cdot\})$$

is univariate regularly varying. Note that this does not contradict [3], Theorem 1.1, where it was proved that a vector $\boldsymbol{X}$ is regularly varying, if and only if *every* linear combination is regularly varying. More precisely, for $z > 0$,

$$\overline{\Gamma}_+(z) = \frac{6+2z\zeta}{z(4+z\zeta)} + \frac{4+2z\zeta}{z(4+z\zeta)\sqrt{z\zeta(4+z\zeta)}}\ln\left(\left|\frac{z\zeta+\sqrt{z\zeta(4+z\zeta)}}{z\zeta-\sqrt{z\zeta(4+z\zeta)}}\right|\right).$$

From this it is easy to see that

$$\overline{\Gamma}_+(z) \sim 2z^{-1} = \overline{\Gamma}_1(z) + \overline{\Gamma}_2(z),$$

which exhibits the same behaviour as for the independence model.

5 Graphical representation of the dependence structure of Lévy processes

For a stable r. v. the spectral measure charaterizes the dependence between the marginals, see [22], which remains true for multivariate regularly varying r. v. in the limit, see Definition 2.1. Consequently, the spectral density has been a popular graphical tool for stable and regularly varying distributions and processes, at least in two dimensions. A 1-homogeneous PLM is the Lévy measure of a standard 1-stable Lévy process and, therefore, we consider its spectral measure where we again restrict the situation to $d = 2$ for presentation purposes.

The Pareto Lévy copula provides a new possibility to visualize the dependence structure between the jump parts of the marginal Lévy processes. As a graphical tool it can also be applied to non-regularly varying PLMs, where no spectral measure exists. Whereas an empirical version of the spectral density estimates a density, immediately by definition, an empirical version of the Pareto Lévy copula estimates a tail integral.

Consequently, one would guess that the empirical spectral density (or some kernel density version) provides more insight into the dependence structure. This is partly true, with the exception that the dependence of joint extremes is estimated by the tail dependence coefficient, which is based on the Pareto Lévy copula and the tail integral as indicated in Definition 3.3 and Corollary 3.2.

In the first subsection we present both graphical representations for bivariate homogeneous Pareto Lévy measures from the previous examples, the spectral density and the Pareto Lévy copula. In the second subsection we visualize the dependence structure of the non-regularly varying PLM given in Example 2.16 only by its PLC, since there exists no spectral measure.

5.1 Homogeneous Pareto Lévy measures

Recall from Theorem 14.3 of [23] that the 1-homogeneous PLM Γ as a 1-stable Lévy measure has for all $B \in \mathcal{B}(\mathbb{R}^d)$ the representation

$$\Gamma(B) = \int_{\mathbb{S}} \int_0^\infty 1_B(r\phi) r^{-2}\, \mathrm{d}r\, \widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi),$$

where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^d$. Since $\widetilde{\mu}_{\mathbb{S}}$ is a finite measure it can be normalized to a probability measure, which means in our situation that

$$\mu_{\mathbb{S}}(\cdot) = \frac{\widetilde{\mu}_{\mathbb{S}}(\cdot)}{\Gamma(\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > 1\})} = \frac{\Gamma(\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > 1, \boldsymbol{x}/|\boldsymbol{x}| \in \cdot\})}{\Gamma(\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > 1\})}\,. \tag{5.1}$$

This representation shows that $\mu_{\mathbb{S}}$ measures the dependence between joint extremes, and that it depends on the chosen norm $|\cdot|$.

Using polar coordinates $r = |\boldsymbol{x}|$ and $\phi = \boldsymbol{x}/|\boldsymbol{x}| \in \mathbb{S}$, the Lévy measure Γ has for the set $A := \{(r, \phi) : 0 \le r_1 < r_2 \le \infty, 0 \le \rho_1 < \rho_2 < 2\pi\}$ the representation

$$\Gamma(A) = \int_{\rho_1}^{\rho_2} \int_{r_1}^{r_2} r^{-2}\, \mathrm{d}r \widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi) \,. \tag{5.2}$$

Note that all sets A of this type form a semi-algebra of subsets of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and, hence, generate the Borel sigma algebra $\mathcal{B}(\mathbb{R}^2 \setminus \{\mathbf{0}\})$. Defining the transformation $T : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$ by $T(r, \phi) = r(\cos\phi, \sin\phi)$, Γ has a density given in polar coordinates as

$$\Gamma \circ T(\mathrm{d}r, \mathrm{d}\phi) = \widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi)\, r^{-2}\mathrm{d}r \,. \tag{5.3}$$

From this we see firstly the well-known fact that the spectral density as the normalized angular measure completely determines the dependence in 1-stable models (as in all other models of arbitrary homogeneous order). We also note that the homogeneity on the whole of $\mathbb{R}^d$ plays an important role for the multiplicative structure in (5.3): for a general PLM this does no longer hold.

Using the notation introduced above we can relate the spectral measure with the PLM as follows. Since the arcs are for any norm given by

$$S_{\rho_1,\rho_2} := \left\{ \frac{(\cos\phi, \sin\phi)}{|(\cos\phi, \sin\phi)|} : \rho_1 < \phi \le \rho_2 \right\},$$

we find from the definition (5.1) by integrating out r over $(1, \infty)$ in (5.2)

$$\mu_{\mathbb{S}}([\rho_1, \rho_2]) = \frac{\int_{\rho_1}^{\rho_2} \widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi)}{\int_0^{2\pi} \widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi)} = \frac{\Gamma(S_{\rho_1,\rho_2})}{\int_0^{2\pi} \widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi)}. \tag{5.4}$$

This relates the spectral measure and the PLM for Borel sets on the sphere. This implies that apart from a normalizing factor both measures carry the same information on the dependence of the model.

We present the spectral measures by plotting the density $\mu_{\mathbb{S}}(\mathrm{d}\phi)/\mathrm{d}\phi$ on $[0, 2\pi)$. Here we take an idea from [2] and visualize $\mu_{\mathbb{S}}$ as a graph such that the area included between two angles (ρ_1 and ρ_2, say) and a solid curve ($s(\rho)$ for $\rho \in [\rho_1, \rho_2]$) represents the spectral measure $\mu_{\mathbb{S}}([\rho_1, \rho_2])$. The uniform distribution corresponds then to the unit circle. These graphs we call *Basrak graphs.*

Whereas polar coordinates are the natural ones for the spectral measure, the natural coordinates for the PLC $\overline{\Gamma}$ are the cartesian coordinates. The natural generator of the Borel sigma algebra $\mathcal{B}(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ are here the sets, which have been used to define the tail integral. It is, however, obvious from the definition of the spectral measure in (5.1) that there is no simple relation between the spectral measure and the PLM on these sets. For specific PLCs it is, however, possible (as shown below) to obtain the form of the corresponding spectral measure, provided the homogeneity property holds.

On the other hand, for non-homogeneous PLCs there exists no spectral measure, more precisely, the factorization of the PLM into a radial part and an angular part is no longer possible: the angular part depends still on r; cf. Example 5.4.

However, the PLC $\overline{\Gamma}$ exists and can also be visualised. Consequently, we also suggest a graphic representation for the PLC. This is done as follows. For a given point (x_1, x_2) not on any axes we plot $|\overline{\Gamma}(x_1, x_2)| = \Gamma(\mathcal{I}(x_1) \times \mathcal{I}(x_2))$ as the L_2-distance of a point to the origin, where we can also use any other distance.

Example 5.1. [Independence PLM, continuation of Examples 2.12 and 4.1]
Since $\Gamma_\perp$ has mass only on the coordinate axes, its spectral measure is for $\phi \in [0, 2\pi)$ given by

$$\mu_{\mathbb{S}}(\mathrm{d}\phi) \;=\; \frac{1}{4}\delta_0(\mathrm{d}\phi) + \frac{1}{4}\delta_{\pi/2}(\mathrm{d}\phi) + \frac{1}{4}\delta_\pi(\mathrm{d}\phi) + \frac{1}{4}\delta_{3\pi/2}(\mathrm{d}\phi)\,.$$

Figure 1 shows the spectral density $\mu_{\mathbb{S}}(\mathrm{d}\phi)/\mathrm{d}\phi$. The PLC Γ of independence is equal to 0; see Figure 3.

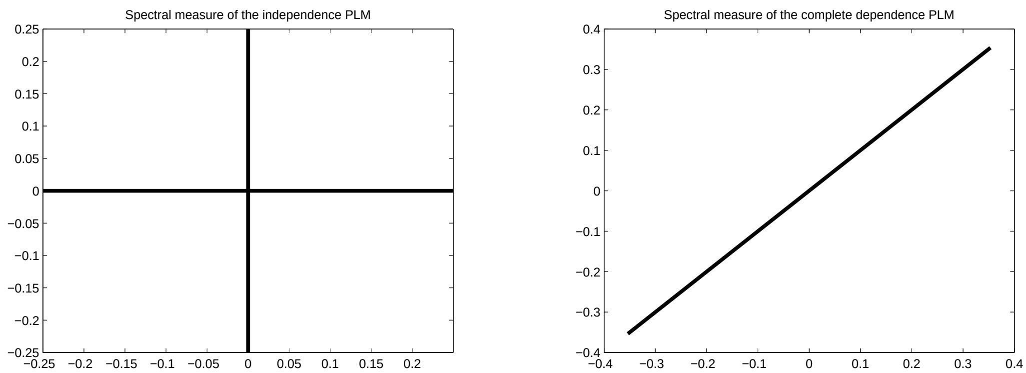

Figure 1: The left figure shows the Basrak graph of the spectral measure $\mu_{\mathbb{S}}$ of the independence PLM in $[0, 2\pi)$ with uniform weights 0.25 on $0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$. The right figure shows the Basrak plot of the spectral measure $\mu_{\mathbb{S}}$ of the complete positive dependence PLM. The length of the rays represents the probability mass on the corresponding angles.

Example 5.2. [Complete positive dependence PLM, continuation of Example 2.13 and 4.2]
$\Gamma_\parallel$ has mass only on $\{\boldsymbol{x} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\} : x_1 = x_2\}$ and its spectral measure is for $\phi \in [0, 2\pi)$ given by

$$\mu_{\mathbb{S}}(\mathrm{d}\phi) \;=\; \frac{1}{2}\delta_{\pi/4}(\mathrm{d}\phi) + \frac{1}{2}\delta_{5\pi/4}(\mathrm{d}\phi)\,.$$

The right Figure 1 shows the spectral density $\mu_{\mathbb{S}}(\mathrm{d}\phi)/\mathrm{d}\phi$ on $[0, 2\pi)$ and as Basrak graph. The PLC $\overline{\Gamma}_{\parallel}$ given as

$$\overline{\Gamma}_{\parallel}(x_1, x_2) = \frac{1}{|x_1| \vee |x_2|} 1_K((x_1, x_2))\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\,.$$

and is visualized in Figure 3.

Example 5.3. [Clayton PLM, continuation of Examples 2.15 and 4.3]
From (2.18) we find

$$\Gamma_{\eta,\theta}(\mathrm{d}x_1, \mathrm{d}x_2) = (1+\theta)\left(|x_1|^\theta + |x_2|^\theta\right)^{-1/\theta-2} |x_1|^{\theta-1}|x_2|^{\theta-1}\left(\eta 1_{\{x_1x_2>0\}} + (1-\eta)1_{\{x_1x_2<0\}}\right) \mathrm{d}x_1\mathrm{d}x_2\,.$$

By transformation to polar coordinate $\Gamma_{\eta,\theta}$ has representation (5.3) with density

$$\begin{aligned} \frac{\widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi)}{\mathrm{d}\phi} &= (1+\theta)\left(|\cos(\phi)|^\theta + |\sin(\phi)|^\theta\right)^{-1/\theta-2} |\cos(\phi)|^{\theta-1}|\sin(\phi)|^{\theta-1} \\ &\quad \left(\eta 1_{\{\cos(\phi)\sin(\phi)>0\}} + (1-\eta)1_{\{\cos(\phi)\sin(\phi)<0\}}\right)\,, \end{aligned} \tag{5.5}$$

and (5.4) applies. These denisities are shown in Figure 2.

We visualize the Clayton PLC $\overline{\Gamma}_{\eta,\theta}$ in Figure 3 for $\eta = 1$ (i.e. joint jumps are always in the same direction) and different parameter values $\theta > 0$. We see that for increasing parameter θ, the PLC values increases. This is reasoned by the increase of mass near $\pi/4$. If θ decreases, the mass of $\widetilde{\mu}_{\mathbb{S}}$ moves near to the axes and the PLC values decrease.

5.2 A non-homogeneous Pareto Lévy measure

Example 5.4. [Non-homogeneous PLM, continuation of Example 2.16]
$\Gamma_{\eta,\zeta}$ has the density

$$\begin{aligned} \Gamma_{\eta,\zeta}(\mathrm{d}x_1, \mathrm{d}x_2) &= \frac{2\mathrm{sgn}(x_1x_2) + \zeta x_1\mathrm{sgn}(x_2) + \zeta x_2\mathrm{sgn}(x_1) + \zeta^2 x_1x_2}{(|x_1| + |x_2| + \zeta|x_1x_2|)^3} \\ &\quad \left(\eta 1_{\{x_1x_2>0\}} - (1-\eta)1_{\{x_1x_2<0\}}\right) \mathrm{d}x_1\mathrm{d}x_2\,. \end{aligned}$$

Transforming $\Gamma_{\eta,\zeta}$ to polar coordinates yields (5.3) where $\widetilde{\mu}_{\mathbb{S}} = \widetilde{\mu}^r_{\mathbb{S}}(\mathrm{d}\phi)$ is given by

$$\begin{aligned} &\frac{\widetilde{\mu}^r_{\mathbb{S}}(\mathrm{d}\phi)}{\mathrm{d}\phi} \\ = &\frac{2\mathrm{sgn}(\cos(\phi)\sin(\phi)) + \zeta r\cos(\phi)\mathrm{sgn}(\sin(\phi)) + \zeta r\sin(\phi)\mathrm{sgn}(\cos(\phi)) + \zeta^2 r^2\cos(\phi)\sin(\phi)}{(|\cos(\phi)| + |\sin(\phi)| + \zeta r|\cos(\phi)\sin(\phi)|)^3} \\ &\left(\eta 1_{\{\cos(\phi)\sin(\phi)>0\}} - (1-\eta)1_{\{\cos(\phi)\sin(\phi)<0\}}\right)\,. \end{aligned} \tag{5.6}$$

In contrast to (5.5) $\widetilde{\mu}^r_{\mathbb{S}}$ depends on the radius and decreases for increasing r, see Figure 4.

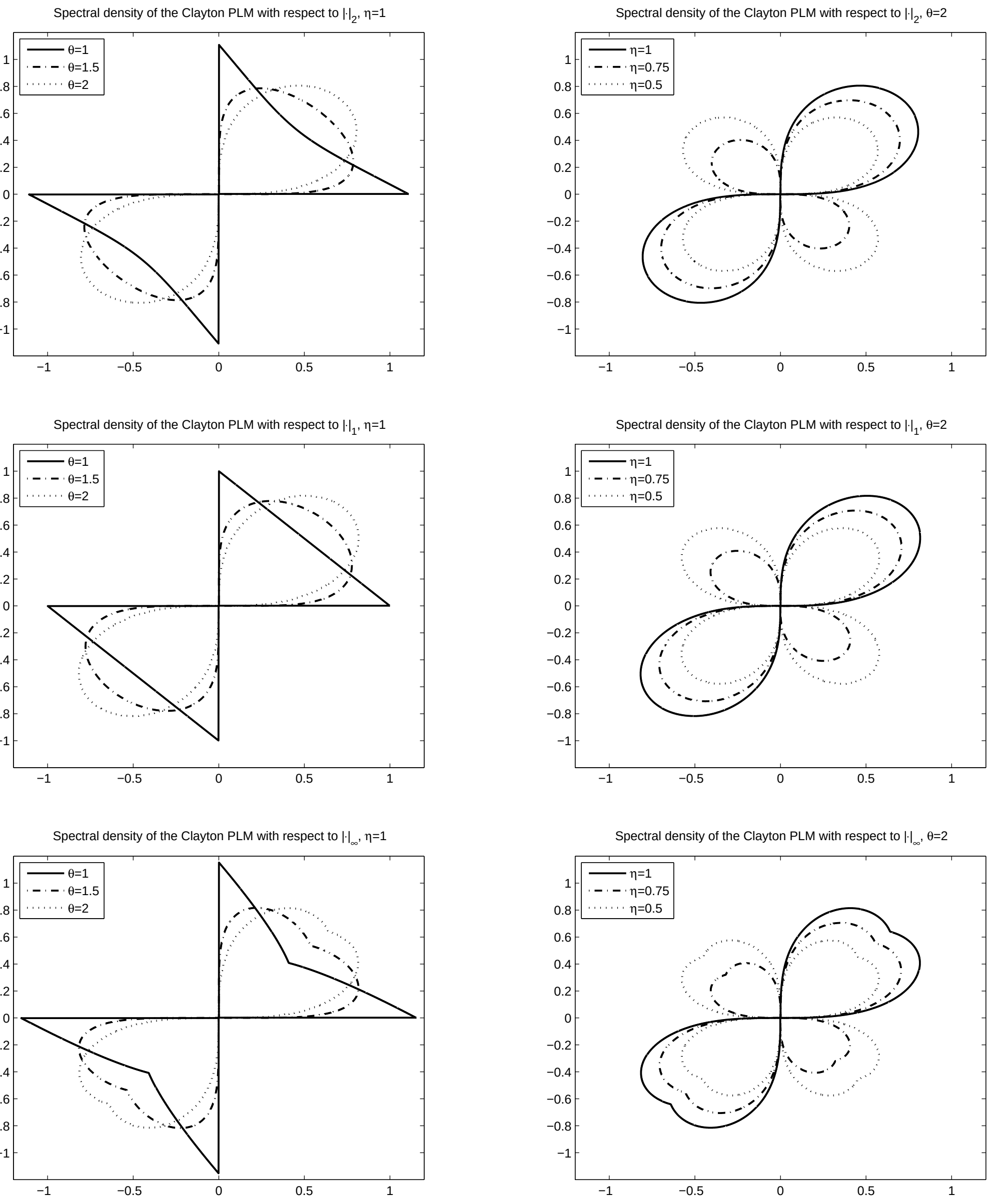

Figure 2: Basrak graphs of the spectral densities $\mu_{\mathbb{S}_2}(\mathrm{d}\phi)/\mathrm{d}\phi$ (first row), $\mu_{\mathbb{S}_1}(\mathrm{d}\phi)/\mathrm{d}\phi$ (second row), and $\mu_{\mathbb{S}_\infty}(\mathrm{d}\phi)/\mathrm{d}\phi$ (third row) of the Clayton PLC for different parameter values of $\theta > 0$ and $\eta \in [0, 1]$ as given in (5.1) on $[0, 2\pi)$. The right figures present them as Basrak graphs.

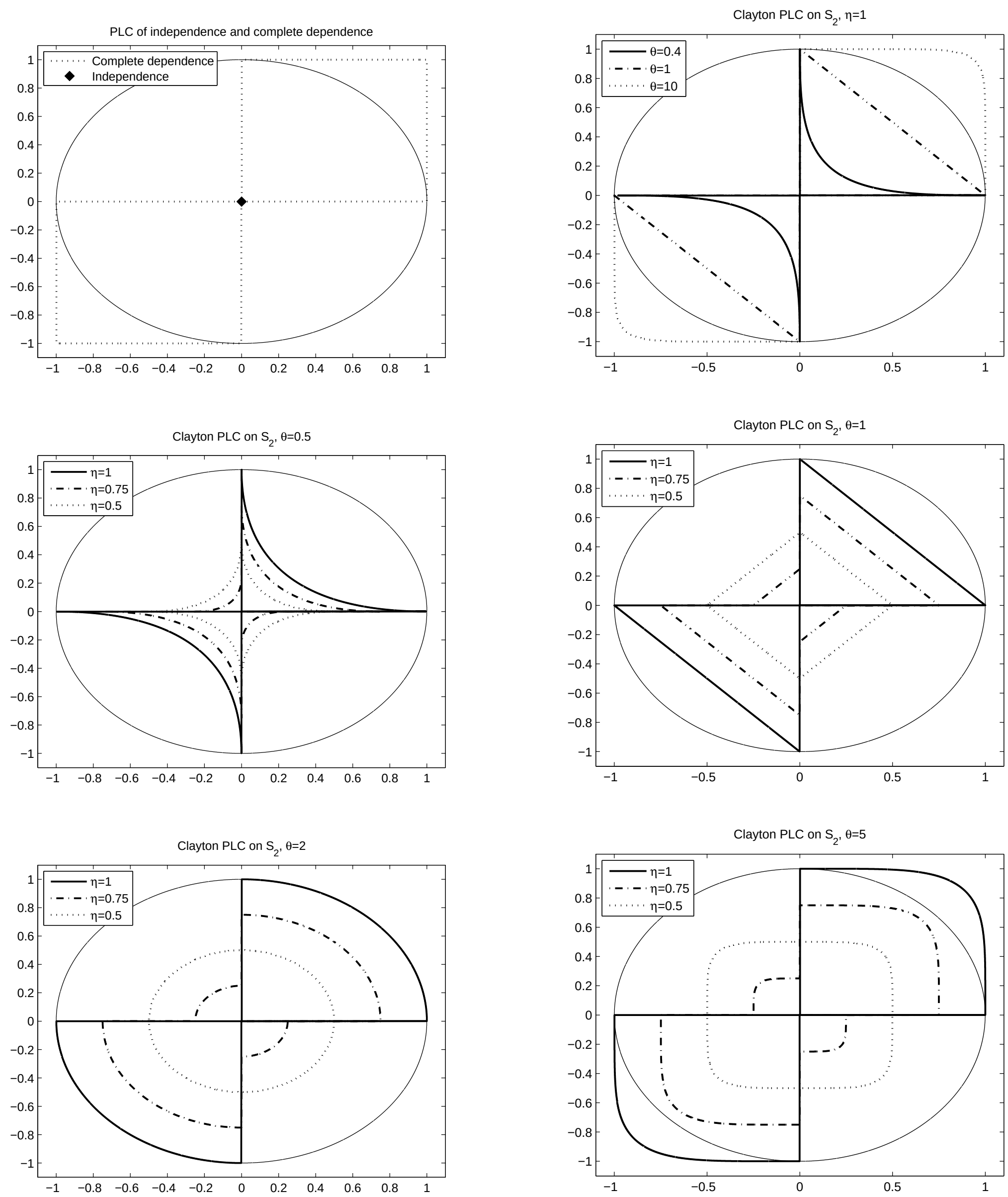

Figure 3: The first row of figures show the Clayton PLC $\overline{\Gamma}_{\perp}$, $\overline{\Gamma}_{\parallel}$ (left) and $\overline{\Gamma}_{\eta,\zeta}$ for $\eta = 1$ and different values $\theta > 0$ (right). The other figures show $\overline{\Gamma}_{\eta,\zeta}$ for different values for θ and $\eta \in [0, 1]$.

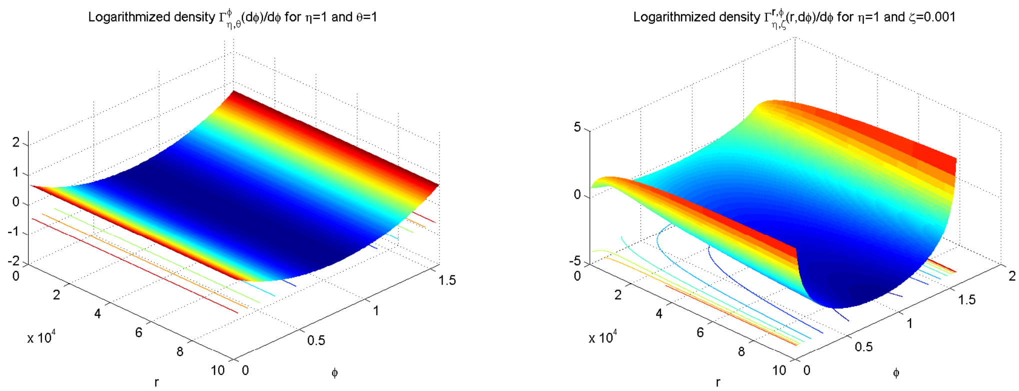

Figure 4: Densities $\widetilde{\mu}_{\mathbb{S}}(\mathrm{d}\phi)/\mathrm{d}\phi$ given in (5.5) and $\widetilde{\mu}^{r}_{\mathbb{S}}(\mathrm{d}\phi)/\mathrm{d}\phi$ given in (5.6).

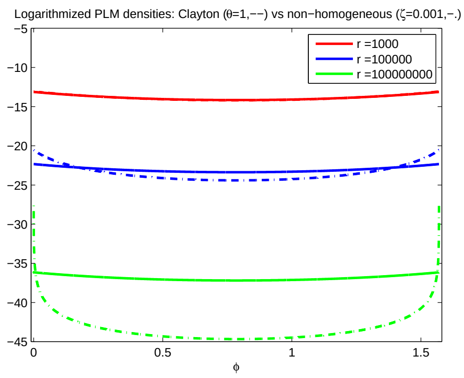

Figure 5: Logarithmic PLM densities of the Clayton PLM $\Gamma_{\eta,\theta}$ for $\eta = 1, \theta = 1$ and the non-homogenous PLM $\Gamma_{\eta,\zeta}$ for $\eta = 1, \zeta = 0.001$ for three different values of the radius r.

Moreover, Figure 5 shows the logarithmic PLM densities $\Gamma_{\eta,\theta} \circ T(r, \mathrm{d}\phi)$ for $(\eta = 1, \theta = 1)$ and $\Gamma_{\eta,\zeta} \circ T(r, \mathrm{d}\phi)$ for $(\eta = 1, \zeta = 0.001)$ for three different values of the radius. We see that for a small radius r both densities are almost identical. But for increasing r, the Clayton density decreases uniformly for all angles ϕ and the non-homogeneous density decreases strongly for angles near $\pi/4$ and weakly for angles near 0 and $\pi/2$. Figure 6 shows the PLC $\overline{\Gamma}_{\|}$ and $\overline{\Gamma}_{\eta,\zeta}$ for different parameter values for $\eta \in [0, 1]$ and $\zeta > 0$. We see that for small values of ζ the PLM $\Gamma_{\eta,\zeta}$ is similar to the Clayton PLM $\Gamma_{\eta,\theta}$ for $\theta = 1$. For increasing parameter ζ the PLC values become smaller and converge to independence.

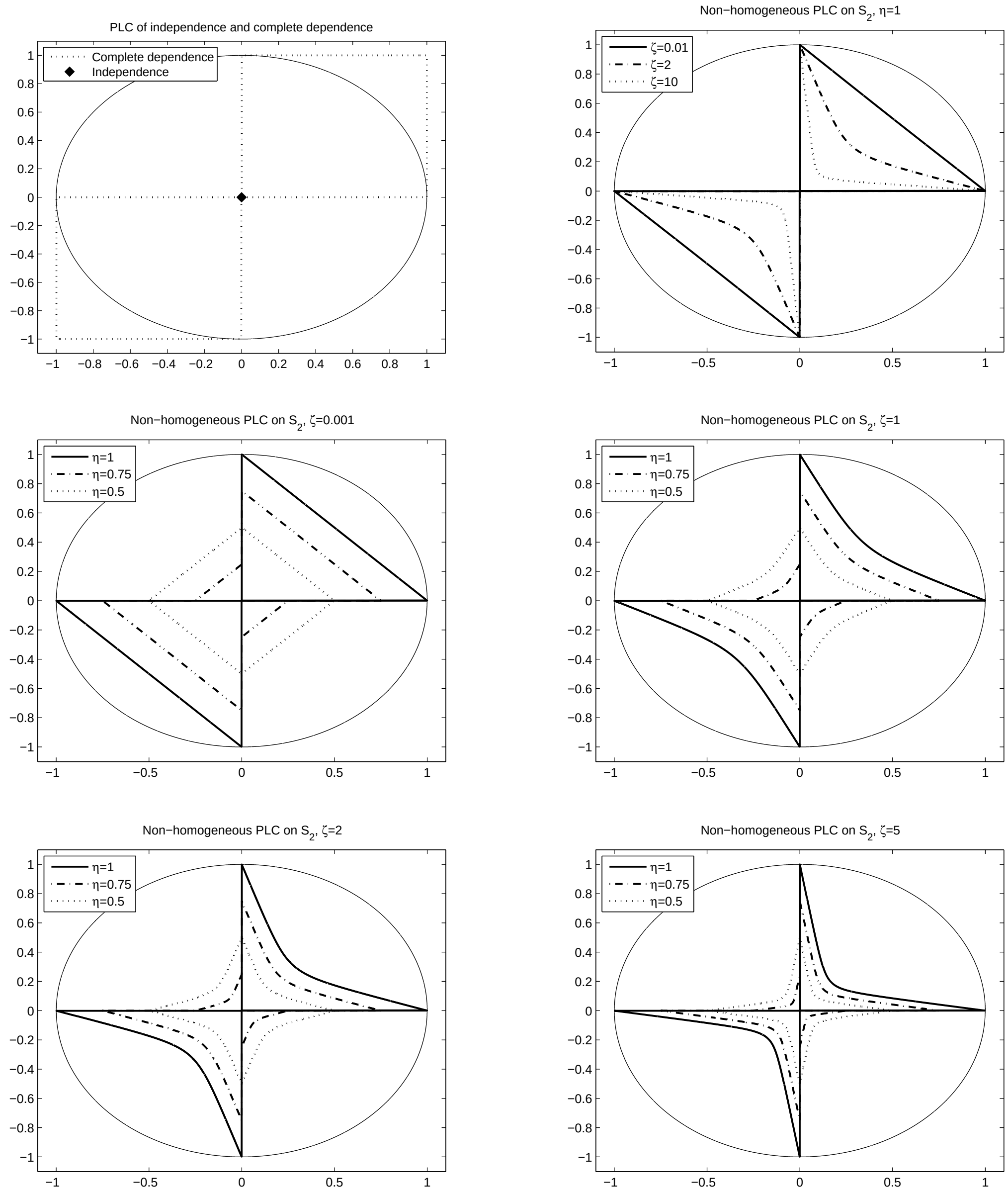

Figure 6: The figures show the PLC $\overline{\Gamma}_\perp$, $\overline{\Gamma}_\parallel$ and $\Gamma_{\eta,\zeta}$ for different parameter values for $\eta \in [0,1]$ and $\zeta \in (0,\infty)$.

A Appendix

A.1 Proofs

Proof of Lemma 2.4

Suppose $i \in \{1, \ldots, d\}$ and $A \in \mathcal{B}(\mathbb{E})$ with $0 \notin \overline{A}$ and $\mu_i(\partial A) = \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in \partial A\}) = 0$. Note that, since $0 \notin \overline{A}$, also $\mathbf{0} \notin \overline{\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}}$. Next we observe that

$$\begin{aligned} \mu(\partial\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}) &= \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in \partial A \text{ for at least one } i \in \{1, \ldots, d\}\}) \\ &\leq \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in \partial A\}) = 0\,. \end{aligned}$$

Consequently,

$$\begin{aligned} n\Pi_i(c_n A) &= n\Pi(c_n\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}) \\ &\to \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}) = \mu_i(A) < \infty\,. \end{aligned}$$

Setting $A = (x, \infty)$ and $A = (-\infty, -x]$ for $x > 0$, the homogeneity property of μ yields (2.5). Since μ is a non-zero measure, there exists at least one index i_* such that μ_{i_*} is a non-zero measure, i.e. $\overline{\mu}_{i_*}(1) - \overline{\mu}_{i_*}(-1) > 0$ and $\Pi_{i_*} \in \mathrm{RV}(\alpha, c_n, \mu_{i_*})$. If μ_i is a non-zero measure, then $\overline{\mu}_i(1) - \overline{\mu}_i(-1) > 0$ and Π_i is tail-equivalent to Π_{i_*}. If μ_i is the zero measure, then $\overline{\mu}_{i_*}(1) - \overline{\mu}_{i_*}(-1) = 0$ and the tail integrals of Π_i are lighter than the tail integrals of Π_{i_*}. $\square$

Proof of Proposition 2.9

(1) Assume that $a_i b_i \leq 0$ for at most $k \in \{0, \ldots, d\}$ indices. We prove (2.12) by induction on $k = 0, \ldots, d$.

Let $k = 0$, i.e. $a_i b_i > 0$ for all $i = 1, \ldots, d$. Define vector $\boldsymbol{u} = (u_i)_{i=1,\ldots,d} \in \prod_{i=1}^d \{a_i, b_i\}$ and set $N(\boldsymbol{u}) := \#\{i : u_i = a_i\}$. Then with Definition 2.2 and (2.6) we obtain

$$\begin{aligned} \Pi\left(\prod_{i=1}^d (a_i, b_i]\right) &= \sum_{\boldsymbol{u} \in \{a_i, b_i\}_{i=1,\ldots,d}} (-1)^{N(\boldsymbol{u})}(-1)^d \overline{\Pi}\,(\boldsymbol{u}) \\ &= \sum_{\boldsymbol{u} \in \{a_i, b_i\}_{i=1,\ldots,d}} (-1)^{N(\boldsymbol{u})}(-1)^d \overline{\Gamma}\left(\left(\frac{1}{\overline{\Pi}_i(u_i)}\right)_{i=1,\ldots,d}\right) \\ &= \Gamma\left(\prod_{i=1}^d \left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\,. \end{aligned}$$

Suppose (2.12) holds for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $a_i b_i \leq 0$ for at most k indices. W.l.o.g. we assume now that $a_i b_i \leq 0$ for $i = 1, \ldots, k+1$. If $a_{k+1} = 0$, then the induction hypothesis results in

$$\begin{aligned}
&\Pi\left(\prod_{i=1}^d (a_i,b_i]\right)\\
&= \lim_{\epsilon\downarrow 0}\Pi\left(\prod_{i<k+1}(a_i,b_i]\times(\epsilon,b_{k+1}]\times\prod_{i>k+1}(a_i,b_i]\right)\\
&= \lim_{\epsilon\downarrow 0}\Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(\frac{1}{\overline{\Pi}_i(\epsilon)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\prod_{i>k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&= \Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(\frac{1}{\overline{\Pi}_i(0+)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\prod_{i>k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right).
\end{aligned}$$

If $a_{k+1}\neq 0$, i.e. $a_{k+1}<0$ and $b_{k+1}\geq 0$, then with the induction hypothesis we get

$$\begin{aligned}
&\Pi\left(\prod_{i=1}^d (a_i,b_i]\right)\\
&= \Pi_{\{1,\ldots,d\}\setminus\{k+1\}}\left(\prod_{i\neq k+1}(a_i,b_i]\right)-\lim_{\beta\downarrow b_{k+1}}\Pi\left(\prod_{i<k+1}(a_i,b_i]\times(\beta,\infty)\times\prod_{i>k+1}(a_i,b_i]\right)\\
&\quad -\Pi\left(\prod_{i<k+1}(a_i,b_i]\times(-\infty,a_k]\times\prod_{i\in I,i>k+1}(a_i,b_i]\right)\\
&= \Gamma_{\{1,\ldots,d\}\setminus\{k+1\}}\left(\prod_{i\neq k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&\quad -\lim_{\beta\downarrow b_{k+1}}\Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(\frac{1}{\overline{\Pi}_{k+1}(\beta)},\infty\right)\times\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&\quad -\Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(-\infty,\frac{1}{\overline{\Pi}_{k+1}(a_{k+1})}\right]\times\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&= \Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(\frac{1}{\overline{\Pi}_{k+1}(a_{k+1})},\frac{1}{\overline{\Pi}_{k+1}(b_{k+1}+)}\right]\times\prod_{i>k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right).
\end{aligned}$$

Recall that for $b_{k+1}>0$ we have by right-continuity of the tail integral $\overline{\Pi}_{k+1}(b_{k+1}+)=\overline{\Pi}_{k+1}(b_{k+1})$. If $b_{k+1}=0$, then $\overline{\Pi}_{k+1}(b_{k+1}+)=\overline{\Pi}_{k+1}(0+)$.

(2) We prove (2.14) by induction on $|K|=1,\ldots,d-1$. For $|K|=1$ we assume w.l.o.g. that $K=\{1\}$. Sklar's Theorem 2.6 implies

$$\begin{aligned}
&\Pi\left(\{0\}\times\prod_{i=2}^d\mathcal{I}(x_i)\right)\\
&= \Pi_{\{2,\ldots,d\}}\left(\prod_{i=2}^d\mathcal{I}(x_i)\right)-\lim_{\epsilon\downarrow 0}\Pi\left(\mathcal{I}(\epsilon)\times\prod_{i=2}^d\mathcal{I}(x_i)\right)-\lim_{\epsilon\uparrow 0}\Pi\left(\mathcal{I}(\epsilon)\times\prod_{i=2}^d\mathcal{I}(x_i)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Gamma_{\{2,\dots,d\}}\left(\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\overline{\Pi}}_i(x_i)}\right)\right) - \Gamma\left(\mathcal{I}\left(\frac{1}{\overline{\overline{\Pi}}_1(0+)}\right)\times\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\overline{\Pi}}_i(x_i)}\right)\right)\\
&\quad -\Gamma\left(\left(-\infty,\frac{1}{\overline{\overline{\Pi}}_1(0-)}\right)\times\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\overline{\Pi}}_i(x_i)}\right)\right)\\
&= \Gamma\left(\left[\frac{1}{\overline{\overline{\Pi}}_1(0-)},\frac{1}{\overline{\overline{\Pi}}_1(0+)}\right]\times\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\overline{\Pi}}_i(x_i)}\right)\right).
\end{aligned}
$$

With induction on $|K|$ equations (2.13) and (2.14) result. $\square$

A.2 Auxiliary results

Proposition A.1. *Let M and $(M_n)_{n\in\mathbb{N}}$ be a measures on $\mathcal{B}(\mathbb{E})$ and $\boldsymbol{a},\boldsymbol{b},(\boldsymbol{a}_n)_{n\in\mathbb{N}},(\boldsymbol{b}_n)_{n\in\mathbb{N}}\in\mathbb{E}$ with $\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]}$ and $\mathbf{0}\notin\overline{(\boldsymbol{a}_n,\boldsymbol{b}_n]}$ for all n. Suppose*

(1) $\boldsymbol{a}_n\to\boldsymbol{a}$ and $\boldsymbol{b}_n\to\boldsymbol{b}$ as $n\to\infty$,

(2) $M(\partial(\boldsymbol{a},\boldsymbol{b}])=0$,

(3) $\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},M(\partial(\boldsymbol{a},\boldsymbol{b}])=0}|M_n((\boldsymbol{a},\boldsymbol{b}])-M((\boldsymbol{a},\boldsymbol{b}])|\to 0$ as $n\to\infty$.

Then $M_n((\boldsymbol{a}_n,\boldsymbol{b}_n])\to M((\boldsymbol{a},\boldsymbol{b}])$ as $n\to\infty$.

Proof. We estimate

$$
\begin{aligned}
|M_n((\boldsymbol{a}_n,\boldsymbol{b}_n])-M((\boldsymbol{a},\boldsymbol{b}])| &\le |M_n((\boldsymbol{a}_n,\boldsymbol{b}_n])-M((\boldsymbol{a}_n,\boldsymbol{b}_n])| + |M((\boldsymbol{a}_n,\boldsymbol{b}_n])-M((\boldsymbol{a},\boldsymbol{b}])|\\
&\le \sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},M(\partial(\boldsymbol{a},\boldsymbol{b}])=0}|M_n((\boldsymbol{a},\boldsymbol{b}])-M((\boldsymbol{a},\boldsymbol{b}])| + |M((\boldsymbol{a}_n,\boldsymbol{b}_n])-M((\boldsymbol{a},\boldsymbol{b}])|.
\end{aligned}
$$

As $n\to\infty$ the first term tends to 0 by condition (3), and the second term by a combination of conditions (1) and (2). $\square$

Proposition A.2. *In the situation of Theorem 3.1 we have*

$$
\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\,\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},\,\nu(\partial(\boldsymbol{a},\boldsymbol{b}])=0}|n\Gamma(n(\boldsymbol{a},\boldsymbol{b}])-\nu((\boldsymbol{a},\boldsymbol{b}])|\to 0 \quad \textit{as } n\to\infty.
$$

Proof. For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{E}$ with $\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]}$ and $\nu(\partial(\boldsymbol{a},\boldsymbol{b}])=0$ define $g_n(\boldsymbol{a},\boldsymbol{b}):=|n\Gamma(n(\boldsymbol{a},\boldsymbol{b}])-\nu((\boldsymbol{a},\boldsymbol{b}])|$. Since Γ and ν have no atoms, g_n is continuous on $\mathbb{E}^2$. So for $\epsilon>0$ the sets $S_n:=\{\boldsymbol{x},\boldsymbol{y}\in\mathbb{E}: g_n(\boldsymbol{x},\boldsymbol{y})<\epsilon\}$ are open. Furthermore, g_n is decreasing for $n\to\infty$ and converges pointwise to 0. Therefore, S_n is increasing and $(S_n)_{n\in\mathbb{N}}$ is an open cover of $\mathbb{E}^2$. Since $\mathbb{R}^2\setminus\{\mathbf{0}\}$ is compact in the here used topology (see [21], p.171), there exists an $N \in \mathbb{N}$ such that $S_N = \mathbb{E}^2$. So for every $n > N$ and every $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{E}^2$ we obtain $|n\Gamma(n(\boldsymbol{x}, \boldsymbol{y}]) - \nu((\boldsymbol{x}, \boldsymbol{y}])| = g_n(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$, where N does not depend on $(\boldsymbol{x}, \boldsymbol{y})$. □

Proposition A.3. *In the situation of Theorem 3.5 we have*

$$\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\, \boldsymbol{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},\, \mu(\partial(\boldsymbol{a},\boldsymbol{b}])=0} |n\Pi(c_n(\boldsymbol{a}, \boldsymbol{b}]) - \mu((\boldsymbol{a}, \boldsymbol{b}])| \to 0 \quad \textit{as } n \to \infty.$$

The proof is analogous to the proof of Proposition A.2.

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