SIGNAL RECONSTRUCTION FROM SINE WAVE CROSSINGS

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ABSTRACT

In this paper we analyze the reconstruction of bandlimited signals from their sine wave crossings by a sampling type reconstruction process. The reconstruction process is highly adapted to the signal which shall be reconstructed, because the reconstruction functions and the sampling points are implicitly generated by the signal. We show that the reconstruction process is uniformly convergent for all signals in the Paley-Wiener spaces $\mathcal{PW}_\pi^p$, $1 < p \le \infty$. However, the adaptivity cannot prevent the peak value of the reconstruction process to diverge for certain signals in $\mathcal{PW}_\pi^1$.

Index Terms— adaptive, reconstruction, sampling, sine type, zero crossings

1. INTRODUCTION

The recovery of bandlimited signals from their zero crossing has been intensively studied, because the zero crossings of a signal are robust under various non-linear disturbances the signal might undergo, for example when it is transmitted over a non-linear channel or in magnetic recording [1]. According to Hadamard's factorization theorem [2, p.26], a bandlimited signal that is bounded on the real axis is—up to some constants—uniquely determined by its zeros [3]. In general the zeros of a bandlimited signal are complex, but in physical systems, complex zeros are not observable [4]. Only the zero crossings, i.e., the locations of the odd order real zeros of a signal, can be easily detected. But, in general, the zero crossings do not contain enough information to reconstruct the signal [5]. As the example $(1 - \mathrm{sinc}(\pi t))/t^2$ shows, there are even bandlimited signals that have no real zero at all. In order to circumvent the problem of complex and multiple zeros, the invertible transformation

$$\psi_f(z) = f(z) - A\sin(\pi z), \quad z \in \mathbb{C}, \tag{1}$$

with $A > \|f\|_\infty$, has been considered [5, 6, 7]. For signals f that are bandlimited with bandwidth not larger than π, and both bounded and real on the real axis, ψ_f has only real and simple zeros [8]. Moreover, given such a signal f, the theory of entire functions shows that ψ_f can be reconstructed from its real zero crossings $\{t_k\}_{k\in\mathbb{Z}}$ by virtue of the infinite product

$$\psi_f(z) = \psi_f(0) \lim_{R\to\infty} \prod_{|t_k|\le R} \left(1 - \frac{z}{t_k}\right) \tag{2}$$

under the assumption that $\psi_f(0) \ne 0$. Thus, ψ_f is determined by its zeros $\{t_k\}_{k\in\mathbb{Z}}$ and $\psi_f(0) = f(0)$. Since $\psi_f(t_k) = f(t_k) - A\sin(\pi t_k) = 0$, $k \in \mathbb{Z}$, the zero crossings $\{t_k\}_{k\in\mathbb{Z}}$ of ψ_f are the sine wave crossings of f. Finally, the signal f can be reconstructed

from ψ_f by using (1). Hence, f is uniquely determined by its sine wave crossings $\{t_k\}_{k\in\mathbb{Z}}$ and $f(0)$.

There is a considerable amount of literature dealing with the reconstruction of signals from its zeros by products [9, 3, 7]. In general the convergence of infinite products is intricate and needs a careful treatment. However, in many publications the argumentation is more or less heuristic, and the results are rather claims without a rigorous mathematical justification. Although, under the assumptions on the signal f from above, the function ψ_f and consequently the signal f are uniquely determined by $\{t_k\}_{k\in\mathbb{Z}}$ and $f(0)$, the question remains whether the reconstruction is stable.

One of the first papers that presents a mathematically rigorous approach is [6]. In [6], Bar-David showed that every bandlimited signal f with bandwidth smaller than π that is bounded and real on the real axis, and that satisfies $f(0) \ne 0$, can be reconstructed according to (1) and (2) from the sine wave crossings of f, i.e., the real and simple zeros of ψ_f, and $f(0)$, provided that $A > \|f\|_\infty$.
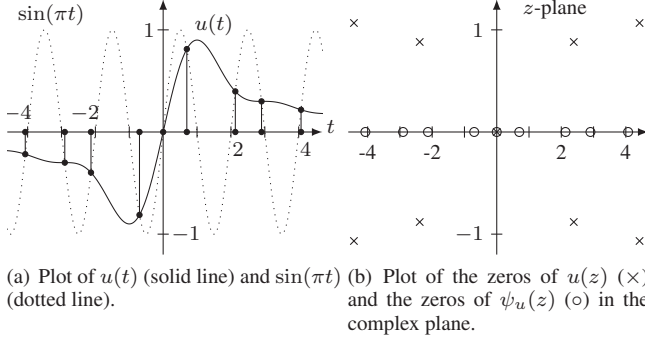
In practice the reconstruction of ψ_f using the infinite product according to (2) has several drawbacks in terms of convergence speed an sensitivity against jitter in the location of the zeros $\{t_k\}_{k\in\mathbb{Z}}$, and thus is not the preferred way to do the reconstruction.

2. NOTATION

Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \le p < \infty$, is the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ is the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite.

For $\sigma > 0$ let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \le C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \le p \le \infty$ [10]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., $\|\cdot\|_{\mathcal{B}_\sigma^p} = \|\cdot\|_p$. A signal in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. By the Paley-Wiener-Schwartz theorem the Fourier transform of a signal bandlimited to σ is supported in $[-\sigma, \sigma]$ [10]. For $1 \le p \le 2$ the Fourier transformation is defined in the classical and for $p > 2$ in the distributional sense.

For $\sigma > 0$ and $1 \le p \le \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of signals f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)e^{iz\omega}\,d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$, $1 \le p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi)\int_{-\sigma}^{\sigma}|\hat{f}(\omega)|^p\,d\omega)^{1/p}$. For $p = 2$ we obtain the Paley-Wiener space $\mathcal{PW}_\sigma^2$, which is nothing else than the space of bandlimited signals with finite energy. Hölder's inequality leads to $\mathcal{PW}_\sigma^s \subset \mathcal{PW}_\sigma^p$ for $1 \le p < s \le \infty$. Moreover, we have $\|f\|_\infty \le \|f\|_{\mathcal{PW}_\sigma^1}$, which implies that every signal in $\mathcal{PW}_\sigma^1$ is bounded on the real line.

(a) Plot of $u(t)$ (solid line) and $\sin(\pi t)$ (dotted line).

(b) Plot of the zeros of $u(z)$ ($\times$) and the zeros of $\psi_u(z)$ ($\circ$) in the complex plane.

Fig. 1. Illustration of the transformation of the complex zeros of a bandlimited signal in real zeros.

3. CONVERGENCE RESULT

3.1. Basic Properties of ψ_f

In the introduction we have seen that the transformation (1) is important, because for signals $f \in \mathcal{B}_\pi^\infty$ that are real on the real axis, ψ_f has only real and simple zeros. Figure 1 illustrates this transformation. The signal $u(t) = (1 - \mathrm{sinc}(\pi t))/t$, which is depicted in Fig. 1(a), has only complex zeros except for one real zero at zero. The zeros of u in the complex plane in the vicinity of the origin are plotted in Fig. 1(b) as crosses. Furthermore, the zeros of ψ_f, which are plotted as circles in Fig. 1(b), are nothing else than the abscissas of the crossings of u and the sine function.

In the following we use the transformation (1) only for signals $f \in \mathcal{PW}_\pi^1$ that are real on the real axis, and we assume that $A = 1$, i.e., we consider the transformation

$$\psi_f(z) = f(z) - \sin(\pi z). \tag{3}$$

The space $\mathcal{PW}_\pi^1$ is interesting, because it is the largest space in the scale of Paley-Wiener spaces.

For signals $f \in \mathcal{PW}_\pi^1$ that are real on the real axis and that satisfy $\|f\|_\infty < 1$, ψ_f is a sine-type function. This fact will be of value for us, because sine-type functions have many pleasant properties.

Definition 1. An entire function f of exponential type π is said to be of sine type if

(i) the zeros of f are separated and simple, and

(ii) there exist positive constants A, B, and H such that $A\,\mathrm{e}^{\pi|y|} \leq |f(x+iy)| \leq B\,\mathrm{e}^{\pi|y|}$ whenever x and y are real and $|y| \geq H$.

Example 1. $\sin(\pi z)$ is a function of sine type and its zeros are $t_k = k$, $k \in \mathbb{Z}$.

In the following discussion we will show that ψ_f is a sine-type function if f is in $\mathcal{PW}_\pi^1$, satisfies $\|f\|_\infty < 1$, and is real on the real axis. From [8] we know that ψ_f has only simple zeros. Moreover, since $\lim_{|t|\to\infty} f(t) = 0$ according to the Riemann-Lebesgue lemma, it follows that the zeros are also separated. Thus, ψ_f satisfies item (i) of Definition 1. Moreover, since ψ_f has the representation $\psi_f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i\omega t}\,\mathrm{d}\mu(\omega)$, where $\mu(\omega)$ is of bounded variation on $[-\pi, \pi]$ and has a jump discontinuity at each endpoint, it follows [11, p. 143] that ψ_f also satisfies item (ii) of Definition 1, and hence is a sine-type function.

For further information about sine-type functions we would like to refer the reader to [11] and [2].

3.2. Adaptive Reconstruction Process

In the previous section we have seen that bandlimited signals $f \in \mathcal{PW}_\pi^1$, $\|f\|_{\mathcal{PW}_\pi^1} < 1$, that are real on the real axis can be reconstructed from their sine wave crossings, i.e., the zero crossings of ψ_f, by using (2) and (3) as reconstruction formulas. In this paper we analyze a different reconstruction process, which has the shape

$$\sum_{k=-\infty}^{\infty} f(\lambda_k) \frac{\psi_f(t)}{\psi_f'(\lambda_k)(t - \lambda_k)}, \tag{4}$$

where ψ_f is defined as in (3), and $\{\lambda_k\}_{k\in\mathbb{Z}}$ denotes the zero sequence of ψ_f. Note that the knowledge of the zero sequence of ψ_f is enough for the reconstruction process (4). The samples $\{f(\lambda_k)\}_{k\in\mathbb{Z}}$ can be calculated from ψ_f: since $\psi_f(\lambda_k) = 0$, $k \in \mathbb{Z}$, it follows from (3) that $f(\lambda_k) = \sin(\lambda_k)$, $k \in \mathbb{Z}$.

Reconstruction processes of the shape (4) are a special case of more general sampling series with non-equidistant sampling patterns of the shape $\sum_{k=-\infty}^{\infty} f(\lambda_k)\phi_k(t)$, where ϕ_k, $k \in \mathbb{Z}$, are certain reconstruction functions, and $\{\lambda_k\}_{k\in\mathbb{Z}}$ is the sequence of sampling points [12]. In our case, the sampling pattern $\{\lambda_k\}_{k\in\mathbb{Z}}$ and the reconstruction functions

$$\phi_{f,k}(t) := \frac{\psi_f(t)}{\psi_f'(\lambda_k)(t - \lambda_k)}$$

are implicitly generated by the signal f, and thus, in a certain sense, adapted to the signal f.

For a large class of signals the reconstruction process (4) converges uniformly on all of $\mathbb{R}$.

Theorem 1. *Let $1 < p \leq \infty$. For all $f \in \mathcal{PW}_\pi^p$ for which ψ_f is a sine-type function we have*

$$\lim_{N\to\infty} \left(\sup_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(\lambda_k) \frac{\psi_f(t)}{\psi_f'(\lambda_k)(t - \lambda_k)} \right| \right) = 0, \tag{5}$$

where $\{\lambda_k\}_{k\in\mathbb{Z}}$ denotes the zero sequence of ψ_f.

The proof of Theorem 1 is based on the following lemma, a proof of which can be found in [2, p. 165].

Lemma 1. *Let ϕ be a function of sine type, and $\{\lambda_k\}_{k\in\mathbb{Z}}$ its zero set. Further, let $1 < p < \infty$. Then there exist two constants $C_L(p) > 0$ and $C_R(p) > 0$, depending only on p, such that for all $f \in \mathcal{B}_\pi^p$*

$$C_L(p)\left(\sum_{k=-\infty}^{\infty} |f(\lambda_k)|^p \right)^{\frac{1}{p}} \leq \|f\|_{\mathcal{B}_\pi^p} \leq C_R(p)\left(\sum_{k=-\infty}^{\infty} |f(\lambda_k)|^p \right)^{\frac{1}{p}}.$$

Proof of Theorem 1. Since $\mathcal{PW}_\pi^p \subset \mathcal{PW}_\pi^2$ for $p \geq 2$, it is clear that if (5) is true for all $f \in \mathcal{PW}_\pi^2$ for which ψ_f is a sine-type function, then (5) is true for all $f \in \mathcal{PW}_\pi^p$, $p \geq 2$, for which ψ_f is a sine-type function. Hence, it suffices to prove the case $1 < p \leq 2$.

Let $1 < p \leq 2$. Further, let $f \in \mathcal{PW}_\pi^p$ be such that ψ_f is a sine-type function. Since $f \in \mathcal{B}_\pi^q$ and $\phi_{f,k} \in \mathcal{B}_\pi^q$, $1/p + 1/q = 1$, we can apply Lemma 1 to obtain

$$\left\| f - \sum_{k=-N}^{N} f(\lambda_k)\phi_{f,k} \right\|_{\mathcal{B}_\pi^q}$$

$$\leq C_R(q)\left(\sum_{l=-\infty}^{\infty} \left| f(\lambda_l) - \sum_{k=-N}^{N} f(\lambda_k)\phi_{f,k}(\lambda_l) \right|^q \right)^{\frac{1}{q}}$$

$$\leq C_R(q)\left(\sum_{|l|>N} |f(\lambda_l)|^q \right)^{\frac{1}{q}}$$

for all $N \in \mathbb{N}$. In the last inequality we used the interpolation property of $\phi_{f,k}$, namely $\phi_{f,k}(\lambda_l) = 1$ if $l = k$, and $\phi_{f,k}(\lambda_l) = 0$ if $l \neq k$. Moreover, since

$$\left(\sum_{l=-\infty}^{\infty} |f(\lambda_l)|^q \right)^{\frac{1}{q}} \leq \frac{\|f\|_{\mathcal{B}_\pi^q}}{C_L(q)} < \infty,$$

it follows that

$$\lim_{N \to \infty} \left(\sum_{|l| > N} |f(\lambda_l)|^q \right)^{\frac{1}{q}} = 0.$$

The proof is complete, because there exists a constant $C(q)$ such that $\|f\|_\infty \leq C(q)\|f\|_{\mathcal{B}_\pi^q}$ for all $f \in \mathcal{B}_\pi^q$. $\qquad \square$

Remark 1. In Theorem 1 we have the requirement that ψ_f is a sine-type function. A simple sufficient condition for ψ_f to be a sine-type function is that the signal $f \in \mathcal{PW}_\pi^p$, $1 < p \leq \infty$, is real on the real axis and $\|f\|_\infty < 1$.

Remark 2. For fixed $t \in \mathbb{R}$, the sum in Theorem 1 is absolutely and unconditionally convergent.

4. DIVERGENCE RESULT

In [13] we conjectured that the sampling series with matched reconstruction function (4) is not uniformly convergent on all of $\mathbb{R}$ for certain $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} < 1$. Next, we will prove that this conjecture is true. This shows that the restriction in Theorem 1 that $f \in \mathcal{PW}_\pi^p$ with p strictly larger than 1 is indeed necessary. Thus, Theorem 1 is sharp in the sense that it cannot be extended to hold for $p = 1$.

Theorem 2. *There exists a function $f_1 \in \mathcal{PW}_\pi^1$, such that ψ_{f_1} is a sine-type function with solely real zeros and*

$$\limsup_{N \to \infty} \left(\sup_{t \in \mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(t_k) \frac{\psi_{f_1}(t)}{\psi_{f_1}'(t_k)(t - t_k)} \right| \right) = \infty,$$

where $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ denotes the zero sequence of ψ_{f_1}.

Theorem 2 shows that even the adaptivity of the reconstruction process (4) to the signals cannot avoid the divergence of the peak value of reconstruction error for certain signals in $\mathcal{PW}_\pi^1$.

For the proof of Theorem 2 we use a recent result by Hryniv and Mykytyuk, which was proved in [14].

Theorem 3 (Hryniv and Mykytyuk).

(i) *Let $w \in \mathcal{PW}_\pi^1$ and $\lambda_k = k + w(k)$, $k \in \mathbb{Z}$. Then there exists a function $f \in \mathcal{PW}_\pi^1$ such that ψ_f has the zero sequence $\{\lambda_k\}_{k \in \mathbb{Z}}$.*

(ii) *Conversely, if $f \in \mathcal{PW}_\pi^1$, then there exists a function $\omega \in \mathcal{PW}_\pi^1$ such that $\{\lambda_k\}_{k \in \mathbb{Z}} = \{k + w(k)\}_{k \in \mathbb{Z}}$ is the zeros sequence of ψ_f.*

This theorem is remarkable, because it completely characterizes the zero sequences of ψ_f for $f \in \mathcal{PW}_\pi^1$ in terms of a disturbance of the integers by a function in $\mathcal{PW}_\pi^1$. In general it is hard to find such descriptions for the zeros of entire functions.

The following corollary is a simple consequence of Theorem 3.

Corollary 1. *Let $w \in \mathcal{PW}_\pi^1$ be real on the real axis with $|w(k)| < 1/2$, $k \in \mathbb{Z}$, and $t_k = k + w(k)$, $k \in \mathbb{Z}$. Then there exists a function $f \in \mathcal{PW}_\pi^1$ that is real on the real axis such that ψ_f is a function of sine type that has the zero sequence $\{t_k\}_{k \in \mathbb{Z}}$.*

Proof. Let $w \in \mathcal{PW}_\pi^1$ be real on the real axis with $|w(k)| < 1/2$, $k \in \mathbb{Z}$, and $t_k = k + w(k)$, $k \in \mathbb{Z}$. According to part (i) of Theorem 3, there exists a function $f \in \mathcal{PW}_\pi^1$ such that ψ_f has the zero sequence $\{t_k\}_{k \in \mathbb{Z}}$. Since $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$, it follows from (2) that ψ_f and consequently f can be chosen to be real on the real axis. It remains to show that ψ_f is a sine-type function. Since $|w(k)| < 1/2$, $k \in \mathbb{Z}$, and $\lim_{|k| \to \infty} w(k) = 0$, the zeros $\{t_k\}_{k \in \mathbb{Z}}$ of ψ_f are separated and simple, hence ψ_f satisfies item (i) of Definition 1. Moreover, by the same argumentations as in Sec. 3.1, ψ_f also satisfies item (ii) of Definition 1, and hence is a sine-type function. $\qquad \square$

For the proof of Theorem 2, we need the following lemmas.

Lemma 2. *Let $f \in \mathcal{PW}_\pi^1$ be such that ψ_f is a sine-type function. Then there exists a constant C_1 such that for all $N \in \mathbb{N}$ sufficiently large we have $\left| \psi_f \left(N + \frac{1}{2} \right) \right| \geq C_1$.*

Proof. Let $\{\lambda_k\}_{k \in \mathbb{Z}}$ be the zero sequence of ψ_f. Since ψ_f is a sine-type function, there exists, for every $\epsilon > 0$, a number $C_2 > 0$ such that $|\psi_f(x + iy)| \geq C_2 \, e^{\pi |y|}$ outside the circles of radius ϵ centered at the zeros $\{\lambda_k\}_{k \in \mathbb{Z}}$ of ψ_f [11, p. 144, Lemma 2]. According to part (ii) of Theorem 3 we have $\lambda_k = k + w(k)$, $k \in \mathbb{Z}$, for some $w \in \mathcal{PW}_\pi^1$. Further, since $\lim_{k \to \infty} w(k) = 0$, there exists a natural number N_0 such that $|N + 1/2 - \lambda_N| \geq 1/4$ for all $N \geq N_0$. $\qquad \square$

Lemma 3. *Let $f \in \mathcal{PW}_\pi^1$, and let $\{\lambda_k\}_{k \in \mathbb{Z}}$ denote the zero sequence of ψ_f. Then for all sequences $\{a_k\}_{k \in \mathbb{Z}} \subset \mathbb{C}$ with $\sup_{k \in \mathbb{Z}} |a_k| < \infty$ there exists a constant C_3 such that for all sufficiently large N we have*

$$\left| \sum_{k=-N}^{N} \frac{a_k}{(N + \frac{1}{2} - \lambda_k)} - \sum_{k=-N}^{N} \frac{a_k}{(N + \frac{1}{2} - k)} \right| \leq C_3.$$

Proof. According to part (ii) of Theorem 3 there exists a function $w \in \mathcal{PW}_\pi^1$ such that $\lambda_k = k + w(k)$, $k \in \mathbb{Z}$. For sufficiently large $N \in \mathbb{N}$ we have $N + 1/2 - \lambda_k \geq 1/4$ for all $k \in \mathbb{Z}$. Further, since $w \in \mathcal{PW}_\pi^1$, there exists a $N_0 \in \mathbb{N}$ such that $|w(k)| < 1/4$ for all $k \geq N_0$. Thus, for sufficiently large N, $N \geq N_0$, we have

$$\left| \sum_{k=-N}^{N} \frac{a_k}{N + \frac{1}{2} - \lambda_k} - \sum_{k=-N}^{N} \frac{a_k}{N + \frac{1}{2} - k} \right|$$

$$\leq \|w\|_{\mathcal{PW}_\pi^1} \sup_{k \in \mathbb{Z}} |a_k| \sum_{k=-N}^{N} \frac{1}{(N + \frac{1}{2} - \lambda_k)(N + \frac{1}{2} - k)}. \quad (6)$$

Finally, using elementary calculations it can be shown that the sum in (6) is bounded from above by some constant. $\qquad \square$

Lemma 4. *There exists a function $w \in \mathcal{PW}_\pi^1$ that is real on the real axis with $\|w\|_{\mathcal{PW}_\pi^1} < 1/2$ and $w(k) \geq 0$, $k \in \mathbb{Z}$, such that*

$$\limsup_{N \to \infty} \sum_{k=-N}^{N} \frac{\sin(\pi w(k))}{N + \frac{1}{2} - k} = \infty.$$

Proof. For $N \in \mathbb{N}$, consider the function

$$f_N(t) := \sum_{k=-2N+1}^{2N-1} f_N(k) \frac{\sin(\pi(t - k))}{\pi(t - k)}$$

where $f_N(k) = 1$ for $|k| \leq N$, and $f_N(k) = 2 - |k|/N$ for $N < |k| < 2N$. A short calculation shows that $\|f_N\|_{\mathcal{PW}_\pi^1} \leq 3$, $N \in \mathbb{N}$. Next, let $M(l) = 2^{(l^3)}$, $l \in \mathbb{Z}$, and choose

$$w(t) = \frac{1}{10} \sum_{l=1}^{\infty} \frac{1}{l^2} f_{M(l)}(t). \quad (7)$$

It follows that $w \in \mathcal{PW}_\pi^1$, because

$$\|w\|_{\mathcal{PW}_\pi^1} \leq \frac{1}{10} \sum_{l=1}^\infty \frac{1}{l^2} \|f_{M(l)}\|_{\mathcal{PW}_\pi^1} \leq \frac{3}{10} \frac{\pi^2}{6} < \frac{1}{2},$$

and that $w(k) \geq 0$, $k \in \mathbb{Z}$. Consequently, we have $0 \leq w(k) < 1/2$, for all $k \in \mathbb{Z}$. Thus, for $r \in \mathbb{N}$, we have

$$\sum_{k=-M(r)}^{M(r)} \frac{\sin(\pi w(k))}{M(r) + \frac{1}{2} - k} \geq 2 \sum_{k=-M(r)}^{M(r)} \frac{w(k)}{M(r) + \frac{1}{2} - k}$$

$$\geq \frac{1}{5r^2} \sum_{k=-M(r)}^{M(r)} \frac{f_{M(r)}(k)}{M(r) + \frac{1}{2} - k} = \frac{1}{5r^2} \sum_{k=0}^{2M(r)} \frac{1}{k + \frac{1}{2}}$$

$$> \frac{1}{5r^2} \log(M(r)) = \frac{1}{5} r \log(2),$$

because $\sin(x) \geq 2x/\pi$ for all $x \in [0, \pi/2]$. $\qquad \square$

Now, we are in the position to prove Theorem 2.

Proof of Theorem 2. Let $w \in \mathcal{PW}_\pi^1$ be the function from Lemma 4 and $\{t_k\}_{k\in\mathbb{Z}} = \{k + w(k)\}_{k\in\mathbb{Z}} \subset \mathbb{R}$. According to Corollary 1 there exists a function $f_1 \in \mathcal{PW}_\pi^1$ that is real on the real axis such that ψ_{f_1} is a function of sine type that has the zero sequence $\{t_k\}_{k\in\mathbb{Z}}$. Therefore, we have $f_1(t_k) = \sin(\pi t_k) = (-1)^k \sin(\pi w(k))$ for all $k \in \mathbb{Z}$. For $N \in \mathbb{N}$, sufficiently large, so that Lemma 2 can be applied, it follows that

$$\left| \sum_{k=-N}^N f_1(t_k) \frac{\psi_{f_1}(N + \frac{1}{2})}{\psi'_{f_1}(t_k)(N + \frac{1}{2} - t_k)} \right|$$

$$= \left| \psi_{f_1}(N + \tfrac{1}{2}) \right| \left| \sum_{k=-N}^N \frac{(-1)^k \sin(\pi w(k))}{\psi'_{f_1}(t_k)(N + \frac{1}{2} - t_k)} \right|$$

$$\geq C_1 \left| \sum_{k=-N}^N \frac{(-1)^k \sin(\pi w(k))}{\psi'_{f_1}(t_k)(N + \frac{1}{2} - t_k)} \right|.$$

Moreover, since ψ_{f_1} is a sine-type function we have either $\psi'_{f_1}(t_k) = (-1)^k c_k$ or $\psi'_{f_1}(t_k) = (-1)^{k+1} c_k$ for some positive c_k, satisfying $\inf_{k\in\mathbb{Z}} c_k > 0$ and $\bar{c} := \sup_{k\in\mathbb{Z}} c_k < \infty$ [2, p. 164]. We continue the proof, assuming that $\psi'_{f_1}(t_k) = (-1)^k c_k$, $k \in \mathbb{Z}$. The other case is treated analogously. Applying Lemma 3, it follows that

$$\left| \sum_{k=-N}^N \frac{(-1)^k \sin(\pi w(k))}{\psi'_{f_1}(t_k)(N + \frac{1}{2} - t_k)} \right| \geq \frac{1}{\bar{c}} \sum_{k=-N}^N \frac{\sin(\pi w(k))}{(N + \frac{1}{2} - k)} - C_3$$

for all sufficiently large $N \in \mathbb{N}$. Combining all partial results, we have

$$\left| \sum_{k=-N}^N f_1(t_k) \frac{\psi_{f_1}(N + \frac{1}{2})}{\psi'_{f_1}(t_k)(N + \frac{1}{2} - t_k)} \right| \geq \frac{C_1}{\bar{c}} \sum_{k=-N}^N \frac{\sin(\pi w(k))}{(N + \frac{1}{2} - k)} - C_4$$

for all $N \in \mathbb{N}$, sufficiently large. Finally, applying Lemma 4 gives

$$\limsup_{N\to\infty} \left(\sup_{t\in\mathbb{R}} \left| \sum_{k=-N}^N f_1(t_k) \frac{\psi_{f_1}(t)}{\psi'_{f_1}(t_k)(t - t_k)} \right| \right)$$

$$\geq \limsup_{N\to\infty} \left| \sum_{k=-N}^N f_1(t_k) \frac{\psi_{f_1}(N + \frac{1}{2})}{\psi'_{f_1}(t_k)(N + \frac{1}{2} - t_k)} \right| = \infty,$$

which completes the proof. $\qquad \square$

5. CONCLUSION

In this paper we have studied the adaptive signal reconstruction scheme (4), where the reconstruction functions and the sampling points are matched to the signal. Due to the transformation (3) that is used, this problem is closely related to the reconstruction of bandlimited signals from their sine wave crossings. Since, under the assumptions that $f \in \mathcal{PW}_\pi^1$ is real valued on the real axis and $\|f\|_\infty < 1$, the sampling points are the zeros of a sine-type function, we could employ the theory of sine-type functions to obtain our results. We have shown the uniform convergence of the reconstruction process for the spaces $\mathcal{PW}_\pi^p$, $1 < p \leq \infty$. However, the adaptivity cannot prevent the divergence of the peak value of the approximation process for certain signals in $\mathcal{PW}_\pi^1$, which is the largest space in the scale of Paley-Wiener spaces.

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