OPTIMIZED CAPACITY BOUNDS FOR THE MIMO RELAY CHANNEL

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ABSTRACT

This paper addresses the optimization of upper and lower bounds on the capacity of the multiple-input multiple-output (MIMO) relay channel. In particular, we show that evaluating the cut-set bound and the maximal achievable decode-and-forward rate is equivalent to solving convex optimization problems, where we assume that perfect channel state information is available at all nodes. Our optimized bounds thus improve on previously published results while they can be efficiently determined using convex programming techniques at the same time.

Index Terms— Relay channel, MIMO, cut-set bound, decode-and-forward, convex optimization.

I. INTRODUCTION

In the standard relay channel, one source wants to transmit information to one destination with the help of a single relay node whose only purpose is to assist this communication. A model for this relay channel was introduced by van der Meulen as early as 1971 [1], but its general capacity remains unknown. In their pioneering work, Cover and El Gamal [2] derived upper and lower bounds on the capacity of the full-duplex relay channel based on a then new cut-set bound (CSB) and two fundamental coding strategies, respectively. The first coding scheme uses a technique called *block Markov superposition encoding* and is now referred to as *decode-and-forward* (DF). In the second strategy, the relay reliably forwards an estimate, a compressed version of its receive signal, to the destination. Therefore, this scheme is usually termed *compress-and-forward* (CF). For wireless networks, DF achieves gains related to multi-antenna *transmission* when the relay is close to the source, whereas CF achieves gains related to multi-antenna *reception* when the relay is close to the destination [3]. Another relaying strategy of lower complexity than both DF and CF is called *amplify-and-forward* (AF), which is considered in [4] for example. When using AF, the relay is restricted to perform linear operations on its receive signal.

The information theoretic results derived for the relay channel apply to both nodes with single and multiple antennas. However, a major difference is that multi-antenna nodes may employ techniques like precoding (beamforming) that devices equipped with a single antenna are not capable of. Hence, the multiple-input multiple-output (MIMO) relay channel offers more degrees of freedom which can be utilized. For example, optimizing the achievable DF rate for the MIMO relay channel with Gaussian source and relay inputs requires to solve an optimization problem with respect to the source covariance matrix $\boldsymbol{R}_{\mathrm{S}}$, the relay covariance matrix $\boldsymbol{R}_{\mathrm{R}}$, and their cross covariance matrix $\boldsymbol{R}_{\mathrm{SR}}$ instead of a scalar correlation coefficient ρ when single antenna nodes are considered.

Upper and lower bounds on the capacity of the MIMO relay channel were first studied in [5]. The authors prove that Gaussian input distributions maximize the CSB and the achievable DF rate. Furthermore, they exploit matrix inequalities to establish a generally loose upper bound on the CSB involving a maximization over $\boldsymbol{R}_{\mathrm{S}}$, $\boldsymbol{R}_{\mathrm{R}}$, and a scalar parameter ρ capturing the cross correlation of the source and relay inputs. Lower bounds are also derived based on point-to-point transmission, the cascaded relay channel, and a suboptimal DF strategy. In [6], two *partial decode-and-forward* (PDF) strategies, where the relay only decodes part of the source message, are presented using superposition or dirty-paper coding at the source. While the achievable rates are shown to improve on the lower bounds of [5], the rate maximization problems are only formulated but not solved for the general case.

Assuming perfect channel state information (CSI) at all nodes, we show that the cut-set bound and the maximal achievable DF rate for the MIMO relay channel can be obtained as the solutions of convex optimization problems. Similar work is presented in [7]. However, it can be verified that the expressions resulting from those derivations are only upper bounds to the optimal solutions given here.

The remainder of this paper is organized as follows. The system model is introduced in Sec. II. Secs. III and IV address the optimization of the CSB and the achievable DF rate, respectively. Numerical results are presented in Sec. V along with a brief discussion before we conclude in Sec. VI.

II. SYSTEM MODEL

Consider the full-duplex MIMO relay channel illustrated in Fig. 1. The signals received at the relay and the destination

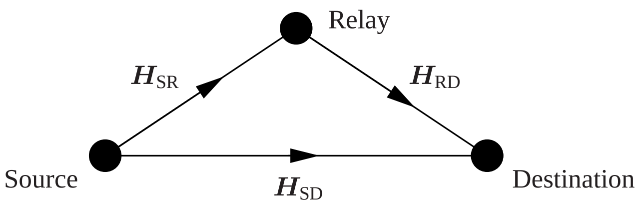

Fig. 1. MIMO relay channel.

can be expressed as

$$\begin{aligned}\boldsymbol{y}_{\mathrm{R}} &= \boldsymbol{H}_{\mathrm{SR}}\boldsymbol{x}_{\mathrm{S}} + \boldsymbol{n}_{\mathrm{R}},\\ \boldsymbol{y}_{\mathrm{D}} &= \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{x}_{\mathrm{S}} + \boldsymbol{H}_{\mathrm{RD}}\boldsymbol{x}_{\mathrm{R}} + \boldsymbol{n}_{\mathrm{D}},\end{aligned} \tag{1}$$

where $\boldsymbol{H}_{\mathrm{SR}} \in \mathbb{C}^{N_{\mathrm{R}}\times N_{\mathrm{S}}}, \boldsymbol{H}_{\mathrm{SD}} \in \mathbb{C}^{N_{\mathrm{D}}\times N_{\mathrm{S}}}, \boldsymbol{H}_{\mathrm{RD}} \in \mathbb{C}^{N_{\mathrm{D}}\times N_{\mathrm{R}}}$ represent the channel gain matrices assumed to be perfectly known at all nodes, and $\boldsymbol{n}_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{I}_{N_{\mathrm{R}}}), \boldsymbol{n}_{\mathrm{D}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{I}_{N_{\mathrm{D}}})$ denote complex white Gaussian noise of unit variance that is independent of the transmit signals $\boldsymbol{x}_{\mathrm{S}}$ and $\boldsymbol{x}_{\mathrm{R}}$. Moreover, the source and the relay are subject to transmit power constraints given by $\mathrm{E}[\boldsymbol{x}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{x}_{\mathrm{S}}] = \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}) \le P_{\mathrm{S}}$ and $\mathrm{E}[\boldsymbol{x}_{\mathrm{R}}^{\mathrm{H}}\boldsymbol{x}_{\mathrm{R}}] = \mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}) \le P_{\mathrm{R}}$, respectively.

The joint transmit covariance matrix of the zero-mean source and relay inputs is determined by

$$\boldsymbol{R} = \mathrm{E}\left[\begin{bmatrix}\boldsymbol{x}_{\mathrm{S}}\\ \boldsymbol{x}_{\mathrm{R}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{S}}\\ \boldsymbol{x}_{\mathrm{R}}\end{bmatrix}^{\mathrm{H}}\right] = \begin{bmatrix}\boldsymbol{R}_{\mathrm{S}} & \boldsymbol{R}_{\mathrm{SR}}\\ \boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} & \boldsymbol{R}_{\mathrm{R}}\end{bmatrix}. \tag{2}$$

By defining the two selection matrices

$$\boldsymbol{D}_{\mathrm{S}} = \begin{bmatrix}\boldsymbol{I}_{N_{\mathrm{S}}} & \boldsymbol{0}_{N_{\mathrm{S}}\times N_{\mathrm{R}}}\end{bmatrix}, \quad \boldsymbol{D}_{\mathrm{R}} = \begin{bmatrix}\boldsymbol{0}_{N_{\mathrm{R}}\times N_{\mathrm{S}}} & \boldsymbol{I}_{N_{\mathrm{R}}}\end{bmatrix}, \tag{3}$$

we see that the source and the relay transmit covariance matrices can be expressed as linear functions of $\boldsymbol{R}$:

$$\boldsymbol{R}_{\mathrm{S}} = \boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}, \quad \boldsymbol{R}_{\mathrm{R}} = \boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}. \tag{4}$$

III. CUT-SET BOUND

In their pioneering work, Cover and El Gamal [2] proved that the capacity of the relay channel is upper bounded by

$$C_{\mathrm{CSB}} = \max\min\left\{I(X_{\mathrm{S}};Y_{\mathrm{R}}Y_{\mathrm{D}}|X_{\mathrm{R}}), I(X_{\mathrm{S}}X_{\mathrm{R}};Y_{\mathrm{D}})\right\}, \tag{5}$$

where the maximization is with respect to the joint distribution of the source and relay signals. In this section, we show that evaluating this cut-set bound is equivalent to solving a convex optimization problem.

The source and relay inputs optimizing the CSB are known to be Gaussian from [5]. With $\boldsymbol{x}_{\mathrm{S}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{S}})$ and $\boldsymbol{x}_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{R}})$, the mutual information expressions defining the cut-set bound read as

$$I(X_{\mathrm{S}};Y_{\mathrm{R}}Y_{\mathrm{D}}|X_{\mathrm{R}}) = \log\det\left(\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}_{\mathrm{S|R}}\boldsymbol{H}_1^{\mathrm{H}}\right), \tag{6}$$

$$I(X_{\mathrm{S}}X_{\mathrm{R}};Y_{\mathrm{D}}) = \log\det\left(\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}\boldsymbol{H}_2^{\mathrm{H}}\right), \tag{7}$$

where $\boldsymbol{H}_1 = \begin{bmatrix}\boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}} & \boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\end{bmatrix}^{\mathrm{H}}, \boldsymbol{H}_2 = \begin{bmatrix}\boldsymbol{H}_{\mathrm{SD}} & \boldsymbol{H}_{\mathrm{RD}}\end{bmatrix}$, and $\boldsymbol{R}_{\mathrm{S|R}} = \boldsymbol{R}_{\mathrm{S}} - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}}$ is the conditional covariance matrix of X_{S} given that $X_{\mathrm{R}} = \boldsymbol{x}_{\mathrm{R}}$. Here $\boldsymbol{R}_{\mathrm{R}}^{\dagger}$ denotes the Moore-Penrose pseudoinverse of $\boldsymbol{R}_{\mathrm{R}}$ which is equal to $\boldsymbol{R}_{\mathrm{R}}^{-1}$ if $\boldsymbol{R}_{\mathrm{R}}$ is non-singular. Hence, computing the cut-set bound requires to solve the following optimization problem:

$$\max_{\boldsymbol{R}\succeq\boldsymbol{0}} C_{\mathrm{CSB}} \tag{8}$$

$$\text{s.t.} \quad C_{\mathrm{CSB}} \le \log\det\left(\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}_{\mathrm{S|R}}\boldsymbol{H}_1^{\mathrm{H}}\right), \tag{9}$$

$$C_{\mathrm{CSB}} \le \log\det\left(\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}\boldsymbol{H}_2^{\mathrm{H}}\right), \tag{10}$$

$$\mathrm{tr}(\boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \le P_{\mathrm{S}}, \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \le P_{\mathrm{R}}. \tag{11}$$

In the present form, this optimization problem is non-convex due to the term $\boldsymbol{R}_{\mathrm{S|R}} = \boldsymbol{R}_{\mathrm{S}} - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}}$ in (9). However, it is possible to reformulate the problematic constraint by means of a slack variable:

$$C_{\mathrm{CSB}} \le \log\det\left(\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{Q}\boldsymbol{H}_1^{\mathrm{H}}\right) \tag{12}$$

$$\text{s.t.} \quad \boldsymbol{0} \preceq \boldsymbol{Q} \preceq \boldsymbol{R}_{\mathrm{S}} - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}}. \tag{13}$$

While (12) is concave in $\boldsymbol{Q} \succeq \boldsymbol{0}$, as we desire, the additional constraints that have been introduced can once more be reformulated by making use of the following lemma:

Lemma 1: Iff $\boldsymbol{R}_{\mathrm{R}} \succeq \boldsymbol{0}, (\boldsymbol{R}_{\mathrm{S}} - \boldsymbol{Q}) - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} \succeq \boldsymbol{0}$, and $(\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{R}_{\mathrm{R}}\boldsymbol{R}_{\mathrm{R}}^{\dagger})\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} = \boldsymbol{0}$, then

$$\begin{bmatrix}\boldsymbol{R}_{\mathrm{S}} - \boldsymbol{Q} & \boldsymbol{R}_{\mathrm{SR}}\\ \boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} & \boldsymbol{R}_{\mathrm{R}}\end{bmatrix} = \boldsymbol{R} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}. \tag{14}$$

Proof: Note that $(\boldsymbol{R}_{\mathrm{S}} - \boldsymbol{Q}) - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}}$ is the (generalized) Schur complement of $\boldsymbol{R}_{\mathrm{R}}$ in $\boldsymbol{R} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{D}_{\mathrm{S}}$. The lemma simply follows from the Schur complement condition for positive semi-definite matrices [8, A.5.5]. ■

Clearly, the first two conditions of the lemma are satisfied as $\boldsymbol{R}_{\mathrm{R}} \succeq \boldsymbol{0}$, since $\boldsymbol{R}_{\mathrm{R}}$ is the relay transmit covariance matrix, and $(\boldsymbol{R}_{\mathrm{S}} - \boldsymbol{Q}) - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} \succeq \boldsymbol{0}$, which follows from (13). It remains to verify that $(\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{R}_{\mathrm{R}}\boldsymbol{R}_{\mathrm{R}}^{\dagger})\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} = \boldsymbol{0}$. To this end, we require another lemma.

Lemma 2: For all possible joint transmit covariance matrices $\boldsymbol{R}$, the following holds:

$$(\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{R}_{\mathrm{R}}\boldsymbol{R}_{\mathrm{R}}^{\dagger})\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} = \boldsymbol{0}. \tag{15}$$

Proof: If $\boldsymbol{R}_{\mathrm{R}}$ is positive definite, $\boldsymbol{R}_{\mathrm{R}}^{\dagger} = \boldsymbol{R}_{\mathrm{R}}^{-1}$ and (15) is trivially satisfied. Hence, suppose that $\boldsymbol{R}_{\mathrm{R}} \in \mathbb{C}^{N_{\mathrm{R}}\times N_{\mathrm{R}}}$ is positive semi-definite with $\mathrm{rank}(\boldsymbol{R}_{\mathrm{R}}) = r < N_{\mathrm{R}}$. In this case, there exists $\boldsymbol{S} \in \mathbb{C}^{N_{\mathrm{R}}\times r}$ such that $\boldsymbol{R}_{\mathrm{R}} = \boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}$. With $\boldsymbol{x}_{\mathrm{R}} = \boldsymbol{S}\boldsymbol{z}, \boldsymbol{z} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{I}_r)$, it follows that $\boldsymbol{x}_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{R}})$ and $\boldsymbol{R}_{\mathrm{SR}} = \mathrm{E}\left[\boldsymbol{x}_{\mathrm{S}}\boldsymbol{z}^{\mathrm{H}}\boldsymbol{S}^{\mathrm{H}}\right] = \boldsymbol{R}_{\mathrm{SZ}}\boldsymbol{S}^{\mathrm{H}}$. Consequently,

$$(\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{R}_{\mathrm{R}}\boldsymbol{R}_{\mathrm{R}}^{\dagger})\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} = (\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}(\boldsymbol{S}\boldsymbol{S}^{\mathrm{H}})^{\dagger})\boldsymbol{S}\boldsymbol{R}_{\mathrm{SZ}}^{\mathrm{H}}.$$

Since $\boldsymbol{S}$ has full column rank, it can be shown that $(\boldsymbol{S}\boldsymbol{S}^{\mathrm{H}})^{\dagger} = \boldsymbol{S}^{\mathrm{H},\dagger}\boldsymbol{S}^{\dagger}$ and $\boldsymbol{S}^{\mathrm{H}}\boldsymbol{S}^{\mathrm{H},\dagger} = \boldsymbol{I}_r$. Thus,

$$\begin{aligned}(\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{R}_{\mathrm{R}}\boldsymbol{R}_{\mathrm{R}}^{\dagger})\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} &= (\boldsymbol{I}_{N_{\mathrm{R}}} - \boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}\boldsymbol{S}^{\mathrm{H},\dagger}\boldsymbol{S}^{\dagger})\boldsymbol{S}\boldsymbol{R}_{\mathrm{SZ}}^{\mathrm{H}}\\ &= (\boldsymbol{S} - \boldsymbol{S}\boldsymbol{S}^{\dagger}\boldsymbol{S})\boldsymbol{R}_{\mathrm{SZ}}^{\mathrm{H}}\\ &= (\boldsymbol{S} - \boldsymbol{S})\boldsymbol{R}_{\mathrm{SZ}}^{\mathrm{H}} = \boldsymbol{0}.\end{aligned}$$

We can conclude that (15) is satisfied for every joint transmit covariance matrix $\boldsymbol{R}$. ∎

Applying Lemma 1 and Lemma 2 to our problem, it is evident that condition (13) may be replaced by

$$\boldsymbol{R} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}} \boldsymbol{Q} \boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}. \tag{16}$$

Therefore, we may substitute constraint (9) in the original optimization problem by conditions (12) and (16) to arrive at the following equivalent formulation:

$$\begin{aligned} C_{\mathrm{CSB}} = \max_{\boldsymbol{Q},\boldsymbol{R}} \min\big\{&\log\det\left(\boldsymbol{I} + \boldsymbol{H}_1 \boldsymbol{Q} \boldsymbol{H}_1^{\mathrm{H}}\right), \\ &\log\det\left(\boldsymbol{I} + \boldsymbol{H}_2 \boldsymbol{R} \boldsymbol{H}_2^{\mathrm{H}}\right)\big\} \end{aligned} \tag{17}$$

$$\text{s.t.} \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{S}} \boldsymbol{R} \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \leq P_{\mathrm{S}}, \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}} \boldsymbol{R} \boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \leq P_{\mathrm{R}}, \tag{18}$$

$$\boldsymbol{Q} \succeq \boldsymbol{0}, \quad \boldsymbol{R} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}} \boldsymbol{Q} \boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}. \tag{19}$$

Note that the objective function is concave since it is the pointwise minimum of two concave functions (in $\boldsymbol{Q} \succeq \boldsymbol{0}, \boldsymbol{R} \succeq \boldsymbol{0}$) [8]. The constraint $\boldsymbol{R} \succeq \boldsymbol{0}$ is redundant here because it is implied by (19). Furthermore, all constraints are affine which means that this optimization problem, which determines the cut-set bound, is convex.

IV. ACHIEVABLE DECODE-AND-FORWARD (DF) RATE

A lower bound on the capacity of the relay channel is given by the rate that can be achieved with the decode-and-forward protocol derived by Cover and El Gamal [2]. If the relay uses DF, all achievable rates satisfy

$$R_{\mathrm{DF}} \leq \max \min\big\{I(X_{\mathrm{S}}; Y_{\mathrm{R}}|X_{\mathrm{R}}), I(X_{\mathrm{S}} X_{\mathrm{R}}; Y_{\mathrm{D}})\big\}, \tag{20}$$

where the maximization is again with respect to the joint distribution of the source and relay signals. Note that R_{DF} differs from C_{CSB} only in the first mutual information term, whereas the second one is the same.

What is more, Gaussian inputs have also been proven to optimize the achievable DF rate [5]. With $\boldsymbol{x}_{\mathrm{S}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{S}})$ and $\boldsymbol{x}_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{R}})$, it hence follows that

$$I(X_{\mathrm{S}}; Y_{\mathrm{R}}|X_{\mathrm{R}}) = \log\det\left(\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SR}} \boldsymbol{R}_{\mathrm{S|R}} \boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}}\right), \tag{21}$$

where the only difference to (6) is that $\boldsymbol{H}_1 = \begin{bmatrix}\boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}} & \boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\end{bmatrix}^{\mathrm{H}}$ is replaced by $\boldsymbol{H}_{\mathrm{SR}}$. Using the same arguments as for the calculation of the cut-set bound, it can thus be shown that

$$\begin{aligned} R_{\mathrm{DF}} \leq \max_{\boldsymbol{Q},\boldsymbol{R}} \min\big\{&\log\det\left(\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SR}} \boldsymbol{Q} \boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}}\right), \\ &\log\det\left(\boldsymbol{I} + \boldsymbol{H}_2 \boldsymbol{R} \boldsymbol{H}_2^{\mathrm{H}}\right)\big\} \end{aligned} \tag{22}$$

$$\text{s.t.} \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{S}} \boldsymbol{R} \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \leq P_{\mathrm{S}}, \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}} \boldsymbol{R} \boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \leq P_{\mathrm{R}}, \tag{23}$$

$$\boldsymbol{Q} \succeq \boldsymbol{0}, \quad \boldsymbol{R} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}} \boldsymbol{Q} \boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}. \tag{24}$$

Consequently, the maximal achievable DF rate is also obtained as the solution of a convex optimization problem.

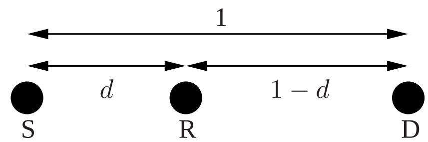

Fig. 2. Line Network.

V. NUMERICAL RESULTS AND DISCUSSION

In this section, numerical results for the achievable DF rate and the cut-set bound are provided, where particular attention is devoted to the comparison of the latter to the upper bound derived in [5]. We discuss under which conditions this upper bound is equal to the cut-set bound, and subsequently, we also touch upon the results presented in [7].

As an example scenario, consider the geometry depicted in Fig. 2, which is also used in [3] and [5]. Here $d_{\mathrm{SD}} = 1$ is fixed, whereas the relay is positioned on the line connecting the source and the destination such that $d_{\mathrm{SR}} = |d|$ and $d_{\mathrm{RD}} = |1-d|$. We assume that the channel gain matrices are random and independent, and the entries of $\boldsymbol{H}_{\mathrm{SR}}, \boldsymbol{H}_{\mathrm{SD}}$, and $\boldsymbol{H}_{\mathrm{RD}}$ are independent and identically distributed complex Gaussian random variables with zero mean and variance $d_{\mathrm{SR}}^{-2}, d_{\mathrm{SD}}^{-2}$, and d_{RD}^{-2}, respectively. All numerical results presented here are averaged over independent channel realizations, where perfect CSI at all nodes is assumed for every realization.

The upper bound on the capacity of the MIMO relay channel derived in [5] is given by

$$\begin{aligned} C_{\mathrm{UB}} = \max_{\boldsymbol{R}_{\mathrm{S}},\boldsymbol{R}_{\mathrm{R}},\rho} \min\big\{&\log\det\left(\boldsymbol{I} + (1-\rho^2)\boldsymbol{H}_1 \boldsymbol{R}_{\mathrm{S}} \boldsymbol{H}_1^{\mathrm{H}}\right), \\ &\inf_{a>0} \log\det\big(\boldsymbol{I} + (1+\tfrac{\rho^2}{a})\boldsymbol{H}_{\mathrm{SD}} \boldsymbol{R}_{\mathrm{S}} \boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}} \\ &\quad + (1+a)\boldsymbol{H}_{\mathrm{RD}} \boldsymbol{R}_{\mathrm{R}} \boldsymbol{H}_{\mathrm{RD}}^{\mathrm{H}}\big)\big\} \end{aligned} \tag{25}$$

$$\text{s.t.} \quad \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}) \leq P_{\mathrm{S}}, \quad \mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}) \leq P_{\mathrm{R}}, \tag{26}$$

$$\boldsymbol{R}_{\mathrm{S}} \succeq \boldsymbol{0}, \quad \boldsymbol{R}_{\mathrm{R}} \succeq \boldsymbol{0}, \quad 0 \leq \rho \leq 1, \tag{27}$$

where the cross correlation between the source and relay inputs is captured only by the scalar parameter ρ instead of the matrix $\boldsymbol{R}_{\mathrm{SR}}$. Several restrictions this bound suffers from are pointed out in [7]. Most importantly, it is only valid for $N_{\mathrm{S}} \leq N_{\mathrm{R}}$ so that our example scenarios are restricted to this case. What is more, the upper bound is strictly larger than the cut-set bound in general. However, note that for scalar channels, i.e., when all nodes have a single antenna, the upper bound is equal to the CSB [5]. Hence, Fig. 3 only compares the cut-set bound to the achievable DF rate and the upper bound $C_{\mathrm{UB}}^{\mathrm{Sim}}$ from [7] for $N_{\mathrm{S}} = N_{\mathrm{R}} = N_{\mathrm{D}} = 1$. It can be observed that the DF scheme achieves the CSB when the relay is very close to the source. The value of ρ that maximizes C_{UB}, and thus C_{CSB}, is close to 1 in this case, which means that source and relay can realize multi-antenna transmission. More interestingly, Fig. 3 also reveals a non-negligible gap between C_{CSB} and $C_{\mathrm{UB}}^{\mathrm{Sim}}$ even in the case when all nodes are equipped with a single antenna.

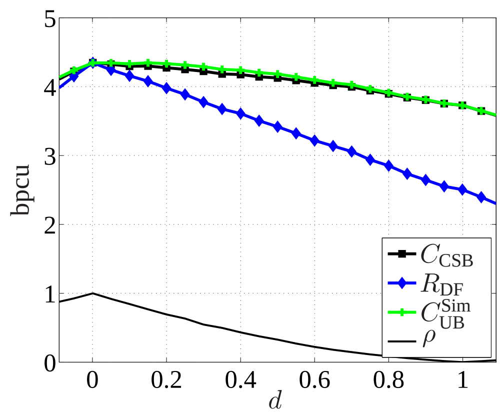

Fig. 3. Comparison of $C_{\text{CSB}}, C_{\text{UB}}^{\text{Sim}}$ (from [7]), and R_{DF} for $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 1$ and $P_{\text{S}} = P_{\text{R}} = 10$ (averaged over 10000 channel realizations).

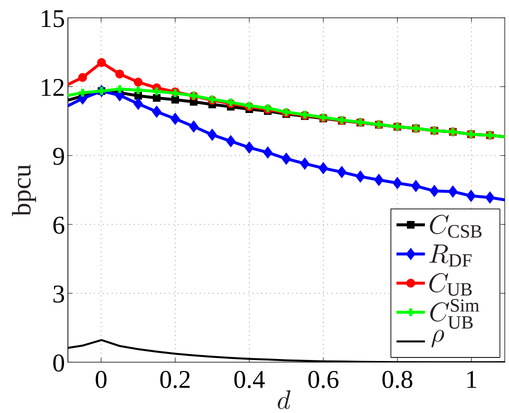

Fig. 4. Comparison of $C_{\text{CSB}}, C_{\text{UB}}$ (from [5]), $C_{\text{UB}}^{\text{Sim}}$ (from [7]), and R_{DF} for $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 2$ and $P_{\text{S}} = P_{\text{R}} = 10$ (averaged over 2000 channel realizations).

If all nodes are now equipped with multiple antennas, there are channel conditions for which C_{UB} is strictly greater than C_{CSB} as well. This is illustrated in Fig. 4 for the case $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 2$. Note that $C_{\text{UB}} = C_{\text{CSB}}$ only if $\rho = 0$. In fact, it can be shown that this bound is equal to the cut-set bound if the optimal solution requires independent source and relay inputs, i.e., if $\rho = 0$ and $\boldsymbol{R}_{\text{SR}} = \boldsymbol{0}$ are optimizers of the respective optimization problems. This result can be explained as follows. In order to arrive at the upper bound, the authors of [5] introduce the scalar parameter ρ to capture the cross correlation between the source and relay inputs normally defined by the matrix $\boldsymbol{R}_{\text{SR}}$. As a result, information about the structure of the cross correlation is lost. However, if the optimal solution is obtained for independent source and relay inputs, the corresponding correlation is completely determined by the scalar $\rho = 0$. Hence, the upper bound is equal to the CSB in this case.

A similar statement applies to $C_{\text{UB}}^{\text{Sim}}$, which is a loose upper bound in general, but tight when independent source and relay inputs are optimal, cf. Figs. 3 and 4. That is because $C_{\text{UB}}^{\text{Sim}}$ is obtained from C_{CSB} by replacing $\boldsymbol{R}_{\text{S|R}}$ in (9) with $\boldsymbol{R}_{\text{S}}$. As $\log\det(\boldsymbol{I} + \boldsymbol{H}\boldsymbol{R}\boldsymbol{H}^{\text{H}})$ is increasing in $\boldsymbol{R} \succeq \boldsymbol{0}$ for given $\boldsymbol{H}$, and since $\boldsymbol{R}_{\text{S|R}} \preceq \boldsymbol{R}_{\text{S}}$, it follows that $C_{\text{CSB}} \leq C_{\text{UB}}^{\text{Sim}}$ with equality if $\boldsymbol{R}_{\text{S|R}} = \boldsymbol{R}_{\text{S}}$, i.e., if $\boldsymbol{R}_{\text{SR}} = \boldsymbol{0}$. Note that a corresponding result for the DF rate presented in [7] is not considered here as any rate $R > R_{\text{DF}}$ is not achievable with the decode-and-forward protocol.

On a different matter, Fig. 4 shows again that the DF scheme approaches the CSB when the relay is close to the source. Furthermore, observe that substantial rate gains can be achieved when multiple antennas are used at each node without increasing the power at the source and the relay.

VI. CONCLUSION

In this paper, we show that the cut-set bound and the achievable DF rate, which constitute upper and lower bounds on the capacity of the relay channel, respectively, can be obtained as the solutions of convex optimization problems. These results improve on previously published upper bounds that are in general strictly larger than the CSB, and in the case of C_{UB} not even valid for all antenna configurations, and several lower bounds derived in [5]. Using our results, it is hence possible to efficiently determine tighter bounds on the capacity of the MIMO relay channel, which may for example serve as benchmarks when studying different relay strategies or the impact of channel estimation errors.

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