Technology University of Munich

Center of Mathematics

Modelling of Liquidity Requirements for Revolving Credit Lines

Diplomarbeit

von

Yi XIE

Themenstellerin:	Prof. Dr. Claudia Czado
Betreuerer:	Dr. Rada Dakovic Dr. Armin Leistenschneider
Abgabetermin:	28. Feb 2008

Hereby I declare that the Diplomarbeit is written independently and only the listed literature is used as reference.

Garching, 31. Aug 2007

Acknowledgments

I would like to express my sincere gratitude to Prof. Dr. Claudia Czado, Dr. Rada Dakovic for the tremendous support for the theoretical research in the liquidity requirement of the revolving credit line. The same thankfulness I like to deliver to my manager Dr. Armin Leistenschneider, Matthias Behrens from Dresdner Bank for leading me into the field of liquidity risk management and their precious help during my simulation work. Last but not least, I like to thank my colleague Bjoern Engelmann, who offered me a large sum of idea during the theoretical derivation and programming for the Monte Carlo simulation.

Contents

Symbol List

$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
t	the time in months, $t = 0, 1, ..., T$, $T \in \mathbb{N}$
i	the index for customer
j	the index for credit line
k	the index for maturity
n	the number of customers
n_i	the number of the credit lines of customer i
l_t^{ij}	the committed limit of credit line (i, j) (given in EUR)
m^{ij}	the maturity of credit line (i, j), $m^{ij} \in M = \{12, 24, 36, 48\}$
R_t^i	the credit rating of customer i at time t, $R_t^i \in S = \{1, 2, ..., 8\}$ in which $1 \triangleq AAA$, $2 \triangleq AA$, $3 \triangleq A$,$4 \triangleq BBB$, $5 \triangleq BB$, $6 \triangleq B$, $7 \triangleq CCC$, $8 \triangleq D$
U_t^{ij}	the proportion of usage of the credit line (i, j) at time t where $U_t^{ij} \in U$, $U = \{u_i : i \in S\}$ and u_s is constant $\forall s \in S$
F_t	the total committed credit limit at time t, $\mathrm{F}_t = \sum_i^n \sum_j^{n_i} l_t^{ij}$
$\mathcal{F}_t$	the total amount of credit loan drawn at time t, $\mathrm{F}_t = \sum_i^n \sum_j^{n_i} l_t^{ij} U_t^{ij}$
f_t	the propotional usage of total committed limit at time t, $f_t = \mathcal{F}_t / \mathrm{F}_t$
$\mathbf{P}_y$	the yearly transition matrix, in which $\mathbf{P}_y \in \mathbb{R}^{8\times 8}$
$\mathbf{P}_m$	the quasi monthly transition matrix
$\mathbf{P}$	the optimal monthly transition matrix
Σ	the correlation matrix of the customers
A_s	the draw decision variable
$L_t^{i,j}$	the time period between credit loan return and the maturity for credit line (i, j)
$E^{i,j}$	the time period between credit loan start and the maturity for credit line (i, j)
$\bar{R}_t^i$	the R_t^i after mapping into credit rating bucket
$\bar{L}_t^{i,j}$	the $L^{i,j}$ after mapping into time bucket
$\bar{E}^{i,j}$	the $E^{i,j}$ after mapping into time bucket
$B_t^{\bar{E}^{i,j}}$	the return decision variable for credit line (i, j) at time t with the start time of the credit loan $\bar{E}^{i,j}$ time buckets before maturity
$Te_t^{i,j}$	the term out decision variable for credit line (i, j) at time t
$Re_t^{i,j}$	the renewal decision variable for credit line (i, j) at time t
$Ex_t^{i,j}$	the expire decision variable for credit line (i, j) at time t

p_s	the draw probability
$\hat{p}_s$	the estimated draw probability
$q(s, l, E)$	the return probability
$q(\bar{s}, \bar{l}, \bar{E})$	the grouped return probability
$\hat{q}(\bar{s}, \bar{l}, \bar{E})$	the estimated return probability
$Q(s, l, E)$	the cumulative return probability
$\hat{Q}(\bar{s}, \bar{l}, \bar{E})$	the estimated cumulative return probability
u_s	the proportional usage of the credit limit
$\hat{u}_s$	the estimated proportional usage of the credit limit

Chapter 1

Introduction

1.1 Background

Major events such as the Asian crisis in 1997, the Russian default on short-term debt in 1998, the downfall of the hedge fund Long-Term Capital Management in 1998 and the disruption in payment systems following the World Trade Center attack in 2001, all resulted in increased management's attention to **liquidity risk**.

Liquidity risk is a secondary risk in the sense that its increase always follows one or more spikes in other financial risks. For this reason, it is often called a consequential risk. Usually a banker's main function is to provide liquidity to the economy and not to generate a liquidity crisis. It is hard to imagine that a bank can have a liquidity problem without having incurred severe losses due to market, credit or operational risk.

There are usually two ways for the bankers to survive during the liquidity crisis. One way is to sell the available liquid securities (e.g. trade assets or involve in repurchase (repo) market to create cash inflow, given that he or she is already an established repo player). The other way refers to the committed lines at other banks, among which the most important is the **revolving credit line (RCL)**. With the credit line reserved, the bank can draw any amount below the limit as the liquidity crisis attacks. Obviously the bank who offers RCL should know, how large amount would be drawn from the commitment limit (the liquidity requirement).

To estimate the liquidity requirement of the revolving credit lines, the research in recent years has indicated that liquidity cost plays an important role (please refer to [Neu et al., 2007, p. 146-169]). The higher the liquidity is reserved, the lower the liquidity risk is, but the higher the liquidity costs. In other words, too low amount of reserved lines may lead to short of liquidity in sudden crisis, but too high may lead to long-term rise of liquidity cost. Thus the commitment line is regarded as a balance between the liquidity cost and liquidity risk and the drawn amount is highly related to the market scenarios. Apart from a liquidity cost model, we can also find models which attribute the usage of revolving credit line to the demand of liquidity to fund unexpected investment opportunities [Spencer and Anthony, 1997, p.1331-1350], etc.

However, it is hard to identify what is the real purpose of the customer to draw the loan, and therefore it is risky to model the behavior of liquidity requirement based on the usage originated from certain purposes. When the market changes, the main original reason to draw from a credit line may also change, and the model loses its accuracy. In this diploma thesis, instead of considering the possible usage of credit loan that gives rise to the liquidity requirement (like investment opportunity) or the possible factor that confines the usage (like the liquidity cost), we build up our model in a statistical way. In our framework, the start and return of a credit line is described by the probability to draw a RCL and to return the drawn amount, which are referred to as draw and return probability respectively. We estimate these probabilities from the empirical data in the nearest past, and derive the expectation of usage in theory. In this way, the market scenarios as well as the different origin of the liquidity requirement are all reflected on the draw and return probabilities, which are to be updated monthly. It will provide us with a more flexible procedure for short-term control and prediction of the liquidity risk for the bank.

1.2 Revolving Credit Line

Revolving credit line is a kind of credit loan product, where the borrower has the right to draw any amount of loan within a maximal limit during a predetermined time period. The difference between the revolving credit loan and the traditional loan is in:

1. Interest start date and interest end date
   For a traditional loan, the start time of the loan is agreed upon before the loan has taken place. On the opposite, for revolving credit line, the bank (and in most cases the customers) are not aware when the loan will be drawn. What has been compromised is a commitment period. During that period, the customer has the right to draw repeatedly any amount of money at any time he or she likes.

2. Size of the credit line
   For the ordinary credit product, the amount of loan is predetermined before the loan is drawn. For a revolving credit loan, there does not exist a certain amount, but only an upper limit (the credit line). Below the specific credit line limit, the customer is allowed to draw a part or the complete amount, as he or she wants.

3. Credit regulation
   For an ordinary credit loan product, the loan is given to the customer only once. When the loan is returned, the contract is closed. For the revolving credit line, the customer has the right to draw the credit loan again even if he or she has already returned the loan from the last draw.

4. Term-out-option
   Once the maturity of the loan is reached for a ordinary credit line, the customer has to repay it. Otherwise the bank will regard the customer as default and exercise

the corresponding procedures. For the RCL, the credit line usually contains a term-out-option. That is the right to prolong the usage of credit loan for another year. This option is only attached to short-term credit lines (with a commitment period less than or equal to 1 year). For example, a customer has reserved a credit line with maturity of 1 year. The line contains a term-out-option. Suppose the customer draws part of the line in the 5-th month. In the 10-th month (two months before maturity), if the customer can not return the loan before the maturity, he or she can exercise the term-out-option, and inform the bank that he or she would like to postpone the maturity to the 24-th month.

5. Renewal rule
   The ordinary credit loan contract is a one time contract. If the current contract has finished and the customer wishes to get another loan, he or she has to run the same application process again, which is time consuming. In order to simplify the process for qualified customers, the bank can exercise a renewal rule for the RCL. If the bank regards the customer as qualified, the bank will roll over the new commitment line. If the customer is not qualified, he or she will be eliminated from the customers group. The criterium for the qualification is called the renewal-rule.

6. The fees
   For the normal credit loan, the customer pays the fee in form of interest on the amount of money borrowed over the usage period. It is constant and predetermined. For RCL, the customer has two fees to cover. The first fee is the commitment (obligation) fee. This fee is independent of the usage of the credit loan. The customer has to pay the obligation fee as long as he or she has the RCL contract signed with the bank. The level of obligation fee is related to the upper limit of the RCL. The higher the limit is, the more the customer has to pay for the obligation fee. The second fee is the interest (usage) fee. The interest rate is usually a daily or monthly based interest rate. The bank starts charging the usage fee when a certain amount of loan is drawn by the customer. The drawn loan amount multiplied by the daily interest rate is called the usage fee for the loan for each day. If the usage exceeds 1 month, 3 months or more, the interest rate is to be changed with respect to the length of the usage.

There are significant advantages of the revolving credit line for both bankers and customers in comparison to ordinary loan. For the bankers, the application procedure of RCL is easier than that of short term loans, which saves management costs. For customers, RCL has a series of advantages that cannot be matched by traditional credit loan. As there is no money borrowed, the customer pays a small obligation fee. When the demand of liquidity rises, the draw of the loan is quick, efficient and the interest is counted flexibly per day.

1.3 Mathematical Modelling of a RCL

We suppose the bank has granted a number of credit lines to n customers. The customer i has n_i credit lines reserved. The j-th credit line of customer i has an upper limit of l_t^{ij} (given in EUR) at time t. Then the total credit limit is

$$\mathrm{F}_t = \sum_{i=1}^{n} \sum_{j=1}^{n_i} l_t^{ij}.$$

Please note that the limit of a single credit line l_t^{ij} is not always constant but maybe changed to 0 if the credit line is not renewed by the bank at maturity. Thus at every time, the total limit F_t is also changing. The easiest way for the bank to cover the liquidity requirement of the credit lines is to assign an amount of F_t aside at time t. However according to empirical experience, not all the customers will fully draw their credit lines simultaneously. If we assume that the proportional usage of the j-th credit line of i-th customer at time t is U_t^{ij}, the total loan drawn at time t

$$\mathcal{F}_t = \sum_{i=1}^{n} \sum_{j=1}^{n_i} l^{ij} U_t^{ij}$$

is below the total credit limit

$$\mathcal{F}_t \leq \mathrm{F}_t, \quad \forall t \in \{1, 2, ..., T\}.$$

This means that the bank can use $\mathrm{F}_t - \mathcal{F}_t$ for other usage (such as investment), which will reduce its liquidity risk and increase profit. Our goal is to investigate the behavior of $\mathcal{F}_t$ both in theory and through simulation. We first simulate the $\mathcal{F}_t$ using Monte Carlo methods to get a brief overview of its characteristics and afterwards develop a theory to explain the simulation results.

To determine the liquidity requirement $\mathcal{F}_t$, with reference to [Duffy et al., 2005, p. 353-369], the following questions are to be answered:

1. **What is the credit rating of the customers in the current month?**
   The credit rating is regarded as a Markov process. The updated credit rating is the basic information we need to study the liquidity requirement.

2. **Whether the customer starts a credit line in the current month?**
   If a credit line is not yet in use, we need to decide whether the customer starts the credit line in the current month. If he or she or she draws the loan, the drawn amount contributes to the total liquidity requirement of the credit line group. The draw decision is, according to our assumption, dependent on the credit rating of the customer. Since the credit rating reflects the financial condition of the customer, our assumption indicates in fact that the decision is dependent on the financial condition of the customer.

3. **Whether a customer decides to return a credit line in the current month?**
   If a credit line is in use, we need to know whether the loan will be returned in the current month. If it is returned, the total liquidity requirement will decrease. The return decision of the customer, according to our assumption, depends on his current credit rating, the time when the loan starts and the time rest till the maturity. The reason why the time also plays an important role in the return decision is that the

longer the credit line is drawn, the heavier the interest is. So the customer should take the time to maturity into consideration while making the return decision.

4. **How large amount of loan is drawn?**
   Once the customer has decided to take the loan, next question is to determine the amount of the loan to be drawn. The amount of money is determined by what is the money used for. The purpose varies widely from the customer to customer and from time to time. For simplification, we assume the usage of an r-rated customer is equal to the average usage of all the r-rated customers for each credit rating $r = 1, ..., 8$. Thus we assumed that the usage is dependent only on the rating of the customer and is a constant for every credit rating.

5. **Whether the customer exercises the term-out-option?**
   As we have discussed, the RCL includes a term-out-option if its maturity is less than or equal to 1 year. Normally the customer should not exercise the option, unless his financial condition turns bad and he or she is not capable to return the loan before the maturity. The term-out-rule provides us with the information about conditions under which the customer will exercise the option. It is an empirical rule and differs among the banks.

6. **Whether the bank renews the credit line?**
   Similar to the term-out-rule, the bank has a renewal rule to decide whether to grant the credit line again to the customer. This rule is also empirical and differs among the banks.

1.4 Monte Carlo Simulation

Monte Carlo simulation is a widely used class of computational algorithms for simulating the behavior of various mathematical and physical systems. The method is distinguished from other simulation methods (such as molecular dynamics) by being stochastic (non-deterministic) as opposed to deterministic algorithms. The Monte Carlo algorithm is often used to find the distribution of stochastic variables whose distribution cannot be easily determined.

We use the Monte Carlo simulation to simulate our liquidity requirement $\mathcal{F}_t$. The generic procedure is as following:

1. **Build up the relationship between target and input stochastic variable:**
   In our case the input stochastic variables are the draw decision variable, the return decision variable, the updated credit rating, etc. The target stochastic variable is $\mathcal{F}_t$.

2. **Define the distribution of the input stochastic variables:**
   Each input stochastic variable has its own distribution. Detailed models are built correspondingly, in which the parameters are estimated by means of empirical analysis on the available data.

3. **Generate the necessary input variables:**
   For example we need to generate an indicator variable X for the decision to draw or not to draw from the credit line. We generate firstly a uniformly distributed random variable U. If we assume a draw probability of 0.8 and $U \leq 0.8$, then the indicator variable X is 1, otherwise 0.

4. **Calculate the observation of target stochastic variable:**
   With the observations of the input variables, we calculate the observation of target variable through the relationship between the input and target stochastic variables we built up in step 1.

5. **Replication:**
   Repeating the simulation from step 3 for K times, we get K observations of the target variable. Further we conduct a value at risk (VaR) analysis on these observations.

1.5 Value at Risk Analysis

Historically seen, the value at risk (VaR) is a measure of how the market value of an asset or of a portfolio is likely to decrease over a certain time period.

Definition 1.1 (Value at Risk (VaR))**.** Suppose $\mathcal{F}_t$ is the liquidity requirement of the credit line group at time t (a random variable) and $\alpha \in [0,1]$. $\mathcal{F}_t^{\alpha}$ is the Value at Risk at the α quantile if

$$P\left\{\mathcal{F}_t \leq \mathcal{F}_t^{\alpha}\right\} = \alpha.$$

The VaR value $\mathcal{F}_t^{\alpha}$ can be interpreted as following. In $100(1-\alpha)\%$ of times, a liquidity reserve $\mathcal{F}_t^{\alpha}$ is not sufficient. The value of α is often chosen as 0.9995, 0.9975 and 0.9950. For $\alpha = 0.9995$ and $\mathcal{F}_t^{\alpha} = 20Mio$, we have that in 5/10000 times the liquidity reserve of $20Mio$ is not sufficient.

Suppose we have 5000 observations of $\mathcal{F}_t$ at time t: $\mathcal{F}_t^1, \mathcal{F}_t^2, ..., \mathcal{F}_t^{5000}$, we estimate the VaR $\mathcal{F}_t^{0.9995}$ as following. We list the 5000 observations in ascending order: $\mathcal{F}_t^{(1)} \leq \mathcal{F}_t^{(2)} \leq ... \leq \mathcal{F}_t^{(5000)}$. Then $\mathcal{F}_t^{(4998)}$ $(5000 \cdot 0.9995 = 4998)$ is the empirical VaR at $\alpha = 0.9995$. For more details, please refer to [Gupton and Finger, 1996, p. 3-36].

1.6 Overview of the thesis

The thesis is organized as follows. In chapter 2, we introduce the credit rating transition model to describe the updated credit rating of the customer in the current month. In chapter 3, we introduce the credit start model, the credit return model, the term-out-rule to describe the decision of the customer as well as the renewal rule to describe the decision of the bank. In chapter 4, we present the results of the simulation.

In chapters 5, 6 and 7, we theoretically derive the explicit distribution and expectation of the liquidity requirement. In chapter 5, we consider the simplest case, the distribution of the usage for a single credit line in single period. In chapter 6, we extend the distribution to the multiple period case. The last chapter contains the derivation of the distribution of several credit lines in multiple period case. In the final chapter we compare the simulation approach to the theoretical one and draw conclusions.

Chapter 2

Modelling Credit Rating Transitions

In this chapter, we describe how a yearly transition matrix can be used to determine monthly transition matrices. We need this step because in most cases only the annual transition matrix is available. We follow the approach taken by Sidelnikova and Krenin [2001].

Afterwards we present an approach to model the transition of the credit rating of a single customer. Instead of separate estimation of the default, downgrade and upgrade probability of the credit rating, these credit rating transitions will be modelled jointly. For this we will use an asset return model, in which two successive steps are involved:

1. Specify a model for the underlying process (in our case, a model for the asset return value), the distribution of which is known to us (here we use the standard normal distribution).

2. Specify the relationship between the underlying process and the credit rating transition process (here between the Markov process and the asset return value).

These two steps allow us to simulate the credit rating process through the underlying process and the mapping which relates the underlying process to the credit rating process.

Finally, the credit rating transition model is extended from a single customer to a group of customers assuming that the correlation between the customers depends on the industry sectors they belong to.

2.1 Regularization of the Transition Matrix

Definition 2.1 (Quasi Monthly Transition Matrix)**.** Suppose the credit rating system has 8 levels: $S = \{1, 2, ..., 8\}$. [1] Denote the yearly transition matrix by $\mathbf{P}_y \in \mathbb{R}^{8\times 8}$. Then $\mathbf{P}_m \in \mathbb{R}^{8\times 8}$ is called quasi monthly transition matrix, if

$$\mathbf{P}_m^{12} = \mathbf{P}_y.$$

[1] $1 \triangleq AAA$, $2 \triangleq AA$, $3 \triangleq A$,$4 \triangleq BBB$, $5 \triangleq BB$, $6 \triangleq B$, $7 \triangleq CCC$, $8 \triangleq D$.

The existence of $\mathbf{P}_m$ is discussed in Sidelnikova and Krenin [2001] as well as the fact that $\mathbf{P}_m$ is not unique. But it is important to point out that $\mathbf{P}_m$ is not necessarily a transition matrix (with all the elements non-negative and the sum of each row equal to 1). We first denote all transition matrices by $\mathcal{A}$:

$$\mathcal{A} = \left\{ \mathbf{A} | \mathbf{A} \in \mathbb{R}^{8\times 8} \;\; , \;\; \sum_{r=1}^{8} \mathbf{A}_{sr} = 1 \;\; \forall s \in S \;\; , \;\; \mathbf{A}_{sr} \geq 0 \;\;\; \forall s, r \in S \right\}.$$

We need to select a matrix $\mathbf{P}$ from $\mathcal{A}$ such that it has minimal distance from $\mathbf{P}_m$ compared to all other transition matrices. $\mathbf{P}$ is called the optimal monthly transition matrix. So an optimization problem - Quasi optimization of the root Matrix (QOM) - is involved.

Problem 1 (QOM, Sidelnikova and Krenin (2001), p.29)**.** *Given* $\mathbf{P}_y \in \mathbb{R}^{8\times 8}$ *and one of its root matrix* $\mathbf{P}_m$ *such that* $\mathbf{P}_m^{12} = \mathbf{P}_y$*, find a matrix* $\mathbf{P}$ *which fulfills the following two conditions:*

(1) $\mathbf{P} \in \mathcal{A}$

(2) $\|\mathbf{P} - \mathbf{P}_m\| = \min_{\mathbf{A} \in \mathcal{A}} \|\mathbf{A} - \mathbf{P}_m\|$

where $\|\cdot\|$ *is the Euclidian norm in the space of* 8×8 *matrices.*

Considering the fact that the set of transition matrices $\mathcal{A}$ can be represented as a Cartesian product of 8 identical 8-dimensional simplices:

$$\mathbb{A}_S = \left\{ \mathbf{a} | \mathbf{a} \in \mathbb{R}^8, \sum_{s=1}^{8} a_s = 1 \;,\; a_s \geq 0 \;\; \forall s \in S \right\},$$

the QOM problem can thus be solved on a row-by-row basis and Problem 1 can be reformulated to the distance minimization problem for root matrix (DMPM).

Problem 2 (DMPM, Sidelnikova and Krenin (2001), p.31)**.** *Given* $\mathbf{p}_m \in \mathbb{R}^8$*, (a row of the root matrix* $\mathbf{P}_m$*), find a vector* $\mathbf{p}$ *such that*

$$\mathbf{p} = \arg\min_{\mathbf{a} \in \mathbb{A}_S} \|\mathbf{p}_m - \mathbf{a}\| = \arg\min_{\mathbf{a} \in \mathbb{A}_S} \sum_{s=1}^{8} \left((p_m)_s - a_s \right)^2.$$

To solve the Problem 2, we need the following Lemma.

Lemma 2.1. *Suppose* $\mathbf{a}, \mathbf{p}_m \in \mathbb{R}^8$*,* S_8 *is the group of permutation of 8 elements and* π *is a permutation,* $\pi \in S_8$*. Denote* $\mathbf{a}' = \pi(\mathbf{a})$*,* $\mathbf{p}'_m = \pi(\mathbf{p}_m)$*. If* $\mathbf{p}' \in \mathbb{A}_S$ *fulfills* $\|\mathbf{p}' - \mathbf{p}'_m\| = \min_{\mathbf{a}' \in \mathbb{A}_S} \|\mathbf{a}' - \mathbf{p}'_m\|$*, then we have that* $\mathbf{p} = \pi^{-1}(\mathbf{p}')$ *fulfills* $\|\mathbf{p} - \mathbf{p}_m\| = \min_{\mathbf{a} \in \mathbb{A}_S} \|\mathbf{a} - \mathbf{p}_m\|$*.*

Proof. It is obvious that we have

$$\|\mathbf{p}' - \mathbf{p}'_m\| = \|\mathbf{p} - \mathbf{p}_m\|$$

and

$$\min_{\mathbf{a}' \in \mathbb{A}_S} \|\mathbf{a}' - \mathbf{p}'_m\| = \min_{\mathbf{a} \in \mathbb{A}_S} \|\mathbf{a} - \mathbf{p}_m\| .$$

Together with the condition

$$\|\mathbf{p}' - \mathbf{p}'_m\| = \min_{\mathbf{a}' \in \mathbb{A}_S} \|\mathbf{a}' - \mathbf{p}'_m\| ,$$

we get

$$\|\mathbf{p} - \mathbf{p}_m\| = \min_{\mathbf{a} \in \mathbb{A}_S} \|\mathbf{a} - \mathbf{p}_m\|$$

in which $\mathbf{p} = \pi^{-1}(\mathbf{p}')$. □

Furthermore, Sidelnikova and Krenin [2001] proposed the following procedure to solve the DMPM.

1. Find a permutation $\pi \in S_8$ that arranges the elements of $\mathbf{p}_m$ in descending order

$$\mathbf{p}'_m = \pi(\mathbf{p}_m) .$$

2. Find the optimal solution $\mathbf{p}'$ that is nearest to the $\mathbf{p}'_m$

$$\|\mathbf{p}' - \mathbf{p}'_m\| = \min_{\mathbf{a}' \in \mathbb{A}_S} \|\mathbf{a}' - \mathbf{p}'_m\| .$$

3. Arrange the $\mathbf{p}'$ using the inverse permutation $\pi^{-1} \in S_8$ and get the final optimal solution

$$\mathbf{p} = \pi^{-1}(\mathbf{p}') .$$

Without loss of generality, Problem 2 can be confined to the following Problem 3.

Problem 3 (Sidelnikova and Krenin (2001), p.30)**.** *Given* $\mathbf{p}_m \in \mathbb{R}^8$ *with* $(p_m)_1 \geq (p_m)_2 \geq \ldots \geq (p_m)_8$*, find a vector* $\mathbf{p}$ *which fulfills*

$$\mathbf{p} = \arg\min_{\mathbf{a} \in \mathbb{A}_S} \sum_{s=1}^{8} (a_s - (p_m)_s)^2 .$$

Lemma 2.2. *Assume* $\mathbf{p}$ *is the optimum solution to Problem 3. Then there exists an index* $d \geq 1$*, such that* $p_s > 0$ *for* $1 \leq s \leq d$ *and* $p_s = 0$ *for* $s > d$.

Proof. Suppose we have a candidate solution $\mathbf{p}$ with $p_s = 0$ and $p_{s+\delta} > 0$ for some $s \in 1, ..., 7$ and $1 \leq \delta \leq 8 - s$. We should have

$$\mathbf{p} = \left(p_1, ..., \underset{\substack{\| \\ 0}}{p_s}, ..., \underset{\substack{\vee \\ 0}}{p_{s+\delta}}, ..., p_8 \right).$$

Consider a new vector $\mathbf{p}^*$ where p_s and $p_{s+\delta}$ in $\mathbf{p}$ are exchanged, namely

$$\mathbf{p}^* = \left(p_1, ..., \underset{\substack{\vee \\ 0}}{p_{s+\delta}}, ..., \underset{\substack{\| \\ 0}}{p_s}, ..., p_8 \right).$$

Then we have

$$\|\mathbf{p} - \mathbf{p}_m\| = (p_1 - (p_m)_1)^2 + \ldots + (p_s - (p_m)_s)^2 + \ldots + (p_{s+\delta} - (p_m)_{s+\delta})^2 + \ldots + (p_n - (p_m)_8)^2$$

$$\|\mathbf{p}^* - \mathbf{p}_m\| = (p_1 - (p_m)_1)^2 + \ldots + (p_{s+\delta} - (p_m)_s)^2 + \ldots + (p_s - (p_m)_{s+\delta})^2 + \ldots + (p_n - (p_m)_8)^2$$

and thus

$$\|\mathbf{p}^* - \mathbf{p}_m\| - \|\mathbf{p} - \mathbf{p}_m\| = 2p_{s+\delta}\left((p_m)_{s+\delta} - (p_m)_s\right) < 0,$$

since $(p_m)_1 \geq (p_m)_2 \geq ... \geq (p_m)_8$ and $p_{s+\delta} \geq 0$ by assumption. Thus $\mathbf{p}$ is more away from $\mathbf{p}_m$ than the constructed new vector $\mathbf{p}^*$, and is not the optimum solution, which conflicts with our assumption. □

Lemma 2.3. *Assume* $\mathbf{p}$ *is the optimal solution to Problem 3, then all the positive elements* p_s *of* $\mathbf{p}$ *should have the form* $p_s = (p_m)_s + \lambda$ *for* $s = 1, 2, ..., d$ *where* $\lambda \in \mathbb{R}$ *is a constant.*

Proof. According to Lemma 2.2, we know that $\mathbf{p}$ should have the following form

$$\mathbf{p} = (p_1, ..., p_d, 0, ..., 0).$$

1. If $d = 1$, $\mathbf{p} = (p_1, 0, ..., 0)$, we just set $\lambda = p_1 - (p_m)_1$.
2. If $d \geq 2$, then for each $1 \leq \tau < \delta \leq d$, we can write $\mathbf{p}$ in the following form

$$\mathbf{p} = (p_1, ..., p_\tau, ..., p_\delta, ..., p_d, 0, ..., 0).$$

Now we construct a new vector by perturbing p_τ, p_δ with $\varepsilon \in \mathbb{R}$

$$\mathbf{p}^\varepsilon = (p_1, ..., p_\tau - \varepsilon, ..., p_\delta + \varepsilon, ..., p_d, 0, ..., 0).$$

Since we assumed that $\mathbf{p}$ is the optimal solution, this means

$$\|\mathbf{p} - \mathbf{p}_m\| - \|\mathbf{p}^\varepsilon - \mathbf{p}_m\| \leq 0 \quad \text{for} \quad \forall\varepsilon \in \mathbb{R}.$$

Consequently we have

$$-2\varepsilon^2 + 2\varepsilon\left(p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta\right) \leq 0 \quad \text{for} \quad \varepsilon \in (-\infty, +\infty). \tag{2.1}$$

Further calculation shows

$$\varepsilon \in \begin{cases} (-\infty, +\infty) & \text{if} \quad (p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta) = 0 \\ (-\infty, 0] \cup [(p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta), +\infty) & \text{if} \quad (p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta) > 0 \\ (-\infty, (p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta)] \cup [0, +\infty) & \text{if} \quad (p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta) < 0 \end{cases}$$

Thus only when $(p_\tau - (p_m)_\tau - p_\delta + (p_m)_\delta) = 0$, we have that the inequality 2.1 is satisfied for $\varepsilon \in (-\infty, +\infty)$. So we set

$$\lambda := p_\tau - (p_m)_\tau = p_\delta - (p_m)_\delta, \;\; \forall 1 \leq \tau < \delta \leq d,$$

and

$$p_s = (p_m)_s + \lambda, \;\; \forall 1 \leq s \leq d.$$

□

Together with Lemma 2.2 and Lemma 2.3, we find that the optimal solution should have following form

$$\mathbf{p} = \left(\overbrace{(p_m)_1 + \lambda, ..., (p_m)_d + \lambda,}^{dPosComponents} \quad \overbrace{0, 0, ..., 0, 0}^{(8-d)ZeroComponents} \right).$$

The Problem 3 can further be reduced to an optimization problem over λ, d.

Problem 4 (Sidelnikova and Krenin (2001), p.32)**.** *Assume* $\mathbf{p}_m \in \mathbb{R}^n$ *with* $(p_m)_1 \geq (p_m)_2 \geq ... \geq (p_m)_8$. *Find* d^* *and* λ^* *such that*

$$(d^*, \lambda^*) = \underset{d \in \{1,...,8\}, \lambda \in \mathbb{R}}{\arg\min} \left(d\lambda^2 + \sum_{s=d+1}^{8} (p_m)_s^2 \right)$$

under the condition

$$d\lambda + \sum_{s=1}^{d} (p_m)_s = 1 \quad \text{and} \quad \lambda \geq -(p_m)_s \quad \forall s \in \{1, 2, ..., d\}.$$

Since we assumed that $(p_m)_1 \geq (p_m)_2 \geq ... \geq (p_m)_8$, so $\lambda \geq -(p_m)_s$ for $s \in \{1, 2, ..., d\}$ is equivalent to $\lambda \geq -(p_m)_d$. Substituting $\lambda = \frac{1}{d}\left(1 - \sum\limits_{s=1}^{d} (p_m)_s\right)$, we reduce Problem 4 to an optimization problem for d only.

Problem 5 (Sidelnikova and Krenin (2001), p.32)**.** *Assume* $\mathbf{p}_m \in \mathbb{R}^8$ *with* $(p_m)_1 \geq (p_m)_2 \geq ... \geq (p_m)_8$. *Find* d^* *such that*

$$d^* = \underset{d \in \{1,2,...,8\}}{\text{argmin}} \; f(d) := \underset{d \in \{1,2,...,8\}}{\text{argmin}} \left(\frac{1}{d} \left(1 - \sum_{s=1}^{d} (p_m)_s \right)^2 + \sum_{s=d+1}^{n} (p_m)_s^2 \right)$$

under the condition

$$S(d) := \sum_{s=1}^{d} ((p_m)_s - (p_m)_d) \leq 1.$$

Solution of Problem 5
(1) Prove that $f(d)$ is non-increasing function in d.

$$f(d) - f(d-1) = \frac{1}{d}\left(1 - \sum_{s=1}^{d}(p_m)_s\right)^2 + \sum_{s=d+1}^{8}(p_m)_s^2 - \frac{1}{d-1}\left(1 - \sum_{s=1}^{d-1}(p_m)_s\right)^2 - \sum_{s=d}^{8}(p_m)_s^2$$

Since

$$\sum_{s=1}^{d}(p_m)_s = \sum_{s=1}^{d}((p_m)_s - (p_m)_d) + d(p_m)_d = S(d) + d(p_m)_d$$

$$\sum_{s=1}^{d-1}(p_m)_s = \sum_{s=1}^{d-1}((p_m)_s - (p_m)_m) + (d-1)(p_m)_d = S(d) + (d-1)(p_m)_d$$

we have

$$f(d) - f(d-1) = -\frac{[1 - S(d)]^2}{d(d-1)} \leq 0.$$

(2) Prove that $S(d)$ is non-decreasing in d and non-negative.

$$S(d) - S(d-1) = (d-1)((p_m)_{d-1} - (p_m)_d) \geq 0$$

$$S(1) = \sum_{s=1}^{1}((p_m)_s - (p_m)_1) = 0.$$

(3) Since $f(d)$ is non-increasing, we just need to find the index d^* that satisfies $S(d^*) \leq 1$ and $S(d^*+1) > 1$ from the series of $S(d)$. Correspondingly we get

$$\lambda^* = \frac{1}{d^*}\left(1 - \sum_{s=1}^{d^*}(p_m)_s\right) = \frac{1 - S(d^*)}{d^*} - (p_m)_{d^*}.$$

Algorithm for determining a monthly transition matrix:
As a summary for this section, we list the necessary steps to find an optimal monthly transition matrix $\mathbf{P}$ given a quasi monthly transition matrix $\mathbf{P}_m$.

1. Select the first row $\mathbf{p}_m \in \mathbb{R}^8$ of the root transition matrix $\mathbf{P}_m$.
2. Permute the elements of $\mathbf{p}_m$ into descending order: $\mathbf{p}'_m = \pi(\mathbf{p}_m)$.
3. Calculate the sequence $S(d)$ based on $\mathbf{p}'_m$ using the iteration formula

$$S(d) - S(d-1) = (d-1)\left((p_m)'_{d-1} - (p_m)'_d\right) \quad \forall d = 2, ..., 8$$

   and initial value

$$S(1) = 0.$$

4. Select d^* such that:
$$S(d^*) \leq 1 \quad \text{and} \quad S(d^*+1) > 1.$$

5. Calculate λ^* using:
$$\lambda^* = \frac{1 - S(d^*)}{d^*} - (p'_m)_{d^*}$$
and set
$$(p')_s := \begin{cases} ((p_m)')_s + \lambda^*, & 1 \leq s \leq d^*, \\ 0, & d^* < s \leq 8. \end{cases}$$

6. Rearrange the $\mathbf{p}'$ to get the final result:
$$\mathbf{p} = \pi^{-1}(\mathbf{p}').$$

Example 2.1.
Suppose we have the yearly transition matrix $\mathbf{P}_y$ and try to calculate the optimal monthly transition matrix $\mathbf{P}$.

$$\mathbf{P}_y = \begin{pmatrix} 0.5308 & 0.3380 & 0.1102 & 0.0142 & 0.0047 & 0.0012 & 0.0004 & 0.0007 \\ 0.0326 & 0.5228 & 0.3486 & 0.0666 & 0.0193 & 0.0062 & 0.0003 & 0.0036 \\ 0.0055 & 0.0834 & 0.6103 & 0.2045 & 0.0630 & 0.0232 & 0.0019 & 0.0082 \\ 0.0037 & 0.0255 & 0.2050 & 0.4451 & 0.1983 & 0.0791 & 0.0073 & 0.0360 \\ 0.0015 & 0.0062 & 0.0394 & 0.1229 & 0.4564 & 0.2226 & 0.0210 & 0.1300 \\ 0.0006 & 0.0026 & 0.0129 & 0.0367 & 0.1688 & 0.3993 & 0.0438 & 0.3355 \\ 0.0002 & 0.0009 & 0.0067 & 0.0265 & 0.0690 & 0.0971 & 0.0862 & 0.7134 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

First of all, we calculate the quasi monthly transition matrix $\mathbf{P}_m = \mathbf{P}_y^{1/12}$.

$$\mathbf{P}_m = \begin{pmatrix} 0.94713 & 0.05165 & 0.00114 & -0.00005 & 0.00013 & -0.00002 & 0.00012 & -0.00007 \\ 0.00485 & 0.94227 & 0.05091 & 0.00103 & 0.00070 & 0.00005 & -0.00006 & 0.00023 \\ 0.00039 & 0.01179 & 0.95091 & 0.03244 & 0.00347 & 0.00095 & 0.00002 & 0.00004 \\ 0.00041 & 0.00175 & 0.03204 & 0.92340 & 0.03491 & 0.00623 & 0.00051 & 0.00074 \\ 0.00015 & 0.00039 & 0.00204 & 0.02166 & 0.92436 & 0.04270 & 0.00236 & 0.00633 \\ 0.00005 & 0.00020 & 0.00079 & 0.00246 & 0.03166 & 0.91584 & 0.01582 & 0.03321 \\ 0.00000 & -0.00001 & -0.00015 & 0.00556 & 0.01541 & 0.03079 & 0.80886 & 0.13954 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 \end{pmatrix}$$

(1) Select the first row of the root matrix

$$\mathbf{p}_m = (0.94713, 0.05165, 0.00114, -0.00005, 0.00013, -0.00002, 0.00012, -0.00007).$$

(2) Permute the elements of $\mathbf{p}_m$ into descending order

$$\mathbf{p}'_m = \pi(\mathbf{p}_m) = (0.94713, 0.05165, 0.00114, 0.00013, 0.00012, -0.00002, -0.00005, -0.00007).$$

(3) Calculate the sequence of $S(m)$

$$S = (0, 0.89548, 0.99650, 0.99953, 0.99956, 1.00029, 1.00045, 1.00055).$$

(4) From the sequence of $S(m)$, we get $m^* = 5$.

(5) Calculate $\lambda^* = \frac{1-S(5)}{5} - (p_m)_5 = -3.3 \times 10^{-5}$ and

$$\mathbf{p}' = (0.94710, 0.05161, 0.00111, 0.00010, 0.00009, 0, 0, 0).$$

(6) Inverse the permutation of the $\mathbf{p}'$ and get the final result

$$\mathbf{p} = (0.94710, 0.05161, 0.00111, 0, 0.00010, 0, 0.00009, 0).$$

(7) Conduct for all rows. We obtain the optimal monthly transition matrix.

$$\mathbf{P} = \begin{pmatrix} 0.94710 & 0.05161 & 0.00111 & 0.00000 & 0.00010 & 0.00000 & 0.00009 & 0.00000 \\ 0.00485 & 0.94227 & 0.05091 & 0.00103 & 0.00070 & 0.00005 & 0.00000 & 0.00023 \\ 0.00039 & 0.01179 & 0.95091 & 0.03244 & 0.00347 & 0.00095 & 0.00002 & 0.00004 \\ 0.00041 & 0.00175 & 0.03204 & 0.92340 & 0.03491 & 0.00623 & 0.00051 & 0.00074 \\ 0.00015 & 0.00039 & 0.00204 & 0.02166 & 0.92436 & 0.04270 & 0.00236 & 0.00633 \\ 0.00005 & 0.00020 & 0.00079 & 0.00246 & 0.03166 & 0.91584 & 0.01582 & 0.03321 \\ 0.00000 & 0.00000 & 0.00000 & 0.00552 & 0.01537 & 0.03076 & 0.80883 & 0.13951 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 \end{pmatrix}$$

Notice that $\mathbf{P}^{12}$ given below is relatively close to $\mathbf{P}_y$.

$$\mathbf{P}^{12} = \begin{pmatrix} 0.53057 & 0.33770 & 0.10995 & 0.01444 & 0.00458 & 0.00124 & 0.00066 & 0.00099 \\ 0.03259 & 0.52276 & 0.34858 & 0.06665 & 0.01937 & 0.00611 & 0.00121 & 0.00317 \\ 0.00550 & 0.08340 & 0.61028 & 0.20450 & 0.06301 & 0.02319 & 0.00199 & 0.00816 \\ 0.00370 & 0.02550 & 0.20499 & 0.44508 & 0.19829 & 0.07910 & 0.00732 & 0.03599 \\ 0.00150 & 0.00620 & 0.03942 & 0.12290 & 0.45640 & 0.22260 & 0.02100 & 0.12999 \\ 0.00060 & 0.00260 & 0.01295 & 0.03670 & 0.16879 & 0.39928 & 0.04380 & 0.33548 \\ 0.00020 & 0.00096 & 0.00719 & 0.02650 & 0.06889 & 0.09698 & 0.08616 & 0.71314 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 \end{pmatrix}$$

While $\mathbf{P}_m$ has several negative entries, all elements of $\mathbf{P}$ are non - negative.

2.2 Modelling Credit Rating Transition for a Single Customer

Suppose the monthly credit rating transition matrix is known, we continue with modelling the transition rating for a single customer using an asset return model. The asset return model consists of three parts:

1. Formulate the credit rating process $(R_t)_{t\in\{1,...,T\}}$, whose distribution is known.

2. Propose an underlying process $\left(\tilde{R}_t\right)_{t\in\{1,...,T\}}$, whose distribution is also known.

3. Build up a relationship between $(R_t)_{t\in\{1,...,T\}}$ and $\left(\tilde{R}_t\right)_{t\in\{1,...,T\}}$.

For the background information, please refer to Greg and Gupton [1997, p. 57-80]. We first give out the first two parts through two assumptions.

Assumption 2.1 (Credit Rating Process)**.** *The credit rating transition process* $(R_t)_{t\in\{1,...,T\}}$ *is Markov process with transition matrix* $\mathbf{P} \in \mathbb{R}^{8\times 8}$, *i.e.* $\mathbf{P}(s, s') := P(R_t = s'|R_{t-1} = s)$, *state space* S *and initial state* s_0.

Assumption 2.2 (Underlying Process)**.** *The credit rating is determined by the asset return* $\left(\tilde{R}_t\right)_{t\in\{1,...,T\}}$ *of the company, which we assume as normally distributed with constant mean* μ *and constant volatility* σ: $\tilde{R}_t \sim N(\mu, \sigma^2)$ *for every* $t \in \mathbb{N}_0$.

In the next step we need to construct a mapping system between the credit rating process and asset return process. Since a company's asset return determines its ability to return the loan, we may regard that there exists a special level such that if the company's asset return falls below this level, he or she will not be able to meet his payment obligations and his credit rating will be regarded as default. For example, if the asset return is less than -0.8: $\tilde{R}_t \in (-\infty, -0.8]$, we can assume the credit rating as default: $R_t = 8$. Similarly, it is reasonable to imagine that there exists a series of interval of asset return which can be mapped into other level of credit rating.

Theorem 2.1. *Given a normally distributed random variable:* $\tilde{R}_t \sim N(\mu, \sigma^2)$ *and a Markov chain* $(R_t)_{t\in\{1,...,T\}}$ *with transition matrix* $\mathbf{P}$ *and* $R_{t-1} = s_0$, *we have*

$$P(R_t = s|R_{t-1} = s_0) = P\left(\tilde{R}_t \in (\sigma z_{s_0,s+1}, \sigma z_{s_0,s})\right), \qquad s = 1$$

$$P(R_t = s|R_{t-1} = s_0) = P\left(\tilde{R}_t \in (\sigma z_{s_0,s+1}, \sigma z_{s_0,s}]\right), \qquad s = 2, 3, ..., 8$$

in which

$$z_{s_0,s} := \begin{cases} \Phi^{-1}\left(\sum\limits_{\varsigma=0}^{8-s} \mathbf{P}(s_0, 8-\varsigma)\right), & s = 1, 2, ..., 8, \\ -\infty, & s = 9. \end{cases}$$

Proof. For $s = 1$, we have

$$\begin{aligned} &P\left(\tilde{R}_t \in (\sigma z_{s_0,2}, \sigma z_{s_0,1})\right) = [\Phi(\sigma z_{s_0,1}/\sigma) + \mu] - [\Phi(\sigma z_{s_0,2}/\sigma) + \mu] \\ &= \Phi(z_{s_0,1}) - \Phi(z_{s_0,2}) = \sum_{\varsigma=0}^{7} \mathbf{P}(s_0, 8-\varsigma) - \sum_{\varsigma=0}^{6} \mathbf{P}(s_0, 8-\varsigma) = \mathbf{P}(s_0, 1) = P(R_t = 1|R_{t-1} = s_0). \end{aligned}$$

Similarly, for $s = 2, 3, ..., 8$ we have

$$P\left(\tilde{R}_t \in (\sigma z_{s_0,s+1}, \sigma z_{s_0,s}]\right) = \left[\Phi\left(\sigma z_{s_0,s}/\sigma\right) + \mu\right] - \left[\Phi\left(\sigma z_{s_0,s+1}/\sigma\right) + \mu\right]$$
$$= \Phi\left(z_{s_0,s}\right) - \Phi\left(z_{s_0,s+1}\right) = \sum_{\varsigma=0}^{8-s} \mathbf{P}(s_0, 8-\varsigma) - \sum_{\varsigma=0}^{7-s} \mathbf{P}(s_0, 8-\varsigma) = \mathbf{P}(s_0, s) = P\left(R_t = s | R_{t-1} = s_0\right).$$

□

From the derivation, we see that the mean value does not have effect on the final result. So we could choose the mean value 0: $\tilde{R}_t \sim N\left(0, \sigma^2\right)$. In next step we prove that the volatility also does not have an impact on the result.

Lemma 2.4. *If $X \sim N(\eta, \kappa^2)$ and $Y = aX + b$ with $a, b \in \mathbb{R}$, then $Y \sim N\left(a\eta + b, a^2\kappa^2\right)$. Especially if $X \sim N(0,1)$ and $Y = aX$, then $Y = aX \sim N\left(0, a^2\right)$.*

For the proof, please refer to **?**, p. 89. Using Lemma 5.1, we can present $\tilde{R}_t \sim N(0, \sigma^2)$ as $\tilde{R}_t = \sigma X$ in which $X \sim N(0,1)$. Thus we have

$$P\left(R_t = s | R_{t-1} = s_0\right) = P\left(\sigma X \in (\sigma z_{s_0,s+1}, \sigma z_{s_0,s}]\right) = P\left(X \in (z_{s_0,s+1}, z_{s_0,s}]\right).$$

So the volatility has no effect on the transition matrix. We can replace $\tilde{R}_t \sim N\left(\mu, \sigma^2\right)$ by $X_t \sim N\left(0,1\right)$ as our underlying process in Assumption 2.2 and simplify the mapping between the underlying process and credit rating process as shown in the following theorem.

Theorem 2.2. *Given $X_t \sim N(0,1)$ and a homogeneous Markov chain $(R_t)_{t\in\mathbb{N}_0}$ with transition matrix $\mathbf{P}$ and $R_{t-1} = s_0$, then we have*

$$P\left(R_t = s | R_{t-1} = s_0\right) = P\left(X_t \in (z_{s_0,s+1}, z_{s_0,s})\right) = P\left(X_t \in (z_{s_0,2}, +\infty)\right), \qquad s = 1$$

$$P\left(R_t = s | R_{t-1} = s_0\right) = P\left(X_t \in (z_{s_0,s+1}, z_{s_0,s}]\right), \qquad s = 2, 3, ..., 8$$

in which

$$z_{s_0,s} := \begin{cases} \Phi^{-1}\left(\sum_{\varsigma=0}^{8-s} \mathbf{P}(s_0, 8-\varsigma)\right), & s = 1, 2, ..., 8, \\ -\infty, & s = 9. \end{cases}$$

If we suppose the initial credit rating is BB, the asset return model will divide $(+\infty, -\infty)$ into 8 sections as shown in the Figure 2.1. If the observation of X_t lies in region B for example, the updated credit rating will be B.

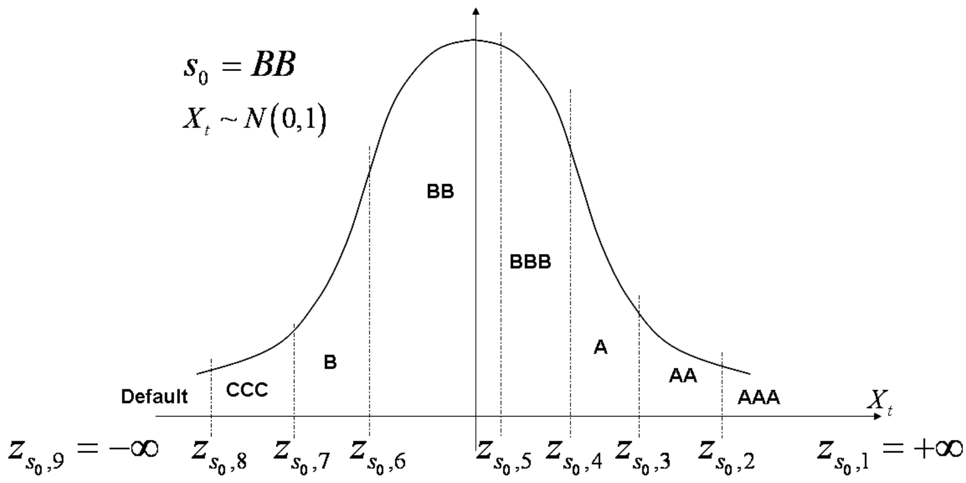

Figure 2.1: Asset Return Model

2.3 Credit Rating Transition of Customer Group

In the simulation of credit migration for a single company, we generate a standard normally distributed stochastic variable $X_t \sim N(0,1)$. If there are n companies at time t, and if the asset return of all the companies has independent identical distribution (standard normal), we need to generate a stochastic vector

$$\mathbf{X}_t \sim N_n(\mathbf{0}, I_n).$$

But in reality the companies are not independent. The companies in the same industry should have higher correlation than those in different industries. So the ratings of companies in the same industry are more likely to shift down or up together according to economic surroundings. So it is reasonable to assume that the customers have a correlation matrix of Σ instead of I_n. So we need to generate

$$\tilde{\mathbf{X}}_t \sim N_n(\mathbf{0}, \Sigma).$$

Suppose there are 2 industries. In the first one, there are two companies, while in the second industry there are 3 companies. Assume the correlation in the same industry is c, the correlation between industry 1 and 2 is c_{12}, $c_{12} = c_{21}$. Then correlation matrix has the following form:

$$\Sigma = \begin{pmatrix} 1 & c & c_{12} & c_{12} & c_{12} \\ c & 1 & c_{12} & c_{12} & c_{12} \\ c_{12} & c_{12} & 1 & c & c \\ c_{12} & c_{12} & c & 1 & c \\ c_{12} & c_{12} & c & c & 1 \end{pmatrix}.$$

In practice the computer can easily generate a multivariate normally distributed variable $\mathbf{X}_t \sim N_n(0, I_n)$ by generating n realizations of a standard normally distributed variable. In order to generate $\tilde{\mathbf{X}}_t \sim N_n(0, \Sigma)$, we need the following lemma.

Lemma 2.5. *If* $\mathbf{X}_t \sim N(0, I_n)$ *and* $\Sigma = BB^T$*, then* $\tilde{\mathbf{X}}_t = B\mathbf{X}_t \sim N_n(0, \Sigma)$*.*

Proof. (1) If $\mathbf{X}$ is multivariate normally distributed, then $\mathbf{X} = BY$ is also multivariate normally distributed. Obviously the mean is 0.

(2) Since Σ is symmetric real matrix, we can decompose the matrix as

$$\Sigma = ADA^T = AEE^T A^T = BB^T$$

in which D is diagonal matrix and $B := AE$.

(3) The covariance matrix of $\tilde{\mathbf{X}}_t$ is

$$Cov\left(\tilde{\mathbf{X}}_t\right) = Cov\left(B\mathbf{X}_t\right) = Cov\left(AE\mathbf{X}_t\right) = \left(AE\right) Cov\left(\mathbf{X}_t\right)\left(E^T A^T\right) = ADA^T = \Sigma.$$

Thus we have

$$\tilde{\mathbf{X}}_t \sim N_n\left(\mathbf{0}, \Sigma\right).$$

□

2.4 Summary for Credit Rating Transition Model

As a summary for this chapter, we list the procedures which is followed in the Credit Rating Transition Model in Monte Carlo simulation.

Input:
Annual Transition Matrix $\mathbf{P}_y$
Initial Credit Rating for customer group $(s_0^1, s_0^2, ..., s_0^n)$
Industry correlation Matrix $\Sigma \in \mathbb{R}^{n\times n}$

Block 1
It needs to be run once to get the data prepared for the following procedure.

1. Regularization of root transition matrix and get $\mathbf{P}$
2. Decompose the correlation matrix $\Sigma = BB^T$

Block 2
For each time t we run Block 2 repeatedly.

1. Generate n independent normally distributed variables

$$X_t^i \sim N(0,1)\ , \quad i = 1, 2, ...n \quad \Rightarrow \quad \mathbf{X}_t = \left(X_t^1, \ldots, X_t^n\right) \sim N_n(\mathbf{0}, I_n).$$

2. Adjust the independent scenarios into correlated scenarios

$$\tilde{\mathbf{X}}_t = B\mathbf{X}_t \sim N_n\left(\mathbf{0}, \Sigma\right).$$

3. For the customer currently under simulation, select the scenario $\tilde{X}_t^i$ from $\tilde{\mathbf{X}}_t$, credit rating s_0^i from $(s_0^1, s_0^2, ..., s_0^n)$ and corresponding row from the monthly transition matrix $\mathbf{P}$:

$$\left(\mathbf{P}(s_0^i, 1), \mathbf{P}(s_0^i, 2), ..., \mathbf{P}(s_0^i, 8)\right)$$

.

4. Calculate the threshold asset return

$$z_{s_0,s} := \begin{cases} \Phi^{-1}\left(\sum_{\varsigma=0}^{8-s} \mathbf{P}(s_0, 8-\varsigma)\right), & s = 1, 2, ..., 8, \\ -\infty, & s = 9. \end{cases}$$

5. Map the generated scenario to credit rating

$$\tilde{X}_t^i \in (z_{s+1}, z_s] \Rightarrow R_t^i = s.$$

6. Go back to step 3, simulate the credit rating for the next customer.

Chapter 3

Credit Start, Return Model and Empirical Rules

In this chapter, we suppose that we already have the updated credit rating, and present the credit loan start, credit loan return model as well as the empirical renewal rule and term-out-rule.

3.1 Modelling the draw decision of customers who have not started using the credit line

Definition 3.1 (Draw Probability)**.** The draw probability p_s is the probability that the customer starts the credit loan in the current month. It only depends on the credit rating.

Let A_s be the stochastic decision variable for a customer with credit rating s, i.e.

$$P(A_s = 1) = p_s \;\text{ and }\; P(A_s = 0) = 1 - p_s \;\text{ for }\; s \in S$$

in which $A_s = 1$ represents that the customer draws the loan and $A_s = 0$ represents not draw.

A realization of A_s can be generated as follows:

- Generate $U \sim U(0, 1)$.
- If $U \leq p_s$, then set $A_s := 1$, otherwise $A_s = 0$.

Note the draw decision probability is independent of the time t. It only depends on the credit rating s.

To estimate the values of p_s, we use the historical bank data. Let $N_{s,start}$ be the number of customers with rating s at time -1 who started their credit line at time 0. Let N_s be the number of customers with rating s at time -1 who have not used their credit line at time -1. Then p_s is estimated as:

$$\hat{p_s} := \frac{N_{s,start}}{N_s}.$$

3.2 Modelling the return decision of customers who have started their credit lines

Definition 3.2 (Return Probability)**.** The return probability is the probability that the customer returns the credit loan within current month.

The decision to return the loan is modelled only for the customers who have already drawn the credit loan. We denote the month when he or she drew the loan as t_{start}. In our model, we assume that the probability for the customer to return the credit loan at current time t depends on three factors:

1. The updated credit rating of the customer at time t: $R_t \in S$.

   R_t is updated through the credit rating transition model.

2. The time (in months) from $t_{start} - 1$ till the maturity date, which we denote by E.

   Given the starting time of a loan $t_{start} \in \{1, ..., T\}$ and the maturity of the credit line $m \in \{12, 24, 36, 48\}$, we can determine E. There are two situations, namely the single-maturity situation and the multi-maturity situation.

- In the single-maturity situation, the time period of our simulation T contains only one maturity m i.e. $T = m$. The customer can start the credit loan only within the first maturity period: $t_{start} \in \{1, ..., m\}$. Under this situation, E is calculated as:

  $$E := m - t_{start} + 1.$$

  If $E = 1$, the customer starts the credit line in the last month before the maturity. If $E = 2$, the last second month and so on. Obviously we have $E \in \{1, 2, ..., m\}$. An example is given in Figure 3.1. Suppose we simulate a time period of 12 months and the credit line has a maturity of 12 months. The customer draws the loan 15 days after the beginning of the 4-th month (marked by the black round point in the figure). Thus we have $t_{start} = 4$ and $E = 12 - 4 + 1 = 9$. It means that the customer starts the credit loan in the 9 months before the maturity.

- Generally the time period of our simulation T contains more than one maturity m, i.e. $T = km, k \in \mathbb{N}$. Suppose the credit line is renewed at maturity, the customer can start the credit loan during the next maturity period. Under this situation, E should be calculated as:

  $$E := m - (t_{start} \bmod m) + 1,$$

  in which

  $$km \bmod m := m, \quad (km + \gamma) \bmod m := \gamma \ \text{ for } \ \gamma = 1, 2, ..., (m-1)\,, \quad k \in \mathbb{N}.$$

  Obviously we have again $E \in \{1, 2, ..., m\}$. An example is given in Figure 3.2. The time period that we simulate for a return is 24 months and the maturity of the credit line is 12 months. Given the credit line is renewed on the first maturity date

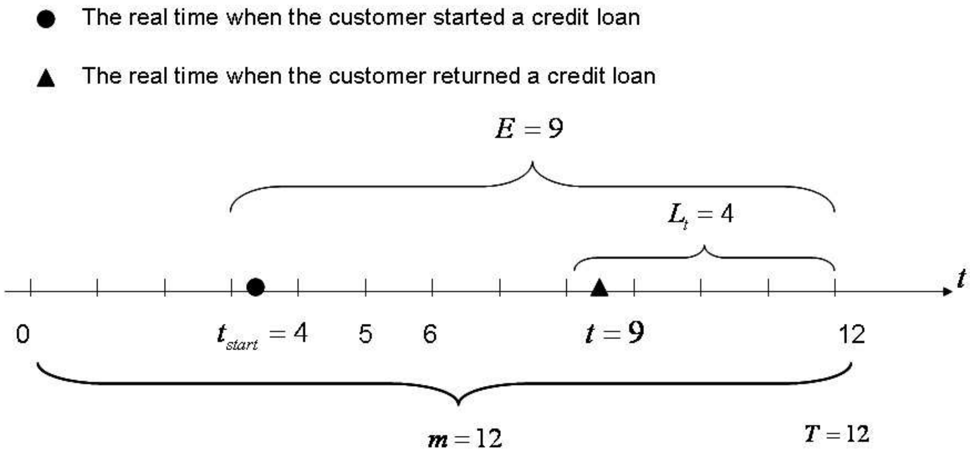

Figure 3.1: Return probability for the line started in single-maturity period

$t = 12$, we have altogether 2 maturities within our simulation. Suppose the customer draws the credit loan in the 3-th day of the 16-th month, we have $t_{start} = 16$, $E = 12 - 16 \bmod 12 + 1 = 9$. It means that the customer starts the credit loan 9-th months before the next maturity date. Since the return probability is used only for the credit line that has already started, so E is always known and can be regarded as a constant.

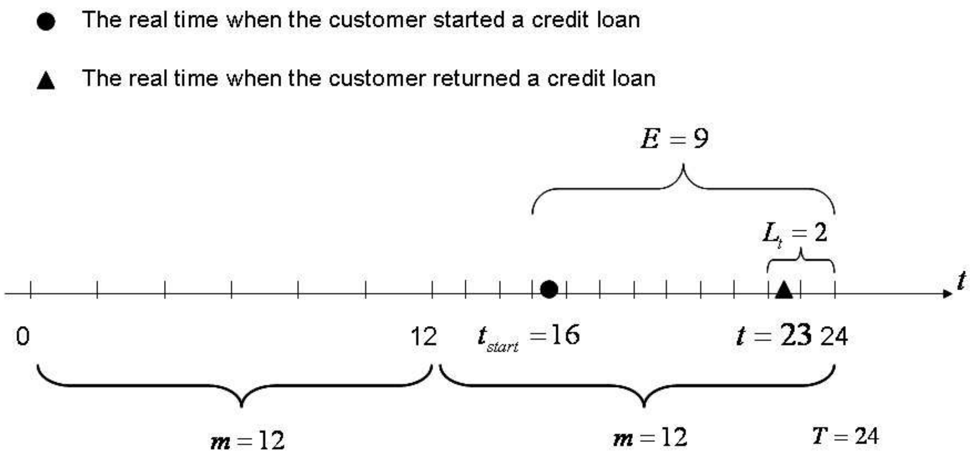

Figure 3.2: Return probability for the line started in the multi-maturity period

3. The time (in months) from $t-1$ until the maturity date, which we denote as L_t. Similar to E, L_t can be calculated as:

$$L_t := m - t + 1,$$

for single-maturity situation. Please consider the example in Figure 3.1. We already know that the customer started the credit line at $t_{start} = 4$ and want to model the probability that the customer returns the loan in each of the following months.

Suppose the customer is in the 20-th day of the 9-th month (marked by the black triangle), we have $t = 9$. And thus $L_9 = 12 - 9 + 1 = 4$. It means that 4 months are left until the maturity date. In the multi-maturity period, we have again a similar formula:

$$L_t := m - (t \bmod m) + 1.$$

Remark 3.1. When $t = km$, we get $L_t = 1$. It means that one month is left for the customer to return the loan. In order to include the situation that the loan is not return till the maturity date, we define $L_t := 0$ in this case. So generally we have $L_t \in \{0, 1, ..., E\}$.

Given E, $L_t = l$ and $R_t = s$, we denote the return probability as

$$q(s, l, E), \quad s \in S, \quad l \in \{0, 1, ..., E\} \quad \text{and} \quad E \in \{1, 2, ..., m\}.$$

Under the condition that $l \neq 0$, the return probability represents the probability that the customer started the credit line in the E-th month before the maturity date and returned the loan in the l-th month before the maturity date with a credit rating of s at the time he or she returns the loan. Under the condition that $l = 0$, the return probability represents the probability that the customer started the credit line in the E-th month before the maturity date and the loan is never returned (default) with the credit rating of s at the maturity date.

Precisely speaking we simulate based on a monthly time unit, so E, l are on a monthly base and s can take values of each credit rating level. If we want to simulate over 48 months, we need to estimate the probability of $q(s, l, E)$ for $s \in \{1, 2, ..., 8\}$, $E = 1, 2, ...48$ and $l = 0, 1, 2, ..., E$. Thus $8 \cdot (2 + 3 + ... + 49) = 9,408$ different probabilities have to be estimated. Our historical data contains approximately 20,000 credit lines. Thus the problem of estimating 9,408 probabilities is not realistic. For this reason, we define a series of credit rating buckets and time buckets.

As introduced before, we have 8 levels of credit ratings. We ignore $s = 8$ because the default company is not allowed to take credit loans and thus there is no need to estimate the return probability for it. The rating levels from 1 to 7 are grouped into 3 credit rating buckets:

$$\bar{s} = c(s) := \begin{cases} 1 & s \in \{1, 2, 3\}, \\ 2 & s \in \{4, 5, 6\}, \\ 3 & s \in \{7\}. \end{cases} \tag{3.1}$$

Using this mapping, we can change the credit rating in our sample into the new value $\bar{R}_t = c(R_t)$. The classification of credit rating may be considered as reasonable, because customers within the same rating class are similar in using the loan and in their capability to return the loan. For example, for A-rated companies, it is easy to get money directly through the money market by issuing bonds or certificates. So their frequency of using the loan and the usage amount may appear less than than the B/C-rated companies.

The maturity of credit lines in our simulation can take four different values, $m \in \{12, 24, 36, 48\}$. For credit line with $m = 12$, we define 4 time buckets. Each bucket has a length of 3 months. For credit line with $m = 24$, we define 8 time buckets. Each bucket has a length of 3 months. For credit line with $m = 36$, we divide the 36 months into two parts. The first part is from $t = 1$ till $t = 12$ and we define 2 time buckets. Each bucket has a length of 6 months. The second part is from $t = 13$ till $t = 36$ and we define 8 buckets. Each bucket has a length of 3 months. For credit line with $m = 48$, we divide the 48 months into two parts. The first part is from $t = 1$ till $t = 24$ and we define 4 time buckets. Each bucket has a length of 6 months. The second part is from $t = 25$ till $t = 48$ and we define 8 buckets. Each bucket has a length of 3 months. For a graphical representation, please see Figure 3.3.

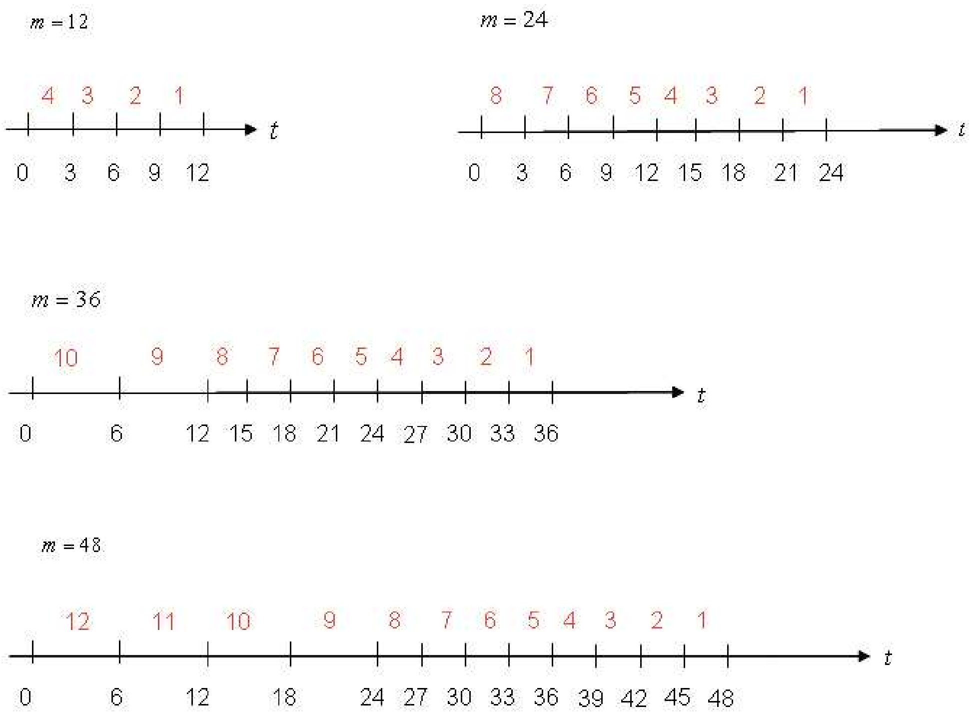

Figure 3.3: Time bucket (month index is below the line while the time bucket index is above the line)

Using the definition of the time bucket, we can map the E and L_t from monthly time unit to the unit in time bucket.

$$\bar{E} := g_m(E) \quad \text{and} \quad \bar{L}_t := g_m(L_t) \quad m = 12, 24, 36, 48.$$

The mapping function is defined in Figure 3.4.

g_{12}	t
1	$t \in \{10,11,12\}$
2	$t \in \{7,8,9\}$
3	$t \in \{4,5,6\}$
4	$t \in \{1,2,3\}$

g_{24}	t
1	$t \in \{22,23,24\}$
2	$t \in \{19,20,21\}$
3	$t \in \{16,17,18\}$
4	$t \in \{13,14,15\}$
5	$t \in \{10,11,12\}$
6	$t \in \{7,8,9\}$
7	$t \in \{4,5,6\}$
8	$t \in \{1,2,3\}$

g_{36}	t
1	$t \in \{31,\ldots,36\}$
2	$t \in \{25,\ldots,30\}$
3	$t \in \{22,23,24\}$
4	$t \in \{19,20,21\}$
5	$t \in \{16,17,18\}$
6	$t \in \{13,14,15\}$
7	$t \in \{10,11,12\}$
8	$t \in \{7,8,9\}$
9	$t \in \{4,5,6\}$
10	$t \in \{1,2,3\}$

g_{48}	t
1	$t \in \{43,\ldots,48\}$
2	$t \in \{37,\ldots,42\}$
3	$t \in \{31,\ldots,36\}$
4	$t \in \{25,\ldots,30\}$
5	$t \in \{22,23,24\}$
6	$t \in \{19,20,21\}$
7	$t \in \{16,17,18\}$
8	$t \in \{13,14,15\}$
9	$t \in \{10,11,12\}$
10	$t \in \{7,8,9\}$
11	$t \in \{4,5,6\}$
12	$t \in \{1,2,3\}$

Figure 3.4: Mapping between time bucket and time (in months)

Such a definition of the time buckets may be considered as reasonable, since most customers tend to keep the loan till the maturity date. So in the time period close to maturity date we observe more frequent return of the credit lines. That is why this time period should consist of smaller time buckets in comparison to the time period not close to maturity date.

Assumption 3.1. *Given two credit lines with maturities $m_1, m_2 \in \{12, 24, 36, 34\}$ (not necessarily equal), the two return probabilities during the usage of two credit loans $q(s, l, E)$ and $q(s', l', E')$ coincide if:*

1. $c(s) = c(s')$
2. $g_{m_1}(E) = g_{m_2}(E')$
3. $g_{m_1}(l) = g_{m_2}(l')$

Definition 3.3 (Grouped Return Probability)**.** The return probability after mapping is called grouped return probability.

$$q(\bar{s}, \bar{l}, \bar{E}) \text{ for } \bar{s} \in \{1, 2, 3\}, \bar{E} \in \{1, 2, ..., 12\}, \bar{l} \in \{0, 1, ..., \bar{E}\}.$$

The number of probabilities needed to be estimated is reduced from $9,408$ down to $3 \cdot (2 + 3 + ... + 13) = 270$. The estimation of 270 probabilities is feasible using our historical data.

Example 3.1.
For each class of credit rating, we get from empirical data a table of simplified return probabilities as shown in Table 3.1 (here only listed till the 4^{th} time bucket for $\bar{s} = 1$).

To simulate the return decision, we need to convert the return probability into the cumulative return probability. It describes the probability for the customer to return the loan by the end of usage period.

Table 3.1: Grouped Return probability for $\bar{s} = 1$

	$\bar{E} = 1$	$\bar{E} = 2$	$\bar{E} = 3$	$\bar{E} = 4$
$\bar{l} = 0$	$q(1,0,1)$	$q(1,0,2)$	$q(1,0,3)$	$q(1,0,4)$
$\bar{l} = 1$	$q(1,1,1)$	$q(1,1,2)$	$q(1,1,3)$	$q(1,1,4)$
$\bar{l} = 2$		$q(1,2,2)$	$q(1,2,3)$	$q(1,2,4)$
$\bar{l} = 3$			$q(1,3,3)$	$q(1,3,4)$
$\bar{l} = 4$				$q(1,4,4)$

Definition 3.4 (Cumulative Return Probability)**.** A cumulative return probability $Q(\bar{s}, \bar{l}, \bar{E})$ is defined as

$$Q(\bar{s}, \bar{l}, \bar{E}) := \sum_{i=\bar{l}}^{\bar{E}} q(\bar{s}, i, \bar{E}) \text{ for } \bar{s} \in \{1,2,3\}, \bar{E} \in \{1,2,...,12\} \text{ and } \bar{l} \in \left\{0,1,...,\bar{E}\right\}.$$

The cumulative return probability represents the probability that the customer with credit rating $\bar{s}$ started the credit line in $\bar{E}$-th month before the maturity date and returned the loan before $\bar{l}$-th month prior to the maturity date.

Example 3.2.
For customers in each credit rating class, we get a table of cumulative return probabilities. Table 3.2 represents cumulative return probability for a customer with $\bar{s} = 1$ (here only listed till the 3^{rd} time bucket).

Table 3.2: Cumulative return probability with $\bar{s} = 1$

	$\bar{E} = 1$	$\bar{E} = 2$	$\bar{E} = 3$
$\bar{l} = 1$	$Q(1,1,1) = \sum_{i=1}^{1} q(1,i,1)$	$Q(1,1,2) = \sum_{i=1}^{2} q(1,i,2)$	$Q(1,1,3) = \sum_{i=1}^{3} q(1,i,3)$
$\bar{l} = 2$		$Q(1,2,2) = \sum_{i=2}^{2} q(1,i,2)$	$Q(1,2,3) = \sum_{i=2}^{3} q(1,i,3)$
$\bar{l} = 3$			$Q(1,3,3) = \sum_{i=3}^{3} q(1,i,3)$

To estimate the cumulative return probability and the grouped return probability, we do the following procedure.

1. For the j-th credit line of i-th customer, we count the number of returns of the credit loans that have started $\bar{E}$ time buckets before maturity and returned $\bar{l}$ time buckets before maturity and had a credit rating in the rating bucket $\bar{s}$ at the time the loan was returned: $N_{i,j,\bar{s},\bar{l},\bar{E}}$.

2. Sum up $N_{i,j,\bar{s},\bar{l},\bar{E}}$ over all the credit lines, so that we get the total number of returns of the credit loans, that have been started $\bar{E}$ time buckets before maturity and returned $\bar{L}$ time buckets before maturity and had a credit rating in the rating bucket $\bar{s}$ at the time the loan was returned: $N_{\bar{s},\bar{l},\bar{E}} = \sum_{i=1}^{n}\sum_{j=1}^{n_i} N_{i,j,\bar{s},\bar{l},\bar{E}}$.

3. Sum up $N_{i,j,\bar{s},\bar{l},\bar{E}}$ over $\bar{l}$, so that we get the number of returns of the credit loans, that have been started $\bar{E}$ time buckets before maturity and had a credit rating in the rating bucket $\bar{s}$ at the time the loan was returned: $N_{\bar{s},\bar{E}} = \sum_{\bar{l}=0}^{\bar{E}} N_{\bar{s},\bar{l},\bar{E}}$.

4. The grouped return probability is estimated by: $\hat{q}\left(\bar{s},\bar{l},\bar{E}\right) = \frac{N_{\bar{s},\bar{l},\bar{E}}}{N_{\bar{s},\bar{E}}}$.

5. The cumulative return probability is estimated through: $\hat{Q}\left(\bar{s},\bar{l},\bar{E}\right) = \sum_{i=\bar{l}}^{\bar{E}} \hat{q}\left(\bar{s},i,\bar{E}\right)$.

Assumption 3.2. *Suppose the credit line has a maturity m. The customer's decision to return the credit loan at time t, which started E months before maturity (E is known) is described by a random variable B_t^E(return decision variable). The distribution of the return decision variable is determined by the cumulative return probability.*

$$P\left(B_t^E = 1\right) = Q(\bar{R}_t, \bar{L}_t, \bar{E}) \;\; \text{and} \;\; P\left(B_t^E = 0\right) = 1 - Q(\bar{R}_t, \bar{L}_t, \bar{E}),$$

in which $\bar{R}_t = c(R_t)$, $\bar{L}_t = g_m(L_t)$ and $\bar{E} = g_m(E)$.

$B_t^E = 1$ represents that the customer returns the loan while $B_t^E = 0$ does not return the loan.

Remark 3.2. The return decision probability is dependent on $\bar{R}_t, \bar{L}_t$ with given E. Since both of the two factors are dependent on t, our distribution of return decision variable is also dependent on t. That is why we denote it as B_t^E.

3.3 Modelling the Amount of Credit Loan drawn by the Customers

In this section we answer the question how large amount of credit loan the customer will take out once he or she decided to start the loan. The usage amount in this chapter is a fraction. It represents the proportion of the committed limit drawn by the customer.

Assumption 3.3. *The proportion of the committed limit to be used at time t is dependent on the credit rating of the customer and is denoted as U_t.*

$$P\left\{U_t = u_s\right\} = 1 \quad \text{if} \quad R_t = s \quad \text{for} \quad s \in \{1, 2, ..., 8\},$$

in which $u_s \in (0, 1)$ is a constant for every $s \in \{1, 2, ..., 8\}$, i.e. the customer uses a fixed proportion of the committed limit corresponding to his or her current credit rating s.

To estimate u_s, we use the empirical mean.

1. For the j-th credit line of i-th customer, we count the number of credit loans started when the customer had a credit rating s: $N_{i,j,s}$, $s \in S$.

2. For the j-th credit line of i-th customer, we sum up the percentage usages of the customer when he or she had a rating s: $\mu_{i,j,s}$, $s \in S$.

3. Sum up $N_{i,j,s}$ over all the credit lines, such that we get the total number of starts of credit line by the customer with credit rating s: $N_s = \sum_{i=1}^{n} \sum_{j=1}^{n_i} N_{i,j,s}$, $s \in S$.

4. Sum up $\mu_{i,j,s}$ over all the credit lines, such that we get the total amount of loan used by customers with a credit rating of s: $\mu_s = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \mu_{i,j,s}$, $s \in S$.

5. u_s is estimated by $\hat{u}_s = \frac{\mu_s}{N_s}$, $s \in S$.

An example is given in Figure 3.5. Suppose for the j-th credit line of i-th customer, we find in the historical data 3 times of the starts of credit loan with a credit rating of $s = 4$ at the time the loans were started. Then we have $N_{i,j,4} = 3$. If the usages are $0.6, 0.33, 0.7$ respectively, we have $\mu_{i,j,4} = 0.6 + 0.33 + 0.7 = 1.63$. Please note that $\mu_{i,j,s}$ can be larger than 1 although each draw u_s had a maximal value 1.

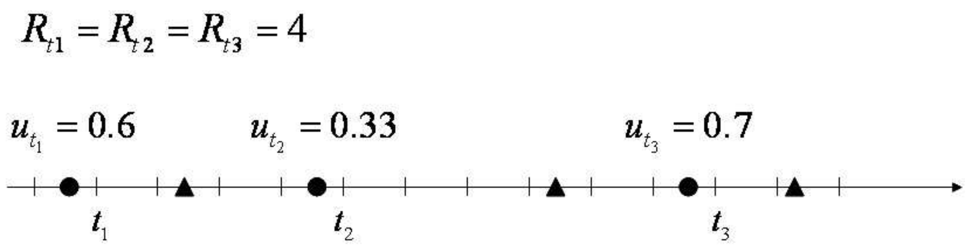

Figure 3.5: Estimation of usage amount

3.4 Modelling the Term-out-rule

For credit lines with maturity equal to 12 months, there exists a term-out-option. It is the right of the customer to prolong the maturity for another year. The question is when will the customer exercise this term-out-option. We assume an empirical expert rule that describes the term-out decision by the customer.

Assumption 3.4 (Term-out-rule)**.** *We assume that the customer will exercise the term-out-option under either of the two conditions:*

1. *The customer's credit rating decreased down to B ($s = 6$) or below.*

2. *The customer's credit rating has been downgraded by 4 levels within the past 12 months.*

Definition 3.5 (Term out variable)**.** The term out decision of the customer can be described using a random variable (term out variable) Te_t. Given the updated credit rating R_t, according to Assumption 3.4, Te_t should have the following form

$$Te_t := \begin{cases} 1, & \{\{R_t \in \{6,7\}\} \cup \{R_{t-11} - R_t \geq 4\}\} \cap \{Te_\tau \neq 1, \tau = 1,2,...,t-1\}, \\ 0, & else. \end{cases}$$

in which $Te_t = 1$ represents that at time t the customer exercises the term-out-option, while $Te_t = 0$ means not exercise.

Once the term-out-option is exercised, the maturity will increase by 12 months:

$$m \to m + 12.$$

3.5 Modelling the Renewal Rule

When the maturity date is reached and the credit loan is already given back to the bank, the bank has to decide whether to grant the line to the customer again or to eliminate the customer out of the credit portfolio.

Assumption 3.5 (Renewal Rule)**.** *The bank will renew the credit line to the customer if the credit loan is totally repaid by the end of last maturity date and the customer has a credit rating higher than B ($s = 6$) at that time.*

Definition 3.6. The renewal decision of a bank can be described by a random variable (renewal variable) Re_t with $t = km\ for\ k \in \mathbb{N}$. Given the updated credit rating of the i-th customer R_t and the amount of j-th credit line in use U_t at the maturity $t = km, k \in \mathbb{N}$, Re_{km} should have following form according to Assumption 3.5.

$$Re_{km} := \begin{cases} 1, & \{R_{km} \in \{1,2,3,4,5\}\} \cap \{U_{km} = 0\} \\ 0, & else \end{cases},$$

where $k \in \mathbb{N}$. Here $Re_{km} = 1$ represents that the bank renews the credit line while $Re_{km} = 0$ means no renewal is allowed.

Once a renewal is denied by the bank, the customer will never get another loan from the revolving credit line, just as if he or she had a credit rating of default. In this case, we set the credit rating of the customer manually as default in our simulation:

$$R_t := 8 \ \text{ for } \ t = km+1, km+2, \ldots \ \text{ if } \ Re_{km} = 0,$$

and the credit limit of the credit line will be set zero:

$$l_t^{i,j} := 0 \ \text{ for } \ t = km+1, km+2, \ldots \ \text{ if } \ Re_{km} = 0.$$

Please note that the limit of other credit lines of the i-th customer are not changed.

3.6 Modelling the Expiration Rule

At the maturity date, if the customer has not returned the credit loan, we have the expiration rule to describe the situation.

Assumption 3.6 (Expiration Rule)**.** *If the customer has not returned the credit loan at the maturity date, the customer will never return the loan and we regard the credit line as expired.*

Definition 3.7. The expiration situation can be described by a random variable called expiration variable Ex_t with $t = km \; for \; k \in \mathbb{N}$. Given the usage proportion of the credit loan U_t at the maturity date km, the expiration variable should have the following form.

$$Ex_{km} := \begin{cases} 1, & U_{km} > 0 \\ 0, & U_{km} = 0 \end{cases},$$

where $k \in \mathbb{N}$. Here $Ex_{km} = 1$ represents that the credit line is expired, while $Ex_{km} = 0$ not expired.

To reflect the expiration rule, we manually set the credit rating of the customer as default,

$$R_t := 8 \;\text{ for }\; t = km + 1, km + 2, \ldots .\text{if }\; Ex_{km} = 1.$$

3.7 Summary for Monte Carlo Simulation

In this section we make a short summary for the simulation part of the thesis. We first make a symbol list for all the variables that we have defined for the simulation. It is shown in Table 3.3. For simplification, we just denote the j-th credit line of i-th customer as credit line (i, j).

Table 3.3: Symbol list of variables used in the model

Symbol	Explanation
n	the number of customers
n_i	the number of the credit lines of customer i
l_t^{ij}	the credit limit of credit line (i, j) at time t (given in EUR)
m^{ij}	the maturity of credit line (i, j), $m^{ij} \in M = \{12, 24, 36, 48\} \; \forall i, j$
R_t^i	the credit rating of customer i at time t, $R_t^i \in S = \{1, 2, ..., 8\}$ in which $1 \triangleq AAA$, $2 \triangleq AA$, $3 \triangleq A$,$4 \triangleq BBB$, $5 \triangleq BB$, $6 \triangleq B$, $7 \triangleq CCC$, $8 \triangleq D$
$L_t^{i,j}$	the time period between the return of loan and maturity for credit line (i, j)
$E^{i,j}$	the time period between the start of loan and maturity for credit line (i, j)
$\bar{R}_t^i$	R_t^i in unit of credit rating bucket
$\bar{L}_t^{i,j}$	$L_t^{i,j}$ in unit of time bucket
$\bar{E}^{i,j}$	$E^{i,j}$ in unit of time bucket
A_s	the draw decision variable for customer with credit rating s
$B_t^{E^{i,j}}$	the return decision variable for credit line (i, j) at time t with the start time of credit loan $E^{i,j}$ months before the maturity
$Te_t^{i,j}$	the term out decision variable for credit line (i, j) at time t
$Ex_t^{i,j}$	the expiration decision variable for credit line (i, j) at time t
$Re_t^{i,j}$	the renewal decision variable for credit line (i, j) at time t

To start the simulation, we need to estimate or input the following parameters.

Table 3.4: Symbol list of input and estimated parameters used in the model

Symbol	Explanation
$\mathbf{P}_y$	the yearly transition matrix $\mathbf{P}_y \in \mathbb{R}^{8\times 8}$
$\mathbf{P}$	the optimal monthly transition matrix
Σ	the correlation matrix of the customers
$\hat{p}_s$	the estimated draw probability for customer with credit rating s
$\hat{Q}(\bar{s}, \bar{l}, \bar{E})$	the estimated cumulative return probability for customer who has a credit rating bucket $\bar{s}$, started the credit loan $\bar{E}$ time buckets before the maturity and returned the loan $\bar{L}$ time buckets before the maturity
$\hat{u}_s$	the estimated proportional usage of the credit limit for customer who has a credit rating s

In the end, the model gives out the following results of simulation.

Table 3.5: Symbol list of outputs

Symbol	Explanation
$\hat{U}_t^{ij}$	the simulated proportional usage for credit line (i, j) at time t
$\hat{\mathrm{F}}_t$	the total committed credit limit at time t, $\hat{\mathrm{F}}_t = \sum_i^n \sum_j^{n_i} l_t^{ij}$
$\hat{\mathcal{F}}_t$	the total amount of credit loan drawn at time t, $\hat{\mathcal{F}}_t = \sum_i^n \sum_j^{n_i} l_t^{ij} U_t^{ij}$
$\hat{\mathrm{f}}_t$	the proportional usage of total committed limit at time t, $\hat{f}_t = \hat{\mathcal{F}}_t / \hat{\mathrm{F}}_t$

Remark 3.3. For the output results, we need to note that:

1. The total committed limit of the credit portfolio is not always constant but decreasing because part of the credit lines are not renewed during our simulation time period, which leads to the reduction of $\hat{\mathrm{F}}_t$. So we need to recalculate the $\hat{\mathrm{F}}_t$ at each time.

2. We replicate simulation 1000 times at each time t and get 1000 values of $\hat{\mathcal{F}}_t$, namely $\hat{\mathcal{F}}_t^1$, $\hat{\mathcal{F}}_t^2$, ... , $\hat{\mathcal{F}}_t^{1000}$. In next step we select the VaR of $\hat{\mathcal{F}}_t$ at 0.75, 0.90 and 0.95 quantile: $\hat{\mathcal{F}}_t^{(750)}$, $\hat{\mathcal{F}}_t^{(900)}$ and $\hat{\mathcal{F}}_t^{(950)}$. (Please refer to our definition of VaR in Chapter 1). The same procedure is run for $\hat{\mathrm{f}}_t$. The plot of $\hat{\mathcal{F}}_t^{\alpha}$ and $\hat{\mathrm{f}}_t^{\alpha}$, $\alpha \in \{750, 900, 950\}$ will be given in the next chapter.

Now we give out a flow chart of the complete simulation process and explain how different models interact and cooperate with each other. Please see Figure 3.6. Consider the credit line (i, j) at time t. Assume we know that the credit rating of the customer in previous month is R_{t-1}^i. The work flow is as follows.

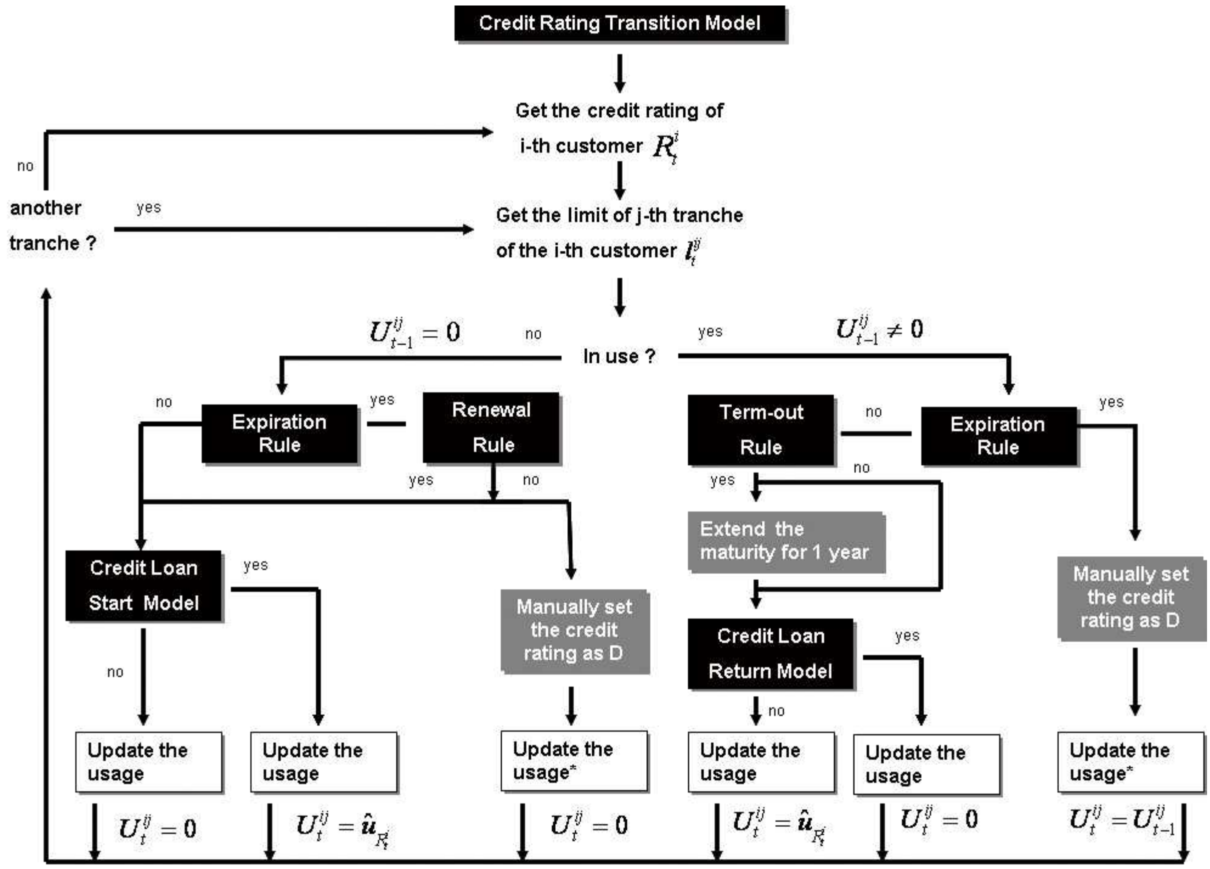

Figure 3.6: Flow chart of the simulation process

1. Run the Credit Rating Transition Model to derive the updated credit rating for all customers and pick out the credit rating: R_t^i.

2. If $U_{t-1}^{ij} = 0$, run the expiration rule.

   (1) If $\exists k \in \mathbb{N}$ such that $t = km$, set $Ex_t^{i,j} = 0$ and the line is not expired, run the renewal rule.

   - If $R_t^i \in \{1, 2, ..., 5\}$, set $Re_t^{i,j} = 1$, the bank renews the credit line. We jump to step (2) and run the credit start model.
   - If $R_t^i \in \{6, 7, 8\}$, set $Re_t^{i,j} = 0$, the bank refuses to renew the credit line. We set $R_t^i = 8$ for $t = km + 1, ...$ and $l_t^{i,j} = 0$. Update the usage $U_t^{i,j} = 0$.

   (2) If $t \neq km$, $\forall k \in \mathbb{N}$, set $Ex_t^{i,j} = 0$, the credit loan not is expired. Run the credit loan start model and generate draw decision variable A_s.

   - If $A_s = 1$, the customer draws the credit loan, update the usage $U_t^{ij} = \hat{u}_{\bar{R}_t^i}$.
   - If $A_s = 0$, the customer does not draw the credit line and the usage $U_t^{ij} = 0$.

3. If $U_{t-1}^{ij} > 0$, run the expiration rule.

(1) If $\exists k \in \mathbb{N}$ such that $t = km$, we set $Ex_t^{i,j} = 1$. (the credit loan expired).

- We regard the customer as default: $R_t^i = 8$ for $t = km + 1, ...T$. Update the usage $U_t^{i,j} = U_{t-1}^{i,j}$.

(2) If $t \neq km$, $\forall k \in \mathbb{N}$, set $Ex_t^{i,j} = 0$, since the credit line has not expired. Run the term-out rule.

- If $\{\{R_t^i \in \{6, 7\}\} \cup \{R_{t-11}^i - R_t^i \geq 4\}\} \cap \{Te_\tau^{i,j} \neq 1, \tau = 1, 2, ..., t-1\}$, set $Te_t^{i,j} = 1$, the credit line is termed out: $m = m + 12$. Go to step (3) and run the credit return model.
- Else, set $Te_t^{i,j} = 0$, the customer does not term out the line. Go to step (3) and run the credit return model.

(3) Generate return decision variable $B_t^{\bar{E}^{i,j}}$.

- If $B_t^{\bar{E}^{i,j}} = 1$, the customer returns the credit loan. Update the usage $U_t^{ij} = 0$.
- If $B_t^{\bar{E}^{i,j}} = 0$, the customer does not return the loan. Update the usage $U_t^{ij} = \hat{u}_{R_t^i}$.

Chapter 4

Monte Carlo Simulation and Results

4.1 Credit Portfolio Parameters

Customer Information
We select randomly 76 customers from the historical data base together with their initial credit ratings R_t^i. The distribution of customers with respect to credit rating is:

Number of Customers	Credit Rating
23	AAA
29	AA
14	A
9	BBB
1	BB
0	B
0	CCC
0	D

Table 4.1: Selected customers grouped by their initial credit ratings R_t^i

The customers are located in seven industries, namely Consuming Goods, Energy, Real Estate, Materials, Oil Services, Financial Service, Technology. The distribution of customers with respect to the industries is as shown in Table 4.2.

Customer Number	Industry	Industry ID
10	Consuming Good	1
6	Energy	2
5	Real Estate	3
11	Materials	4
3	Oil Services	5
15	Financial	6
26	Technology	7

Table 4.2: Selected customers grouped by their industries

We use the correlation among the exchange traded funds (ETFs) of different industries (Table 4.3) to represent the correlation among the industries as well as among the customers in different industries. These correlations are listed in Table 4.3.

Correlation Between U.S. Sector ETFs - Daily Returns								
		Cons. Good	Energy	Real Estate	Materials	Oil Service	Financial	Technology
	$x \backslash y$	1	2	3	4	5	6	7
Cons. Good	1	1.00						
Energy	2	0.38	1.00					
Real Estate	3	0.42	0.30	1.00				
Materials	4	0.56	0.55	0.47	1.00			
Oil Service	5	0.15	0.58	0.16	0.36	1.00		
Financial	6	0.56	0.41	0.49	0.67	0.23	1.00	
Technology	7	0.37	0.31	0.32	0.55	0.18	0.65	1.00

Table 4.3: The correlation c_{xy} between the industries. Source: S&P Index Month 2006.

In this way, we get the correlation between the customers within different industries: c_{xy} for $x, y = 1, ..., 7$ and $x \neq y$. For companies within the same industry, we use a correlation of $c_{xx} = 0.8$ for $x = 1, ..., 7$.

Credit Line Information

For 76 customers, we have in all 120 revolving credit lines selected from the historical database. Each of the customers has maximum 3 credit lines committed. The total committed limit is EUR 83.37 Mio. The distribution of credit lines with respect to maturity is shown in Table 4.4.

Number of Credit Lines	Maturity
47	12
50	24
16	36
7	48

Table 4.4: Credit lines grouped by maturities m

If we sum up all the committed lines within the same credit rating s and maturity m, we get the distribution of the total amount of the committed lines in TEUR (thousand EUR) with respect to maturity and credit rating as shown in Table 4.5.

	AAA	AA	A	BBB	BB	B	CCC	D
m \ s	1	2	3	4	5	6	7	8
12	10500	10610	2490	900	0	0	0	0
24	9280	13260	3300	4850	0	0	0	0
36	4100	9080	2200	1500	0	0	0	0
48	2000	800	3000	2500	3000	0	0	0

Table 4.5: Credit lines grouped by credit rating s and maturity m

4.2 Input Parameters

Transition Matrix

From Bloomberg Server, we obtain the yearly credit rating transition matrix $\mathbf{P}_y$, see Table 4.6.

		AAA	AA	A	BBB	BB	B	CCC	D
	s \ r	1	2	3	4	5	6	7	8
AAA	1	0.8866	0.1029	0.0102	0.0003	0.0000	0.0000	0.0000	0.0000
AA	2	0.0108	0.8871	0.0955	0.0034	0.0015	0.0010	0.0005	0.0003
A	3	0.0006	0.0288	0.9021	0.0592	0.0074	0.0018	0.0001	0.0001
BBB	4	0.0005	0.0034	0.0707	0.8524	0.0605	0.0101	0.0009	0.0016
BB	5	0.0003	0.0008	0.0056	0.0568	0.8357	0.0808	0.0054	0.0146
B	6	0.0001	0.0004	0.0017	0.0065	0.0660	0.8270	0.0676	0.0306
CCC	7	0.0000	0.0000	0.0066	0.0105	0.0305	0.0611	0.8297	0.0616
D	8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Table 4.6: Credit rating transition matrix $\mathbf{P}_y$. Source: S&P Credit Week (2002).

After the regularization procedure we have described in Chapter 2, we get the optimal monthly credit rating transition matrix $\mathbf{P}$ as shown in Table 4.7.

		AAA	AA	A	BBB	BB	B	CCC	D
	s \ r	1	2	3	4	5	6	7	8
AAA	1	0.9920	0.0076	0.0003	0.0000	0.0001	0.0000	0.0000	0.0000
AA	2	0.0006	0.9917	0.0071	0.0004	0.0000	0.0002	0.0000	0.0000
A	3	0.0001	0.0021	0.9920	0.0051	0.0006	0.0001	0.0000	0.0000
BBB	4	0.0000	0.0002	0.0055	0.9880	0.0052	0.0009	0.0001	0.0001
BB	5	0.0000	0.0001	0.0004	0.0076	0.9816	0.0088	0.0009	0.0006
B	6	0.0000	0.0001	0.0002	0.0001	0.0064	0.9844	0.0045	0.0043
CCC	7	0.0002	0.0000	0.0002	0.0013	0.0022	0.0123	0.9642	0.0196
D	8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Table 4.7: Monthly credit rating transition matrix $\mathbf{P}$

Credit Draw Probability

We use the historical data in Jan 2007 and Feb 2007 (20,000 credit lines) to estimate the draw probability. Using the empirical mean, we get the estimated draw probability as in Table 4.8. The default customers are not allowed to get a credit loan, so we have to set their draw probability 0.

	AAA	AA	A	BBB	BB	B	CCC	D
s	1	2	3	4	5	6	7	8
$\hat{p}_s$	0.0725	0.1745	0.1258	0.3554	0.3726	0.3223	0.2160	0.0000

Table 4.8: Estimated draw probability $\hat{p}_s$ for $s \in S$

Credit Line Usage

To estimate the usage proportion and the return probability, we need more historical data. Here we retrieved from the database the usage information of revolving credit lines from 1999 till 2007, with around 20,000 credit lines in each month. We need to estimate 7 usage proportion in all. The estimation result is given in Table 4.9.

	AAA	AA	A	BBB	BB	B	CCC	D
s	1	2	3	4	5	6	7	8
$\hat{u}_s$	0.5056	0.3433	0.5073	0.3966	0.5144	0.5542	0.5838	0.0000

Table 4.9: Estimated credit line usage $\hat{u}_s$ for $s \in S$

Credit Return Probability

As we have calculated, we need to estimate 270 grouped return probabilities. With 20,000 historical credit lines, we have approximately 75 data to estimate one single probability. After getting the grouped return probability, we use that to calculate the estimated cumulative return probability. Tables 4.10, 4.11, 4.12 and 4.13 contain the cumulative return probability for customers with $\bar{s} = 1, 2, 3, 4$ correspondingly. In Table 4.13, all the cumulative return probabilities are 0, since the default customers are regarded not to return the loan.

$\bar{l}$ \ $\bar{E}$	1	2	3	4	5	6	7	8	9	10	11	12
1	0.9600	0.9656	0.9536	0.9294	0.9389	0.9867	0.9623	1.0000	0.9818	0.9750	0.9800	1.0000
2		0.3292	0.4725	0.4969	0.6870	0.6933	0.7358	0.8602	0.8000	0.9375	0.9600	0.9667
3			0.2957	0.3957	0.5802	0.5733	0.6415	0.7312	0.7273	0.9000	0.9600	0.9333
4				0.2730	0.4580	0.4933	0.5660	0.6774	0.5818	0.9000	0.9600	0.9000
5					0.2061	0.3867	0.4717	0.5914	0.5091	0.8125	0.9400	0.9000
6						0.2933	0.3774	0.4839	0.4364	0.6125	0.9200	0.9000
7							0.3208	0.4301	0.3818	0.5625	0.8800	0.8667
8								0.2151	0.2727	0.5125	0.5600	0.8000
9									0.1818	0.4250	0.5000	0.7667
10										0.2250	0.4400	0.5667
11											0.1600	0.4333
12												0.1667

Table 4.10: Estimated cumulative return probability $\hat{Q}(\bar{s}, \bar{l}, \bar{E})$ for customer with $\bar{s} = 1$

$\bar{l}$ \ $\bar{E}$	1	2	3	4	5	6	7	8	9	10	11	12
1	0.9596	0.9613	0.9543	0.9433	0.9787	0.9657	0.9706	0.9697	0.9816	0.9787	1.0000	0.9630
2		0.3483	0.5239	0.5490	0.7707	0.7399	0.8015	0.7814	0.9118	0.9268	0.9823	0.9444
3			0.3239	0.4536	0.7005	0.6815	0.7684	0.7532	0.8566	0.9055	0.9735	0.9444
4				0.2892	0.6015	0.6129	0.7279	0.7186	0.7904	0.8780	0.9735	0.9444
5					0.4048	0.5121	0.6360	0.6602	0.7426	0.7683	0.9646	0.8704
6						0.2863	0.5625	0.5541	0.6691	0.7104	0.9204	0.7963
7							0.3713	0.4351	0.6213	0.6616	0.8850	0.7037
8								0.3074	0.5441	0.5884	0.8496	0.6481
9									0.4118	0.5061	0.8230	0.6296
10										0.2957	0.7345	0.5185
11											0.5044	0.3704
12												0.2963

Table 4.11: Estimated cumulative return probability $\hat{Q}(\bar{s}, \bar{l}, \bar{E})$ for customer with $\bar{s} = 2$

$\bar{l}$ \ $\bar{E}$	1	2	3	4	5	6	7	8	9	10	11	12
1	0.9140	0.9563	0.9438	0.9660	0.3250	0.7500	0.9623	0.5000	1.0000	1.0000	1.0000	1.0000
2		0.4000	0.5618	0.3401	0.2750	0.5000	0.7358	0.5000	1.0000	1.0000	1.0000	0.8571
3			0.2809	0.2585	0.2750	0.5000	0.6415	0.5000	0.9333	1.0000	1.0000	0.8571
4				0.1497	0.1000	0.5000	0.5660	0.5000	0.9333	1.0000	1.0000	0.8571
5					0.0500	0.5000	0.4717	0.5000	0.9333	0.7500	1.0000	0.8571
6						0.2500	0.3774	0.5000	0.9333	0.5000	1.0000	0.8571
7							0.3208	0.5000	0.9333	0.5000	1.0000	0.8571
8								0.5000	0.9333	0.2500	1.0000	0.8571
9									0.8667	0.2500	0.0000	0.5714
10										0.2500	0.0000	0.5714
11											0.0000	0.4286
12												0.2857

Table 4.12: Estimated cumulative return probability $\hat{Q}(\bar{s}, \bar{l}, \bar{E})$ for customer with $\bar{s} = 3$

$\bar{l}$ \ $\bar{E}$	1	2	3	4	5	6	7	8	9	10	11	12
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3			0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4				0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5					0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6						0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7							0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8								0.0000	0.0000	0.0000	0.0000	0.0000
9									0.0000	0.0000	0.0000	0.0000
10										0.0000	0.0000	0.0000
11											0.0000	0.0000
12												0.0000

Table 4.13: Estimated cumulative return probability $\hat{Q}(\bar{s}, \bar{l}, \bar{E})$ for customer with $\bar{s} = 4$

Furthermore in order to illustrate the cumulative return probability, we plot the probability $\hat{Q}(\bar{s}, \bar{l}, \bar{E})$ against $\bar{l}$ for $\bar{E} \in \{1, 2, ..., 12\}$ and $\bar{s} = 1$. We get 12 decreasing curves given in Figure 4.1. The curves are decreasing since customers tend to keep the loan till the maturity. As $\bar{l}$ decreases, we expect more frequent return of credit loan, which is reflected in a higher return probability.

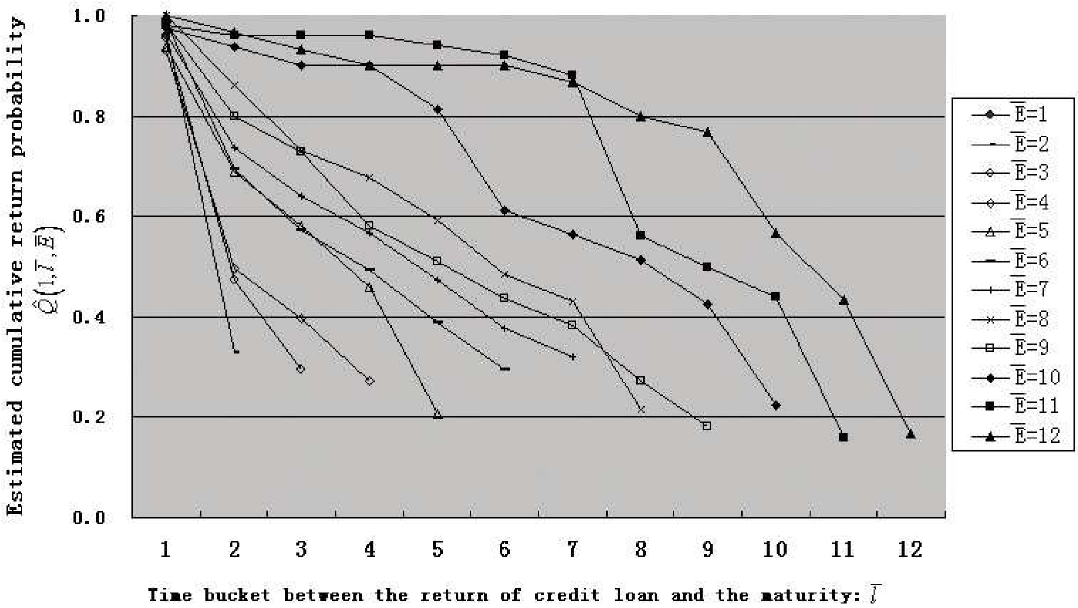

Figure 4.1: Estimated cumulative return probability $\hat{Q}(\bar{s}, \bar{l}, \bar{E})$ for customer with $\bar{s} = 1$

4.3 Simulation Parameters and Results

With all the input data, we start our Monte Carlo simulation. We simulate within a period of 48 months, $t = 0, 1, ..., 48$. For each credit line, at each time, we replicate the simulation $K = 1000$ times to calculate 1000 realizations of the total liquidity requirement $\hat{\mathcal{F}}_t$. For VaR analysis, we select the time series (path) in $\alpha = 0.75, 0.9, 0.95$ quantile of total usage amount (given in TEUR). Moreover, we obtain the $\alpha = 0.75, 0.9, 0.95$ quantile of proportion $\hat{\mathrm{f}}_t$ of total committed limit which was in use at each time point t. The simulation result is shown in Figure 4.2 and Figure 4.3.

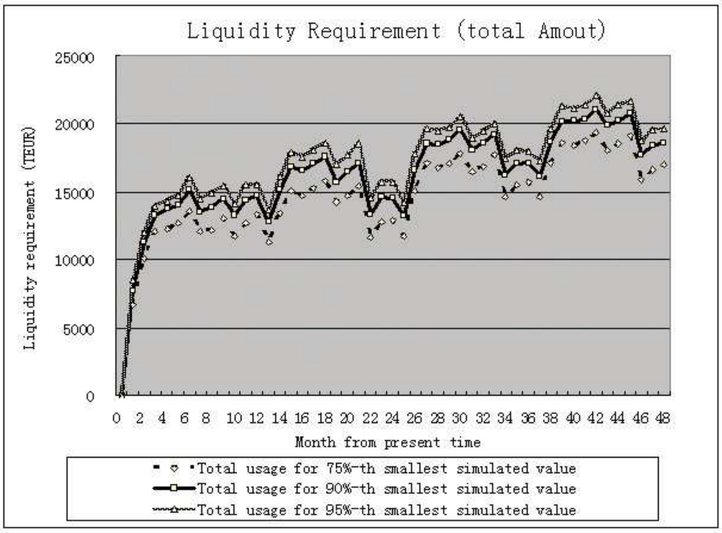

Figure 4.2: Estimated liquidity requirement $\hat{\mathcal{F}}_t$ in TEUR

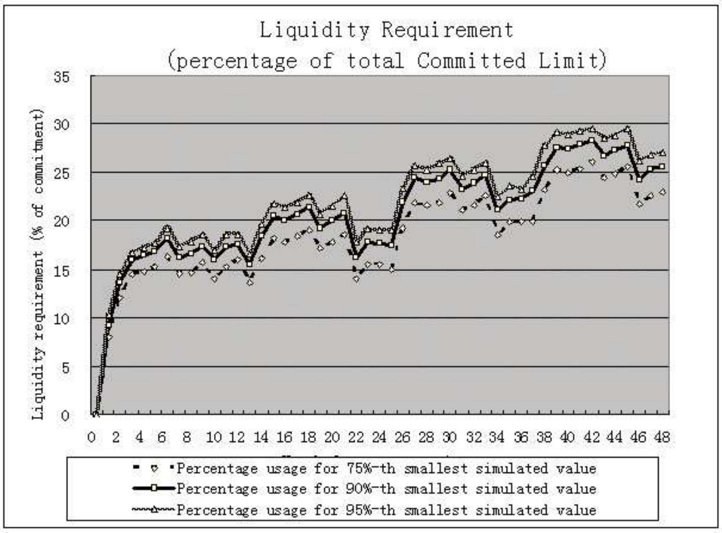

Figure 4.3: Estimated liquidity requirement $\hat{\mathrm{f}}_t$ as percentage of total committed limit

4.4 Sensitivity Analysis

After we get the simulation result of our liquidity requirement, we may ask whether this result is stable related to the parameters like the estimated usage $\hat{u}_s$ and estimated draw probability $\hat{p}_s$. Thus we carry out a sensitivity analysis. In the analysis, the credit portfolio is not changed (76 customers with 120 credit lines). We observe how the total liquidity requirement changes according to the change of u_s and p_s at a certain time point e.g. $t = 1$ here.
We have introduced the estimated proportional usage $\hat{u}_s$ for $s \in S$ in Section 3.3. Consider a vector

$$\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_8).$$

The usage is changed by multiplying $\hat{u}$ with a constant c, i.e. we define $\mathbf{u} := c\hat{\mathbf{u}}$. Changing the constant $c = 0.6, 0.7, ..., 1.4$, we get different total simulated usage $\hat{\mathcal{F}}_t$ of credit loan drawn by the customer group. The plot of the $\hat{\mathcal{F}}_t$ against c is given in Figure 4.4.

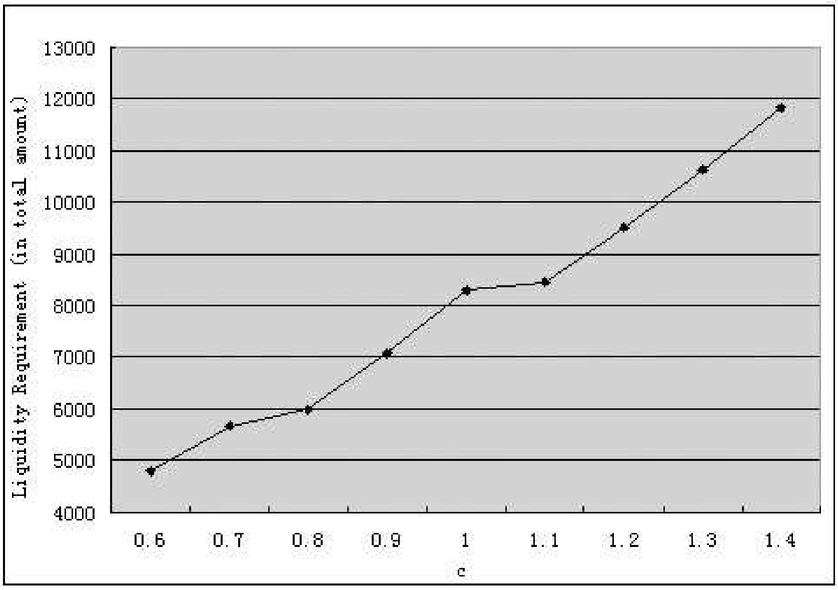

Figure 4.4: Sensitivity of $\hat{\mathcal{F}}_t$ with respect to c (in units of $\hat{\mathbf{u}}_s$) at time $t = 1$

The analysis of sensitivity of $\hat{\mathcal{F}}_t$ to the draw probability is similar. We consider

$$\mathbf{p} := c\hat{\mathbf{p}} = c(\hat{p}_1, \hat{p}_2, ..., \hat{p}_8).$$

Changing $c = 0.6, ..., 1.4$, we obtain $\hat{\mathcal{F}}_t$ corresponding to $\hat{\mathbf{p}}$. The plot is given in Figure 4.5.

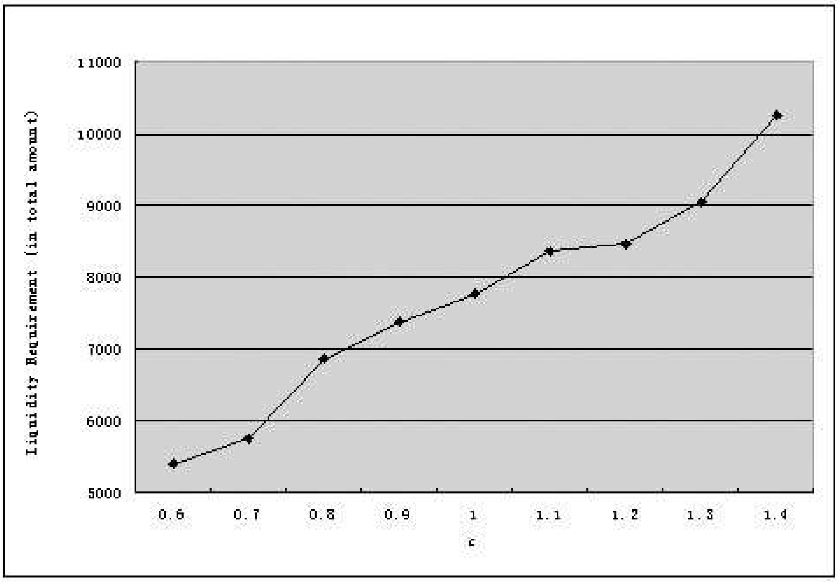

Figure 4.5: Sensitivity of $\hat{\mathcal{F}}_t$ with respect to c (in units of $\hat{\mathbf{p}}_s$) at time $t = 1$

Similarly we get the sensitivity analysis at time $t = 20$. See the Figures below.

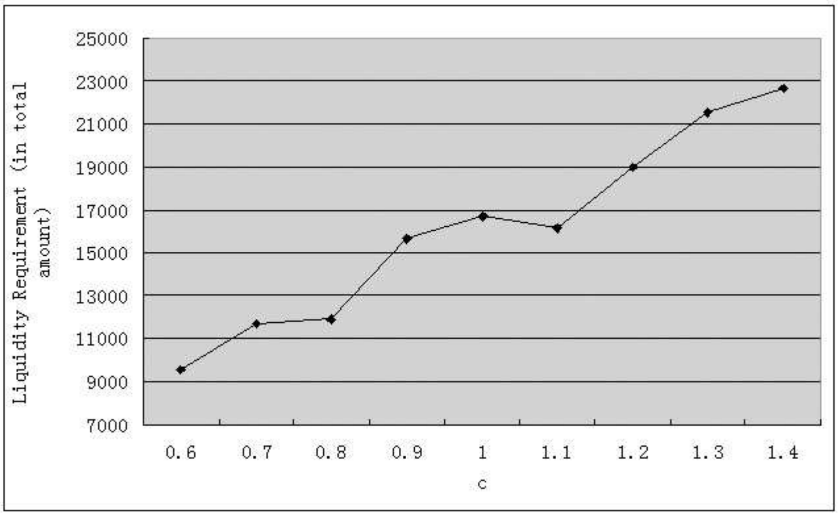

Figure 4.6: Sensitivity of $\hat{\mathcal{F}}_t$ with respect to c (in units of $\hat{\mathbf{u}}_s$) at time $t = 20$

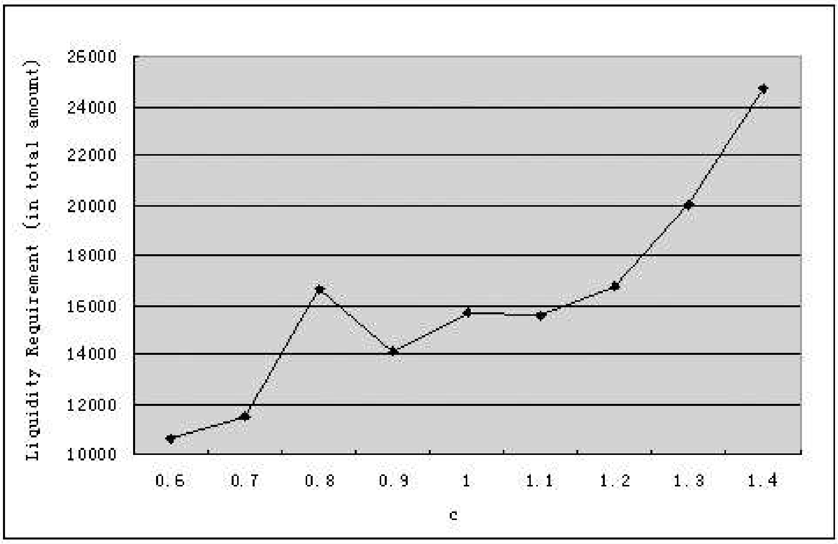

Figure 4.7: Sensitivity of $\hat{\mathcal{F}}_t$ with respect to c (in units of $\hat{\mathbf{p}}_s$) at time $t = 20$

Chapter 5

Distribution of the Liquidity Requirement for a Single Credit Line with Single Maturity

In this chapter, we derive the distribution of the liquidity requirement $\mathcal{F}_t$:

$$\mathcal{F}_t = \sum_{i=1}^{n} \sum_{j=1}^{n_i} l_t^{ij} U_t^{ij}.$$

For simplification, we assume the committed limit $l_t^{i,j}$ unchanged over the time. The expectation of the liquidity requirement $\mathrm{E}\left(\mathcal{F}_t\right)$ is

$$\mathrm{E}\left(\mathcal{F}_t\right) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} l_t^{ij} \mathrm{E}\left(U_t^{ij}\right).$$

We consider a single credit line and can therefore ignore the indices in this chapter. Denote

$$R_t^i \rightarrow R_t \quad U_t^{ij} \rightarrow U_t \quad l_t^{ij} \rightarrow l \quad m_t^{ij} \rightarrow m.$$

Since l is a constant, our key problem is to derive the distribution of U_t for $t \in \{0, 1, 2, ..., km\}$, $k \in \mathbb{N}$. Here k represents the number of maturities we have during the simulated period. Suppose we are simulating a time period of 48 months and the maturity of the credit line is 12 months, then we have $k = 4$. For simplification, we investigate in this chapter a time period with $k = 1$. We call it a single-maturity (SM) situation. In the next chapter, we extend our distribution of U_t to multi-maturity (MM) situation ($k > 1$). The difference between SM and MM situation is in the presence of renewal rule when $k \geq 2$. For detailed description, please refer to Chapter 6.

5.1 Distribution of Usage in Single Period

Before we start the derivation, we give an overview of the notations we are going to use.

1. **Credit Rating**
   The credit rating R_t for $t \in \{0, 1, 2, ..., m\}$ is a Markov process with initial state $R_0 = s_0$, state space $S = \{1, 2, ...7, 8\}$ and transition matrix $\mathbf{P}$. Thus

$$P\left(R_t = s | R_{t-1} = s', R_{t-2} = s'', ...\right) = P\left(R_t = s | R_{t-1} = s'\right) = \mathbf{p}_{s's}, \tag{5.1}$$

   where $\mathbf{p}_{s's}$ is the element of optimal monthly transition matrix.

2. **Draw Probability**
   The draw probability is denoted as p_s and it represents the probability that the customer has not drawn the credit line in last month and starts the loan this month under the condition that he or she has a credit rating of s, i.e.

$$p_s := P\left(U_t > 0, U_{t-1} = 0 | R_t = s\right), \tag{5.2}$$

   with $s \in S$. This definition is the same as in the simulation. That is why we use the same notation. But be careful that the return probability in theory and in simulation are different. Please refer to the Remark 5.1.

3. **Joint Return Probability**
   Suppose at time t, the customer has a credit rating $R_t = s$ and uses the credit loan, which started at $t - l + 1$ ($l - 1$ is the number of the months that the credit loan is kept in use). The probability of the customer to return the loan at time t is $P\left(U_t = 0, U_{t-1} > 0, U_{t-2} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0 | R_t = s\right)$. On the other hand, we have the time period between the start time of the credit loan and the maturity $E = m - t + l$. The time period between the return of the credit loan and maturity is $L_t = m - t + 1$. Denote the return probability as $\tilde{q}(R_t, E, L_t) = \tilde{q}(s, (m - t + l), (m - t + 1))$, namely

$$\tilde{q}(s, (m-t+l), (m-t+1)) := P\left(U_t = 0, U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0 | R_t = s\right), \tag{5.3}$$

   where $s \in S$, $l \in \{1, 2, ..., t-1\}$ and $t \in \{1, 2, ..., m\}$. If at the maturity the credit loan is still in use, we regard the credit loan as expired. The expiration probability is considered as a special case of return probability with $t = m + 1$ in (5.3), namely

$$\tilde{q}(s, l+1, 0) := P\left(U_m > 0, ..., U_{t-1} > 0, U_{t-2} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0 | R_m = s\right), \tag{5.4}$$

   So in general we have $s \in S$, $l \in \{1, 2, ..., t-1\}$ and $t \in \{1, 2, ..., m+1\}$ for (5.3). An explanation of the difference between the joint return probability we have defined here and the return probability we have used in simulation is given in Remark 5.1.

4. **Usage**
   The usage at time t is denoted as U_t with $t \in \{0, 1, 2, ..., m\}$, initial state $U_0 = 0$ and state space $U = \{u_1, u_2, ..., u_7, u_8\}$ where $u_8 = 0$.

Remark 5.1. In simulation, we consider the return decision at time t under the condition that the usage information in the past $U_{t-1}, U_{t-2}, ..., U_1$ and R_t are already known. The probability of return is thus $P\left(U_t = 0|U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0, R_t = s\right)$. Comparing this form to the right hand side of (5.3), we find out that two return probabilities are different. According to our notation in Chapter 4, the return probability in simulation is denoted as $q(s, m-t+l, m-t+1)$ ($R_t = s$, $E = m-t+l$ and $L_t = m-t+1$) and we can call it **conditional return probability** more precisely. In contrast, $\tilde{q}(s, m-t+l, m-t+1)$ is called **joint return probability**.

Remark 5.2. In the following calculation, we need a short notation for $\tilde{q}(s, m-t+l, m-t+1)$ in order to make our formulae more compact. We denote from now on $\tilde{q}(R_t, E, L_t) \equiv {}_{R_t}\tilde{q}^{E}_{L_t}$.

Remark 5.3. As in the simulation, we will assume that the u_s is a fraction that represents the proportion of the credit limit drawn by the customer. It can take only 8 values u_s for $s \in S$ according to the credit rating of the customer (see Assumption 1 below).

In this chapter we obtain the distribution of U_t at each time point: $P(U_t = u_s)$ for $s \in S$ and $t \in \{1, ..., m\}$. Now we give the assumptions under which we determine the $P(U_t = u_s)$.

Assumption 5.1. *If the credit rating of the customer is not default ($R_t \neq 8$) and the customer starts to use the credit line ($U_t > 0$), the proportion of the loan drawn by the customer is u_s, i.e.*

$$P\left(U_t = u_s|R_t = s, U_t > 0\right) = 1, \tag{5.5}$$

where $t \in \{1, 2, ..., m\}$ and $s \in S\backslash\{8\}$.

Assumption 5.2. *If the credit rating of the customer is default ($R_t = 8$) and the customer has already started using the credit line ($U_{t-1} > 0$), then the usage is not changed ($U_t = U_{t-1}$), i.e.*

$$P\left(U_t = u_s|R_t = 8, U_{t-1} = u_s\right) = 1, \tag{5.6}$$

where $t \in \{1, 2, ..., m\}$ and $s \in S\backslash\{8\}$.

Assumption 5.3. *The usage of previous months has no impact on the credit rating of the current month, i.e.*

$$P\left(R_t = s|U_{t-\delta} = u_{s'}\right) = P\left(R_t = s\right), \quad \delta = 1, 2, ..., t, \tag{5.7}$$

and

$$P\left(R_t = s|U_{t-\alpha} = u_{s_\alpha}, U_{t-2} = u_{s_2}, ..., U_{t-\delta} = u_{s_\delta}\right) = P\left(R_t = s\right), \quad \alpha \leq \delta. \tag{5.8}$$

Assumption 5.4. *There is no term-out-option for the credit line.*

Suppose the initial credit rating is s_0, and define

$$P_{s_0}(U_t = u_s) := P(U_t = u_s|R_0 = s_0).$$

We first give out three lemmas before we formulate our theorem.

Lemma 5.1. *Suppose the initial credit rating s_0 is given. Then*

$$P_{s_0}(U_t = u_s) = \begin{cases} P_{s_0}(R_t = s, U_t > 0) + P_{s_0}(U_t = u_s, R_t = 8), & s \in S \setminus \{8\}, \\ 1 - \sum_{\rho=1}^{7} P_{s_0}(U_t = u_\rho), & s \in \{8\}. \end{cases} \tag{5.9}$$

Proof. Assume $s \in S \setminus \{8\}$. We have

$$\begin{aligned} & P_{s_0}(U_t = u_s) \\ & = P_{s_0}(U_t = u_s | R_t = s, U_t > 0)\, P_{s_0}(R_t = s, U_t > 0) \\ & + P_{s_0}(U_t = u_s | R_t = 8, U_t > 0)\, P_{s_0}(R_t = 8, U_t > 0) \\ & + \sum_{\rho \in S\setminus\{s,8\}} P_{s_0}(U_t = u_s | R_t = \rho, U_t > 0)\, P_{s_0}(R_t = \rho, U_t > 0) \\ & + P_{s_0}(U_t = u_s | R_t = s, U_t = 0)\, P_{s_0}(R_t = s, U_t = 0) \\ & + P_{s_0}(U_t = u_s | R_t = 8, U_t = 0)\, P_{s_0}(R_t = 8, U_t = 0) \\ & + \sum_{\rho \in S\setminus\{s,8\}} P_{s_0}(U_t = u_s | R_t = \rho, U_t = 0)\, P_{s_0}(R_t = \rho, U_t = 0) \end{aligned} \tag{5.10}$$

According to Assumption 5.1, we have

$$P_{s_0}(U_t = u_s | R_t = s, U_t > 0) = 1,$$

we get

$$P_{s_0}(U_t = u_\rho | R_t = s, U_t > 0) = 0 \quad \text{for} \quad \rho \in S \setminus \{s\}.$$

Expression (5.10) can thus be reduced to

$$\begin{aligned} & P_{s_0}(U_t = u_s) \\ & = P_{s_0}(U_t = u_s | R_t = s, U_t > 0)\, P_{s_0}(R_t = s, U_t > 0) \\ & + P_{s_0}(U_t = u_s | R_t = s, U_t = 0)\, P_{s_0}(R_t = s, U_t = 0) \\ & + P_{s_0}(U_t = u_s | R_t = 8, U_t = 0)\, P_{s_0}(R_t = 8, U_t = 0) \\ & + \sum_{\rho \in S\setminus\{s,8\}} P_{s_0}(U_t = u_s | R_t = \rho, U_t = 0)\, P_{s_0}(R_t = \rho, U_t = 0). \end{aligned} \tag{5.11}$$

On the other hand, we have that $u_s > 0$ for $s \in S \setminus \{8\}$, thus $P_{s_0}(U_t = u_s | R_t = s, U_t = 0) = 0$ for $s \in S \setminus \{8\}$. Expression (5.11) can thus be reduced to

$$\begin{aligned} & P_{s_0}(U_t = u_s) \\ & = P_{s_0}(U_t = u_s | R_t = s, U_t > 0)\, P_{s_0}(R_t = s, U_t > 0) \\ & + P_{s_0}(U_t = u_s | R_t = 8, U_t = 0)\, P_{s_0}(R_t = 8, U_t = 0) \\ & = P_{s_0}(U_t = u_s, U_t > 0, R_t = s) + P_{s_0}(U_t = u_s, U_t > 0, R_t = 8) \\ & = P_{s_0}(U_t = u_s, R_t = s) + P_{s_0}(U_t = u_s, R_t = 8). \end{aligned} \tag{5.12}$$

□

Expression (5.12) tells us that the probability of the usage u_s consists of two parts. The first part ($P_{s_0}(R_t = s, U_t > 0)$) is the probability that the customer is not default and has a usage of u_s. The second part ($P_{s_0}(U_t = u_s, R_t = 8)$) is the probability that the customer is default and has a usage of u_s. In order to refer to these two probabilities simply later on, we add the following definition.

Definition 5.1. No default usage probability is the probability that the customer has not defaulted at time t when using u_s proportion of credit limit given an initial rating s_0 and the maturity of the credit line m. We denote it as ${}^{m}_{s_0}b^{s}_{t} := P_{s_0}(R_t = s, U_t > 0)$, where s_0 is the initial rating of the customer, m is the maturity of the credit line, t is the current time and s is the current credit rating of the customer. The ${}^{m}_{s_0}b^{s}_{t}$ is dependent on m through the joint return probability ${}_{s}\tilde{q}^{m-t+l}_{m-t+1}$.

Definition 5.2. Default usage probability is the probability that the customer defaults at time t when using u_s proportion of credit limit given an initial rating s_0 and the maturity of the credit line m. We denote it as ${}^{m}_{s_0}d^{s}_{t} := P_{s_0}(U_t = u_s, R_t = 8)$, where s_0 is the initial rating of the customer, m is the maturity of the credit line, t is the current time and s is the current credit rating of the customer.

Lemma 5.2 (Calculation of no default usage probability ${}^{m}_{s_0}b^{s}_{t}$)**.** *Suppose the draw probability p_s, the return probability ${}_{s}\tilde{q}^{E}_{L_t}$, the transition matrix $\mathbf{P}$ are given. The credit line has a maturity of m and the customer has an initial credit rating of s_0. Then*

$$
{}^{m}_{s_0}b^{s}_{t} = \begin{cases} p_s\left(\mathbf{p}^t\right)_{s_0 s}, & t = 1, \\ \left(\mathbf{p}^t\right)_{s_0 s}\left[\left(p_s - {}_{s}\tilde{q}^{m-t+1}_{m-t+1}\right) + \sum\limits_{l=2}^{t}\left({}_{s_0}y^{l}_{t} - {}_{s}\tilde{q}^{m-t+l}_{m-t+1}\right)\right], & t = 2, 3, ..., m, \end{cases}
$$

where $s \in S\backslash\{8\}$ and

$$
{}^{m}_{s_0}y^{l}_{t} = \sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}^{m-t+l}_{0}\left(\mathbf{p}^m\right)_{s_0\rho} + \sum_{\tau=1}^{m-t+1}{}_{\rho}\tilde{q}^{m-t+l}_{m-t-\tau+2}\left(\mathbf{p}^{t+\tau-1}\right)_{s_0\rho}\right), \quad l = 2, 3, ..., t. \tag{5.13}
$$

Here $\left(\mathbf{p}^t\right)_{s_0 s}$ is the corresponding element of the t^{th}-power of transition matrix $\mathbf{P}$.

Proof.

(1) We first show $P_{s_0}(R_t = s, U_t > 0) = \sum\limits_{l=1}^{t} P_{s_0}(R_t = s, U_t > 0, ..., U_{t-l} = 0)$, namely

$$
\begin{aligned}
&P_{s_0}(R_t = s, U_t > 0) \\
&= P_{s_0}(R_t = s, U_t > 0, U_{t-1} > 0) + P_{s_0}(R_t = s, U_t > 0, U_{t-1} = 0) \\
&= P_{s_0}(R_t = s, U_t > 0, U_{t-1} = 0) + P_{s_0}(R_t = s, U_t > 0, U_{t-1} > 0, U_{t-2} = 0) \\
&+ P_{s_0}(R_t = s, U_t > 0, U_{t-1} > 0, U_{t-2} > 0) \\
&\cdots \\
&= P_{s_0}(R_t = s, U_t > 0, U_{t-1} = 0) + P_{s_0}(R_t = s, U_t > 0, U_{t-1} > 0, U_{t-2} = 0) + ... \\
&+ P_{s_0}(R_t = s, U_t > 0, U_{t-1} > 0, U_{t-2} > 0, ..., U_1 > 0, U_0 = 0)
\end{aligned} \tag{5.14}
$$

For more practical explanation, please refer to Remark 5.4.
(2) Further we show that

$$
\begin{aligned}
&P_{s_0}(R_t = s, U_t > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0) \\
&= \begin{cases} p_s\left(\mathbf{p}^t\right)_{s_0 s}, & l = 1, \\ \left(\mathbf{p}^t\right)_{s_0 s}\left[P_{s_0}(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0) - {}_{s}\tilde{q}^{m-t+l}_{l-1}\right], & l = 2, ..., t. \end{cases}
\end{aligned}
$$

If $l = 1$, we have

$$\begin{aligned} & P_{s_0}\left(R_t = s, U_t > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) = P_{s_0}\left(R_t = s, U_t > 0, U_{t-1} = 0\right) \\ & = P_{s_0}\left(U_t > 0, U_{t-1} = 0 | R_t = s\right) P_{s_0}\left(R_t = s\right) = p_s \left(\mathbf{p}^t\right)_{s_0 s}. \end{aligned} \tag{5.15}$$

If $l = 2, 3, ..., t$, we get

$$\begin{aligned} & P_{s_0}\left(R_t = s, U_t > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & = P_{s_0}\left(R_t = s, U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & - P_{s_0}\left(R_t = s, U_t = 0, U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right). \end{aligned} \tag{5.16}$$

From Assumption 5.3 and properties of Markov processes, we have

$$\begin{aligned} & P_{s_0}\left(R_t = s, U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & = P_{s_0}\left(R_t = s | U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & \times P_{s_0}\left(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & = P_{s_0}\left(R_t = s\right) P_{s_0}\left(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & = \left(\mathbf{p}^t\right)_{s_0 s} P_{s_0}\left(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right). \end{aligned} \tag{5.17}$$

From (5.3), we have

$$\begin{aligned} & P_{s_0}\left(R_t = s, U_t = 0, U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & = P_{s_0}\left(U_t = 0, U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0 | R_t = q\right) P_{s_0}\left(R_t = s\right) \\ & = {}_s\tilde{q}_{m-t+1}^{m-t+l} \left(\mathbf{p}^t\right)_{s_0 s}. \end{aligned} \tag{5.18}$$

Using expressions (5.16), (5.17) and (5.18), we get

$$\begin{aligned} & P_{s_0}\left(R_t = s, U_t > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) \\ & = \left(\mathbf{p}^t\right)_{s_0 s} \left[P_{s_0}\left(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) - {}_s\tilde{q}_{M-t+1}^{M-t+l}\right]. \end{aligned} \tag{5.19}$$

Thus

$$P_{s_0}\left(R_t = s, U_t > 0\right) = \left(\mathbf{p}^t\right)_{s_0 s} \sum_{l=1}^{t} \left[P_{s_0}\left(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) - {}_s\tilde{q}_{m-t+1}^{m-t+l}\right]. \tag{5.20}$$

(3) Further we show that

$$P_{s_0}\left(U_{t-\delta} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) = \sum_{\rho=1}^{8} \left({}_\rho\tilde{q}_0^{m-t+l} \left(\mathbf{p}^m\right)_{s_0 \rho} + \sum_{\tau=1}^{m-t+\delta} {}_\rho\tilde{q}_{m-t+\delta-\tau+1}^{m-t+l} \left(\mathbf{p}^{t-\delta+\tau}\right)_{s_0 \rho}\right), \tag{5.21}$$

where $\delta = 1, 2, ..., t-1$ and $l = \delta + 1, \delta + 2, ..., t$.

$$
\begin{aligned}
&P_{s_0}\left(U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&=P_{s_0}\left(U_{t-\delta+1}=0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+P_{s_0}\left(U_{t-\delta+1}>0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&=P_{s_0}\left(U_{t-\delta+1}=0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+P_{s_0}\left(U_{t-\delta+2}=0,U_{t-\delta+1}>0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+P_{s_0}\left(U_{t-\delta+2}>0,U_{t-\delta+1}>0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&\cdots\\
&=P_{s_0}\left(U_{t-\delta+1}=0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+P_{s_0}\left(U_{t-\delta+2}=0,U_{t-\delta+1}>0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+......\\
&+P_{s_0}\left(U_m=0,U_{m-1}>0,...,U_{t-\delta+1}>0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+P_{s_0}\left(U_m>0,U_{m-1}>0,...,U_{t-\delta+1}>0,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right).
\end{aligned}
\tag{5.22}
$$

Expression (5.22) can be rewritten as following:

$$
\begin{aligned}
&P_{s_0}\left(U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)=P_{s_0}\left(U_m>0,...,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&+\sum_{\tau=1}^{m-t+\delta}P_{s_0}\left(U_{t-\delta+\tau}=0,U_{t-\delta+\tau-1}>0...,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right).
\end{aligned}
\tag{5.23}
$$

For a more detailed explanation of the meaning of (5.23), please refer to Remark 5.5. Furthermore, we have

$$
\begin{aligned}
&P_{s_0}\left(U_{t-\delta+\tau}=0,...,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&=\sum_{\rho=1}^{8}P_{s_0}\left(R_{t-\delta+\tau}=\rho\right)P_{s_0}\left(U_{t-\delta+\tau}=0,...,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0|R_{t-\delta+\tau}=\rho\right)\\
&=\sum_{\rho=1}^{8}{}_{\rho}\tilde{q}^{m-t+l}_{m-t+\delta-\tau+1}\left(\mathbf{p}^m\right)_{s_0\rho}\qquad \tau=1,...,(m-t+\delta),
\end{aligned}
\tag{5.24}
$$

and

$$
\begin{aligned}
&P_{s_0}\left(U_m>0,...,U_{i-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)\\
&=\sum_{\rho=1}^{8}P_{s_0}\left(U_m>0,...,U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0|R_m=\rho\right)P_{s_0}\left(R_m=\rho\right)\\
&=\sum_{\rho=1}^{8}{}_{\rho}\tilde{q}^{m-t+l}_{0}\left(\mathbf{p}^m\right)_{s_0\rho},
\end{aligned}
\tag{5.25}
$$

where the definition of return probability in (5.3) and Assumption 5.3 were used. Using expression (5.22), (5.24) and (5.25), we get

$$
P_{s_0}\left(U_{t-\delta}>0,...,U_{t-l+1}>0,U_{t-l}=0\right)=\sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}^{m-t+l}_{0}\left(\mathbf{p}^m\right)_{s_0\rho}+\sum_{\tau=1}^{m-t+\delta}{}_{\rho}\tilde{q}^{m-t+l}_{m-t+\delta-\tau+1}\left(\mathbf{p}^{t-\delta+\tau}\right)_{s_0\rho}\right),
\tag{5.26}
$$

where $l=\delta+1,\delta+2,...,t$.

For $\delta = 1$ we define

$$ {}^{m}_{s_0}y^{l}_{t} := P_{s_0}\left(U_{t-1} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right) = \sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}_{0}^{m-t+l}\left(\mathbf{p}^{m}\right)_{s_0\rho} + \sum_{\tau=1}^{m-t+1} {}_{\rho}\tilde{q}_{m-t-\tau+2}^{m-t+l}\left(\mathbf{p}^{t+\tau-1}\right)_{s_0\rho}\right), \tag{5.27} $$

where $l = 2, 3, ..., t$.
Here ${}^{m}_{s_0}y^{l}_{t}$ stands for the probability that the customer starts a credit line at $t-l+1$, and till time $t-1$ he is still in use of the credit line. Using (5.27) and (5.16), we get the no default usage probability.

$$ {}^{m}_{s_0}b^{s}_{t} = P_{s_0}\left(R_t = s, U_t > 0\right) = \left(\mathbf{p}^{t}\right)_{s_0 s}\left[\left(p_s - {}_{s}\tilde{q}_{m-t+1}^{m-t+1}\right) + \sum_{l=2}^{t}\left({}^{m}_{s_0}y^{l}_{t} - {}_{s}\tilde{q}_{m-t+1}^{m-t+l}\right)\right], \tag{5.28} $$

$$ t = 2, 3, 4, ..., m. $$

which concludes the proof of the Lemma. □

Remark 5.4. $P_{s_0}\left(R_t = s, U_t > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right)$ is the probability that the customer with credit rating s has started the credit line at $t-l+1$ and keeps it in use till current time t. Expression (5.14) means that the probability that the customer is in use of the credit line at time t is equal to the sum of probabilities that the customer starts the credit line at each time point in the past from 1 till $t-1$ and keeps in use till current time.

Remark 5.5. $P_{s_0}\left(U_{t-\delta+\tau} = 0, ..., U_{t-\delta} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right)$ is the probability that the customer with initial credit rating s_0 started the credit line at time $t-l+1$ and returned the loan at $t-\delta+\tau$. $P_{s_0}\left(U_m > 0, U_{m-1} > 0, ..., U_{t-\delta+1} > 0, U_{t-\delta} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0\right)$ is the probability that the customer started the credit line at time $t-l+1$ and defaulted at the maturity. The expression (5.23) describes the probability that the customer started using a credit line and kept it in use till current time as the sum of probabilities that he or she returns the loan at any time in the future or defaults.

Lemma 5.3 (Calculation of default usage probability ${}^{m}_{s_0}d^{s}_{t}$). *Suppose the draw probability p_s, the return probability ${}_{s}\tilde{q}^{E}_{L_t}$ and the transition matrix $\mathbf{P}$ are given. The maturity of the credit line is m and the customer has an initial rating of s_0. The default usage probability ${}^{m}_{s_0}d^{s}_{t}$ is*

$$ {}^{m}_{s_0}d^{s}_{t} = \begin{cases} 0, & t = 1, \\ \left(\mathbf{p}^{t}\right)_{s_0 8} P_{s_0}\left(U_{t-1} = u_s\right), & t = 2, 3, ..., m, \end{cases} $$

where $s \in S \setminus \{8\}$.

Proof.
For $t = 1$, we have

$$ P_{s_0}\left(U_1 = u_s, R_1 = 8\right) = \sum_{\rho=1}^{8} P_{s_0}\left(U_1 = u_s, R_1 = 8 | U_0 = u_\rho\right) P_{s_0}\left(U_0 = u_\rho\right). \tag{5.29} $$

Since the initial usage is 0 and $u_8 = 0$, we have $P_{s_0}(U_0 = u_\rho) = 0$ for $\rho \in S \backslash \{8\}$. We get

$$\begin{aligned}
&P_{s_0}(U_1 = u_s, R_1 = 8) \\
&= P_{s_0}(U_1 = u_s, R_1 = 8 | U_0 = u_8) P_{s_0}(U_0 = u_8) \\
&= P_{s_0}(U_1 = u_s, R_1 = 8, U_0 = u_8) \\
&= P_{s_0}(U_1 = u_s | R_1 = 8, U_0 = u_8) P_{s_0}(R_1 = 8, U_0 = u_8).
\end{aligned} \tag{5.30}$$

Using Assumption 5.2, we get

$$P_{s_0}(U_1 = u_s | R_1 = 8, U_0 = u_8) = 0, \tag{5.31}$$

and

$$P_{s_0}(U_1 = u_s, R_1 = 8) = 0, \tag{5.32}$$

where $s \in S \backslash \{8\}$.
For $t = 2, 3, 4, ..., m$, we have

$$\begin{aligned}
&P_{s_0}(U_t = u_s, R_t = 8) \\
&= \sum_{\rho=1}^{8} P_{s_0}(U_t = u_s, R_t = 8, U_{t-1} = u_\rho) \\
&= \sum_{\rho=1}^{8} P_{s_0}(U_t = u_s | R_t = 8, U_{t-1} = u_\rho) P_{s_0}(R_t = 8, U_{t-1} = u_\rho) \\
&= P_{s_0}(U_t = u_s | R_t = 8, U_{t-1} = u_s) P_{s_0}(R_t = 8, U_{t-1} = u_s) \\
&= P_{s_0}(R_t = 8, U_{t-1} = u_s, U_t = u_s) \\
&= P_{s_0}(U_t = u_s | R_t = 8, U_{t-1} = u_s) P_{s_0}(R_t = 8, U_{t-1} = u_s) \\
&= P_{s_0}(R_t = 8, U_{t-1} = u_s) = P_{s_0}(R_t = 8) P_{s_0}(U_{t-1} = u_s) \\
&= (\mathbf{p}^t)_{s_0 8} P_{s_0}(U_{t-1} = u_s).
\end{aligned} \tag{5.33}$$

□

With Lemma 5.1, 5.2 and 5.3, we can formulate Theorem 5.1. For simplicity, we denote ${}^{m}_{s_0}x^s_t := P_{s_0}(U_t = u_s) = {}^{m}_{s_0}b^s_t + {}^{m}_{s_0}d^s_t$ and formulate the following theorem.

Theorem 5.1 (Distribution of Usage for SM Situation)**.** *Given the initial credit rating of the customer s_0, the maturity of the credit line as m, the distribution of usage ${}^{m}_{s_0}x^s_t$ is given by*

$$ {}^{m}_{s_0}x^s_t = \begin{cases} {}^{m}_{s_0}b^s_t + {}^{m}_{s_0}d^s_t, & s \in S \setminus \{8\}, \\ 1 - \sum_{\rho=1}^{7} {}^{m}_{s_0}x^\rho_t, & s \in \{8\}. \end{cases} \tag{5.34}$$

where $t = 0, 1, ..., m$. Here

$$ {}^{m}_{s_0}b^s_t = \begin{cases} 0, & t = 0, \\ (\mathbf{p})_{s_0 s}\, p_s, & t = 1, \\ (\mathbf{p}^t)_{s_0 s} \left(p_s - {}_s\tilde{q}^{m-t+1}_{m-t+1} + \sum_{l=2}^{t} \left({}^{m}_{s_0}y^l_t - {}_s\tilde{q}^{m-t+l}_{m-t+1} \right) \right), & t = 2, ..., m, \end{cases} \tag{5.35}$$

and

$$ {}_{s_0}^{m}d_t^s = \begin{cases} 0, & t=0,1, \\ p_s\left(\mathbf{p}\right)_{s_0 s}\left(\mathbf{p}^2\right)_{s_0 8}, & t=2, \\ p_s\left(\mathbf{p}\right)_{s_0 s}\prod_{\delta=0}^{t-2}\left(\mathbf{p}^{t-\delta}\right)_{s_0 8} + \sum_{\tau=0}^{t-3}\left({}_{s_0}^{m}b_{t-\tau-1}^s\prod_{\delta=0}^{\tau}\left(\mathbf{p}^{t-\delta}\right)_{s_0 8}\right), & t=3,...,m, \end{cases} \tag{5.36} $$

where

$$ {}_{s_0}^{m}y_t^l = \sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}_0^{m-t+l}\left(\mathbf{p}^m\right)_{s_0\rho} + \sum_{\tau=1}^{m-t+1}{}_{\rho}\tilde{q}_{m-t-\tau+2}^{m-t+l}\left(\mathbf{p}^{t+\tau-1}\right)_{s_0\rho}\right). \tag{5.37} $$

Proof.
(1) Consider the case $t=0$.

Since the usage for $t=0$ is zero, we have

$$ {}_{s_0}^{m}x_0^s = P_{s_0}\left(U_0 = u_s\right) = 0 \quad \text{for} \quad s \in S\backslash\{8\}. $$

and

$$ {}_{s_0}^{m}x_0^s = {}_{s_0}^{m}b_0^s + {}_{s_0}^{m}d_0^s = 0. $$

(2) Assume $t=1$.
From Lemma 5.2 and 5.3, we get

$$ {}_{s_0}^{m}x_1^s = {}_{s_0}^{m}b_1^s + {}_{s_0}^{m}d_1^s \quad \text{for} \quad s \in S\backslash\{8\} \tag{5.38} $$

with

$$ {}_{s_0}^{m}b_1^s = p_s\left(\mathbf{p}\right)_{s_0 s} \quad \text{and} \quad {}_{s_0}^{m}d_1^s = 0. $$

(3) For $t=2,...,m$, using Lemma 5.1 and 5.3, we get

$$ {}_{s_0}^{m}x_t^s = \left(\mathbf{p}^t\right)_{s_0 8}\left({}_{s_0}^{m}x_{t-1}^s\right) + {}_{s_0}^{m}b_t^s \quad \text{for} \quad s \in S\backslash\{8\}, \tag{5.39} $$

$$ {}_{s_0}^{m}x_t^s = 1 - \sum_{\rho=1}^{7}{}_{s_0}^{m}x_t^{\rho} \quad \text{for} \quad s \in \{8\}. \tag{5.40} $$

Define ${}_{s_0}^{m}\mathbf{x}_t = ({}_{s_0}^{m}x_t^1, ..., {}_{s_0}^{m}x_t^8)'$, ${}_{s_0}^{m}\mathbf{b}_t = ({}_{s_0}^{m}b_t^1, ..., {}_{s_0}^{m}b_t^7, 1)'$ and matrix ${}_{s_0}\mathbf{A}_t$:

$$ {}_{s_0}\mathbf{A}_t = \begin{pmatrix} (\mathbf{p}^t)_{s_0 8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\mathbf{p}^t)_{s_0 8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mathbf{p}^t)_{s_0 8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\mathbf{p}^t)_{s_0 8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\mathbf{p}^t)_{s_0 8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\mathbf{p}^t)_{s_0 8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{p}^t)_{s_0 8} & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}. $$

Then (5.39) and (5.40) can be rewritten as

$$ {}_{s_0}^{m}\mathbf{x}_t = {}_{s_0}\mathbf{A}_t \, {}_{s_0}^{m}\mathbf{x}_{t-1} + {}_{s_0}^{m}\mathbf{b}_t, $$

with initial value ${}_{s_0}^{m}\mathbf{x}_1$ and $t = 2, 3, 4, ..., m$.
For $t = 2$, from (5.39) and (5.38), we can easily get

$$ {}_{s_0}^{m}x_2^s = p_s\left(\mathbf{p}\right)_{s_0 s}\left(\mathbf{p}^2\right)_{s_0 8} + {}_{s_0}^{m}b_2^s. \tag{5.41} $$

Since on the other hand we have

$$ {}_{s_0}^{m}x_2^s = {}_{s_0}^{m}b_2^s + {}_{s_0}^{m}d_2^s, $$

we have that

$$ {}_{s_0}^{m}d_2^s = p_s\left(\mathbf{p}\right)_{s_0 s}\left(\mathbf{p}^2\right)_{s_0 8}. \tag{5.42} $$

For $t = 3, 4, ..., m$

$$ \begin{aligned} {}_{s_0}^{m}\mathbf{x}_t &= {}_{s_0}\mathbf{A}_t \, {}_{s_0}^{m}\mathbf{x}_{t-1} + {}_{s_0}^{m}\mathbf{b}_t \\ &= {}_{s_0}\mathbf{A}_t \left({}_{s_0}\mathbf{A}_{t-1} \, {}_{s_0}^{m}\mathbf{x}_{t-2} + {}_{s_0}^{m}\mathbf{b}_{t-1}\right) + {}_{s_0}^{m}\mathbf{b}_t = {}_{s_0}\mathbf{A}_t \, {}_{s_0}\mathbf{A}_{t-1} \, {}_{s_0}^{m}\mathbf{x}_{t-2} + {}_{s_0}\mathbf{A}_t \, {}_{s_0}^{m}\mathbf{b}_{t-1} + {}_{s_0}^{m}\mathbf{b}_t \\ &= {}_{s_0}\mathbf{A}_t \, {}_{s_0}\mathbf{A}_{t-1} \left({}_{s_0}\mathbf{A}_{t-2} \, {}_{s_0}^{m}\mathbf{x}_{t-3} + {}_{s_0}^{m}b_{t-2}\right) + {}_{s_0}\mathbf{A}_t \, {}_{s_0}^{m}\mathbf{b}_{t-1} + {}_{s_0}^{m}\mathbf{b}_t \\ &= {}_{s_0}\mathbf{A}_t \, {}_{s_0}\mathbf{A}_{t-1} \, {}_{s_0}\mathbf{A}_{t-2} \, {}_{s_0}^{m}\mathbf{x}_{t-3} + {}_{s_0}\mathbf{A}_t \, {}_{s_0}\mathbf{A}_{t-1} \, {}_{s_0}^{m}\mathbf{b}_{t-2} + {}_{s_0}\mathbf{A}_t \, {}_{s_0}^{m}\mathbf{b}_{t-1} + {}_{s_0}^{m}\mathbf{b}_t \\ &= \ldots\ldots \\ &= \left(\prod_{\delta=0}^{t-2} {}_{s_0}\mathbf{A}_{t-\delta}\right) {}_{s_0}^{m}\mathbf{x}_1 + \sum_{\tau=0}^{t-3}\left(\left(\prod_{\delta=0}^{\tau} {}_{s_0}\mathbf{A}_{t-\delta}\right) {}_{s_0}^{m}\mathbf{b}_{t-\tau-1}\right) + {}_{s_0}^{m}\mathbf{b}_t. \end{aligned} \tag{5.43} $$

Since

$$ \prod_{\delta=0}^{0} {}_{s_0}\mathbf{A}_{t-\delta} = {}_{s_0}\mathbf{A}_t \quad \text{and} \quad \prod_{\delta=0}^{\tau} {}_{s_0}\mathbf{A}_{t-\delta} = {}_{s_0}\mathbf{A}_t \left(\prod_{\delta=1}^{\tau}\left(\mathbf{p}^{t-\delta}\right)_{s_0 8}\right), \quad \tau \geq 1, $$

expression (5.43) can be rewritten as

$$ {}_{s_0}^{m}x_t^s = {}_{s_0}^{m}b_t^s + p_s\left(\mathbf{p}\right)_{s_0 s}\left(\prod_{\delta=0}^{t-2}\left(\mathbf{p}^{t-\delta}\right)_{s_0 8}\right) + \sum_{\tau=0}^{t-3}\left({}_{s_0}^{m}b_{t-\tau-1}^s \prod_{\delta=0}^{\tau}\left(\mathbf{p}^{t-\delta}\right)_{s_0 8}\right). \tag{5.44} $$

□

The expectation of usage amount of liquidity requirement of the single credit line is

$$ {}_{s_0}^{m}E_t := \mathrm{E}\left(\mathcal{F}_t\right) = \mathrm{E}\left(lU_t\right) = l\mathrm{E}\left(U_t\right) = l\left({}_{s_0}^{m}\mathbf{x}_t\right)' \mathbf{u}, $$

where $\mathbf{u} = (u_1, u_2, ..., u_8)$. The expectation of the liquidity requirement at time t is dependent on the initial rating of the customer s_0 and the maturity of the credit line m.

5.2 Estimation for Input Parameters

To plot the expectation of the liquidity requirement ${}^{m}_{s_0}E_t$, we need to estimate the following input parameters.

1. Estimated draw probability $\hat{p}_s$,
2. Estimated proportional usage $\hat{u}_s$,
3. Estimated joint return probability ${}_s\hat{\tilde{q}}^{m-t+l}_{m-t+1}$.

The estimated draw probability and the estimated proportional usage are the same as we have got in the simulation part. We only need to estimate the joint return probabilities ${}_s\hat{\tilde{q}}^{m-t+l}_{m-t+1}$. The joint return probability is defined as

$$ {}_s\tilde{q}^{m-t+l}_{m-t+1} = P\left(U_t = 0, U_{t-1} > 0, U_{t-2} > 0, ..., U_{t-l+1} > 0, U_{t-l} = 0 | R_t = s\right). \tag{5.45} $$

Notice that these probabilities are not grouped as those in Section 3.2, i.e. the probabilities are on a monthly basis instead of time bucket basis and on a normal credit rating units instead of credit rating buckets. To estimate the joint return probabilities, we first group them as described in Section 3.2. We estimate the grouped joint return probabilities ${}_s\hat{\tilde{q}}^{m-t+l}_{m-t+1}$ and map them inversely to joint return probabilities ${}_s\hat{\tilde{q}}^{m-t+l}_{m-t+1}$ as described below.

Step 1: Estimation of the grouped joint return probability.

1. For the j-th credit line of i-th customer, we count the number of the returns of the credit loans that have started $\bar{E}$ time buckets before maturity and returned $\bar{l}$ time buckets before the maturity and had a credit rating in the rating bucket $\bar{s}$ at the time the loan was returned: $N_{i,j,\bar{s},\bar{l},\bar{E}}$.
2. Sum up $N_{i,j,\bar{s},\bar{l},\bar{E}}$ over all the credit lines, so that we get the total number of the returns of the credit loans that have been started $\bar{E}$ time buckets before maturity and returned $\bar{l}$ time buckets before the maturity and had a credit rating in the rating bucket $\bar{s}$ at the time the loan was returned: $N_{\bar{s},\bar{l},\bar{E}} = \sum_{i=1}^{n}\sum_{j=1}^{n_i} N_{i,j,\bar{s},\bar{l},\bar{E}}$.
3. Sum up $N_{\bar{s},\bar{l},\bar{E}}$ over $\bar{l},\bar{E}$, so that we get the number of the returns of the credit loans with a credit rating in the rating bucket $\bar{s}$ at the time the loan was returned: $N_{\bar{s}} = \sum_{\bar{E}=1}^{12}\sum_{\bar{l}=0}^{\bar{E}} N_{\bar{s},\bar{l},\bar{E}}$.
4. The grouped joint return probability is estimated by: ${}_{\bar{s}}\hat{\tilde{q}}^{\bar{E}}_{\bar{l}} := \frac{N_{\bar{s},\bar{l},\bar{E}}}{N_{\bar{s}}}$.

Using the estimator, we can get the grouped joint return probability as following.

$\bar{l}$	$\bar{E}=1$	$\bar{E}=2$	$\bar{E}=3$	$\bar{E}=4$	$\bar{E}=5$	$\bar{E}=6$	$\bar{E}=7$	$\bar{E}=8$	$\bar{E}=9$	$\bar{E}=10$	$\bar{E}=11$	$\bar{E}=12$
0	0.024593	0.007149	0.002288	0.003289	0.001144	0.000143	0.000286	0.000000	0.000143	0.000286	0.000143	0.000000
1	0.590363	0.132399	0.023735	0.020160	0.004718	0.003146	0.001716	0.001859	0.001430	0.000429	0.000143	0.000143
2		0.068487	0.008722	0.004718	0.002002	0.001287	0.000715	0.001716	0.000572	0.000429	0.000000	0.000143
3			0.014584	0.005719	0.002288	0.000858	0.000572	0.000715	0.001144	0.000000	0.000000	0.000143
4				0.012725	0.004718	0.001144	0.000715	0.001144	0.000572	0.001001	0.000143	0.000000
5					0.003860	0.001001	0.000715	0.001430	0.000572	0.002288	0.000143	0.000000
6						0.003146	0.000429	0.000715	0.000429	0.000572	0.000286	0.000143
7							0.002431	0.002860	0.000858	0.000572	0.002288	0.000286
8								0.002860	0.000715	0.001001	0.000429	0.000143
9									0.001430	0.002288	0.000429	0.000858
10										0.002574	0.002002	0.000572
11											0.001144	0.001144
12												0.000715

Figure 5.1: Grouped joint return probability for customer with $\bar{s} = 1$.

$\bar{l}$	$\bar{E}=1$	$\bar{E}=2$	$\bar{E}=3$	$\bar{E}=4$	$\bar{E}=5$	$\bar{E}=6$	$\bar{E}=7$	$\bar{E}=8$	$\bar{E}=9$	$\bar{E}=10$	$\bar{E}=11$	$\bar{E}=12$
0	0.027602	0.006853	0.002076	0.003189	0.000231	0.000231	0.000109	0.000190	0.000068	0.000095	0.000000	0.000027
1	0.655783	0.108510	0.019569	0.022161	0.002253	0.001520	0.000624	0.001181	0.000258	0.000231	0.000027	0.000014
2		0.061651	0.009092	0.005360	0.000760	0.000394	0.000122	0.000176	0.000204	0.000095	0.000014	0.000000
3			0.014724	0.009242	0.001072	0.000461	0.000149	0.000217	0.000244	0.000122	0.000000	0.000000
4				0.016258	0.002131	0.000679	0.000339	0.000366	0.000176	0.000489	0.000014	0.000054
5					0.004383	0.001520	0.000271	0.000665	0.000271	0.000258	0.000068	0.000054
6						0.001927	0.000706	0.000746	0.000176	0.000217	0.000054	0.000068
7							0.001371	0.000801	0.000285	0.000326	0.000054	0.000041
8								0.001927	0.000489	0.000366	0.000041	0.000014
9									0.001520	0.000936	0.000136	0.000081
10										0.001316	0.000353	0.000109
11											0.000774	0.000054
12												0.000217

Figure 5.2: Grouped joint return probability for customer with $\bar{s} = 2$.

$\bar{l}$	$\bar{E}=1$	$\bar{E}=2$	$\bar{E}=3$	$\bar{E}=4$	$\bar{E}=5$	$\bar{E}=6$	$\bar{E}=7$	$\bar{E}=8$	$\bar{E}=9$	$\bar{E}=10$	$\bar{E}=11$	$\bar{E}=12$
0	0.048697	0.009602	0.003429	0.003429	0.018519	0.000686	0.000000	0.000686	0.000000	0.000000	0.000000	0.000000
1	0.517833	0.122085	0.023320	0.063100	0.001372	0.000686	0.000000	0.000000	0.000000	0.000000	0.000000	0.000686
2		0.087791	0.017147	0.008230	0.000000	0.000000	0.000000	0.000000	0.000686	0.000000	0.000000	0.000000
3			0.017147	0.010974	0.004801	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4				0.015089	0.001372	0.000000	0.001372	0.000000	0.000000	0.000686	0.000000	0.000000
5					0.001372	0.000686	0.000686	0.000000	0.000000	0.000686	0.000000	0.000000
6						0.000686	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7							0.000000	0.000000	0.000000	0.000686	0.000000	0.000000
8								0.000686	0.000686	0.000000	0.000686	0.001372
9									0.008916	0.000000	0.000000	0.000000
10										0.000686	0.000000	0.000686
11											0.000000	0.000686
12												0.001372

Figure 5.3: Grouped joint return probability for customer with $\bar{s} = 3$.

$\bar{l}$	$\bar{E}=1$	$\bar{E}=2$	$\bar{E}=3$	$\bar{E}=4$	$\bar{E}=5$	$\bar{E}=6$	$\bar{E}=7$	$\bar{E}=8$	$\bar{E}=9$	$\bar{E}=10$	$\bar{E}=11$	$\bar{E}=12$
0	0.566529	0.219479	0.061043	0.100823	0.027435	0.002743	0.002058	0.001372	0.010288	0.002743	0.000686	0.004801
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3			0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4				0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5					0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6						0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7							0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
8								0.000000	0.000000	0.000000	0.000000	0.000000
9									0.000000	0.000000	0.000000	0.000000
10										0.000000	0.000000	0.000000
11											0.000000	0.000000
12												0.000000

Figure 5.4: Grouped joint return probability for customer with $\bar{s} = 4$.

Step 2: Mapping between the grouped joint return probability and the joint return probability. The grouped joint return probabilities in Table 5.1, 5.2, 5.3 and 5.4 are to be mapped to joint return probabilities. Three different situations described below are possible.

1. Default situation ($\bar{l} = 0$ and $\bar{E} = 1, 2, ..., 8$, or $\bar{l} = 0$ and $\bar{E} = 9, ..., 12$).
   The default situation corresponds to the first line in the table of grouped joint return probabilities. Each combination of $\bar{l}, \bar{E}$ can represent 3 sub-situations (for $\bar{l} = 0$ and $\bar{E} = 1, ..., 8$) or 6 sub-situations (for $\bar{l} = 0$ and $\bar{E} = 9, ..., 12$). In order to describe the pattern that can arise for different combination of $\bar{l}$ and $\bar{E}$, we consider the case $\bar{l} = 0$, $\bar{E} = 1$ and $\bar{s} = 1$. Following sub-situations may occur.

1) The customer starts the credit line 3 months before the maturity and the credit loan is not returned till the maturity.

2) The customer starts the credit line 2 months before the maturity and the credit loan is not returned till the maturity.

3) The customer starts the credit line 1 month before the maturity and the credit loan is not returned till the maturity.

   Please refer to Figure 5.5.

	$\bar{E}=1$
$\bar{l}=0$	${}_1\hat{\tilde{q}}_0^{1}$

⇩

	$E=1$	$E=2$	$E=3$
$l=0$	${}_1\hat{\tilde{q}}_0^{1} = \frac{1}{3}\,{}_1\hat{\tilde{q}}_0^{1}$	${}_1\hat{\tilde{q}}_0^{2} = \frac{1}{3}\,{}_1\hat{\tilde{q}}_0^{1}$	${}_1\hat{\tilde{q}}_0^{3} = \frac{1}{3}\,{}_1\hat{\tilde{q}}_0^{1}$

Figure 5.5: Mapping between the grouped joint return probability and joint return probability for $\bar{l} = 0$ and $\bar{E} = 1$.

For simplification, we assume that the joint probabilities in each of the sub-situations are equal (${}_{\bar{s}}\hat{\tilde{q}}_0^{1} = {}_{\bar{s}}\hat{\tilde{q}}_0^{2} = {}_{\bar{s}}\hat{\tilde{q}}_0^{3}$) and the sum of the probabilities in each sub-situation is equal to the probability of the grouped situation ($\sum_{E=1}^{3} {}_{\bar{s}}\hat{\tilde{q}}_0^{E} = {}_{\bar{s}}\hat{q}_0^{\bar{E}}$).

So we set

$$ {}_{\bar{s}}\hat{\tilde{q}}_l^{E} := \frac{1}{3}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}} \quad \text{for} \quad l = 0, E = 1, 2, 3. $$

In general, we set

$$ {}_{\bar{s}}\hat{\tilde{q}}_l^{E} := \begin{cases} \frac{1}{3}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l = 0, E = 1, ..., 8, \\ \frac{1}{6}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l = 0, E = 9, ..., 12. \end{cases} $$

2. Diagonal situation ($\bar{l} = \bar{E} = 1, ..., 8$ or $\bar{l} = \bar{E} = 9, ..., 12$).
   The diagonal situation corresponds the diagonal of the table of grouped joint return probabilities without the first line. Each combination of $\bar{E}, \bar{l}$ contains 6 sub-situations for $\bar{l} = \bar{E} = 1, ..., 8$ or 21 sub-situations for $\bar{l} = \bar{E} = 9, ..., 12$. We take $\bar{l} = \bar{E} = 1$ with $\bar{s} = 1$ as example. It represents the following sub-situations.

1) The customer starts the credit line 3 months before the maturity and returns it 3, 2 or 1 month before maturity.

2) The customer starts the credit line 2 months before the maturity and returns it 2 or 1 month before the maturity.

3) The customer starts the credit line in the last month and returns it in the same month.

   Please refer to Figure 5.6. The general inverse mapping is:

$$
{}_{\bar{s}}\hat{\tilde{q}}_{l}^{E} := \begin{cases} \frac{1}{6}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l = E = 1, ..., 8, \\ \frac{1}{21}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l = E = 9, ..., 12. \end{cases}
$$

	$\bar{E}=1$
$\bar{l}=1$	${}_1\hat{q}_1^1$

	$E=1$	$E=2$	$E=3$
$l=1$	${}_1\hat{\tilde{q}}_1^1 = \frac{1}{6}\,{}_1\hat{q}_1^1$	${}_1\hat{\tilde{q}}_1^2 = \frac{1}{6}\,{}_1\hat{q}_1^1$	${}_1\hat{\tilde{q}}_1^3 = \frac{1}{6}\,{}_1\hat{q}_1^1$
$l=2$		${}_1\hat{\tilde{q}}_2^2 = \frac{1}{6}\,{}_1\hat{q}_1^1$	${}_1\hat{\tilde{q}}_2^3 = \frac{1}{6}\,{}_1\hat{q}_1^1$
$l=3$			${}_1\hat{\tilde{q}}_3^3 = \frac{1}{6}\,{}_1\hat{q}_1^1$

Figure 5.6: Mapping between the grouped joint return probability and joint return probability for $\bar{l} = 1$ and $\bar{E} = 1$.

   Similarly, for $\bar{l} = \bar{E} = 9, ..., 12$, the total number of sub-situations for each combination is $1 + 2 + ... + 6 = 21$.

3. Normal situation ($\bar{E} = 1, ..., 8$ and $\bar{l} = 1, ..., E - 1$, or $\bar{E} = 9, ..., 12$ and $\bar{l} = 1, ..., E - 1$).
   Each combination of $\bar{E}, \bar{l}$ contains 9 sub-situations ($\bar{E} = 1, ..., 8$ and $\bar{l} = 1, ..., E - 1$) or 36 sub-situations ($\bar{E} = 9, ..., 12$ and $\bar{l} = 1, ..., E - 1$). We take $\bar{l} = 1$ and $\bar{E} = 2$ with $\bar{s} = 1$ as example. It represents the following sub-situations.

1) The customer starts the credit line 6 months before the maturity and returns it 3, 2 or 1 month before the maturity.

2) The customer starts the credit line 5 months before the maturity and returns it 3, 2 or 1 month before the maturity.

3) The customer starts the credit line 4 months before the maturity and returns it 3, 2 or 1 month before the maturity.

   Please refer to Figure 5.7. The general mapping is:

$$
{}_{\bar{s}}\hat{\tilde{q}}_l^{E} := \begin{cases} \frac{1}{9}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & \bar{E}=1,...,8 \text{ and } \bar{l}=1,...,E-1, \\ \frac{1}{36}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & \bar{E}=9,...,12 \text{ and } \bar{l}=1,...,E-1. \end{cases}
$$

	$\bar{E}=2$
$\bar{l}=1$	${}_1\hat{\tilde{q}}_1^2$

	$E=4$	$E=5$	$E=6$
$l=1$	${}_1\hat{\tilde{q}}_1^4=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$	${}_1\hat{\tilde{q}}_1^5=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$	${}_1\hat{\tilde{q}}_1^6=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$
$l=2$	${}_1\hat{\tilde{q}}_2^4=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$	${}_1\hat{\tilde{q}}_2^5=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$	${}_1\hat{\tilde{q}}_2^6=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$
$l=3$	${}_1\hat{\tilde{q}}_3^4=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$	${}_1\hat{\tilde{q}}_3^5=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$	${}_1\hat{\tilde{q}}_3^6=\frac{1}{9}{}_1\hat{\tilde{q}}_1^2$

Figure 5.7: Mapping between the grouped joint return probability and joint return probability for $\bar{l}=1$ and $\bar{E}=2$.

Furthermore, we assume that ${}_{\bar{s}}\hat{\tilde{q}}_l^{E} = {}_{s}\hat{\tilde{q}}_l^{E}$ if $\bar{s}=c(s)$ (customer within the same credit rating buckets share the same joint return probability given that they have the same E, l). So we can summarize the mapping between the grouped joint return probability and joint return probability as following.

$$
{}_{s}\hat{\tilde{q}}_l^{E} := \begin{cases} \frac{1}{3}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l=0, E=1,...,8, \\ \frac{1}{6}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l=0, E=9,...,12. \\ \frac{1}{6}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l=E=1,...,8, \\ \frac{1}{21}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & l=E=9,...,12. \\ \frac{1}{9}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & \bar{E}=1,...,8 \text{ and } \bar{l}=1,...,E-1, \\ \frac{1}{36}\,{}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}, & \bar{E}=9,...,12 \text{ and } \bar{l}=1,...,E-1. \end{cases} \tag{5.46}
$$

where $s \in S$ with credit rating bucket $\bar{s}$, and ${}_{\bar{s}}\hat{q}_{\bar{l}}^{\bar{E}}$ is the grouped joint return probability that we have estimated in step 1.

5.3 Empirical analysis

In this section, we plot the time series of the expectation of the total usage, the default usage and the no default usage.

Suppose our customer has an initial credit rating of $s_0 = 7$ (CCC) and the maturity of the credit line is 1 year ($m = 12$). The expectation of usage U_t (or ${}^{m}_{s_0}x^{s}_{t}$), the default usage ${}^{m}_{s_0}d^{s}_{t}$ and the no default usage ${}^{m}_{s_0}b^{s}_{t}$ are given in Figure 5.8.

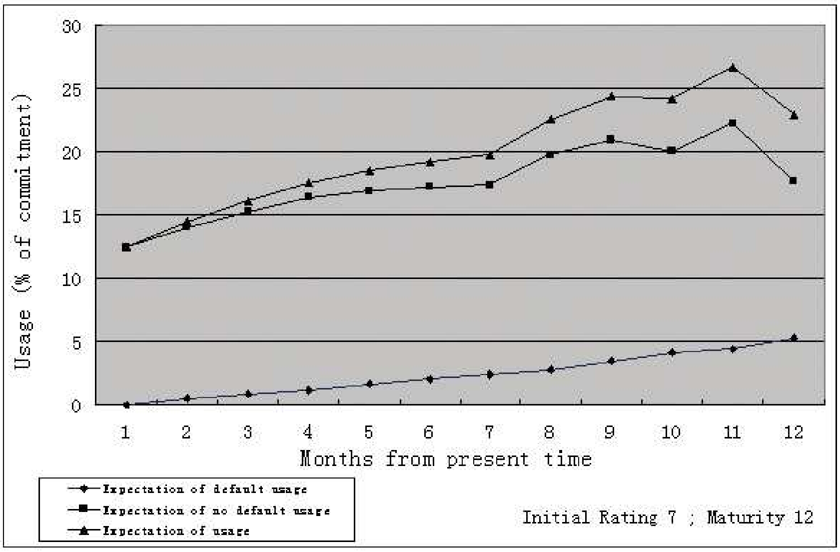

Figure 5.8: Expectation of U_t(or ${}^{m}_{s_0}x^{s}_{t}$), ${}^{m}_{s_0}b^{s}_{t}$ and ${}^{m}_{s_0}d^{s}_{t}$.

The default usage is caused by the customers who default during the usage of their credit lines never returning the loan. Thus the default usage always increases in time. The no default part is to be repaid and we notice that it decreases as the maturity comes near. In Fig 5.9, the usage of customers with different initial credit ratings are plotted, in Fig 5.10 the default usage and in Fig 5.11 the no default usage.

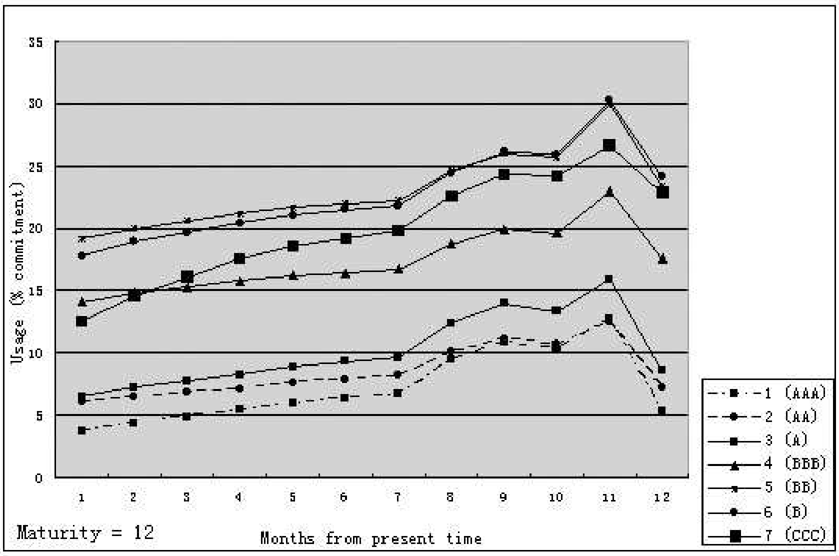

Figure 5.9: Expectation of U_t (or ${}^{m}_{s_0}x^{s}_{t}$) for different initial credit rating levels.

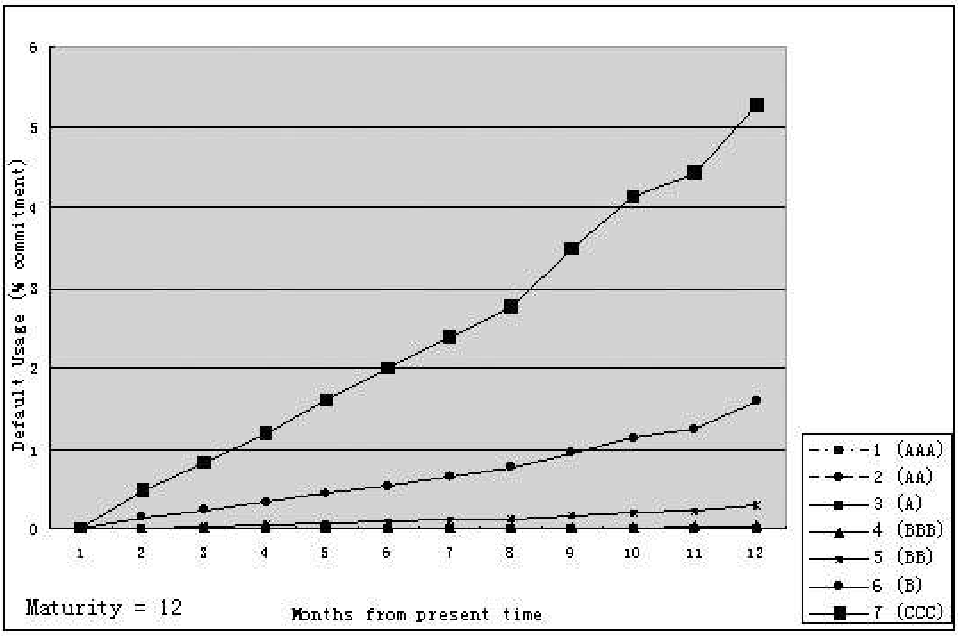

Figure 5.10: Expectation of default usage as proportion of total committed limit (for different credit rating levels).

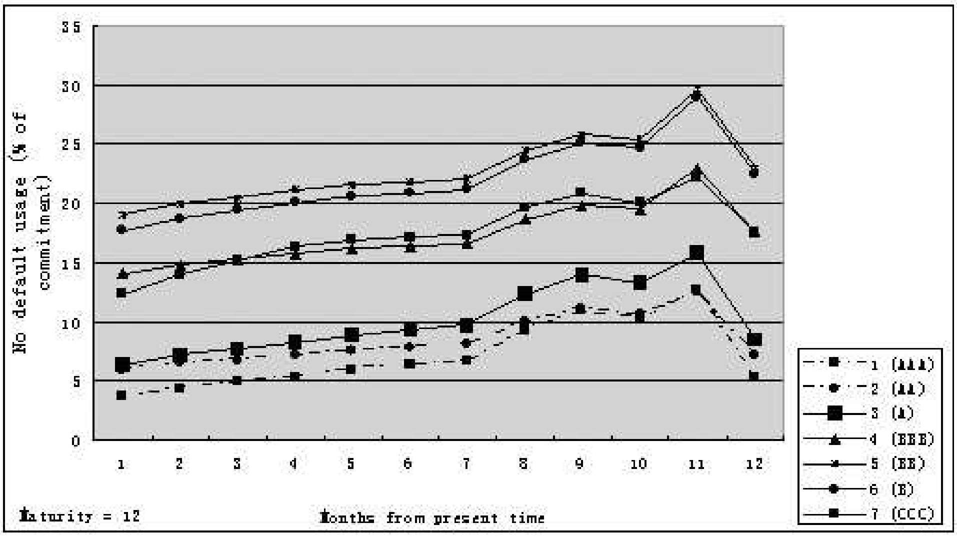

Figure 5.11: Expectation of no default usage as proportion of total committed limit (for different credit rating levels).

Chapter 6

Distribution of Liquidity Requirement for a Single Credit Line in Multiple Maturity Situation

6.1 Distribution of Usage in Multiple Period

The distribution of usage in multiple periods is different from that in single period in that the renewal rule and the expiration rule take effect every time the credit line is matured.

Assumption 6.1. *If the customer still has credit loan u_s in use with $s \in S\backslash\{8\}$ at the maturity $t = km$, $k \in \mathbb{N}$, the customer never pays back the loan in the following periods. So the usage at each time point in the following time period is the same as that at time $t = km$.*

$$P_{s_0}\left(U_t = u_s | U_{km} = u_s\right) = 1,$$
$$P_{s_0}\left(U_t = u_s, U_{t-1} = u_s, ..., U_{km+1} = u_s | U_{km} = u_s\right) = 1,$$

for $t = km+1, km+2, ..., (k+1)m$ where $s \in S\backslash\{8\}$ and s_0 is the initial credit rating.

Assumption 6.2. *If the customer is not in use of credit loan at the maturity $U_t = u_8 = 0$ and $t = km$, $k \in \mathbb{N}$ having a credit rating $s_k = 6, 7, 8$ at maturity $t = km$, then the customer will not be allowed to reserve the credit line in the future and the usage of the credit loan will always be zero.*

$$P_{s_0}\left(U_t = u_8 | U_{km} = u_8, R_{km} = s_k\right) = 1, \quad t = km+1, km+2, ..., (k+1)m$$

where $s_k \in \{6, 7, 8\}$ and s_0 is the initial credit rating.

Assumption 6.3. *If the customer in not in use of credit loan at the maturity $U_t = u_8 = 0$, $t = km$, $k \in \mathbb{N}$, and has a credit rating $s_k = 1, 2, 3, 4, 5$ at $t = km$, then the customer will be allowed to reserve the credit line from $t = km+1$ and the following is assumed.*

$$P_{s_0}\left(U_t = u_s | U_{km} = u_8, R_{km} = s_k\right) = {}^{m}_{s_k}x^{s}_{t-km}, \quad t = km+1, km+2, ..., (k+1)m$$

where $s_k \in \{1, 2, 3, 4, 5\}$ and s_0 is the initial credit rating.

Lemma 6.1. *Suppose the initial credit rating of the customer is s_0, the maturity of the credit line is m. Last maturity is $t = km$, $k \in \mathbb{N}$ and we know the transition matrix $\mathbf{P}$. Then we have*

$$P_{s_0}\left(U_t = u_s\right) = {}^{m}_{s_0}x^{s}_{km} + \sum_{s_k=1}^{5} {}^{m}_{s_k}x^{s}_{t-km}\left(\left(\mathbf{p}^{km}\right)_{s_0 s_k} - P_{s_0}\left(U_{km} > 0, R_{km} = s_k\right)\right),$$

for $t = km+1, ..., (k+1)m$ where $s \in S \setminus \{8\}$.

Proof. Obviously we have for $s \in S \setminus \{8\}$,

$$P_{s_0}\left(U_t = u_s\right) = P_{s_0}\left(U_t = u_s, U_{km} = u_s\right) + P_{s_0}\left(U_t = u_s, U_{km} = u_8\right) + \sum_{\rho \neq s, 8} P_{s_0}\left(U_t = u_s, U_{km} = u_\rho\right).$$

According to Assumption 6.1, we have $P_{s_0}\left(U_t = u_s, U_{km} = u_\rho\right) = 0$, $\rho \in S \setminus \{s_0, 8\}$. Thus

$$\begin{aligned}
&P_{s_0}\left(U_t = u_s\right)\\
&= P_{s_0}\left(U_t = u_s, U_{km} = u_s\right) + P_{s_0}\left(U_t = u_s, U_{km} = u_8\right)\\
&= P_{s_0}\left(U_t = u_s | U_{km} = u_s\right) P_{s_0}\left(U_{km} = u_s\right) + P_{s_0}\left(U_t = u_s, U_{km} = u_8\right).
\end{aligned}$$

Again using Assumption 6.1, we get $P_{s_0}\left(U_t = u_s | U_{km} = u_s\right) = 1$ for $s \in S \setminus \{8\}$, so

$$\begin{aligned}
&P_{s_0}\left(U_t = u_s\right) = P_{s_0}\left(U_{km} = u_s\right)\\
&+ \sum_{s_k=6}^{8} P_{s_0}\left(U_t = u_s, U_{km} = u_8, R_{km} = s_k\right) + \sum_{s_k=1}^{5} P_{s_0}\left(U_t = u_s, U_{km} = u_8, R_{km} = s_k\right).
\end{aligned}$$

(1) For the first term on the right hand side, we have $P_{s_0}\left(U_{km} = u_s\right) = {}_{s_0}x^{s}_{km}$.
(2) For the second term on the right hand side, we use Assumption 6.2 and get

$$\sum_{s_k=6}^{8} P_{s_0}\left(U_t = u_s, U_{km} = u_8, R_{km} = s_k\right) = 0.$$

(3) For the third term on the right hand side, we use the Assumption 6.3 and get

$$\begin{aligned}
&\sum_{s_k=1}^{5} P_{s_0}\left(U_t = U_s, U_{km} = U_8, R_{km} = s_k\right)\\
&= \sum_{s_k=1}^{5} P_{s_0}\left(U_t = U_s | U_{km} = U_8, R_{km} = s_k\right) P_{s_0}\left(U_{km} = U_8, R_{km} = s_k\right)\\
&= \sum_{s_k=1}^{5} {}^{m}_{s_k}x^{s}_{t-km}\left(P_{s_0}\left(R_{km} = s_k\right) - P_{s_0}\left(U_{km} > 0, R_{km} = s_k\right)\right)\\
&= \sum_{s_k=1}^{5} {}^{m}_{s_k}x^{s}_{t-km}\left(\left(\mathbf{p}^{km}\right)_{s_0 s_k} - P_{s_0}\left(U_{km} > 0, R_{km} = s_k\right)\right).
\end{aligned}$$

□

For simplicity, we denote ${}_{s_0}^{m}b_{km}^{s} = P_{s_0}\left(U_{km} > 0, R_{km} = s_k\right)$. (Please compare it with the definition of no default usage probability in single period ${}_{s_0}^{m}b_{t}^{s} = P_{s_0}\left(U_t > 0, R_t = s\right)$, $t = 1, ..., m$). It represents the probability that the customer is not default at the k-th maturity $t = km$, but still in use of the credit loan. The explicit form of ${}_{s_0}^{m}b_{km}^{s}$ is derived in following lemma.

Lemma 6.2. *Let k, s_k and s_0 be given, then*

$$\begin{aligned}
&{}_{s_0}^{m}b_{km}^{s_k} = P_{s_0}\left(U_{km} > 0, R_{km} = s_k\right)\\
&= \left(\mathbf{p}^{km}\right)_{s_0 s_k}\left(p_{s_k} + \sum_{\delta=2}^{m}\left({}_{s_0}^{m}y_m^{\delta} - {}_{s_k}q_1^{\delta}\right) + \sum_{\tau=1}^{k-1}\sum_{\delta=\tau m+1}^{(\tau+1)m}\sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}_0^{\delta-\tau m}\left(\mathbf{p}^{(k-\tau)m}\right)_{s_0\rho}\right)\right),
\end{aligned} \tag{6.1}$$

for $s_k \in S\backslash\{8\}$ where ${}_{s_0}^{m}y_m^{\delta}$ is defined in (5.27) with $t = m$ and $l = \delta$. It stands for the probability that the customer starts a credit line at time $m - \delta + 1$ and till time $m - 1$, he is still in use of the credit line.

Proof. Obviously for $k \geq 2$, we have

$$\begin{aligned}
&P_{s_0}\left(R_{km} = s_k, U_{km} > 0\right)\\
&= \sum_{\delta=1}^{km} P_{s_0}\left(R_{km} = s_k, U_{km} > 0, ..., U_{km-\delta+1} = 0, U_{km-\delta} = 0\right)\\
&= P_{s_0}\left(R_{km} = s_k, U_{km} > 0, U_{km-1} = 0\right)\\
&+ \sum_{\delta=2}^{m} P_{s_0}\left(R_{km} = s_k, U_{km} > 0, ..., U_{km-\delta+1} = 0, U_{km-\delta} = 0\right)\\
&+ \sum_{\delta=m+1}^{2m} P_{s_0}\left(R_{km} = s_k, U_{km} > 0, ..., U_{km-\delta+1} = 0, U_{km-\delta} = 0\right)+\\
&\ldots\ldots\\
&+ \sum_{\delta=(k-1)m+1}^{km} P_{s_0}\left(R_{km} = s_k, U_{km} > 0, ..., U_{km-\delta+1} = 0, U_{km-\delta} = 0\right).
\end{aligned}$$

(1) For $\delta = 1$,

$$\begin{aligned}
&P_{s_0}\left(R_{km} = s_k, U_{km} > 0, U_{km-1} = 0\right)\\
&= P_{s_0}\left(U_{km} > 0, U_{km-1} = 0 | R_{km} = s_k\right) P_{s_0}\left(R_{km} = s_k\right) = p_{s_k}\left(\mathbf{p}^{km}\right)_{s_0 s_k}.
\end{aligned} \tag{6.2}$$

(2) For $\delta = 2, 3, ..., m$,

$$\begin{aligned}
&P_{s_0}\left(R_{km} = s_k, U_{km} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right)\\
&= P_{s_0}\left(R_{km} = s_k, U_{km-1} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right)\\
&-P_{s_0}\left(R_{km} = s_k, U_{km} = 0, U_{km-1} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right).
\end{aligned} \tag{6.3}$$

For the first term on the right hand side, we have

$$\begin{aligned}
&P_{s_0}\left(R_{km} = s_k, U_{km-1} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right)\\
&= P_{s_0}\left(R_{km} = s_k\right) P_{s_0}\left(U_{km-1} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right)\\
&= \left(\mathbf{p}^{km}\right)_{s_0 s_k} P_{s_0}\left(U_{km} = 0, U_{km-1} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right) +\\
&\left(\mathbf{p}^{km}\right)_{s_0 s_k} P_{s_0}\left(U_{km} > 0, U_{km-1} > 0, ..., U_{km-\delta+1} > 0, U_{km-\delta} = 0\right)\\
&= \left(\mathbf{p}^{km}\right)_{s_0 s_k} \sum_{\rho=1}^{8}\left(\mathbf{p}^{km}\right)_{s_0\rho}\left({}_{\rho}\tilde{q}_1^{\delta} + {}_{\rho}\tilde{q}_0^{\delta}\right) = \left(\mathbf{p}^{km}\right)_{s_0 s_k}\, {}_{s_0}^{m}y_m^{\delta},
\end{aligned}$$

in which we used Assumption 5.3 in the first step and the definition of joint return probability (5.3), (5.4) in the third step, and the definition of ${}^{m}_{s_0}y^{\delta}_{m}$ in (5.27) in the last step.

For the second term on the right hand side, we have

$$\begin{aligned}
&P_{s_0}\left(R_{km}=s_k,U_{km}=0,U_{km-1}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\
&=P_{s_0}\left(U_{km}=0,U_{km-1}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0|R_{km}=s_k\right)P_{s_0}\left(R_{km}=s_k\right)\\
&={}_{s_k}\tilde{q}^{\delta}_{1}\left(\mathbf{p}^{km}\right)_{s_0s_k}.
\end{aligned}$$

So we get for $\delta=2,3,...,m$,

$$P_{s_0}\left(R_{km}=s_k,U_{km}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)=\left(\mathbf{p}^{km}\right)_{s_0s_k}\left({}^{m}_{s_0}y^{\delta}_{m}-{}_{s_k}\tilde{q}^{\delta}_{1}\right).$$

(3) For $\delta\geq m+1$

$$\begin{aligned}
&\sum_{\delta=\tau m+1}^{(\tau+1)m}P_{s_0}\left(R_{km}=s_k,U_{km}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\
&=\sum_{\delta=\tau m+1}^{(\tau+1)m}\begin{pmatrix}P_{s_0}\left(R_{km}=s_k,U_{km-1}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\-P_{s_0}\left(R_{km}=s_k,U_{km}=0,U_{km-1}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\end{pmatrix}\\
&=\sum_{\delta=\tau m+1}^{(\tau+1)m}P_{s_0}\left(R_{km}=s_k,U_{km-1}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\
&=\sum_{\delta=\tau m+1}^{(\tau+1)m}P_{s_0}\left(R_{km}=s_k\right)P_{s_0}\left(U_{km-1}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\
&=\sum_{\delta=\tau m+1}^{(\tau+1)m}\begin{pmatrix}P_{s_0}\left(R_{km}=s_k\right)\\\times P_{s_0}\left(U_{km-1}>0,...,U_{(k-\tau)m+1}>0|U_{(k-\tau)m}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\\times P_{s_0}\left(U_{(k-\tau)m}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\end{pmatrix}\\
&=\sum_{\delta=\tau m+1}^{(\tau+1)m}P_{s_0}\left(R_{km}=s_k\right)P_{s_0}\left(U_{(k-\tau)m}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0\right)\\
&=\sum_{\delta=\tau m+1}^{(\tau+1)m}\begin{pmatrix}\left(\mathbf{p}^{km}\right)_{s_0s_k}\\\times\sum_{\rho=1}^{8}P_{s_0}\left(U_{(k-\tau)m}>0,...,U_{km-\delta+1}>0,U_{km-\delta}=0|R_{(k-\tau)m}=\rho\right)P_{s_0}\left(R_{(k-\tau)m}=\rho\right)\end{pmatrix}\\
&=\left(\mathbf{p}^{km}\right)_{s_0s_k}\sum_{\delta=\tau m+1}^{(\tau+1)m}\left(\sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}^{\delta-\tau m}_{0}\left(\mathbf{p}^{(k-\tau)m}\right)_{s_0\rho}\right)\right).
\end{aligned}\tag{6.4}$$

With (6.2) , (6.3) and (6.4), we get

$${}^{m}_{s_0}b^{s_k}_{km}=\left(\mathbf{p}^{km}\right)_{s_0s_k}\left(p_{s_k}+\sum_{\delta=2}^{m}\left({}^{m}_{s_0}y^{\delta}_{m}-{}_{s_k}\tilde{q}^{\delta}_{1}\right)+\sum_{\tau=1}^{k-1}\sum_{\delta=\tau m+1}^{(\tau+1)m}\sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}^{\delta-\tau m}_{0}\left(\mathbf{p}^{(k-\tau)m}\right)_{s_0\rho}\right)\right),$$

where $s_k\in S\backslash\{8\}$. □

From Lemma 6.1 and 6.2, we can easily get distribution of usage for multiple period.

Theorem 6.1 (Distribution of usage for multiple maturity case)**.** *Suppose the distribution of usage* ${}^{m}_{s_0}x^{s}_{t}$ *for* $t = 1, ..., km$*, the draw probability* p_s*, the joint return probability* ${}_{s_0}\tilde{q}^{E}_{l}$ *are known, where* $s_0 \in S$ *is the initial credit rating,* $s \in S$ *is the credit rating of customer at time* t *and* $m \in M$ *is the maturity of the credit line. Then*

$$
{}^{m}_{s_0}x^{s}_{t} = \begin{cases} {}^{m}_{s_0}x^{s}_{km} + \sum\limits_{s_k=1}^{5} {}^{m}_{s_k}x^{s}_{t-km}\left(\left(\mathbf{p}^{km}\right)_{s_0 s_k} - {}^{k}_{s_0}b^{s_k}_{m}\right), & s \in S \backslash \{8\}, \\ 1 - \sum\limits_{\rho=1}^{7} {}^{m}_{s_0}x^{\rho}_{t}, & s \in \{8\}, \end{cases}
$$

where

$$
{}^{m}_{s_0}b^{s_k}_{km} = \left(\mathbf{p}^{km}\right)_{s_0 s_k}\left(p_{s_k} + \sum_{\delta=2}^{m}\left({}^{m}_{s_0}y^{\delta}_{m} - {}_{s_k}\tilde{q}^{\delta}_{1}\right) + \sum_{\tau=1}^{k-1}\sum_{\delta=\tau m+1}^{(\tau+1)m}\sum_{\rho=1}^{8}\left({}_{\rho}\tilde{q}^{\delta-\tau m}_{0}\left(\mathbf{p}^{(k-\tau)m}\right)_{s_0 \rho}\right)\right)
$$

for $t = km+1, km+2, ..., (k+1)m$ *and* $k \geq 2$*.*

Remark 6.1. Theorem 6.1 gives us an iterative way to calculate the distribution of usage period by period. Suppose we already know the initial credit rating s_0 of a customer and the maturity of the credit line m, we can summarize our calculation of distribution of usage of a single credit line in multiple period as following.

1. Calculate ${}^{m}_{s_0}y^{\delta}_{m}$ for $\delta = 1, ..., m$.
2. Calculate the no default probability ${}^{m}_{s_0}b^{s}_{t}$ for $s \in S$ and $t = 1, ..., m$.
3. Calculate the default probability ${}^{m}_{s_0}d^{s}_{t}$ for $s \in S$ and $t = 1, ..., m$.
4. Calculate the distribution of the usage in the first period: ${}^{m}_{s_0}x^{s}_{t}$ for $s \in S$ and $t = 1, ..., m$.
5. Calculate the no default usage probability at each following maturity ${}^{m}_{s_0}b^{s_k}_{km}$ where $s \in \{1,2,3,4,5\}$ and $k = 2, 3, ...$
6. Use ${}^{m}_{s_0}b^{s_k}_{m}$, $s_k \in \{1,2,3,4,5\}$ and ${}^{m}_{s_0}x^{s}_{m}$, $s \in S$ to calculate the distribution of the usage in the second period ${}^{m}_{s_0}x^{s}_{t}$ for $s \in S$ and $t = m+1, ..., 2m$.
7. Use ${}^{m}_{s_0}b^{s_k}_{2m}$, $s_k \in \{1,2,3,4,5\}$ and ${}^{m}_{s_0}x^{s}_{2m}$, $s \in S$ to calculate the distribution of the usage in the third period ${}^{m}_{s_0}x^{s}_{t}$ for $s \in S$ and $t = 2m+1, ..., 3m$.
8. Repeat till the last period.

6.2 Simulation I

Assume the transition matrix, draw probability and joint return probability are the same as in the single period situation. Our worst credit rating to be renewed at maturity is $s_c = 5$ (BB). The credit line has a maturity of 12 months: $m = 12$. Our simulation period is 48 months, which covers 4 maturities: $k = 4$. We run Monte Carlo simulation with these input parameters.

From Theorem 6.1 we get the curve of the expectation of usage $E[U_t]$ directly. For the Monte Carlo simulation, we replicate 1000 realizations of the usage at each time and get the average usage $\bar{U}_t$. Both curves are plotted in Figure 6.1.

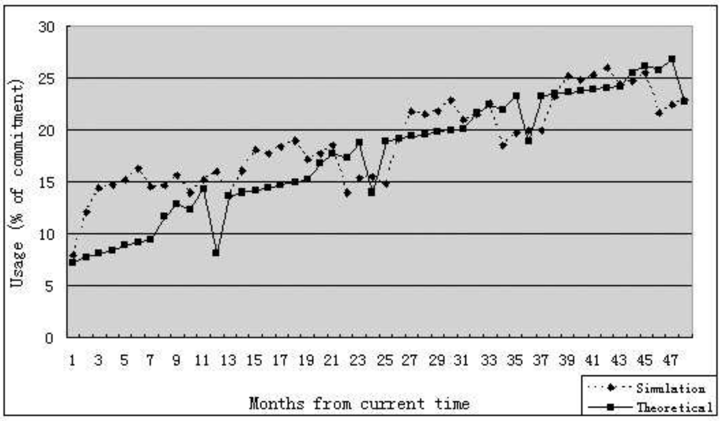

Figure 6.1: Comparison between the Monte Carlo simulation and theoretical analysis.

Remark 6.2. From the comparison, we can see a consistence between the theoretical and simulation result in the sense of trend, model and magnitude level.

Remark 6.3. There exists a bias between the two curves. The curve resulting from simulation is above the curve that comes from theory. The reason lies in the return probabilities. In theorem we use the joint return probability, which is mapped from the grouped return probability. And for simplicity, we have assumed that the joint return probabilities in each month in the same time bucket are the same. We use this simplification due to lack of historical data and consequently, it leads to the under-estimation or over-estimation of the return probability in comparison to the return probability used in the simulation.

Based on the model with multiple maturity for a single credit line, we can calculate the expectation of usage of each customer. The expectation is determined by two factors: the initial credit rating of the customer s_0 and the maturity of the credit line m. Two different customers will have the same expectation of usage if they have the same initial rating and maturity of the credit loan. We denote the expectation as ${}^{m}_{s_0}\mathrm{E}_t$.

Furthermore, we define $\Omega \in \mathbb{R}^{4\times 8}$, where the element $(\Omega)_{m\in M, s_0\in S}$ is as following.

$$(\Omega)_{m\in M, s_0\in S} = \sum_{(i,j)\in C_{s_0,m}} l^{ij}, \quad C_{s_0,m} := \{(i,j) \in \mathbb{N} | i \le n \ , \ j \le n_i \ , s_i = s_0, m_{ij} = m\} ,$$

where l^{ij} is the credit limit of the j-th credit line of i-th customer, m^{ij} is the corresponding maturity of the credit line, s_i is the initial rating of i-th customer, n_i is the number of credit lines that customer i has reserved and n is the number of customers. It represents the sum of all the commitment of credit lines with maturity m that is reserved by customers with credit rating s_0. Ω is called the credit distribution matrix.

Theorem 6.2 (Expectation of Usage of Several Credit Lines in Multiple Maturity Situation)**.** *Suppose* ${}^{m}_{s_0}\mathrm{E}_t$ *for* $s_0 \in S$*,* $m \in M$ *and* Ω *are given. The expectation of the total amount of liquidity requirement at time* t *will be:*

$$\Xi_t = \sum_{s_0 \in S} \sum_{m \in M} \left({}^{m}_{s_0}\mathrm{E}_t\right) ((\Omega)_{m,s_0}), \tag{6.5}$$

Remark 6.4. The proportional usage ξ_t equals

$$\xi_t = \sum_{\rho \in S} \sum_{m \in M} \left({}^{m}_{\rho}\mathrm{E}_t\right) ((\Omega)_{m,\rho}/\mathcal{L}), \tag{6.6}$$

where the total commitment $\mathcal{L} := \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n_i} l_0^{ij}$ is set constant. l_0^{ij} is the initial commitment limit of the j-th credit line of i-th customer at time $t = 0$.

6.3 Simulation II

We use Theorem 6.4 to calculate the theoretical expectation of usage for a group of credit lines. Grouping the 120 credit lines that we use in the Monte Carlo simulation, we get $\Omega/\mathcal{L}$.

$$\Omega/\mathcal{L} = \begin{pmatrix} 0.1259 & 0.1273 & 0.0299 & 0.0108 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1113 & 0.1591 & 0.0396 & 0.0582 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0492 & 0.1089 & 0.0264 & 0.0180 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0240 & 0.0096 & 0.0360 & 0.0300 & 0.0360 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}$$

From 1000 replications in our Monte Carlo simulation, we obtain the VaR at 0.75, 0.90, 0.975 and 0.9995 quantile together with the mean value. The results are plotted in Figure 6.2 and Figure 6.3.

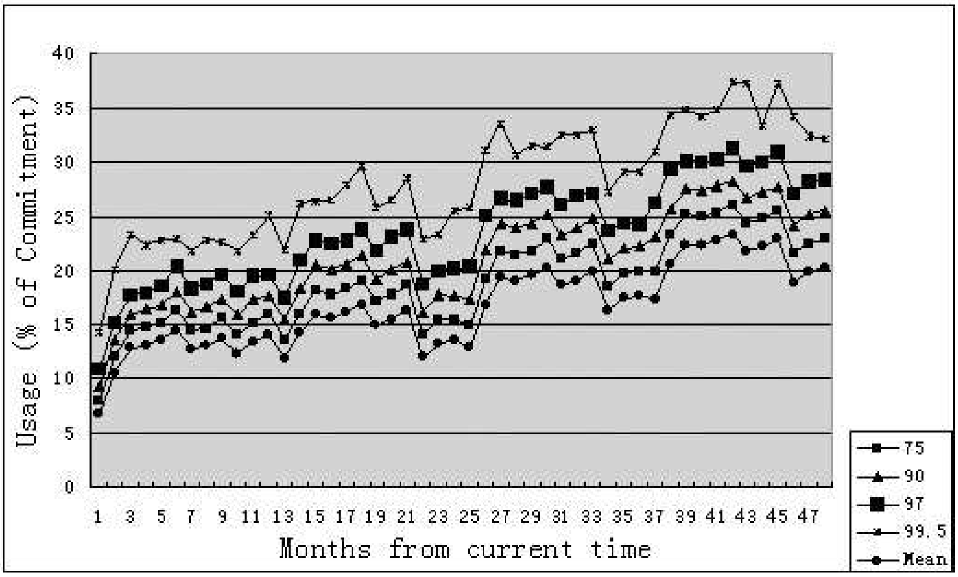

Figure 6.2: Simulated liquidity requirement as percentage of total committed limit

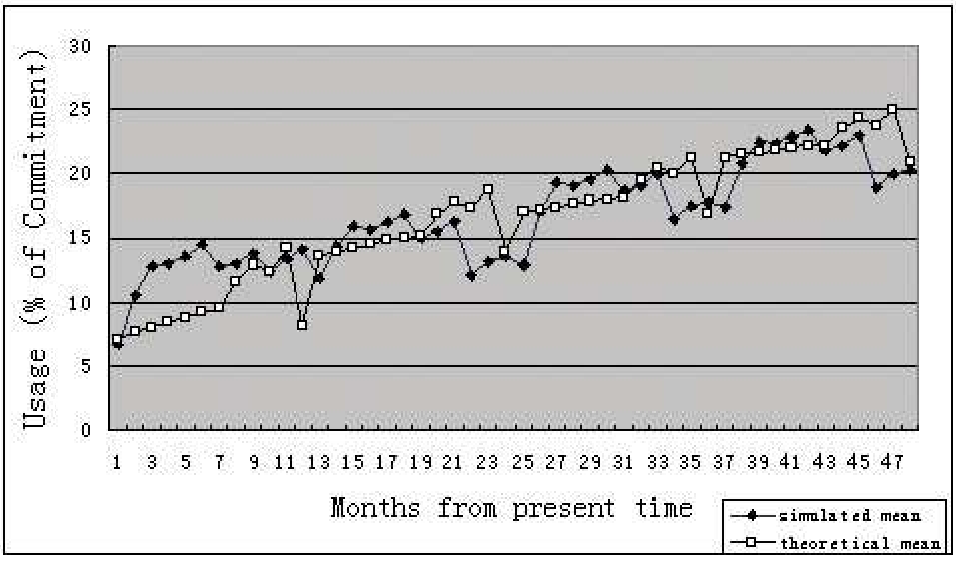

Figure 6.3: Comparison between Monte Carlo simulation and theoretical analysis

Conclusion

1. The 99.95% -percentile does not exceed 40% of the total committed limit within 48 months. Therefore the bank may set aside 60% of the committed capital for other usage.

2. The theoretical expectation of the liquidity requirement is close to the simulated mean. Therefore the theoretical analysis can be used instead of the simulation.

3. The result from Monte Carlo simulation is more fluctuant during each year. The reason is that, in the theorem, we have ignored the correlation between the customers within the same and across different industries, while in the simulation the correlation is taken into consideration.

4. Another difference between the two curves is that the simulation result is above the theoretical curve. The reason is that we have term-out-option in the simulation, which is ignored in the theoretical model. When there is term-out-option, the customer is allowed to keep the usage of loan after the maturity, which increases the liquidity requirement.

5. Simulation method allows for quantile estimation, but with the theoretical derived model, high quantiles are not estimable since the model is discrete.

Chapter 7

Summary of the Thesis

In this thesis, we did the analyzed the liquidity requirement of the revolving credit lines in two ways: Monte Carlo simulation and theoretical model.

In the Monte Carlo simulation, we built up a series of models to describe the business circle of the revolving credit lines including the transition of credit rating of the customer (Chapter 2), the draw and return of the credit loan and the drawn amount of the customer, the renewal decision of the bank etc. (Chapter 3). The credit rating transition is the most important part of our simulation, in which we applied the asset return model. The updated credit rating serves as the input information to the other models. The estimation of the parameters (the draw probability, return probability and the drawn amount) are given in Chapter 4, together with the result of the simulated liquidity requirement for a sample group of real customers of the bank. The sensitivity analysis convinces us of the stability and reliability of the simulation.

In the theoretical analysis, we first started with the analysis of a single credit line in single maturity situation (Chapter 5). In this simplest theoretical model, we have taken into consideration of the credit rating transition, the start and return of the credit loan. Taking additionally the renewal rule and expiration rule of the bank at each maturity into consideration, we extend our result in single maturity situation to multiple maturities (Chapter 6). In the end, we apply the result of single credit line analysis to describe the liquidity requirement of a group of credit lines. To illustrate the consistence between the simulation and theorem, we have used the same sample customer group as input and compared the output liquidity requirement from both simulation and theorem.

Outlook and suggestions for future analysis

The following suggestions are made for further analysis of the modelling of the liquidity requirement for revolving credit line.

In simulation, to ensure a more accurate result, we suggest to do the statistics on the historical data on monthly bases. This would allow us to avoid the error induced by mapping of return probability.

In theory, to develop the model further, we suggest following improvements.

1. Introduce the industry correlation among the customers in the theoretical model.

2. Introduce the term-out-option in the theoretical model.

3. Introduce more sophisticated assumption for the usage of credit loan. Instead of constant usage for customers with certain credit rating, we can replace it with an appropriate probability distribution.

Bibliography

Duffy, Hatzakis, and Hsu. Merrill lynch improves liquidity risk management for revolving credit lines. *Banking and Risk Management*, (3), 2005.

Greg and Gupton. *CreditMetrics - Technical Document*. Reuter, 1997.

Gupton and Finger. *RiskMatrics - Technical Document*. Morgan Guaranty Trust Company, Reuters, 1996.

Neu, Knippschild, Wondrak, and Leistenschneider. *Liquidity Risk Measurement and Management*. Wiley and Sons, Asia Pte Ltd, 1th edition, 2007.

Sidelnikova and Krenin. Regularization algorithms for transition matrices. *Algo research quarterly*, (4), 2001.

Spencer and Anthony. Investment opportunities and corporate demand for lines of credit. *Journal of Banking and Finance*, (3), 1997.