TITELBLATT

Technische Universität München

Zentrum Mathematik

Collateralized Debt Obligations Pricing Using CreditRisk$^+$

Diplomarbeit

von

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Abgabetermin:	08. Juni 2007

Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

Garching, den 08. Juni 2007

Acknowledgments

I would like to take this opportunity to give my thanks to all the people who gave me their help and support over the last years.

First of all my thanks go to Professor Doctor Claudia Klüppelberg for her kindness and support to make this thesis possible.

Furthermore, I am most grateful to my supervisor Doctor Martin Hilleband, who contributed to this thesis by giving me lasting patience, numerous helpful comments and kind support over these months. Without his help this thesis would not have been completed so successfully. I am deeply indebted to him for his great job.

My special thanks to Mr. Jürgen Brückner for bring the topic about Collateralized Debt Obligations to my attention. For all the helpful things that I learned from the interesting work together with them in Frankfurt I express my sincere thanks to Mr. Norbert Benedik and Ms. Susanna Schäfer.

Last but not at least I thank my parents for their love and support, Jun for his encouragement and support during these years, and all my friends in Munich and Shanghai simply for everything.

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Chapter 1

Introduction

Credit derivatives are probably one of the most important types of new financial products introduced during the last decade. The market for credit derivatives was created in the early 1990s in London and New York. It is the market segment of derivative securities which is growing fastest at the moment. Particularly Credit Default Swaps (CDS) and Collateralized Debt Obligations (CDO) have gained interest not only from the market side because of a dramatic rise in traded contracts but also from an academic side because the pricing of such contracts is difficult and still an open issue.

In 1995 first Credit Default Swaps and Collateralized Debt Obligation structures were created by JPMorgan for higher returns without assuming buy and hold risk. Two of the most attractive features of these products can be summarized:

- Accounting and regulatory arbitrage generate significant revenues
- Shifting of credit risk off bank balance sheets by pooling credits and re-marketing portfolios, and buying default protection after syndicating loans for clients.

The CDS market is a large and fast-growing market that allows investors to trade credit risk. Since the late 1990s the CDS indices have increasingly become standardized, liquid, and high-volume. The market originally started as an inter-bank market to exchange credit risk without selling the underlying loans but now involves financial institutions from insurance companies to hedge funds. The British Banker Association (BBA) and the International Swaps and Derivatives Association (ISDA) estimate that the market has grown from 180 billion dollar in notional amount in 1997 to 5 trillion dollar by 2004 and the Economist ("On Top of the World", Economist, April 27, 2006) estimates that the market is currently 17 trillion dollar in notional amount. End of the dotcom boom caused waves of company defaults, which made investors realize the increasing importance of credit protection. Because of significant counterparty risks due to defaults, systematic risks become highly evident and fear of future financial crises rises. Therefore, special purpose vehicles are used to securitize assets.

CDO is one of such credit derivative risk transfer products. At a very simple level a CDO is a transaction that transfers the credit risk of a reference portfolio of assets. The defining feature of a CDO structure is the tranching of credit risk. The risk of loss on the reference portfolio is divided into tranches of increasing seniority. Losses will first affect the "equity" tranche, next the "mezzanine" tranches, and finally the "senior" tranches. In recent years the CDO market has expanded by packaging illiquid private company loans and selling off tranches to investors. In addition, *CDX* and *iTraxx* indices are becoming now standard pricing sources, which enable further broad trading of credit derivatives. In this thesis the pricing of CDO tranches of synthetic CDOs is studied. In a synthetic CDO the reference portfolio consists of CDS.

A brief outline of the thesis is as follows. We introduce the most liquid credit derivative, the credit default swap (CDS) and the most prominent credit correlation product, the collateralized debt obligation (CDO) in Chapter 2 as well as the general approach for pricing synthetic CDO. It shows that the CDO pricing problem can be solved as long as the loss distribution of the reference portfolio is calculated. It should be noted that the modeling of default dependence is crucial when calculating loss distributions.

Chapter 3 deals with the structural model, the one-factor Gaussian model, which has been the standard model in practice for its simplicity, to pricing a CDO tranche. Assuming that the correlation of defaults on the reference portfolio is driven by common factors, defaults are independent conditional on these common factors. By integrating over the common factors we can compute the unconditional loss distribution. Based on the firm-value model of Merton, default occurrences can be modeled. Using the large homogenous portfolio (LHP) approximation approach common factors can be reduced to one factor and correlation is the single implicit parameter of dependence to be estimated. Although being the primary model for the valuation of CDO tranches, the one-factor model fails to fit the market prices of CDO tranches. Some issues arising by applying this method for CDO tranche pricing are discussed. Since the reference portfolio in a synthetic CDO consists of CDS, the individual default intensities are calibrated from CDS prices. Thus we give a short look at the intensity based model, which is also called the reduced form approach and introduce how it can be used to calibrate individual default intensities.

Chapter 4 introduces an alternative model, the CreditRisk$^+$ model, created by Credit Suisse Financial Products (CSFP), which is more or less based on a typical insurance mathematics approach. It is a representative of the group of Poisson mixture models. The most important reason for the popularity of CreditRisk$^+$ is that the portfolio loss distribution function can be computed analytically, not by using Monte Carlo simulations. Using probability-generating functions, the CreditRisk$^+$ model offers a explicit description of the portfolio loss of any given credit portfolio. This enables users to compute loss distributions in a quick manner. Besides the original CreditRisk$^+$ model, some expanding approaches are investigated. The Fast Fourier Transform provides a stable numerical computation in inverting the characteristic function to obtain the portfolio loss distribution function. Additionally, it requires no basic loss unit, which is a critical choice for the calculation. It provides a possibility to relax the requirement for loss discretization by computing the characteristic function of the portfolio loss instead of the probability-generating function for calculating the loss distribution of the reference portfolio. CreditRisk$^+$ allows the losses of an obligor are affected by a number of systematic factors, which are assumed to be independent in the CreditRisk$^+$. But in the reality industries are correlated with each other. From empirical studies, the consequences of neglecting these industry default rate correlations might lead to significant underestimation of unexpected losses. Therefore, the correct modeling of the dependence structure is very important. We present two approaches to model correlated default events. One is Merton-type asset value threshold model, the other one is based on the reduced form model. The estimated dependency is as input information into the factor analysis. The Principal Component Analysis provides a framework, which allows for simultaneous identifying the independent latent random variables as the estimation of sector weights as well. The obligors sharing the same industrial sector have the common characters, e.g. default rate, default rate volatility. Sector weights reflect the interdependency among the industries. Based on the idea of Lehnert and Rachev [2005], we give numerical implementations for calibration of the standard model as well as our investigation results and remarks. The static nature of the CreditRisk$^+$ framework is a major drawback when we work with portfolio exposures having different maturities and when pricing credit derivative instruments where the term structure of default rates matters. From this point, we introduce the approach of Hillebrand and Kadam, which allows even for modeling heterogeneous credit portfolios, where time varying default rates and volatilities may differ across names. The application of this dynamic model on CDO tranche pricing is the focus of ongoing work.

Chapter 2

Credit Derivatives Background Knowledge

Most credit derivatives have a default-insurance feature. A credit derivative contract provides protection against the default of a reference entity or a portfolio of reference entities. The protection seller in the contract compensates the protection buyer for any default losses incurred in the reference assets and in return receives a periodical fee from the protection buyer for the provided protection. One of the attractions of credit derivatives is the large degree of flexibility in their specifications. In this chapter some basics of two of the most prominent credit derivative products, Credit Default Swaps (CDS) and Collateralized Debt Obligations (CDO) will be presented. Chacko [2006] provides simple, yet rigorous explanations about essential principals, models, techniques and widely used credit instruments, especially about CDS and CDO. For more details the reader can refer to Bluhm et al. [2003] and Schönbucher [2003].

2.1 Credit Default Swaps

CDS are bilateral contracts in which the protection buyer pays a fee termed CDS spread periodically, typically expressed in basis points (bps) on the notional amount, in return for a contingent payment by the protection seller following a credit event of a reference security. The credit event could be either default or downgrade; the credit event and the settlement mechanism used to determine the payment are flexible and negotiated between the counterparties. A CDS is triggered by a credit event. If there is no default of the reference security until the maturity, the protection seller pays nothing. If a default occurs between two fee payment dates, the protection buyer has to pay the fraction of the next fee payment that has accrued until the time of default. CDS are almost exclusively interprofessional transactions, and range in nominal size of reference assets from a few millions to billions of euros, with smaller sizes for lower credit quality. Maturities usually run from one to ten years.
CDS allow users to reduce credit exposure without physically removing an asset from the balance sheet. More precisely, following a default event the protection seller makes a

payment equal to $(1 - R)$ times CDS notional, where R is the recovery rate. Recovery rate represents in the event of a default, what fraction of the exposure may be recovered through bankruptcy proceedings or some other form of settlement. In this thesis we consider only the deterministic case. The payment stream from the protection seller to the protection buyer is called the *protection leg* and the payment stream from the protection buyer to the protection seller is known as the *premium leg*. The CDS spread is determined at the initiation of the trade. It is fixed such that the value of the protection leg equals the value of the premium leg. Figure 2.1 shows payment streams of a CDS contract.

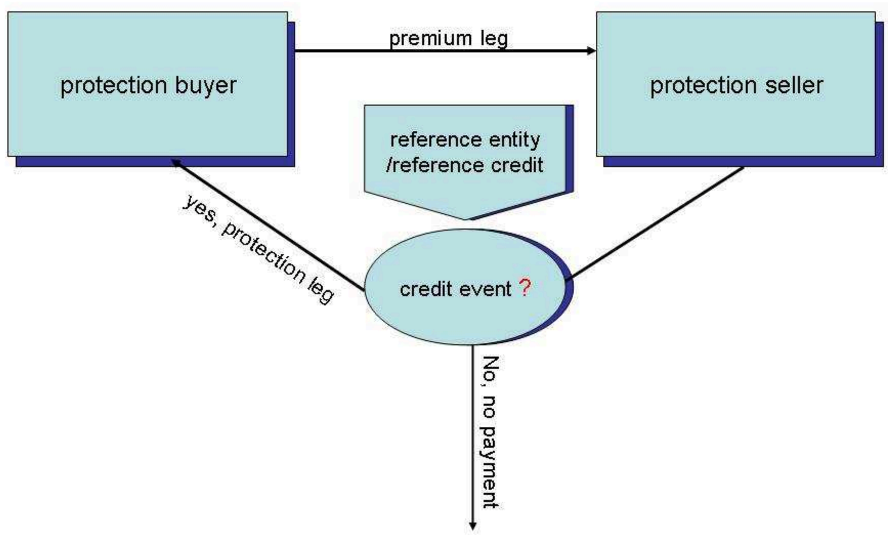


Fig. 2.1: payment streams of a credit default swaps contract

For a better understanding let us look at an illustrative example. In Figure 2.2 we can see how a CDS looks like. The counterparty buys 10 million Euro *iTraxx Europe* exposure with maturity 5 years. Details can be listed as follows:

- CDS references the credit spread (premium) of the most current series at launch
- Premium of the iTraxx Europe is 30 bps
- After two days, the market price is 28 bps and counterparty wants to buy 10 million Euro iTraxx Europe exposure in CDS
- CDS is executed at the premium level. Market maker pays 30 bps per annum quarterly to counterparty on notional amount of €10m
- Present value of difference between premium and fair value of the CDS is settled through an upfront payment

- Counterparty pays the present value of 2 bps plus accrued interest to market maker (€ 9,493.28)
- Present value is for example calculated via the CDSW function on Bloomberg

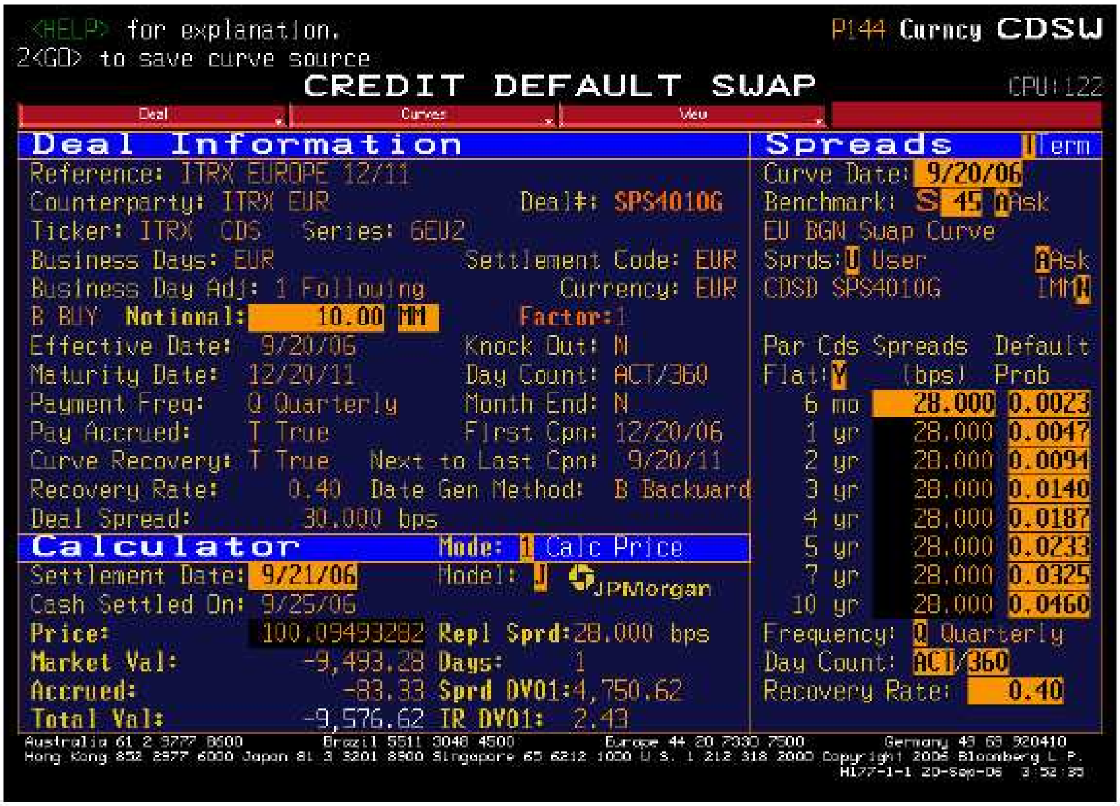


Fig. 2.2: Credit Default Swap Contract, Source: Bloomberg

Market maker pays 30 bps per annum quarterly on notional amount of € 10 m to the counterparty, and in the case of no credit event the counterparty will continue to receive the premium on the original notional amount until maturity.
What happens if a credit event occurs? For example a credit event occurs on the reference entity in year 3 and the reference entity weighting is 0.8%. The counterparty pays to the market maker € 80,000 ($0.8\% \cdot 10,000,000$) and the market maker delivers € 80,000 nominal face value of deliverable obligations of the reference entity to the counterparty. Meanwhile, the notional amount on which the premium is paid reduces by 0.8% to 99.2%, € 9,920,000. After the credit event, the counterparty receives the premium of 30 bps on € 9.92m until maturity subject to any further credit events.
From above explanations we can see the basic property of CDS, transferring the credit risk of an entity from one party to another where the possession of the reference entity does not change hands.

2.2 Collateralized Debt Obligations

CDOs are Special Purpose Vehicles (SPV) that invest in a diversified pool of assets (collateral pool) and a financial innovation to securitize portfolios of these defaultable assets (loans, bonds or credit default swaps). The investments are financed by issuing several tranches of financial instruments. The repayment of the tranches depends on the performance of the underlying assets in the collateral pool. The rating of the single tranches is determined by the rank order they are paid off with the interest and nominal payments that are generated from the cash flows in the collateral pool. So called senior notes are usually rated between AAA and A and have the highest priority in interest and nominal payments, i.e. they are paid off first. Mezzanine notes are typically rated between BBB and B. They are subordinated to senior notes, i.e. they are only paid off if the senior notes have already been serviced. And so on to the equity notes. In another words, the risk of losses on the reference portfolio is divided into tranches of increasing seniority. Losses will first affect the 'equity' tranche, next the 'mezzanine' tranches, and finally the 'senior' tranches. The prices of the respective tranches depend critically on the perceived likelihood of joint default of the underlying pool, or default dependency. The collateral pool of a CDO may consist of bonds, collateral bond obligation (CBO); loans, collateral loan obligation (CLO); credit derivatives, like credit default swaps; and asset backed securities. The key idea behind this instrument is to pool assets and transfer specific aspects of their overall credit risk to new investors.

2.2.1 Market Indices

One of the latest developments in the credit derivatives market is the availability of liquidly traded standardized tranches on CDS indices. In June 2004, the DJ iTraxx Europe index family was created by merging existing credit indices, thereby providing a common platform to all credit investors. The most popular examples are the *iTraxx Europe* and the *CDX IG*.

The *iTraxx Europe* Index is the most widely traded of the indicies. It is composed of an equally weighted portfolio of 125 most liquidly traded European CDS referencing European investment grade credits, subject to certain sector rules as determined by the International Index Company (IIC). A new series of *iTraxx Europe*, agreed by participating dealers, is issued every six months, a process known as "rolling" the index. The roll dates are 20 March and 20 September each year. It is published online for transparency. The latest series is Series 7 launched on 20 March 2007. This standardization led to a major increase in transparency and liquidity of the credit derivatives market. Figure 2.3 is the *iTraxx Europe* Series 6, which was issued on 20. September 2006. The *iTraxx Europe HiVol* is a subset of the main index involving the top 30 highest spread names from the *iTraxx Europe*. The *iTraxx Europe Crossover* is constructed in a similar way but is composed of 45 sub-investment grade credits. The maturities for the *iTraxx Europe* and *HiVol* are 3 years, 7 years and 10 years, the *Crossover* only traded at 5 and 10 years.

Analogously, the *CDX IG* is an equally weighted portfolio of 125 CDS on investment grade North American companies. The new index allows for a cost efficient and timely access to diversified credit market and is therefore attractive for portfolio managers, as a hedging tool for insurances and corporate treasuries as well as for credit correlation trading desks. Besides a direct investment in the *iTraxx Europe* index via a CDS on the index or on a subindex, it is also possible to invest in standardized tranches of the indices via the tranched *iTraxx* and the *CDX IG*, which are nothing else but synthetic CDO on a static portfolio. At present trading the indices is limited to the over-the-counter market. Table 2.1 lists the market agreeing quoted standard tranches. This means that the *iTraxx*

iTraxx Europe Series 6 Coupons			
	Years	Maturity	Coupon bps
Europe	3	20-Dec-09	20
	5	20-Dec-11	30
	7	20-Dec-13	40
	10	20-Dec-16	50
HiVol	3	20-Dec-09	35
	5	20-Dec-11	55
	7	20-Dec-13	70
	10	20-Dec-16	85
Non-Financials	5	20-Dec-11	35
	10	20-Dec-16	55
Senior Financials	5	20-Dec-11	10
	10	20-Dec-16	20
Subordinated Financials	5	20-Dec-11	20
	10	20-Dec-16	30
Crossover	5	20-Dec-11	280
	10	20-Dec-16	345

iTraxx Europe Series 6 Recovery Rates	
	Recovery Rates %
Europe	40
HiVol	40
Non-Financials	40
Senior Financials	40
Subordinated Financials	20
Crossover	40

Fig. 2.3: *iTraxx Europe* Series 6

Europe equity tranche bears the first 3% of the total losses, the second tranche bears 3% to 6% of the losses and so on. When tranches are issued, they usually receive a rating from an independent agency.

Figure 2.4 shows cash flows of a CDO contract. By tranching the losses different classes of securities are created, which have varied degrees of seniority and risk exposures. Therefore, they are able to meet very specific risk return profiles of investors. Investors take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer, and in turn collecting the premium. The premium is a percentage of the outstanding notional amount of the transaction and is paid periodically, generally quarterly. The fixed rate day count fraction is *actual*/360. The outstanding notional amount is the original

	iTraxx Europe		**CDX IG**	
Tranche	K_L	K_U	K_L	K_U
Equity	0%	3%	0%	3%
Junior Mezzanine	3%	6%	3%	7%
Senior Mezzanine	6%	9%	7%	10%
Senior	9%	12%	10%	15%
Junior Super Senior	12%	22%	15%	30%
Super Senior	22%	100%	30%	100%

Table 2.1: Standard synthetic CDO structure on iTraxx Europe and CDX IG North American. With K_L lower attachment point and K_U upper attachment point.

tranche size reduced by the losses that have been covered by the tranche. More information is available in **www.itraxx.com** and **www.mark-it.com**.
It is common to distinguish between cash CDOs and synthetic CDOs. Cash CDOs have a reference portfolio made up of cash assets, such as bonds or loans. In a synthetic CDO the reference portfolio contains synthetically created credit risk, such as a portfolio of credit default swap contracts. Synthetic arbitrage CDOs also have a significant effect on the underlying CDS markets, because they form an important channel through outside investors, who can sell default protection in the CDS market on a diversified basis. If a reference credit is included in a synthetic arbitrage CDO, the CDO manager will be able to offer protection on this name relatively cheaply. The presence of protection sellers is of central importance to the functioning of the CDS market, and the volume of synthetic CDOs issuance is an important indicator of the current supply of credit protection in the single-name CDS market.

2.2.2 Why are CDOs issued?

The possibility to buy CDO tranches is very interesting for investors to manage credit risk. The investment in a CDO tranche with a specific risk-return profile is much more attractive for a credit investor or a hedger than to achieve the same goal via the rather illiquid bond and loan market. First, the CDO's spread income from the reference portfolio can compensate investors in the CDO tranches and also cover transactions costs. Second, the rapid adoption of CDO technology by credit investors suggests that the cost of creating a CDO is less than the cost a credit investor would incur to assemble a portfolio of bonds and loans to meet the investor's diversification and risk-return targets. Since the costs of lawyers, issuers, assets managers and rating agencies encountered when setting up a CDO can be very high, there are three main reason why CDO are issued.

- Spread arbitrage opportunity
  Profit from price differences between the components included in the CDO and the sale price of the CDO tranches, i.e. the total spread collected on single credit risky instruments at the asset side of the transaction exceeds the total "diversified" spread to be paid to investors on the tranched liability side of the structure. There

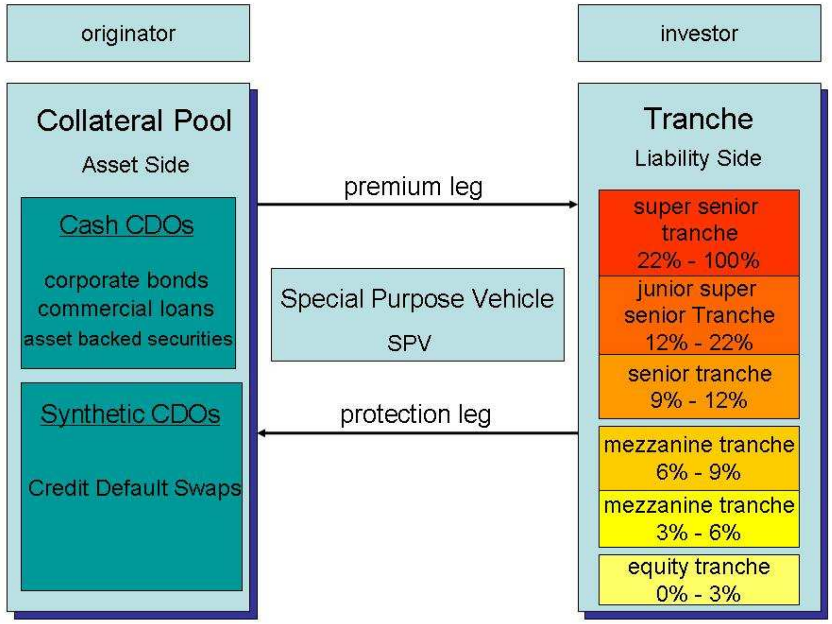


Fig. 2.4: cash flows of collateralized debt obligations

are many transactions motivated by spread arbitrage opportunities in the CDO market. In some cases, structures involve a so-called rating arbitrage which arises whenever spreads increase quickly and rapidly but the corresponding ratings do not react fast enough to reflect the increased risk of the instruments. Rating arbitrages as a phenomenon is an important reason why a serious analysis of arbitrage CDO should not rely on ratings alone but also considers all kinds of other sources of information.

- Balance sheet transaction
  The collateral pool is not actively managed. Changes in the collateral pool only arise from instruments that have already matured. By using balance sheet transactions financial institutions can remove loans or bonds from their balance sheet in order to obtain capital, to increase liquidity or to earn higher yields. Therefore, CDO transfers outstanding money of obligors into liquidity. This helps to reduce economic and regulatory capital. In addition, they are a good supplement to the classical instruments for asset liability management as they allow for active risk management and are an alternative for financing and refinancing.

- Regulatory capital relief
  Regulatory capital relief is another major motivation why banks issue CDO. Let us briefly outline what a CDO or most often CLO transaction means for the regulatory capital requirement of the underlying reference pool. In general, loan pools require regulatory capital in size of 8% times risk-weighted assets (RWA) of the reference pool, according to Basel II standard model. Ignoring collateral eligible for a risk

weight reduction, regulatory capital equals 8% of the pool's notional amount. After the synthetic securitization of the pool, the only regulatory capital requirement the originator has to fulfil regarding the securitized loan pool is holding capital for retained pieces. For example if the originator retained the equity tranche, the regulatory capital required on the pool would have been reduced from 8% to 50bps, which is the size of the equity tranche. The 50bps come from the fact that retained equity pieces typically require a full capital deduction.

2.2.3 The General Approach for Pricing Synthetic CDOs

Throughout the thesis the framework is set by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{Q})$. All subsequently introduced filtration are subsets of $\mathcal{F}$ and complete. Since all models are applied for the valuation of default contingent claims, the full specification of the models take place under the equivalent martingale measure, the pricing measure $\mathbf{Q}$. And all probabilities and expectations in the calculations are defined with respect to $\mathbf{Q}$.
We start with the general approach for pricing synthetic CDOs. Consider a synthetic CDO with a reference portfolio consisting of credit default swaps only instead of bonds or loans. A tranche only suffers losses, if the total portfolio loss exceeds the lower attachment point of the tranche. The maximum loss a tranche can suffer is its tranche size $K_U - K_L$, where K_U is upper attachment point of the tranche and K_L lower attachment point. As long as no default event has happened, the CDO issuer pays a regular premium to the tranche investor (usually quarterly). The premium is a percentage of the outstanding notional amount of the transaction. The outstanding notional amount is the original tranche size reduced by the losses that have been covered by the tranche. To illustrate this point, let us assume that the subordination of tranche is 56m Euro and the tranche size is 10m Euro. If 60m Euro of credit losses have occurred, the premium will be paid on the outstanding amount of 6m Euro (tranche size of 10m Euro - 4m Euro that represents the amount of losses which exceeds the subordination of 56m Euro). If the total loss of the reference credit portfolio exceeds the notional of the subordinated tranches, the investor (protection seller) has to make compensation payments for these losses to the CDO issuer (protection buyer). The next premium is paid on the new reduced notional.

Definition 2.1 (Tranche Loss distribution) *Denote by $L_{(K_L,K_U)}(t)$ the cumulative loss on a given tranche (K_L, K_U) with the lower attachment point K_L and the upper attachment point K_U at time t, and by $L(t)$ the cumulative loss on the reference portfolio at time t:*

$$L_{(K_L,K_U)}(t) = \begin{cases} 0, & L(t) \leq K_L; \\ L(t) - K_L, & K_L \leq L(t) \leq K_U; \\ K_U - K_L, & K_U \leq L(t). \end{cases}$$

We can easily see that the "payoff" in terms of loss on the reference portfolio, has option-like features with both upper and lower attachment points as strike prices. So we can say that the loss of a given tranche is an option with tranche upper and lower attachment points of the total portfolio loss.
The determination of the incurred portfolio loss L(t) is the essential part in order to calculate the cash flows between protection seller and buyer and hence also in pricing the

CDO tranches.

Definition 2.2 (Portfolio loss) *Consider N reference obligors, each with a nominal amount N_i and recovery rate R_i for $i = 1, 2, \ldots, N$. Let $L_i = (1 - R_i) \cdot N_i$ denote the loss given default of obligor i. Let τ_i be the default time of obligor i and $D_i(t) = 1_{\{\tau_i < t\}}$ be the counting process. The portfolio loss is given by:*

$$L(t) = \sum_{i=1}^{N} L_i \cdot D_i(t) \tag{2.1}$$

Note that L(t) and therefore also $L_{(K_L,K_U)}(t)$ are pure jump processes. At every jump of $L_{(K_L,K_U)}(t)$ a default payment has to be made from the protection seller to the protection buyer.

The notional amount N_i and the recovery rate R_i are assumed to be same for all obligors. In discrete time we can write the percentage expected loss of a given tranche as:

$$\begin{aligned} EL_{(K_L,K_U)}(t_j) &= \frac{\mathbf{E}[L_{(K_L,K_U)}(t_j)]}{K_U - K_L} \\ &= \frac{1}{K_U - K_L} \sum_{i=1}^{N} \Big(\min(L_i(t_j), K_U) - K_L \Big)^{+} \cdot p_i \end{aligned} \tag{2.2}$$

where p_i is the probability that $(K_U - K_L)$ tranche suffers a loss of $\Big(\min(L_i(t_j), K_U) - K_L \Big)^{+}$.

Lemma 2.3 *Given a continuous portfolio loss distribution function F(x), the percentage expected loss of the $(K_U - K_L)$ CDO tranche can be computed as:*

$$EL_{(K_L,K_U)} = \frac{1}{K_U - K_L} \left(\int_{K_L}^{1} (x - K_L) dF(x) - \int_{K_U}^{1} (x - K_U) dF(x) \right) \tag{2.3}$$

Proof: Omitting the time index t_j

$$\begin{aligned} EL_{(K_L,K_U)} &= \frac{1}{K_U - K_L} \sum_{i=1}^{N} \Big(\min(L_i, K_U) - K_L \Big)^{+} \cdot p_i \\ &= \frac{1}{K_U - K_L} \sum_{i=1}^{N} \left(L_i \mathbf{1}_{\{L_i < K_U\}} + K_U \mathbf{1}_{\{L_i \geq K_U\}} - K_L \right) \mathbf{1}_{\{\min(L_i,K_U) > K_L\}} \cdot p_i \\ &= \frac{1}{K_U - K_L} \sum_{i=1}^{N} \Big(\left(L_i \mathbf{1}_{\{L_i < K_U\}} - K_L \right) \mathbf{1}_{\{\min(L_i,K_U) > K_L\}} \Big) \cdot p_i \\ &+ \frac{1}{K_U - K_L} \sum_{i=1}^{N} \Big(\left(K_U \mathbf{1}_{\{L_i \geq K_U\}} - K_L \right) \mathbf{1}_{\{\min(L_i,K_U) > K_L\}} \Big) \cdot p_i \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{K_U - K_L} \sum_{i=1}^{N} \Big((L_i - K_L)\, \mathbf{1}_{\{K_L < L_i < K_U\}} + (K_U - K_L)\, \mathbf{1}_{\{L_i \geq K_U\}} \Big) \cdot p_i \\
&= \frac{1}{K_U - K_L} \Big(\int_{K_L}^{K_U} (x - K_L) dF(x) + \int_{K_U}^{1} (K_U - K_L) dF(x) \Big) \\
&= \frac{1}{K_U - K_L} \Big(\int_{K_L}^{1} (x - K_L) dF(x) - \int_{K_U}^{1} (x - K_L) dF(x) + \int_{K_U}^{1} (K_U - K_L) dF(x) \Big) \\
&= \frac{1}{K_U - K_L} \Big(\int_{K_L}^{1} (x - K_L) dF(x) - \int_{K_U}^{1} (x - K_L - K_U + K_L) dF(x) \Big) \\
&= \frac{1}{K_U - K_L} \Big(\int_{K_L}^{1} (x - K_L) dF(x) - \int_{K_U}^{1} (x - K_U) dF(x) \Big) \qquad \square
\end{aligned}
$$

Assume that $0 = t_0 < \ldots < t_{n-1}$ denote the spread payment dates, and T with $t_{n-1} < t_n = T$ is the maturity of the synthetic CDO.
The value of the premium leg (PL) of the tranche depends on the outstanding tranche notional $N_{out}(t)$ at time t of the tranche $(K_U - K_L)$. The outstanding tranche notional at time t is defined by the initial tranche notional N_{tr} subtracted by any expected loss $N_{tr} EL_{(K_L, K_U)}(t)$ in the tranche up to time t, which can be formulated as:

$$
N_{out}(t) = N_{tr} \Big(1 - EL_{(K_L, K_U)}(t) \Big) \tag{2.4}
$$

This means that at any default date in the reference portfolio that affects the tranche, the outstanding tranche notional is reduced.
With knowledge of the fundamental pricing rule of the capital market, the price of a contingent claim is given by the expected value of its discounted expected payoff under a martingale measure, the equivalent martingale measure, the pricing measure $\mathbf{Q}$. Thus, the present value of all expected premium payments can be defined as:

$$
\begin{aligned}
PL(t_0) &= \mathbf{E} \left[\sum_{i=1}^{n} e^{-\int_{t_0}^{t_{i-1}} r(u) du} s N_{out}(t_{i-1}) \Delta t_i \right] \\
&= \sum_{i=1}^{n} B(t_0, t_{i-1}) \Delta t_i s N_{tr} \left(1 - EL_{(K_L, K_U)}(t_{i-1}) \right)
\end{aligned} \tag{2.5}
$$

where $\Delta t_i = t_i - t_{i-1}$, is the discretized time interval, $B(t_0, t_i)$ denotes the discount factor at time t_i, which is defined as $B(t_0, t) := \mathbf{E}[e^{-\int_{t_0}^{t} r(u) du}]$ with the default-free continuous short rate $r(u)$ and s is the predetermined premium.
Similarly, the value of the protection leg, also called default leg (DL) is given by the discounted expected default losses in the tranche,

$$
DL(t_0) = \mathbf{E} \left[\int_{t_0}^{T} e^{-\int_{t_0}^{s} r(u) du} N_{tr} dEL_{(K_L, K_U)}(s) \right] \tag{2.6}
$$

In case the tranche loss is independent of the short rate process, equation (2.6) can be rewritten as

$$DL(t_0) = \int_{t_0}^{T} \mathbf{E}\left[e^{-\int_{t_0}^{s} r(u)du}\right] N_{tr} dEL_{(K_L,K_U)}(s) = \int_{t_0}^{T} B(t_0,s) N_{tr} dEL_{(K_L,K_U)}(s) \quad (2.7)$$

which can be approximated by

$$DL(t_0) \approx \sum_{i=1}^{n} B(t_0,t_i) N_{tr} \left(EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1})\right) \quad (2.8)$$

Thus the fair price of the CDO tranche is defined as the present value of premium leg is equal to the present value of the default payment.

$$PL(s^*) \stackrel{!}{=} DL$$

Solving the equation we can get the fair premium:

$$s^* = \frac{\sum_{i=1}^{n} B(t_0,t_i) \left(EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1})\right)}{\sum_{i=1}^{n} B(t_0,t_{i-1}) \Delta t_i \left(1 - EL_{(K_L,K_U)}(t_{i-1})\right)} \quad (2.9)$$

Hence, for the evaluation of the premium and default payment leg of a CDO tranche it suffices to calculate the expected percentage loss $EL_{(K_L,K_U)}(t)$ on the tranche for each time t.

Following from (2.3) this can be done by deriving the loss distribution of the reference portfolio which is unfortunately not trivial. This is mainly due to the fact that we have to consider the dependency structure between obligors. Depending on the dependence between obligors the portfolio loss distribution can look completely different. The modeling of default dependence between obligors is therefore crucial when calculating loss distributions. Therefore, by pricing a CDO tranche one has to consider not only joint defaults but also the timing of defaults, since the premium payment depends on the outstanding notional which is reduced during the lifetime of the contract if obligors default.

Chapter 3

Valuation for the Large Homogenous Portfolio Model

As shown in the previous chapter, the probability distribution of default losses on the reference portfolio is a key input when pricing a CDO tranche. In the following the current market standard model, the large homogenous portfolio model (LHP), is presented. The most important references can be found in Schönbucher [2003] and Bluhm et al. [2003]. Similar approaches have been followed by Andersen and Basu, Li [2000] and Laurent and Gregory [2003]. It employs the following assumptions:

- There exists risk-neutral martingale measure and recovery rates derived from market spreads.
- A default of an obligor is triggered when its asset value X_n of obligor n falls below a certain threshold TH_n, $X_n < TH_n$. Specially in the Merton model default occurs when the value of assets of a firm X_n falls below the firm's liabilities TH_n.
- The asset value is driven by one standard normally distributed factor. The factor both incorporates the market by a systematic risk component and the firm specific risk by an idiosyncratic risk component.
- The portfolio consists of a large number of credits of uniform size, uniform recovery rate and uniform probability of default, which means the reference portfolio consists of an infinite number of firms each with the same characteristics, i.e. "large homogeneous".

3.1 The One-Factor Gaussian Model

The one-factor model in CreditMetrics is completely described in the case of only one single factor common to all counterparties, hereby assuming that the asset correlation among all obligors is uniform. And the normalized asset value of the ith obligor X_i can be described by the one-factor model, in which the values of the assets of the obligors are driven by a single common factor and an idiosyncratic noise component:

$$X_i = \sqrt{\rho_i}M + \sqrt{1-\rho_i}Z_i \tag{3.1}$$

where M denotes the common market factor and Z_i the idiosyncratic risk factor.
M is a standard normally distributed random variable and Z_i are independent univariate standard normally distributed random variables, which are also independent of M. Due to the stability of the normal distribution under convolution, the X_i are also standard normally distributed. ρ_i is the correlation of obligor i with the market factor.
Default of firm i occurs when its asset value X_i falls below a threshold TH_i, which can be represented as a default indicator function: $D_i = \mathbf{1}_{\{X_i \leq TH_i\}}$. Using this approach the value of the assets of two obligors are correlated with linear correlation coefficient ρ. The important point is that conditional on the realization of the systematic factor M, the firm's value and the defaults are independent, i.e. *conditional independence*.
This works because as soon as we condition on the common factor the X_i only differ by their individual noise term Z_i which was defined to be independently distributed for all i and also independent of M. Therefore, after conditioning on the common factor M the critical random variables X_i and therefore also defaults are independent.
Let p_i denote the probability of default of obligor i, then the default event can be modeled as:

$$p_i = \mathbf{Q}\left(X_i \leq TH_i\right) = \Phi(TH_i) \tag{3.2}$$

so

$$TH_i = \Phi^{-1}(p_i).$$

the conditional default probability Conditional on the common factor $M = m$ we can calculate the conditional default probability $p_i(m)$ for each obligor. This can be done easily according to equation (3.1).

$$\begin{aligned} p_i(m) &= \mathbf{Q}\left(X_i \leq TH_i \mid M = m\right) \\ &= \mathbf{Q}\left(\sqrt{\rho_i}M + \sqrt{1-\rho_i}Z_i \leq TH_i \mid M = m\right) \\ &= \mathbf{Q}\left(Z_i \leq \frac{TH_i - \sqrt{\rho_i}M}{\sqrt{1-\rho_i}} \mid M = m\right) \\ &= \Phi\left(\frac{TH_i - \sqrt{\rho_i}m}{\sqrt{1-\rho_i}}\right) \end{aligned} \tag{3.3}$$

If we assume that the portfolio is homogeneous, i.e. $\rho_i = \rho$ and $TH_i = TH$ for all obligors and the notional amounts and recovery rate R are the same for all issuers, then the default probability of all obligors in the portfolio conditional on $M = m$ is given by

$$p(m) = \mathbf{Q}(X < TH \mid M = m) = \Phi\left(\frac{TH - \sqrt{\rho}m}{\sqrt{1-\rho}}\right) \tag{3.4}$$

Assume that the probability of the percentage portfolio loss L being $L_k = \frac{k}{N}(1-R)$ is equal to the probability that exactly k out of N issuers default, the loss distribution of the portfolio can be computed as:

$$\mathbf{Q}(L = L_k \mid M = m) = \begin{pmatrix} N \\ k \end{pmatrix} \Phi\left(\frac{TH - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)^k \left(1 - \Phi\left(\frac{TH - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)\right)^{N-k} \tag{3.5}$$

Conditional on the general state of the economy, the individual defaults occur indepently from each other due to the conditional independency. There are only two possible states (default or not), so the conditional loss distribution is binomial. The unconditional loss distribution $\mathbf{Q}(L = L_k)$ can be obtained by integrating equation (3.5) with the distribution of the factor M, which is normally distributed:

$$\mathbf{Q}(L = L_k) = \left(\begin{array}{c} N \\ k \end{array} \right) \int_{-\infty}^{\infty} \Phi\left(\frac{TH - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)^k \left(1 - \Phi\left(\frac{TH - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)\right)^{N-k} d\Phi(m). \tag{3.6}$$

Since the calculation of the loss distribution in (3.6) is quite computationally intensive for large N, namely $\frac{1}{2}N(N-1)$ times, it is desirable to use some approximation. The large portfolio limit approximation by Vasicek [1987], Vasicek [1991] is a very simple but powerful method.

Theorem 3.1 (Large Portfolio Approximation) *Assume that the portfolio consists of very large number of obligors, i.e. $N \to \infty$. Then*

$$F_\infty(x) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - TH}{\sqrt{\rho}}\right)$$

Proof: For simplicity let us first assume a zero recovery rate.
We consider the cumulative probability of the percentage portfolio loss not exceeding $x \in [0, 1]$,

$$F_N(x) = \sum_{k=0}^{[Nx]} \mathbf{Q}(L = L_k)$$

Substituting $s = \Phi\left(\frac{TH - \sqrt{\rho}\cdot u}{\sqrt{1-\rho}}\right)$ and plugging in equation (3.6) we get the following expression for $F_N(x)$:

$$F_N(x) = \sum_{k=0}^{[Nx]} \left(\begin{array}{c} N \\ k \end{array} \right) \int_0^1 s^k (1-s)^{N-k} d\Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(s) - TH}{\sqrt{\rho}}\right). \tag{3.7}$$

By the law of large numbers,

$$\lim_{N\to\infty} \sum_{k=0}^{[Nx]} \left(\begin{array}{c} N \\ k \end{array} \right) s^k (1-s)^{N-k} = \begin{cases} 0, & \text{if } x < s; \\ 1, & \text{if } x > s. \end{cases}$$

the cumulative distribution of losses of a large portfolio equals

$$F_\infty(x) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - TH}{\sqrt{\rho}}\right) \quad \square \tag{3.8}$$

Therefore, in the case of large homogeneous portfolio assumption it is possible to compute the integrals in (2.9) analytically.

Substituting $TH = \Phi^{-1}(p)$ and taking the derivative of (3.8) with respect to x yields the corresponding probability density function $f(x)$, which is also called Vasicek density:

$$f(x) = \sqrt{\frac{1-\rho}{\rho}} \exp\left\{\frac{1}{2}\left(\Phi^{-1}(x)\right)^2 - \frac{1}{2\rho}\left(\Phi^{-1}(p) - \sqrt{1-\rho}\,\Phi^{-1}(x)\right)^2\right\} \tag{3.9}$$

It is documented in Schönbucher [2003] that large portfolio limit distributions are often remarkably accurate approximations for finite size of the portfolios, especially in the upper tail. Given the uncertainty about the correct value for asset correlation the small error generated by the large portfolio assumption is negligible.

Now, let us assume that assets have the same (maybe non-zero) recovery rate R. Then the total loss of the equity tranche of K will occur only when assets of the total amount of $\frac{K}{1-R}$ have defaulted. Thus, the expected loss of the tranche between K and 1 is given by

$$EL^R_{(K,1)} = \int_{\frac{K}{1-R}}^{1} (1-R)\left(x - \frac{K}{1-R}\right) dF_\infty(x) = (1-R)\cdot EL_{\left(\frac{K}{1-R},1\right)}$$

where EL^R denotes the expected loss under the recovery rate R. Finally, it is easy to see that the expected percentage loss of the mezzanine tranche taking losses from K_L to K_U under the assumption of a constant recovery rate R is

$$EL^R_{(K_L,K_U)} = EL_{\left(\frac{K_L}{1-R},\frac{K_U}{1-R}\right)} \tag{3.10}$$

After calibration of the input parameters TH and ρ it is straightforward to calculate the CDO premium. The threshold TH can be obtained by calibration of the individual default probabilities from observed market CDS spreads. More details can be found in Arvantis and Gregory [2001] and we will also discuss it in the following section.

How are expected tranche losses, thereby tranche prices, sensitive to the correlation in the LHP model? In order to see the effect of correlation on the expected tranche losses we calculate expected percentage losses for given correlation from 1% to 90%. Results are listed in Table 3.1. From the calculated expected tranche losses with corresponding

	$\rho = 1\%$	$\rho = 5\%$	$\rho = 10\%$	$\rho = 20\%$	$\rho = 30\%$	$\rho = 50\%$	$\rho = 90\%$
Equity	92.38%	81.80%	74.05%	62.64%	53.50%	38.85%	17.98%
3%-6%	7.62%	16.85%	20.47%	22.08%	21.64%	19.07%	13.68%
6%-9%	0%	1.27%	4.39%	8.7%	10.84%	12.09%	11.81%
9%-12%	0%	0.07%	0.91%	3.63%	5.84%	8.25%	10.51%
12%-22%	0%	0%	0.07%	0.77%	1.92%	4.19%	8.53%

Table 3.1: expected percentage losses for given correlations

correlations we can see the monotonic character of the equity and senior tranche, but not in mezzanine tranche. That can be seen more clearly by plotting in Figure 3.1

More precisely, when correlation goes up, the expected loss decreases in the equity tranche. This is also the fact that the equity tranche absorbs any losses below K_U, 3% for

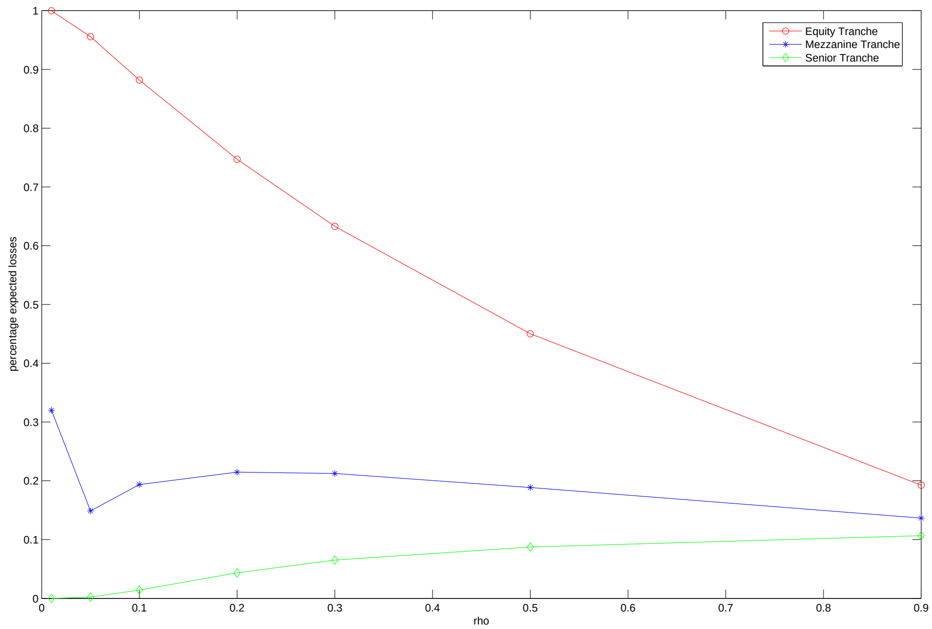


Fig. 3.1: sensitivity of the expected tranche losses to correlation

example, and the more senior tranches absorb losses above K_U. Increased default correlation among the firms referenced by the CDS, keeping the marginal default probabilities fixed, means that it becomes more likely to observe many or few defaults. Because of the upper limit on losses, the equity tranche is not affected much by occurrences with many defaults. On the other hand, there is upside in occurrences with few defaults, as the payments of the tranche holder would then decrease. This reduces the expected loss in the tranche and in turn, the fair spread. Thus, when correlation goes up, the expected loss decreases in an equity tranche.

Focusing instead on the senior tranche, we have the reverse relationship. Only losses above for K_L, 22% for example, of the pool affects this tranche. Thus, many defaults have to occur before it is affected. The probability of this event icreases with increased correlation so the expected loss and the fair spread of the senior tranche increase monotonically with correlation.

For mezzanine tranches, we do not have the monotonicity. The loss in the tranche is

$$L = \min\left(L_{portfolio}, K_U\right) - \min\left(L_{portfolio}, K_L\right)$$

For both components in the expression above, the expected value is decreasing in the correlation in the loss portfolio. Since the components enter the expression with opposite signs, we cannot generally be sure that the expected loss in the tranche is monotonic in the correlation. This means that we cannot expect fair spreads of mezzanine tranches to be monotonic in correlation.

Meanwhile, the relationship between expected tranche losses and correlations can be used to calibrate the market quotes. For a correlation of 0, each name behaves independently from each other and the total expected percentage portfolio loss converges by the application of the law of large numbers to the probability of default almost surely. From Table 3.1 we can see that for a correlation of 1% the whole expected loss is concentrated on the first two tranches. For increasing correlation, the mass of the loss distribution is shifted to the tails and therefore expected losses of the equity tranche decrease and expected losses of senior tranches increase.

The limitation of the large homogenous pool model is the application to relatively small portfolios. There will be non-diversified idiosyncratic risk left because the law of large numbers does not fully apply.

In order to calculate the CDO premium we need to calculate the time-dependent expected tranche losses first. The time dependency hides in the input parameter threshold TH, which is the inverse function of the individual default probabilities from the observed CDS market. The individual default probabilities from the observed CDS markets have close relationship with the default intensities. Therefore, in the following section the intensity model will be introduced. The other parameter, correlation which is also named *compound correlation* in order to differ from the other correlations can be implied from observed CDO tranche prices. This will be discussed in Section 3.4.

3.2 CDS Valuation and Intensity Calibration

Since the individual default probabilities have close relationship with the default intensities, we introduce now briefly how the default time distribution, i.e. the intensity in a reduced form model, can be calibrated from individual CDS quotes. The intensity discussed here is assumed to be deterministic, not stochastic. The stochastic intensity model will be discussed in Section 4.4. Since in the LHP approximation we assume the default time distribution and thus the intensity to be homogenous over the obligors, it is intuitive to derive the intensity as a constant.

3.2.1 The Intensity-Based Model

From the growing credit derivatives market the time of default can be modeled as an exogenous random variable, which could be fit to market data, such as the prices for defaultable bonds or credit default swaps. In comparison with the firm-value model of Merton this model is known as intensity-based model, also called the reduced-form approach, which defines the time of default as a continuous stopping time driven by a Poisson process. More precisely, the time of default is determined as the time of the first jump of a Poisson process with intensity process (doubly stochastic). As the model is calibrated from market data and is applied for the valuation of default contingent claims, the full specification of the model takes place under the equivalent martingale measure, the pricing measure $\mathbf{Q}$. Thus all probabilities and all expectations in the calculations for this model are defined with respect to $\mathbf{Q}$.

In this thesis the focus is not on the intensity-based model, only some important terminologies used in survival analysis will be laid out. More interested details about this model can be found in following literatures. The initial model was introduced by Jarrow and Turnbull [1995]. They modeled the time of default as the first jump of a homogenous Poisson process with a constant intensity λ. In their model the investor receives a predetermined recovery payment in case of default. Under the assumption that this recovery payment equals zero, it can be shown that a defaultable claim can be priced similar to the corresponding default-free claim when adjusting the discount factor for the probability of default. The default-adjusted short rate is equal to the sum of the default-free short rate $r(t)$ and the constant default intensity λ. Thus under zero recovery the default intensity can be interpreted as a credit spread accounting for the default possibility. But from historical spread data it becomes clear that spreads are not constant functions over time. Lando [1998a] generalized the Jarrow and Turnbull approach by allowing for stochastic intensities without loosing the attractive features. In Lando's model the time of default is driven by a doubly-stochastic Poisson process (Cox process). We restrict our analysis based on Lando's model.

Definition 3.2 (default indicator process) *The time of default τ is defined to be the time of the first jump of the doubly stochastic process $N = (N(t))_{t\geq 0}$ with an $\mathcal{F}_t$-adapted càdlàg intensity process $\lambda = (\lambda(t))_{t\geq 0}$ under $\mathbf{Q}$.*

$$\tau := \inf\{t : N(t) = 1\}$$

The stopped indicator process $N_\tau(t) := \mathbf{1}_{\{\tau\leq t\}}$ is equal to the doubly stochastic Poisson process $(N(t))_{t\geq 0}$ stopped at the time of default τ, i.e. $N_\tau(t) = N(t \wedge \tau)$.

Based on the default indicator process the information setup $(\mathcal{F}_t)_{t\geq 0}$ can be specified precisely.

Definition 3.3 (Information Setup) *The information setup is defined by the following filtrations, which are all assumed to be complete subsets of $\mathcal{F}$.*

- *$(\mathcal{G}_t)_{t\geq 0}$ contains all background information determining the market up to time t, excluding information on default behavior. Thus $r(t)$ and $\lambda(t)$ are $\mathcal{G}_t$-adapted and $\mathcal{G} = \bigcup_{t\geq 0} \mathcal{G}_t$ combines all the information about the market. It is not essential but is is more convenient to think of the background filtration that is generated by a state vector of economy.*

- *$(\mathcal{H}_t)_{t\geq 0}$ contains information whether default has occurred or not up to time t.*

$$\mathcal{H}_t = \sigma\left(\mathbf{1}_{\{\tau\leq s\}} : 0 \leq s \leq t\right)$$

- *$(\tilde{\mathcal{F}}_t)_{t\geq 0}$ contains information whether default has occurred or not up to time t and full market information, i.e. $\tilde{\mathcal{F}}_t = \mathcal{H}_t \vee \mathcal{G}$*

- *$(\mathcal{F}_t)_{t\leq 0}$ is the full filtration by combining $(\mathcal{G}_t)_{t\geq 0}$ and $(\mathcal{H}_t)_{t\geq 0}$, i.e. $\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{G}_t$*

To set up the framework we need to make some assumptions.

Assumption 3.1 *:*

- *Information: At time t, the defaultable contingent claims and default-free short rate prices of all maturities $T \geq t$ are known.*
- *Absence of arbitrage*
- *Independence: Under pricing measure $\mathbf{Q}$ the default-free interest rate dynamics are independent of the default probability.*

Based on the definition of the default model, the probability of survival of the obligor can be calculated.

Definition 3.4 (survival probability) *From the Definition 3.2 the probability of survival up to time t, given survival up to time s and market information up to time s, is denoted by*

$$P_{surv}(t|s) = \mathbf{Q}\left(\tau > t|\mathcal{F}_s \wedge \{\tau > s\}\right)$$

Using $\mathbf{Q}\left((N(t) - N(s)) = k|\mathcal{F}_s\right) = \frac{\left(\int_s^t \lambda(u)du\right)^k}{k!} e^{-\int_s^t \lambda(u)du}$ the survival probabilities can be calculated.

Theorem 3.5 (survival probability) *The probability of survival up to time t, conditional on survival up to time s, $s < t$, and full market information, is given by*

$$\tilde{P}_{surv}(t|s) = e^{-\int_s^t \lambda(u)du}.$$

The probability of survival up to time t, conditional on survival up to time s and market information up to time s, is given by

$$P_{surv}(t|s) = \mathbf{E}\left[e^{-\int_s^t \lambda(u)du}|\mathcal{F}_s\right]$$

Proof: For $\tau > s$, $\mathcal{G}_t \subset \mathcal{G}$

$$\begin{aligned}
\tilde{P}_{surv}(t|s) &= \mathbf{Q}\left(\tau > t|\tilde{\mathcal{F}}_s \wedge \{\tau > s\}\right) \\
&= \mathbf{Q}\left(N(t) = 0|\tilde{\mathcal{F}}_s \wedge \{N(s) = 0\}\right) \\
&= e^{-\int_s^t \lambda(u)du}.
\end{aligned}$$

Using iterated expectations and $\mathcal{F}_s \subset \tilde{\mathcal{F}}_s$

$$\begin{aligned}
P_{surv}(t|s) &= \mathbf{E}\left[\mathbf{1}_{\tau>t}|\mathcal{F}_s \wedge \{\tau > s\}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\mathbf{1}_{\tau>t}|\tilde{\mathcal{F}}_s \wedge \{\tau > s\}\right]|\mathcal{F}_s\right] \\
&= \mathbf{E}\left[e^{-\int_s^t \lambda(u)du}|\mathcal{F}_s\right]. \quad \square
\end{aligned}$$

Analogously we can calculate the default probability.

Corollary 3.6 *For the default model the probability of default up to time t, conditional on survival up to time s and market information up to time s, is given by*

$$\begin{aligned} P_{def}(t|s) &= \mathbf{Q}\left(\tau \leq t|\mathcal{F}_s \wedge \{\tau > s\}\right) \\ &= 1 - P_{surv}(t|s) \\ &= 1 - \mathbf{E}\left[e^{-\int_s^t \lambda(u)du}|\mathcal{F}_s\right]. \end{aligned}$$

Given information up to time t and non-occurrence of defaults up to time t, i.e.$\tau > t$, let us see what is the possibility that a default may occur in $[t, t + \Delta t]$, that can be also called the instantaneous default probability over the next small time interval Δt.

Lemma 3.7 (instantaneous default probability) *For $\tau > t$, conditional on information up to time t the instantaneous default probability is given by*

$$\lim_{\Delta t \to 0} \mathbf{Q}\left(\tau \leq t + \Delta t|\{\tau > t\} \wedge \mathcal{F}_t\right) = \lambda(t)\Delta t$$

Proof:

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\mathbf{Q}\left(\tau \leq t + \Delta t|\{\tau > t\} \wedge \mathcal{F}_t\right)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{\mathbf{Q}\left(t < \tau \leq t + \Delta t|\mathcal{F}_t\right)}{\mathbf{Q}\left(\tau > t|\mathcal{F}_t\right)\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\mathbf{Q}(\tau \leq t + \Delta t|\mathcal{F}_t) - \mathbf{Q}(\tau \leq t|\mathcal{F}_t)}{\mathbf{Q}(\tau > t|\mathcal{F}_t)\Delta t} \\ &= \frac{1}{\mathbf{Q}(\tau > t|\mathcal{F}_t)} \frac{\partial}{\partial t} \mathbf{Q}(\tau \leq t|\mathcal{F}_t) \\ &= \frac{1}{e^{-\int_0^t \lambda_s ds}} \frac{\partial}{\partial t}\left(1 - e^{-\int_0^t \lambda(s)ds}\right) \\ &= \frac{\lambda(t)e^{-\int_0^t \lambda(s)ds}}{e^{-\int_0^t \lambda(s)ds}} \\ &= \lambda(t). \end{aligned}$$

□

This intensity can be understood as the rate at which defaults occur. This links very closely to *hazard rate*. We can also call it *pre-default intensity* under $\mathbf{Q}$ in the interval $[t, t + \Delta t]$ conditional on the survival up to t. We can say it is an interpretation of the default intensity.
From the definition of the probability density function $f(t)$ of a distribution function $F(t) = \mathbf{Q}(\tau \leq t)$ for $t \geq 0$, we obtain:

$$f(t) = \lim_{\Delta t \to 0} \frac{\mathbf{Q}(t < \tau \leq t + \Delta t)}{\Delta t}$$

If the limits exist, we can define the hazard rate as follows.

Definition 3.8 *With the definition of the local arrival probability of the stopping time per time interval, hazard rate is defined as:*

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{Q}(\tau \leq t + \Delta t | \tau > t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\mathbf{Q}(t < \tau \leq t + \Delta t)}{\Delta t} \cdot \frac{1}{\mathbf{Q}(\tau > t)} \\ &= \frac{f(t)}{1 - F(t)} \end{aligned}$$

For a meaningful definition $F(t) < 1$, $f(t) \geq 0$ and $t \geq 0$ should be assumed.

Equivalently we can define the conditional hazard rate function $h(t|s)$, $t \geq s$ and $\tau > s$.

$$h(t|s) = \frac{f(t|s)}{1 - F(t|s)},$$

with the definition of conditional distribution function $F(t|s) = \mathbf{Q}(\tau \leq t|\mathcal{F}_s)$, $F(t|s) < 1$ and the corresponding conditional density function $f(t|s)$.
Comparing this with Lemma 3.7 we can see the relationship between the hazard rate and the default intensity. Schönbucher [2003] states that under some regularity conditions the default intensity coincides with the conditional hazard rate before the time of default. One of the useful tools to prove this is the theorem of Aven [1985]. We conclude the result in the following Theorem.

Theorem 3.9 *Let the time of default be defined as in definition 3.2 with the $\mathcal{G}_t$-adapted càdlàg intensity process $\lambda(t)$ and $P_{surv}(t|s)$ be differentiable from the right with respect to t at t=s. Let the difference quotients that approximate the derivative satisfy the regularity conditions in Aven's theorem. Then for $\tau > s$ the intensity of N is given by*

$$\lambda(s) = -\frac{\partial}{\partial t}\Big|_{t=s} P_{surv}(t|s) = h(s|s), \quad \forall \tau \geq s.$$

Proof: Schönbucher [2003] page 90.

In Definition 3.3 we define the background filtration $\mathcal{G} = \bigcup_{t \geq 0} \mathcal{G}_t$, which would be presented for the equivalent default-free model, i.e. a model in which all the same stochastic process (interest rates, exchange rates, share prices etc.) are modeled, with the exception of the default arrivals and the recovery rates. In particular, the default-free interest rate and intensity process are part of this model, just like in Definition 3.3 defined. Note that, although $(\mathcal{G}_t)_{t \geq 0}$ was generated without using the default indicator process, $N(t)$, it may be possible that $N(t)$ is measurable with respect to $(\mathcal{G}_t)_{t \geq 0}$ or that knowledge of the background information gives us some information on the realization of $N(t)$. We take this into consideration in the modeling environment in the following way: The jumps in $N(t)$ are caused by a background process, e.g. $N(t)$ jumps whenever a background process hits a prespecified barrier. This is also the case in the firm-value model. We will enlarge the equivalent default-free model to incorporate defaults as follows.
Recall Theorem 3.5 the time of default is characterized by its survival distribution function, $\tilde{P}_{surv}(t|s) = e^{-\int_s^t \lambda(u)du}$.

Definition 3.10 *As usual the default intensity process is a non-negative càdlàg process $(\lambda(t))_{t\geq 0}$ adapted to the filtration $\mathcal{G}_t$. Simulate a uniformly distributed on [0,1] random variable U under $(\mathcal{F}_0, \mathbf{Q})$, and independent of $\mathcal{G} = \bigcup_{t\geq 0} \mathcal{G}_t$*
The time of default is defined as the first time when the process $e^{-\int_0^t \lambda(s)ds}$ hits the level U:

$$\tau := \inf\{t : e^{-\int_0^t \lambda(s)ds} \leq U\}$$

This uniformly distributed on [0,1] random variable can be written as $U = e^{-E}$, with a standard exponentially distributed random variable E, i.e. $E \sim \mathcal{E}(1)$.

For the same conditions of Definition 3.10 instead of U with a standard exponentially distributed random variable E, the time of default can be redefined as:

$$\tau := \inf\{t : \int_0^t \lambda(s)ds \geq E\}$$

In the following we will present a general pricing formula for a default contingent claim valuation based on Lando's model and the recovery of market value (RMV) assumption from Duffie and Singleton [1999b]. The RMV assumption reduces the technical difficulties of defaultable claim valuation and leads to pricing formulas of great intuitive appeal. More precisely, under this assumption for an exogenously determined recovery rate the valuation of defaultable claims allows for the application of standard default-free pricing formulas where the default-free short rate is substituted with a default-adjusted short rate, which equals the sum of default-free short rate and intensity rate. Mentionable sources are Schönbucher [2000], Rutkowski and Bielecki [2000] and Casarin [2005].
We give the following specifications that define the default contingent claims.

Definition 3.11 (default contingent claim) *A default contingent claim with maturity T is defined by the following payment streams.*

- *The claim promises a payment of X, which is a $\mathcal{G}_T$-measurable random variable, at maturity T if no default has occurred before T.*
- *In case of a default at time $\tau \leq T$, the claim ceases to exist and the investor receives a compensatory recovery payment $R(\tau)$. The recovery payment takes place immediately at the time of default. In additional, $R = (R(t))_{t\geq 0}$ is a $\mathcal{G}_t$-adapted stochastic process. Thus $R(\tau)$ is known at the time of default τ and $R(t) = 0$ for $t > \tau$.*

Based on the intensity model above we can derive now the price of a default contingent claim following Lando [1998a] and Duffie [2001].

Theorem 3.12 (default contingent claim valuation) *Consider a default contingent claim with promised payment X at maturity T where X is $\mathcal{F}_T$-measurable. The recovery process $(R(t))_{t\geq 0}$ is $\mathcal{G}_t$-adapted. Assume that the time of default defined in Definition 3.2 follows the intensity-based model with a risk-neutral intensity process $(\lambda(t))_{t\geq 0}$. The following integrability conditions shall be satisfied for all $t \leq T$.*

$$\mathbf{E}\left[\exp\left(-\int_t^T r(s)ds\right)|X|\right] < \infty,$$

and

$$\mathbf{E}\left[\int_t^T |R(s)\lambda(s)| \exp\left(-\int_t^s (r(u)+\lambda(u))du\right) ds\right] < \infty.$$

For $\tau > t$ let $V(t)$ denote the value of the defaultable claim with maturity T. Let $\tilde{V}_X(t)$ denote the value of the discounted payout X at time t and $\tilde{V}_R(t)$ the discounted recovery payment $R(\tau)$. $V(t)$ is the sum of $\tilde{V}_X(t)$ and $\tilde{V}_R(t)$. For $\tau > t$

$$\begin{aligned}
V(t) &= \tilde{V}_X(t) + \tilde{V}_R(t) \\
\tilde{V}_X(t) &= \mathbf{E}\left[e^{-\int_t^T r(s)ds} X \mathbf{1}_{\{\tau > T\}} | \mathcal{F}_t\right] \\
&= \mathbf{E}\left[e^{-\int_t^T (r(s)+\lambda(s))ds} X | \mathcal{F}_t\right]; \qquad (3.11) \\
\tilde{V}_R(t) &= \mathbf{E}\left[e^{-\int_t^\tau r(s)ds} R_\tau | \mathcal{F}_t\right] \\
&= \mathbf{E}\left[\int_t^T R(s)\lambda(s) e^{-\int_t^s (r(u)+\lambda(u))du} ds | \mathcal{F}_t\right]. \qquad (3.12)
\end{aligned}$$

Proof: Using the law of iterated expectations and the measurability of the short rate with respect to $\mathcal{G}_t$ we can get for $\tau > t$

$$\begin{aligned}
\tilde{V}_X(t) &= \mathbf{E}\left[e^{-\int_t^T r(s)ds} X \mathbf{1}_{\{\tau > T\}} | \mathcal{F}\right] \qquad (3.13) \\
&= \mathbf{E}\left[\mathbf{E}\left[e^{-\int_t^T r(s)ds} X \mathbf{1}_{\{\tau > T\}} | \mathcal{G}_T \vee \mathcal{H}_t\right] | \mathcal{F}_t\right] \\
&= \mathbf{E}\left[e^{-\int_t^T r(s)ds} X \mathbf{E}\left[\mathbf{1}_{\{\tau > T\}} | \mathcal{G}_T \vee \mathcal{H}_t\right] | \mathcal{F}_t\right].
\end{aligned}$$

Recall that the σ-algebra $\mathcal{H}_t$ is generated by the default indicator process, using the complementary property of σ-algebra and the doubly stochasticity of the default time we can compute $\mathbf{E}\left[\mathbf{1}_{\{\tau \geq T\}} | \mathcal{G}_T \vee \mathcal{H}_t\right]$ for $\tau > t$ in the following way.

$$\begin{aligned}
\mathbf{E}\left[\mathbf{1}_{\{\tau \geq T\}} | \mathcal{G}_T \vee \mathcal{H}_t\right] &= \frac{\mathbf{Q}\left(\tau \geq T, \tau > t | \mathcal{G}_T\right)}{\mathbf{Q}\left(\tau > t | \mathcal{G}_T\right)} \\
&= \frac{\mathbf{Q}\left(\tau \geq T | \mathcal{G}_T\right)}{\mathbf{Q}\left(\tau > t | \mathcal{G}_T\right)} \\
&= \frac{\exp\left(-\int_0^T \lambda(s)ds\right)}{\exp\left(-\int_0^t \lambda(s)ds\right)} \\
&= \exp\left(-\int_t^T \lambda(s)ds\right).
\end{aligned}$$

By setting this into (3.13) we get

$$\tilde{V}_X(t) = \mathbf{E}\left[e^{-\int_t^T (r(s)+\lambda(s))ds} X | \mathcal{F}_t\right]$$

So we have had (3.11). Then for (3.12)

$$\tilde{V}_R(t) = \mathbf{E}\left[e^{-\int_t^\tau r(s)ds} R(\tau) | \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \mathbf{E}\left[\mathbf{E}\left[e^{-\int_t^\tau r(s)ds} R(\tau)|\mathcal{G}_T \vee \mathcal{H}_t\right]|\mathcal{F}_t\right] \\
&= \mathbf{E}\left[\int_t^T p_{def}(s|\mathcal{G}_T) e^{-\int_t^s r(u)du} R(s)ds|\mathcal{F}_t\right] \\
&= \mathbf{E}\left[\int_t^T R(s)\lambda(s) e^{-\int_t^s (r(u)+\lambda(u))du} ds|\mathcal{F}_t\right]
\end{aligned}
$$

where

$$
\begin{aligned}
P_{def}(s|\mathcal{G}_T) &= \frac{\partial}{\partial s}\mathbf{Q}\left(\tau \le s|\mathcal{G}_T\right) \\
&= \frac{\partial}{\partial s}\left(1 - e^{-\int_0^s \lambda(u)du}\right) \\
&= \lambda(s) e^{-\int_0^s \lambda(u)du}. \quad \square
\end{aligned}
$$

Under a zero recovery $V(t) = \tilde{V}_X(t) = \mathbf{E}\left[e^{-\int_t^T (r(s)+\lambda(s))ds} X|\mathcal{F}_t\right], \forall\tau \ge t$. The price of the corresponding default-free claim is $\mathbf{E}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right], \forall t \le T$. Thus we see that under a zero recovery assumption the price of the defaultable claim is equal to the price of the corresponding default-free claim with risk-adjusted short rate, $r(t) + \lambda(t)$. This also shows the reason why we refer often to λ as the credit spread (under zero recovery) compensating for the risk of loss through default. We use this formula for CDS valuation to get information from market data.

3.2.2 CDS Valuation

Now we go to the details for CDS valuation. For modeling purpose let us reiterate some basic terminology. We consider a frictionless economy with finite time horizon $[0, T]$. We assume that there exists a unique martingale measure $\mathbf{Q}$ making all the default-free and risky security prices martingales, after renormalization by the money market account. This assumption is equivalent to the statement that the markets for the riskless and credit-sensitive debt are complete and arbitrage-free. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{Q})$ is given and all processes are assumed to be defined on this space and adapted to the filtration $\mathcal{F}_t$.
Analogous to CDO pricing, in order to determine the CDS spread, the protection leg and the premium leg (as a function of the spread) are set to be equal.
The money market account that accumulates return at the spot rate $r(s)$ is defined as $A(t) = e^{\int_0^t r(s)ds}$.
Under above assumptions, we recall the discount factor as the expected discount value of a sure currency unit received at time T, that is,

$$
B(t, T) = \mathbf{E}\left[e^{-\int_t^T r(s)ds}\right]
$$

We consider in this thesis only the deterministic recovery rate and the intensity process $(\lambda(t))_{t\ge 0}$ is $\mathcal{G}_t$-adapted. In the case of a default before maturity the protection seller has to make the compensatory payment $(1-R)N$, $\tau \leq T$. The expected value today of this protection payment is:

$$V_{prot}(0) = \mathbf{E}\left[e^{-\int_0^{\tau} r(s)ds}\mathbf{1}_{\{\tau\leq T\}}(1-R)N\right]$$

With the results of theorem 3.12 this can be written as:

$$V_{prot}(0) = \mathbf{E}\left[\int_0^T (1-R)N\lambda(t)e^{-\int_0^t (r(u)+\lambda(u))du}dt\right]$$

Recall the probability of default from Corollary 3.6 and under the assumption of the independence between the short rate and the intensity process, the protection leg valuation is given by:

$$\begin{aligned} V_{prot}(0) &= \int_0^T (1-R)NB(0,t)\mathbf{E}\left[\lambda(t)e^{-\int_0^t \lambda(u)du}\right] dt \\ &= \int_0^T (1-R)NB(0,t)dP_{def}(0,t). \end{aligned}$$

The valuation of the premium leg is slightly more complicated since the accrued premium has to be considered. At each premium payment date, t_i, $i = 1,\ldots,n$ the protection buyer has to make a premium payment to the protection seller in case no default has occurred before the premium payment date. In case of a default event, at the default date the protection buyer has the pay the accrued premium since the last premium payment date to the protection seller. The valuation can be separated into two parts, one is the valuation for payment dates which no defaults has occurred, the other is the valuation between the time interval $[t_i-1, t_i]$, in which the default event is triggered:

$$V_{prem}(0) = \mathbf{E}\left[\sum_{i=1}^{n} e^{-\int_0^{t_i} r(u)du} sN\Delta t_i \mathbf{1}_{\{\tau>t_i\}}\right] + \mathbf{E}\left[\sum_{i}^{n} e^{-\int_0^{\tau} r(u)du}\mathbf{1}_{\{t_{i-1}<\tau\leq t_i\}} sN(\tau-t_{i-1})\right]$$

where $\Delta t_i = t_i - t_{i-1}$ is the year fraction between premium dates.
Analogously we can calculate this further using the valuation formulas.

$$\begin{aligned} V_{prem}(0) &= \mathbf{E}\left[\sum_{i=1}^{n} e^{-\int_0^{t_i}(r(t)+\lambda(t))dt} sN\Delta t_i\right] \\ &+ \mathbf{E}\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} sN(t-t_{i-1})\lambda(t)e^{-\int_0^t (r(u)+\lambda(u))du}dt\right] \end{aligned}$$

Under the independence assumption for the short rate and the intensity this follows

$$\begin{aligned} V_{prem}(0) &= \sum_{i=1}^{n} B(0,t_i)\mathbf{E}\left[e^{-\int_0^{t_i}\lambda(t)dt} sN\Delta t_i\right] \\ &+ \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} B(0,t)\mathbf{E}\left[sN(t-t_{i-1})\lambda(t)e^{-\int_0^t \lambda(u)du}\right] dt. \end{aligned}$$

where $(\lambda(t))_{t\geq 0}$ is the default intensity process of the reference entity and s is the annual CDS spread.
The CDS spread is determined such that the present values of the two legs are equal. From the above valuation formulas we can derive the following lemma.

Lemma 3.13 (CDS spread) *For a credit default swap with maturity date T, premium payment dates $0 < t_1 < t_2 < \ldots < t_n = T$ and notional N, the CDS spread is given by*

$$s = \frac{\mathbf{E}\left[\int_0^T (1-R)\lambda(t)e^{-\int_0^t (r(u)+\lambda(u))du}dt\right]}{\mathbf{E}\left[\sum_{i=1}^n e^{-\int_0^{t_i}(r(t)+\lambda(t))dt}\Delta t_i + \sum_{i=1}^n \int_{t_{i-1}}^{t_i}(t-t_{i-1})\lambda(t)e^{-\int_0^t (r(u)+\lambda(u))du}dt\right]} \tag{3.14}$$

$$= \frac{\int_0^T (1-R)B(0,t)dP_{def}(0,t)}{\sum_{i=1}^n B(0,t_i)\mathbf{E}\left[e^{-\int_0^{t_i}\lambda(t)dt}\Delta t_i\right] + \int_{t_{i-1}}^{t_i} B(0,t)\mathbf{E}\left[(t-t_{i-1})\lambda(t)e^{-\int_0^t \lambda(u)du}\right]dt} \tag{3.15}$$

3.2.3 Calibration of Default Intensity

In the following we will show how the default intensity can be derived from CDS quotes for the individual entities. The calibration method is mainly based on **?**, O'Kane and Schlögle [2001], Garcia and Ginderen [2001] and Elizalde [2005]. Hoefling [2006] gives a good brief summary. From Lemma 3.13 the CDS spread is a function of the default intensity if the short rate process and recovery rate are known. Thus, numerically we can invert this function to get the default intensity of the reference entity as a function of CDS spreads. But for a stochastic intensity a higher amount of CDS market quotes is needed and the calibration of the default intensity is more complex and time consuming. For ease we assume here the intensity to be deterministic. Moreover, we consider a special case of a constant default intensity such that a single CDS quote is sufficient to determine the intensity of the reference entity. In practice it is common to assume the recovery rate as constant, which is approximated by the average historical US corporate recovery rate ($\approx 40\%$). Under this assumption the enumerator in (3.15) can be approximated by:

$$\begin{aligned}\int_0^T (1-R)B(0,t)dP_{def}(0,t) &\approx \sum_{i=1}^n (1-R)B(0,t_i)\Big(P_{def}(0,t_i) - P_{def}(0,t_{i-1})\Big) \\ &= (1-R)\sum_{i=1}^n B(0,t_i)\left(e^{-\int_0^{t_{i-1}}\lambda(s)ds} - e^{-\int_0^{t_i}\lambda(s)ds}\right)\end{aligned}$$

If we do not consider the accrued premium term of (3.15), the denominator will become

$$\sum_{i=1}^n B(0,t_i)\mathbf{E}[e^{-\int_0^{t_i}\lambda(t)dt}\Delta t_i]$$

Under the constant intensity assumption it becomes much easier:

$$\sum_{i=1}^n B(0,t_i)e^{-\lambda t_i}\Delta t_i$$

The following lemma gives us the relationship between the CDS spread and the default intensity under the assumption of constant intensity and recovery rate, and without the consideration of accrued premium.

Lemma 3.14 (default intensity calibration from CDS spreads) *For a credit default swap with maturity date T, premium payment dates $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ and notional N, under the assumption of a constant intensity λ and a constant recovery rate R for the reference entity, the CDS spread is approximated by:*

$$s = (1-R)\frac{e^{\lambda\Delta}-1}{\Delta}, \tag{3.16}$$

where $\Delta = \Delta t_i = t_i - t_{i-1}$, $\forall 1 \leq i \leq n$, i.e. equivalent time interval of premium payment dates. Thus the intensity is calculated as:

$$\lambda = \frac{1}{\Delta}\ln\left(\frac{s\Delta}{1-R}+1\right). \tag{3.17}$$

Proof: Under the assumptions above the equation (3.13) can be approximated as:

$$\begin{aligned}
s &= \frac{(1-R)\sum_{i=1}^{n} B(0,t_i)\left(e^{-\lambda t_{i-1}} - e^{-\lambda t_i}\right)}{\sum_{i=1}^{n} B(0,t_i)e^{-\lambda t_i}\Delta t_i} \\
&= (1-R)\frac{\sum_{i=1}^{n} B(0,t_i)e^{-\lambda t_i}\left(e^{-\lambda(t_{i-1}-t_i)} - 1\right)}{\sum_{i=1}^{n} B(0,t_i)e^{-\lambda t_i}\Delta t_i} \\
&= (1-R)\frac{\sum_{i=1}^{n} B(0,t_i)e^{-\lambda t_i}\left(e^{\lambda\Delta} - 1\right)}{\sum_{i=1}^{n} B(0,t_i)\left(e^{-\lambda t_i}\right)\Delta} \\
&= (1-R)\frac{e^{\lambda\Delta}-1}{\Delta}.
\end{aligned}$$

This expression can be inverted to derive the constant default intensity λ as a function of only one given CDS spread,

$$\lambda = \frac{1}{\Delta}\ln\left(\frac{s\Delta}{1-R}+1\right). \quad \square$$

Under less simplifying assumptions for the default intensity, the intensity can be calibrated from CDS quotes in this manner. Since in the LHP model the default intensity is assumed to be homogeneous over the obligors, it is intuitive to derive the homogeneous default intensity from the average CDS spread of the reference portfolio from (3.17). Then the individual default time distribution of any obligor in the reference portfolio is given by $p = 1 - e^{-\lambda t}$. From $1 - p = e^{-\lambda t}$ one can see the approximate relationship between default intensity λ and default probability p. Using (3.2) we can get the homogeneous threshold value.

3.3 CDO Valuation

In this section we will perform the valuation of the tranches of a CDO contract based on the one-factor Gaussian model. As usual we assume that we are given a filtered probability

space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{Q})$ with the usual condition satisfied as above. $\mathbf{Q}$ is the equivalent martingale measure under which pricing takes place.
Consider a tranche initiated at time 0 with lower attachment point K_L and upper attachment point K_U. Let N_p denote the notional of the reference portfolio, s the tranche spread and $N_{tr} = (K_U - K_L)N_p$ the notional of the tranche. Denote the fixed premium payment dates of the contract are quarterly and the fixed rate day count fractions are actual/360. Recalling section 2.2.3 $EL_{(K_L,K_U)}(t)$ denote the expected percentage default loss in the (K_L, K_U)-th tranche up to time t.
At each default date the protection seller has to compensate the protection buyer with a payment equal to the change in the tranche loss (which is equal to zero if the tranche is not affected by a default loss in the reference portfolio). The present value of the protection leg (default leg) is given by the discounted expected default losses in the tranche,

$$DL(0) = \int_0^T B(0,s) N_{tr} dEL_{(K_L,K_U)}(s) \tag{3.18}$$

The above integral can be approximated by a discrete sum, where the premium dates are chosen as the grid for approximation. We perform a midpoint approximation, i.e. in each time grid interval $[t_{i-1}, t_i]$ the expected tranche loss $EL(t_i) - EL(t_{i-1})$ is discounted by the average discount factor of this interval, $(B(0,t_{i-1}) + B(0,t_i))/2$, thus (3.18) can be written as:

$$DL(0) \approx \sum_{i=1}^{n} \frac{B(0,t_{i-1}) + B(0,t_i)}{2} N_{tr} \left(EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1}) \right) \tag{3.19}$$

In return the protection buyer applies the tranche spread on the outstanding tranche notional $N_{out}(t)$ at time t, which is $N_{out}(t) = N_{tr}\left(1 - EL_{(K_L,K_U)}(t)\right)$.
Then for the year fraction between the default date and the next premium date (or the next default date, whichever comes first), the tranche spread is only applied to this new outstanding tranche notional. To approximate this we apply the tranche spread to the average outstanding notional of the tranche for each payment interval $\Delta t_i = t_i - t_{i-1}$,

$$\frac{N_{out}(t_{i-1}) + N_{out}(t_i)}{2} = N_{tr}\left(1 - \left(\frac{EL_{(K_L,K_U)}(t_{i-1}) + EL_{(K_L,K_U)}(t_i)}{2}\right)\right)$$

Thus the present value of the premium leg is approximated by :

$$PL(0) \approx \sum_{i=1}^{n} B(0,t_i) s \Delta t_i N_{tr}\left(1 - \left(\frac{EL_{(K_L,K_U)}(t_{i-1}) + EL_{(K_L,K_U)}(t_i)}{2}\right)\right) \tag{3.20}$$

Note that for the equity tranche (where the spread is fixed at 5%) we would have to add the upfront fee u to the premium leg and get

$$PL^{equity}(0) \approx uN_{tr} + \sum_{i=1}^{n} B(0,t_i) 5\% \Delta t_i N_{tr}\left(1 - \left(\frac{EL_{(K_L,K_U)}(t_{i-1}) + EL_{(K_L,K_U)}(t_i)}{2}\right)\right) \tag{3.21}$$

At the initiation of the trade the tranche spread s (or the tranche upfront u in case of the equity tranche) is fixed such that values of two legs are equal. Therefore

$$
\begin{aligned}
s &= \frac{\int_0^T \mathbf{E}\left[e^{-\int_0^s r(u)du} dEL_{(K_L,K_U)}(s)\right]}{\sum_{i=1}^n \mathbf{E}\left[e^{-\int_0^{t_i} r(u)du} \Delta t_i \left(1 - \frac{EL_{(K_L,K_U)}(t_{i-1}) + EL_{(K_L,K_U)}(t_i)}{2}\right)\right]} \\
&= \frac{\sum_{i=1}^n \frac{B(0,t_{i-1})+B(0,t_i)}{2} \left(EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1})\right)}{\sum_{i=1}^n B(0,t_i)\Delta t_i \left(1 - \frac{EL_{(K_L,K_U)}(t_{i-1}) + EL_{(K_L,K_U)}(t_i)}{2}\right)}
\end{aligned}
$$

The upfront fee of the equity tranche is calculated as

$$
\begin{aligned}
u &= \sum_{i=1}^n \frac{B(0,t_{i-1}) + B(0,t_i)}{2} \left(EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1})\right) \\
&- \sum_{i=1}^n B(0,t_i) 5\% \Delta t_i \left(1 - \frac{EL_{(K_L,K_U)}(t_{i-1}) + EL_{(K_L,K_U)}(t_i)}{2}\right)
\end{aligned}
$$

The tranche spread and the upfront fee of the equity tranche are functions of the expected tranche loss only. Thus we have to use a default model to calculate the expected tranche loss. In this section we have derived the expected tranche loss in an analytical form using the one-factor Gaussian model approximation. We derive the homogeneous default intensity from the average CDS spread of the portfolio using Lemma 3.13 and with $TH = \Phi^{-1}(p)$, where $p = 1 - e^{-\lambda t}$. Using (3.9) we can derive the expected tranche loss by the corresponding Vasiceck denstiy function. It is clear that the value of a CDO tranche is a function of the correlation parameter ρ of the one-factor Gaussian model. This correlation can be implied from observed CDO tranche prices. In the next section we will point out why the development of better models for CDO pricing is essential.

3.4 Drawbacks of the LHP Approach and the implied Correlations

Because of convenience and simplicity of the one-factor Gaussian model, it serves as a benchmark model in practice, but one major drawback of it is that it fails to fit the market prices of different tranches of a CDO reference portfolio correctly. More precisely, if we calculate the implied correlation parameter ρ (also termed compound correlation) from the market value for different tranches of a reference portfolio, those compound correlations are not unique for all tranches. For example we take the market prices of the *CDX.IG* series 5 5-year index issued on September 20th, 2005 to demonstrate the correlation smile. The quotes are listed in the first two columns of Table 3.2. Using the CDO valuation formulas in the one-factor Gaussian model we can calculate the compound correlation implied from the market quotes for each tranche. The recovery rate was assumed to be 40% for any obligor and the individual default probabilities were derived by the average CDS spread of the reference portfolio (47 bps). For simplicity the discount factor was set to 1. Given all those input parameters except for the correlation we can numerically

	Upfront fee	Market Spread	implied compound correlation
Equity	37.75%	5%	18.988%
3%-7%	0%	1.2%	4.786%
7%-10%	0%	0.3%	11.337%
10%-15%	0%	0.17%	17.504%
15%-30%	0%	0.08%	28.643%

Table 3.2: implied compound correlations for the given CDX tranches on September 20th, 2005

invert the tranche valuation formulas to derive the implied compound correlation. The calculated results are listed in the last column of Table 3.2.

Plotting the implied compound correlations by the corresponding tranches shows a smile curve. This observation became known as the correlation smile, also termed as *correlation skew*. Different correlations correspond to different loss distributions. It would mean that we assume different loss distributions for the same portfolio depending on which tranche we look at. This is definitively nonsense. It is clear that if the one-factor Gaussian model were correct, then the implied tranche correlations should be unique over different tranches if all tranches refer to the same underlying CDO portfolio. The conclusion is that either the market is not pricing accurately or that the assumed model to calculate implied default correlation, the one-factor Gaussian model, is wrong. Up to today, it is not yet clear whether this failure is due to technical issues or due to informational or liquidity effects. Note that the Gaussian family does not admit tail dependence and may fail to sufficiently create default clusters. There are some extensions of the LHP approach by using other distributional assumptions that produce heavy tails. For example the double t one-factor model proposed by Hull and White [2004] assumes Student t distributions for the common market factor as well as for the individual factors and a further modification, the one-factor normal inverse Gaussian (NIG) model by Kalemanova and Werner [2005]. Having a wrong assumption about default correlation values can be fatal, because the CDO trades are actually correlation trades.

In Torreseti et al. [2006] more flaws of the compound correlation are highlighted. First of all, for some market CDO tranche spreads compound correlation cannot be implied. The authors looked for the ten past years tranche spreads and found out that especially for *iTraxx* the 6%-9% tranche and for *CDX* the 7%-10% tranche the market spreads were too small to be inverted for compound correlation. Secondly, because of the non-monotonicity of mezzanine tranches more than one compound correlation can be implied from the unique market spread. And the procedure with compound correlation corresponding to the tranche is difficult to value off-market tranches, which are not liquidly traded in the market.

Because of above significant weaknesses of the compound correlation Ahluwalia and McGinty [2004a] and Ahluwalia and McGinty [2004b] of JPMorgan have developed a new type of implied correlation called *base correlation*, which is the correlation required to match quoted spreads for a sequence of first loss tranches of a standardized CDO

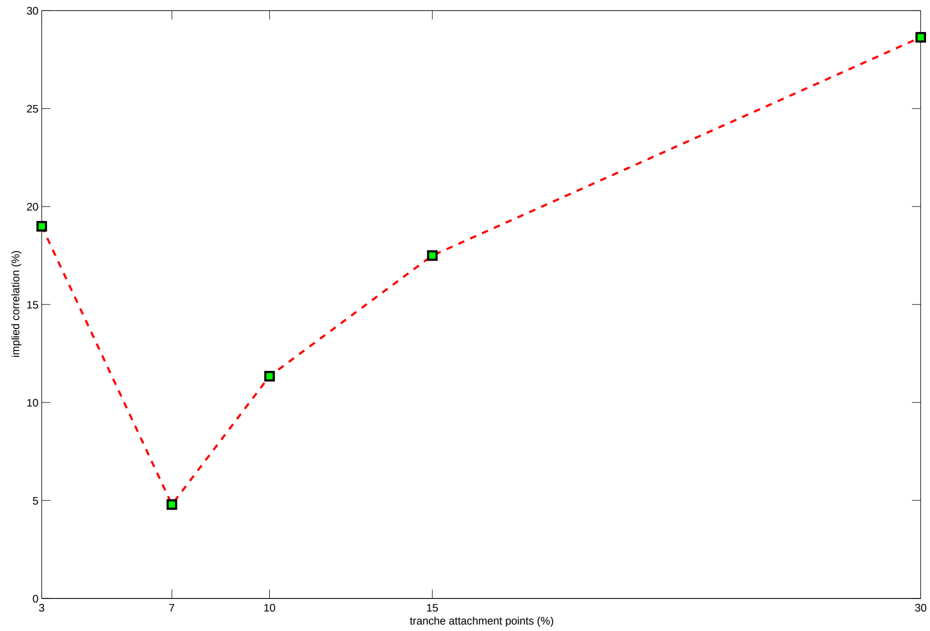


Fig. 3.2: the compound correlation skew for the given CDX tranches on September 20th, 2005

structure. From the point of view of the authors, this mechanism produces a meaningful and well-defined correlation skew, and avoids the difficulties associated with quoting correlation tranche-by-tranche, which can lead to meaningless implied correlations for mezzanine tranches. More precisely, this type takes its foundation in the monotonicity of equity tranche and extends this by including additional, fictive, equity tranches, which can be used to construct the traded mezzanine tranches. Formally, consider a tranche with lower and upper attachment points K_L and K_U and assume that an equity tranche with upper attachment point K_L is traded. Then,

$$EL_{(K_L,K_U)} = EL_{(0,K_U)} - EL_{(0,K_L)} \tag{3.22}$$

illustrates how the expected loss of a mezzanine tranche can be decomposed into the expected loss of two equity tranches.

For the $(0, K_L)$ tranche, we can, given the fair spread, invert the unique correlation, which produces this spread. We denote this correlation with base correlation for attachment point K_L. Note that this is the same as the compound correlation. Given the K_L base correlation we fix the expected loss in the equity tranche $EL_{(0,K_L)}$ of equation (3.22) and given the fair spread of the (K_L, K_U) tranche we iterate over the correlation parameter ρ_{base} in equation (3.1) which generates an expected loss of a fictive $(0, K_U)$ tranche such that the expected loss of the (K_L, K_U) tranche via equation (3.22) implies the given spread. This is denoted the *base correlation* of attachment point K_U and it is thus the unique correlation of a $(0, K_U)$ tranche which is consistent with the quoted spreads given

the base correlation of attachment point K_L. Proceeding in this fashion we can extract the base correlation for all attachment points of traded tranches.
Briefly speaking, the base correlation approach seeks to exploit the monotonicity of equity tranches to construct fictive equity tranches, consistent with the observed tranche spreads. This is done via a bootstrapping mechanism through equation (3.22) and results in a set of unique correlations. Embedded in the base correlation framework is a very convenient method to value off-market tranches based on the traded tranches. To understand the base correlation more let us look at a numerical example: *DJ tranched TRAC-X Europe* 5-year on May 4th, 2004. With given market spread 49 bps we can calculate the base correlation for each tranche (Table 3.3). By plotting (Figure 3.3) they show a clearly monotonic feature. Using the base correlation framework we can use the market standard liquid tranches to calibrate the model for base correlation inputs, and then interpolate from these to value off-the-run tranches (Table 3.4) with the same collateral pool. In spite of the flexibility of base correlation it has also some flaws. According to Willemann [2004] base correlation has also some flaws:

	Upfront fee	Market Spread	Base Correlation
Equity	32.30%	5%	27.06%
3%-6%	0%	2.67%	33.07%
6%-9%	0%	1.14%	37.69%
9%-12%	0%	0.61%	41.85%
12%-22%	0%	0.26%	54.11%

Table 3.3: the implied base correlations of DJ tranched TRAC-X Europe on May 4th, 2004

	Spread	Base Correlation
0%-1%	31.06%	23.05%
2%-3%	6.25%	27.06%
3%-4%	3.81%	29.07%
4%-5%	2.50%	31.07%
5%-6%	1.46%	34.61%
7%-8%	1.11%	36.15%
8%-9%	0.86%	37.69%
9%-10%	0.72%	39.07%

Table 3.4: the interpolated base correlations for off-market tranches based on DJ TRAC-X Europe on May 4th, 2004

- From increasing intensity correlations base correlations for some tranches may actually decrease. But for equity tranche the intensity model produces base correlation which is monotonic in the intensity correlation. Thus, the non-monotonic relationship is due to the bootstrapping process.

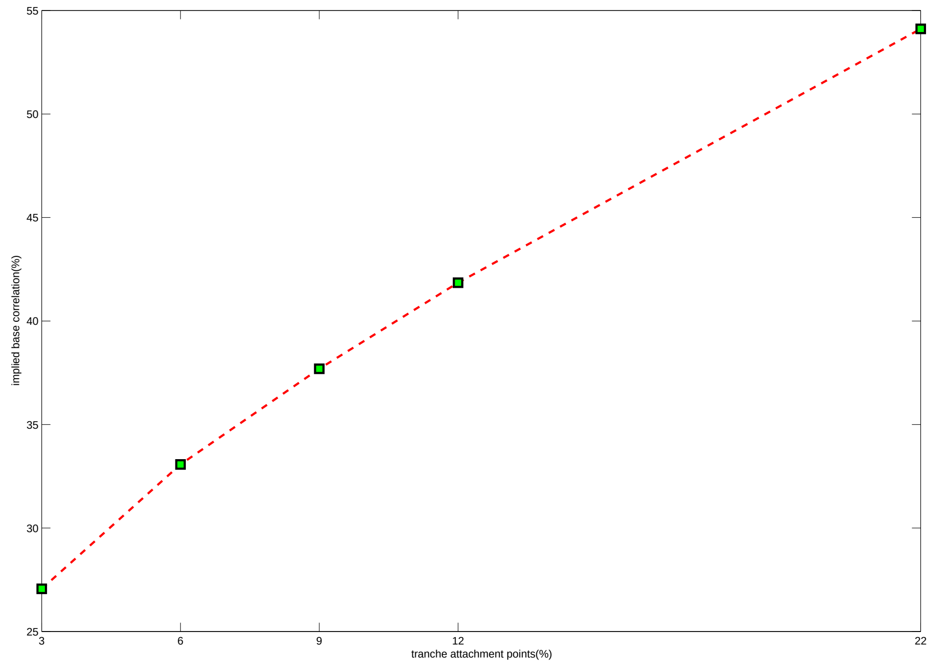


Fig. 3.3: base correlation skew for the given DJ TRAC-X Europe on May 4th, 2004

- In the relative valuation framework expected losses can go negative for steep correlation skews. We know that the equity tranche is monotonically decreasing in increasing correlation. From (3.22) we can see that the slope of the correlation skew can be so steep that the expected loss of the $(0, K_L)$ tranche becomes larger than the expected loss of the $(0, K_U)$ tranche. Thus, the expected loss of the (K_L, K_U) tranche becomes negative.

A main shortcoming of the implied correlation approach is that quoting a single correlation number per tranche for the whole portfolio. This means different correlation parameters for different parts of the same payoff. But this method does not account for the correlation heterogeneity between the single names. A lot of information, which influences the fair value of a portfolio is neglected. An alternative implied correlation measure, the implied *correlation bump* for relative value analysis of alternative tranched investments can be found in Mashal et al. [2004]

Chapter 4

CreditRisk$^+$ Model

CreditRisk$^+$ is a credit risk model developed by Credit Suisse Financial Products (CSFP). It is more or less based on a typical insurance mathematics approach, so it is sometimes classified in actuarial models. We introduce in this chapter basics of CreditRisk$^+$, based on the technical documentation in Wilde [1997] and some implementations based on Gundlach and Lehrbass [2004] for efficient and more stable computation of the loss distribution. We present two approaches for modeling the correlated default events and perform the estimation of default correlation from the equity market as well as from the credit market. Based on the idea in Lehnert and Rachev [2005] we investigate calibration of the original model by increasing default rate volatility, which produces fatter tails to meet market tranche losses. Additionally, similar to the correlation skew in the LHP model, different default rate volatilities for each tranche have to be used to meet market quotes. At last a dynamic version for heterogeneous credit portfolios will be introduced.

4.1 CreditRisk$^+$ Basics

The fundamental ideas for the original CreditRisk$^+$ model, which can be summarized as follows:

- **No model for default event:** No assumptions about the causes of default. Instead the default is described as a purely random event, characterized by a probability of default.
- **Stochastic probability of default and incorporating default rate volatilities:** The probability of default of an obligor is not seen as a constant, but a randomly varying quantity, driven by one or more (systematic) risk factors, the distribution of which is usually assumed to be a gamma distribution. Default rates are considered as continuous random variables and the volatility of default rates is incorporated in order to capture the uncertainty in the level of default rates. With default rate volatility the tail of the default loss distribution becomes fatter, while the expected loss remains unchanged. The effect of background factors, such as the

state of the economy, are incorporated into the model through the use of default rate volatilities and sector analysis rather than using default correlations as explicit inputs into the model.

- **Conditional independence:** Given the risk factors defaults of obligors are independent.

- **Only implicit correlations via risk drivers:** Correlations among obligors are not explicit, but arise only implicitly due to common risk factors which drive the probability of defaults.

- **Discretization of losses:** In order to aggregate losses in a portfolio in a comfortable way, they are represented as multiples of a common loss unit.

- **Sector analysis:** The model allows the portfolio of exposures to be allocated to sectors to reflect the degree of diversification and concentration present. The most diversified portfolio is obtained when each exposure is in its own sector and the most concentrated is obtained when the portfolio consists of a single sector. As the number of sectors is increased, the impact of concentration risk is reduced.

In the following section we will go deeply into mathematical modeling backgrounds.

4.1.1 Data Inputs for the Model

The inputs used by CreditRisk$^+$ Model are:

- **Credit Exposures** The CreditRisk$^+$ model is capable of handling all types of instruments that give rise to credit exposure. For some of these transaction types, it is necessary to make an assumption about the level of exposure in the event of a default: for example, a financial letter of credit will usually be drawn down prior to default and therefor the exposure at risk should be assumed to be the full nominal amount. In addition, if a multi-year time horizon is being used, it is important that the changing exposures over time are accurately captured.

- **Default Rates** A default rate, which represents the likelihood of a default event occurring within one year, should be assigned to each obligor. This can be obtained in a number of ways, including:
  Observed credit spreads from traded instruments can be used to provide market-assessed probabilities of default.
  Alternatively, obligor credit rating, together with a mapping of a credit rating to default rate, provides a convenient way of assigning probability of defaults to obligors. The rating agencies publish historic default statistics by rating category for the population of obligors that they have rated. It should be noted that one-year default rates show significant variation from year to year. During periods of economic recession, the number of defaults can be many times of the level observed at other times.

- **Default Rate Volatilities** Published default statistics include average default rates over many years. Actual observed default rates vary from these averages. The amount of variation in default rates about these averages can be described by the volatility (standard deviation) of default rates. The standard deviation of default rates can be significant compared to actual default rates, reflecting the high fluctuations observed during economic cycles. For example, in Figure 4.1 standard deviations of default rates were calculated over the period from 1970 to 1996 and therefore included the effect of economic cycles. As described above, the default rate volatility is used to model the effects of background factors rather than default correlations.

- **Recovery rates** In the event of a default of an obligor, a firm generally incurs a loss equal to the amount owed by the obligor less a recovery amount, which the firm recovers as a result of foreclosure, liquidation or restructuring of the defaulted obligor or the sale of the claim. Recovery rates should take account of the seniority of the obligation and any collateral or security held. Publicly available recovery rate data indicate that there can be significant variation in the level of loss, given the default of an obligor. Therefore, a careful assessment of recovery rate assumptions is required. But in this thesis we consider the average recovery rate.

Credit rating	One-year default rate (%)	
	Average	Standard deviation
Aaa	0.00	0.0
Aa	0.03	0.1
A	0.01	0.0
Baa	0.12	0.3
Ba	1.36	1.3
B	7.27	5.1

Source: Carty & Lieberman, 1996, Moody's Investors Service Global Credit Research

Fig. 4.1: average one-year default rates (%) from 1970 to 1996

Let us consider a portfolio consisting of K obligors. We denote by $\widetilde{p}_A$ the expected probability of default of obligor A. In general this quantity is the output of a rating process. Furthermore we denote by $\mathcal{E}_A$ the outstanding exposure of obligor A. This is assumed to be constant. In the case of a default, parts of $\mathcal{E}_A$ can be recovered and we only have to consider the potential loss $\widetilde{\nu}_A$ for obligor A. Thus the expected loss for obligor A is

$$EL_A = \widetilde{p}_A \widetilde{\nu}_A$$

Since it is one of the features of CreditRisk$^+$ to work with discretized losses, for this purpose one fixes a loss unit L_0 and chooses a positive integer ν_A as a rounded version of

$\widetilde{\nu}_A/L_0$. In order to compensate for the error due to the rounding process, one adjusts the expected probability of default rates to keep the expected loss unchanged, i.e. instead of $\widetilde{p}_A$ one considers

$$p_A = \frac{EL_A}{\nu_A L_0} = \frac{\widetilde{\nu}_A}{\nu_A L_0}\widetilde{p}_A$$

Therefore we consider from now on the expected probability of default rates p_A and integer-valued losses ν_A.

It should be noted that p_A describes an expectation rather than an actual value. The expectation usually represents a mean over time and economic situations in different countries or industries for obligors with similar risk profiles. There is a general agreement that the state of the economy in a country has a direct impact on observe default rates. Furthermore, for each year, different sectors will be affected to differen degrees by the state of economy. The magnitude of the impact depends on how sensitive an obligor's earnings are to various economic factors, such as the growth rate of the economy and the level of interest rates. In order to take into account the economy of a specific industry or country the obligor is active in, one can try to adjust the default probability via positive random scalar factors to the respective current risk situation. The latter can be seen to be determined by certain risk drivers according to countries or industries, for example. This corresponds to the introduction of sector variables in the terminology of CreditRisk$^+$ for the consideration of systematic default risk.

We consider N independent sectors $\mathcal{S}_1, \ldots, \mathcal{S}_N$. Each obligor A can be active in more than one sector, therefore we introduce sector weights ω_{Ak}, to fix shares ω_{Ak} of obligor A in the sector k. Such sector weights have to satisfy the condition:

$$0 \leq \omega_{Ak} \leq 1, \quad and \quad \sum_{k=1}^{N} \omega_{Ak} \leq 1.$$

Thus $\sum_{k=1}^{N} \omega_{Ak}$ specifies the systematic default risk share of obligor A, while $\omega_{A0} := 1 - \sum_{k=1}^{N} \omega_{Ak}$ represents the share of the obligor specific or idiosyncratic risk.

The sectors $\mathcal{S}_i$ are usually assumed to be independent Gamma-distributed random variables. Conditional on $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_N)$ with $\mathcal{S}_0 \equiv 1$ the probability of defaults is defined as:

$$p_A^S := p_A \left(1 + \sum_{k=1}^{N} \omega_{Ak} \left(\frac{\mathcal{S}_k}{\mathbf{E}[\mathcal{S}_k]} - 1\right)\right) = p_A \sum_{k=0}^{N} \omega_{Ak} \frac{\mathcal{S}_k}{\mathbf{E}[\mathcal{S}_k]} \tag{4.1}$$

with the expectation and variance:

$$\mathbf{E}[p_A^S] = p_A, \quad \mathrm{var}[p_A^S] = p_A^2 \sum_{k=1}^{N} \frac{\omega_{Ak}^2}{\mathbf{E}[\mathcal{S}_k]^2} \mathrm{var}[\mathcal{S}_k]$$

We can see from the definition that the sector variables affect probability of default of obligor through sector weights.

It is one of the main assumptions that, conditional on the sector $\mathcal{S}$, all obligors are independent. This feature is known as the conditional independence framework, the same as it in the large homogenous portfolio model.

4.1.2 Determining the Distribution of Default Loss using the Probability-Generating Function

The main object of interest is the distribution of the portfolio loss. We denote this loss by the random variable L. Expressing the default event of obligor A by $\mathbf{1}_A$, the portfolio loss is given by the default indicator and non-negative integer-valued losses ν_A

$$L = \sum_A \mathbf{1}_A \nu_A$$

Therefore the expected portfolio loss conditional on $\mathcal{S}$ and the corresponding conditional variance can be presented as:

$$\begin{aligned} EL^{\mathcal{S}} &= \mathbf{E}[L|\mathcal{S}] = \sum_A p_A^{\mathcal{S}} \nu_A \overset{4.1}{=} \sum_{k=0}^{N} \frac{\mathcal{S}}{\mathbf{E}[\mathcal{S}_k]} \sum_A \omega_{Ak} p_A \nu_A, \\ \mathrm{var}[X|\mathcal{S}] &= \sum_A \nu_A^2 \mathrm{var}[\mathbf{1}_A] \end{aligned}$$

As L attains only values in non-negative integers, an efficient way to calculate the loss distribution for L is via its probability-generating function (PGF).

The Probability-Generating Function in the Model

One of the features of the CreditRisk+ model is to derive the distribution of losses from probability-generating function (PGF), which can be used if L attains only values in the non-negative integers. (If one prefers to work with the true potential loss $\widetilde{\nu}_A$ and dispenses with the use of PGF, there is an option to use characteristic functions. This will be shown in Section 4.2)
We define the PGF in terms of an auxiliary variable z.

Definition 4.1 *If a random variable Y attains only values in the non-negative integers, its probability-generating function is defined as:*

$$G_Y(z) = \mathbf{E}[z^Y] = \sum_{n=0}^{\infty} p(n) z^n$$

with $p(n) = \mathbf{P}(Y = n)$.

The conditional PGF is

$$G_Y(z|\cdot) = \mathbf{E}[z^Y|\cdot] \tag{4.2}$$

Recall that this expectation exists only for $|z| \leq 1$.

Moreover, we recall the rules for calculation:

- If $G_Y(s) = G_Z(s)$, $\forall |s| < 1$, we can say that Y and Z have the same distribution.
- For independent random variables $Y_1, \cdots, Y_n$,

$$G_{Y_1+\cdots+Y_n}(s) = G_{Y_1}(s) \cdots G_{Y_n}(s)$$

- For any natural number k,

$$G_{kY}(z) = G_Y(z^k)$$

Theorem 4.2 *The probability-generating function for the portfolio loss is given as*

$$G_L(z) = \exp\left(\sum_A p_A \omega_{A0}(z^{\nu_A} - 1)\right) \exp\left(-\sum_{k=1}^{N} \frac{1}{\sigma_k^2} \ln\left(1 - \sigma_k^2 \sum_A p_A \omega_{Ak}(z^{\nu_A} - 1)\right)\right)$$

conditional on $\mathbf{E}[\mathcal{S}_k] = 1$*, and* $\sigma_k^2 = \mathrm{var}[\mathcal{S}_k]$.

Proof: We consider first for the Bernoulli variable $\mathbf{1}_A$, its probability-generating function is:

$$G_{\mathbf{1}_A}(z) = (1-p)z^0 + pz^1 = 1 + p(z-1),$$

where p is the probability of default rate of obligor A. Furthermore we denote the PGF $G_{\mathbf{1}_A}$ for the loss of obligor A by G_{L_A}, then we can determine G_{L_A} conditional on $\mathcal{S}$ using (4.2) by

$$G_{L_A}(z|\mathcal{S}) = 1 + p_A^{\mathcal{S}}(z^{\nu_A} - 1) \tag{4.3}$$

As a consequence of independence of obligors conditional on $\mathcal{S}$, we deduce via (4.3) the PGF for the whole portfolio conditional on $\mathcal{S}$ as the product of the individual PGFs conditional on $\mathcal{S}$ with the second calculation rule. Therefore

$$G_L(z|\mathcal{S}) = \prod_A G_{L_A}(z|\mathcal{S}) = \prod_A \left(1 + p_A^{\mathcal{S}}(z^{\nu_A} - 1)\right) \tag{4.4}$$

Knowing the probability density function $f_{\mathcal{S}_k}$ for the random risk factors $\mathcal{S}_k$ we can determine the unconditional probability-generating function for L by integrating

$$G_L(z) = \int G_L(z|\mathcal{S} = s) f_{\mathcal{S}}(s) ds \tag{4.5}$$

where $f_{\mathcal{S}}$ can be written as a product of $f_{\mathcal{S}_i}(s_i)$ for realization s_i of $\mathcal{S}_i$ due to the independence of sectors.
Rewrite (4.3) as

$$G_{L_A}(z|\mathcal{S}) = \exp\left(\ln\left(1 + p_A^{\mathcal{S}}(z^{\nu_A} - 1)\right)\right)$$

Given that the probabilities of default rates are small, the logarithms can be approximated, i.e.

$$\ln\left(1 + p_A^{\mathcal{S}}(z^{\nu_A} - 1)\right) \approx p_A^{\mathcal{S}}(z^{\nu_A} - 1),$$

which yields

$$G_{L_A}(z|\mathcal{S}) \approx \exp\left(p_A^{\mathcal{S}}(z^{\nu_A} - 1)\right)$$

which is the PGF for a Poisson-distribution random variable with parameter $p_A^{\mathcal{S}}$
Hence we can replace $\mathbf{1}_A$ with D_A which is a Poisson-distributed random variable with parameter $p_A^{\mathcal{S}}$. Then we obtain the PGF of L conditional on $\mathcal{S}$ as:

$$G_L(z|\mathcal{S}) = \exp\left(\sum_A p_A^{\mathcal{S}}(z^{\nu_A} - 1)\right) \tag{4.6}$$

Using (4.1) for $p_A^{\mathcal{S}}$ and assuming without loss of generality that $\mathbf{E}[\mathcal{S}_i] = 1$ (a natural condition that ensures $\mathbf{E}[p_A^{\mathcal{S}}] = p_A$) we obtain:

$$G_L(z|\mathcal{S}) = \exp\left(\sum_A \sum_{k=0}^N p_A \omega_{Ak} \mathcal{S}_k (z^{\nu_A} - 1)\right)$$

Rearranging the terms

$$\mu_k = \sum_A \omega_{Ak} p_A, \quad \mathcal{P}_k(z) = \frac{1}{\mu_k} \sum_A \omega_{Ak} p_A z^{\nu_A}$$

yields

$$\begin{aligned} G_L(z|\mathcal{S}) &= \exp\left(\sum_{k=0}^N \mathcal{S}_k \left(\sum_A p_A \omega_{Ak} (z^{\nu_A} - 1)\right)\right) \\ &= \exp\left(\sum_{k=0}^N \mathcal{S}_k \mu_k \left(\mathcal{P}_k(z) - 1\right)\right) \end{aligned} \tag{4.7}$$

Assumed that sectors $\mathcal{S}_i$ are Gamma-distributed we get in this case according to (4.5)

$$G_L(z) = \int G_L(z|\mathcal{S} = s) f_{\mathcal{S}}^{\alpha,\beta}(s) ds \tag{4.8}$$

using multiple integration, $s_0 \equiv 1$ and parameter vectors $\alpha = (\alpha_1, \ldots, \alpha_N)^T$ and $\beta = (\beta_1, \ldots, \beta_N)^T$.
Recalling the probability density function of a Gamma-distributed random variable,

$$f_{\mathcal{S}}^{\alpha,\beta}(s) = \frac{s^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{s}{\beta}}$$

for $s \geq 0, \alpha > 0, \beta > 0$ and the gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, its expectation and variance are given by

$$\mathbf{E}[\mathcal{S}_i] = \alpha_i \beta_i, \quad \mathrm{var}[\mathcal{S}_i] = \alpha_i \beta_i^2 \tag{4.9}$$

From the condition $\mathbf{E}[\mathcal{S}_i] = 1$ we get for (4.8)

$$\begin{aligned} G_L(z) &= \int \exp\left(\sum_{k=0}^N s_k \mu_k \left(\mathcal{P}_k(z) - 1\right)\right) \prod_{l=1}^N \frac{s_l^{\alpha_l - 1} \alpha_l^{\alpha_l}}{\Gamma(\alpha_l)} e^{-\alpha_l s_l} ds_l \\ &= e^{\mu_0 (\mathcal{P}_0(z) - 1)} \prod_{k=1}^N \frac{\alpha_k^{\alpha_k}}{\Gamma(\alpha_k)} \int_0^\infty e^{s_k [\mu_k (\mathcal{P}_k(z) - 1) - \alpha_k]} s_k^{\alpha_k - 1} ds_k \end{aligned}$$

as $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x} dx$,

$$\begin{aligned} G_L(z) &= e^{\mu_0 (\mathcal{P}_0(z) - 1)} \prod_{k=1}^N \frac{\alpha_k^{\alpha_k}}{\Gamma(\alpha_k)} \left(-\frac{1}{\mu_k \left(\mathcal{P}_k(z) - 1\right) - \alpha_k}\right)^{\alpha_k} \int_0^\infty e^{-s_k} s_k^{\alpha_k - 1} ds_k \\ &= e^{\mu_0 (\mathcal{P}_0(z) - 1)} \prod_{k=1}^N \left(\frac{\alpha_k}{\alpha_k - \mu_k \left(\mathcal{P}_k(z) - 1\right)}\right)^{\alpha_k} \end{aligned}$$

From (4.9) we can get $\alpha_k = 1/\mathrm{var}[\mathcal{S}_k]$ and replace $1/\alpha_k$ by $\mathrm{var}[\mathcal{S}_k] = \sigma_k^2$ we can easily get our wanted formula. $\square$

The only variable to specify is α_k which can be determined from the relation between α_k and σ_k^2. In the optimal case there exists a way to estimate the variance of sector σ_k^2. The CreditRisk$^+$ model regards each sector as driven by a single underlying factor, which explains the variability over time in the average total default rate measured for that sector. The underlying factor influences the sector through the total expected rate of defaults in the sector, which is modeled as a random variable with mean μ_k and standard deviation σ_k specified for each sector. The sector standard deviation σ_k may be estimated from the set σ_A of obligor standard deviation by averaging process, for example, by choosing the standard deviation σ_k for $\mathcal{S}_k$ as the sum of weighted estimated standard deviations for each obligor in the sector, i.e.

$$\sigma_k = \sum_{A \in \mathcal{S}_k} \omega_{Ak} \sigma_A$$

In Wilde [1997] section A 7.3 it is suggested that the standard deviation of default rate is roughly equal to the probability of default rate for each name.In absence of detailed data, the obligor specific estimates of the ratio of standard deviation to mean, σ_k/μ_k, for each obligor can be replaced by a single flat ration instead of the order of one. By setting $\alpha_k = \sigma_k^{-2}$ we can obtain the PGF of the portfolio loss. It was suggested in Lehrbass and Thierbach [2001] to use $\sigma_k = \sum_A \sqrt{\omega_{Ak}} \sigma_A$ for generating fatter tails in the loss distribution.

In order to determine the portfolio loss distribution via its PGF G_L in an efficient way, it was based on Panjer recursion, see Panjer and Willmot [1992] and Binnenhei [2000]. But it has turned out that the Panjer recursion is numerically unstable. Thus numerically stable computation methods should be searched. One of the alternatives is to use the implementation of the characteristic function that will be discussed in the following section. In addition, the PGF cannot only be used to determine the distribution of a random variable, but also for calculating its variance.

$$\begin{aligned} G'_Y(z) &= \frac{1}{z} \mathbf{E}\left[Y z^Y\right], \\ G''_Y(z) &= \frac{1}{z^2} \mathbf{E}\left[\left(Y^2 - Y\right) z^Y\right]. \end{aligned}$$

in particular,

$$\begin{aligned} G'_Y(1) &= \mathbf{E}[Y], \\ G''_Y(1) &= \mathbf{E}[Y^2] - \mathbf{E}[Y]. \end{aligned}$$

Hence one has for the variance

$$\sigma_Y^2 = \mathbf{E}[Y^2] - \mathbf{E}[Y]^2 = G''_Y(1) + G'_Y(1) - G'_Y(1)^2.$$

Note that in the current set-up the default correlation between any two obligors A, B is given by

$$\rho_{AB} = \frac{\mathbf{E}[\mathbf{1}_A \mathbf{1}_B] - \mathbf{E}[\mathbf{1}_A]\mathbf{E}[\mathbf{1}_B]}{\sqrt{\mathbf{E}[\mathbf{1}_A]\mathbf{E}[1 - \mathbf{1}_A]\mathbf{E}[\mathbf{1}_B]\mathbf{E}[1 - \mathbf{1}_B]}}$$

As

$$\begin{aligned}\mathbf{E}[\mathbf{1}_A\mathbf{1}_B] &= \mathbf{E}[\mathbf{E}[\mathbf{1}_A|\mathcal{S}]\mathbf{E}[\mathbf{1}_B|\mathcal{S}]] \\ &= \mathbf{E}[p_A^{\mathcal{S}}p_B^{\mathcal{S}}] \\ &= p_A p_B \sum_{k,l=0}^{N} \omega_{Ak}\omega_{Bl}\frac{\mathbf{E}[\mathcal{S}_k\mathcal{S}_l]}{\mathbf{E}[\mathcal{S}_k]\mathbf{E}[\mathcal{S}_l]}\end{aligned}$$

we obtain, due to the independence of the risk factors,

$$\begin{aligned}\mathbf{E}[\mathbf{1}_A\mathbf{1}_B] &= p_A p_B \left(\sum_{k\neq l}\omega_{Ak}\omega_{Bl} + \sum_{k=0}^{N}\omega_{Ak}\omega_{Bl}\frac{\mathbf{E}[\mathcal{S}_k^2]}{\mathbf{E}[\mathcal{S}_k]^2}\right) \\ &= p_A p_B \left(1 - \sum_{k=0}^{N}\omega_{Ak}\omega_{Bk} + \sum_{k=0}^{N}\omega_{Ak}\omega_{Bk}\frac{\mathrm{var}[\mathcal{S}_k] + \mathbf{E}[\mathcal{S}_k]^2}{\mathbf{E}[\mathcal{S}_k]^2}\right),\end{aligned}$$

which finally yields

$$\rho_{AB} = \frac{\sqrt{p_A p_B}}{\sqrt{(1-p_A)(1-p_B)}}\sum_{k=1}^{N}\frac{\omega_{Ak}\omega_{Bk}}{\alpha_k}$$

4.2 CreditRisk$^+$ in terms of the Characteristic Function

Analytical tractability is perhaps the most compelling advantage of CreditRisk$^+$ over competing models of portfolio credit risk. The algorithm eliminates the need for Monte Carlo simulation, computation time is reduced dramatically, and simulation error is avoided entirely.

However, as is well known in Gordy [2002b] and Wilde [2000], the standard recursion relation for the loss distribution in CreditRisk$^+$, which goes back to Panjer recursion (Panjer and Willmot [1992]), tends to be numerically unstable for large portfolios, and the need to round loss exposures to integer multiples of the so-called basic loss unit L_0 may introduce a trade-off between speed and accuracy. Up to now, some algorithms such as nested evaluation of the moment-generating function (Giese [2003]), saddlepoint approximation (Gordy [2002a]) or Fast Fourier Transform (FFT) (Reiss [2003]), respectively, instead of Panjer recursion have been investigated.

Since the proper choice of the basic loss unit may be crucial, it is an advantage of FFT algorithms by relaxing the requirement for loss discretization. The result of this analysis is stated how to obtain the distribution of a random variable from its characteristic function. This general technique, which is based on Fourier Transform, is applied to the CreditRisk$^+$ model and yields efficient and numerically stable algorithm. In addition, this algorithm is easy to implement and is numerically stable for large portfolios.

First of all, let us recall some basics about the characteristic function of a real-valued random variable.

4.2.1 From the Characteristic Function to the Probability Density Function via Fourier Inversion

Let X be a real-valued random variable. Then the characteristic function of X is defined by

$$\varphi_X(z) := \mathbf{E}\left[e^{izX}\right]$$

Note that the characteristic function $\varphi_X(z)$ exists for any real-valued random variable X and for all $z \in \mathbb{R}$. This is one reason why this function is a powerful tool in stochastics. There is a close connection between the characteristic function and the Fourier Transform. Recall that functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ are noted as in $\mathbf{L}^1$.
Let f be in $\mathbf{L}^1$. Then the **Fourier Transform** of f exists and is defined by

$$\mathcal{F}f(z) := \int_{-\infty}^{\infty} e^{izx} f(x)dx.$$

If f is the probability density of a random variable X, then the characteristic function of X is given by the Fourier Transform of f,

$$\varphi_X(z) = \mathcal{F}f(z).$$

From the Fourier inversion theorem we can get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \mathcal{F}f(z)dz.$$

assumed that f is in $\mathbf{L}^1$ and also $\mathcal{F}f$ is in $\mathbf{L}^1$.
Assume that the characteristic function $\varphi_X(z)$ of a random variable X is given, and $\varphi_X(z) \in \mathbf{L}^1$. Then the density of X can be computed. Unfortunately, there is the precondition that the characteristic function is in $\mathbf{L}^1$. Before presenting a method for the case that the characteristic function is not integrable, we recall the relationship between the probability-generating function and the characteristic function of a random variable.
Let X be a discrete random variable with values in $\mathbb{N}$. The probability-generating function (PGF) of X is given by

$$G_X(z) := \mathbf{E}\left[z^X\right],$$

and note that this function exists at least for $|z| \leq 1$. The name and the relevance of this function is based on the property

$$\mathbf{Q}\left(X = n\right) = \frac{1}{n!}\frac{d^n G_X}{dz^n}(0).$$

From $G_X(z)$ one can obtain the corresponding characteristic function easily

$$\varphi_X(z) = G_X\left(e^{iz}\right). \tag{4.10}$$

4.2.2 Portfolio Loss in terms of the Characteristic Function

Consider a portfolio with K obligors. If obligor A defaults, the corresponding loss given default is given by ν_A, which should be positive. For the dependencies among obligors, we assume that there are N sectors and the affiliation of each obligor to these sectors is described by the factor loadings ω_{Ak}. The share of idiosyncratic risk is denoted by ω_{A0}. These N sectors are modeled as independent Gamma-distributed random variables S_k that have expectation 1 and variance σ_k^2. It is clear that the default probability p_A of the obligor A is dependent on sector variables in which A is active. With the additional normalization condition, the default probability conditional on $\mathcal{S}$ in (4.1) can be rewritten as

$$\mathbf{Q}(\mathbf{1}_A|\mathcal{S}) = p_A^S = p_A\left(\omega_{A0} + \sum_{k=1}^{N}\omega_{Ak}S_k\right) \tag{4.11}$$

where p_A is the average probability of default within one year.
For a given state of the sector variables S_k default events of obligors are independent of each other, known as conditional independence. The portfolio loss denoted by L is given by

$$L = \sum_{k=1}^{N} L_k = \sum_{k=1}^{N}\sum_{A\in\mathcal{S}_k}\mathbf{1}_A\nu_A$$

and the expected loss and the variance of L are given by

$$EL = \mathbf{E}[L] = \sum_{k=1}^{N}\sum_{A\in\mathcal{S}_k} p_A\nu_A, \tag{4.12}$$

$$\sigma_L^2 = \mathrm{var}[L] = \sum_{k=1}^{N}\sigma_{L_k}^2 = \sum_{k=1}^{N}\mathrm{var}[L_k] = \sum_{k=1}^{N}\left(\sum_{A\in\mathcal{S}_k} p_A\nu_A^2 + \sigma_k^2 EL_k^2\right) \tag{4.13}$$

Under the assumption of independence, σ_L^2 is simply the sum of the variances of each sector losses $\sigma_{L_k}^2$.

Recalling Theorem 4.2 the PGF of the portfolio loss L is known as:

$$G_L(z) = e^{\sum_A p_A\omega_{A0}(z^{\nu_A}-1)}\prod_{k=1}^{N}\left(\frac{1}{1+\sigma_k^2\sum_A p_A\omega_{Ak}(1-z^{\nu_A})}\right)^{\frac{1}{\sigma_k^2}}$$

From the general relationship between the PGF and the corresponding characteristic function in (4.10), one can obtain the characteristic function, which is valid without introduction of a basic loss unit and relaxing the requirement for loss discretization:

$$\varphi_L(z) = e^{\sum_A p_A\omega_{A0}(e^{iz\nu_A}-1)}\prod_{k=1}^{N}\left(\frac{1}{1+\sigma_k^2\sum_A p_A\omega_{Ak}(1-e^{i\nu_A z})}\right)^{\frac{1}{\sigma_k^2}}. \tag{4.14}$$

4.2.3 Applying the Fourier Transform in the CreditRisk$^+$ Model

From the Fourier inversion theorem one can obtain a continuous density of the distribution of the portfolio loss L from its characteristic function. But in general, the characteristic function may be not integrable. The method following aims to find an approximative density using the special structure of the Fast-Fourier-Transform (henceforth FFT) algorithm.

The FFT-based Fourier Transform

Let $\Phi_X(z)$ be the known characteristic function of a real-valued random variable X. Perhaps this random variable has no density, but let us assume that the distribution is "almost" continuous, that is to say, X may be approximated by a random variable that has a density f. Further it is assumed that f vanishes outside a given interval $[a, b]$. Its Fourier transform can be numerically computed by the following algorithms. More details can be found in Press et al. [1992] and Reiss et al. [2003].
Let Z be the number of sample points (usually a power of 2) and define $\Delta x := \frac{b-a}{Z-1}$, $\Delta z := \frac{2\pi}{Z\Delta x}$ as well as three Z-dimensional vectors and a $Z \times Z$ matrix M for $j, k = 0, \ldots, Z-1$ by:

$$x_k := a + k\Delta x$$

$$f_k := f(x_k)$$

$$z_k := \begin{cases} k\Delta z, & \text{if } k < \frac{Z}{2}; \\ (k-Z)\Delta z, & \text{else.} \end{cases}$$

$$M_{jk} := e^{2\pi i \frac{jk}{Z}}$$

Define the set $\mathcal{Z} := z_k, k = 0, \ldots, Z-1$. The Fourier transform of f at the points z_k will be denoted by $\varphi_k := \varphi_X(z_k)$. Then the Fourier integral can be computed by the approximation

$$\int_{-\infty}^{\infty} e^{izx} f(x)dx \approx \Delta x e^{iza} \sum_{k=0}^{Z-1} e^{izk\Delta x} f(a + k\Delta x). \tag{4.15}$$

Using the vector and matrix notations, equation (4.15) can be rewritten as

$$\varphi_k = \Delta x e^{iaz_k} \sum_{j=0}^{Z-1} M_{kj} f_j. \tag{4.16}$$

Having chosen a suitable interval $[a, b]$ and the number of Fourier steps Z, the density f_j is the only unknown in equation (4.16). Since the inverse of M_{jk} is given by

$$M_{jk}^{-1} = \frac{1}{Z} e^{-2\pi i \frac{jk}{Z}},$$

equation (4.16) can be easily solved for f_j. A short calculation shows that this is in fact the inverse:

$$\sum_{k=0}^{Z-1} M_{jk} M_{kl}^{-1} = \frac{1}{Z} \sum_{k=0}^{Z-1} \left(e^{2\pi i \frac{j-1}{Z}} \right)^k = \begin{cases} 1, & \text{j=l;} \\ \frac{1}{Z} \frac{1-e^{2\pi i (j-1)}}{1-e^{2\pi i (j-1)/Z}}, & j \neq l. \end{cases} \tag{4.17}$$

Hence the inversion formula based on the linear equations is given by

$$f_j = \frac{1}{Z\Delta x} \sum_{k=0}^{Z-1} e^{-2\pi i \frac{jk}{Z}} e^{-iaz_k} \varphi_k. \tag{4.18}$$

So even if the characteristic function is not integrable, the Fourier inversion by FFT works. Since the algorithm does not compute an integral, but it solves a set of linear equations that describe the Fourier transform of the unknown density f.

Although the portfolio loss is a discrete random variable, it is intuitively obvious that there is an approximative density for the distribution if the number of obligors is large. This density can be determined using the FFT algorithm. The portfolio loss lies between 0, in case that no obligor defaults, and $\sum_A \nu_A$, which corresponds to the worst case that all obligor default. Hence the density f resides between 0 and $\sum_A \nu_A$.

Let Z be the number of sample points; then the distance between two adjacent sampling points is given by $\Delta x = \frac{1}{Z-1} \sum_A \nu_A$ and $\Delta z = \frac{2\pi}{Z\Delta x}$. The choice of Z depends on the accuracy one needs for further computations using the density. We set $Z = 2^n$ and n lies round about 10.

To compute $\varphi_L(z)$ one may adapt the equation (4.14):

$$\varphi_L(z) = e^{\xi_0(z)} \prod_{k=1}^{N} \left(\frac{1}{1-\sigma_k^2 \xi_k(z)} \right)^{\frac{1}{\sigma_k^2}} \tag{4.19}$$

where $\xi_k(z) := \sum_A p_A \omega_{Ak} \left(e^{i\nu_A z} - 1 \right)$, $k = 0, \ldots, N$.

Using the natural logarithm one can rewrite the power expression as:

$$\left(\frac{1}{1-\sigma_k^2 \xi_k(z)} \right)^{\frac{1}{\sigma_k^2}} = \exp\left(-\frac{1}{\sigma_k^2} \ln\left(1 - \sigma_k^2 \xi_k(z)\right) \right) \tag{4.20}$$

Hence the characteristic function of portfolio loss L can be evaluated using the equation:

$$\varphi_L(z) = \exp\left(\xi_0(z) - \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \ln\left(1 - \sigma_k^2 \xi_k(z)\right) \right). \tag{4.21}$$

This formula is numerically more stable than the representation (4.19). The Fourier inversion of this function using the FFT algorithm yields the probability density function of L.

4.3 Sector Weights Estimation

The CreditRisk$^+$ Model measures the benefit of portfolio diversification and the impact of concentrations through the use of sector analysis. Concentration risk results from having a number of obligors whose fortunes are affected by a common factor in the portfolio. In order to quantify concentration risk, the concepts of systematic factors and specific factors are necessary. Systematic factors are background factors that affect the fortunes (asset values, for example) of a proportion of the obligors in the portfolio. The fortunes of any one obligor can be affected by a number of systematic factors. Additionally, the fortunes of an obligor are affected to some extent by specific factors unique to the obligor. It can be assumed that each obligor is subject to only one systematic factor, which is responsible for all of the uncertainty of the obligor's default rate. Once the obligor is allocated to a sector, the default rate and default rate volatility are set individually. In this case, a sector can be thought of as a collection of obligors having the common property and influenced by the same single systematic factor.

A more generalized approach is to assume that the fortunes of an obligor are affected by a number of systematic factors. This situation can be expressed by apportioning an obligor across several sectors rather than by allocating an obligor to a single sector.

In CreditRisk$^+$ each obligor of a given portfolio of credits can be allocated to a set of independent risk factors represented by a continuous random variable of known distribution. The obligors sharing a sector are dependent on a common random factor and therefore their default events can be modeled as being correlated random variables. The definition and parameterizations of sectors play a key role in model implementation and moreover the distribution of losses is heavily influenced by the chosen sector composition and its parameters.

First of all, the dependence structure of the correlated default events should be modeled. We introduce two methods. The first one is based on the Merton threshold model for modeling correlated default events. Taking the equity prices as a proxy for asset returns, the asset correlations can be transformed to default correlations. The seconde one is based on the reduced form model, in which default events are modeled as correlated default intensity processes. From the relationship between CDS spreads and hazard rates, the hazard rate correlation, (therefore, default correlation), can be estimated from the credit market data. Since the estimated default correlation matrix describes the dependency of the default events from the considered industries, we perform a principal component analysis (PCA) on the correlation matrix. Through an orthogonal transformation, the dependent risk factors with certain dependence structure are transformed to some latent independent sectors. Meanwhile, the corresponding sector weights are also calculated.

4.3.1 Correlated Default Events Modeling

Correlated Defaults Modeling based on the Asset-Value Threshold Model

As widely used in practice, we introduce a method for estimating default correlation by using equity prices as a proxy for asset returns, whose correlations are in turn transformed to default correlations. The pairwise asset correlations are usually estimated from historical time series of asset values. However, such data may not be readily available. Since equity can be viewed as a European call option on a firm's assets with a strike price of the face value of its debt, the value of the equity option should correspond to the value of

the firm's assets. Therefore, equity price data, which are far more readily available than asset values, can be used as a proxy for asset correlations.
The inspiration for structural models is provided by Merton [1974]. Based on the Merton-type threshold value model, in the default-only mode, this model is of Bernoulli type by deciding about default or survival of a firm. The firm's asset value is compared at a certain horizon with the face value of its liabilities, also referred as default threshold value. If the firm's asset value at the horizon falls below this threshold, then the firm is considered to be insolvent. Otherwise, the firm survived for the considered time period. Referred to the Chapter 3, the default threshold value is the inverse of the standard normal cumulative distribution of its probability of default. Thus a default event of obligor i is modeled as

$$D_i = \mathbf{1}_{\{A_i < TH_i\}} = \mathbf{1}_{\{A_i < \Phi^{-1}(p_i)\}} \sim B\left(1; \mathbf{Q}(A_i < \Phi^{-1}(p_i))\right)$$

which is a Bernoulli random variable, where p_i is usually the one-year probability of default of firm i.
It is assumed that for every company i there is a critical threshold TH_i such that the firm defaults in the period $[0, T]$ if and only if $A_i < TH_i$, where A_i denote the asset value of firm i at the considered valuation horizon T. In the framework of Bernoulli, the asset value of firm i, A_i, can be viewed as a latent variable driving the default event of firm i.
In the classical Merton model, where asset value process are described as correlated geometric Brownian motions, the log-returns of asset values $r_i = \log(A_i(t))/\log(A_i(t-1))$ are normally distributed, so that the joint distribution of two asset value log-returns at the considered horizon is a bivariate normal distribution with a correlation equal to the asset correlation of the processes.
Generalizing the one-factor model we can identify underlying drivers of correlated defaults by assuming that the asset value process is dependent on the underlying factors that reflect industrial and regional influences, thereby driving the economic future of the firm. In mathematical term, the multi-factor model can be written as

$$r_i = \sum_k \beta_{ik} Y_k + \sqrt{1 - \sum_k \beta_{ik}^2}\,\epsilon_i \tag{4.22}$$

Y_k denotes a standard normally distributed systematic risk factor with coefficient β_{ik} and ϵ_i denotes the error term, which is assumed to be independent of Y_{ik}, standard normally distributed, and pairwise independent. The returns r_i are correlated by means of their systematic factors.
Via the asset value approach we can resort the equity market information for equity correlation as a proper proxy for asset correlations, i.e. $\mathrm{corr}(A_i, A_j) \approx \mathrm{corr}(b_i, b_j)$, where b_i is the time series data of equity indices for entity i. Since the equity information of large corporates is well available, this approach is suited for portfolios with large corporates, but may be not adapted for a middle market portfolio with mostly non-quoted firms.
Since the asset values of two obligors are correlated with a linear correlation coefficient $\sum_k \beta_{ik}\beta_{jk}$ we can derive the default correlation between two obligors, i and j, i.e.

$$\begin{aligned}
\mathrm{cov}[\mathbf{1}_i, \mathbf{1}_j] &= \mathbf{E}[\mathbf{1}_i \mathbf{1}_j] - \mathbf{E}[\mathbf{1}_i]\mathbf{E}[\mathbf{1}_j] \\
&= \Phi_2\left(\Phi^{-1}(p_i), \Phi^{-1}(p_j), \sum_k \beta_{ik}\beta_{jk}\right) - p_i p_j
\end{aligned} \tag{4.23}$$

$\mathbf{E}[\mathbf{1}_i\mathbf{1}_j]$ represents the joint default probability of two obligors i and j, which can be formulated as

$$\mathbf{E}[\mathbf{1}_i\mathbf{1}_j] = \mathbf{Q}\left[r_i < \Phi^{-1}(p_i), r_j < \Phi^{-1}(p_j)\right],$$

and Φ_2 denotes the bivariate normal distribution function.
From the construction of log-returns (4.22), the correlation between two asset value log-returns r_i and r_j, which are assumed to be standard normally distributed, is $\sum_k \beta_{ik}\beta_{jk}$, and $\mathbf{E}[\mathbf{1}_i] = \mathbf{Q}[r_i < \Phi^{-1}(p_i)] = p_i$.

Assuming that the covariance in CreditRisk$^+$ is the same as (4.23), we can derive the default correlation between any tow obligors as

$$\begin{aligned} \rho_{ij} &= \frac{\mathrm{cov}[\mathbf{1}_i, \mathbf{1}_j]}{\sqrt{\mathbf{E}[\mathbf{1}_i]\mathbf{E}[1-\mathbf{1}_i]\mathbf{E}[\mathbf{1}_j]\mathbf{E}[1-\mathbf{1}_j]}} \\ &= \frac{\Phi_2\Big(\Phi^{-1}(p_i), \Phi^{-1}(p_j), \sum_k \beta_{ik}\beta_{jk}\Big) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \end{aligned} \tag{4.24}$$

For public traded companies ρ_{ij} is the correlation between the equity prices, as introduced above. When this is not the case, we can use other proxies. For example, when i is a private company, we can replace it by a public traded company that is in the same industry and geographical region for the purpose of calculating ρ_{ij}. When i is a sovereign entity, we can use the exchange rate of the currency issued by the sovereign entity as a substitute for equity price. These proxies are less than ideal, but are widely used in practice.

Modeling Correlated Defaults from Credit Market Data

The main drawback of traditional structural models is that they are not consistent with the risk-neutral probabilities of default estimated from corporate bond prices or CDS spreads.
The reduced form models focus on the risk-neutral hazard rate, $h(t)$. As defined in Section 3.2, $h(t)\Delta t$ is the probability of default between time t and $t + \Delta t$ as seen at time t assuming no earlier defaults. These models can incorporate correlations between defaults by allowing hazard rates to be stochastic and correlated with macroeconomic variables. They can be made consistent with the risk-neutral probabilities of default estimated from corporate bond prices or CDS spreads. Their main disadvantage is that the range of default correlations that can be achieved is limited. Even in the case of a perfect correlation between two hazard rates, the corresponding correlation between defaults is usually very low.
For interested readers we give some references. Examples of research following this approach are Duffie and Singleton [1999a] and Lando [1998a]. Jarrow and Yu [1999] provides a way of overcoming this weakness of the reduced-form model by generalizing the existing reduced-form models to include default intensities dependent on the default of a counterparty, i.e. firms have correlated defaults due to not only an exposure to common risk

factors, but also to firm-specific risks that are termed "counterparty risks". Numerical examples are also given for the illustration of the effect of counterparty risk not only on the pricing of defaultable bonds but also on the pricing of CDS. Hull and White [2000] present an alternative approach that is a natural development of the structural models of Merton [1974].

In the original CreditRisk$^+$ framework a Gamma distribution is defined to model the stochastic behavior of the risk factors driving individual default probabilities. A more general way of obtaining the conditional default probabilities can be pursued from empirical market data. The aim is to develop a general framework, which allows us to determine a set of appropriate risk factors $\mathcal{S}$ which drive the dependence structure as well as the corresponding conditional default probabilities. Since the market for CDS has now reached the stage where CDS on reference entities with a particular credit rating are often more actively traded than bonds issued by the reference entities, risk-neutral default probabilities can be easily estimated from CDS spreads and these risk-neutral probabilities can be used further to value CDOs. Since the market for CDS becomes sufficiently liquid, pairwise correlations could be implied from the prices of the CDS.

Estimation of Hazard Rates Recalling Section 3.2, the risk-neutral survival and default probabilities corresponding to a reference entity i are determined by its hazard rate function $\lambda_i(t)$, $0 \le t \le T$. Each individual hazard rate function can be estimated with the help of Lemma 3.12 and 3.13 through observable CDS market quotes and through an assumption on the recovery rate of the reference entity.

In order to estimate hazard rate functions in their most general form we have to consider all market information for each single reference entity. Suppose that we have found market prices for different tenors $T_1 < T_2 \ldots < T_m$, $m \ge 1$ for reference entity i, i.e. we have the price vector $\left(S^i_{T_1}, \ldots, S^i_{T_m}\right)^T$ of CDS spreads for the reference entity i. Let T_0=0, we define a piecewise constant hazard rate function

$$\lambda_i(t) = \sum_{j=1}^{m} \lambda_{i,j} \mathbf{1}_{[T_{j-1},T_j)}(t)$$

for each reference entity i.

For the ease of presentation we restrict the considerations to the case of one liquid market quote for each single reference entity.

Denote the tenor of the liquid contract for reference entity i by T^i. In general, the most liquid tenor is $T^i = 5$ years for almost all traded names. Thus we obtain a flat hazard rate function for each reference entity, i.e. $\lambda_i(t) = \lambda_i$, $0 \le t \le T$, which implied the following survival and default probabilities

$$P^i_{surv}(t) = e^{-\lambda_i t},$$
$$P^i_{def}(t) = 1 - e^{-\lambda_i t}.$$

Under the assumption of flat riskless interest rate curve we can get the following approximation from (3.16) (3.17):

$$\lambda_i = \frac{S^i_T}{1 - Rev_i}, \;\; \Delta\lambda_i = \frac{\Delta S^i_T}{1 - Rev_i}, \tag{4.25}$$

where Rev is the recovery rate.
Both equation are very helpful for estimating default correlations from time series data of CDS spreads without having to resort to equity market data and corresponding asset value approaches.

Risk Factors In Duffie and Singleton [1999b], Lando [1994] and Lando [1998b] the intensity of default is assumed as a stochastic process that derives its randomness from a set of macroeconomic state variables, such as the short-term interest rate, unemployment rate, etc. Conditional on the macroeconomic states defaults are independent events and correlation arises due to the common influence of these macroeconomic state variables.
In contrast to the structural model, we explain the interdependencies between two reference entities by systematic risk factors that affect individual default rates. Denote the set of systematic risk factors by $R = (R_1, \dots, R_K)$. The credit spread S_T^i of entity i with tenor T can be described by the following linear model:

$$\frac{\lambda_i}{\sigma_i} = \frac{S_T^i}{\sigma_{S_T^i}} = a_{i,0}\epsilon_i + \sum_{k=1}^{K} a_{i,k} R_k, \;\; 1 \le i \le n, \tag{4.26}$$

where $\sigma_{S_T^i}$ is the credit spread volatility and σ_i the volatility of the hazard rate λ_i. Note that $\sigma_{S_T^i}$ and σ_i are related to each other by the estimation procedure of the hazard rate function. An approximate relationship could be deduced from (4.25).
In general, we assume that the systematic risk factors are pairwise uncorrelated and that the idiosyncratic risks are pairwise independent as well as independent from the systematic factors just like in the framework of the structural model. Then it depends on the distributional assumption on the random vector $\mathcal{S} = (\epsilon_1, \dots, \epsilon_n, R_1, \dots, R_K)$.
Based on Section 3.2.1, we can form the conditional default and survival probabilities

$$\mathbf{Q}(\tau_i \le t|\mathcal{S}) = P_{def}(t)^{i|\mathcal{S}} = 1 - e^{-\lambda_i t} \tag{4.27}$$

$$\mathbf{Q}(\tau_i > t|\mathcal{S}) = P_{surv}(t)^{i|\mathcal{S}} = e^{-\lambda_i t} \tag{4.28}$$

Furthermore, we have the following expression

$$\begin{aligned} P^i_{surv}(t) &= \mathbf{E}[P^i_{surv}(t)|\mathcal{S}] = \mathbf{E}[e^{-\lambda_i t}] = \mathbf{E}\left[\exp\left\{-\left(a_{i,0}\sigma_i\epsilon_i + \sum_{k=1}^{K} a_{i,k}\sigma_i R_k\right)t\right\}\right] \\ &= \mathbf{E}\left[\exp\{-(a_{i,0}\sigma_i\epsilon_i t)\}\right]\mathbf{E}\left[\exp\left\{-\sum_{k=1}^{K} a_{i,k}\sigma_i R_k t\right\}\right] \\ &= M_{\epsilon_i}(-a_{i,0}\sigma_i t) M_R(-a_{i,1}\sigma_i t, \dots, -a_{i,K}\sigma_i t) \end{aligned} \tag{4.29}$$

where $M_X(u) = \mathbf{E}[e^{Xu}]$, $u \in \mathbb{R}^d$, denotes the moment-generating function (MGF) of a d-dimensional random variable $X = (X_1, \dots, X_d)$.
Because of the conditional independence and the nature of MGF, we can get the joint survival probability from the individual survival probability (4.29):

$$P_{surv}(t_1, \dots, t_n) = \prod_{i=1}^{n} M_{\epsilon_i}\left(-a_{i,0}\sigma_i t\right) M_R\Big(-\sum_{i=1}^{n} a_{i,1}\sigma_i t_i, \dots, -\sum_{i=1}^{n} a_{i,K}\sigma_i t_i\Big)$$

Without loss of generality we could assume in (4.26) that the risk factors R_k and ϵ have variance 1. This implies the following relationship of CDS spread, respecting hazard rate correlation

$$\begin{aligned}
\mathrm{corr}(\lambda_i, \lambda_j) &= \mathrm{cov}\left(\frac{\lambda_i}{\sigma_i}, \frac{\lambda_j}{\sigma_j}\right) = \mathrm{cov}\left(\frac{S_T^i}{\sigma_{S_T}^i}, \frac{S_T^j}{\sigma_{S_T}^j}\right) \\
&= \mathrm{cov}\left(a_{i,0}\epsilon_i + \sum_{k=1}^{K} a_{i,k} R_k, a_{j,0}\epsilon_j + \sum_{k=1}^{K} a_{j,k} R_k\right) \\
&= \sum_{k=1}^{K} a_{i,k} a_{j,k} \qquad (4.30)
\end{aligned}$$

In particular, for $i = j$

$$\sum_{k=1}^{K} a_{i,k}^2 = 1 \qquad (4.31)$$

The advantage of this approach is the flexibility regarding how market information enters into the modeling. It turns out that the linear model (4.26) essentially describes the interrelations between the different reference entities. There are several methods to establish this linear model, regarding choosing the systematic risk factors. One possible approach is to choose the systematic component of the credit spread evolution by macroeconomic factors. The main problem is to find appropriate factors that explain the systematic part of the credit spread sufficiently well such that the remaining individual risks ϵ_i, $i = 1, \ldots, n$, are independent.

This approach is call *exogenous modeling* in contrast to the Merton-based modeling, which is calibrated using historical data. The exogenous modeling has the advantage that only current market information enters into the model.

In the next section we introduce the principal component analysis, through which the correlated risk factors could be orthogonalized to some latent independent sectors, and the corresponding sector weights are obtained simultaneously.

4.3.2 Factor Analysis

As dependence structure the estimated default correlations are input into factor analysis. Our first goal is to identify appropriate independent risk factors. Secondly, the influence of these risk factors on the respective industry, and therefore, on the obligors that belong to this industry, has to be estimated. This can be accomplished by the principal component analysis, which allows for the simultaneous identification of independent latent risk factors as well as the calculation of the corresponding risk factor loadings.

Relevant references are: Boegelein and Roesch [2004] introduce econometric methods for a factor analysis based on regressions. Lesko and Vorgrimler [2004] give estimation approaches for sector weights from real-world data.

Principle Component Analysis

In Lesko and Vorgrimler [2001] the authors have proposed the estimation of risk factor weights by principal component analysis (PCA). A comprehensive introduction about PCA can be found Jolliffe [2002] and for more details about PCA and the factor analysis we refer to Fahrmeir and Tutz [1996].
PCA is a method that transforms a number of correlated observable variables into a smaller number of uncorrelated variables, called principal components. Thus it can be used for dimensionally reduction in a data set while retaining those characteristics of the data set that contribute most to its variance and identify new meaningful underlying variables. The reduced variables should incorporate overall variance as much as possible. These principle components are uncorrelated with each other and ordered in descending variances. Technically speaking, PCA is an orthogonal linear transformation that transforms the data to a new coordinate system such that the greatest variance by any projection of the data comes to lie on the first coordinate (the first principal component), the second greatest variance on the second coordinate, and so on.
In mathematical words, the goal is to find an orthogonal matrix that consists of the normalized eigenvectors with descending ordered eigenvalues (empirical variances) of empirical correlation matrix R. This can be formulated in the following proposition.

Theorem 4.3 *Let Z be a $\mathbb{R}^{N\times p}$ matrix with rank p, then the matrix $H = (H_1, \ldots, H_p)$ is the principle axis of Z, $H = ZT$, where the orthogonal matrix T consists of the normalized eigenvectors to descending ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$ of $R = Z^T Z$. And*

$$H^T H = diag\{\lambda_1, \ldots, \lambda_p\} = \Lambda$$

The principle axis H_i are orthogonal to each other and have the sequence after the square sums (empirical variances) $\lambda_1, \ldots, \lambda_p$.
For matrix Z

$$Z = HT^T$$

$$Z^T Z = R = T\Lambda T^T$$

A rescale provide a orthonormal representation of factors $F = (F_1, \ldots, F_p)$

$$Z = FL^T$$

with $F = H\Lambda^{-1/2}$, $L = T\Lambda^{1/2}$, $F^T F = I$, and

$$Z^T Z = R = LL^T$$

The normalized principle axis $F_1, \ldots, F_p$ are called principle components. The decomposition $Z = F\Lambda^{1/2}T^T$ is called singular value decomposition of Z. Solution for F we can get

$$F = ZT\Lambda^{-1/2} = ZT\Lambda^{-1}T^T T\Lambda^{1/2} = ZR^{-1}L$$

The proof is referred to Fahrmeir and Tutz [1996], pp. 663-668. Geometric representation is also presented.
Thereby latent independent risk factors can be obtained as assumed in the original

CreditRisk$^+$ model. All obligors that have the same industry classification are assumed to have the same sector weights within this industry.
In general, the number of systematic risk factors can be chosen arbitrarily between 1 and the number of given industries. When choosing this parameter, it should be noted that for a higher number of abstract risk factors, the accuracy of the resulting risk factor weights is also higher in terms of the implicit correlation structure represented by these weights. The criteria for the choice of the number of systematic risk factors are listed as follows. Interested readers can refer to Fahrmeir and Tutz [1996].

- Only the principle components with eigenvalues $\lambda_i \geq 1$ are considered, i.e. the components that explain at least variance 1 are chosen.

$$k = \max\{j|\lambda_j \geq 1\}$$

- Extract so many principle components that a arbitrarily settled percentage $c\%$ of the total variance can be explained.

$$k = \min\{r|\lambda_1 + \ldots + \lambda_r \geq c\%p\}$$

- Scree-Test. A scree plot of eigenvalues is made to get an indication of the importance of each eigenvalue. The exact contribution of each eigenvalue to the explained variance can also be queried. If the points on the graph tend to level out (show an elbow), these eigenvalues can be ignored.

- Bartlett-Test for significance of principal components.

It should be noted that the principal axis and the columns of orthogonal matrix is only unique except for sign, when the eigenvalues of correlation matrix R are pairwise different. In CreditRisk$^+$ model only non-negative factor weights are allowed.
Once the systematic risk factors are determined, an idiosyncratic risk factor is included by adding a column to the truncated matrix of eigenvectors such that the sum for each row of the resulting matrix equals 1. Then the final output is the resulting sector weights matrix (each row i contains the sector weights of an obligor belonging to industry i) and the diagonal matrix of eigenvalues containing the volatilities of the risk factors with additional zero for the idiosyncratic risk factor.

Remark: Although most investigations use the direct principal components of correlation matrix as sector weights, it should be remarked that if we look exactly at the correlation matrix in CreditRisk$^+$ with normalized risk factors:

$$R = \frac{\sqrt{p_A p_B}}{\sqrt{(1-p_A)(1-p_B)}} \sum_k \omega_{Ak}\omega_{Bk}\text{var}(\mathcal{S}_k)$$

the resulting principal components of the correlation matrix R are actually sector weights multiplied with the factor $\sqrt{\frac{p_i}{1-p_i}}$.

4.4 Empirical Calibration Methods

In the following we investigate some studies on empirical data, including equity market data and CDS market data, using the approaches introduced in Chapter 4.3. For comparison the estimation is also performed on the one-sector assumption.

Initially, the default rate volatility is set to be equal to the default rate. According to Lehnert and Rachev [2005], such parameter set in the initial CreditRisk$^+$ model underestimates senior tranche losses. The authors suggest to increase the default rate volatility up to about 3 times of the probability of default for producing tails, which are fat enough to meet market tranche losses. Based on this idea, we perform model calibration and give our remarks.

4.4.1 Sector Weights Estimation

Based on Section 4.3 we implement two empirical studies. One is performed on the Merton-based asset value threshold model with the information from historical equity market data. The other one is based on the reduced form model with CDS market information.

Default Rate Calibration

There are two essentially approaches to estimate the default probabilities.

- *Calibration of default probabilities from ratings.*
  In this approach, default probabilities are associated with ratings, and ratings are assigned to customers either by external rating agencies like *Moody's Investors Services, Standard & Poor's, or Fitch,* or by bank-internal rating methodologies.

- *Calibration of default probabilities from market data.*
  Calibrating default probabilities from market data is based on credit spreads of traded products bearing credit risk, e.g. corporate bonds and credit derivatives.

The process of assigning a default probability to a rating is called a *calibration*. Bluhm et al. [2003] demonstrates how such a calibration works. The end product of a calibration of default probabilities to ratings is a mapping, i.e. Rating → PD, e.g. $\{AAA, AA, \ldots, D\}$ $\mapsto [0, 1]$, R → PD(R), such that to every rating R a certain default probability PD(R) is assigned.
We get rating information for all 125 companies in *iTraxx* Series 6 from the Standard & Poor's, and calibrate the default probabilities to the ratings. In *iTraxx* each name is equally weighted (0.8%), thus, we can obtain the average default rates for each industrial branch. It should be noted that these average annual default probabilities for European firms are very low.
In the second approach, we calibrate the default probabilities from CDS market data. Based on the reduced form model, we refer to the Theorem 3.14 for the default intensity

calibration from CDS spreads, and based on the relationship between the default intensity and conditional default probability, the default probabilities for each sector can be estimated.

Default Correlation Estimation

Based on Section 4.3, we model correlated default events respecting equity market data and CDS market information. Our research data are the historical annual EuroStoxx indices from 30.09.1996 to 28.02.2007 for the analogous six sectors in *iTraxx*, industrial, bank, auto, utility, telecom and food and beverage/consumer.
Firstly, the asset correlations are estimated. The results are listed in Table 4.1.
We can see from Table 4.1 that on average bank is the most positively correlated with

	industrial	bank	auto	utility	telecom	food & beverage
industrial	1	80.91%	50.62%	76.30%	76.24%	63.73%
bank	80.91%	1	67.18%	94.75%	36.51%	88.72%
auto	50.62%	67.18%	1	66.83%	41.21%	77.57%
utility	76.31%	94.75%	66.83%	1	36.28%	86.90%
telecom	76.24%	36.51%	41.21%	36.28%	1	21.69%
food & beverage	63.73%	88.72%	77.57%	86.90%	21.69%	1

Table 4.1: estimated asset correlations from annual EuroStoxx indices from 30.09.1996 to 28.02.2007 for 6 sectors.

other sectors and telecom is the lowest correlated with other sectors.
Based on the asset correlations, and the calibrated average annual default rates from S&P's ratings, we estimate the default correlations according to the Merton-type structural model, (4.24). The results are presented in Table 4.2.

	industrial	bank	auto	utility	telecom	food & beverage
industrial	1	30.41%	6.12%	19.24%	24.54%	13.00%
bank	30.41%	1	14.71%	40.39%	3.26%	42.63%
auto	6.12%	14.71%	1	11.87%	3.60%	23.79%
utility	19.24%	40.39%	11.87%	1	2.07%	32.49%
telecom	24.54%	3.26%	3.60%	2.07%	1	0.98%
food & beverage	13.00%	42.63%	23.79%	32.49%	0.98%	1

Table 4.2: estimated default correlations based on Merton-type asset value threshold model with calibrated average annual default probabilities from S&P's ratings

For the second approach, the reduced form based model, risk-neutral probabilities of defaults are estimated from CDS market data. The credit market information is available only for two years, from 30.06.2004 to 03.10.2006. Thus, in contrast to the annual equity prices, monthly CDS spreads are used for calibration of the default rates and for further estimation of default correlation.

	industrial	financial	auto	energy	TMT	consumers
industrial	1	34.56%	58.51%	30.46%	71.25%	28.59%
financial	34.56%	1	83.29%	78.44%	-12.51%	87.62%
auto	58.51%	83.29%	1	70.46%	15.87%	80.53%
energy	30.46%	78.44%	70.46%	1	-4.09%	77.88%
TMT	71.25%	-12.51%	15.87%	-4.09%	1	3.23%
consumers	28.59%	87.62%	80.53%	77.88%	3.23%	1

Table 4.3: estimated default correlations based on reduced form model from monthly CDS market spreads from 30.06.2004 to 03.10.2006

In Table 4.3 we find that the default correlations estimated from the CDS market are greater than those estimated from the equity market data and there appear some negative correlations. It may be due to the CDS market has shorter time series than the equity market. Compared to the annual equity market data, the monthly CDS market data are more volatile. But the financial sector is still on average mostly correlated with other sectors and TMT the lowest correlated with other sectors.

Principal Component Analysis

We have estimated the asset correlations from the time series of the annual equity prices, and have transformed them to the default correlations according to (4.24). For calculating the sector weights we perform PCA on these two estimated default correlation matrices as listed in Table 4.2 and 4.3

From the first criterion for the choice of the number of systematic risk factors in our introduction of principle component analysis, we present only the principal components with eigenvalues which are greater than 1 are considered. Thus, we consider the first two principal components as systematic risk factors.

It should be noted that sector weights should not be negative. But this condition does not always hold in the application. As mentioned, the principal axis and the columns of the orthogonal matrix is only unique except for sign, when the eigenvalues of the correlation matrix are pairwise different. Since the eigenvalues of the input correlation matrix are pairwise different, it is intuitive to change the sign of the negative weights. The negative sector weights that are greater than -0.1 are ignored, and the other negative weights that are smaller than -0.1 are replaced with their absolute values.

Recall that the sector weights should be not negative, be less than 1, and the sum of them should not be over 1 either. Therefore it should be examined whether the sum of each row of the sector weight matrix (each row i of the sector weight matrix represents the uncorrelated underlying variables for the sector i) is greater than 1. If it is, then the sector weights should be normalized.

The coefficients for idiosyncratic risk factors follow from the setup of the standard CreditRisk$^+$ model, i.e.

$$\omega_{A0} = 1 - \sum_{k=1}^{N} \omega_{A,k}$$

On the other hand, based on the monthly CDS spreads from 30.06.2004 to 03.10.2006 the principal component analysis is performed as well relating to (4.30).

4.4.2 Default Rate Volatility Calibration

As in the section A 7.3 of Wilde [1997] it is suggested that default rate volatilities are roughly equal to the probability of default for each name. This suggestion is adopted for the initial parameter calibration.

By performing the principal component analysis on historical equity market information and on credit market information as well, we have obtained the sector weights matrix of systematic and idiosyncratic risk factors. The model accounts for the heterogeneity of the underlying portfolio and the idiosyncratic risk by incorporating a sector weights matrix and default rate volatilities in contrast to the large homogenous pool model.

The features of the example portfolio are listed in Table 4.4. The market quotes are given in the second columns. For CreditRisk$^+$, the portfolio loss distribution using the above mentioned input parameters is calculated analytically. Given the loss distribution as an output of CreditRisk$^+$, we calculate the percentage losses for each tranche.

For the LHP model, we calculate the base correlations for each tranche, and the expected tranche loss which corresponds to the tranche base correlation is also calculated.

	Market	LHP		CreditRisk$^+$
Tranche	Market Quotes	Base Corr.	percent. loss	percent. loss
Equity	24.08	20.13%	74.20%	80.73%
3%-6%	132.250	28.27%	10.59%	15.81%
6%-9%	46.000	34.70%	3.77%	2.86%
9%-12%	31.625	39.08%	2.60%	0.50%
12%-22%	15.750	49.59%	4.34%	0.10%
22%-100%			4.50%	0.00%
Total			100%	100%

Table 4.4: *iTaxx Europe* market quotes on 03.11.2004, calculated base correlation and percentage losses implied by market quotes in the large homogenous portfolio model and calculated by CreditRisk$^+$ with the initial parameter calibration.

In CreditRisk$^+$, default rate volatility σ_A was set initially equal to be $1 \cdot p_A$. From the calculated portfolio losses in Table 4.4 with the initial input parameters, one can see that the losses are highly concentrated on the first two tranches, the equity tranche and the 3%-6% tranche. Referring to Lehnert and Rachev [2005], we perform calibrations of the default rate volatility by factors, i.e. $\sigma_A = f \cdot p_A$, on the models based on the sector weights matrix derived from equity market information as well as from credit market to produce fatter tailed tranche losses. As a purpose for comparison, the case of a single risk factor is also considered.

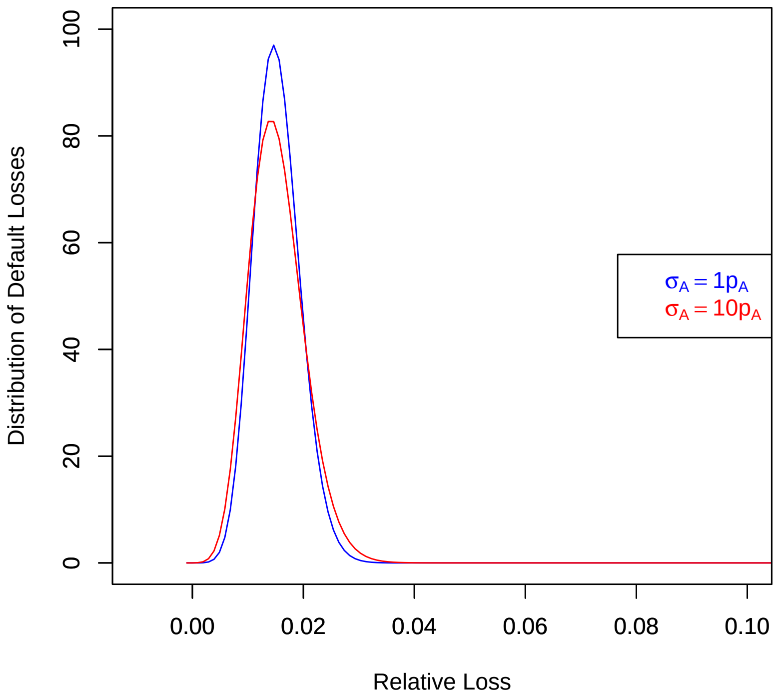
As output the portfolio loss distribution is presented graphically as well as the expected losses for each iTraxx tranche, which are calculated both in absolute and percentage means. For comparing the shape of the loss distribution, we calculate also the value-at-risk at 99.9% level.

Increase in the default rate volatility can result in the increase the variance of the portfolio loss. As we know, increasing variance with unchanged expected loss leads to fatter tails. Since the authors of Lehnert and Rachev [2005] mentioned that the sensitivities of correlation are too low to generate a fatter tail loss distribution ("even in the case of a correlation of 1, still 95.24% of total expected losses concentrate on the equity and 3%-6% tranche."), we calibrate only the default rate volatility input.

From our research we can conclude that the tranche losses are shifted from the equity tranche to more senior tranches by increasing default rate volatilities. Since the annual average probabilities of default for investment grade obligors are so small (less than 2.00%) that even to the calibration factor f=9 we cannot get a fatter tailed portfolio loss distribution curve to match the senior tranche losses (Figure 4.2) and still 99.98% losses concentrate on the first two tranches (Table 4.5).

	$\sigma_A = 1 * p_A$		$\sigma_A = 3 * p_A$		$\sigma_A = 9 * p_A$		$\sigma_A = 18 * p_A$	
Tranche	absolute	percent	absolute	percent	absolute	percent	absolute	percent
Equity	0.9887	83.53%	0.9868	83.37%	0.9706	82.00%	0.9265	78.28%
3%-6%	0.1949	16.47%	0.1968	16.63%	0.2128	17.98%	0.2477	20.92%
6%-9%	0.0000	0.00%	0.0000	0.00%	0.0002	0.02%	0.0093	0.79%
9%-12%	0.0000	0.00%	0.0000	0.00%	0.0000	0.00%	0.0001	0.01%
12%-22%	0.0000	0.00%	0.0000	0.00%	0.0000	0.00%	0.0184	0.00%
22%-100%	0.0000	0.00%	0.0000	0.00%	0.0000	0.00%	0.0009	0.00%
Total	1.1836	100%	1.1836	100%	1.1836	100%	1.1836	100%
	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$
	0.0582	0.0861	0.0582	0.0880	0.0582	0.1037	0.0582	0.1438
	$\sigma_A = 30 * p_A$		$\sigma_A = 40 * p_A$		$\sigma_A = 50 * p_A$		$\sigma_A = 55 * p_A$	
Tranche	absolute	percent	absolute	percent	absolute	percent	absolute	percent
Equity	0.8568	72.43%	0.8000	67.95%	0.7480	64.31%	0.7241	62.81%
3%-6%	0.2715	22.95%	0.2725	23.14%	0.2643	22.72%	0.2583	22.40%
6%-9%	0.0474	4.01%	0.0794	6.75%	0.1018	8.75%	0.1093	9.49%
9%-12%	0.0068	0.58%	0.0227	1.93%	0.0409	3.51%	0.0490	4.25%
12%-22%	0.0003	0.03%	0.0027	0.23%	0.0081	0.70%	0.0118	1.02%
22%-100%	0.0000	0.00%	0.0000	0.00%	0.0001	0.01%	0.0003	0.02%
Total	1.1828	100%	1.1773	100%	1.1632	100%	1.1528	100%
	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$
	0.0582	0.2182	0.0582	0.2955	0.0582	0.3845	0.0582	0.4325

Table 4.5: expected tranche losses for calibration of default rate volatilities based on annual historical EuroStoxx indices from 30.09.1996 to 28.02.2007



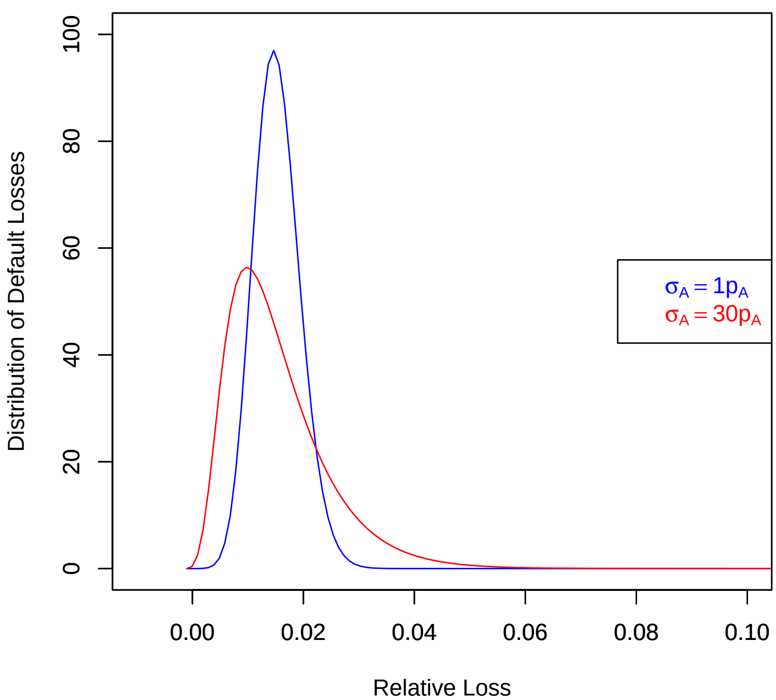


Fig. 4.2: calibration of default rate volatilities based on annual historical EuroStoxx indices from 30.09.1996 to 28.02.2007 with calibration factor 1, 10, and 30

In comparison with the calibrations based on the annual equity market information the calibration of default rate volatilities based on monthly CDS spreads are more sensitive to the calibration factor f. In Figure 4.3 we can see "some pretty" distribution curves. The expected tranche losses for each tranche are calculated and listed in Table 4.6. With factor 9 there emerge already some losses in the 12%-22% tranche, while in the calibration based on the annual equity data, even in the case that factor equals 18, there is still no loss in the 12%-22% tranche.
If we compare the default correlation matrix in Table 4.2 and 4.3, we can find that the default events based on the monthly CDS data are more correlated than they based on the annual equity information. Thus we can conclude that the inputs of default probability are very critical for the calculation of loss distributions and the time-varying effect should be thought as relevant. This may be one of the motivations for dynamic model setup.

Under the assumption that there is only one systematic background risk factor, i.e. the general state of the whole economy, the risk factor weights of all obligors are assumed to be 1. The same calibration of default rate volatility is performed. This choice of the risk factor weights and the calculation of a standard deviation for the single risk factor is similar to an implicit correlation between the default rates of 1 for each pair of branches, so it leads to a overestimation of the portfolio risk if the true correlation is lower. In other words, this approach ignores possible diversifications by investing in different branches.
We can see in Figure 4.4 that by doubling the initial input the loss distribution is fat tailed enough, which corresponds with the calculated expected loss in Table 4.7. Since the calibration factor equals 1.5, there is already loss in the 22%-100% tranche, which is considered as an extreme case.

Remark: The results in Lehnert and Rachev [2005] seem to be questionable. Without question on the estimated risk-neutral probability of default (2.89% as the average level is a little too much for annual default probability) and with the assumption of the same average level as given in Lehnert and Rachev [2005] 2.89%, even for the single risk factor the expected loss in the equity tranche could not be only 40.47%. The authors show a result that by increasing $\sigma_A = 3p_A$ the tranche losses are 40.47%, 19.74%, 12.23%, 8.11%, 13.04%, 6.04% for increasing seniority. The question is, are these losses in the last three senior tranches a bit too high?

	$\sigma_A = 1 * p_A$		$\sigma_A = 3 * p_A$		$\sigma_A = 9 * p_A$		$\sigma_A = 18 * p_A$	
Tranche	absolute	percent	absolute	percent	absolute	percent	absolute	percent
Equity	0.9752	82.23%	0.9556	80.58%	0.8517	71.92%	0.7138	63.05%
3%-6%	0.2106	17.76%	0.2290	19.31%	0.2678	22.61%	0.2317	20.47%
6%-9%	4e-05	0.01%	0.0013	0.11%	0.0540	4.56%	0.1099	9.71%
9%-12%	0.0000	0.00%	0.0000	0.00%	0.0101	0.85%	0.0574	5.07%
12%-22%	0.0000	0.00%	0.0000	0.00%	0.0007	0.06%	0.0184	1.62%
22%-100%	0.0000	0.00%	0.0000	0.00%	0.0000	0.00%	0.0009	0.08%
Total	1.1859	100%	1.1859	100%	1.1843	100%	1.1321	100%
	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$
	0.0583	0.0978	0.0583	0.1154	0.0583	0.2387	0.0583	0.5391

Table 4.6: expected tranche losses for calibration of default rate volatilities based on CDS monthly spreads from 30.06.2004 to 03.10.2006

	$\sigma_A = 1 * p_A$		$\sigma_A = 1.5 * p_A$		$\sigma_A = 2 * p_A$		$\sigma_A = 2.5 * p_A$	
Tranche	absolute	percent	absolute	percent	absolute	percent	absolute	percent
Equity	0.8839	74.53%	0.8112	68.43%	0.7441	63.21%	0.6828	58.49%
3%-6%	0.2760	23.27%	0.2945	24.84%	0.2952	25.08%	0.2888	24.74%
6%-9%	0.0249	2.10%	0.0665	5.61%	0.1005	8.54%	0.1253	10.73%
9%-12%	0.0011	0.10%	0.0122	1.03%	0.0322	2.74%	0.0545	4.67%
12%-22%	0.0000	0.00%	0.0007	0.06%	0.0042	0.35%	0.0117	1.01%
22%-100%	0.0000	0.00%	0.0004	0.03%	0.0009	0.08%	0.0042	0.36%
Total	1.1859	100%	1.1853	100%	1.1771	100%	1.1673	100%
	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$	Mean	$VaR_{99.9\%}$
	0.0583	0.1702	0.0583	0.2397	0.0583	0.3219	0.0583	0.4383

Table 4.7: expected tranche losses for single risk factor based on CDS monthly spreads from 30.06.2004 to 03.10.2006.

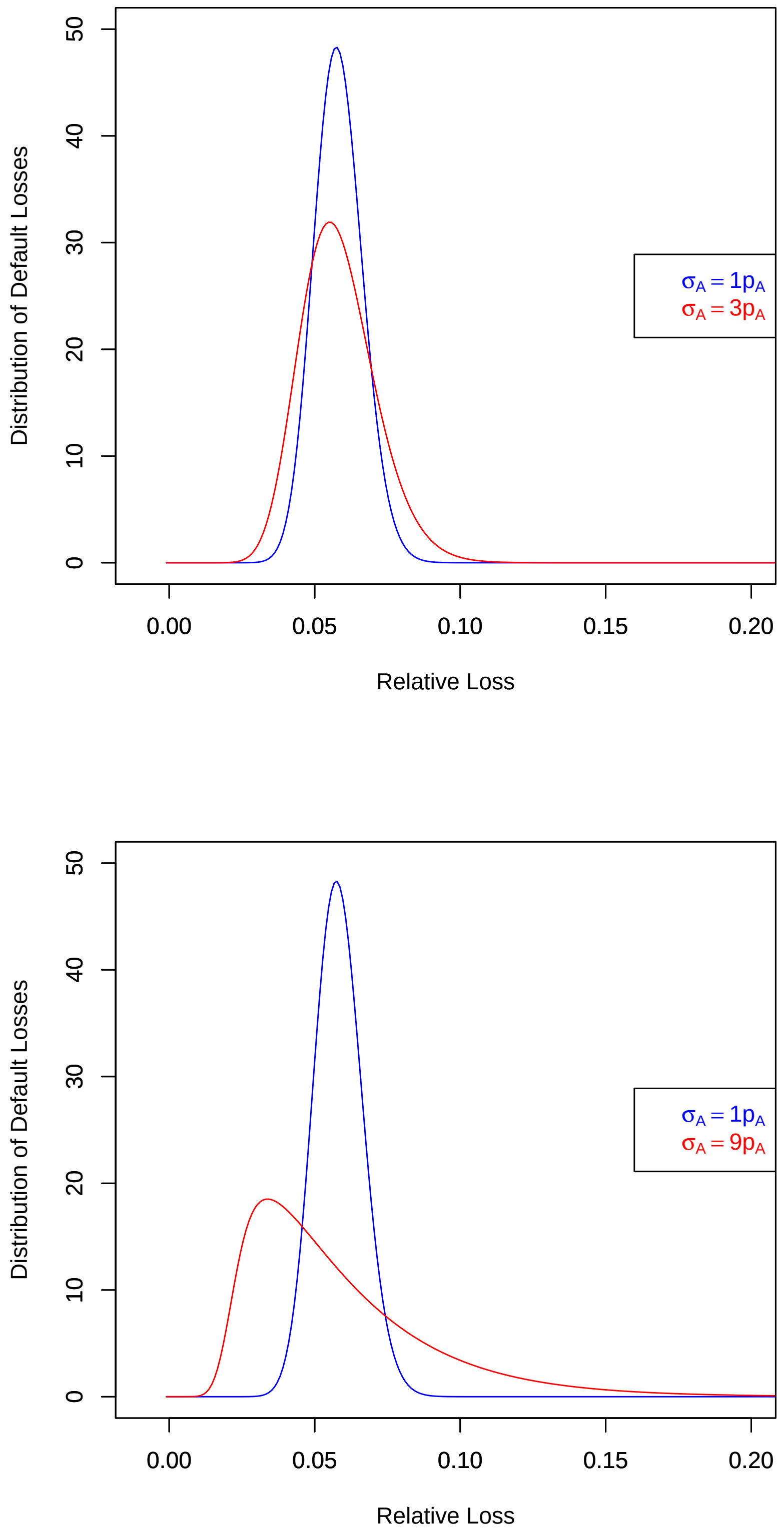


Fig. 4.3: calibration of default rate volatilities based on CDS monthly spreads from 30.06.2004 to 03.10.2006 with calibration factor 1, 3, and 9.

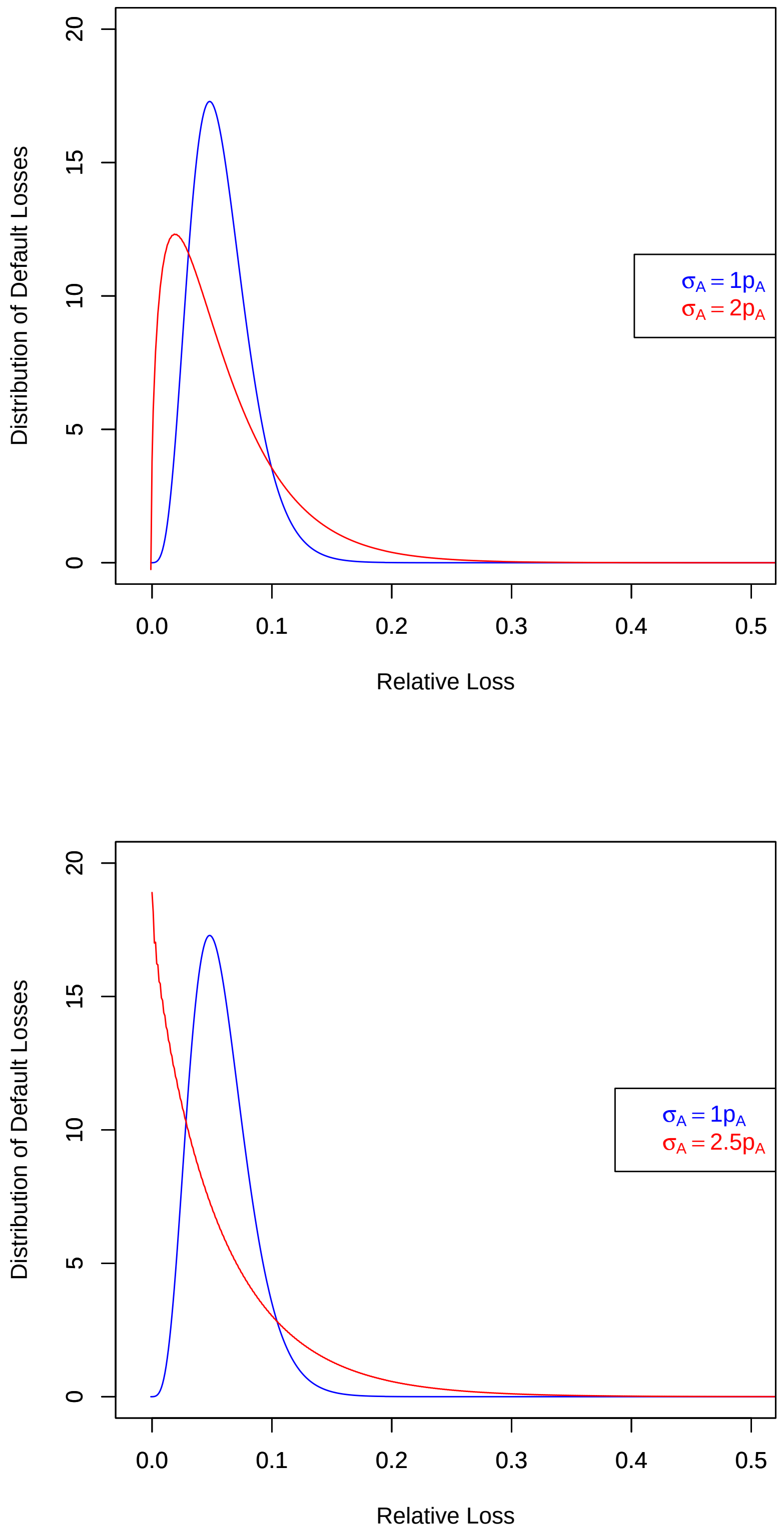


Fig. 4.4: calibration for single risk factor based on CDS monthly spreads from 30.06.2004 to 03.10.2006 with factor 1, 2, and 2.5.

Although we get different results, analogous conclusion can be drawn. By increasing default rate volatility one can obtain heavy tails and more tranche losses in the senior tranches. However, in order to meet the tranche losses implied by market quotes different default rate volatilities have to be used for each tranche.
The calibration factors, for which the tranche losses are implied by market quotes, are f=(12.5, 6.9, 11.2, 14.9, 18.9, 27.3) for each tranche, ordered by seniority for *iTraxx Europe* markets quotes on 13.11.2004. These results can be described as a "PD volatility skew", similar to the compound correlation skew in the large homogeneous portfolio model.

4.5 Dynamizing the CreditRisk$^+$ Model

In this section we generalize the previous framework by incorporating the time dimension into credit risk analyzing. The following model incorporates time-varying default rates and default rate volatilities that may differ across names as well. The sector variables are described by stochastic processes instead of discrete random variables. Although Reiss et al. [2003], Reiss [2003] introduce also a dynamic version by assuming the risk factors as dependent geometric Brownian motions, it loses the analytical tractability of calculating the credit loss distribution, but has to apply to time consuming simulations. We use Cox-Ingersoll-Ross processes (sometimes also called *square root processes*), as latent macroeconomic processes driving the dynamic hazard rates. Available references can be found in Cox et al. [1985a], Cox et al. [1985b]
The integrated Cox-Ingersoll-Ross process (henceforth CIR process) can be computed explicitly with the help of the Laplace transform.

4.5.1 The Approach of Hillebrand and Kadam

From the growing credit derivatives market, time to default can be modeled as an exogenous random variable, which could be fit to market data, such as the prices for defaultable bonds or credit default swaps. In comparison with the firm value model of Merton, this model is known as the intensity model, which defines the time to default as a continuous stopping time driven by a Poisson process. More precisely, the time of default is determined as the time of the first jump of a Poisson process with stochastic intensity process (doubly stochastic). As the model is calibrated from market data and is applied for the valuation of default contingent claims, the full specification of the model takes place under the equivalent martingale measure, the pricing measure $\mathbf{Q}$. Thus all probabilities and all expectations in the calculations for the model are defined with respect to $\mathbf{Q}$.

Model Introduction

Let τ_A denote the default time of obligor A. Linking τ_A to the information structure revealed in the filtration $(\mathcal{F}_t)_{t\geq 0}$, means mathematically that at every time t, whether τ_A has already occurred or not, is known, i.e. $\{\tau_A \leq t\} \in \mathcal{F}_t$, $\forall t \geq 0$. This property defines that the random time τ is a stopping time. Then the default indicator becomes a process, denoted as $\mathbf{1}_A(t)$.

$$\mathbf{1}_A(t) := \begin{cases} 1, & \text{if } \tau_A \leq t; \\ 0, & \text{if } \tau_A > t. \end{cases}$$

Recalling Theorem 3.7, Definition 3.8 and Theorem 3.9, we know that an alternative way of characterizing the distribution of the default time is the hazard rate function $h(t)$, which gives the instantaneous probability of default at time $t + \Delta t$ conditional on the survival up to t. Lemma 3.6 and Definition 3.10 represent the time of default as a exponentially distributed random variable with the parameter $h(t)$. In Theorem 3.9 we have introduced that under some regularity condition and with the help of Aven's theorem (Aven [1985], Schönbucher [2003] pp.88-90) intensity and local conditional hazard rate coincide. Thus, if the default intensity process $\lambda(t)$ is a deterministic function of time,

then the future path of the default intensity is given by the forward hazard rates.
Understanding the time of default τ as a Poisson process, the constant average arrival rate h can be understood as default intensity and is denoted by λ, $\lambda(t)$ is an expression with consideration of the time variation.
With a default indicator process $\mathbf{1}_A(t)$ and the loss given default, ν_A, assumed to be deterministic, the process of cumulated default losses can be presented as:

$$L(t) := \sum_A \nu_A \mathbf{1}_A(t)$$

The default probability of obligor A conditional on $\mathcal{S}$ is now also a stochastic process:

$$\begin{aligned} p_A^{\mathcal{S}}(t) = \mathbf{Q}[\mathbf{1}_A(t) = 1|\mathcal{F}_t] = \mathbf{Q}[\tau_A \leq t|\mathcal{F}_t] &= 1 - \exp\left(-\int_0^t \lambda_A^{\mathcal{S}}(u)du\right) \\ &\approx \int_0^t \lambda_A^{\mathcal{S}}(u)du. \qquad (4.32) \end{aligned}$$

The above approximation is acceptable because $\int_0^t \lambda_A^{\mathcal{S}}(u)du$ is rather small if t is not too large. This is equivalent to a Poisson approximation of the default process.
Analogously to the static model, we consider factor loadings ω_{Ak} with the usual conditions and the one period default probability of obligor A, p_A, rather small. Instead of random variables the risk factors are now developing to be time-varying factor processes $(\mathcal{S}_k(t))_{t\geq 0}$ that generate the filtration $\mathcal{F}_t$.
The natural transition from the static model to a dynamic one is to model the instantaneous default probability being linearly dependent on factor processes, since they drive the individual default probabilities. The time-varying default rate can be characterized as a product of a baseline one period default rate and a time dependent adjustment process, which is a weighted sum of factor processes $\mathcal{S}_k(t)$. Thus the hazard rate of obligor A conditional on $\mathcal{S}$ can be defined as:

$$\lambda_A^{\mathcal{S}}(t) := p_A \left(\omega_{A0} + \sum_{k=1}^N \omega_{Ak}\mathcal{S}_k(t)\right). \qquad (4.33)$$

where $\mathcal{S}_0(t) \equiv 1$.

With (4.32) and (4.33) the conditional default probability of obligor A can be computed as:

$$p_A^{\mathcal{S}}(t) = \int_0^t \lambda_A^{\mathcal{S}}(u)du = p_A\omega_{A0}t + p_A \sum_{k=1}^N \omega_{Ak} \int_0^t \mathcal{S}_k(u)du \qquad (4.34)$$

For further analysis the characteristic function need to be computed in this time-continuous model.

Let $\mathcal{F}_t$ be the filtration induced by the risk factor processes $(\mathcal{S}_k(t))_{t\geq 0}$ with $\mathcal{S}_k(0) = 1$. As in the static case we assume obligor defaults are independent conditional on the evolution of the sector specific risk factor. Thus the conditional PGF of the portfolio loss can be computed as $G_{L(t)}(z|\mathcal{S}) = \prod_A G_{L_A(t)}(z|\mathcal{S})$, and the PGF of the portfolio loss can be computed by taking the expectation. Referring to the proof of Theorem 4.2 and under the assumption that $p_A^{\mathcal{S}}(t)$ is rather small

$$\begin{aligned}
G_{L(t)}(z) &= \mathbf{E}\left[\prod_A G_{L_A(t)}(z|\mathcal{S})\right] \\
&= \mathbf{E}\left[\prod_A \left(1 + p_A^{\mathcal{S}}(t)(z^{\nu_A} - 1)\right)\right] \\
&\approx \mathbf{E}\left[\prod_A \exp\left(p_A^{\mathcal{S}}(t)(z^{\nu_A} - 1)\right)\right] \\
&= \mathbf{E}\left[\exp\left(\sum_A p_A^{\mathcal{S}}(t)(z^{\nu_A} - 1)\right)\right]
\end{aligned}$$

Using equation (4.34) we can get the PGF of the portfolio loss:

$$\begin{aligned}
G_{L(t)}(z) &= \mathbf{E}\left[\exp\left(\sum_A \left(p_A \sum_{k=0}^N \omega_{Ak} \int_0^t \mathcal{S}_k(u)du\right)(z^{\nu_A} - 1)\right)\right] \\
&= \mathbf{E}\left[\exp\left(\sum_{k=0}^N \sum_A (z^{\nu_A} - 1) p_A \omega_{Ak} \int_0^t \mathcal{S}_k(u)du\right)\right] \\
&= \mathbf{E}\left[\exp\left(\sum_{k=0}^N \xi_k(z) \cdot I_k(t)\right)\right]
\end{aligned}$$

where $\xi_k(z) = \sum_A (z^{\nu_A} - 1) p_A \omega_{Ak}$ is deterministic and $I_k(t)$ is the integrated risk factor process for sector k, $\int_0^t \mathcal{S}_k(u)du$.

Based on the relationship between the PGF and the characteristic function (4.10), one can obtain the characteristic function of the portfolio loss:

$$\begin{aligned}
\varphi_{L(t)}(z) &= G_{L(t)}(e^{iz}) \\
&= \mathbf{E}\left[\exp\left(\sum_{k=0}^N \sum_A (e^{i\nu_A z} - 1) p_A \omega_{Ak} \int_0^t \mathcal{S}_k(u)du\right)\right] \\
&= \mathbf{E}\left[\exp\left(\sum_{k=0}^N \xi_k(z) \int_0^t \mathcal{S}_k(u)du\right)\right] \qquad (4.35) \\
&= \mathbf{E}\left[\exp\left(\sum_{k=0}^N \xi_k(z) \cdot I_k(t)\right)\right] \\
&= \mathbf{E}\left[\prod_{k=0}^N \exp\left(\xi_k(z) \cdot I_k(t)\right)\right] \\
&= \prod_{k=0}^N \mathbf{E}\left[\exp\left(\xi_k(z) \cdot I_k(t)\right)\right] \qquad (4.36)
\end{aligned}$$

where now $\xi_k(z) := \sum_A (e^{i\nu_A z} - 1) p_A \omega_{Ak}$ for $k = 0, \ldots, N$.
The last expectation is exactly the Laplace transform of the integrated risk factor process, $I_k(t)$, with respect to the complex parameter $\xi_k(z)$. Fortunately, we know that the Laplace transform of the integrate CIR process can be computed explicitly, without simulations. With the assumption that $(\mathcal{S}_k(t))_{t\geq 0}$ are independent CIR process, the characteristic function of the portfolio loss is the product of the Laplace transform of the integrated CIR processes. Thus we can compute the characteristic function not only effectively but also analytically. This keeps the analytical tractability of CreditRisk$^+$, and the further main task in the dynamic modeling is to parameterize the integrated factor processes, $I_k(t) = \int_0^t \mathcal{S}_k(u)du$, in proper manner. The intuitive idea is coming from Hillebrand and Kadam [2007].

The Integrated Cox-Ingersoll-Ross Process

First of all we introduce the *Cox-Ingersoll-Ross Process* in an abbreviated manner. More details can be found in Cox et al. [1985a] and Cox et al. [1985b].
The CIR process, which is also called the *square root process*, is defined as the solution of the following stochastic differential equation:

$$dr(t) = \lambda(\eta - r(t))dt + \theta\sqrt{r(t)}dW(t) \tag{4.37}$$

The parameters λ, η and θ are positive constants. The initial value $r(0)$ is generally assumed to be positive and $W(t)$ is a Wiener process. This process is continuous and positive because if it ever touches zero, the diffusion term disappears and the drift factor $\lambda(\eta - r(t))$ pushes the process in the positive direction. It ensures mean reversion of the process towards the long run value η, with speed of adjustment governed by the strictly positive parameter λ.
The original CIR process is often used to model interest rate dynamics, with the parameter λ the speed of adjustment, η, the central location or long-term value, and θ, the square root coefficient. The standard deviation factor, $\theta\sqrt{r(t)}$, ensures that the interest rate cannot become negative. Thus, at low values of the interest rate, the standard deviation becomes close to zero, canceling the effect of the random shock on the interest rate.
We define the risk factor processes $(\mathcal{S}_k(t))_{t\geq 0}$ as independent CIR processes with parameters λ_k, η_k, θ_k as:

$$d\mathcal{S}_k(t) = \lambda_k\left(\eta_k - \mathcal{S}_k(t)\right)dt + \theta_k\sqrt{\mathcal{S}_k(t)}dW(t) \tag{4.38}$$

The dynamic nature of the default processes is now modeled via integrated CIR processes as latent macroeconomic risk processes driving the dynamic default intensities with respect to (4.34).
Integrated CIR processes are used to scale the one period default probability up or down as time evolves. With respect of (4.36) using Laplace transform of the integrated CIR process, $I_k(t) := \int_0^t \mathcal{S}_k(u)du$, the expectation can be computed in a closed-form.
The density of $\mathcal{S}(t)$ is derived in Dufresne [2001] in terms of of the modified Bessel function. Refer to Revuz and Yor [1999] the relationship between the solution of (4.37) and Bessel process can be used to obtain the MGF and the probability density function of

$\mathcal{S}(t)$.

Since the Laplace transform of the integral of the squared Bessel process may be derived explicitly (see Revuz and Yor [1999], p.445), it is not surprising that the Laplace transform of the square root process also has an explicit expression. A quick way to find the Laplace transform of the integrated CIR process, $\mathbf{E}\left(e^{-uI_k(t)}\right)$, is to modify the CIR formula suitably for the price of a zero-coupon bond. (Cox et al. [1985a]). Suppose that the short rate follows a process of type (4.37), the bond price is known to be the expectation of the exponential of minus the integral of the short rate. Multiplying the spot rate by a positive number yields another process which also satisfies (4.37), but with different parameters. Rewriting the bond price formula for this new process immediately yields the Laplace transform of $I_k(t)$.

Theorem 4.4 *The Laplace transform of the integrated CIR process, $I_k(t) = \int_0^t \mathcal{S}_k(u)du$ can be computed as follows.*

$$\mathbf{E}\left[e^{-uI_k(t)}\right] = \frac{\exp\left(\frac{\lambda_k^2\eta_k t}{\theta_k^2}\right)}{\left(\cosh\frac{\gamma_k t}{2} + \frac{\lambda_k}{\gamma_k}\sinh\frac{\gamma_k t}{2}\right)^{\frac{2\lambda_k\eta_k}{\theta_k^2}}}\exp\left(-\frac{2\mathcal{S}_k(0)u}{\lambda_k + \gamma_k\coth\frac{\gamma_k t}{2}}\right).$$

Proof: Suppose $\mathcal{S}_k(t)$ is the solution of (4.38). Let $u > 0$, and define

$$\tilde{\mathcal{S}}_k(t) = u\mathcal{S}_k(t),\ \ \tilde{\mathcal{S}}_k(0) = u\mathcal{S}_k(0)$$

Then

$$\tilde{\mathcal{S}}_k(t) = \tilde{\mathcal{S}}_k(0) + \lambda_k\int_0^t\left(\tilde{\eta}_k - \tilde{\mathcal{S}}_k(s)\right)ds + \tilde{\theta}_k\int_0^t\sqrt{\tilde{\mathcal{S}}_k(s)}dW(s)$$

where $\tilde{\eta}_k = u\eta_k$ and $\tilde{\theta}_k = \theta_k\sqrt{u}$. The formula for the price of a zero-coupon bond maturing in t years is Cox et al. [1985a]

$$\begin{aligned}
& \mathbf{E}\left[e^{-uI_k(t)}\right] \\
= \ & \mathbf{E}\left[e^{-\int_0^t \tilde{\mathcal{S}}_k(u)du}\right] \\
= \ & \left(\frac{2\sqrt{\lambda_k^2 + 2\tilde{\theta}_k^2}\,e^{(\sqrt{\lambda_k^2+2\tilde{\theta}_k^2}+\lambda_k)t/2}}{\left(\sqrt{\lambda_k^2 + 2\tilde{\theta}_k^2} + \lambda_k\right)\left(e^{\sqrt{\lambda_k^2+2\tilde{\theta}_k^2}t} - 1\right) + 2\sqrt{\lambda_k^2 + 2\tilde{\theta}_k^2}}\right)^{\frac{2\lambda_k\tilde{\eta}_k}{\tilde{\theta}_k^2}} \\
& \cdot\exp\left(-\mathcal{S}_k(0)\frac{2(e^{\sqrt{\lambda_k^2+2\tilde{\theta}_k^2}t} - 1)}{\left(\sqrt{\lambda_k^2 + 2\tilde{\theta}_k^2} + \lambda_k\right)\left(e^{\sqrt{\lambda_k^2+2\tilde{\theta}_k^2}t} - 1\right) + 2\sqrt{\lambda_k^2 + 2\tilde{\theta}_k^2}}\right) \\
= \ & \left(\frac{e^{\frac{\lambda_k t}{2}}}{\cosh(\frac{\gamma_k t}{2}) + \frac{\lambda_k}{\gamma_k}\sinh(\frac{\gamma_k t}{2})}\right)^{\frac{2\lambda_k\eta_k}{\theta_k^2}}\exp\left(-\frac{u\mathcal{S}_k(0)}{\gamma_k}\frac{2\sinh(\frac{\gamma_k t}{2})}{\cosh(\frac{\gamma_k t}{2}) + \frac{\lambda_k}{\gamma_k}\sinh(\frac{\gamma_k t}{2})}\right)
\end{aligned}$$

$$= \frac{\exp\left(\frac{\lambda_k^2\eta_k t}{\theta_k^2}\right)}{\left(\cosh\frac{\gamma_k t}{2} + \frac{\lambda_k}{\gamma_k}\sinh\frac{\gamma_k t}{2}\right)^{\frac{2\lambda_k\eta_k}{\theta_k^2}}} \exp\left(-\frac{2\mathcal{S}_k(0)u}{\lambda_k + \gamma_k \coth\frac{\gamma_k t}{2}}\right). \tag{4.39}$$

where $\gamma_k = \gamma_k(u) = \sqrt{\lambda_k^2 + 2\theta_k^2 u}$ and $\mathcal{S}_k(0)$ is the initial value of the integrated CIR process for sector k. □

The Laplace transform of $I_k(t)$ is finite in a neighborhood of the origin. This is seen by noting that $\cosh(\frac{\gamma_k t}{2})$, $\frac{\lambda_k}{\gamma_k}\sinh(\frac{\gamma_k t}{2})$ are both analytic functions of γ_k, and that their difference does not vanish at $\gamma_k(0) = |\lambda_k|$, for any values of the parameters λ_k, θ_k considered, and for any $t > 0$.
Substituting $-u = \xi_k(z)$ and because of independency among sectors we can compute (4.36) as $\exp(\xi_0(z)t)\prod_{k=1}^{N}\mathbf{E}[\exp(\xi_k(z)I_k(t))]$, which can be computed explicitly in the following way with (4.39):

$$\begin{aligned}
&\exp(\xi_0(z)t)\prod_{k=1}^{N}\mathbf{E}\left[\exp(\xi_k(z)I_k(t))\right] \\
=\; &\exp(\xi_0(z)t)\sum_{k=1}^{N}\left(\frac{\exp\left(\frac{\lambda_k^2\eta_k t}{\theta_k^2}\right)}{\left(\cosh\frac{\gamma_k t}{2} + \frac{\lambda_k}{\gamma_k}\sinh\frac{\gamma_k t}{2}\right)^{\frac{2\lambda_k\eta_k}{\theta_k^2}}}\exp\left(\frac{2\mathcal{S}_k(0)\xi_k(z)}{\lambda_k + \gamma_k\coth\frac{\gamma_k t}{2}}\right)\right)
\end{aligned} \tag{4.40}$$

where $\gamma_k = \sqrt{\lambda_k^2 - 2\theta_k^2\xi_k(z)}$ and $\xi_k(z) = \sum_A \left(e^{i\nu_A z} - 1\right)\omega_{Ak}p_A$.

Using FFT technique the density function can be computed explicitly.

Moments of the integrated CIR Process

To choose plausible parameters we are interested in the first two moments of the integrated CIR process, $I_k(t)$ to match the mean and variance of the sector variables.

Lemma 4.5 *The first two moments of integrated CIR process are given by*

$$\begin{aligned}
\mathbf{E}[I_k(t)] &= \eta t + \frac{(s_0-\eta)(1-e^{-\lambda t})}{\lambda} \\
\mathbf{E}\left[I_k(t)^2\right] &= \left(\frac{s_0-\eta}{\lambda}\right)^2 + \frac{s_0\theta^2}{\lambda^3} - \frac{5\eta\theta^2}{2\lambda^3} \\
&+ t\left(\frac{2\eta(s_0-\eta)}{\lambda} + \frac{\eta\theta^2}{\lambda^2}\right) + t^2\eta^2 \\
&+ e^{-t\lambda}\left[-2\left(\frac{s_0-\eta}{\lambda}\right)^2 + \frac{2\eta\theta^2}{\lambda^3} + t\left(\frac{-2\eta(s_0-\eta)}{\lambda} - \frac{2\theta^2(s_0-\eta)}{\lambda^2}\right)\right] \\
&+ e^{-2t\lambda}\left[\frac{(s_0-\eta)^2}{\lambda^2} - \frac{\theta^2(2s_0-\eta)}{2\lambda^3}\right] \\
\mathrm{var}[I_k(t)] &= \frac{s_0\theta^2}{\lambda^3} - \frac{5\eta\theta^2}{2\lambda^3} + t\frac{\eta\theta^2}{\lambda^2} + e^{-\lambda t}\left[\frac{2\eta\theta^2}{\lambda^3} - t\frac{2\theta^2(s_0-\eta)}{\lambda^2}\right] - e^{-2\lambda t}\frac{\theta^2(2s_0-\eta)}{2\lambda^3}
\end{aligned}$$

Proof: Since the Laplace transform of $I_k(t)$ is finite, all the moments of $I_k(t)$ are finite. For simplicity denote $\mathcal{S}_k(0) = s_0$ and omit index k. With the relation between moments and Laplace transform we can get:

$$\mathbf{E}\left[I(t)^n\right] = (-1)^n \frac{d^n}{ds^n} \mathbf{E}\left[e^{-sI(t)}\right]|_{s=0}, \;\; n \in \mathbb{N}. \tag{4.41}$$

Related to Dufresne [2001] the quickest way to compute the moments of $I_k(t)$ is to use the recursive method, especially for higher moments. Since we need only the first two moments, I compute them with the general symbolic formulas in Matlab.
The first moment can be given simply using Fubini. Because $\mathcal{S}_k(t)$ is positive, the process $I_k(t) = \int_0^t \mathcal{S}_k(u)du$ is an increasing process and we can compute its first moment by:

$$\begin{aligned}
\mathbf{E}\left[I_k(t)\right] &= \int I_k(t) d\mathbf{Q} \\
&= \int \int_0^t \mathcal{S}_k(u) du d\mathbf{Q} \\
&= \int_0^t \int \mathcal{S}_k(u) d\mathbf{Q} du \\
&= \int_0^t \mathbf{E}\left[\mathcal{S}_k(u)\right] du \\
&= \int_0^t \left[\eta + (s_0 - \eta) e^{-\lambda u}\right] du \\
&= \eta t + \frac{(s_0 - \eta)(1 - e^{-\lambda t})}{\lambda}
\end{aligned} \tag{4.42}$$

where the mean of CIR process can be computed by taking the expectation of (4.38):

$$\mathbf{E}\left[\mathcal{S}(t)\right] = s_0 + \lambda \int_0^t \left(\eta - \mathbf{E}\left[\mathcal{S}(u)\right]\right) du$$

The solution of this equation is given by:

$$\mathbf{E}\left[\mathcal{S}(t)\right] = \eta + (s_0 - \eta)\, e^{-\lambda t}$$

Other moments of $\mathcal{S}(t)$ can be computed in a similar way. In particular,

$$\mathrm{var}[\mathcal{S}(t)] = \frac{\theta^2 \eta}{2\lambda} + \frac{\theta^2 (s_0 - \eta)}{\lambda} e^{-\lambda t} + \frac{\theta^2 (\eta - 2s_0)}{2\lambda} e^{-2\lambda t}$$

As introduced above, η represents the long-term mean of the CIR process and λ is responsible for the rate of mean-reversion. In the limit $t \to \infty$ (or in stationary case) the last two terms of the variance die out and the variance becomes $\frac{\theta^2 \eta}{2\lambda}$. This explains the role of θ in the model setting.
With respect to (4.41) we get the second moment of $I_k(t)$ as:

$$
\begin{aligned}
\mathbf{E}\left[I_{k,t}^2\right] &= \frac{s_0^2}{\lambda^2} - \frac{2s_0\eta}{\lambda^2} + \frac{\eta^2}{\lambda^2} + \frac{s_0\theta^2}{\lambda^3} - \frac{5\eta\theta^2}{2\lambda^3} \\
&+ t\left(\frac{2s_0\eta}{\lambda} - \frac{2\eta^2}{\lambda} + \frac{\eta\theta^2}{\lambda^2}\right) + t^2\eta^2 \\
&+ e^{-\lambda t}\left[-\frac{2s_0^2}{\lambda^2} + \frac{4s_0\eta}{\lambda^2} - \frac{2\eta^2}{\lambda^2} + \frac{2\eta\theta^2}{\lambda^3} + t\left(-\frac{2s_0\eta}{\lambda} + \frac{2\eta^2}{\lambda} - \frac{2s_0\theta^2}{\lambda^2} + \frac{2\eta\theta^2}{\lambda^2}\right)\right] \\
&+ e^{-2\lambda t}\left(\frac{s_0^2}{\lambda^2} - \frac{2s_0\eta}{\lambda^2} + \frac{\eta^2}{\lambda^2} - \frac{s_0\theta^2}{\lambda^3} + \frac{\eta\theta^2}{2\lambda^3}\right) \\
&= \left(\frac{s_0-\eta}{\lambda}\right)^2 + \frac{s_0\theta^2}{\lambda^3} - \frac{5\eta\theta^2}{2\lambda^3} \\
&+ t\left(\frac{2\eta(s_0-\eta)}{\lambda} + \frac{\eta\theta^2}{\lambda^2}\right) + t^2\eta^2 \\
&+ e^{-t\lambda}\left[-2\left(\frac{s_0-\eta}{\lambda}\right)^2 + \frac{2\eta\theta^2}{\lambda^3} + t\left(\frac{-2\eta(s_0-\eta)}{\lambda} - \frac{2\theta^2(s_0-\eta)}{\lambda^2}\right)\right] \\
&+ e^{-2t\lambda}\left[\frac{(s_0-\eta)^2}{\lambda^2} - \frac{\theta^2(2s_0-\eta)}{2\lambda^3}\right] \qquad (4.43)
\end{aligned}
$$

Using (4.42) (4.43) we can get the variance of $I_k(t)$

$$
\begin{aligned}
\operatorname{var}[I_k(t)] &= E[I_k(t)^2] - E[I_k(t)]^2 \\
&= \frac{s_0\theta^2}{\lambda^3} - \frac{5\eta\theta^2}{2\lambda^3} + t\frac{\eta\theta^2}{\lambda^2} \\
&+ e^{-\lambda t}\left[\frac{2\eta\theta^2}{\lambda^3} - t\frac{2\theta^2(s_0-\eta)}{\lambda^2}\right] - e^{-2\lambda t}\frac{\theta^2(2s_0-\eta)}{2\lambda^3} \quad \square \qquad (4.44)
\end{aligned}
$$

With the normalization condition $\mathbf{E}[\mathcal{S}_k(t)] = 1$ we can set $s_0 = \eta = 1$. That means the initial value of the course is one, which corresponds also with the long-term mean value. This is a meaningful choice. From (4.42) we can get

$$\mathbf{E}[I_k(t)] = t,$$

$$\mathbf{E}[L(t)] = t\sum_A p_A\nu_A.$$

For $t = 1$:

$$\sigma_{\mathcal{S}_k}^2 := \operatorname{var}[\mathcal{S}_k(1)] = \frac{\theta^2}{2\lambda}(1 - e^{-2\lambda})$$

and

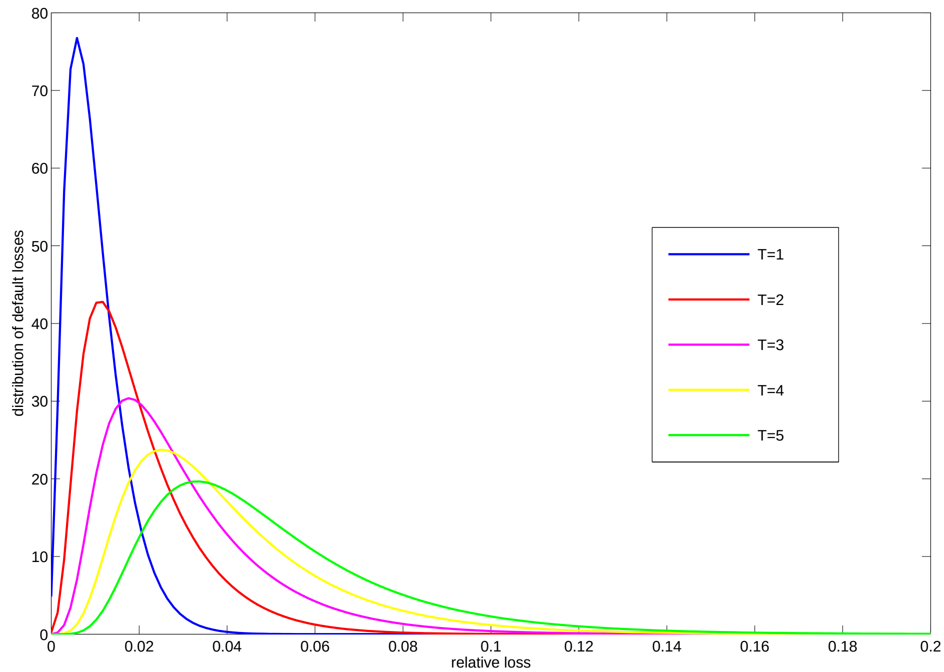
$$
\begin{aligned}
\operatorname{var}[I_k(1)] &= -\frac{3\theta^2}{2\lambda^3} + \frac{\theta^2}{\lambda^2} + e^{-\lambda}\frac{2\theta^2}{\lambda^3} - e^{-2\lambda}\frac{\theta^2}{2\lambda^3} \\
&= -\frac{\theta^2}{\lambda^3} + \frac{\theta^2}{\lambda^2} + e^{-\lambda}\frac{2\theta^2}{\lambda^3} - \frac{\theta^2}{2\lambda^3}(1 - e^{-2\lambda})
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\theta^2}{\lambda^3} + \frac{\theta^2}{\lambda^2} + e^{-\lambda}\frac{2\theta^2}{\lambda^3} - \frac{\sigma_{\mathcal{S}}^2}{\lambda^2} \\
&= \frac{\theta^2}{\lambda^3}(2e^{-\lambda} - 1) + \frac{\theta^2}{\lambda^2} - \frac{\sigma_{\mathcal{S}}^2}{\lambda^2}
\end{aligned}
$$

Parameter θ can be chosen such that the one-period variance of the integrated process matches the variance of the corresponding sector variables in the static model. The additional parameter λ steers the speed of increase of the integrated CIR process variance and hence of the portfolio loss variance, in time, like the calibration factor in Section 4.4.2.

Parameter Calibration and Numerical Example

In the following figures we show how the speed parameters affect on the loss distributions (Figure 4.5 and 4.6) and how the loss distribution looks like for different sector variances (Figure 4.7 and 4.8). The example portfolio has a average default rate of 1% and two latent independent systematic risk factors and one idiosyncratic risk factor. They are set simply equally weighted. From the visible results we can conclude that the variance of the portfolio loss distribution is monotonic to the sector variance, i.e. the portfolio loss distribution with smaller sector variance has also smaller variance. And the speed parameter affects the loss distribution variance in a reverse manner, i.e. the higher the speed of increase of the integrated CIR process is, the smaller variance the loss distribution has. Therefore, both parameters can be used to steer the variance of the whole portfolio loss distribution.



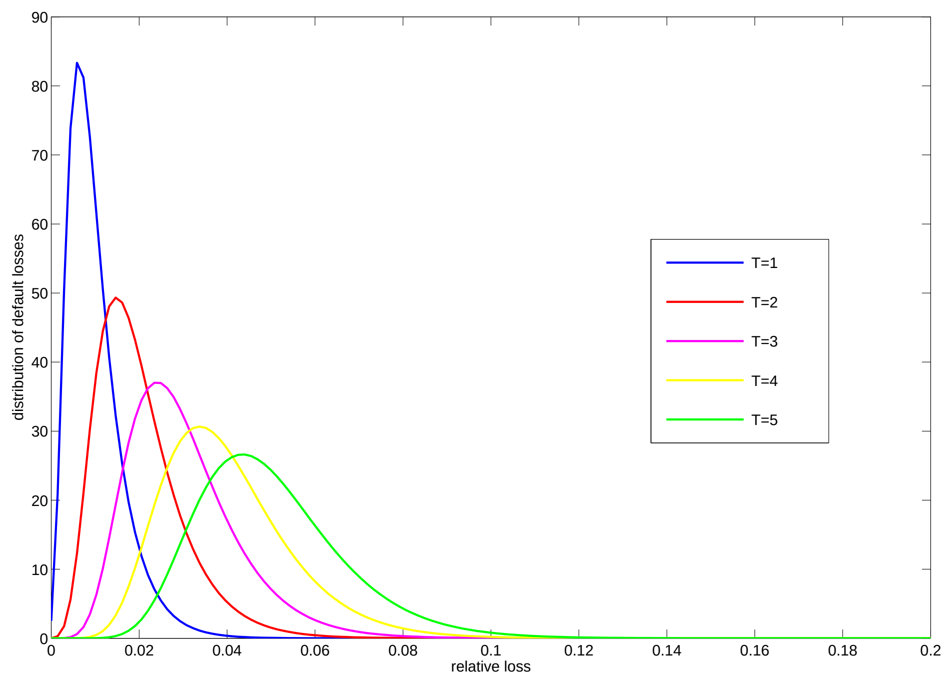


Fig. 4.5: $s_0 = \eta = 1$, $\sigma_{\mathcal{S}}^2 = 1$ and the speed parameter $\lambda = 1$ upper, $\lambda = 4$ lower

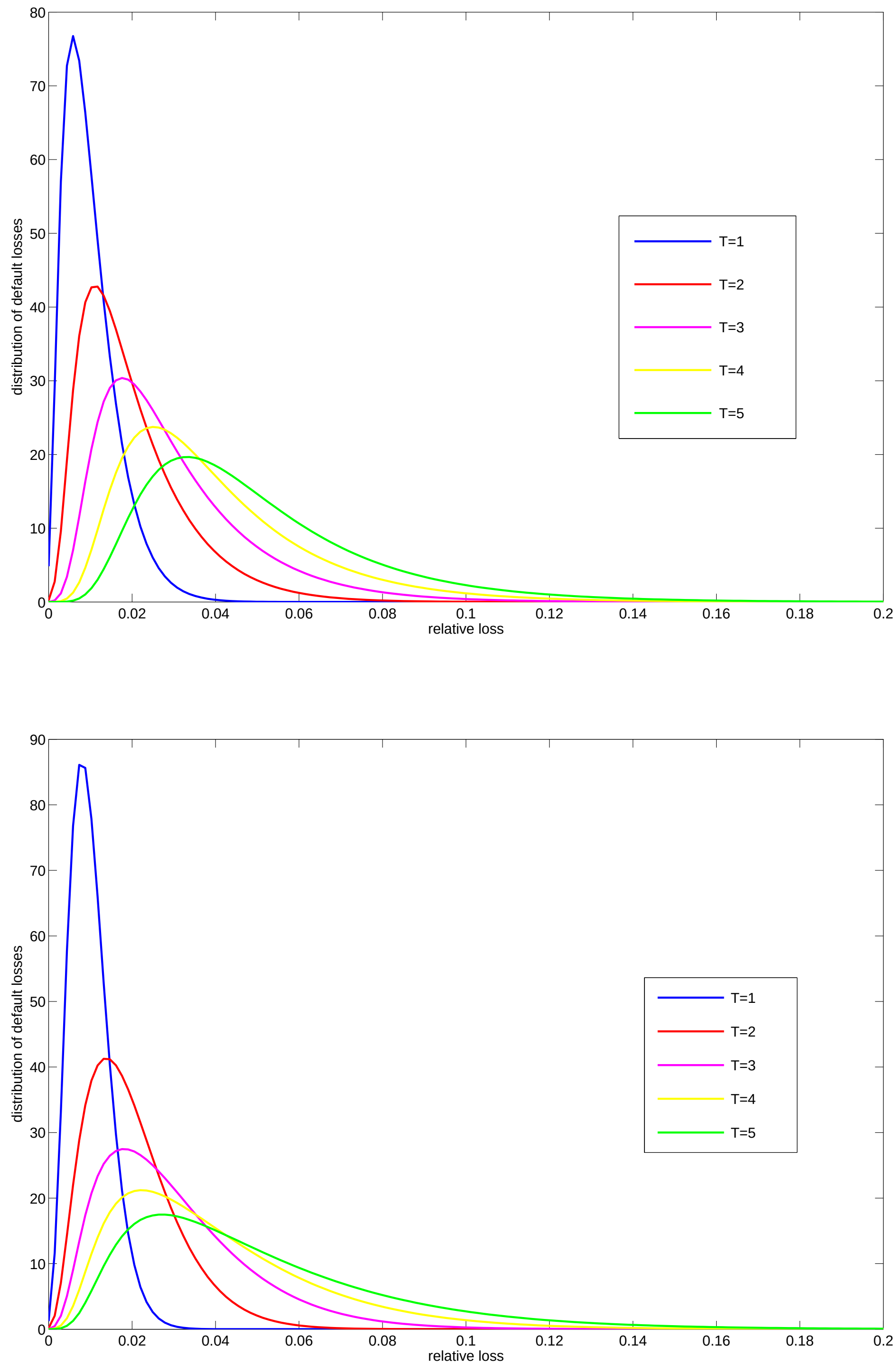


Fig. 4.6: $s_0 = \eta = 1$, $\sigma_{\mathcal{S}}^2 = 1$ and the speed parameter $\lambda = 1$ upper, $\lambda = 0.3$ lower

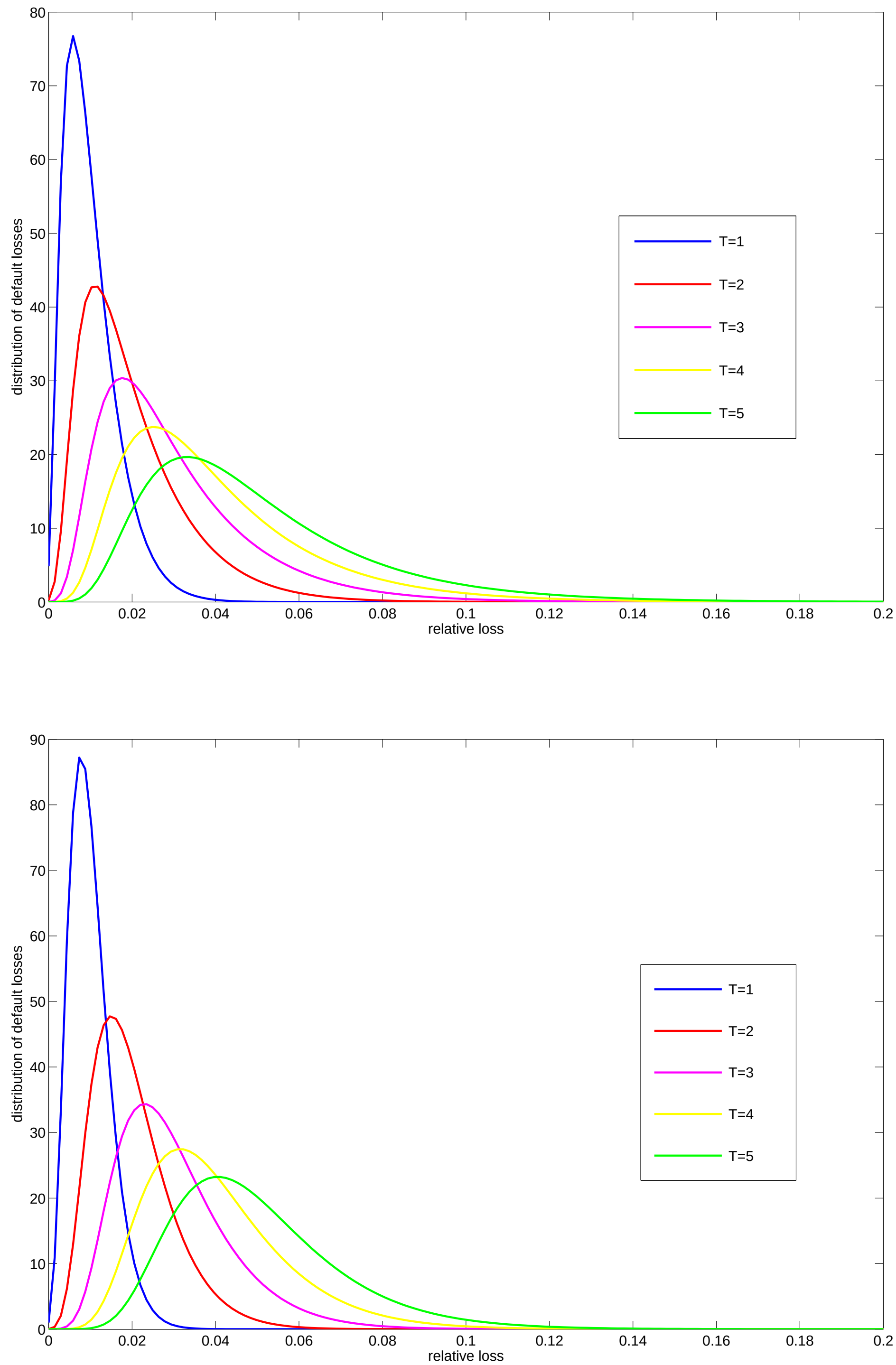
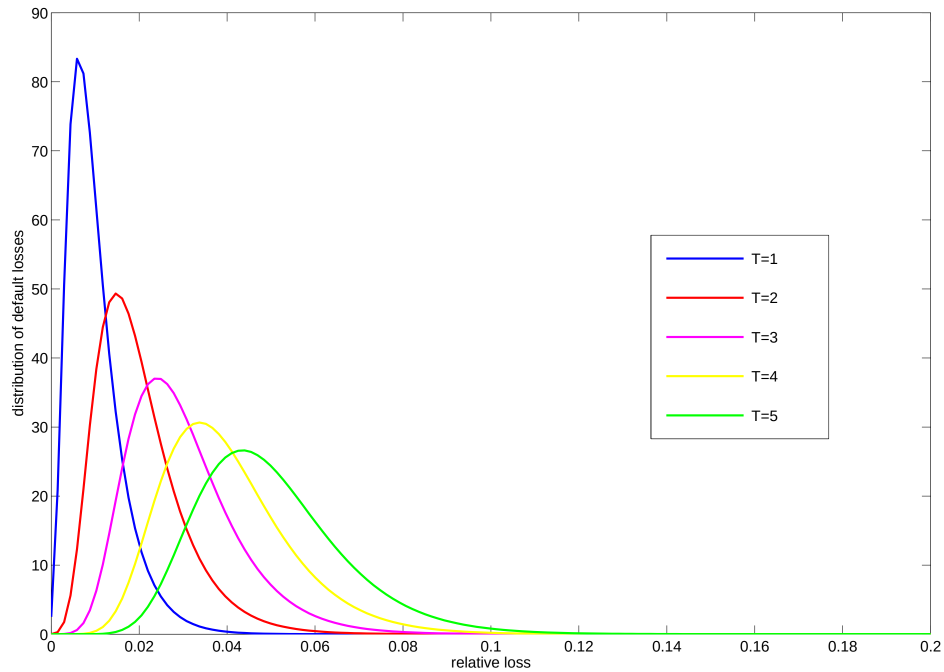


Fig. 4.7: $s_0 = \eta = 1$, $\lambda = 1$, $\sigma_{\mathcal{S}}^2 = 1$ upper, and $\sigma_{\mathcal{S}}^2 = 0.5$ lower



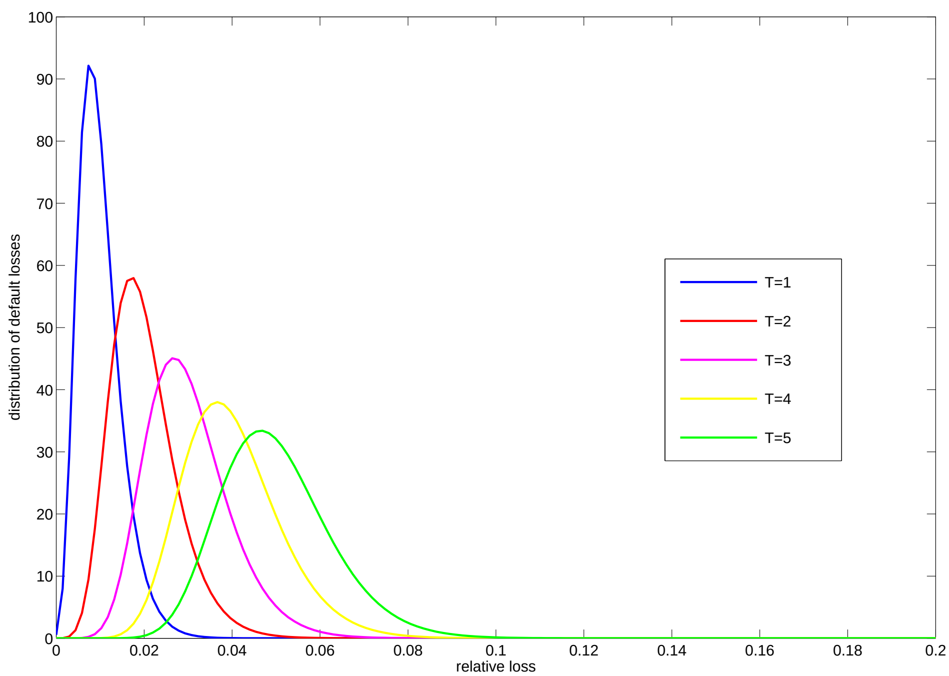


Fig. 4.8: $s_0 = \eta = 1$, $\lambda = 4$, $\sigma_{\mathcal{S}}^2 = 1$ upper, and $\sigma_{\mathcal{S}}^2 = 0.5$ lower

Chapter 5

Conclusion

In the previous chapters we introduced the Large Homogenous Portfolio model and the CreditRisk$^+$ model for pricing synthetic collateralized debt obligations. It was also demonstrated how the standard CreditRisk$^+$ model can be extended by relaxing the condition of independent sector variables and by dynamizing the static property of the model.

The LHP model provides an effective approach, in which the valuation of CDO tranches takes place. Its convenience in numerical calculations contributes to its appeal for practical applications. The approaches to implied correlations for CDO tranches include the compound correlation and the base correlation framework. As we presented, the one-factor Gaussian model is inadequate for the valuation of CDO tranches. Despite this knowledge the approach is still the standard CDO pricing model in practice today.

We present the CreditRisk$^+$ model in terms of the characteristic function instead of the usual approach by the probability-generating function. One advantage of this approach is that no basic loss unit has to be determined, since the proper choice of the basic loss unit may be critical. Based on the Fast Fourier Transform the distribution function of the portfolio loss in CreditRisk$^+$ can be obtained quite fast and numerically stably.

Using initial parameters, the CreditRisk$^+$ model produces much less heavy tails compared to the losses of senior tranches traded in the market. By increasing the volatility of default probability one can obtain tails which are fat enough to meet market tranche losses. However, in order to fit each tranche loss, different calibration factors have to be used. These results can be interpreted as being similar to the idea of correlation skew in the LHP model.

Default rates show a high degree of variability, which can be partially explained by the dynamics of macroeconomic risk factors. Thus, it seems plausible to model latent macroeconomic risk factors by stochastic processes. Modeling in a static single period is a drawback of the CreditRisk$^+$ model, which can be modified by a dynamic modeling. This leads to a version of CreditRisk$^+$ in continuous time without losing the analytical tractability. Parameter calibration of the sector processes is the topic ongoing research in the dynamic modeling.

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