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# A Fractional Heath-Jarrow-Morton Approach For Interest Rate Markets

Diplomarbeit

von

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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

Garching, den 16. September 2010

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# Contents

# List of Figures

# Abbreviations

| | |
|---|---|
| Bm | Brownian motion |
| fBm | fractional Brownian motion |
| $H$-sssi | $H$-self similar with stationary increments |
| a.s. | almost surely |
| sde | stochastic differential equation |
| CIR | Cox-Ingersoll-Ross |
| HJM | Heath-Jarrow-Morton |

# Notations

| | |
|---|---|
| $P(t,T)$ | bond price at time $t$ with maturity $T$ |
| $B_0(t)$ | money market account at time $t$ |
| $Z_t(T)$ | discounted bond price at time $t$ with maturity $T$ |
| $R(t,T)$ | zero rate at time $t$ with maturity $T$ |
| $r(t)$ | short rate at time $t$ |
| $f(t,T)$ | forward rate at time $t$ with maturity $T$ |
| $T^*$ | upper limit of trading interval |
| log, exp | natural logarithm, exponential function |
| sup | supremum |
| $\{B(t)\}_{t\in\mathbb{R}}$ | Brownian motion |
| $H$ | Hurst parameter |
| $\{B^H(t)\}_{t\in\mathbb{R}}$ | fractional Brownian motion |
| Var, Cov | variance, covariance |
| $\mathbb{P}$, $\mathbb{Q}$ | real-world measure, risk-neutral measure |
| $\mathbb{E}$, $\mathbb{E}_{\mathbb{Q}}$ | expectation, expectation with respect to the risk-neutral measure |
| $\stackrel{d}{=}$ | equality in distribution |
| sign | signum function |
| $\mathbf{1}_B$ | indicator function of the set $B$ |
| $\chi_B$ | indicator function of the set $B$ used for trading strategy |
| $L^p(\mathbb{R})$ | space of $p$-integrable functions in $\mathbb{R}$ |
| $a \wedge b$ | minimum of $a, b \in \mathbb{R}$ |
| $\lfloor x \rfloor$ | greatest integer not exceeding $x \in \mathbb{R}$ |
| $\mathcal{C}^2(\mathbb{R},\mathbb{R})$ | space of all two times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ |
| $\mathcal{C}_\Delta$ | space of all real-valued continuous functions on a metric space $\Delta$ |
| Dom(A) | domain of the set $A$ |
| $[X,X]_t$ | quadratic variation of the stochastic process $X$ |
| $K_H$ | square integrable kernel function |

# 1 Introduction

One dollar today is better than one dollar tomorrow. And one dollar tomorrow is obviously better than one dollar in a year. The question that captures us is what we should pay today for a guaranteed cash payment of one dollar at some specified time in the future. This is the question we deal with when pricing a zero-coupon bond. It has been one of the main challenges in interest-rate theory to find the driving factors of these zero-coupon bond prices. Therefore many different interest-rate models evolved such as the models by [Vasicek [1977]], [Brennan and Schwartz [1979]] or [Ho and Lee [1979]] and many modifications have been made as well, e.g. by [Cox et al. [1985]].

Another important reason for developing interest-rate models is the pricing of interest rate derivatives, i.e. financial instruments whose payoffs depend in some way on the level of the underlying interest rates. The notional amount of interest rate derivatives globally outstanding at the end of 2009 increased by 6% from the year before to an estimated $426.8 trillion after a year of a decline due to the financial crisis.[1] Hence the interest rate derivatives market can be considered the world´s biggest market.

Interest rate derivatives are more difficult to evaluate than equity or foreign exchange derivatives which is due to a number of reasons. First of all the behaviour of an individual interest rate is more complicated than that of a stock price or an exchange rate, i.e. interest rates are driven by macroeconomic factors such as gross domestic products or volatilities, which exhibit long-range dependence. We refer to [Henry and Zaffaroni [2003]] for empirical evidence of this finding and will get back to it in detail later on. Secondly, for many products it is necessary to develop a model that describes the entire zero-coupon yield curve in order to valuate them. Moreover, the volatilities of different points on the yield curve are different. And most obviously, interest rates are used both for discounting and as the underlying defining the payoff of the derivative, which is different to the valuation of stock options for example.

Due to these complexities new approaches for interest-rate models had to be developed. For instance, amongst other models, [Heath et al. [1992]] came up with a more general and unifying framework for interest-rate models. The aim of this thesis is to implement the above mentioned long-range dependence of interest rates into the Heath-Jarrow-Morton interest-rate model. The commonly used Brownian motion does not reflect this long-range dependence due to its independent increments. Therefore we will embed fractional Brownian motion into a Heath-Jarrow-Morton model following an approach by [Ohashi [2009]] in

[1] Numbers from [International Swaps and Derivatives Association [2009]]

order to capture this dependence in our model. A line of argumentation for this approach will be given. Therefore our mathematical focus is targeted at fractional Brownian motion, where a lot of theory exists for, e.g. by [Samorodnitsky and Taqqu [2000]], [Pipiras and Taqqu [2000]] or [Duncan et al. [2002]]. We will work through this theory always focused on the purpose of interest-rate modelling. There has already been some research done with embedding fractional Brownian motion into interest-rate models such as in [Fink et al. [2010], Section 4] for a fractional Vasicek model. Even for a Heath-Jarrow-Morton model other fractional approaches exist, for instance in [Gapeev [2004]] whereas his paper only focuses on the Markovian case.

This thesis is organized as follows. We will start with some basic definitions used in the course of this thesis and introduce the important notions for interest-rate markets in Section 2.1. In Section 2.2 we will give a definition of fractional Brownian motion and analyze its most relevant properties. Moreover we come up with an integration theory with respect to fractional Brownian motion, which is different to ordinary stochastic calculus and a bit more sophisticated. We will build our fractional Heath-Jarrow-Morton model upon those preliminaries later on.

We will introduce some famous interest-rate models in Chapter 3, i.e. short-rate models that have been developed some time ago already, but which are still important for interest-rate modelling theory. Hereby we will get an idea of what interest-rate modelling is about and where problems may arise. Furthermore, this illustrates the evolution of those interest-rate models. Additionally to our summary in the end, we will always provide a conclusion for every chapter in order to point out the main results.

Afterwards we will go into detail with the Heath-Jarrow-Morton model in Chapter 4, a model that takes on a different and more general approach by modelling forward rates. We will give a closed form solution of bond prices in Section 4.1 and derive a no-arbitrage condition in Section 4.2. This chapter is a very important step in order to understand the differences to the short-rate models mentioned above and for providing the basics and a thorough background for the fractional Heath-Jarrow-Morton model in Chapter 5, our main focus of this thesis.

In Chapter 5 we start with the set-up of the fractional Heath-Jarrow-Morton model in order to derive a closed form solution for the bond price process. We will need a lot of maths in order to come up with a no-arbitrage framework and the change of measure for deriving the no-arbitrage drift condition, which is a lot more sophisticated than in the classical case as it will turn out. Consequently, this will enable us to state the bond price in a conditional expectation form as well.

In Chapter 6 we will run simulations for the paths of fractional Brownian motion, the underlying stochastic differential equations of some interest-rate models and finally for the bond prices, too. This will necessitate mathematical prework as well and we will need to specify our economic environment. The purpose of this chapter is to illustrate our results. Therefore we will present several graphs for models under different assumptions.

# 2 Preliminaries

## 2.1 Interest-Rate Markets

We will start off with some basic knowledge of interest-rate markets and introduce terminology we will need throughout this thesis.
The guaranteed cash payment of one dollar in the future is one of the basic assets in the interest-rate market. It is called a *zero-coupon bond*. At the evaluation time $t \in \mathbb{R}_+$ we have to pay a price $P(t,T)$ to receive that one dollar at the *maturity date* $T \in \mathbb{R}_+$, $T \geq t$, denoted in years. Zero-coupon bonds are traded in *face value*, also called *nominal value*. 'Zero-coupon' refers to the fact that there will be no payments during the lifetime of this contract. Conversely, in interest-rate markets there are so-called coupon bonds as well. For those the holder of the coupon bond receives some specified periodic payments from the issuer of the coupon bond during the lifetime of the bond - the coupons. In European markets coupons are usually paid once a year, whereas in the United States coupon bonds may have semi-annual payments. Throughout this thesis we will focus on the pricing of zero-coupon bonds since coupon bonds can be considered as a portfolio of many zero-coupon bonds. In order to illustrate this let $C(T_i)$, $i = 1, ..., N$, $N \in \mathbb{N}$, with $0 \leq T_1 < T_2 < ... < T_N = T$, denote the coupons of the bond paid at time $T_i$ and $T$ the maturity of the bond. For this matter we denote the price of the coupon bond by $P_C(t,T)$. Then we can write

$$P_C(t,T) = \sum_{i=1}^{N} C(T_i)P(t,T_i),$$

where the last coupon payment $C(T_N)$ includes the face value of the bond as well.
The time $T-t$ refers to the period of time $t$ to $T$ and is called *time to maturity*. Moreover we focus on the case of non-defaultable bonds, assuming the issuer of the bond can always meet its liabilities and will not go bankrupt. Formally this can be expressed by $P(T,T) = 1$. We will touch on the much more sophisticated case of defaultable bonds in brief later on.
Now we will state some important definitions, we will need in the following chapters of this thesis.

**Definition 2.1** (The short rate)**.** *The short rate is the interest rate at time $t$ for an infinitesimal period of time given the limit exists, which is why we refer to it as the instantaneous short rate. Formally this is*

$$r(t) := R(t,t) := - \lim_{\Delta t \searrow 0} \frac{\log P(t, t + \Delta t)}{\Delta t},$$

*where $R(t,T)$ is the zero rate (or spot rate) that denotes the appreciation of one unit at time $t$ in an interval $[t,T]$ and $P(t,T)$ is the price of the zero-coupon bond at time $t$ with maturity $T$ and face value (or liability) $L=1$. [Zagst [2007]]*

We repeat that the term 'instantaneous' refers to the fact that at time $t$ one borrows money at a certain interest rate and pays it back just one instant later.

**Remark 2.2.** *The mapping $T \mapsto R(t,T)$ is the zero curve (or spot curve), which describes the evolution of spot rates for different maturities. The mapping $T \mapsto P(t,T)$ is called discount curve at time $t$. So, a simple approach to a bond price would be*

$$P(t,T) = e^{-R(t,T)(T-t)}.$$

*The zero curve is called normal if its slope is positive, i.e. the mapping $T \mapsto R(t,T)$ is increasing whereas the zero curve is called inverse if its slope is negative, i.e. the mapping $T \mapsto R(t,T)$ is decreasing. A flat zero curve would be characterized by a slope of zero.*

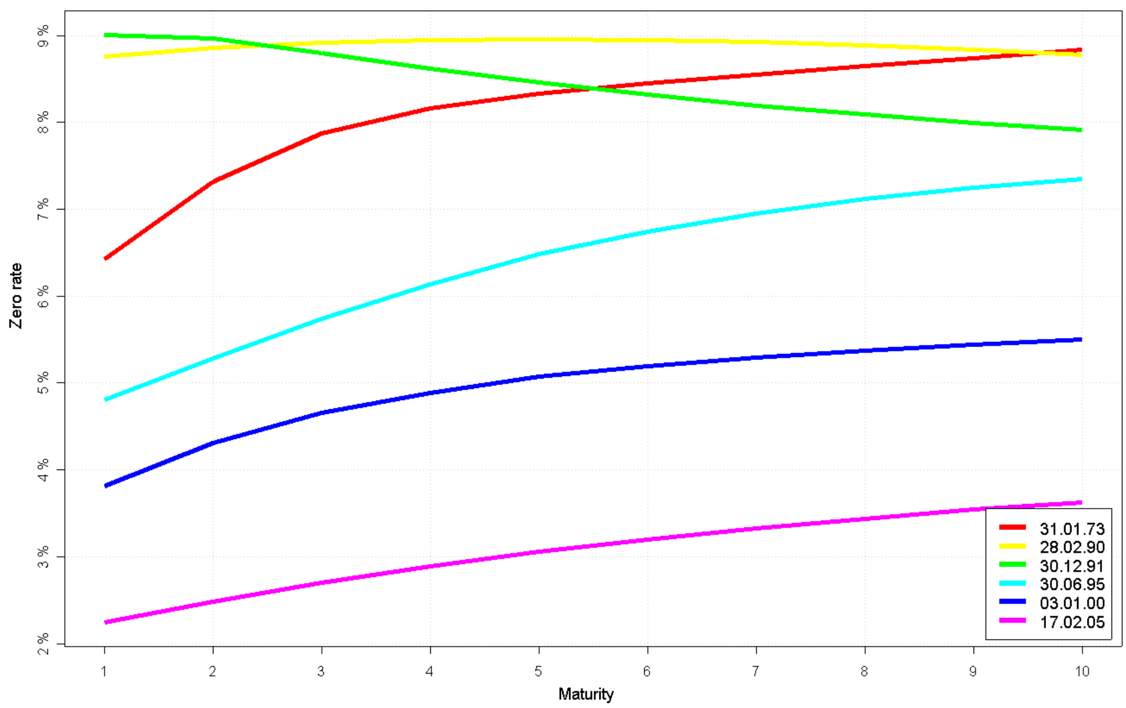



Figure 2.1: Zero curves of the German Bundesbank at different times in history[2]

*We observe many normal zero curves in Figure 2.1 except of the yellow curve in 1990, which can be considered flat and the green curve in 1991, which is inverse. This development was caused by the German reunification, an extraordinary economic circumstance.*

[2]Numbers from [Deutsche Bundesbank]

**Remark 2.3.** *Moreover the short rate can be viewn in a different way, that is*

$$
\begin{aligned}
-\frac{\partial}{\partial T}\log P(t,t) :&= -\frac{\partial}{\partial T}\log P(t,T)\Big|_{T=t} \\
&= -\lim_{\Delta t \searrow 0} \frac{\log P(t,T+\Delta t) - \log P(t,T)}{\Delta t}\Big|_{T=t} \\
&= -\lim_{\Delta t \searrow 0} \frac{\log P(t,T+\Delta t)}{\Delta t}\Big|_{T=t} + \lim_{\Delta t \searrow 0} \frac{\log P(t,T)}{\Delta t}\Big|_{T=t} \\
&= -\lim_{\Delta t \searrow 0} \frac{\log P(t,t+\Delta t)}{\Delta t} = r(t),
\end{aligned}
$$

*where the last equation stems from the fact that the second part of the sum equals zero since $P(t,t) = 1$.*

**Definition 2.4** (The forward short rate)**.** *The instantaneous forward short rate (from now on only referred to as forward rate) is the interest rate for an infinitesimal period of time at time $T$ measured at time $t \leq T$, i.e.*

$$
f(t,T) := -\lim_{\Delta t \searrow 0} \frac{\log P(t,T+\Delta t) - \log P(t,T)}{\Delta t} = -\frac{\partial}{\partial T}\log P(t,T). \tag{2.1}
$$

*The mapping $T \mapsto f(t,T)$ is called forward curve. Obviously $r(t) = f(t,t)$ holds. [Zagst [2007]]*

There is also a different, more intuitive approach to the forward rate in [Zagst [2002], chapter 4]. We imagine a contract (a so-called forward zero-coupon bond) in which, at time $t$, we agree at no cost to exchange a zero-coupon bond at a future time $T_1 \geq t$ with maturity $T_2 \geq T_1$ for a cash payment denoted by $P(t,T_1,T_2)$. The question is how large the price $P(t,T_1,T_2)$ has to be. Therefore we come up with a simple no-arbitrage argument. We sell a number $P(t,T_1,T_2)$ of the zero-coupon bonds with maturity $T_1$ at time $t$ and we agree to invest at the future time $T_1$ the amount $P(t,T_1,T_2)P(t,T_1)$, that we receive from this sale, in a zero-coupon bond with maturity $T_2$. By a simple no-arbitrage reasoning this portfolio has to be identical to a zero-coupon bond with maturity $T_2$. Otherwise there would be an opportunity for a risk-less profit. Therefore the price of the portfolio has to equal the price of the $T_2$-zero-coupon bond at all times. Formally this is

$$
\begin{aligned}
& P(t,T_1,T_2)P(t,T_1) = P(t,T_2) \\
\Leftrightarrow \quad & P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}.
\end{aligned} \tag{2.2}
$$

Analogously to the zero rate in Definition 2.1 we denote the corresponding forward zero rate by $R(t,T_1,T_2)$. Then we know

$$
P(t,T_1,T_2) = e^{-R(t,T_1,T_2)(T_2-T_1)} \quad \Leftrightarrow \quad R(t,T_1,T_2) = -\frac{\log P(t,T_1,T_2)}{T_2 - T_1},
$$

which yields by 2.2

$$R(t,T_1,T_2) = -\frac{\log P(t,T_2) - \log P(t,T_1)}{T_2 - T_1}.$$

We let $T_2 - T_1$ approach zero and $T_1 = T$ to come up with the instantaneous forward short rate

$$f(t,T) := R(t,T,T) := -\lim_{\Delta t \to 0} \frac{\log P(t,T+\Delta t) - \log P(t,T)}{\Delta t} = -\frac{\partial}{\partial T} \log P(t,T).$$

**Remark 2.5.** *If the zero curve is normal then the forward curve lies above the zero curve, since it has to balance the gap between short maturity and long maturity (see Figure 2.2 below). Conversely, if the zero curve is inverse then the forward curve lies below the zero curve. Moreover the zero curve and the forward curve coincide for $t = T$. Formally this can be easily verified by*

$$f(t,T) = -\frac{\partial}{\partial T} \log P(t,T) = \frac{\partial}{\partial T}\big(R(t,T)(T-t)\big) = R(t,T) + (T-t)\frac{\partial}{\partial T} R(T,t),$$

*where $\frac{\partial}{\partial T} R(T,t)$ is positive for a normal zero curve and negative for an inverse zero curve respectively. For $t = T$ we obviously get $f(T,T) = R(T,T) = r(T)$.*

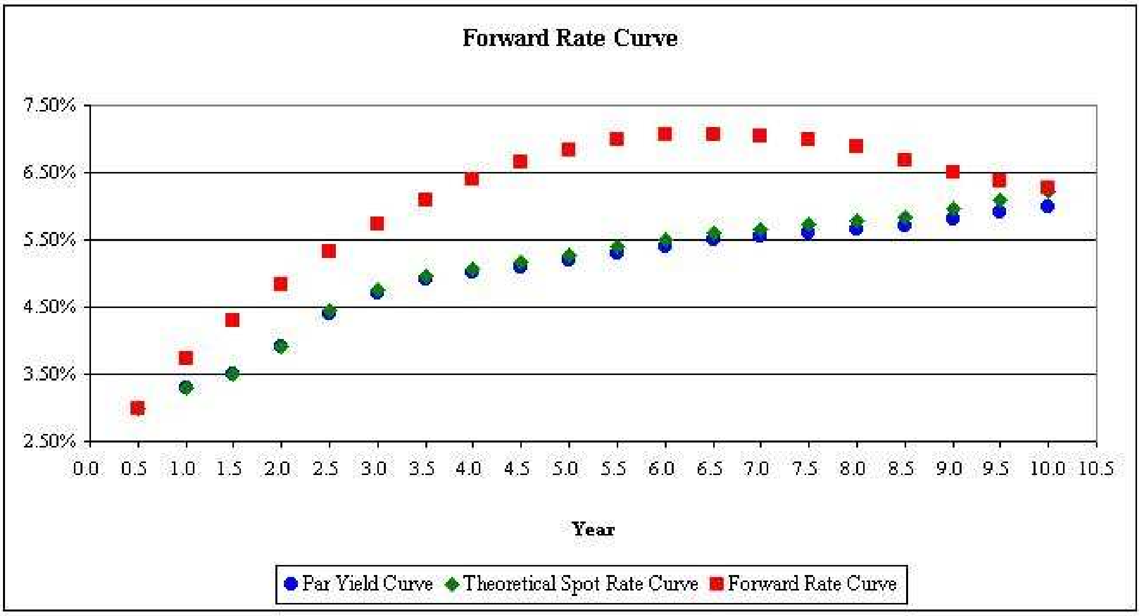



Figure 2.2: Zero curve and forward curve[3]

*In Figure 2.2 we can easily see how forward rate and zero rate drift apart, then move back towards each other and finally coincide at $t = T$ in the case of a normal zero curve.*

In the following chapters we will see that both the short rate $r(t)$ and the forward rate $f(t,T)$ are starting points for many interest rate models in order to describe the evolution of interest rates over time that is the term structure of interest rates.

[3]Figure from [Murray State University]

## 2.2 Fractional Brownian Motion

In this section we will give some important mathematical background knowledge - definitions and conclusions we will need throughout this thesis. The main parts of this section on fractional Brownian motion are based on the chapter about self-similar processes in [Samorodnitsky and Taqqu [2000]].

**Definition 2.6** (Fractional Brownian motion)**.** *A fractional Brownian motion (fBm) $\{B^H(t)\}_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0,1)$ is a Gaussian zero-mean process with $B^H(0) = 0$, stationary increments and covariance function*

$$R_H(t_1,t_2) := \operatorname{Cov}\big(B^H(t_1), B^H(t_2)\big) = \frac{1}{2}\big(\,|t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H}\,\big)\operatorname{Var}(B^H(1)), \quad (2.3)$$

*for $t_1, t_2 \in \mathbb{R}$.*

From now on we will always deal with the standard fractional Brownian motion, that is $\operatorname{Var}(B^H(1)) = 1$. In this case and for $H = \frac{1}{2}$ we get a standard Brownian motion, since the increments will be independent in this case.
Recall that a real valued process $\{X(t)\}_{t\in\mathbb{R}}$ has stationary increments if for $h \in \mathbb{R}_+$ we have $\{X(t+h) - X(h)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t) - X(0)\}_{t\in\mathbb{R}}$.
There are some important properties of fractional Brownian motion, which we will discuss in the following.

**Definition 2.7** (Self-similarity)**.** *A real valued process $\{X(t)\}_{t\in\mathbb{R}}$ is self-similar with index $H > 0$ (H-ss) if for all $a > 0$ the finite-dimensional distributions of $\{X(at)\}_{t\in\mathbb{R}}$ are identical to the finite-dimensional distributions of $\{a^H X(t)\}_{t\in\mathbb{R}}$, i.e. if for any $d \geq 1, t_1, ..., t_d \in \mathbb{R}$ and any $a > 0$*

$$\big(X(at_1), ..., X(at_d)\big) \stackrel{d}{=} \big(a^H X(t_1), ..., a^H X(t_d)\big). \quad (2.4)$$

**Lemma 2.8.** *(i) For every H-self-similar process $X$ we have $X(0) = 0$ a.s., since for each $a > 0$ we get $X(0) = X(a0) \stackrel{d}{=} a^H X(0)$.*

*(ii) Every H-self similar process $X$ with stationary increments (H-sssi) is symmetric, i.e.*

$$X(-t) = X(-t) - X(0) \stackrel{d}{=} X(0) - X(t) \stackrel{(i)}{=} -X(t) \quad \text{for all } t \in \mathbb{R}. \quad (2.5)$$

**Theorem 2.9.** *Fractional Brownian motion exists and is the only Gaussian process that is self-similar with index $H \in (0,1)$ and has stationary increments.*

*Proof.* Existence: Let $X$ be a zero-mean Gaussian random variable whose characteristic function is given by

$$\varphi_X(\theta) := \mathbb{E}[e^{i\theta X}] = \exp(-\sigma^2\theta^2), \quad \theta \in \mathbb{R}, \sigma \in \mathbb{R}_+.$$

The finite-dimensional distributions of a Gaussian process $\{X(t)\}_{t\in\mathbb{R}}$ satisfy

$$\mathbb{E}[e^{i\sum_{j=1}^{m}\theta_j X(t_i)}] = \exp\Big( -\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} A(t_j,t_k)\theta_j\theta_j + \sum_{j=1}^{m}\mu(t_j)\theta_j\Big),$$

where $\theta_1,...\theta_m \in \mathbb{R}, m \geq 1$, $\mu(t_j)$ is a real-valued function for all $j = 1,...,m$, $t \in \mathbb{R}$ and $\{A(t_1,t_2) : t_1,t_2 \in \mathbb{R}\}$ is non-negative definite.
Conversely, to each $\mu$ and $A$ corresponds a Gaussian process with $\mu$ being its mean and $A$ being its autocovariance function.
Now fix $0 < H < 1$:
Since the function $\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H} : t_1, t_2 \in \mathbb{R}\}$ is non-negative definite (for a proof see [Samorodnitsky and Taqqu [2000], Lemma 2.10.8]), there exists a Gaussian process $\{X(t)\}_{t\in\mathbb{R}}$ with mean zero and covariance function

$$R_H(t_1,t_2) = \frac{1}{2}\big( |t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H} \big), \tag{2.6}$$

$t_1, t_2 \in \mathbb{R}$.
We still need to show that this process is self-similar: We get

$$\begin{aligned}\mathrm{Cov}\big(X(at_1), X(at_2)\big) &= \frac{1}{2}\big( |at_1|^{2H} + |at_2|^{2H} - |at_1 - at_2|^{2H} \big)\\ &= \frac{a^{2H}}{2}\big( |t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H} \big),\end{aligned}$$

where the self-similarity index is $H' := 2H > 0$. The mean is zero on both sides. Since the Gaussian distribution is completely determined by its mean and covariance, we can conclude that this process is self-similar.
Similarly the stationarity of the increments can be shown:

$$\begin{aligned}&\mathrm{Cov}\big(X(t_1+h) - X(h), X(t_2+h) - X(h)\big) = \frac{1}{2}\big( |t_1+h|^{2H} + |t_2+h|^{2H} - |t_1-t_2|^{2H} \big)\\ &\quad - \frac{1}{2}\big( |t_1+h|^{2H} + |h|^{2H} - |t_1|^{2H} \big) - \frac{1}{2}\big( |h|^{2H} + |t_2+h|^{2H} - |-t_2|^{2H} \big)\\ &\quad + \frac{1}{2}\big( |h|^{2H} + |h|^{2H} \big) = \frac{1}{2}\big( |t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H} \big)\\ &\quad = \mathrm{Cov}\big(X(t_1) - X(0), X(t_2) - X(0)\big)\end{aligned}$$

Since the mean is zero in either case, the same reasoning for a Gaussian distribution as for self-similarity yields stationarity of the increments.
Uniqueness: Let $\{Y(t)\}_{t\in\mathbb{R}}$ be another $H$-sssi Gaussian process and $\mathrm{Var}(Y(1)) = 1$ as for the standard fractional Brownian motion. We utilize these properties to derive the covariance as

$$\begin{aligned}\mathbb{E}[Y(t_1)Y(t_2)] &= \frac{1}{2}\big(\mathbb{E}[Y^2(t_1)] + \mathbb{E}[Y^2(t_2)] - \mathbb{E}[(Y(t_1) - Y(t_2))^2]\big)\\ &= \frac{1}{2}\big(\mathbb{E}[Y^2(t_1)] + \mathbb{E}[Y^2(t_2)] - \mathbb{E}[(Y(t_1-t_2) - Y(0))^2]\big)\\ &= \frac{1}{2}\big( |t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H} \big).\end{aligned}$$

The first step is a simple rearrangement. The second step is due to the stationary increments where we use the result of Lemma 2.8, $(i)$ for a H-self-similar process, that is $Y(0) = 0$ a.s. The last step utilizes the $H$-self similarity of $Y$ and $\mathbb{E}[Y^2(1)] = 1$.
Moreover we will have to calculate the mean of $Y(t)$: Since $Y(t)$ is $H$-sssi and $Y(0) = 0$, we know that $\mathbb{E}[Y(1)] = \mathbb{E}[Y(2) - Y(1)] = 2^H \mathbb{E}[Y(1)] - \mathbb{E}[Y(1)] = (2^H - 1)\mathbb{E}[Y(t)]$. We can conclude $\mathbb{E}[Y(1)] = 0$ and hence $\mathbb{E}[Y(t)] \equiv 0$, because $\mathbb{E}[Y(-1)] = -\mathbb{E}[Y(1)]$ due to Lemma 2.8, $(ii)$ and $\mathbb{E}[Y(t)] = |t|^H \mathbb{E}[Y(\text{sign}(t))]$, which is obvious by self-similarity.
So all $H$-sssi Gaussian processes have the covariance function from above and mean zero. For a given $H$ these processes only differ by a multiplicative constant. This proves that $Y(t) \stackrel{d}{=} B^H(t)$ and so uniqueness is proved. □

**Remark 2.10.** *The increments of fractional Brownian motions are called fractional Gaussian noise. One can show that for $H \in (\frac{1}{2}, 1)$ an fBm displays long-range dependence, that is its autocovariance function $\gamma_{B^H}(h) := \text{Cov}(B^H(t+h), B^H(t))$ decreases so slowly at large lags that $\sum_{h=-\infty}^{\infty} \gamma_{B^H}(h) = \infty$ as $\gamma_{B^H}(h) \to 0$ holds. Intuitively, when long-range dependence is present, high-lag correlations may be individually small, but their cumulative effect is significant.*

**Proof:** Considering the covariance function $R_H(t_1, t_2)$ the autocovariance function of an fBm is given by

$$\gamma_{B^H}(h) = \frac{1}{2}\big(|h-1|^{2H} - 2|h|^{2H} + |h+1|^{2H}\big).$$

Define the function $g(x) = (1-x)^{2H} - 2 + (1+x)^{2H}$ and note that $\gamma_{B^H}(h) = \frac{1}{2} h^{2H} g(1/h)$, for $h \geq 1$. Using a Taylor expansion at the origin of $g(1/h)$ to the second degree one can see that

$$\gamma_{B^H}(h) \sim H(2H-1)h^{2H-2}$$

for $h \to \infty$ and so for $H > \frac{1}{2}$ the series $\sum_{h=-\infty}^{\infty} \gamma_{B^H}(h)$ obviously diverges. □

Long-range dependence is one of the key facts why fractional Brownian motions are more reasonable for financial modelling compared to ordinary Brownian motions in some areas. Dependent on the data the independent increments of Brownian motions may not be very realistic when observing time series into the past. In empirical studies of financial time series, for instance, [Mandelbrot [1997]] demonstrated that log-returns exhibit this long-range dependence, a finding that is very controversial and he supports solely. More interestingly for our examination later on, [McCarthy et al. [2004]] and many others have detected long-range dependence for macroeconomic data, i.e. interest rates or volatilities. This is where our new modelling approach for interest-rate markets in Chapter 5 applies.

**Remark 2.11.** *Instead of analyzing a stochastic process in the time domain, processes can also be analyzed in the so-called frequency or spectral domain. A stationary time-domain series can be transformed into a frequency-domain series without loss of information by the so-called Fourier transform, defined by $\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-itx}dx$ [Rudin [2005]]. This means that the time-domain series is perfectly recovered from the frequency-domain series* *by the so-called inverse Fourier transform. A deterministic function or a realization of a stochastic process can be thought of to consist of trigonometric functions with different frequencies. The information to which extent each frequency is present in the signal is then summarized in the so-called spectral density, which is defined as the square of the magnitude of the Fourier transform, that is* $\Phi(\theta) = |\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\theta x}dx|^2$. *So it captures the frequency content of stochastic processes and helps identify periodicities. In the case of long-range dependence the spectral density increases like a power function at low frequencies and explodes at the origin.*

Moreover, as Theorem 2.9 points out, there are alternative ways to define a fractional Brownian motion, which we will summarize in the following corollary:

**Corollary 2.12.** *Fix* $0 < H < 1$ *and let* $\sigma^2 = \mathbb{E}[X^2(1)] = 1$.
*The following statements are equivalent:*

*(i)* $X(t), t \in \mathbb{R}$, *is Gaussian and H-self-similar with stationary increments*

*(ii)* $X(t), t \in \mathbb{R}$, *is an fBm with self-similarity index H*

*(iii)* $X(t), t \in \mathbb{R}$, *is Gaussian, has mean zero and covariance*

$$\mathrm{Cov}(X(t_1), X(t_2)) = \frac{1}{2}\big(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\big), \quad t_1, t_2 \in \mathbb{R}.$$

### 2.2.1 Integral Representations of Fractional Brownian Motion

In the following we will present two integral representations of a fractional Brownian motion - the time representation and the spectral representation. Each representation will take the integral form $\int_{-\infty}^{\infty} f_t(x)M(dx)$ but with its own set of deterministic functions $f_t, t \in \mathbb{R}$, and its own random measure $M$. For our means it is sufficient to consider $M$ as a Brownian motion $B$ for the time representation and a Gaussian measure $\tilde{B}$ for the spectral representation, respectively. These integral representations are another way to characterize fractional Brownian motion, but more importantly we will need them in order to define an integration theory with respect to fractional Brownian motion, which we will outline in the subsequent subsection 2.2.2.

#### The time representation

This representation is also called the *moving average representation* of fractional Brownian motion.

**Proposition 2.13.** *Let $(\Omega, \mathcal{F})$ be a measure space and let $B$ be a standard Brownian motion defined on $\mathbb{R}$. Let $H \in (0,1)$. Then the standard fractional Brownian motion $B^H(t), t \in \mathbb{R}$ has the integral representation*

$$B^H(t) \overset{d}{=} \frac{1}{C_1(H)} \int_{-\infty}^{\infty} \big(((t-s)_+)^{H-\frac{1}{2}} - ((-s)_+)^{H-\frac{1}{2}}\big) dB(s), t \in \mathbb{R},$$

*where* $C_1(H) = \big(\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})^2 ds + \frac{1}{2H}\big)^{\frac{1}{2}}$.

See [Samorodnitsky and Taqqu [2000], Prop. 7.2.6] for a proof.

#### The spectral representation

This integral representation is also known as the *harmonizable representation* and is of the form $\int_{-\infty}^{\infty} \widetilde{f}_t(x)\widetilde{M}(dx)$, where $\widetilde{f}_t$ is a complex and deterministic function and $\widetilde{M}$ is a specific complex measure. We will focus on a simplified special case which will be sufficient for our purposes. We will integrate with respect to a complex Gaussian measure $\tilde{B} = B^1 + iB^2$ such that $B^1(A) = B^1(-A)$, $B^2(A) = -B^2(-A)$ and $\mathbb{E}[B^1(A)]^2 = \mathbb{E}[B^2(A)]^2 = \frac{1}{2}|A|$, for a Borel set $A$ of finite Lebesgue measure $|A|$.

**Proposition 2.14.** *Let $0 < H < 1$. Then the standard fractional Brownian motion $\{B^H(t), t \in \mathbb{R}\}$ has the integral representation*

$$B^H(t) \overset{d}{=} \frac{1}{C_2(H)} \int_{-\infty}^{\infty} \frac{e^{ixt}-1}{ix} |x|^{-(H-\frac{1}{2})} d\widetilde{B}(x), t \in \mathbb{R}, \tag{2.7}$$

*where* $C_2(H) = \left(\frac{\pi}{H\Gamma(2H)\sin(H\pi)}\right)^{\frac{1}{2}}$.

For a proof of this see Samorodnitsky and Taqqu [2000].

### 2.2.2 Integration With Respect to Fractional Brownian Motion

In our subsequent analysis of the fractional Heath-Jarrow-Morton model in chapter 5 we will make use of the time representation when we come up with an integration approach with respect to fractional Brownian motion. These insights are based on [Pipiras and Taqqu [2000]].
In order to explain the difficulty of defining a stochastic integral with respect to fractional Brownian motion we contrast it with the standard Brownian motion case. Therefore we define $\mathscr{E}$ as the set of all elementary functions

$$f(u) := \sum_{k=1}^{n} f_k \mathbf{1}_{[u_k, u_{k+1})}(u),\ u \in \mathbb{R}, \tag{2.8}$$

where $f_k$ and $u_k < u_{k+1}$ are real numbers. Moreover we need to define an integral of $f \in \mathscr{E}$ with respect to the fBm $B^H$ for $H \in (0,1)$ as

$$\mathscr{I}^H(f) := \int_{\mathbb{R}} f(u) dB^H(u).$$

Then denote the closed span of $B^H$ by

$$\overline{Sp}(B^H) := \{X : \mathscr{I}^H(f_n) \xrightarrow{L^2} X \text{ for some } (f_n) \subseteq \mathscr{E}\}.$$

An element $X \in \overline{Sp}(B^H)$ is a zero-mean Gaussian random variable with variance $\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}(\mathscr{I}^H(f_n))$. Let $f_X$ denote the equivalence class of sequences of elementary functions $(f_n)$ such that $\mathscr{I}^H(f_n) \xrightarrow{L^2} X$ and write the integral with respect to fBm on the real line as

$$X = \int_{\mathbb{R}} f_X dB^H. \tag{2.9}$$

We recall that the characterization for the standard Brownian motion $B^{\frac{1}{2}}$ simplifies due to its independent increments and so $\mathrm{Var}(\mathscr{I}^{\frac{1}{2}}(f)) = \int_{\mathbb{R}} f^2(u)du,\ f \in \mathscr{E}$.
Hence, if $(f_n) \subseteq \mathscr{E}$ and if $\mathscr{I}^{\frac{1}{2}}(f_n)$ converges to $X \in \overline{Sp}(B^{\frac{1}{2}})$ in the $L^2$-sense, there is a unique function $f_X \in L^2(\mathbb{R})$ such that

$$\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}(\mathscr{I}^{\frac{1}{2}}(f_n)) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n^2(u)du = \int_{\mathbb{R}} f_X^2(u)du,$$

due to the fact that $L^2(\mathbb{R})$ is a complete space. So, in contrast to the general case in (2.9), $X \in \overline{Sp}(B^{\frac{1}{2}})$ can more easily be characterized by a single function $f_X \in L^2(\mathbb{R})$ as

$$X = \int_{\mathbb{R}} f_X dB^{\frac{1}{2}}(u). \tag{2.10}$$

The crucial fact is that for $X, Y \in \overline{Sp}(B^{\frac{1}{2}})$ we can state that

$$\mathbb{E}[XY] = \int_{\mathbb{R}} f_X(u) f_Y(u) du$$

and therefore we can say that $\overline{Sp}(B^{\frac{1}{2}})$ and $L^2(\mathbb{R})$ are isometric, i.e. there is a linear and injective mapping between the spaces that preserves the inner products [Kallsen [2007], Theorem 5.1.2].
Therefore our goal is to find a Hilbert space $\mathscr{C}$ of functions on the real line that is isometric to $\overline{Sp}(B^H)$ as well in order to come up with an integral form in the spirit of (2.10). When we proceed to our fractional Heath-Jarrow-Morton model in Chapter 5 we will only consider the long-range dependent case where $1/2 < H < 1$ and that is why we can focus on this case. Unfortunately, spaces isometric to $\overline{Sp}(B^H)$ could not be found, yet. But we can come up with spaces that are isometric to linear subspaces of $\overline{Sp}(B^H)$. Therefore we cite a proposition that shows us how to construct those classes of integrands:

**Proposition 2.15.** *Let $\mathscr{E}$ be the set of elementary functions as in (2.8). Let $\mathscr{I}^H(f) := \int_{\mathbb{R}} f(u)dB^H(u)$ be an integral of $f \in \mathscr{E}$ with respect to the fBm $B^H$ for $H \in (0,1)$. Suppose that $\mathscr{C}$ is a set of deterministic functions on the real line such that*

*(i) $\mathscr{C}$ is an inner product space with an inner product $(f,g)_{\mathscr{C}}$, for $f,g \in \mathscr{C}$,*

*(ii) $\mathscr{E} \subseteq \mathscr{C}$ and $(f,g)_{\mathscr{C}} = \mathbb{E}[\mathscr{I}^H(f)\mathscr{I}^H(g)]$, for $f,g \in \mathscr{E}$ and*

*(iii) the set $\mathscr{E}$ is dense in $\mathscr{C}$.*

*Then there is an isometry between the space $\mathscr{C}$ and a linear subspace of $\overline{Sp}(B^H)$, which is an extension of the mapping $f \to \mathscr{I}^H(f)$, for $f \in \mathscr{E}$.*
*Moreover $\mathscr{C}$ is isometric to $\overline{Sp}(B^H)$ itself if and only if $\mathscr{C}$ is complete.*

For a proof of this proposition we refer to [Pipiras and Taqqu [2000]].

There are several ways to come up with classes of integrands both for the time representation and for the spectral representation of fBm. We will focus on the class of integrands in the time domain, because we will use this integration approach in chapter 5.
So by *(ii)* of Proposition 2.15 we start with the calculation of the covariance

$$\mathbb{E}[\mathscr{I}^H(f)\mathscr{I}^H(g)] = \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(\nu)d^2\mathbb{R}_H(u,\nu),$$

where $R_H$ is the covariance function of a fractional Brownian motion. The double integral is defined to be linear and to satisfy

$$\int_{[a,b]}\int_{[c,d]} d^2R_H(u,\nu) = R_H(d,b) - R_H(d,a) - (R_H(c,b) - R_H(c,a)), \quad u,\nu \in \mathbb{R}$$

for any real numbers $a < b$ and $c < d$. This suggests that we can define our class of integrands as

$$|\Lambda|^H := \{f : \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(\nu)|d^2|R_H|(u,\nu) < \infty\},$$

where $|R_H|$ is the *total variation measure* of $R_H$, which is only defined in the case $1/2 < H < 1$ as

$$|\mathbb{R}_H|(E) = \sup_{\Pi} \sum_i |\,\mathbb{R}_H(E_i)| \quad \text{for all } E \in \mathcal{F},$$

where $\Pi := \cup_i E_i$ is an arbitrary partition of $E$ with measurable subsets $E_i$ and $\mathcal{F}$ is the $\sigma$-algebra of the measure space $(\Omega, \mathcal{F})$.
By differentiating the covariance function $R_H$ as defined in (2.3) with respect to $u$ and with respect to $\nu$ we get

$$d^2 R_H(u,\nu) = H(2H-1)|u-\nu|^{2H-2} du\, d\nu.$$

Hence we can finally define our class of integrands that satisfies all conditions of Proposition 2.15 and accordingly an isometry between this class and a linear subspace of $\overline{Sp}(B^H)$ exists.

**Definition 2.16.**

$$|\,\Lambda|^H := \{f : \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(\nu)||u-\nu|^{2H-2} du\, d\nu < \infty\},$$

*for* $\frac{1}{2} < H < 1$, *whereas the inner product on* $|\,\Lambda|^H$ *can be expressed as*

$$(f,g)_{|\Lambda|^H} = H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) g(\nu) |u-\nu|^{2H-2} du\, d\nu.$$

We state an important result, that characterizes the functions in the space $|\,\Lambda|^H$. We provide a large subset of that space.

**Proposition 2.17.** *Let* $\frac{1}{2} < H < 1$. *Then*

$$L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subseteq |\,\Lambda|^H.$$

**Remark 2.18.** *We can give an analogon to Itô's isometry in the fractional case with the help of Proposition 2.13. Therefore let* $I^{\alpha}_{-}\phi$ *denote a fractional integral of order* $\alpha > 0$ *of a function* $\phi$ *defined by*

$$(I^{\alpha}_{-}\phi)(s) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)(s-u)_{-}^{\alpha-1} du, \quad s \in \mathbb{R}.$$

*Hence, for* $f \in \mathscr{E}$ *and* $\frac{1}{2} < H < 1$, *[Pipiras and Taqqu [2000]] find the isometry*

$$\int_{\mathbb{R}} f(u) dB^H(u) \stackrel{d}{=} \frac{\Gamma(H+1/2)}{C_1(H)} \int_{\mathbb{R}} (I^H_{-} f)(s) dB(s).$$

*This gives rise to a class of functions of the form*

$$\{f : \int_{\mathbb{R}} \left((I^H_{-} f)(s)\right)^2 ds < \infty\}$$

*as a step prior to our Definition 2.16.*
*We will specify this isometry in Chapter 5, when we get to the change of measure.*

# 3 Short-Rate Models

In this chapter we want to give a brief overview about some popular interest-rate models and their evolution. Interest-rate modelling in theory was originally based on the assumptions of specific one-dimensional dynamics for the instantaneous spot rate process $r$. For this direct modelling approach all fundamental quantities are defined by no arbitrage arguments. In particular, the existence of a risk-neutral measure $\mathbb{Q}$ implies that the arbitrage-free price of a zero-coupon bond at time $t$ with maturity $T$ is given by the conditional expectation

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\right]. \tag{3.1}$$

This formula calls for some general definitions, which we will assume to hold for all of the three models covered in this chapter. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space that models the uncertainty in our economy, where $\Omega$ is the state space, $\mathcal{F}$ is the $\sigma$-algebra representing measurable events and $\mathbb{P}$ is the real-world probability measure. Moreover, let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete, right-continuous filtration defined by $\mathcal{F}_t := \sigma(B_i(s) : s \leq t)_{t\in[0,T],i=1,\ldots,d}$, where $B_i, i = 1, ..d$, are $d$ independent Brownian motions. We refer to it as the filtration generated by the Brownian motions, which includes all information from the past.

In contrast to $\mathbb{P}$, $\mathbb{Q}$ is the associated risk-neutral probability measure that can be attained by a change of measure. We will cover this more precisely in chapters 4 and 5, because in this chapter we will start modelling right away with the dynamics under the risk-neutral measure and so we will not need this approach. For the pricing of interest-rate derivatives it is totally sufficient to directly model the dynamics under the risk-neutral measure.

We will cover some of the most popular early interest-rate models and give their stochastic differential equations (sde) representing their dynamics. We will explain the Vasicek model and then compare it to the Cox-Ingersoll-Ross model and the Hull-White model, which introduce certain modifications.

All of the three models belong to the group of affine term-structure models where the continuously compounded zero rate $R(t,T)$ is an affine function in the short rate $r(t)$, i.e.

$$R(t,T) = a(t,T) + b(t,T)r(t),$$

where $a$ and $b$ are deterministic functions of time. Then the model is said to possess an affine term structure. This is always satisfied if the zero-coupon bond price can be written in the form

$$P(t,T) = D(t,T)e^{-A(t,T)r(t)}.$$

This chapter is based on the one-factor short-rate models in the book of [Brigo and Mercurio [2007]].

## 3.1 The Vasicek Model

We assume that the instantaneous spot rate under the risk-neutral measure evolves as an Ornstein-Uhlenbeck process with constant coefficients, drift term $k(\theta - r(t))$ and diffusion term $\sigma$, i.e.

$$dr(t) = k(\theta - r(t))dt + \sigma dB(t), \quad r(0) = r_0, \tag{3.2}$$

where $k$, $\theta$, $\sigma$ and $r_0$ are positive real-valued constants. Moreover $\theta$ is considered as mean. We can observe that the drift is positive whenever the short rate is below $\theta$ and negative otherwise. It can also be formally shown that $r$ is mean-reverting, that is $\mathbb{E}[r(t)|\mathcal{F}_t] \to \theta$ for $t \to \infty$, which is obvious by looking at (3.3) underneath. Hence we can conclude that $r$ is pushed to be closer to the level $\theta$ with every time step. We refer to $\sigma$ as the volatility of $B$, i.e. taking into account the sensitivity of $r$ with respect to random shocks, represented by the standard Brownian motion $B$. We call $k$ the speed of adjustment.
The simulations of the Vasicek sde in chapter 6 illustrate the dynamics of this model very well.

In order to come up with the spot-rate process as a solution of the stochastic differential equation (3.2) we use the integrating factor $e^{kt}$ and obtain

$$\begin{aligned} d(e^{kt}r(t)) &= e^{kt}dr(t) + ke^{kt}r(t)dt \\ &= e^{kt}\big(k(\theta - r(t))dt + \sigma dB(t)\big) + ke^{kt}r(t)dt \\ &= e^{kt}k\theta dt + e^{kt}\sigma dB(t). \end{aligned}$$

By integration we get

$$e^{kt}r(t) = r(s)e^{ks} + \theta k \int_s^t e^{ku}du + \sigma \int_s^t e^{ku}dB(u), \quad 0 \le s \le t,$$

which resolves to

$$r(t) = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)}dB(u), \quad 0 \le s \le t.$$

Due to the driving Brownian motion $B(t)$, $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance respectively given by

$$\begin{aligned} \mathbb{E}[r(t)|\mathcal{F}_s] &= r(s)e^{-k(t-s)} + \theta\big(1 - e^{-k(t-s)}\big) \quad \text{and} \\ \mathrm{Var}(r(t)|\mathcal{F}_s) &= \mathbb{E}\left[\left(\sigma \int_s^t e^{-k(t-u)}dB(u)\right)^2\right] \\ &= \sigma^2 \mathbb{E}\left[\int_s^t e^{-2k(t-u)}du\right] \qquad \text{by Itô's Isometry} \\ &= \frac{\sigma^2}{2k}\big(1 - e^{-2k(t-s)}\big). \end{aligned} \tag{3.3}$$

From the expectation and variance in (3.3) we can easily see that the rate $r(t)$ can be negative with positive probability at each time $t$, which is a major drawback of the Vasicek model, because this is usually not a realistic event, although there might be negative interest rates in times of deflation. For a suitable choice of the parameters, however, the probability of negative values can be kept marginally small. On the other hand, an advantage of the model is its analytical tractability implied by a Gaussian density, which is hardly achieved when assuming other distributions for $r$. This property is very helpful for historical estimation.

The bond price for the Vasicek model can be derived by starting with (3.1) and hence computing the condititional expectation under $\mathbb{Q}$. Hence we come up with the formula

$$P(t,T) = D(t,T)e^{-A(t,T)r(t)},$$

where

$$A(t,T) = \frac{1}{k}\big(1 - e^{-k(T-t)}\big)$$

and

$$D(t,T) = \exp\left(\big(\theta - \frac{\sigma^2}{2k^2}\big)\big(A(t,T) - T + t\big) - \frac{\sigma^2}{4k}A(t,T)^2\right).$$

**Proof:** Write $X(u) = r(u) - \theta$. So, $X(u)$ is the solution of the Ornstein-Uhlenbeck equation

$$dX(t) = -kX(t)dt + \sigma dB(t) \quad \text{with } X(0) = r_0 - \theta.$$

By applying exactly the same procedure as for the sde of the spot-rate process $r(t)$ before, we obtain a solution to this sde by using the integrating factor $e^{ku}$ again, i.e. the process

$$X(u) = e^{-ku}\left(X(0) + \int_0^u \sigma e^{as} dB(s)\right). \tag{3.4}$$

Obviously, $X(u)$ is a Gaussian process with continuous sample paths and therefore $\int_0^t X(u)du$ is Gaussian, too. Hence we have $\mathbb{E}[X(u)] = X(0)e^{-ku}$ and thus

$$\mathbb{E}\left[\int_0^t X(u)du\right] = \frac{X(0)}{k}\big(1 - e^{-kt}\big). \tag{3.5}$$

In this chapter all expectations are taken under the risk-neutral measure without further mentioning, since we model the dynamics under this measure. Similarly,

$$\begin{aligned}
\operatorname{Cov}(X(t), X(u)) &= \sigma^2 e^{-k(u+t)}\mathbb{E}\left[\int_0^t e^{ks} dB(s) \int_0^u e^{ks} dB(s)\right] \\
&= \sigma^2 e^{-k(u+t)} \int_0^{u\wedge t} e^{2ks} ds \\
&= \frac{\sigma^2}{2k} e^{-k(u+t)}\big(e^{2k(u\wedge t)} - 1\big)
\end{aligned}$$

and so

$$
\begin{aligned}
&\mathrm{Var}\Big(\int_0^t X(u)du\Big) = \mathrm{Cov}\Big(\int_0^t X(u)du, \int_0^t X(s)ds\Big)\\
&= \mathbb{E}\bigg[\bigg(\int_0^t X(u)du - \mathbb{E}\Big[\int_0^t X(u)du\Big]\bigg)\bigg(\int_0^t X(s)ds - \mathbb{E}\Big[\int_0^t X(s)ds\Big]\bigg)\bigg]\\
&= \int_0^t \int_0^t \mathbb{E}\big[\big(X(u) - \mathbb{E}[X(u)]\big)\big(X(s) - \mathbb{E}[X(s)]\big)\big]du\, ds \qquad (3.6)\\
&= \int_0^t \int_0^t \mathrm{Cov}\big(X(u), X(s)\big)du\, ds = \int_0^t \int_0^t \frac{\sigma^2}{2k} e^{-k(u+s)}\big(e^{2k(u\wedge t)} - 1\big)du\, ds\\
&= \frac{\sigma^2}{2k^3}\big(2kt - 3 + 4e^{-kt} - e^{-2kt}\big).
\end{aligned}
$$

Since we have $X(u) = r(u) - \theta$, we get

$$
\mathbb{E}\Big[-\int_0^t r(u)du\Big] = \mathbb{E}\Big[-\int_0^t (X(u) + \theta)du\Big]
$$

and so, together with expression (3.5) we obtain

$$
\mathbb{E}\Big[-\int_t^T r(u)du\Big] = \frac{r(t) - \theta}{k}\big(1 - e^{-k(T-t)}\big) - \theta(T - t). \qquad (3.7)
$$

Moreover, by result (3.6) we derive

$$
\begin{aligned}
\mathrm{Var}\Big(-\int_t^T r(u)du\Big) &= \mathrm{Var}\Big(-\int_t^T X(u)du\Big)\\
&= \frac{\sigma^2}{2k^3}\big(2k(T-t) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}\big).
\end{aligned} \qquad (3.8)
$$

From the Itô integral representation of $r(t)$, we conclude that the defining process for the short rate is Markovian (for a proof see [Karatzas and Shreve [1988], p.355]). Hence

$$
P(t,T) = \mathbb{E}\Big[e^{-\int_t^T r(u)du} \big| \mathcal{F}_t\Big] = \mathbb{E}\Big[e^{-\int_t^T r(u)du} \big| r(t)\Big],
$$

where we can write $r(u)$ as a function of $r(t)$, i.e. $r(u, r(t))$, so that

$$
P(t,T) = \mathbb{E}\Big[e^{-\int_t^T r(u)du} \big| r(t)\Big] := \mathbb{E}\Big[e^{-\int_t^T r(u,r(t))du}\Big].
$$

In the Gaussian case we know that

$$
P(t,T) = \exp\bigg\{\mathbb{E}\Big[-\int_t^T r(u,r(t))du\Big] + \frac{1}{2}\mathrm{Var}\Big(-\int_t^T r(u,r(t))du\Big)\bigg\}
$$

and we can conclude the proof by plugging in (3.7) and (3.8):

$$
\begin{aligned}
P(t,T) = \exp\bigg\{ & \frac{r(t)-\theta}{k}\big(1-e^{-k(T-t)}\big) - \theta(T-t) \\
& + \frac{1}{2}\frac{\sigma^2}{2k^3}\big(2k(T-t) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}\big)\bigg\} \\
= \exp\bigg\{ & -\bigg(\frac{1-e^{-k(T-t)}}{k}\bigg)r(t) + \theta\bigg(\frac{1-e^{-k(T-t)}}{k} - (T-t)\bigg) \\
& - \frac{\sigma^2}{2k^2}\bigg(\frac{1-e^{-k(T-t)}}{k}\bigg) + \frac{\sigma^2}{2k^2}(T-t) - \frac{\sigma^2}{4k}\bigg(\frac{1-2e^{-k(T-t)}+e^{-2k(T-t)}}{k^2}\bigg)\bigg\} \\
= \exp\bigg\{ & -A(t,T)r(t) + \theta A(t,T) - \theta(T-t) \\
& - \frac{\sigma^2}{2k^2}A(t,T) + \frac{\sigma^2}{2k^2}(T-t) - \frac{\sigma^2}{4k}A(t,T)^2\bigg\} \\
= D(t,T) & \exp\big(-A(t,T)r(t)\big).
\end{aligned}
$$

□

## 3.2 The Cox-Ingersoll-Ross Model

The Cox-Ingersoll-Ross (CIR) interest dynamics are formulated under the risk-neutral measure $\mathbb{Q}$ with a modification of the diffusion term in comparison to the Vasicek model, in order to avoid negative values for $r(t)$ given that reasonable values for the parameters are chosen. Therefore a square-root term is introduced into the diffusion term. So the risk-neutral interest rate dynamics are assumed to be

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dB(t), r(0) = r_0, \tag{3.9}$$

where $k$, $\theta$, $\sigma$ and $r_0$ are positive constants with the same interpretations as for the Vasicek model. Once again $B$ denotes a standard Brownian motion. In order to ensure the positivity of $r(t)$ we postulate $2k\theta > \sigma^2$. The process $r$ features a non-central chi-squared distribution.

The zero-coupon bond price is given by the same form as for the Vasicek model, that is

$$P(t,T) = D(t,T)e^{-A(t,T)r(t)},$$

but with different specifications for $A(t,T)$ and $D(t,T)$, i.e.

$$\begin{aligned} A(t,T) &= \frac{2\big(\exp((T-t)h) - 1\big)}{2h + (k+h)\big(\exp((T-t)h) - 1\big)}, \\ D(t,T) &= \left(\frac{2h\exp\left(\frac{(k+h)(T-t)}{2}\right)}{2h + (k+h)\big(\exp((T-t)h) - 1\big)}\right)^{\frac{2k\theta}{\sigma^2}}, \\ h &= \sqrt{k^2 + 2\sigma^2}. \end{aligned}$$

We forego a derivation in this case, since our focus is not upon the short-rate models in the first place and so we refer to [Cox et al. [1985]].

## 3.3 The Hull-White Model

Hull and White tried to improve the poor fitting of the initial term structure of interest rates implied by the Vasicek model. Therefore they introduced time-varying parameters into the Vasicek model, i.e. deterministic functions instead of the constants in the short-rate dynamics. They assume that the risk-neutral short rate evolves according to

$$dr(t) = (\nu(t) - a(t)r(t))dt + \sigma(t)dB(t), \quad r(0) = r_0, \tag{3.10}$$

where $a$, $\sigma$ and $\nu$ are deterministic and positive functions of time that are chosen so as to exactly fit the term structure of interest rates being currently observed in the market.
However, we choose $a$ and $\sigma$ as positive constants because quotes of market volatilities are not always significant due to liquidity issues in some markets and so perfect fitting can be dangerous. Thus we can speak of a Vasicek model with a time-dependent reversion level of the form

$$dr(t) = (\nu(t) - ar(t))dt + \sigma dB(t), \quad r(0) = r_0.$$

In the Hull-White model $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed and again the zero-coupon bond price takes on the form

$$P(t,T) = D(t,T)e^{-A(t,T)r(t)},$$

where this time

$$\begin{aligned} A(t,T) &= \frac{1}{a}\big(1 - e^{-a(T-t)}\big), \\ D(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left(A(t,T)f(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2a(T-t)}) - \frac{3}{2a}\right), \end{aligned}$$

with $P(0,\cdot\,)$ denoting the initial bond price given by the market.

## 3.4 Conclusion

The Vasicek model and the CIR model are called endogenous term-structure models, which refers to the fact that the current term structure of interest rates is an output and not an input of the models. We can illustrate that by setting the evaluation time $t = 0$, which yields the initial interest-rate curve as an output. In practice one has to find suitable parameters that force the initial bond price curve to be as close as possible to the market curve, but usually three parameters are not enough to reproduce the market term structure satisfactorily. This is contrasted by the so-called no-arbitrage models like the Hull-White model or the more general Heath-Jarrow-Morton approach, on which we will focus in the following chapters. They are called exogenous and today's term structure of interest rates is an input.

A clear drawback of the Vasicek model is that interest rates can assume negative values with positive probability, a problem that is tackled by the CIR model. On the other hand its linearity enables an explicit solution which makes the model attractive from an analytical point of view. As a consequence several expressions and distributions of useful quantities related to the interest-rate world, for instance volatility, can be easily obtained. This is not given for every model, e.g. the CIR model with its non-centrally chi-squared distributed interest rate is less analytically tractable.

As we have already seen by the different pros and cons one has to weigh up what is more important when choosing a certain model. It is a matter of positive interest rates, distribution of the process and explicitly computable bond prices or option prices. Moreover, questions about mean reversion, implied volatility structures, computation and estimation techniques arise - in a nutshell: a very difficult choice for which many factors have to be considered.

# 4 The Heath-Jarrow-Morton Model

The Heath-Jarrow-Morton (HJM) model is a more general approach to an interest-rate model than what we have already seen for the short-rate models. We will cover this model more precisely because understanding of this classical case is essential for a detailed analysis of the fractional Heath-Jarrow-Morton model in chapter 5, the main focus of this thesis. The HJM approach starts off by modelling the forward rate instead of the short rate which allows to capture the evolution of the entire forward rate curve. This will facilitate a more precise calibration of the model to the inital forward rate curve. As it will turn out in the arbitrage-free framework, the forward-rate dynamics will be fully specified by their instantaneous volatility structures. This is a major difference to the one-factor short-rate models covered in the previous chapter, where also the drift has to be specified in order to characterize the relevant interest-rate model. We will get to the advantages and disadvantages of this approach in our conclusion later on.
This chapter is based on the original article of [Heath et al. [1992]] and adds to it.

## 4.1 The Set-Up

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space that characterizes the uncertainty in our economy and let $T^* > 0$ so that $[0, T^*]$ is the trading interval in our continuous trading economy. Let $(\mathcal{F}_t)_{t\in[0,T^*]}$ be a complete, right-continuous and augmented filtration defined by $\mathcal{F}_t := \sigma(B_i(s) : s \leq t)_{t\in[0,T],i=1,\dots,d}$, where $B_i, i = 1, ..., d$, $d \in \mathbb{N}$, are $d$ independent standard Brownian motions.
There is a continuum of default-free discount bonds, each bond trading with a different maturity, one for each date $T \in [0, T^*]$. Hence for the price of the T-maturity bond $P(t, T)$ we require:

(i) $P(T, T) = 1$ for all $T \in [0, T^*]$;

(ii) $P(t, T) > 0$ for all $T \in [0, T^*], t \in [0, T]$;

(iii) $\frac{\partial}{\partial T} \log P(t, T)$ exists for all $T \in [0, T^*], t \in [0, T]$.

The first equation implies that this is a default-free market, because the bond payoff equals 100% at maturity. The second equation exludes trivial arbitrage opportunities. The last equation ensures that the forward rates are well-defined.

Recall the definition of the instantaneous forward rate $f(t,T)$ at time $t \in \mathbb{R}_+$ for maturity $T > t$ from (2.1), that is

$$f(t,T) = -\frac{\partial}{\partial T} \log P(t,T) \quad \text{for all } T \in [0,T^*], t \in [0,T].$$

Conversely, this yields for the bond price

$$P(t,T) = \exp\left( - \int_t^T f(t,s)ds \right) \quad \text{for all } T \in [0,T^*], t \in [0,T].$$

We observe the major difference between the Heath-Jarrow-Morton model and the short-rate models, that is the HJM model is based on forward rates and not on short rates. As a consequence we will see that in the HJM framework we can derive arbitrage-free conditions for the stochastic evolutions of the entire yield curve.

*Condition 1 - A family of forward-rate processes*

We will start off with an assumption on the family of stochastic processes for the forward rate movements, that is

$$f(t,T) - f(0,T) = \int_0^t \alpha(\nu,T)d\nu + \sum_{i=1}^d \int_0^t \sigma_i(\nu,T)dB_i(\nu) \quad \text{for all} \quad 0 \le t \le T, \tag{4.1}$$

where $T \in [0,T^*]$ is fixed, but arbitrary and $B_i, i = 1,..d$ are $d$ independent Brownian motions that model the stochastic fluctuation of the entire forward rate curve starting from a fixed, non-random initial forward rate curve $\{f(0,T) : T \in [0,T^*]\}$ given by the market. Furthermore we impose the following conditions:

(i) $f(0,\cdot) : ([0,T^*], \mathcal{B}[0,T^*]) \to (\mathbb{R}, \mathcal{B})$ is measurable with $\mathcal{B}[0,T^*]$ denoting the Borel-$\sigma$-algebra restricted to $[0,T^*]$.

(ii) $\alpha : \{(t,s) : 0 \le t \le s \le T\} \times \Omega \to \mathbb{R}$ is a $\mathcal{B}\{(t,s) : 0 \le t \le s \le T\} \times \mathcal{F}$-jointly measurable and adapted stochastic process itself. Moreover $\alpha$ satisfies $\int_0^T |\alpha(t,T)|\, dt < \infty$ $\mathbb{P}$-a.s.

(iii) The volatilites $\sigma_i : \{(t,s) : 0 \le t \le s \le T\} \times \Omega \to \mathbb{R}$ are $\mathcal{B}\{(t,s) : 0 \le t \le s \le T\} \times \mathcal{F}$-jointly measurable and adapted stochastic processes.
They satisfy $\int_0^T \sigma_i^2(t,T)dt < \infty$ $\mathbb{P}$-a.s. for $i = 1,...,d$.

The differing volatility coefficients reflect the sensitivity of a particular maturity forward rate's change to each Brownian motion. The $d$ independent Brownian motions imply that the restrictions for our economy are the continuous sample paths of the forward-rate processes and a finite number of random shocks across the entire forward rate curve, i.e. for all maturities. This is a major difference to short-rate models where different maturities are not taken into account when modelling the spot rate.

Now, given the forward-rate process in (4.1) we can easily compute the spot-rate process as

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(\nu,t)d\nu + \sum_{i=1}^{d} \int_0^t \sigma_i(\nu,t)dB_i(\nu) \quad \text{for all} \quad t \in [0,T]. \quad (4.2)$$

*Condition 2 - Regularity of the money market account*

**Notation:** For convenience we define

$$B_0(t) = \exp\left(\int_0^t r(y)dy\right) \quad \text{for all} \quad t \in [0,T^*] \quad (4.3)$$

as our *money market account* or *numeraire.*

We need to make sure that the value of the money market account is finite. Therefore we postulate for the money market account to hold

$$\int_0^{T^*} |f(0,\nu)|d\nu < \infty$$

and

$$\int_0^{T^*} \left(\int_0^t |\alpha(\nu,t)|d\nu\right)dt < \infty \quad \mathbb{Q}\text{-a.s.}$$

**The dynamics of the bond price process**

*Condition 3 - Regularity of the bond price process*

In order to ensure that the bond price process is well-behaved, i.e. that integrals are well-defined, we impose three conditions on the bond price process:

(i) $\int_0^t \left(\int_\nu^t \sigma_i(\nu,y)dy\right)^2 d\nu \quad \mathbb{P}\text{-a.s. for all } t \in [0,T^*], i = 1,...,d;$

(ii) $\int_0^t \left(\int_t^T \sigma_i(\nu,y)dy\right)^2 d\nu \quad \mathbb{P}\text{-a.s. for all } t \in [0,T], i = 1,...,d;$

(iii) $t \mapsto \int_t^T \left(\int_0^t \sigma_i(\nu,y)dB_i(\nu)\right)dy \quad$ is continuous $\mathbb{P}$-a.s. for all $\quad T \in [0,T^*], i = 1,...,d.$

**Theorem 4.1.** *Let Conditions 2 and 3 hold. Then the dynamics of the bond price process are*

$$\begin{aligned}\log P(t,T) &= \log P(0,T) + \int_0^t (r(\nu) + b(\nu,T))d\nu \\ &\qquad - \frac{1}{2}\sum_{i=1}^d \int_0^t a_i(\nu,T)^2 d\nu + \sum_{i=1}^d \int_0^t a_i(\nu,T)dB_i(\nu) \quad \mathbb{P}\text{-a.s.}, \\ \textit{where} \quad a_i(t,T) &= -\int_t^T \sigma_i(t,\nu)d\nu \quad \textit{for } i=1,...,d \\ \textit{and} \quad b(t,T) &= -\int_t^T \alpha(t,\nu)d\nu + \frac{1}{2}\sum_{i=1}^d a_i(t,T)^2.\end{aligned} \tag{4.4}$$

**Proof:** In order to prove Theorem 4.1 we will need the following lemma:

**Lemma 4.2** (Generalized form of Fubini's theorem for stochastic integrals)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t\in[0,T^*]}$ be a filtration satisfying the usual conditions (i.e. complete and right-coninuous), which is generated by the Brownian motion $B$.*
*Let $\{\Phi(t,a,\omega) : (t,a,\omega) \in [0,T^*] \times [0,T^*] \times \Omega\}$ be a family of real random variables such that:*

*(i) $((t,\omega),a) \in \{([0,T^*] \times \Omega) \times [0,T^*]\} \to \Phi(t,a)$ is $L \times \mathcal{B}[0,T^*]$-measurable where $L$ is the predictable $\sigma$-field;*

*(ii) $\int_0^t \Phi^2(s,a)ds < \infty$ a.s. for all $t \in [0,T^*]$;*

*(iii) $\int_0^t \left(\int_0^{T^*} \Phi(s,a)da\right)^2 ds < \infty$ a.s. for all $t \in [0,T^*]$.*

*If $t \mapsto \int_0^{T^*} \left(\int_0^t \Phi(s,a)dB(s)\right)da$ is continuous a.s., then:*

$$\int_0^t \left(\int_0^{T^*} \Phi(s,a)da\right)dB(s) = \int_0^{T^*} \left(\int_0^t \Phi(s,a)dB(s)\right)da \quad \textit{for all } t \in [0,T^*].$$

Moreover we will apply the following corollary that we will state without proof since it is just one straightforward rearrangement.

**Corollary 4.3.** *Let the hypotheses of Lemma 4.2 hold. Define*

$$\Phi(s,a) := \begin{cases} \sigma(s,a)\mathbf{1}_{\{s\le a\}} & \textit{if } (s,a) \in [0,t]\times[0,t], \\ 0 & \textit{if } (s,a) \notin [0,t]\times[0,t]. \end{cases}$$

*Then*

$$\int_0^y \left(\int_s^t \sigma(s,a)da\right)dB(s) = \int_0^t \left(\int_0^{s\wedge y} \sigma(s,a)dB(s)\right)da \quad \textit{for all } y \in [0,t].$$

Now we can start with our proof by using Conditions 1-3:

$$\begin{aligned}\log P(t,T) &= -\int_t^T f(t,s)ds\\ &= -\int_t^T f(0,y)dy - \int_t^T \Big(\int_0^t \alpha(\nu,y)d\nu\Big)dy - \sum_{i=1}^d \int_t^T \Big(\int_0^t \sigma_i(\nu,y)dB_i(\nu)\Big)dy\\ &= -\int_t^T f(0,y)dy - \int_0^t \Big(\int_t^T \alpha(\nu,y)dy\Big)d\nu - \sum_{i=1}^d \int_0^t \Big(\int_t^T \sigma_i(\nu,y)dy\Big)dB_i(\nu),\end{aligned}$$

where all integrals are well-defined due to Conditions 1 and 2 and where we used Lemma 4.2. We are now splitting up the integrals by changing their limits and then rearrange and apply Corollary 4.3. Moreover recall expression (4.2) where $r(t) = f(0,t) + \int_0^t \alpha(\nu,y)d\nu + \sum_{i=1}^d \int_0^t \sigma_i(\nu,y)dB_i(\nu)$. So we can rearrange

$$\begin{aligned}\log P(t,T) &= -\int_0^T f(0,y)dy - \int_0^t \Big(\int_\nu^T \alpha(\nu,y)dy\Big)d\nu - \sum_{i=1}^d \int_0^t \Big(\int_\nu^T \sigma_i(\nu,y)dy\Big)dB_i(\nu)\\ &\quad + \int_0^t f(0,y)dy + \underbrace{\int_0^t \Big(\int_\nu^t \alpha(\nu,y)dy\Big)d\nu}_{=\int_0^t\int_0^{t\wedge\nu}\alpha(\nu,y)d\nu dy} + \sum_{i=1}^d \underbrace{\int_0^t \Big(\int_\nu^t \sigma_i(\nu,y)dy\Big)dB_i(\nu)}_{=\int_0^t\int_0^{t\wedge\nu}\sigma_i(\nu,y)dydB_i(\nu)}\\ &= \log P(0,T) + \int_0^t r(y)dy - \int_0^t \Big(\underbrace{\int_\nu^T \alpha(\nu,y)dy}_{=-b(\nu,T)+\frac{1}{2}\sum_{i=1}^d a_i(\nu,T)^2}\Big)d\nu\\ &\quad - \sum_{i=1}^d \int_0^t \Big(\underbrace{\int_\nu^T \sigma_i(\nu,y)dy}_{=-a_i(\nu,T)}\Big)dB_i(\nu)\\ &= \log P(0,T) + \int_0^t (r(y)+b(y,T))dy - \frac{1}{2}\sum_{i=1}^d \int_0^t a_i(\nu,T)^2 d\nu + \sum_{i=1}^d \int_0^t a_i(\nu,T)dB_i(\nu).\end{aligned}$$

□

In order to come up with the stochastic differential equation (sde) that represents the bond price dynamics and of which the bond price process $P(t,T)$ is a strong solution, we apply Itô's lemma. First recall:

**Lemma 4.4** (Itô's lemma). *Let $X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s)$ be an Itô process with drift $b$ and diffusion $\sigma$, $t \in \mathbb{R}_+$ and $B$ a Brownian motion.*
*Let $f \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$. Then $(f(X(t)))_{t\in\mathbb{R}_+}$ is an Itô process of the form:*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[X,X]_s,$$

*where* $\{[X,X]_t\}_{t\in\mathbb{R}_+}$ *with* $[X,X]_t := \int_0^t \sigma^2(s)ds$ *is called the quadratic variation of* $X$. *[Kallsen [2007], Theorem 5.4.8]*

So, in our case we define

$$X(t) = \int_0^t (r(\nu) + b(\nu,T))d\nu - \frac{1}{2}\sum_{i=1}^{d}\int_0^t a_i(\nu,T)^2 d\nu + \sum_{i=1}^{d}\int_0^t a_i(\nu,T)dB_i(\nu).$$

Then Itô's lemma with $f(x) = e^x$ yields:

$$\begin{aligned} dP(t,T) = df(X(t)) &= 0 + e^{X(t)}dX(t) + e^{X(t)}\frac{1}{2}\sum_{i=1}^{d} a_i(t,T)^2 dt \\ &= e^{X(t)}\left((r(t)+b(t,T))dt - \frac{1}{2}\sum_{i=1}^{d} a_i(t,T)^2 dt + \sum_{i=1}^{d} a_i(t,T)dB_i(t)\right) \\ &\quad + e^{X(t)}\frac{1}{2}\sum_{i=1}^{d} a_i(t,T)^2 dt \\ &= e^{X(t)}\left((r(t)+b(t,T))dt + \sum_{i=1}^{d} a_i(t,T)dB_i(t)\right) \\ &= (r(t)+b(t,T))P(t,T)dt + \sum_{i=1}^{d} a_i(t,T)P(t,T)dB_i(t). \end{aligned} \tag{4.5}$$

Our bond price process is in general non-Markov because its drift term $(r(t) + b(t,T))$ and its volatility coefficients $a_i(t,T)$ can depend on the history of the Brownian motions $B_i, i = 1, ..., d$.

**The relative bond price process**

We can easily determine the relative (or discounted) bond price process for a $T$-maturity bond. Let $Z_t(T) = \frac{P(t,T)}{B_0(t)}$ denote the relative bond price for a $T$-maturity bond at time $t$ for $T \in [0,T^*]$ and $t \in [0,T]$, where the numeraire $B_0(t)$ was defined earlier in expression (4.3). $Z_t(T)$ is the bond's value expressed in units of the accumulation factor and not in dollars, which is particularly useful for analysis and no-arbitrage theory which we will see later on. Analogously to Theorem 4.1 we get the relative bond price process as

$$\begin{aligned} \log Z_t(T) = \log Z_0(T) &+ \int_0^t b(\nu,T)d\nu - \frac{1}{2}\sum_{i=1}^{d}\int_0^t a_i(\nu,T)^2 d\nu \\ &+ \sum_{i=1}^{d}\int_0^t a_i(\nu,T)dB_i(\nu) \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.6}$$

with $a_i(t,T)$ and $b(t,T)$ defined as in Theorem 4.1.

## 4.2 Arbitrage Free Bond Pricing

Similarly to the original no-arbitrage theory of [Harrison and Pliska [1981]] we derive necessary and sufficient conditions on the forward-rate process in order to ensure existence and uniqueness of an equivalent martingale measure. We proceed step by step.

*Condition 4 - Existence of the market prices for risk*

Fix $S_1, ..., S_d \in [0, T^*]$ such that $0 < S_1 < ... < S_d \leq T^*$. Assume there exist solutions

$$\gamma_i((\cdot\,,\cdot\,); S_1, ..., S_d) : \Omega \times [0, S_1] \to \mathbb{R} \qquad \text{for } i = 1, ..., d \quad \mathbb{P} \times \lambda \text{ - a.s.},$$

where $\lambda$ is the Lebesgue measure, to the following system of equations:

$$\begin{bmatrix} b(t, S_1) \\ \vdots \\ b(t, S_d) \end{bmatrix} + \begin{bmatrix} a_1(t, S_1) & \cdots & a_d(t, S_1) \\ \vdots & & \vdots \\ a_1(t, S_d) & \cdots & a_d(t, S_d) \end{bmatrix} \begin{bmatrix} \gamma_1(t; S_1, ..., S_d) \\ \vdots \\ \gamma_d(t; S_1, ..., S_d) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.7}$$

which satisfy

(i) $\int_0^{S_1} \gamma_i(\nu; S_1, ..., S_d)^2 d\nu < \infty \qquad \mathbb{P}\text{-a.s. for } i = 1, ..., d;$

(ii) $\mathbb{E}\left[\exp\left(\sum_{i=1}^d \int_0^{S_1} \gamma_i(\nu; S_1, ..., S_d) dB_i(\nu) - \frac{1}{2}\sum_{i=1}^d \int_0^{S_1} \gamma_i(\nu; S_1, ..., S_d)^2 d\nu\right)\right] = 1;$

(iii) $\mathbb{E}\Bigg[\exp\Bigg(\sum_{i=1}^d \int_0^{S_1} [a_i(\nu, y) + \gamma_i(\nu; S_1, ..., S_d)] dB_i(\nu)$
$\qquad - \frac{1}{2}\sum_{i=1}^d \int_0^{S_1} [a_i(\nu, y) + \gamma_i(\nu; S_1, ..., S_d)]^2 d\nu\Bigg)\Bigg] = 1 \quad \text{for } y \in \{S_1, ..., S_d\}.$

This condition will ensure the existence of an equivalent martingale measure as we will see in Proposition 4.5. Solving the system of equations in expression (4.7) provides us with $\gamma_i(t; S_1, ..., S_d)$ for $i = 1, ..., d$. We can interpret the $\gamma_i$'s as the market prices for risk associated with the random factors $B_i(t)$ for $i = 1, ..., d$, respectively. In order to point this out it helps to rearrange expression (4.7) to

$$b(t, T) = \sum_{i=1}^d a_i(t, T)(-\gamma_i(t; S_1, ..., S_d)) = \sum_{i=1}^d \int_t^T \sigma_i(t, \nu) d\nu \; \gamma_i(t; S_1, ..., S_d), \tag{4.8}$$

where $b(t, T)$ is the instantaneous excess expected return on the T-maturity bond exceding the risk-free rate $r(t)$, since the drift of our bond price dynamics under the real-world measure as in expression (4.5) is $(r(t)+b(t, T))$. Obviously the right-hand side of expression (4.8) is the sum of all market prices of risk $\gamma_i$ times the forward rate volatility coefficients $\sigma_i$ for all factors $i = 1, ..., d$. The volatility coefficients can also be considered as the instantaneous covariance between the T-maturity bond's return and the $i$th random factor $B_i$. By looking at expression (4.7) we can see that the solutions of this equation system depend on the vector of bonds $[S_1, ..., S_d]$ chosen (or more precisely the bonds associated with those maturities). However, $b$ is the premium an investor would expect for taking over the risk of the bond.

**Proposition 4.5** (Existence of an equivalent martingale measure)**.** *Fix $S_1, ..., S_d \in [0, T^*]$ such that $0 < S_1 < ... < S_d \leq T^*$. Let $[\alpha(\cdot\,, S_1), ..., \alpha(\cdot\,, S_d)]$ be a vector of forward rate drifts and let $[\sigma_i(\cdot\,, S_1), ..., \sigma_i(\cdot\,, S_d)]$ be a vector of forward rate volatilities that both satisfy conditions 1-3.*
*Then Condition 4 holds if and only if there exists an equivalent probability measure $\mathbb{Q}_{S_1,...,S_d}$ such that $[Z_t(S_1), ..., Z_t(S_d)]$ are martingales with respect to $(\mathcal{F}_t)_{t\in[0,S_1]}$.*

**Proof:** We need the following two lemmas to prove the proposition:

**Lemma 4.6.** *Let Conditions 1-3 hold for fixed $S_1, ..., S_d \in [0, T^*]$ such that $0 < S_1 < ... < S_d \leq T^*$. Define*

$$X(t, y) = \int_0^t b(\nu, y)d\nu + \sum_{i=1}^{d} \int_0^t a_i(\nu, y)dB_i(\nu) \quad \text{for all } t \in [0, y] \text{ and } y \in \{S_1, ..., S_d\}.$$

*Then $\gamma_i : \Omega \times [0, T^*] \to \mathbb{R}$ for $i = 1, ..., d$ satisfies condition 4 if and only if there exists a probability measure $\mathbb{Q}_{S_1,...,S_d}$ such that*

*(a)* $d\mathbb{Q}_{S_1,...,S_d}/d\mathbb{P} = \exp\left(\sum_{i=1}^d \int_0^{S_1} \gamma_i(\nu)dB_i(\nu) - \frac{1}{2}\sum_{i=1}^d \int_0^{S_1} \gamma_i(\nu)^2 d\nu\right)$;

*(b)* $\tilde{B}_i^{S_1,...,S_d}(t) = B_i(t) - \int_0^t \gamma_i(\nu)d\nu$ *are independent Brownian motions on* $(\Omega, \mathcal{F}, \mathbb{Q}_{S_1,...,S_d}, (\mathcal{F}_t)_{t\in[0,S_1]})$ *for* $i = 1, ..., d$;

*(c)* $$\begin{bmatrix} dX(t, S_1) \\ \vdots \\ dX(t, S_d) \end{bmatrix} = \begin{bmatrix} a_1(t, S_1) & \cdots & a_d(t, S_1) \\ \vdots & & \vdots \\ a_1(t, S_d) & \cdots & a_d(t, S_d) \end{bmatrix} \begin{bmatrix} d\tilde{B}_1^{S_1,...,S_d}(t) \\ \vdots \\ d\tilde{B}_d^{S_1,...,S_d}(t) \end{bmatrix} \quad \text{for } t \in [0, S_1];$$

*(d)* $Z_t(S_i)$ *are martingales on* $(\Omega, \mathcal{F}, \mathbb{Q}_{S_1,...,S_d}, (\mathcal{F}_t)_{t\in[0,S_1]})$ *for* $i = 1, ..., d$.

The proof of this lemma is straightforward. Obviously expression (4.7) holds if and only if $(c)$ holds. $(i)$ holds if and only if the integral in $(a)$ is well-defined as well. $(b)$ is a $\mathbb{Q}_{S_1,...,S_d}$-Brownian motion according to the Girsanov theorem where $\mathbb{Q}_{S_1,...,S_d}$ is given by Girsanov in $(a)$. $(ii)$ und $(iii)$ coincide with the martingale property in $(d)$.

**Lemma 4.7.** *The same setting as in Lemma 4.6 is given.*
*Then there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $Z_t(S_i)$ are martingales on $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\in[0,S_1]})$ for all $i = 1, ..., d$ if and only if there exist $\gamma_i : \Omega \times [0, T^*] \to \mathbb{R}$ for $i = 1, ..., d$ and a probability measure $\mathbb{Q}_{S_1,...,S_d}$ such that (a), (b), (c) and (d) of Lemma 4.6 hold.*

This obviously proves the equivalence in Proposition 4.5, since condition 4 is equivalent to (a)-(d). □

As we have seen earlier the market prices of risk depend on the vector of bonds $[S_1, ..., S_d]$ and hence so does the martingale measure. In order to come up with a no-arbitrage property we need to impose a uniqueness condition for an equivalent martingale measure, that is one single measure that simultaneously makes all vectors of relative bond prices $[Z_t(S_1), ..., Z_t(S_d)]$, $0 < S_1 < ... < S_d \leq T^*$, martingales. In Proposition 4.8 we will show that the following condition is both necessary and sufficient for the uniqueness of an equivalent martingale measure.

*Condition 5 - Uniqueness of the equivalent martingale measure*

Fix $S_1, ..., S_d \in [0, T^*]$ such that $0 < S_1 < ... < S_d \leq T^*$. Assume that

$$\begin{bmatrix} a_1(t, S_1) & \cdots & a_d(t, S_1) \\ \vdots & & \vdots \\ a_1(t, S_d) & \cdots & a_d(t, S_d) \end{bmatrix} \text{ is nonsingular } \mathbb{Q} \times \lambda\text{-a.s.} \tag{4.9}$$

**Proposition 4.8.** *Fix $S_1, ..., S_d \in [0, T^*]$ such that $0 < S_1 < ... < S_d \leq T^*$. Let the vectors of forward rate drifts $[\alpha(\cdot\,, S_1), ..., \alpha(\cdot\,, S_d)]$ and volatilities $[\sigma_i(\cdot\,, S_1), ..., \sigma_i(\cdot\,, S_d)], i = 1, ..., d$, respectively, satisfy conditions 1-4.*
*Then condition 5 holds if and only if the (equivalent) martingale measure is unique.*

For a detailed proof of this proposition we refer to the appendix of [Heath et al. [1992]].

In the following proposition we will show that with a unique equivalent martingale measure all relative bond prices really are martingales.

**Proposition 4.9** (Absence of arbitrage)**.** *Let the vectors of the forward rate drifts $[\alpha(\cdot\,, S_1), ..., \alpha(\cdot\,, S_d)]$ and volatilities $[\sigma_i(\cdot\,, S_1), ..., \sigma_i(\cdot\,, S_d)], i = 1, ..., d$, respectively, satisfy conditions 1-5.*
*Then the following are equivalent:*

*(i)* $\mathbb{Q} := \mathbb{Q}_{S_1,...,S_d}$ *for any $S_1, ..., S_d \in (0, T^*]$ is the unique equivalent martingale measure such that $Z_t(T)$ is a martingale for all $T \in [0, T^*]$ and $t \in [0, S_1]$;*

*(ii)* *we have $\gamma_i(t; S_1, ..., S_d) = \gamma_i(t; T_1, ..., T_d)$ for $i = 1, ..., d$ and all $S_1, ..., S_d, T_1, ..., T_d \in [0, T^*], t \in [0, T^*]$ such that $0 \leq t < S_1 < ... < S_d \leq T^*$ and $0 \leq t < T_1 < ... < T_d \leq T^*$;*

*(iii)* $\alpha(t, T) = -\sum_{i=1}^d \sigma_i(t, T)(\phi_i(t) - \int_t^T \sigma_i(t, \nu) d\nu)$ *for all $T \in [0, T^*]$ and $t \in [0, T]$, where for $i = 1, ..., d$, $\phi_i(t) = \gamma_i(t; S_1, ..., S_d)$ for any $S_1, ..., S_d \in (t, T^*]$ and $t \in [0, S_1]$.*

**Proof:** By using Proposition 4.8 we can conclude that $\mathbb{Q}_{S_1,...,S_d}$ is the unique equivalent martingale measure, for each vector $[S_1, ..., S_d]$ with $S_1 < ... < S_d \leq T^*$, that makes $Z_t(S_i)$

a martingale over $t \leq S_1$ for $i = 1, ..., d$.
"$\Leftrightarrow$" Since $\mathbb{Q} := \mathbb{Q}_{S_1,...,S_d}$ is defined in [Lemma 4.6, (a)] via $\gamma_i$, these measures are all equal to $\mathbb{Q}$ if and only if $\gamma_i(t; S_1, ..., S_d) = \gamma_i(t; T_1, ..., T_d)$ for all $i = 1, ..., d$ and all $S_1, ..., S_d, T_1, ..., T_d \in [0, T^*]$, $t \in [0, T^*]$, such that $0 \leq t < S_1 < ... < S_d \leq T^*$ and $0 \leq t < T_1 < ... < T_d \leq T^*$.
"$\Leftrightarrow$" Due to $(ii)$ it is obvious that $[\phi_1(t), ..., \phi_d(t)]$ is independent of $T$. We will start off with expression 4.8, where $\gamma_i(t; S_1, ..., S_d) = \phi_i(t)$, substitute for $b(t,T) = -\int_t^T \alpha(t,\nu)d\nu + \frac{1}{2}\sum_{i=1}^d a_i(t,T)^2$ and $a_i(t,T) = -\int_t^T \sigma_i(t,\nu)d\nu$, $i = 1, ..., d$, and take the partial derivative with respect to $T$:

$$b(t,T) = -\sum_{i=1}^d a_i(t,T)\phi_i(t)$$

$$\Leftrightarrow \quad -\int_t^T \alpha(t,\nu)d\nu + \frac{1}{2}\sum_{i=1}^d a_i(t,T)^2 = \sum_{i=1}^d \int_t^T \sigma_i(t,\nu)d\nu\, \phi_i(t)$$

$$\Rightarrow \quad \frac{\partial}{\partial T}\Big(\int_t^T \alpha(t,\nu)d\nu\Big) = \sum_{i=1}^d \sigma_i(t,T)\int_t^T \sigma_i(t,\nu)d\nu - \sum_{i=1}^d \sigma_i(t,T)\phi_i(t),$$

which proves $(iii)$. $\square$

There are major implications of this proposition for arbitrage theory and financial markets, i.e. the first equivalence shows that the existence of a unique equivalent probability measure $\mathbb{Q}$ (expression $(i)$) making relative bond prices martingales is equivalent to the fact that market prices of risk $\gamma_i$, $i = 1, ..., d$, are independent of the vector of bonds $[S_1, ..., S_d]$ (expression $(ii)$). Moreover this is also equivalent to a restriction on the family of forward rate drifts $\alpha$ (expression $(iii)$), known as the *Heath-Jarrow-Morton drift condition*. This condition will be mostly used in contingent claims valuation in order to ensure the existence of a unique equivalent martingale measure, since not all potential forward-rate processes satisfy this restriction.
From the martingale property in $(i)$ we also get the bond price as a conditional expectation that is

$$P(t,T) = B_0(t)\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(\sum_{i=1}^d \int_0^T \phi_i(t)dB_i(t) - \frac{1}{2}\sum_{i=1}^d \int_0^T \phi_i(t)^2 dt\Big)/B_0(T)\big|\mathcal{F}_t\Big].$$

From this expression we can easily see the factors that impact the bond price, i.e. the forward rate drifts $(\alpha(\cdot\,,T))_{T\in[0,T^*]}$ and the forward rate volatilities $(\sigma_i(\cdot\,,T))_{T\in[0,T^*]}$ for $i = 1, ..., d$ implicitly through the market prices for risk $\phi_i(t)$, $i = 1, ..., d$, since they are obtained by the equation system (4.7). Another price-influencing factor is the initial forward rate curve $(f(0,T))_{T\in[0,T^*]}$ implicitly given by the money market account $B_0(T)$.

## 4.3 Conclusion

Consequently we will come up with a short analysis of the HJM model in comparison to the short-rate models. An obvious advantage of the HJM model is that they capture the full dynamics of the entire forward rate curve, while the short-rate models only capture the dynamics of a point on the curve, the short rate. As already pointed out this allows for a better fitting to the initial forward rate curve. Another advantage of directly modelling the forward rate is that the current term structure of rates $f(0,T)$ is an input of the selected model, which is not given when modelling the short rate. This enables a better fitting than in the Vasicek or the CIR model as well. These are very strong arguments in favour of this approach.
On the other hand the short-rate approach provides a larger liberty in choosing the related dynamics due to the specific choice of its drift.
Consequently, as we already pointed out in the conclusion of Chapter 3 there are a number of factors that have to be considered when choosing a model appropriate to a certain purpose.
One can say that the HJM model is a very general model. Many short-rate models can be derived within the more general Heath-Jarrow-Morton framework and we will provide one example of that, i.e. we will derive the Hull-White model.

As we have emphasized many times, the short-rate process $r$ in the HJM model is not Markovian in general, which is a more realistic approach but more difficult to implement and work with. However, with a suitable specification of $\sigma$ the short-rate process can be made Markovian (as proved by [Carverhill [1994]]), which is

$$\sigma_i(t,T) = \xi_i(t)\nu_i(T) \quad \text{for each } i = 1, ..., d,$$

where $\xi_i$ and $\nu_i$ are strictly positive and deterministic functions of time. We recall the short-rate process in our HJM framework from expression (4.2) and combine it with the HJM drift condition from Proposition 4.9, $(iii)$. So we obtain

$$\begin{aligned}
r(t) &= f(0,t) + \int_0^t \alpha(\nu,t)d\nu + \sum_{i=1}^d \int_0^t \sigma_i(\nu,t)dB_i(\nu) \\
&= f(0,t) + \int_0^t \Big( -\sum_{i=1}^d \sigma_i(\nu,t)\big(\phi_i(\nu) - \int_\nu^t \sigma_i(\nu,u)du\big)\Big)d\nu + \sum_{i=1}^d \int_0^t \sigma_i(\nu,t)dB_i(\nu) \\
&= f(0,t) - \sum_{i=1}^d \int_0^t \xi_i(\nu)\psi_i(t)\Big(\phi_i(\nu) - \int_\nu^t \xi_i(\nu)\psi_i(u)du\Big)d\nu + \sum_{i=1}^d \int_0^t \xi_i(\nu)\psi_i(t)dB_i(\nu) \\
&= f(0,t) - \sum_{i=1}^d \psi_i(t) \int_0^t \xi_i(\nu)\phi_i(\nu)d\nu + \sum_{i=1}^d \psi_i(t) \int_0^t \xi_i^2(\nu) \int_\nu^t \psi_i(u)du\, d\nu \\
&\qquad + \sum_{i=1}^d \psi_i(t) \int_0^t \xi_i(\nu)dB_i(\nu).
\end{aligned}$$

From now on we only focus on the one-factor case $d = 1$ in order to arrive at the one-factor Hull-White model. Hence we define the deterministic function $A$ by

$$A(t) := f(0,t) - \psi(t) \int_0^t \xi(\nu)\phi(\nu)d\nu + \psi(t) \int_0^t \xi^2(\nu) \int_\nu^t \psi(u)du\, d\nu.$$

So $r(t)$ simplifies to

$$r(t) = A(t) + \psi(t) \int_0^t \xi(\nu)dB(\nu).$$

We assume differentiability of $A$ and so we are able to derive

$$\begin{aligned} dr(t) &= A'(t)dt + \psi'(t) \int_0^t \xi(s)dB(s) + \psi(t)\xi(t)dB(t) \\ &= \left(A'(t) + \psi'(t)\frac{r(t) - A(t)}{\psi(t)}\right)dt + \psi(t)\xi(t)dB(t) \\ &= \big(a(t) + b(t)r(t)\big)dt + c(t)dB(t), \end{aligned}$$

where

$$\begin{aligned} a(t) &:= A'(t) - \frac{\psi'(t)}{\psi(t)}A(t) \\ b(t) &:= \frac{\psi'(t)}{\psi(t)} \\ c(t) &:= \psi(t)\xi(t). \end{aligned}$$

This expression is obviously of the same form as the dynamics in the Hull-White model given by equation (3.10).

# 5 The Fractional Heath-Jarrow-Morton Model

In the preceding chapter we studied the classical Heath-Jarrow-Morton model, in which Brownian motion was the driving factor for randomness in the forward-rate process. But many empirical studies (e.g. [McCarthy et al. [2004]]) propose a long-memory behaviour in bond markets that is in particular inherent in interest rates and originated by the fundamentals of an economy e.g. gross domestic product. One major drawback of the classical HJM approach is that this long-memory behaviour is not captured by Brownian motion due to its independent increments. Therefore we will try to incorporate this long-range dependence and model our forward rates with fractional Brownian motion as the driving noise factor. As we have already shown in chapter 2, fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ exhibits long-range dependence over time and so we hope to get a more realistic forward-rate process. On the other hand this causes a new problem. The existing arbitrage theory from the classical case is now not valid anymore, since for fractional Brownian motion with $H \neq \frac{1}{2}$ one major condition, the semimartingale condition for the process, is not satisfied anymore. As [Gapeev [2004]] has outlined this implies arbitrage opportunities in the absence of transaction costs.
So we will have to come up with a new framework in order to introduce an arbitrage-free interest-rate model. This will turn out to be a drift restriction, too, but a drift restriction that differs from the classical case. Consequently, the non-Markovianity property of fBm makes predictions more complicated.
This chapter is mainly based on the article of [Ohashi [2009]].

## 5.1 The Set-Up

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, i.e. right-continuous and complete. Let $B^H = (B_i^H)_{i=1,\ldots,d}$, $d \in \mathbb{N}$, be a $d$-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ whereas the $d$ fractional Brownian motions are independent of each other.
In order to state our forward-rate dynamics we need to introduce several definitions in advance.

**Definition 5.1** (Semigroup). *A one-parameter semigroup of contractions on the Hilbert space $E$ is a family of bounded, linear operators $\{S(t) : t \in \mathbb{R}_+\}$ on $E$ for which*

*(1)* $S(s+t) = S(s)S(t)$ *for all* $s, t \geq 0$,

*(2)* $S(0) = 1$,

*(3)* $\|S(t)\| \leq 1$ *for all* $t \geq 0$,

*(4) the mapping* $\mathbb{R}_+ \to \mathcal{L}(U, E)$, $t \mapsto S(t)$ *is strongly continuous at zero, that is* $\lim_{t \searrow 0} \|S(t)\psi - \psi\| = 0$ *for all* $\psi \in E$.

*We consider the special case of the right-shift semigroup $\{S(t) : t \in \mathbb{R}_+\}$ on $E$, which is defined by the operator $S(t)g(x) := g(t+x)$ for any function $g : \mathbb{R}_+ \to \mathbb{R}$ and satisfies the above conditions. [Applebaum [2004], Section 3.2]*

As the second step we will specify the separable Hilbert space $E$ in concurrence with [Filipovic [2001], Sections 4 and 5] such that three minimum criteria are fulfilled:

(i) The functions $h \in E$ are continuous and the pointwise evaluation $\mathcal{J}_x(h) := h(x)$ is a continuous linear functional on $E$, for all $x \in \mathbb{R}_+$.

(ii) The semigroup $\{S(t) : t \in \mathbb{R}_+\}$ is strongly continuous in $E$, that is $\lim_{s \to t} \|S_t\psi - S_s\psi\| = 0$ for all $t \geq 0$, $\psi \in E$.

(iii) There exists a constant $K$ such that $\|Sh\|_E \leq K \|h\|_E^2$ for all $h \in E$ with $Sh \in E$, whereas the $E$-norm is specified below.

From now on let $U$ denote a $d$-dimensional vector space with an orthonormal basis $(e_i)_{i=1,\dots,d}$ and let $\mathcal{L}(U, E)$ be the space of bounded linear operators from $U$ into $E$ with the norm $\|\cdot\|_E$ defined by

$$\| h\|_E = | h(0)|^2 + \int_{\mathbb{R}_+} | h'(x)|^2 w(x) dx, \quad h \in E,$$

where $w : \mathbb{R}_+ \to [1, \infty)$ is a non-decreasing $\mathcal{C}^1$-function such that $w^{-\frac{1}{3}} \in L^1(\mathbb{R}_+)$. We will conveniently denote this norm by $\|\cdot\|$ from now on.

Throughout this chapter we will work with the Musiela reparametrization of the forward rate, that is $r_t(x) := f(t, t+x)$, $(t, x) \in \mathbb{R}_+^2$, where $T \in \mathbb{R}_+$, $T \geq t$, denotes the maturity as usual and hence $x = T - t$ is the time to maturity. There is a precise analysis of the advantages of this representation in [Filipovic [2001], p.5 ff], e.g. this modification eliminates the difficulty that the state space is a space of functions on an interval varying with $t$. Now that we have set all the groundwork we assume the forward curve $x \mapsto r_t(x)$ to be a Hilbert space-valued stochastic process under the real-world measure described by a linear stochastic partial differential equation

$$dr_t = \big(Ar_t + \alpha_t\big)dt + \sum_{i=1}^{d} \sigma_t^i dB_i^H(t), \quad r_0(\cdot) = \xi \in E, \tag{5.1}$$

where we assume that the coefficient functions are defined as $\alpha_{\cdot} : \mathbb{R}_+^2 \to E$ and $\sigma_{\cdot}^i : \mathbb{R}_+^2 \to E$, $i = 1, ..., d$, and the first-order derivative operator $A : \mathrm{Dom}(A) \to E$, $A := \frac{\partial}{\partial x}$ is the infinitesimal generator of the right-shift semigroup, which we will have to outline according to [Applebaum [2004], Section 3.2]. Basically we look for a linear operator $A$ for which $S_t = e^{tA}$ can be given meaning in view of the stochastic differential equation in (5.1). Therefore we define the linear space

$$D_A = \left\{ \psi \in E : \exists\, \phi_\psi \in E \text{ such that } \lim_{t \searrow 0} \left\| \frac{S_t \psi - \psi}{t} - \phi_\psi \right\| = 0 \right\}.$$

This enables us to define the infinitesimal generator $A$ in $E$, with domain $D_A$, by $A\psi := \phi_\psi$, so that for each $\psi \in D_A$ we have

$$A\psi = \lim_{t \searrow 0} \frac{S_t \psi - \psi}{t}.$$

The assumption for the dynamics in (5.1) is equivalent to assuming the forward rate analogously to chapter 4 as

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T) ds + \sum_{i=1}^d \int_0^t \sigma^i(s,T) dB_i^H(s), \tag{5.2}$$

where the drift function $\alpha : \mathbb{R}_+^2 \to E$ and the volatility functions $\sigma^i : \mathbb{R}_+^2 \to \mathcal{L}(U,E)$, for all $i = 1, ..., d$, are deterministic. They concur with the coefficient functions in (5.1), where just a different notation is used. The $d$ fractional Brownian motions represent different random noises with different sensitivities $\sigma^i$, $i = 1, ..., d$. $f(0, \cdot\,)$ is a given non-random inital forward rate curve.
In order for the integrals in equation (5.2) to be well-defined we postulate

$$\int_0^T |\,\alpha(s,T)|\, ds < \infty \qquad \text{and}$$
$$\int_0^T \int_0^T |\,\sigma^i(s,T)||\sigma^i(t,T)|\,\phi_H(t-s)\, ds\, dt < \infty$$

for all $i = 1, ..., d$ and all $T \in (0, \infty)$ and where $\phi_H(u) := H(2H-1)|u|^{2H-2}$, $u \in \mathbb{R}$. The latter condition stems directly from the integration with respect to fBm (see the definition of the inner product in Definition 2.16).

Applying the operator $S(t)$ on $r$, $\alpha$ and $\sigma^i$, $i = 1, ..., d$, and recalling $r_t(x) = f(t, t+x)$, we can rewrite the forward-rate process using the Musiela reparametrization as

$$r_t(x) = S(t) r_0(x) + \int_0^t S(t-s)\alpha(s, s+x) ds + \sum_{i=1}^d \int_0^t S(t-s)\sigma^i(s, s+x) dB_i^H(s). \tag{5.3}$$

Moreover we define $\Delta^2 := \{(t,T) \in \mathbb{R}^2 |\ 0 \le t \le T < \infty\}$ as the subset of $\mathbb{R}^2$, on which all the action takes place. This enables us to come up with the term structure of interest rates

$$\{r_t(x) : (t,T) \in \Delta^2\}.$$

So, we are finally set up to introduce a term structure of bond prices $\{P(t,T) : (t,T) \in \Delta^2\}$ with $P(t,T)$ being a zero coupon bond at time $t$ with maturity $T$. It is given by

$$P(t,T) = \exp\Big( - \int_t^T f(t,u)du\Big) = \exp\Big( - \int_0^{T-t} r_t(x)dx\Big), \tag{5.4}$$

using the reparametrization $u = t + x$. The usual normalization condition $P(t,t) = 1$ for all $t > 0$ holds, which implies that all bonds are non-defaultable. Moreover $P(t,T)$ is a.s. continuously differentiable. Moreover, we introduce a convenient notation. Set $\alpha_t(\,\cdot\,) := \alpha(t, t+\,\cdot\,)$ and $\sigma = (\sigma^i)_{i=1,\ldots,d}$, where $\sigma^i_t(\,\cdot\,) := \sigma^i(t, t+\,\cdot\,)$ for all $i = 1, ..., d$. This notation also concurs with the notation of the Musiela reparametrization for the forward-rate dynamics in (5.1).

We suppose that these coefficient functions satisfy the following four assumptions (5.5)-(5.8), which we will need for Theorem 5.2 and where the last three assumptions stem from the theory of integration with respect to fractional Brownian motion (see Section 2.2.2):

$$\int_0^T \|\alpha_s\| ds + \int_0^T \|\sigma_s\|^2 ds < \infty \quad \text{for every } 0 < T < \infty. \tag{5.5}$$

A straightforward conclusion of (5.5) is $\int_0^T \|S(t-s)\sigma_t\|^2 dt < \infty$ for every $0 < T < \infty$, and therefore the stochastic convolution defined as

$$\sum_{i=1}^d \int_0^t S(t-s)\sigma^i_s dB^H_i(s), \quad t > 0,$$

which appears in the forward-rate process (5.3), is a well-defined $E$-valued Gaussian process. The following assumption will be essential for the existence of a continuous version for the mild solution of the sde in (5.1). We assume that there is a $\gamma \in (0, \frac{1}{2})$ such that

$$\int_0^T \int_0^T u^{-\gamma}\nu^{-\gamma} \|S(u)\sigma_u\| \, \|S(\nu)\sigma_\nu\| \, \phi_H(u-\nu) du \, d\nu < \infty \quad \text{for every } 0 < T < \infty. \tag{5.6}$$

We will make two further assumptions, which we will need for the stochastic Fubini theorem, in order to get well-defined bond prices $\{P(t,T); (t,T) \in \Delta^2\}$ in Theorem 5.2:

$$\int_{[0,T]^4} \|\sigma_u(s)\| \, \|\sigma_\nu(r)\| \, \phi_H(u-\nu) du \, d\nu \, ds \, dr < \infty \quad \text{for every } 0 < T < \infty \tag{5.7}$$

and

$$\int_{[0,T]^3} \|\sigma_u(t)\| \, \|\sigma_\nu(t)\| \, \phi_H(u-\nu) d\nu \, du \, dt < \infty \quad \text{for every } 0 < T < \infty. \tag{5.8}$$

Analogously to the classical Heath-Jarrow-Morton model in the previous chapter we define the numeraire as

$$B_0(t) := \exp\Big( \int_0^t r_s(0) ds\Big) < \infty.$$

So we can also define the relative (or discounted) bond prices as

$$Z_t(T) := \frac{P(t,T)}{B_0(t)}, \quad (t,T) \in \Delta^2.$$

We introduce a convenient notation before we will state our bond price theorem:

**Notation:** Let $\nu : [0,T] \times R_+ \rightarrow \mathbb{R}$ be a function that is locally integrable in $\mathbb{R}_+$. Then define

$$\mathcal{I}_\nu(s,T) := \int_0^{T-s} \nu_s(x)dx.$$

**Theorem 5.2.** *Assume that the coefficients $\alpha$ and $\sigma$ satisfy the assumptions (5.5), (5.6), (5.7) and (5.8). Then the forward rate $r_t$ given by the Musiela reparametrization in (5.3) is the continuous mild solution of the sde in (5.1).*
*Moreover the term structure of bond prices is given by the continuous process*

$$P(t,T) = P(0,T)\exp\left(\int_0^t (r_s(0) - \mathcal{I}_\alpha(s,T))ds + \sum_{i=1}^d \int_0^t -\mathcal{I}_{\sigma^i}(s,T)dB_i^H(s)\right) \tag{5.9}$$

*for* $(t,T) \in \Delta^2$.

**Proof:** The first statement of Theorem 5.2 can be proved in two steps. At first one will show that the forward rate (5.3) is a mild solution of the sde in (5.1) [see Duncan et al. [2002], Proposition 3.1]. In a second step we can prove that if condition (5.6) holds, then there is a version of $r_t(x)$ with continuous sample paths [see Duncan et al. [2002], Proposition 3.2].
In order to come up with the bond price we recall equation (5.4)

$$P(t,T) = \exp\left(-\int_0^{T-t} r_t(x)dx\right) = \exp\big(-\mathcal{I}_r(t,T)\big).$$

We will compute $\mathcal{I}_r(t,T)$. Fix $(t,T) \in \Delta^2$ and subsitute $r_t(x)$ by the Musiela equation (5.3), where we make use of the notation $\alpha_t(\,\cdot\,) := \alpha(t, t{+}\cdot\,)$ and $\sigma_t^i(\,\cdot\,) := \sigma^i(t, t{+}\cdot\,)$ for all $i = 1, ..., d$:

$$\mathcal{I}_{r_t}(t,T) = \mathcal{I}_{S(t)r_0}(t,T) + \int_0^t \mathcal{I}_{S(t-s)\alpha_s}(t,T)ds + \sum_{i=1}^d \int_0^t \mathcal{I}_{S(t-s)\sigma_s^i}(t,T)dB_i^H(s).$$

Note that $\mathcal{I}(t,T) \circ S(t) = \mathcal{I}(0,T) - \mathcal{I}(0,t)$. This yields

$$\begin{aligned}\mathcal{I}_{r_t}(t,T) = \mathcal{I}_{r_0}(0,T) - \mathcal{I}_{r_0}(0,t) + \int_0^t (\mathcal{I}_{\alpha_s}(s,T) - \mathcal{I}_{\alpha_s}(s,t))ds \\ + \sum_{i=1}^d \int_0^t (\mathcal{I}_{\sigma_s^i}(s,T) - \mathcal{I}_{\sigma_s^i}(s,t))dB_i^H(s).\end{aligned}$$

We will split up the integrals into two parts, that is $\mathcal{I}_{r_t}(t,T) := I_1 - I_2$, where

$$I_1 := \mathcal{I}_{r_0}(0,T) + \int_0^t \mathcal{I}_{\alpha_s}(s,T)ds + \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma_s^i}(s,T)dB_i^H(s),$$
$$I_2 := \mathcal{I}_{r_0}(0,t) + \int_0^t \mathcal{I}_{\alpha_s}(s,t)ds + \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma_s^i}(s,t)dB_i^H(s).$$

We need to transform the integrals in $I_2$ using

$$\tilde{\sigma}_s^i(u) = \begin{cases} \sigma_s^i(u-s), & \text{if } s \le u \\ 0 & \text{otherwise.} \end{cases}$$

Substituting $u = x + s$, we get

$$\mathcal{I}_{\sigma_s^i}(s,t) = \int_0^{t-s} \sigma_s^i(x)dx = \int_s^t \sigma_s^i(u-s)du = \int_0^t \tilde{\sigma}_s^i(u)du.$$

Since the assumptions (5.7) and (5.8) hold, we can apply the stochastic Fubini theorem to the fractional Brownian motion term [see Krvavich and Mishura [2001], Theorem 1] in $I_2$ and substitute backwards to get

$$\begin{aligned}\sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma_s^i}(s,t)dB_i^H(s) &= \sum_{i=1}^d \int_0^t \left(\int_0^t \tilde{\sigma}_s^i(u)du\right)dB_i^H(s) \\ &= \int_0^t \left(\sum_{i=1}^d \int_0^t \tilde{\sigma}_s^i(u)dB_i^H(s)\right)du \\ &= \int_0^t \left(\sum_{i=1}^d \int_0^u \sigma_s^i(u-s)dB_i^H(s)\right)du.\end{aligned} \tag{5.10}$$

With the ordinary Fubini theorem we similarly find

$$\int_0^t \mathcal{I}_{\alpha_s}(s,t)ds = \int_0^t \left(\int_0^u \alpha_s(u-s)ds\right)du. \tag{5.11}$$

Results (5.10) and (5.11) combined we can derive

$$\begin{aligned} I_2 &= \int_0^t \left(S(u)r_0(0) + \int_0^u S(u-s)\alpha_s(0)ds + \sum_{i=1}^d \int_0^u S(u-s)\sigma_s^i(0)dB_i^H(s)\right)du \\ &= \int_0^t r_u(0)du.\end{aligned}$$

In $I_1$ we observe that $\mathcal{I}_{r_0}(0,T) = -\log P(0,T)$ and left with the remaining terms we come up with

$$\log P(t,T) = \log P(0,T) + \int_0^t (r_s(0) - \mathcal{I}_\alpha(s,T))ds + \sum_{i=1}^d \int_0^t -\mathcal{I}_{\sigma_s^i}(s,T)dB_i^H(s),$$

which proves our expression for the bond price in (5.9). □

## 5.2 Arbitrage Free Bond Pricing

In the presence of proportional transaction costs, semimartingales are not the only arbitrage-free assets anymore. At the same time, not all strategies are permitted or meaningful, as trading volume must remain finite. In other words, as the class of reasonable integrators enlarges, the set of admissible integrands shrinks, i.e. we focus on a particular set of strategies, the measure-valued elementary processes. First of all we will have to come up with a definition of our wealth process also known as portfolio value process and with the notion of admissible trading strategies. Therefore we introduce a setting that takes transaction costs into account which is essential for a no-arbitrage framework in the fractional case.

**The portfolio setting**

Let $\mathcal{M}_{T^*}$ denote the space of all finite signed measures on $[0, T^*]$ endowed with the total variation norm $\| \cdot \|_{TV}$, where $T^*$ is the upper limit of our trading interval.

**Definition 5.3** (Finite signed measure)**.** *Let $\mathcal{F}$ be a $\sigma$-Algebra.*
*A function $\mu : \mathcal{F} \to (-\infty, \infty)$ is called a (finite) signed measure on $\mathcal{F}$, if*

*(i)* $\mu(\varnothing) = 0$,

*(ii) for any pairwise disjoint sequence* $(A_n)_{n=1}^{\infty} \subseteq \mathcal{F}$*, we have*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

*[Klenke [2006], Definition 7.40]*

Basically a signed measure is a measure that takes on negative values, too.

**Definition 5.4** (Total variation norm)**.** *Let $\mu$ be a signed measure on a measurable space $(\Omega, \mathcal{F})$ and the Jordan decomposition is given by $\mu = \mu^+ - \mu^-$. Let $\Omega = \Omega^+ \uplus \Omega^-$ be the Hahn decomposition. Then the total variation norm is defined by*

$$\begin{aligned} \| \mu \|_{TV} &: = \sup\{\mu(A) - \mu(\Omega \backslash A) : A \in \mathcal{F}\} \\ &= \mu(\Omega^+) - \mu(\Omega^-) \\ &= \mu^+(\Omega) - \mu^-(\Omega). \end{aligned}$$

*[Klenke [2006], Corollary 7.45]*

The Jordan decomposition and the Hahn decomposition exist by the sole existence of a signed measure according to [Klenke [2006], Corollary 7.44] and [Klenke [2006], Theorem 7.43].

Let $\mu$ be a measure-valued elementary process that is defined as

$$\mu_t(\omega, \cdot\,) := \sum_{i=0}^{N-1} \chi_{F_i \times (t_i, t_{i+1}]}(\omega, t) m_i(\cdot\,), \tag{5.12}$$

where $m_i \in \mathcal{M}_{T^*}$, $0 = t_0 < ... < t_N \leq T^*$, $N \in \mathbb{N}$ and $F_i \in \mathcal{F}_{t_i}$. $\omega$ emphasizes the impact of coincidence within this process. $\mu$ will later function as our trading strategy.
Let $\mathcal{S}_b$ denote the set of all elementary processes of the form (5.12) endowed with the following norm

$$\|\mu\|_V^2 := \mathbb{E}\Big[\sup_{(t,T)\in[0,T^*]^2} \|\mu_t\|_{TV}^2\Big] \tag{5.13}$$

and $V$ being the completion of $\mathcal{S}_b$ with respect to this norm.

We assume that all economic activity takes place on the bounded set $[0, T^*]^2$ and hence in the following we have $Z_t(T) = 0$ if $(t, T) \notin \Delta^2$. As a direct consequence of Theorem 5.2 the discounted bond price process $Z_t(T)$ satisfies the following mild integrability assumption if the assumptions (5.5)-(5.8)hold:

*Condition A*

$\{Z_t(T) : (t, T) \in [0, T^*]^2\}$ is a jointly continuous real-valued stochastic process such that

$$\mathbb{E}\Big[\sup_{(t,T)\in[0,T^*]^2} |Z_t(T)|^2\Big] < \infty.$$

Let $\mu \in \mathcal{S}_b$ as in (5.12). Then the integral with respect to the discounted price process, that represents the capital gain of the portfolio $\mu$ (which will be the same as the capital value of the portfolio for zero inital capital), is defined as

$$\int_0^t \mu_s dZ_s := \sum_{i=0}^{N-1} \chi_{F_i}(Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t})m_i, \tag{5.14}$$

where $Z_{t_i}m_i$ is considered as the dual action of a stochastic process $Z$ and a measure $m$. This dual action term can be interpreted by the following form:

$$(Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t})m_i := \int_0^{T^*} (Z_{t_{i+1}\wedge t}(x) - Z_{t_i\wedge t}(x))\, m_i(dx). \tag{5.15}$$

The expression $\int_0^\cdot \mu_s dZ_s$ is well-defined for every $\mu \in V$ by the definition of V and the following lemma:

**Lemma 5.5.** *Let $\|\cdot\|_\infty$ denote the uniform topology norm on the space of real-valued bounded functions defined on $[0, T^*]$ given by the supremum, i.e. $\|Z_t\|_\infty := \sup\{Z_t : t \in [0, T^*]\}$. Then we have*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T^*}\Big|\int_0^t \mu_s dZ_s\Big|\Big] \leq \|\mu\|_V\, \mathbb{E}\Big[\sup_{0\leq s,t\leq T^*} \|Z_s - Z_t\|_\infty^2\Big]^{1/2} < \infty \quad \textit{for all } \mu \in V.$$

**Proof:** We will prove this inequality by applying some estimates and the Hölder inequality. First of all we consider the easiest case, that is $N = 1$. So the elementary process simplifies to $\mu_t(\omega, \cdot\,) = \chi_{F_0\times(0,t_N]}(\omega, t)m_0(\cdot\,)$ and the integral with respect to $Z$ simplifies to

$$\begin{aligned}\int_0^t \mu_s dZ_s = \chi_{F_0}(Z_{t_N\wedge t} - Z_{0\wedge t})m_0 &= \chi_{F_0}\int_0^{T^*}(Z_{t_N\wedge t}(x) - Z_{0\wedge t}(x))m_0(dx)\\ &= \chi_{F_0}\int_{t_N\wedge t}^{T^*}(Z_{t_N\wedge t}(x) - Z_{0\wedge t}(x))m_0(dx),\end{aligned}$$

where we use the fact that $Z_{t_N\wedge t}(x) = 0 \Leftrightarrow x < t_N \wedge t$. Hence we obtain

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{0\le t\le T^*}\Bigg|\int_0^t \mu_s dZ_s\Bigg|\Bigg] &= \mathbb{E}\Bigg[\sup_{0\le t\le T^*}\Bigg|\chi_{F_0}\int_{t_N\wedge t}^{T^*}(Z_{t_N\wedge t}(x) - Z_{0\wedge t}(x))m_0(dx)\Bigg|\Bigg]\\ &\le \mathbb{E}\Bigg[\sup_{0\le t\le T^*}\chi_{F_0}\Bigg|\int_{t_N\wedge t}^{T^*}(Z_{t_N\wedge t}(x) - Z_{0\wedge t}(x))m_0(dx)\Bigg|\Bigg]\\ &\le \mathbb{E}\Bigg[\sup_{0\le t\le T^*}\Bigg(\chi_{F_0}\underbrace{\sup_{x\in[t_N,T^*]}\big|Z_{t_N\wedge t}(x) - Z_{0\wedge t}(x)\big|}_{\le\left\|Z_{t_N\wedge t}-Z_{0\wedge t}\right\|_\infty \text{ by def of } \|\cdot\|_\infty}\underbrace{\int_{t_N\wedge t}^{T^*}|\,m_0|(dx)}_{=|m_0|([t_N\wedge t,T^*])\le\|m_0\|_{TV}}\Bigg)\Bigg].\end{aligned}$$

We observe that $\|Z_{t_N\wedge t} - Z_{0\wedge t}\|_\infty = 0$, if $t \notin (0, t_N]$ and so we can just insert $\chi_{(0,t_N]}$ into the equation. This yields

$$\begin{aligned}&\le \mathbb{E}\Bigg[\sup_{0\le t\le T^*}\Bigg(\|Z_{t_N\wedge t} - Z_{0\wedge t}\|_\infty\left\|\chi_{F_0\times(0,t_N]}m_0\right\|_{TV}\Bigg)\Bigg]\\ &= \mathbb{E}\Bigg[\sup_{0\le t\le T^*}\Bigg(\|Z_{t_N\wedge t} - Z_{0\wedge t}\|_\infty\|\mu_t\|_{TV}\Bigg)\Bigg]\\ &\le \mathbb{E}\Bigg[\sup_{0\le s,t\le T^*}\|Z_s - Z_t\|_\infty\sup_{0\le t\le T^*}\|\mu_t\|_{TV}\Bigg]\\ &\overset{\text{Hölder}}{\le} \mathbb{E}\Bigg[\sup_{0\le s,t\le T^*}\|Z_s - Z_t\|_\infty^2\Bigg]^{1/2}\mathbb{E}\Bigg[\sup_{0\le t\le T^*}\|\mu_t\|_{TV}^2\Bigg]^{1/2}\\ &= \mathbb{E}\Bigg[\sup_{0\le s,t\le T^*}\|Z_s - Z_t\|_\infty^2\Bigg]^{1/2}\|\mu\|_V\,,\end{aligned}$$

where we apply the definition of the norm from (5.13). Hölder holds, since for positive functions $f$ we have $(\sup f)^2 \le \sup(f^2)$, where in our case $\|\cdot\|$ is the positive function.

We can now proceed to the general case, that is $\mu_t(\omega, \cdot\,) := \sum_{i=0}^{N-1} \chi_{F_i \times (t_i, t_{i+1}]}(\omega, t) m_i(\cdot\,)$, and derive the inequality analogously.

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T^*}\Big|\int_0^t \mu_s dZ_s\Big|\Big] &\overset{(5.15)}{=} \mathbb{E}\Big[\sup_{0\le t\le T^*}\Big|\sum_{i=0}^{N-1}\chi_{F_i}\int_0^{T^*} (Z_{t_{i+1}\wedge t}(x) - Z_{t_i\wedge t}(x)) m_i(dx)\Big|\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T^*}\Big|\sum_{i=0}^{N-1}\chi_{F_i}\int_{t_i\wedge t}^{T^*} (Z_{t_{i+1}\wedge t}(x) - Z_{t_i\wedge t}(x)) m_i(dx)\Big|\Big] \\
&\le \mathbb{E}\Big[\sup_{0\le t\le T^*}\Big(\sum_{i=0}^{N-1}\chi_{F_i}\underbrace{\Big|\int_{t_i\wedge t}^{T^*} (Z_{t_{i+1}\wedge t}(x) - Z_{t_i\wedge t}(x)) m_i(dx)\Big|}_{\le \sup\limits_{x\in[t_i\wedge t, T^*]}|Z_{t_{i+1}\wedge t}(x) - Z_{t_i\wedge t}(x)|\int\limits_{t_i\wedge t}^{T^*}|m_i|(dx)}\Big)\Big] \\
&\le \mathbb{E}\Big[\sup_{0\le t\le T^*}\Big(\sum_{i=0}^{N-1}\chi_{F_i}\left\|Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}\right\|_\infty \|m_i\|_{TV}\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T^*}\Big(\sum_{i=0}^{N-1}\left\|Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}\right\|_\infty \chi_{F_i\times(t_i,t_{i+1}]}\|m_i\|_{TV}\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T^*}\Big(\sum_{i=0}^{N-1}\left\|Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}\right\|_\infty \left\|\chi_{F_i\times(t_i,t_{i+1}]}m_i\right\|_{TV}\Big)\Big] \\
&\le \mathbb{E}\Big[\sup_{0\le s,t\le T^*}\|Z_s - Z_t\|_\infty \sup_{0\le t\le T^*}\Big(\underbrace{\sum_{i=0}^{N-1}\left\|\chi_{F_i\times(t_i,t_{i+1}]}m_i\right\|_{TV}}_{=\|\mu_t\|_{TV}\ \ \text{since all summands are positive}}\Big)\Big] \\
&\overset{\text{Hölder}}{\le} \mathbb{E}\Big[\sup_{0\le s,t\le T^*}\|Z_s - Z_t\|_\infty^2\Big]^{1/2} \mathbb{E}\Big[\sup_{0\le t\le T^*}\|\mu_t\|_{TV}^2\Big]^{1/2} \\
&= \mathbb{E}\Big[\sup_{0\le s,t\le T^*}\|Z_s - Z_t\|_\infty^2\Big]^{1/2} \|\mu\|_V\,.
\end{aligned}
$$

The inequality follows for all $\mu \in V$ by classical limit argument.
Moreover, the expression is finite due to condition A. □

The liquidation value of a portfolio with zero initial capital consists of the capital gain of an elementary portfolio $\mu$ as defined in (5.12) from time 0 to $t$ minus the transaction costs of all transactions incurred minus the final cost of liquidation, that is

$$
V_t^k(\mu) = \sum_{t_i<t}\chi_{F_i}(Z_{t_i\wedge t} - Z_{t_{i+1}\wedge t})m_i - k\sum_{t_i<t} Z_{t_i}|\mu_{t_{i+1}} - \mu_{t_i}| - kZ_t|\mu_t|, \tag{5.16}
$$

where $k$ is an arbitrary positive number, which stands for the proportional transaction costs in the bond market, and $|\cdot|$ denotes the total variation measure which we will define in the following:

**Definition 5.6** (Total variation measure)**.** *Let $\mu$ be a signed measure on a measurable space $(\Omega, \mathcal{F})$. Let $E$ be a measurable subset of $\mathcal{F}$. Let $\Pi := \cup_i E_i$ be an arbitrary partition of $E$ with measurable subsets $E_i$. Then the total variation measure $|\mu|$ is defined by*

$$|\mu|(E) = \sup_{\Pi} \sum_i |\mu(E_i)| \quad \textit{for all } E \in \mathcal{F}.$$

*In words, the supremum over all partitions $\Pi$ of a measurable set $E$ into a finite number of disjoint measurable subsets is taken. The total variation measure can be interpreted as the infinitesimal version of the absolute value.*

Consequently, we will extend the wealth process from a finite number of transactions to continuous trading:

*Condition B*

Let $\mathcal{P}_{T^*}$ be the set of all partitions of $[0, T^*]$. We assume that

$$\Pi_{T^*}(\mu) := \sup_{\pi \in \mathcal{P}_{T^*}} \sum_{t_i \in \pi} \left\| \mu_{t_{i+1}} - \mu_{t_i} \right\|_{TV} \quad \text{is square integrable.}$$

**Lemma 5.7** (Convergence of the wealth process)**.** *Let $\mu \in V$ satisfy Condition B. Then the wealth process in (5.16) converges to*

$$V_t^k(\mu) = \int_0^t \mu_s dZ_s - k \int_0^t Z_s d|\mu_s| - k Z_t |\mu_t|, \tag{5.17}$$

*where $\int_0^t Z_s d\mu_s$ is defined as*

$$\int_0^t Z_s d\mu_s := \lim_{n\to\infty} \int_0^t \overline{Z}_s^n d\mu_s = \lim_{n\to\infty} \int_0^t \underline{Z}_s^n d\mu_s \tag{5.18}$$

*with $\overline{Z^n}$ and $\underline{Z^n}$ being the upper and the lower approximations of $Z$, respectively, along the partition $t_i^n := \frac{iT^*}{2^n}$.*

For a proof of this lemma see [Guasoni [2002]].

**The no-arbitrage framework**

No-arbitrage theory is a lot more complicated in the fractional case and we will demonstrate an approach completely different to the classical case. First, there are three important expressions which are commonly used and which we need to define for no-arbitrage theory under transaction costs - an admissible trading strategy, an arbitrage opportunity and a $k$-arbitrage free bond market.

**Definition 5.8.** *$\mu \in V$ is an admissible trading strategy if it satisfies condition B, it is weakly $\mathcal{F}_t$-adapted and there exists a constant $M > 0$ such that $V_t^k(\mu) \geq -M$ a.s. for every $t \leq T^*$. The last assumption excludes the possibility of infinite losses.*
*An admissible trading strategy is called an arbitrage opportunity with transaction costs $k > 0$ on $[0, T^*]$, if $V_{T^*}^k(\mu) \geq 0$ a.s. and $\mathbb{P}(V_{T^*}^k(\mu) > 0) > 0$.*
*The bond market is called $k$-arbitrage free on $[0, T^*]$ if no such strategy exists.*

We will need the following lemma in order to prove the important upcoming proposition:

**Lemma 5.9.** *Let $Z$ and $\widetilde{Z}$ be càdlàg functions such that $|Z_t - \widetilde{Z}_t| < kZ_t$ a.s. for all $t \in (0, \infty)$. If $\theta : [0, \infty) \to \mathbb{R}$ is a left-continuous function of bounded variation, then*

$$V_t^k(\theta) \leq \int_0^t \theta_s d\widetilde{Z}_s \quad \text{for all} \quad t \in (0, \infty)$$

*and equality holds for $t$ if and only if $\theta_s = 0$ for all $s \leq t$. [Guasoni [2006], Lemma 2.1]*

Now we can come up with a very general no-arbitrage criterion.

**Proposition 5.10.** *Fix $k > 0$. If for every $(\mathcal{F}_t)_{t \geq 0}$-stopping time $\tau$, that satisfies $\mathbb{P}(\tau < T^*) > 0$, we have*

$$\mathbb{P}\left( \sup_{\tau \leq t \leq T \leq T^*} \left| \frac{Z_\tau(\tau)}{Z_t(T)} - 1 \right| < k, \tau < T^* \right) > 0,$$

*then the bond market is arbitrage free on $[0, T^*]$ with transaction costs $k$.*

**Proof:** Let $\theta$ be a strategy that is not identically zero. Define the stopping time $\tau$ and the event $A$, respectively, as

$$\tau := T \wedge \inf\{t : \theta_t \neq 0\} \quad \text{and} \quad A := \left\{ \sup_{\tau \leq t \leq T \leq T^*} \left| \frac{Z_\tau(\tau)}{Z_t(T)} - 1 \right| < k, \tau < T^* \right\}.$$

Since $\theta_t \neq 0$ for at least one $t \in [0, T]$, we obviously have $\mathbb{P}(\tau < T^*) > 0$. Now we will assume $\mathbb{P}(A) > 0$ to hold.
We define $\widetilde{Z}_t = Z_{t \wedge \tau}$ and apply Lemma 5.9 on the event $A$. The assumptions of the lemma are satisfied due to the choice of $A$. This yields

$$\begin{aligned} V_T^k(\theta) &= \int_0^T \theta_s dZ_s - k \int_0^T Z_s d|\theta_s| - kZ_T|\theta_T| \\ &\leq \int_0^T \theta_s d\widetilde{Z}_s = 0 \end{aligned}$$

and since $\theta_s \neq 0$ by assumption, we get $V_T^k(\theta) < 0$ and so $\theta$ is not an arbitrage. $\square$

Intuitively, we have chosen $\tau$ as the first trading point in time, which generates transaction costs $k$. For an arbitrage opportunity this will have to be made up for in the future by a price movement larger than $k$. So we choose $A$ as the event for which the price movement is not large enough. If there is a positive probability for this event to occur, we cannot avoid the risk of a loss and so arbitrage is impossible.

**Remark 5.11.** *We have seen so far that for every $k > 0$ we can come up with a no-arbitrage criterion and for $k = 0$ there is an arbitrage strategy due to [Gapeev [2004]]. One might ask what would happen for the limit $k \to 0$. Due to the choice of the event $A$ Proposition 5.10 would not make sense anymore and so we cannot make any statement concerning this case. This is not surprising, since intuitively at $k = 0$ we have a discontinuity.*

Now we will come up with conditions which ensure a $k$-arbitrage free bond market for every $k > 0$. Therefore we will combine the full-support property on $\mathcal{C}_{\Delta^2_{T^*}}$ with a suitable choice on the drift $\alpha$ of the forward-rate process. We define $\Delta^2_{T^*} := \{(t,T) : 0 \leq t \leq T \leq T^*\}$ and denote by $\mathcal{C}_{\Delta^2_{T^*}}$ the space of all real-valued continuous functions on the metric space $\Delta^2_{T^*}$. We start off with the definition of the important full-support property:

**Definition 5.12** (Full support)**.** *Let $\mathcal{X}$ be a Polish space, i.e. a metric, separabel and complete topological space. A random element $\xi : \Omega \to \mathcal{X}$ has $\mathbb{P}$-full support if $\mathbb{P}_\xi := \mathbb{P} \circ \xi^{-1}(\mathcal{U}) > 0$ for every non-empty open set $\mathcal{U}$ in $\mathcal{X}$.*

**Example 5.13.** *In order to clarify the notion of full support we provide an easy example for what is described by "support". The exponential distribution is given by the density $f(x) = \lambda e^{-\lambda x}$ for all $x \geq 0$ and $f(x) = 0$ for all $x < 0$. We say that the exponential distribution has support on $\mathbb{R}_+$.*
*In contrast to that the normal distribution given by $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)$ for all $x \in \mathbb{R}$ has support on the complete real line $\mathbb{R}$.*

By the full-support property we can give a condition when Proposition 5.10 can be used:

**Lemma 5.14.** *Let $Y : \Omega \to \mathcal{C}_{\Delta^2_{T^*}}$ be a measurable map such that $X := \log Y$ has $\mathbb{P}$-full support. Then $Y$ satisfies the assumption in Proposition 5.10, that is the probability of the event $A$ is positive.*

**Proof:** Let $\varepsilon > 0$ and let $\tau$ be an $\mathcal{F}_t$-stopping time such that $\mathbb{P}(\tau < T^*) > 0$. Since

$$\begin{aligned} \log Y_t(T) - \log Y_\tau(\tau) = \log \frac{Y_t(T)}{Y_\tau(\tau)} &< \varepsilon \\ \Leftrightarrow \qquad \frac{Y_t(T)}{Y_\tau(\tau)} - 1 &< \varepsilon, \end{aligned}$$

it is sufficient to check that

$$\mathbb{P}\bigg( \sup_{\tau \leq t \leq T \leq T^*} |X(t,T) - X(\tau,\tau)| < \varepsilon, \tau < T^* \bigg) > 0.$$

Applying the triangle inequality we can easily derive for $p \in \mathcal{C}_{\Delta^2_{T^*}}$:

$$\begin{aligned}&\left\{ \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - p(t,T)| < \frac{\varepsilon}{2}, \tau < T^* \right\} \\ &\subseteq \left\{ \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - X(\tau,\tau)| < \varepsilon, \tau < T^* \right\}.\end{aligned} \tag{5.19}$$

We denote $\mathcal{P}$ the set of all polynomials $p$ on $\Delta^2_{T^*}$ with rational coefficients such that $p(0,0) = 0$. We claim that there exists $p \in \mathcal{P}$ such that

$$\mathbb{P}\left( \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - X(\tau,\tau)| < \frac{\varepsilon}{2}, \tau < T^* \right) > 0. \tag{5.20}$$

If we can prove (5.20), the actual statement of Lemma 5.14 follows due to "$\subseteq$" in (5.19). Therefore we are going to make a proof by contradiction, i.e. assume that (5.20) is violated, that is $\mathbb{P}\big( \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - X(\tau,\tau)| < \frac{\varepsilon}{2}, \tau < T^* \big) = 0$ for every $p \in \mathcal{P}$. So we easily get

$$\left\{ \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - p(t,T)| < \frac{\varepsilon}{2}, \tau < T^* \right\} \subseteq \{\tau \geq T^*\} \quad \mathbb{P}\text{-a.s. for all } p \in \mathcal{P}$$

and hence

$$\bigcup_{p\in\mathcal{P}} \left\{ \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - p(t,T)| < \frac{\varepsilon}{2} \right\} \subseteq \{\tau \geq T^*\} \quad \mathbb{P}\text{-a.s.}. \tag{5.21}$$

The density of $\mathcal{P}$ in $\mathcal{C}_{\Delta^2_{T^*}}$ and the full support property of $X$ combined, we can conclude that

$$\mathbb{P}\left( \bigcup_{p\in\mathcal{P}} \left\{ \sup_{(t,T)\in\Delta^2_{T^*}} |X(t,T) - p(t,T)| < \frac{\varepsilon}{2} \right\} \right) = 1$$

and due to the "$\subseteq$" in (5.21) we get $\mathbb{P}(\tau \geq T^*) = 1$. So we have $\mathbb{P}(\tau < T^*) = 0$ which is a contradiction to the assumption in the very beginning. $\square$

**Remark 5.15.** *Our next step towards the absence of arbitrage makes use of the Hölder continuity of the paths of a fractional Brownian motion for any order $\gamma < H$. (For a proof of this fact see [Decreusefond and Üstünel [1999], Theorem 3.1].)*

**Definition 5.16** ($\gamma$-Hölder continuous)**.** *Let $(E,d)$ and $(E',d')$ be metric spaces and let $\gamma \in (0,1]$. A mapping $\varphi : E \to E'$ is called Hölder continuous in $r \in E$ of order $\gamma$, if there are constants $\varepsilon > 0$ and $C < \infty$ such that for every $s \in E$ with $d(s,r) < \varepsilon$ we have $d'(\varphi(r), \varphi(s)) \leq C\, d(r,s)^\gamma$. [Klenke [2006], Definition 21.2]*

We observe that for $\gamma = 1$ this definition is equivalent to the definition of a Lipschitz continuous function.

**Remark 5.17.** *Moreover there is an fBm Wiener measure on a separable Banach space $\mathbb{W}$ continuously embedded on the space $\mathcal{C}_{\mathbb{R}_+}$ such that the elements of $\mathbb{W}$ are $\gamma$-Hölder continuous functions on any compact interval. (For a proof of this finding see [Hairer and Ohashi [2007], Lemma 4.1].)*

We will make use of this remark in the following lemma, which states the crucial volatility conditions in order to come up with absence of arbitrage in the bond market.

**Lemma 5.18.** *Assume that $I_{\sigma^i}(t,T)$ is $\gamma$-Hölder continuous on $\Delta^2_{T^*}$ for every $i \geq 1$ where $\frac{1}{2} < \gamma < 1$. Then the process $\sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)dB_i^H(s)$ has $\mathbb{P}$-full support on $\mathcal{C}_{\Delta^2_{T^*}}$.*

A detailed proof of this lemma is outlined in [Ohashi [2009], Lemma 3.2].

This lemma almost completes our framework in order to conclude the absence of arbitrage, which we will briefly summarize. Combining Lemma 5.14 and Proposition 5.10 we observe that if $\log Z$ has $\mathbb{P}$-full support, then the bond market is $k$-arbitrage free for every $k > 0$. We ensure this full support property for $\log Z$ by imposing conditions on the volatilities $\sigma^i$, $i = 1, ..., d$, in Lemma 5.18, i.e. the Hölder continuity.
There is still one more step to go, because so far there are still infinitely many choices of $\alpha$ that ensure the full support property and so absence of arbitrage in the fractional bond market. We will see that there is a unique choice for the drift that will guarantee the existence of a *quasi-martingale measure.*

**Definition 5.19** (Quasi-martingale measure)**.** *An equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ is called a quasi-martingale measure if the discounted bond price process $Z_t(T)$ has $\mathbb{Q}$-constant expectation, i.e. for every $0 < T < \infty$, we have*

$$\mathbb{E}_{\mathbb{Q}}[Z_t(T)] = P(0,T) \quad \text{for } t \in [0,T].$$

But first of all we derive the drift condition under the real-world measure $\mathbb{P}$. We will later convert this to the risk-neutral case using the change of measure.

**Theorem 5.20.** *The $\mathbb{P}$-constant expectation $\mathbb{E}\left[Z_t(T)\right] = P(0,T)$ holds for every $T \in (0,\infty)$, $t \in [0,T]$, if and only if the drift $\alpha$ satisfies*

$$\begin{aligned} \alpha_t(\,\cdot\,) = \sum_{i=1}^d \Bigg( & \sigma_t^i(\,\cdot\,) \int_0^t \mathcal{I}_{\sigma^i}(\theta, \cdot + t)\phi_H(t-\theta)\, d\theta \\ & + \int_0^{\cdot} \sigma_t^i(y)dy \int_0^t \sigma_\theta^i(\cdot + t - \theta)\phi_H(t-\theta)\, d\theta \Bigg). \end{aligned} \tag{5.22}$$

**Proof:** We start with the discounted bond price process easily derived from Theorem 5.2

$$Z_t(T) = \frac{P(t,T)}{B_0(t)} = P(0,T)\exp\left( -\int_0^t \mathcal{I}_\alpha(s,T)ds - \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)dB_i^H(s) \right),$$

$(t,T) \in \Delta^2$. Since we need the constant expectation to hold, we need to prove that

$$\mathbb{E}\left[ \exp\left( -\int_0^t I_\alpha(s,T)ds - \sum_{i=1}^d \int_0^t I_{\sigma^i}(s,T)dB_i^H \right) \right] = 1. \tag{5.23}$$

First we will examine the latter part of the exponential. So we define
$y(t,T) := \mathbb{E}\left[ \exp\left( -\sum_{i=1}^d \int_0^t I_{\sigma^i}(s,T)dB_i^H \right) \right]$ for $0 \leq t \leq T$.

Now we need the Itô formula in the fractional Brownian motion case:

**Lemma 5.21.** *Let $f$ be a function of class $C^2(\mathbb{R})$. Assume that $u = \{u_t, t \in [0,T]\}$ is a process such that the indefinite integral $X(t) = \int_0^t u_s dB^H(s)$ is a.s. continuous. Then $f(X(t))$ is a process of the following form*

$$\begin{aligned} f(X(t)) =& f(0) + \int_0^t f'(X(s)u_s dB^H(s) \\ &+ H(2H-1)\int_0^t f''(X(s))u_s\Big(\int_0^T |s-\sigma|^{2H-2}\Big(\int_0^s \frac{\partial}{\partial\sigma}u_\theta dB^H(s)\Big)d\sigma\Big)ds \\ &+ H(2H-1)\int_0^t f''(X(s))u_s\Big(\int_0^s u_\theta(s-\sigma)^{2H-2}d\theta\Big)ds \end{aligned}$$

*for $0 \le t \le T$. [For a detailed analysis and proof see Alòs and Nualart [2003]]*

Applying the Itô formula with $f(x) = e^x$ yields for $T \in (0,\infty)$

$$y(t,T) = 1 + \sum_{i=1}^d \int_0^t y(s,T)\mathcal{I}_{\sigma^i}(s,T)\Big(\int_0^s \mathcal{I}_{\sigma^i}(\theta,T)\phi_H(s-\theta)d\theta\Big)ds, \tag{5.24}$$

where we recall $\phi_H(s-\theta) = H(2H-1)|s-\theta|^{2H-2}$. Thereafter applying the variation of constants formula for differential equations [Pontryagin [1962]] yields

$$y(t,T) = \exp\Big(\int_0^t e(s,T)ds\Big),$$

where $e(t,T) := \sum_{i=1}^d \mathcal{I}_{\sigma^i}(t,T)\int_0^t \mathcal{I}_{\sigma^i}(\theta,T)\phi_H(t-\theta)d\theta$ for $0 \le t \le T$.
So, obviously equation (5.23) holds if and only if

$$\mathcal{I}_\alpha(t,T) = \int_0^{T-t} \alpha_t(y)\ dy = e(t,T) \quad \text{for every } T \in (0,\infty), t \in [0,T]. \tag{5.25}$$

We will now differentiate expression (5.25) with respect to $y$ on both sides and make a change of variables $x = T-t$:

$$\begin{aligned} \alpha_t(x) = \frac{de(t,T)}{dy} =& \sum_{i=1}^d \sigma_t^i(x)\int_0^t \mathcal{I}_{\sigma^i}(\theta, x+t)\phi_H(t-\theta)\,d\theta \\ &+ \sum_{i=1}^d \int_0^x \sigma_t^i(y)dy\int_0^t \sigma_\theta^i(x+t-\theta)\phi_H(t-\theta)\,d\theta, \end{aligned} \tag{5.26}$$

which exactly coincides with expression (5.22) in the theorem. □

### The change of measure

We are now going to face the change of measure. We need to begin with an important result obtained by a detailed analysis of [Alòs and Nualart [2003]]. Therefore we specify the $d$-dimensional vector space $U$ from the beginning of our set-up in Section 5.1 as $U = \mathbb{R}^d$. Let $\mathcal{H}$ be the Cameron-Martin space associated to the fractional Brownian motion. This is a subspace of the Wiener space with absolutely continuous paths $\omega : [0, \infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$ and $\int_0^\infty |\, \omega'(s)|^2 ds < \infty$. Hence it is also a subspace of $L^2$.
We define $\mathcal{H} = \text{Image } \mathcal{K}$, where

$$\mathcal{K}h(t) := \int_0^t K_H(t,s)h(s)ds, \quad h \in L^2([0,T^*])$$

and the square integrable kernel function $K_H$ for $\frac{1}{2} < H < 1$ is given by

$$K_H(t,s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad t > s, \tag{5.27}$$

with $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and where $\beta$ denotes the beta function defined as $\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt$, for $x > 0$, $y > 0$ [Rudin [2005]].
In [Alòs and Nualart [2003]] it turns out that this is a wisely chosen kernel since we can verify that

$$\int_0^{t \wedge s} K_H(t,u)K_H(s,u)du = R_H(t,s),$$

where $R_H$ is the covariance function of a fractional Brownian motion with Hurst parameter $H$ as defined in chapter 2, expression (2.6). We introduce a linear operator $K^*$ defined by $(K^*\varphi)(s) := \int_s^T \varphi_r \frac{\partial K}{\partial r}(r,s)dr$, where $\varphi$ is a step function. One can show that $K^*$ provides an isometry between $\mathcal{H}$ and $L^2([0,T^*])$, which makes the process $B$ defined by

$$B(t) = B^H((K^*)^{-1}(\mathbf{1}_{[0,t]})), \quad t \in [0,T], \tag{5.28}$$

a standard Brownian motion. Consequently the fractional Brownian motion can be represented by a standard Brownian motion, i.e.

$$B^H(t) = \int_0^t K_H(t,s)dB(s), \tag{5.29}$$

which is essential for the Girsanov theorem in the fractional case:

**Theorem 5.22.** *Let $B^H = (B_1^H, ..., B_d^H)$ be a d-dimensional fractional Brownian motion and let $\{\gamma(t) : 0 \leq t \leq T^*\}$ be an $\mathbb{R}^d$-valued measurable function such that $\int_0^{T^*} \|\gamma(t)\|_{\mathbb{R}^d} dt < \infty$ and $R(\cdot) := \int_0^{\cdot} \gamma(s)ds \in \mathcal{H}$.*
*Then $\tilde{B}^H(t) := B^H(t) - \int_0^t \gamma(s)ds$ is a d-dimensional $\mathbb{Q}_{T^*}$-fBm on $[0, T^*]$ such that*

$$\frac{d\mathbb{Q}_{T^*}}{d\mathbb{P}} = \mathcal{E}(\mathcal{K}^{-1}R \cdot B)_{T^*}, \tag{5.30}$$

*where*

$$\mathcal{E}(\mathcal{K}^{-1}R \cdot B)_{T^*} := \exp\left((\mathcal{K}^{-1}R \cdot B)_{T^*} - \frac{1}{2}\int_0^{T^*} \left\|\mathcal{K}^{-1}R(t)\right\|_{\mathbb{R}^d}^2 dt\right) \tag{5.31}$$

*is the stochastic exponential of $(\mathcal{K}^{-1}R \cdot B)_{T^*}$ and $(\mathcal{K}^{-1}R \cdot B)_{T^*}$ is the usual Itô stochastic integral with respect to the Brownian motion $B$ associated to $B^H$ as outlined in (5.28). In this case we may write*

$$\tilde{B}^H(t) = \sum_{i=1}^{d} \tilde{B}_i^H(t) e_i,$$

*where $\tilde{B}_i^H(t) := B_i^H(t) - \int_0^t \gamma_s^i ds$ is a real-valued independent $\mathbb{Q}_{T^*}$-fBm for each $i = 1, ..., d$. and $(e_i)_{i=1}^d$ is an orthonormal basis.*

This is the usual Girsanov theorem [see Filipovic [2001], Theorem 2.3.3] applied to $(\mathcal{K}^{-1}R \cdot B)_{T^*}$. Hence, with the help of relation (5.29) the change of measure can also be expressed for an fBm $B^H$.

### The no-arbitrage drift condition in the fractional HJM model

We can now go on with the main result for the absence of arbitrage, sort of an equivalent to the famous fundamental theorem of asset pricing by [Harrison and Pliska [1981]]. Therefore we recall that all economic activity is restricted to $[0, T^*]$. We fix the proportional transaction costs $k > 0$. And we define

$$\mathcal{S}_H\sigma_t(\,\cdot\,) := \sum_{i=1}^{d}\bigg(\sigma_t^i(\,\cdot\,)\int_0^t \mathcal{I}_{\sigma^i}(\theta, \cdot + t)\phi_H(t-\theta)\,d\theta \\ + \int_0^{\cdot} \sigma_t^i(y)dy \int_0^t \sigma_\theta^i(\cdot + t - \theta)\phi_H(t-\theta)\,d\theta\bigg),$$

which corresponds to $\alpha_t$ in equation (5.22). We will assume regularity of the volatilities, i.e.

$$\int_0^T \|\mathcal{S}_H\sigma_t\|\,dt < \infty \tag{5.32}$$

for every $T \in (0, \infty)$.

**Theorem 5.23.** *Assume that $\mathcal{I}_{\sigma^i}(t,T)$ is $\lambda$-Hölder continuous on $\Delta^2_{T^*}$ for every $i = 1, ..., d$, where $\frac{1}{2} < \lambda < 1$ (as in Lemma 5.18). Let $\{\gamma(t) : 0 \le t \le T^*\}$ be an $\mathbb{R}^d$-valued measurable function such that $\int_0^{T^*} \|\gamma(t)\|_{\mathbb{R}^d}dt < \infty$ (as in Theorem 5.22) so that*

$$\sigma_t\gamma_t = \mathcal{S}_H\sigma_t - \alpha_t, \quad t \ge 0, \tag{5.33}$$

*holds. Then there exist a quasi-martingale measure for the bond market.*
*Moreover the market is $k$-arbitrage free on $[0, T^*]$.*

We have already seen that if for the drift $\alpha_t$ relation (5.22) holds, the constant expectation condition is satisfied. But now $\alpha_t$ is arbitrary and we define $\mathcal{S}_H\sigma_t$ as in (5.22). In Theorem 5.23 we define $\sigma_t\gamma_t$ exactly as the difference of the two and then construct a new measure $\mathbb{Q}_{T^*}$ using the Girsanov Theorem 5.22. Under this new measure the drift is $\mathcal{S}_H\sigma_t$, which due to definition obviously satisfies (5.22) and so $\mathbb{Q}_{T^*}$-constant expectation holds, which proves the existence of a quasi-martingale measure. We will see in the proof how this works out in detail. By rearranging equation (5.33) to $\alpha_t + \sigma_t\gamma_t = \mathcal{S}_H\sigma_t$ we can interpret $-\gamma$ as the market price of risk, similarly to the classical HJM approach in the preceding chapter.
An important fact we can observe is that the change of measure from the real world to the risk-neutral world only affects the rate of return of the bond but not its volatility. Furthermore, in the real world an investor expects a higher return the riskier the bond is. A most recent example has been the issuance of Greek government bonds, for which Greece has to pay its investors higher coupons due to its higher default risk compared to for example Germany, whose government bonds are used as the benchmark. Since we dispense with incorporating default risk, this example refers to interest rate risk in our case, but with the same reasoning. Opposed to that all bonds have the same expected rate of return under the risk-neutral quasi-martingale measure $Q_{T^*}$, no matter how risky the bonds are.

**Proof:** We recall from Theorem 5.2 that the forward rate $r_t$ is the continuous mild solution of the sde

$$dr_t = (Ar_t + \alpha_t)dt + \sum_{i=1}^{d} \sigma_t^i dB_i^H(t), \quad \text{where } A := \frac{\partial}{\partial x},$$

under the real-world measure $\mathbb{P}$. We also have $d\tilde{B}_i^H(t) = dB_i^H(t) - \gamma_t dt$ from Theorem 5.22 and $\alpha_t = \sigma_t \gamma_t + \mathcal{S}_H \sigma_t$ from (5.33). So by Theorem 5.22 we get

$$dr_t = (Ar_t + \mathcal{S}_H \sigma_t)dt + \sum_{i=1}^{d} \sigma_t^i d\tilde{B}_i^H(t)$$

under the equivalent probability measure $\mathbb{Q}_{T^*}$ given in Theorem 5.22 by equations (5.30) and (5.31). Due to the regularity of the volatilities according to expression (5.32) the sde is well-defined under $\mathbb{Q}_{T^*}$. Now obviously for the new drift $\mathcal{S}_H \sigma_t$ equation (5.22) from Theorem 5.20 holds and hence we have

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[Z_t(T)] = P(0,T), \quad \text{for all } t \in [0,T] \text{ and } T \in (0,\infty).$$

Therefore $\mathbb{Q}_{T^*}$ is a quasi-martingale measure.
Consequently by Lemma 5.18 $\sum_{i=1}^{d} \int_0^t \mathcal{I}_{\sigma^i}(s,T) d\tilde{B}_i^H(s)$ has $\mathbb{Q}_{T^*}$-full support and so $\log Z$ obviously has $\mathbb{Q}_{T^*}$-full support as well. We can conclude absence of arbitrage by Lemma 5.14 and Proposition 5.10 as we have already outlined earlier. $\square$

Note that this implies that if a quasi-martingale measure exists, then it is of the form (5.30), (5.31).

### The bond price as a conditional expectation

In the following theorem we explicitly express the bond price as a conditional expectation using the results from Theorem 5.2. For instance, this can be very useful for pricing contingent claims.

**Theorem 5.24.** *Assume that* $\mathbb{Q}$ *is a quasi-martingale measure. Then the bond price can be expressed by*

$$P(t,T) = e^{\xi(t,T)}\,\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s(0)ds\Big)\big|\mathcal{F}_t\Big],$$

*where the kernel* $\xi(t,T)$ *is given by*

$$\xi(t,T) = -\int_0^t \mathcal{I}_{\mathcal{S}_H\sigma}(s,T)ds - \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s) + G(t,T)$$

*and where*

$$G(t,T) = \sum_{i=1}^d \int_0^t \int_r^T \theta^i(r,u)du\, dM_r^i - \frac{1}{2}\sum_{i=1}^d \int_t^T \Big(\int_r^T \theta^i(r,u)du\Big)^2 d[M^i]_r.$$

$[M^i]$ *denotes the usual quadratic variation of the martingale* $M^i$ *and*

$$\theta^i(r,t) := \int_r^t \sigma^i(s,t)s^{H-\frac{1}{2}}(s-r)^{H-\frac{3}{2}}ds, \quad i=1,...,d \quad \text{for } 0<r<t<\infty.$$

**Proof:** We start backwards and assume the bond price to be

$$P(t,T) = e^{\xi(t,T)}\,\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s(0)ds\Big)\big|\mathcal{F}_t\Big], \tag{5.34}$$

where

$$\begin{aligned}\xi(t,T) = &-\int_0^t \mathcal{I}_{\mathcal{S}_H\sigma}(s,T)ds - \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\\ &- \log \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Big)\big|\mathcal{F}_t\Big].\end{aligned}$$

We will prove that expression (5.34) is equal to the bond price (5.9) in Theorem 5.2 by using $d\tilde{B}_i^H(t) = B_i^H(t) - \int_0^t \gamma_s ds$ and $\mathcal{S}_H\sigma_t = \sigma_t\gamma_t + \alpha_t$:

$$\begin{aligned}P(t,T) = \exp\Big(&-\int_0^t \mathcal{I}_{\mathcal{S}_H\sigma}(s,T)ds - \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\\ &- \log \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Big)\big|\mathcal{F}_t\Big]\Big)\\ \times\, \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(&\underbrace{-\int_t^T r_s(0)ds}_{-\int_0^T r_s(0)ds+\int_0^t r_s(0)ds}\Big)\big|\mathcal{F}_t\Big]\end{aligned}$$

$$
\begin{aligned}
&= \exp\Bigg(-\int_0^t \mathcal{I}_{\mathcal{S}_H\sigma}(s,T)ds - \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)dB_i^H(s) + \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i\gamma}(s,T)ds \\
&\qquad + \log \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Bigg)\Big|\mathcal{F}_t\Bigg]^{-1}\Bigg) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\int_0^T r_s(0)ds + \underbrace{\int_0^t r_s(0)ds}_{\in\mathcal{F}_t}\Bigg)\Big|\mathcal{F}_t\Bigg] \\
&= \exp\Bigg(\int_0^t r_s(0)ds\Bigg)\exp\Bigg(-\int_0^t \mathcal{I}_{\alpha}(s,T)ds - \sum_{i=1}^d \int_0^t \mathcal{I}_{\sigma^i}(s,T)dB_i^H(s)\Bigg) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Bigg)\Big|\mathcal{F}_t\Bigg]^{-1} \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\int_0^T r_s(0)ds\Bigg)\Big|\mathcal{F}_t\Bigg].
\end{aligned}
$$

Now we will show that

$$
\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Bigg)\Big|\mathcal{F}_t\Bigg]^{-1} \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\int_0^T r_s(0)ds\Bigg)\Big|\mathcal{F}_t\Bigg] = P(0,T).
$$

Therefore we plug in the forward-rate process from equation (5.3), isolate the deterministic parts and get

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Bigg)\Big|\mathcal{F}_t\Bigg]^{-1} \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\int_0^T \Bigg(r_0(0) + \int_0^s \underbrace{S(s-u)\alpha(u,u)}_{\alpha_u(s-u)}du \\
&\qquad\qquad + \sum_{i=1}^d \int_0^s \underbrace{S(s-u)\sigma^i(u,u)}_{\sigma_u^i(s-u)}d\tilde{B}_i^H(u)\Bigg)ds\Bigg)\Big|\mathcal{F}_t\Bigg] \\
&= \exp\Bigg(-\int_0^T \Big(r_0(0) + \int_0^s \underbrace{S(s-u)\alpha(u,u)}_{\alpha_u(s-u)}du\Big)ds\Bigg) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\Bigg)\Big|\mathcal{F}_t\Bigg]^{-1} \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(-\int_0^T \sum_{i=1}^d \int_0^s \underbrace{S(s-u)\sigma^i(u,u)}_{\sigma_u^i(s-u)}d\tilde{B}_i^H(u)\, ds\Bigg)\Big|\mathcal{F}_t\Bigg].
\end{aligned}
$$

We will apply the stochastic Fubini theorem for fBm [Krvavich and Mishura [2001], Theorem 1] to the integrals in the second conditional expectation, which causes the two conditional expectations to exactly cancel out. Hence we are left with

$$
\exp\Bigg(-\int_0^T \Big(r_0(0) + \int_0^s \underbrace{S(s-u)\alpha(u,u)}_{\alpha_u(s-u)}du\Big)ds\Bigg) = P(0,T).
$$

All those parts put back together yields the bond price (5.9) from Theorem 5.2, that is

$$P(t,T) = P(0,T)\exp\left(\int_0^t (r_s(0) - \mathcal{I}_\alpha(s,T))ds + \sum_{i=1}^d \int_0^t -\mathcal{I}_{\sigma^i}(s,T)dB_i^H(s)\right) \tag{5.35}$$

and so the equality to expression (5.34) is shown.
From this we will only have to show that

$$-\log \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\right)\big|\mathcal{F}_t\right] = G(t,T) \tag{5.36}$$

in order to finish the proof. Therefore we make use of the fact that a fractional Brownian motion can be represented in terms of a stochastic convolution with respect to a Gaussian martingale:

**Lemma 5.25.** *Let*

$$w(t,s) := \begin{cases} c_1 s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} & \text{for } s \in (0,t), \\ 0 & \text{for } s \notin (0,t), \end{cases}$$

*where* $c_1 = \left(2H\beta(\frac{3}{2}-H, H+\frac{1}{2})\right)^{-1}$. *Then the centered Gaussian process*

$$M_t = \int_0^t w(t,s)dB^H(s)$$

*has independent increments and is a martingale. [Norros et al. [1999], Theorem 3.1]*

Applying this lemma we find that

$$\int_0^t \sigma^i(s,T)d\tilde{B}_i^H(s) = \int_0^t \theta^i(r,t)dM_r^i.$$

Then we will use conditions (5.5),(5.7) and (5.8) and change the order of integration in order to derive

$$\begin{aligned}
\int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s) &= \int_0^T \int_0^{T-s} \sigma^i(u)du\, d\tilde{B}_i^H(s) \\
&= \int_0^{T-s} \int_0^T \sigma^i(u)d\tilde{B}_i^H(s)\, du = \int_r^t \int_0^T \theta^i(r,u)dM_r^i\, du \\
&= \int_0^T \int_r^t \theta^i(r,u)du\, dM_r^i, \quad i = 1, ..., d.
\end{aligned}$$

Hence we can compute the remaining conditional expectation by the characteristic function of a Gaussian distribution, which is in general given by $\varphi_X(t) := \mathbb{E}[e^{-itX}] = e^{it\mu + \frac{t^2}{2}\sigma^2}$, $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+$. So, for $t = -i$ we can conclude

$$\begin{aligned}
\log \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^d \int_0^T \mathcal{I}_{\sigma^i}(s,T)d\tilde{B}_i^H(s)\right)\big|\mathcal{F}_t\right] &= -\sum_{i=1}^d \int_0^t \int_r^T \theta^i(r,u)du\, dM_r^i \\
&\quad + \frac{1}{2}\sum_{i=1}^d \int_t^T \left(\int_r^T \theta^i(r,u)du\right)^2 d[M^i]_r,
\end{aligned}$$

which proves (5.36) and hence the theorem. □

**Remark 5.26.** *As we have already pointed out the conditional expectation formula for bond prices can be used for pricing contingent claims in a formal way. Therefore we assume that* $X \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_T)$ *is a claim which is due at time* $T$. $\mathbb{Q}$ *denotes the quasi-martingale-measure as usual. This gives rise to*

$$P(t,T)\mathbb{E}_{\mathbb{Q}_T}[X|\mathcal{F}_t] = B_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{B_0(T)}|\mathcal{F}_t\right], \tag{5.37}$$

*where* $\mathbb{Q}_T \sim \mathbb{Q}$ *is defined by Girsanov as*

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{P(T,T)}{B_0(T)P(0,T)}.$$

*The formula in (5.37) is very useful as long as one can compute the distribution of* $X$ *under* $\mathbb{Q}_T$ *and its right-hand side can be formally interpreted as the price of the claim at time* $t$.

**Remark 5.27.** *As a final remark we want to state that the bond prices given in Theorem 5.24 or equivalently in Theorem 5.2 are obviously not depending on the transaction costs* $k$. *This is only a matter for the value of the portfolio given in Lemma 5.7. Moreover, prices of contingent claims will be affected by transaction costs as well.*

## 5.3 Conclusion

It is very difficult to compare the classical HJM approach to the fractional one in this chapter, since there are a lot of differences. Due to the specifics of fractional Brownian motion such as the non-existence of the semimartingale property, it is much more difficult to come up with an appropriate set-up. Moreover, the no-arbitrage framework is completely different due to the lack of the semimartingale property of fBm in contrast to standard Brownian motion. Whereas in the Brownian motion case the existence of a unique martingale measure leads to the absence of arbitrage, in the fractional case we come up with a multi-stage process, which we will summarize in the following.
First of all the forward-rate process is given by the Musiela reparametrization in (5.3), a modification we work with throughout this chapter. From this we derive a closed form solution for zero-coupon bond prices in Theorem 5.2. After introducing our portfolio setting, the no-arbitrage framework starts with a quite general no-arbitrage criterion for a bond market with transaction costs in Proposition 5.10. Thereafter we impose a volatility condition in Lemma 5.18, which guarantees the full-support property for the logarithm of our discounted bond price process to hold. By Lemma 5.14 this can be led back, so that the mentioned Proposition 5.10 can be applied and the absence of arbitrage holds. Consequently, a unique choice of the drift of the fractional HJM sde in (5.1) guarantees the existence of a quasi-martingale measure by Theorem 5.23, which is very important for the change of measure in our case.
Finally the zero-coupon bond price is given in a conditional expectation form, which is very useful for pricing contingent claims.
We have already shown in Chapter 2 how to price a coupon bond, i.e. by considering it as a portfolio of zero-coupon bonds. Opposed to that the pricing of defaultable bonds causes much more problems. This is where the approach by Ohashi faces its limitations. For the pricing of defaultable bonds one would have to incorporate credit risk, which is usually done by multi-factor models, where credit risk is modelled by one of the factors. It is an interesting question what a defaultable HJM approach looks like. In the classical case this problem has been faced already, e.g. by [Schönbucher [2006]], but the fractional case still awaits subsequent research.

# 6 Simulations of Interest-Rate Models

## 6.1 Simulations of a Fractional Brownian Motion

We will start with the simulation of a fractional Brownian motion. Therefore we need to define an fBm on a compact interval. Consider the interval $[0, a]$ and let $s \in [0, a]$. An integral over $[0, s]$ is called left-sided, an integral over $[s, a]$ is called right-sided.

**Definition 6.1.** *(Fractional integral and derivative) The right-sided fractional integral of order $\alpha > 0$ on an interval $[0, a]$ of a function $f \in L^1[0, a]$ is defined by*

$$(I_{a-}^{\alpha} f)(s) = \frac{1}{\Gamma(\alpha)} \int_0^a f(u)(u-s)_+^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_s^a f(u)(u-s)^{\alpha-1} du, \; s \in (0, a), \quad (6.1)$$

*where $\Gamma$ denotes the gamma function defined as*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

*for $z \in \mathbb{C}$ with $Re(z) > 0$ [Rudin [2005], Definition 8.17].*
*The right-sided fractional derivative of order $0 < \alpha < 1$ on an interval $[0, a]$ of a function $\phi$ is defined by*

$$(\mathcal{D}_{a-}^{\alpha} \phi)(s) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_0^a \phi(s)(s-u)_+^{-\alpha} ds, \; u \in (0, a). \quad (6.2)$$

*[Samko et al. [1993]]*

**Remark 6.2.** *If $0 < \alpha < 1$ and $\phi(s) = (I_{a-}^{\alpha} f)(s)$, $s \in (0, a)$, then $\mathcal{D}_{a-}^{\alpha}$ can be viewed as an inverse of $I_{a-}^{\alpha}$, since then*

$$f(u) = (\mathcal{D}_{a-}^{\alpha} \phi)(u), \; u \in (0, a).$$

*[Samko et al. [1993]]*

Now we can represent a fractional Brownian motion in terms of a fractional integral on the interval $[0, a]$ with respect to an ordinary Brownian motion.

**Proposition 6.3.** *Let $a > 0$ and let $B^H$ be a fractional Brownian motion with index $H \in (0,1)$ and $B$ a Brownian motion. Then*

$$\big(B^H(t)\big)_{t\in[0,a]} \stackrel{d}{=} \left(\sigma_1(H)\int_0^a s^{H-\frac{1}{2}}\big(I_{a-}^{H-\frac{1}{2}}u^{H-\frac{1}{2}}\mathbf{1}_{[0,t)}(u)\big)(s)dB(s)\right),$$

*where*

$$\sigma_1(H)^2 = \frac{\Gamma(H-1/2)H(2H-1)}{\beta(H-1/2,2-H)} = \frac{\pi H(2H-1)}{\Gamma(2-H)\sin(\pi(H-1/2))}.$$

Since we will focus on the simulations in practice, we do not want to outline the proof of this proposition and refer to the article of [Pipiras and Taqqu [2001], Proposition 3.1].

From this we continue by coming up with a discrete approximation scheme of a fractional Brownian motion. This can be seen as a fractional analogue to the Donsker theorem, e.g. in [Kallsen [2007], Theorem 4.3.10]. We recall the kernel representation of the fBm $B^H$ with respect to the standard Brownian motion $B$ from equation (5.29) as

$$B^H(t) = \int_0^t K_H(t,s)dB(s)$$

with the kernel function $K_H$ as in (5.27)

$$K_H(t,s) := c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, \quad t > s$$

and $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}}$.
In order to come up with an approximation we let $\xi_i^{(n)}$ be i.i.d. random variables with $\mathbb{E}[\xi_i^{(n)}] = 0$ and $\mathrm{Var}(\xi_i^{(n)}) = 1$. Denote

$$B(t)^{(n)} := \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i^{(n)}$$

with $\lfloor x\rfloor$ denoting the greatest integer not exceeding $x$. We know by Donsker's theorem that $B^{(n)}$ converges weakly to $B$. So now we can formulate the most important result for our purposes.

**Theorem 6.4.** *The random walk $B^{H\,(n)}$ is defined by*

$$B^H(t)^{(n)} := \int_0^t K_H^{(n)}(t,s)dB(s)^{(n)} = \sum_{i=1}^{\lfloor nt\rfloor} n\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_H\left(\frac{\lfloor nt\rfloor}{n},s\right)ds\,\frac{1}{\sqrt{n}}\,\xi_i^{(n)},$$

*where the function $K_H^{(n)}(t,\cdot)$ is an approximation to $K_H(t,\cdot)$, i.e.*

$$K_H^{(n)}(t,s) := n\int_{s-\frac{1}{n}}^{s} K_H\left(\frac{\lfloor nt\rfloor}{n},u\right)du.$$

*$B^{H\,(n)}$ converges weakly to the fractional Brownian motion $B^H$.*

For a proof see [Sottinen [2001], Theorem 1].

As we move on with our simulation we will have to outline how to implement the integrals of the random walk $B^{H\,(n)}$. Therefore we work analogously to the approximation in the paper of [Fink et al. [2010], Example 4.2]. First, altogether we get

$$B^H(t)^{(n)} = c_H \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{\frac{1}{2}-H} \int_s^{\frac{\lfloor nt \rfloor}{n}} (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du\, ds\, \frac{1}{\sqrt{n}}\, \xi_i^{(n)}.$$

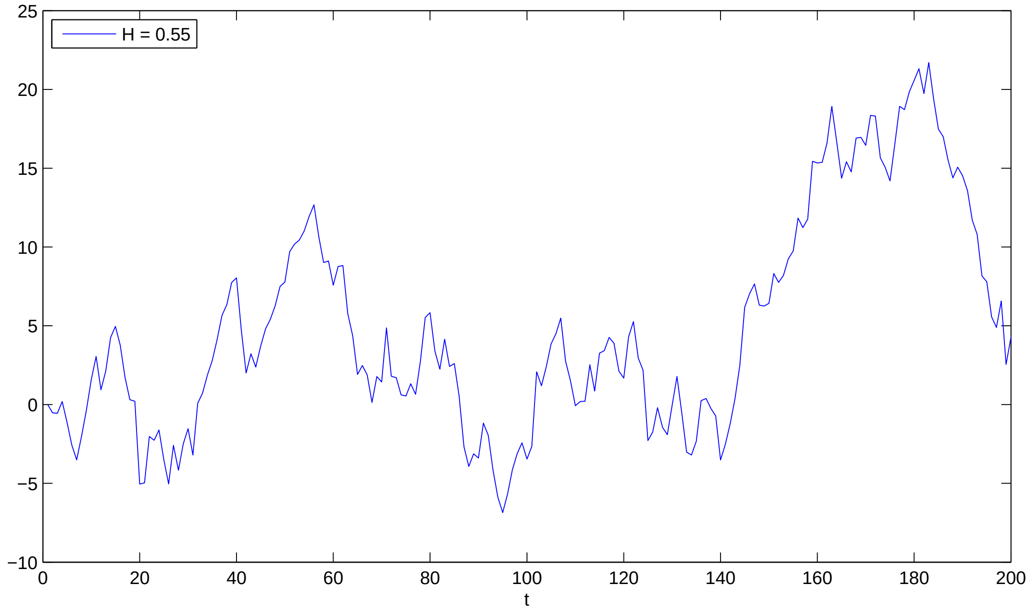
We will obviously have to approximate two integrals for our simulation. We split the interval $[0,t]$ by Theorem 6.4 into sufficiently small intervals $[s_i, s_{i+1}] = [\frac{i-1}{n}, \frac{i}{n}]$, $i = 0, ..., \lfloor nt \rfloor - 1$, with $0 = s_0 \leq s_1 \leq ... \leq s_{\lfloor nt \rfloor} = t$. Moreover, for $i = 0, ..., \lfloor nt \rfloor - 1$, we split the intervals $[s_i, s_{i+1}]$ in order to calculate the inner integral. The partition is $s_i = v_0^i \leq v_1^i \leq ... \leq v_{m_i}^i = s_{i+1}$ for some $m_i \in \mathbb{N}$, which denotes the number of single parts for the partition. Since we consider $n$ very large, it is appropriate to substitute $s$ in the function by $s_i$, because they approximately coincide. We demonstrate the decomposition at first for the inner integral in the following:

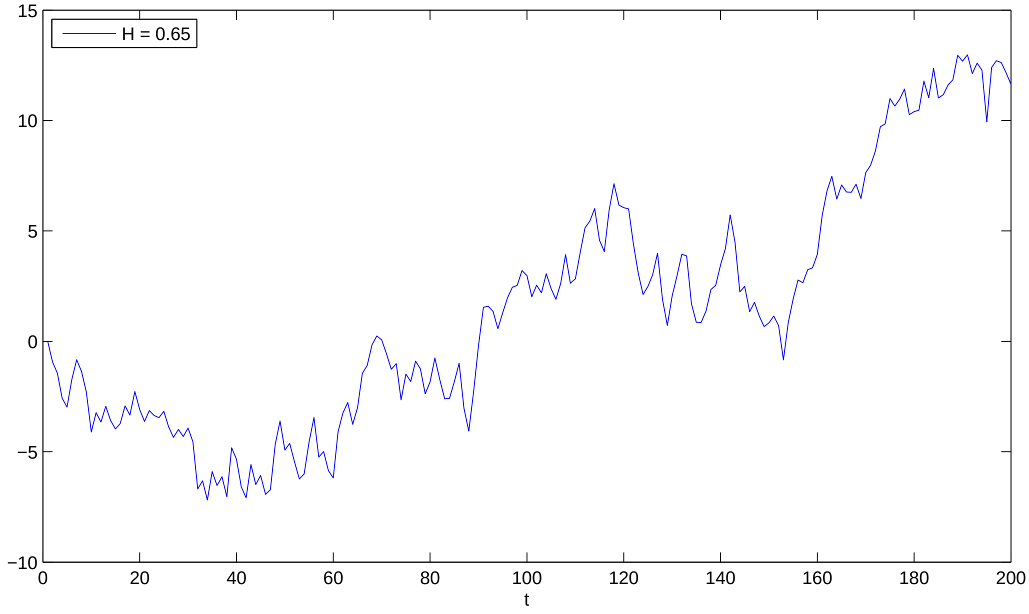
$$\begin{aligned}
&\int_{s_i}^{s_{i+1}} u^{H-\frac{1}{2}} (u-s_i)^{H-\frac{3}{2}} du \\
&\quad = \sum_{j=0}^{m_i-1} \int_{v_j^i}^{v_{j+1}^i} u^{H-\frac{1}{2}} (u-s_i)^{H-\frac{3}{2}} du \\
&\quad \approx \sum_{j=0}^{m_i-1} \frac{(v_j^i)^{H-\frac{1}{2}} + (v_{j+1}^i)^{H-\frac{1}{2}}}{2} \big((v_{j+1}^i - s_i)^{H-\frac{1}{2}} - (v_j^i - s_i)^{H-\frac{1}{2}}\big) \frac{1}{H-\frac{1}{2}},
\end{aligned}$$

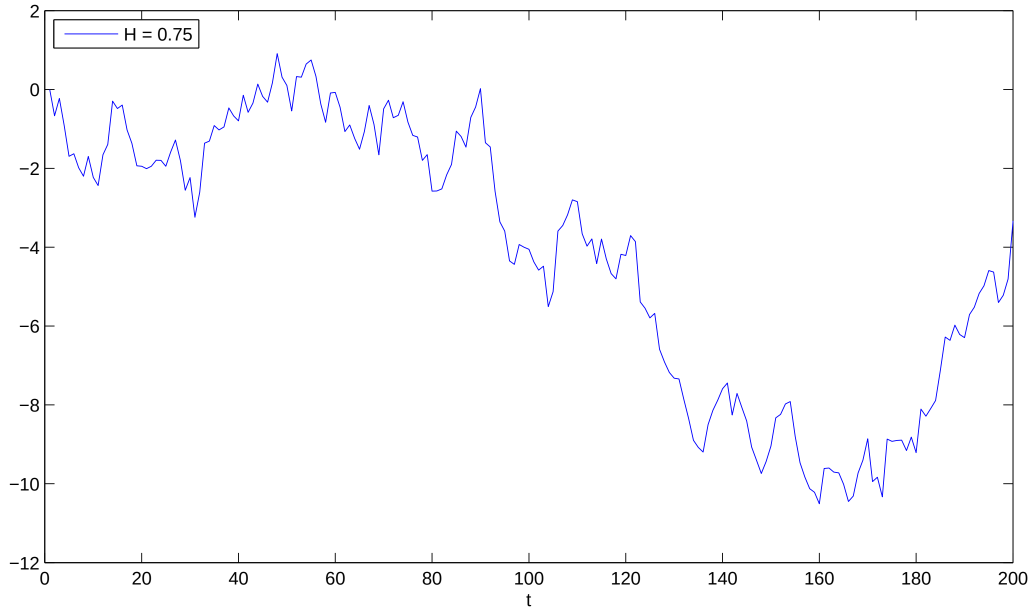
where we use the arithmetic mean for our approximation. All combined and put together this yields

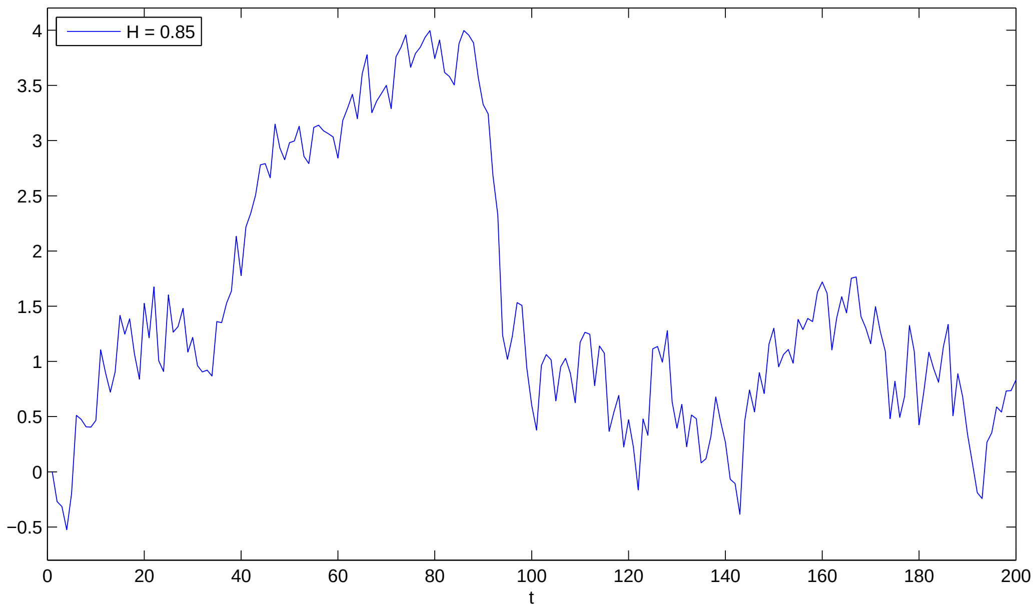
$$\begin{aligned}
B^H(t)^{(n)} \approx & \frac{c_H}{(\frac{3}{2}-H)(H-\frac{1}{2})} \sum_{i=1}^{\lfloor nt \rfloor} n \left( \left(\frac{i}{n}\right)^{\frac{3}{2}-H} - \left(\frac{i-1}{n}\right)^{\frac{3}{2}-H} \right) \\
& \times \left( \sum_{j=0}^{m_i-1} \frac{(v_j^i)^{H-\frac{1}{2}} + (v_{j+1}^i)^{H-\frac{1}{2}}}{2} \big((v_{j+1}^i - \frac{i}{n})^{H-\frac{1}{2}} - (v_j^i - \frac{i}{n})^{H-\frac{1}{2}}\big) \right) \frac{1}{\sqrt{n}} \xi_i^{(n)}.
\end{aligned}$$

By this procedure we simulate the fBm for different Hurst parameters, where we only focus on the long-range dependent case, i.e. $\frac{1}{2} < H < 1$. We choose $n$ large enough, i.e. $n = 400$, and a time horizon of $T = 200$ for demonstration purposes. In our implementation we choose $m_i = 100$ equally for all $i = 0, ..., n-1$. In order to point out the differences for different indices, we simulate the fBm for Hurst parameters close to 0.5 and 1 and some in between, i.e. $H = 0.55, 0.65, 0.75, 0.85, 0.95$. This provides us with the following plots of the paths of fractional Brownian motions.


H = 0.55
25
20
15
10
5
0
−5
−10
0
20
40
60
80
100
120
140
160
180
200
t


H = 0.65
15
10
5
0
−5
−10
0
20
40
60
80
100
120
140
160
180
200
t


H = 0.75
2
0
−2
−4
−6
−8
−10
−12
0
20
40
60
80
100
120
140
160
180
200
t


H = 0.85
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
0
20
40
60
80
100
120
140
160
180
200
t

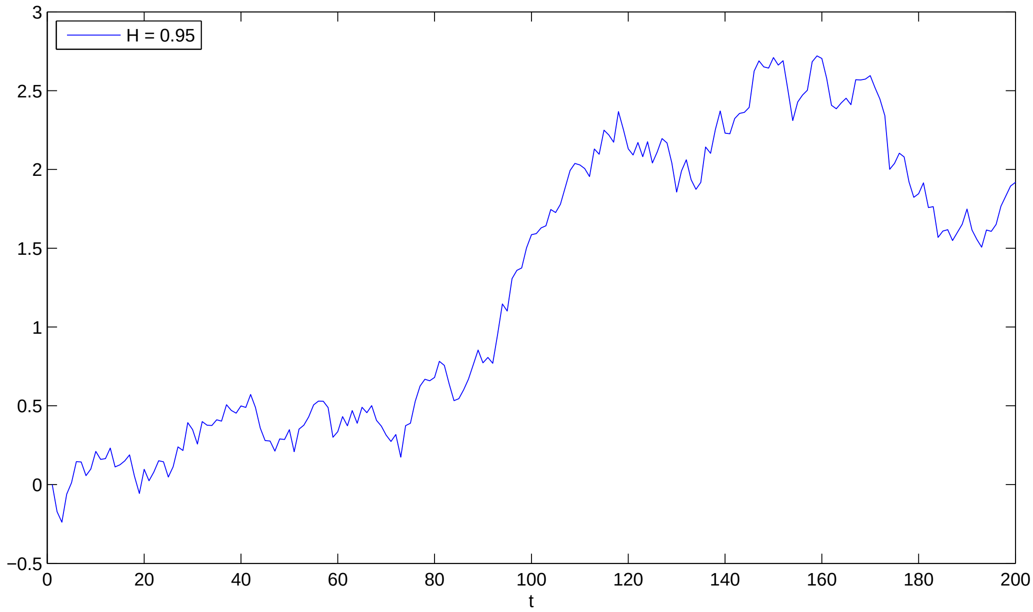
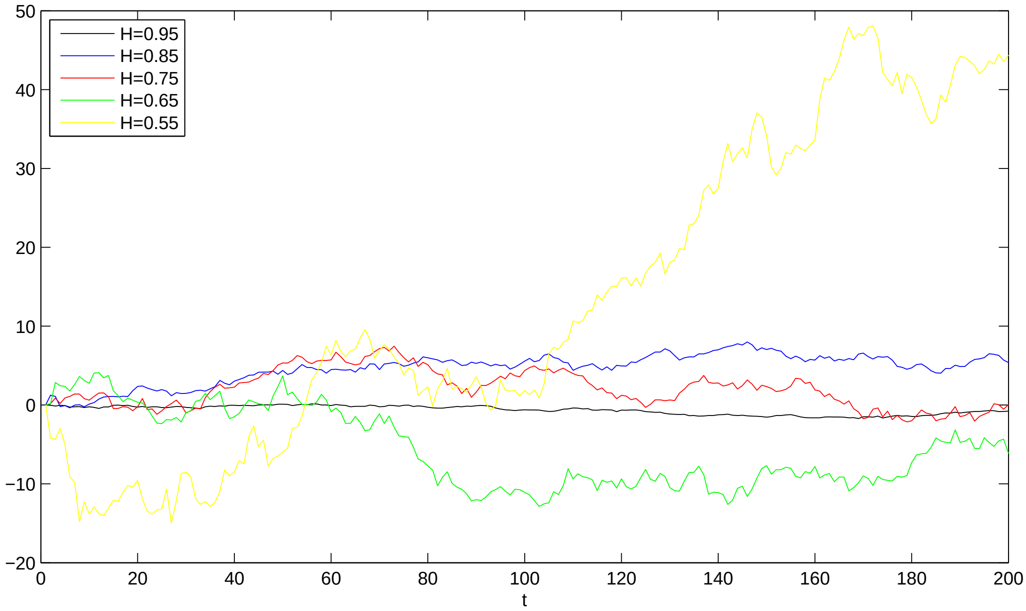



Figure 6.1: Simulations of the paths of fractional Brownian motions for various Hurst parameters

By looking at the scales of the y-axes of our plots we can easily determine that the paths are much more volatile the lower the Hurst parameters are. That is, for instance, for $H = 0.95$ the path is between values of $-0.5$ and 3 whereas for $H = 0.55$ the paths range from $-10$ up to 25. We point out that for $H = 0.55$ there are only slight correlations of the increments compared to the independent increments of a Brownian motion, because with a limit argument of $H \to 0.5$ we arrive at the ordinary Brownian motion as we pointed out in the preliminaries already. For larger Hurst parameters correlations increase.
Moreover, the paths are smoother for larger $H$, which can be explained by the concept of $p$-variation by [Mikosch and Norvaisa [2000], Proposition 2.2], which follows by a combination of the results in [Fernique [1964]] and [Kawada and Kôno [1973]].

In order to emphasize the various characteristics of the path movements regarding the different Hurst parameters we will plot another five simulated paths with the different Hurst parameters into one single graph. One can compare the amplitudes of the several paths more easily in here, which illustrates the differences we have already outlined above.



## 6.2 Simulations of Stochastic Differential Equations

In order to simulate the model-specific stochastic differential equations we will introduce the Euler-Maryuama procedure for an sde with an fBm as the driving noise analogously to the general form with a standard Brownian motion [see Pulch [2007]]. Consider the general sde

$$\begin{aligned} dX(t) &= a(t, X(t))dt + b(t, X(t))dB^H(t), \quad t \in [0, T], \\ X(0) &= X_0, \end{aligned}$$

where $a, b : [0, T] \times \mathbb{R}^l \to \mathbb{R}^l$, $l \geq 1$, and $(B^H(t))_{t \geq 0}$ is a fractional Brownian motion as usual. The interval $[0, T]$, with $T$ being the final point in time that we can simulate, is split up into equidistant parts with steps $h = \Delta t = \frac{T}{n}$. Since we will interpret our results on interest-rate markets from now on, we consider $T$ as given in years.
Let $\Delta B_j^H := B^H(t_{j+1}) - B^H(t_j)$ denote the increments of the fBm. Consequently, for the approximations $\tilde{X}_j = X_{t_j}$ we get the recursive formula

$$\tilde{X}_{j+1} = \tilde{X}_j + a(t, \tilde{X}_j)h + b(t_j, \tilde{X}_j)\Delta B_j^H, \quad j = 0, 1, ..., n-1, \tag{6.3}$$

with $\tilde{X}_0 = X_0$ and $B^H(0) = 0$ by definition.

In order to provide examples in practice for the various stochastic differential equations of the models and to compare their differences we start off with the fractional Vasicek model and specify our economy. Afterwards we will continue with the dynamics of our fractional HJM model. We will then compare those to the ones of the classical HJM model.

**The Vasicek model**

We recall the short-rate dynamics for the Vasicek model given by the sde (3.2) and modify it by including an fBm $B^H$ as the driving noise factor, that is

$$dr(t) = k(\theta - r(t))dt + \sigma\, dB^H(t), \quad r(0) = r_0.$$

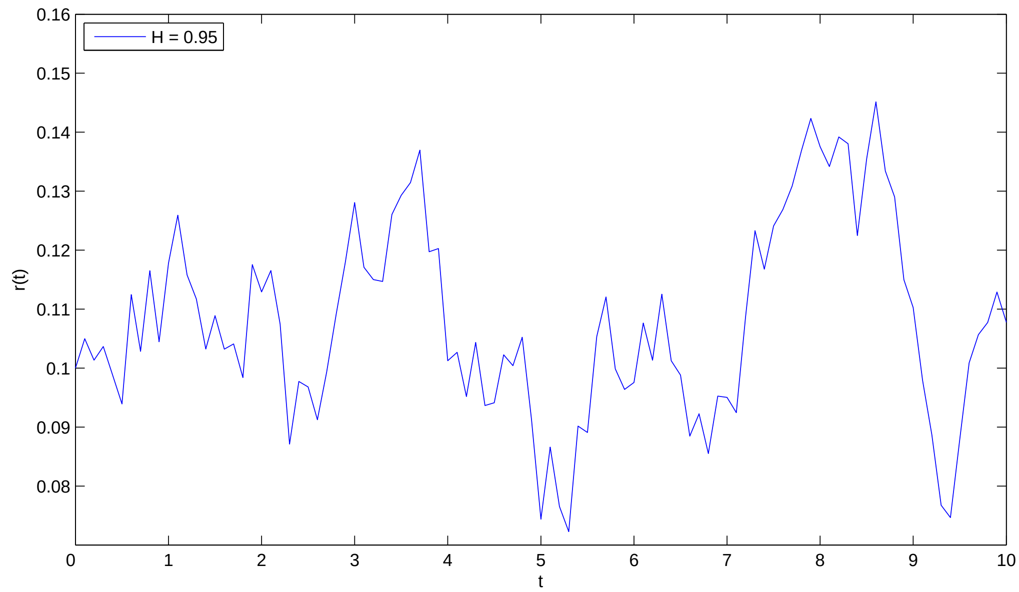
We assume for the fractional Vasicek model $k = 0.7$, mean $\theta = 0.1$, diffusion coefficient $\sigma = 0.15$ and an initial value of $r(0) = 0.1$. With these specifications we have

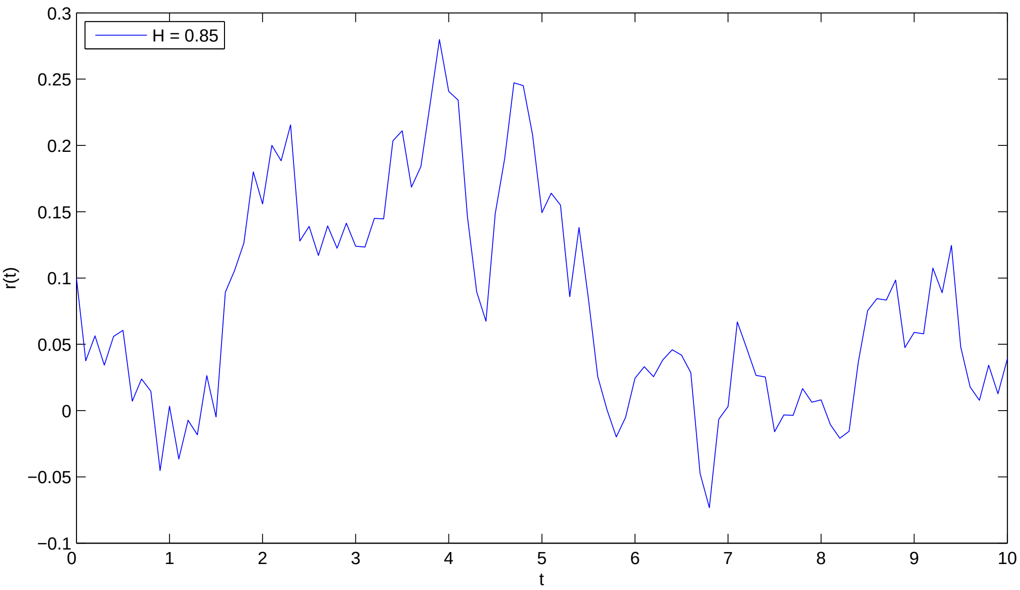
$$dr(t) = 0.7(0.1 - r(t))dt + 0.15dB^H(t), \quad r(0) = 0.1.$$

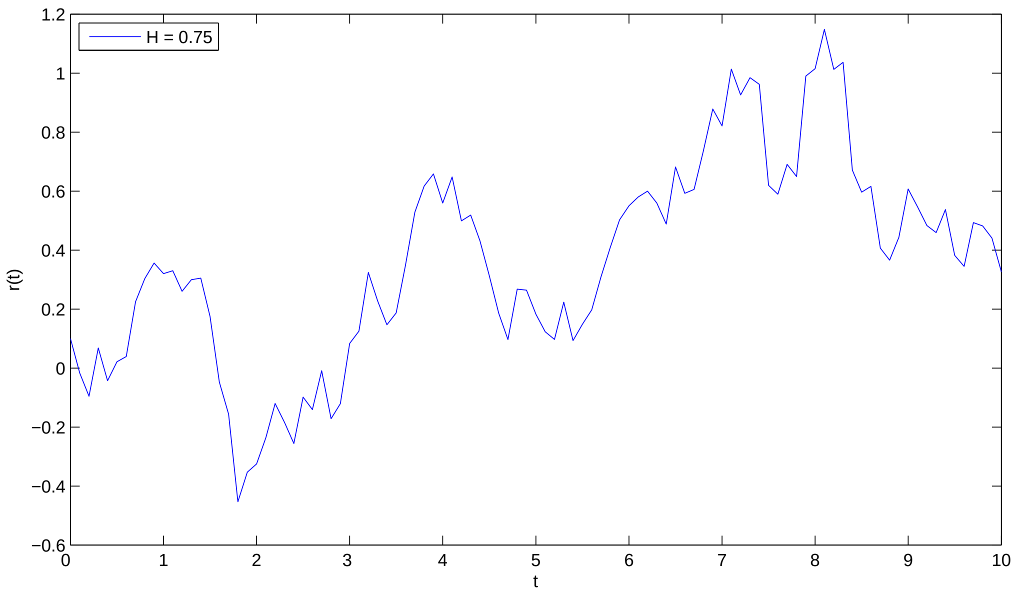
In order to implement this sde by applying the Euler-Maryuama procedure as in equation (6.3), we choose the number of discretization steps to be $n = 100$ and the final simulation point $T = 10$. By Euler-Maryuama this yields the approximation

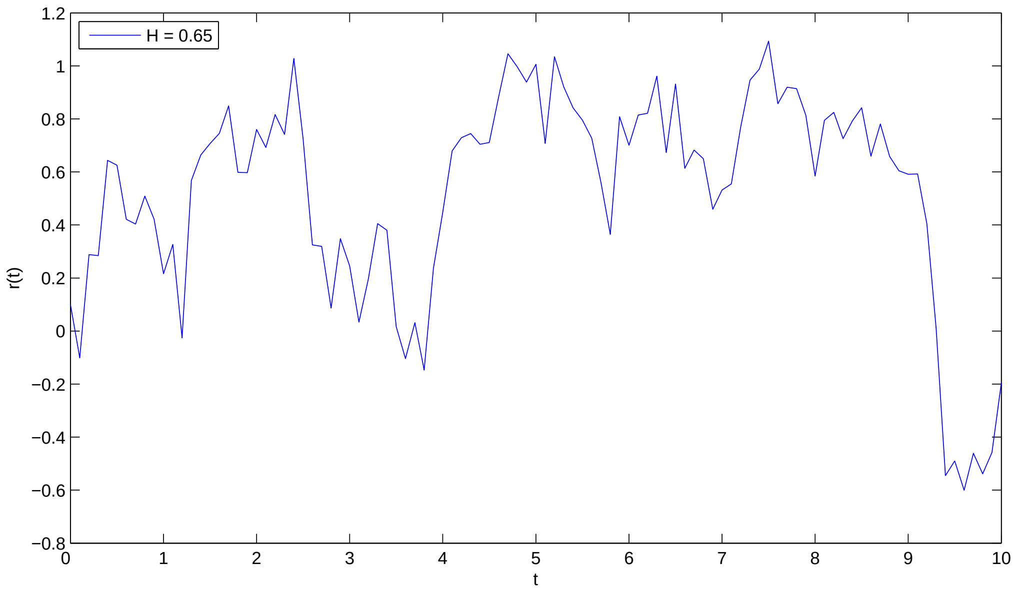
$$\begin{aligned} \tilde{r}_{j+1} &= \tilde{r}_j + k(\theta - \tilde{r}_j)h + \sigma\Delta B_j^H \\ &= \tilde{r}_j + 0.7(0.1 - \tilde{r}_j)h + 0.15\Delta B_j^H, \quad j = 0, 1, ..., 99, \end{aligned}$$

with the initial value $\tilde{r}_0 = 0.1$ and steps $h = \frac{1}{10}$. Hence we get the simulated short-rate dynamics for different Hurst parameters, i.e. $H = 0.95$, $0.85$, $0.75$, $0.65$, $0.55$, pictured by the graph in figure 6.2.


H = 0.95
r(t)
t
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0
1
2
3
4
5
6
7
8
9
10


H = 0.85
r(t)
t
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0
1
2
3
4
5
6
7
8
9
10


H = 0.75
r(t)
t
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0
1
2
3
4
5
6
7
8
9
10


H = 0.65
r(t)
t
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
1
2
3
4
5
6
7
8
9
10

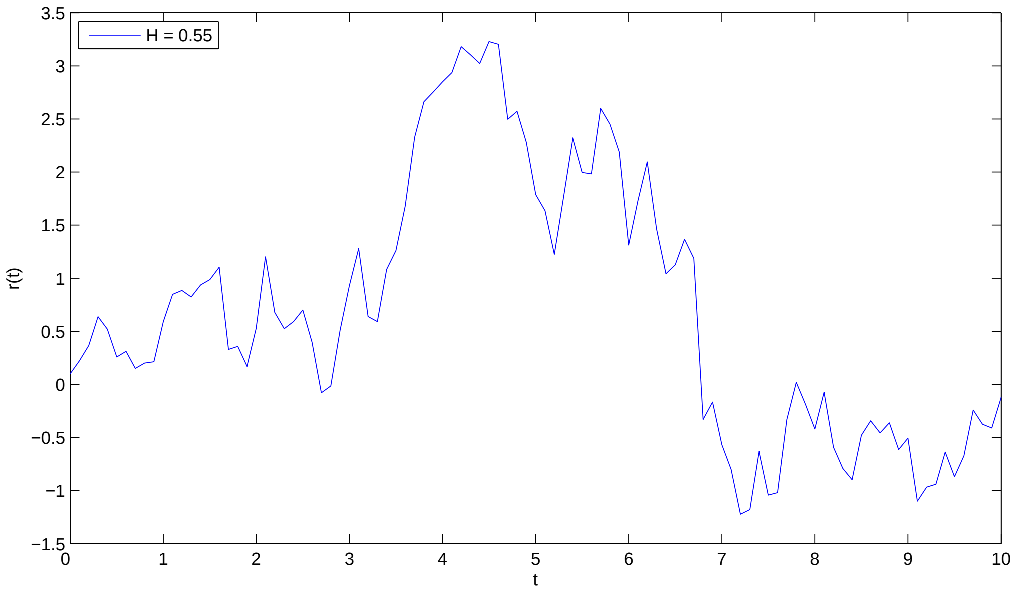



Figure 6.2: Simulations of the fractional Vasicek interest-rate dynamics for various Hurst parameters

We observe negative values for the interest rates, which is not conform with reality in most cases, even though in times of deflation we might get negative interest rates for short periods of time. However, this is a general problem of the Vasicek model as outlined in Section 3.1. By comparing the realizations of the stochastic differential equations given by the plots in the figures above we see that the most realistic results for common economic scenarios are achieved for a large Hurst index, i.e. $H = 0.95$. In this case interest rates are mostly positive and not too volatile. They are limited to values between $-0.15$ and $0.3$ in this case, which is quite reasonable given an initial value of $\tilde{r}_0 = 0.1$. Opposed to that we get different results for a choice of $H = 0.55$. The values of the interest rates in this case are very volatile and lie within an interval of $[-1.5, 3.5]$, which is not realistic in most cases taking into account an initial value of $\tilde{r}_0 = 0.1$. Still, we have to consider that we have chosen the parameters of the sde with respect to $H = 0.95$. For lower Hurst parameters better results could be achieved by a different choice of the parameters of the sde, i.e. a choice that suits a model with these specific Hurst parameters better. Anyways, our tests with many different parameters have shown that in most cases we get the best and most realistic results by choosing a Hurst parameter close to one. In times of extraordinary economic evolutions or crises, however, a different choice for $H$ has to be considered depending on historical estimations.
Furthermore we point out that one can see the mean reversion inherent in the dynamics very well. It is apparent that on average the values level out at the mean 0.1.

### The fractional vs. the classical Heath-Jarrow-Morton model

We will now focus on our fractional HJM model. We already know that we will have to simulate the forward-rate dynamics in this case, opposed to the short-rate dynamics in the Vasicek model. We recall the stochastic differential equation representing these dynamics, but for this purpose with the usual notation and not in the Musiela reparametrization form, i.e.

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{d} \sigma^i(t,T)dB^H(t), \quad t \in [0,T].$$

We are going to implement a model proposed by [Schönbucher [2006], p.168]. In this model drift and diffusion coefficients are given by

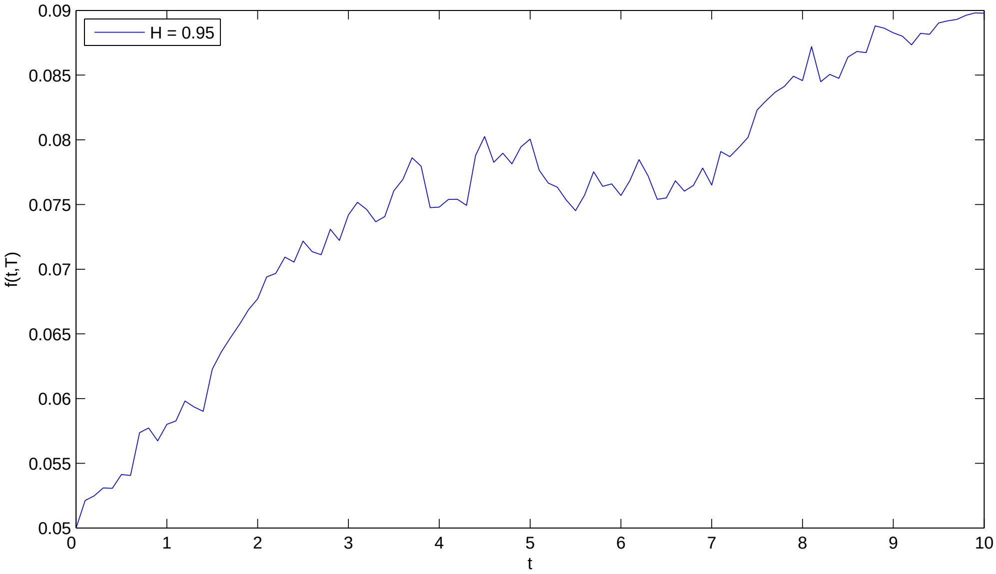
$$\begin{aligned} \alpha(t,T) &= \sigma(t)^2 e^{-a(T-t)}\big(1 - e^{-a(T-t)}\big) \quad \text{and} \\ \sigma(t,T) &= \sigma(t)e^{-a(T-t)}, \end{aligned} \tag{6.4}$$

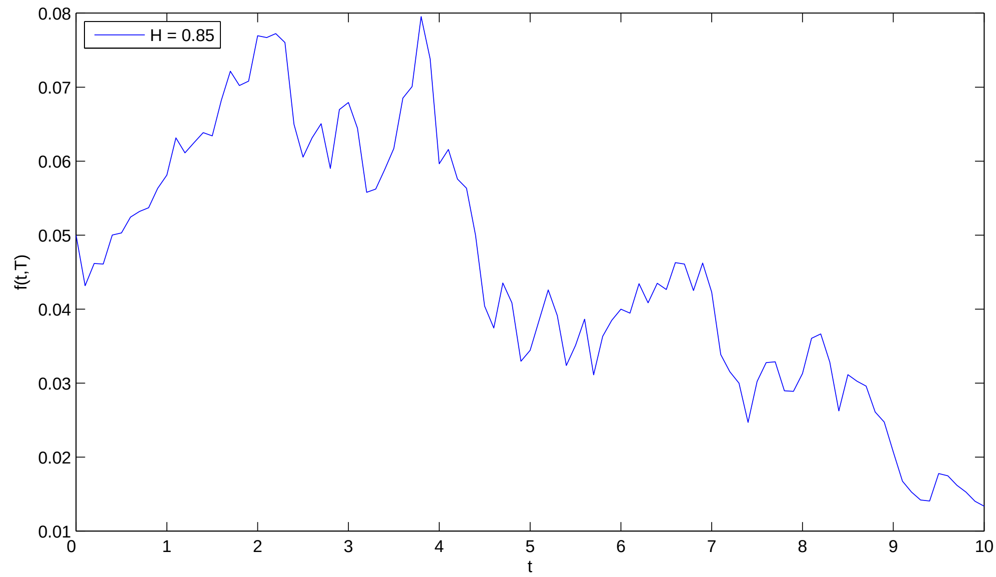
where $a \geq 0$ and $\sigma(t)$ is a time-dependent, deterministic function. Since the drift approaches zero for $t \to T$ and $e^{-a(T-t)}$ in the diffusion approaches one, we will choose $\sigma(t)$ as a decreasing function in order to avoid an overly strong influence of the fBm. This would cause large jumps towards the end of our dynamics. This is also in line with reality. The greater the time to maturity, the more sensitive it will be to changes in interest rates, in our case the forward rate. Thus, a 1-year bond will change less than a 10-year bond or a 30-year bond, but a 1-year bond will have the same sensitivity to interest rates as a 30-year bond with 1 year to maturity. Thus, bonds with longer remaining terms will be more volatile than those with less time to maturity.
Intuitively this can be easily illustrated by the following reasoning. The present value of the interest payments and of the principal diminish as interest rates rise. Likewise, the present value increases when interest rates decrease. Equivalently for the time to maturity - the greater the bond's time to maturity, the lower the present value of the bond's payments. Because the present value of any future payment is inversely proportional to the length of time and to interest rates, rising interest rates will cause the prices of bonds with long remaining terms to drop more than those with shorter remaining terms. On the other hand, if interest rates drop, then the present value of each payment increases proportionately.

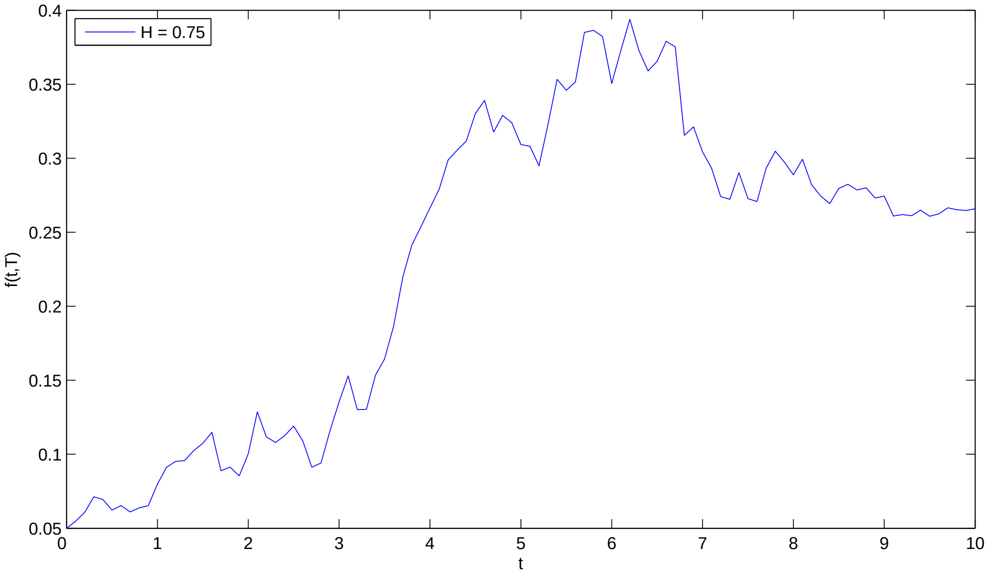
Additionally to Schönbucher's model we have to realize our approach to the model incorporating $d$ fractional Brownian motions. Therefore we will simulate $d$ stochastic differential equations with $d$ fractional Brownian motions as outlined above and then combine them by introducing weights $w_i$, $i = 1, ..., d$, so that $\sum_{i=1}^{d} w_i = 1$. Formally that is

$$\begin{aligned} & df(t,T) = \sum_{i=1}^{d} df_i(t,T), \\ \text{where} \quad & df_i(t,T) = \frac{\sigma_i(t)^2}{w_i} e^{-a(T-t)}\big(1 - e^{-a(T-t)}\big)dt + \sigma_i(t)e^{-a(T-t)}dB^H(t), \quad i = 1, ..., d. \end{aligned}$$

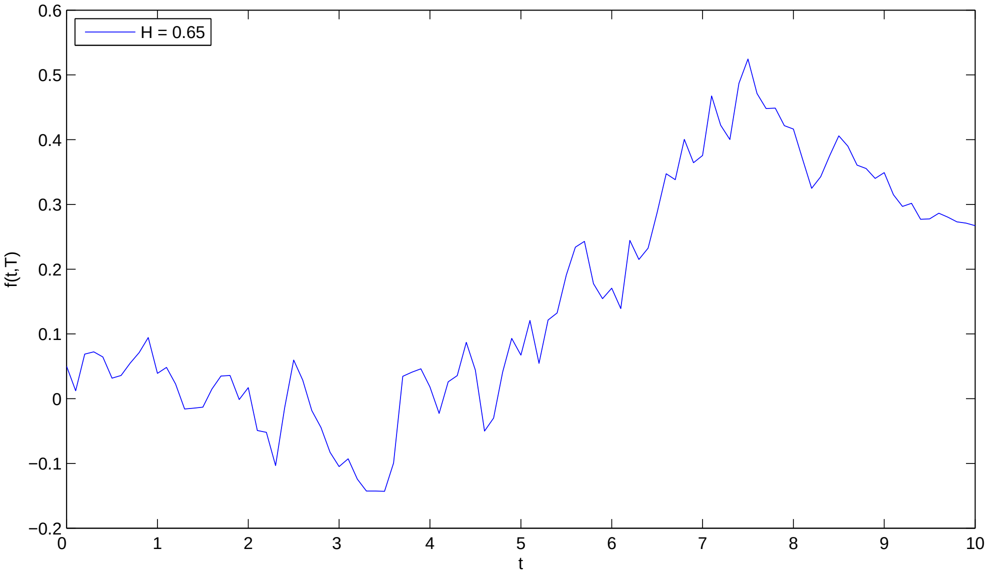
In order to implement the sde, once again we make use of the Euler-Maryuama procedure and assume $n = 100$ and $T = 10$. We start with an inital value of $f(0, T) = 0.05$. In a second step we set $d = 1$, as a simplification for now, and we set $\sigma(x) = 0.12-0.12x/(n+1)$, $x = 0, ..., n$, as a linearly decreasing function, which does not arrive at zero. We choose $a = 0.2$, since for any other choice the forward rates decrease or increase too fast given this volatility function. We carry out these simulations for various Hurst parameters, i.e. $H = 0.95$, $0.85$, $0.75$, $0.65$, $0.55$.





H = 0.75
f(t,T)
t
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6
7
8
9
10

H = 0.65
f(t,T)
t
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
0
1
2
3
4
5
6
7
8
9
10

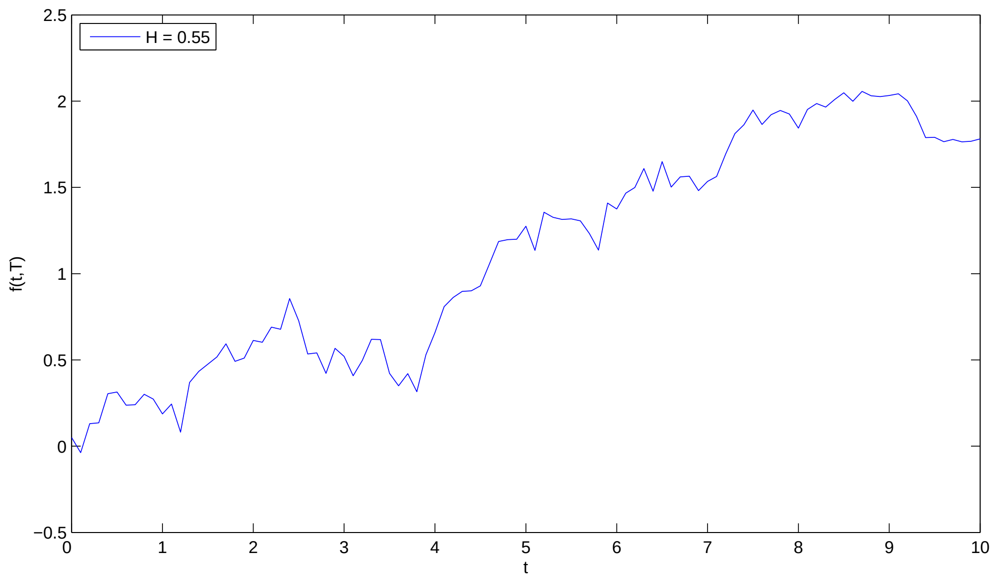
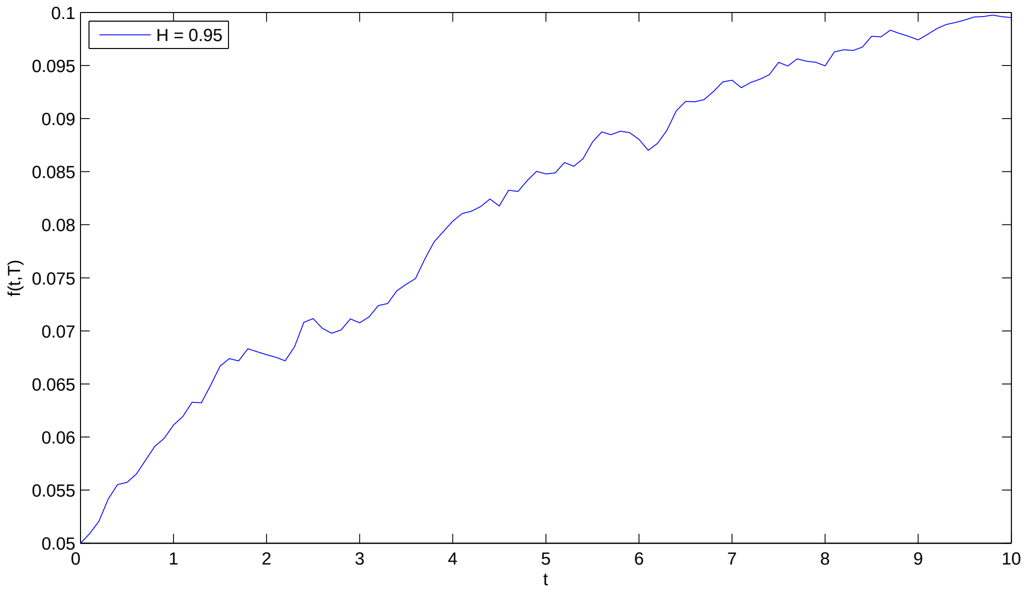



Figure 6.3: Simulations of the fractional HJM interest-rate dynamics for various Hurst parameters

Similarly to our simulations of the Vasicek dynamics, we observe that for $H = 0.95$ the forward rate movements are not as volatile as for lower Hurst parameters, which is more realistic when compared to forward rates in the markets under common economic scenario assumptions. For $H = 0.95$ we only observe positive forward rate values whereas for lower Hurst parameters we also get negative values. Once again we have to mention that we might be able to fix this problem by another choice of the parameter $a$, since $a = 0.2$ has been chosen with respect to the presumably best Hurst parameter $H = 0.95$, similarly to what we pointed out in the Vasicek dynamics above. However, we usually get the most reasonable results given an initial forward rate value of $f(0, T) = 0.05$ for the Hurst parameter $H = 0.95$.
In order to fit the model to the initial forward rate curve, one would estimate correlations and volatilities empirically and then implement the observed long-range dependence. This will influence the choice of the Hurst parameter. Thereafter a model can be fitted by a different choice of the parameter $a$.

We can extend this notion and include more independent fractional Brownian motions, i.e. $d = 3$ and $\sigma_1(x) = 0.1 - 0.1x/(n+1)$, $\sigma_2(x) = 0.12 - 0.12x/(n+1)$ and $\sigma_3(x) = 0.15 - 0.15x/(n+1)$, $x = 0, ..., n$. In the following we will always allocate equal weights to the various noise factors, but we could also use different weights for each fBm depending on their empirically estimated influence. We stick to $a = 0.2$ and this time we only consider $H = 0.95$, which obviously proved to yield the best results anyways.




We can see a finer dynamics structure that is not quite as volatile as in the case with only one fBm driving the dynamics, which naturally stems from the fact that the three noise factors level each other out. The approach of including several fractional Brownian motions into our dynamics enables us to incorporate different noise factors such as macroeconomic factors like gross domestic product or volatilities into our forward rate modelling. Hence, we are more flexible in our modelling.

In order to demonstrate the superiority of the fractional HJM model compared to the classical HJM model we will contrast their dynamics. We will simulate the evolution of the sde of the fractional model with Hurst index $H = 0.95$ and the evolution of the sde of the classical model with a concurrent choice of coefficients. We choose $d = 2$ and work with the volatility functions $\sigma_1(x) = 0.1 - 0.1x/(n+1)$, $\sigma_2(x) = 0.12 - 0.12x/(n+1)$, $x = 0, ..., n$. We maintain $a = 0.2$ and print both the forward rate evolution driven by fBm and the forward rate evolution driven by Bm into one single plot.

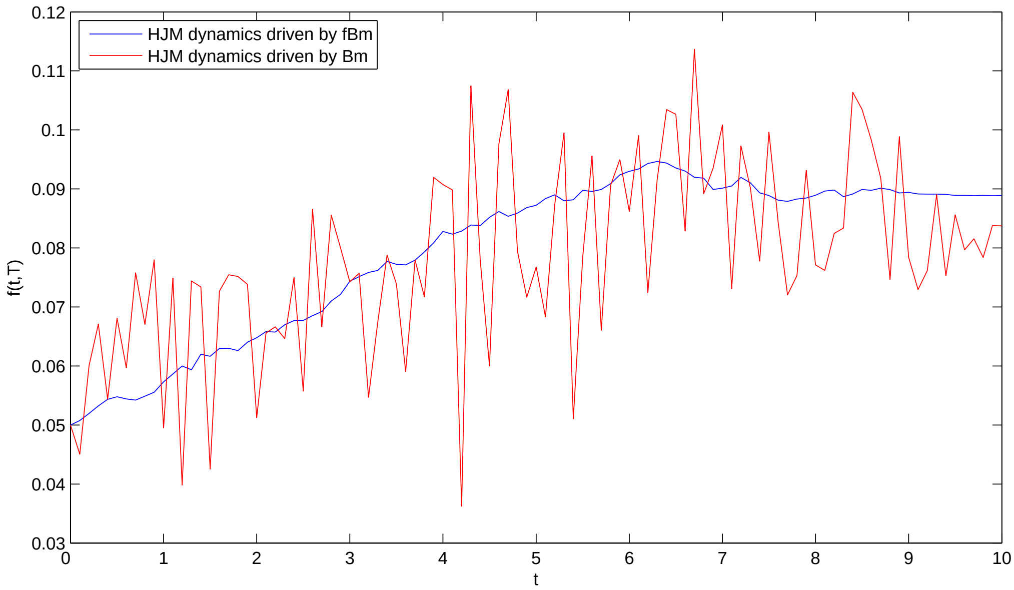



Figure 6.4: HJM dynamics driven by fBm with $H = 0.95$ vs. HJM dynamics driven by Bm

We can see that the fluctuations in the fractional case are by far not as volatile as for the forward rates driven by standard Brownian motion. The smoother evolution is in most cases a more realistic approach to a HJM model considering forward rates in the markets and as we have already outlined we can fit the fractional model better to estimations by an appropriate choice of the Hurst parameter. However, we want to stress that the forward rate values driven by Brownian motion are not as volatile as they might appear at first glance, which can be verified by a look at the scales of the plot. The Bm-driven forward rates lie in the interval $[0.03, 0.12]$, whereas the forward rates driven by fractional Brownian motion are within $[0.05, 0.1]$.

## 6.3 Simulations of HJM Bond Prices

Finally, as the most complex task, we simulate prices of zero-coupon bonds by implementing equation (5.9) from Theorem 5.2. However, we will have to establish a slight modification due to our choice of the coefficients in (6.4). This will force us to implement the following formula:

$$
\begin{aligned}
P(t,T) &= P(0,T)\exp\left(\int_0^t (r_s(0) - \mathcal{I}_\alpha(s,T))ds + \sum_{i=1}^d \int_0^t -\mathcal{I}_{\sigma^i}(s,T)dB_i^H(s)\right) \\
&= P(0,T)\exp\left(\int_0^t \left(r_s(0) - \sum_{i=1}^d \int_0^{T-s} \frac{\sigma_i(x)^2}{w_i} e^{-ax}\left(1-e^{-ax}\right)dx\right)ds \right. \\
&\qquad \left. - \sum_{i=1}^d \int_0^t \int_0^{T-s} \sigma_i(x)e^{-ax}dx\, dB_i^H(s)\right), \quad (t,T)\in\Delta^2.
\end{aligned}
\tag{6.5}
$$

We fix the maturity at $T = 10$ and set the discretization frequency to $n = 100$. As an inital bond price at time $t = 0$ we use $P(0,T) = 0.5$. We use the same simulated fractional Brownian motion terms for the noise factors in the bond-price formula (6.5) as for the forward rates $r_s(0)$ used in this formula as well. We choose two fractional Brownian motions as driving noises with volatility functions $\sigma_1(x) = 0.1 - 0.1x/(n+1)$ and $\sigma_2(x) = 0.12 - 0.12x/(n+1)$, $x = 0, ..., n$. Since we have seen the best results for $H = 0.95$, we will focus on this case for our bond price simulation. We will model the bond prices for all times $t \in [0, 10]$.

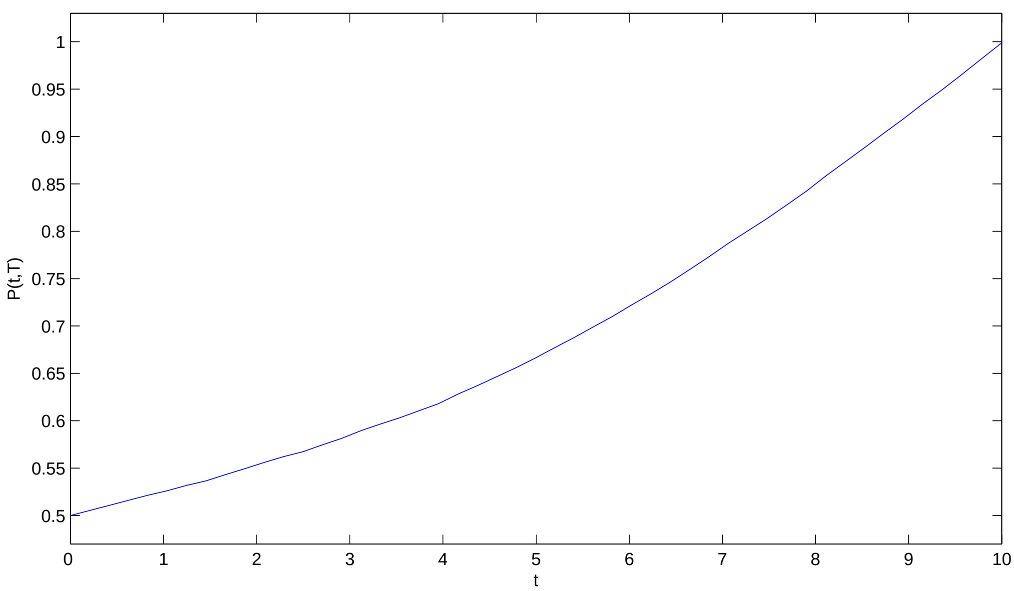



Figure 6.5: Bond price simulation for $t \in [0, 10]$

We can see that for $t = 10$ we get $P(10, 10) = 1$ as demanded for default-free bond prices.

In order to compare those bond prices driven by fractional Brownian motion to bond prices driven by standard Brownian motion we plot them into one graph for $t \in [0, 10]$.

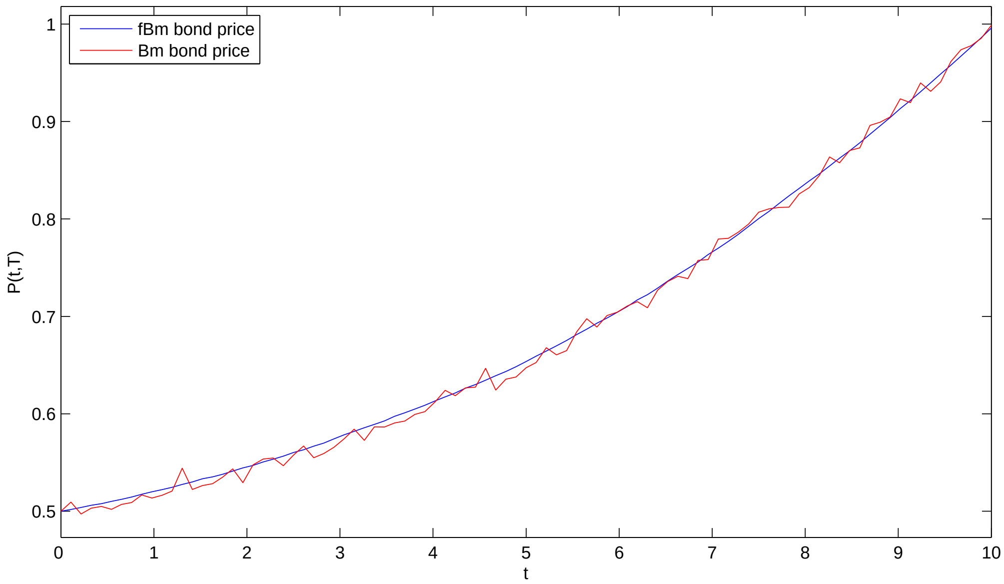



Figure 6.6: Bond price driven by fBm vs. bond price driven by Bm

We observe that bond prices driven by fractional Brownian motion are less volatile than bond prices driven by standard Brownian motion, as one might assume taking into account the results for the simulations of the paths of the fractional Brownian motions and the forward-rate dynamics as well. The fBm-bond price curve is very smooth compared to the Bm-bond price curve. Moreover, we can see that the prices are on average at about the same levels. The less volatile fractional prices can be considered better simulations of real market prices of zero-coupon bonds under most economic scenarios, since we only consider interest rate risk when modelling our bond prices and we dispense with incorporating default risk.

## 6.4 Conclusion

In contrast to [Fink et al. [2010]] we get a different result when varying Hurst parameters. In the mentioned paper bond prices are calculated in a fractional Vasicek model at time $t = 0$ and compared for different Hurst parameters. Larger bond prices are calculated the larger the Hurst parameters are, which can be considered a more realistic approach in that case. However, a comparison to our fractional Heath-Jarrow-Morton model is difficult, since we work with forward rates whereas the Vasicek model is based on short rates. In our summary we will refer to this by explaining how this could be interesting for further research.

Our simulations show that bond prices are less volatile the larger the Hurst parameters are, but on average they lead to about the same level of bond prices as for lower Hurst parameters. However, a less volatile bond price curve can be considered a more realistic scenario under certain economic specifications. By all means a better fitting to historical estimates can be achieved by our fractional approach.

Moreover, our results for bond prices can be led back to the simulations of fBm paths, since the model is driven by those fractional Brownian motions. We already concluded that their paths are less volatile than those of Bm, which consequently influences bond prices in the same manner. The same finding holds for the simulations of the forward-rate dynamics, which are driven by fBm, too.

For our results we have experimented a lot with many different assumptions on the model parameters such as Hurst parameters, volatilities or number of noise factors, but we only pictured the best and most reasonable results and provided graphs where we could illustrate the particular differences very well.

# 7 Summary

This thesis mainly focuses on the fractional approach for the Heath-Jarrow-Morton model and all the preceding chapters aim at a better understanding of this model. Therefore we introduced into interest-rate markets with some basic definitions and ideas. Subsequently we worked through the necessary mathematical theory in order to create a background for being able to understand the sophisticated fractional HJM framework. After having given the basic definitions and properties of fractional Brownian motion we explained an integration theory with respect to fBm, which was much more sophisticated than the ordinary stochastic calculus with respect to Brownian motion.

Thereafter we explained some important short-rate models as an introduction into interest rate modelling. As a very simple model we outlined the Vasicek model, which was already established in 1977. We mentioned two of its extensions namely the Cox-Ingersoll-Ross model and the Hull-White model, which tackle two of the problems of the Vasicek model, i.e. negative interest rates and the poor fitting of the initial term structure, respectively.

Afterwards we derived the classical Heath-Jarrow-Morton model in Chapter 4, a more general interest-rate model. We found that it is possible to derive the short-rate models from this HJM framework, which we examplarily conducted for the Hull-White model. A big difference in contrast to the short-rate models was that we had to come up with a no-arbitrage theory due to the change of measure from the real world to the risk-neutral world. This was not necessary for the short-rate models since we started modelling under the risk-neutral measure straight away. The classical HJM model led us to the fractional one, the most important part of this thesis.

We found that there are big differences between those two, most importantly the no-arbitrage framework. This necessitated a completely different approach, since we could not use the semimartingale property of Brownian motions anymore. In contrast we made use of the full-support property in order to give conditions for the absence of arbitrage and finally end up with the fractional analogue to the Heath-Jarrow-Morton no-arbitrage condition. We have summarized the line of argumentation in the conclusion of Chapter 5 in more detail already. Based on this new framework we provided a closed form solution for zero-coupon bond prices depending only on observables and the forward rate volatilities and we derived a conditional expectation form of zero-coupon bond prices, which is useful for pricing contingent claims.

Finally, the simulations in Chapter 6 illustrated our results and the particular differences between the models very well. We came up with the finding that interest rate modelling in the fractional Heath-Jarrow-Morton framework is a more realistic and flexible approach than modelling with ordinary Brownian motions, due to a number of reasons provided in Chapter 6. Most importantly our approach provides the opportunity to incorporate the long-range dependence which is inherent in macroeconomic data. The larger we chose our Hurst parameter $H$, the less volatile the paths of the fractional Brownian motions turned out to be and vice versa. This was also the result of both modelling stochastic differential equations and bond prices, where we consequently got less volatile interest rate values and bond prices, respectively, as well.

We could continue this thesis with many applications and extensions. Moreover, there remain many possible tasks for further research, for example the pricing of interest-rate sensitive contingent claims similarly to the work of [Heath et al. [1992]] in the classical case. These derivatives might include interest caps, swaptions, callable bonds, bond options and many more. The pricing of options in fractional Brownian markets has already been explicitly investigated by [Rostek [2009]].
A very interesting extension to his book could deal with the question of embedding our fractional term structure model into a pricing model for stock options.

Since the Heath-Jarrow-Morton model can be considered a more general model than the short-rate models, one could investigate whether it is possible to derive a fractional Vasicek model starting from our fractional Heath-Jarrow-Morton model, similarly to what we did in the classical HJM framework for the Hull-White model in the conclusion of Chapter 4. We assume that the solution to this problem will be much more sophisticated than in the classical case.

# Bibliography


E. Alòs and D. Nualart. Stochastic integration with respect to the fractional Brownian motion. *Stochastics Reports*, 75:129–152, 2003.

D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, UK, 2004.

M.J. Brennan and E.S. Schwartz. A continuous approach to the pricing of bonds. *Journal of Banking and Finance*, 3:133–155, 1979.

D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice*. Springer Finance, Berlin Heidelberg New York, 2007.

Deutsche Bundesbank. Zinsstrukturkurven, Zeitreihen.
http://www.bundesbank.de/statistik/statistik_zeitreihen.php.

A. Carverhill. When is the Short Rate Markovian? *Mathematical Finance*, 4:305–312, 1994.

J. C. Cox, J. E. Ingersoll Jr., and S. A. Ross. An Intertemporal General Equilibrium Model of Asset Prices. *Econometrica*, 53:385–407, 1985.

L. Decreusefond and A.S. Üstünel. Stochastic Analysis of the Fractional Brownian Motion. *Potential Analysis*, 10:177–214, 1999.

T.E. Duncan, B. Pasik-Duncan, and B. Maslowski. Fractional Brownian Motion and Stochastic Equations in Hilbert Spaces. *Stochastics and Dynamics*, 2:225–250, 2002.

X. Fernique. Continuité des processus gaussiens. *C. R. Acad. Sci. Paris*, 258:6058 – 6060, 1964.

D. Filipovic. Consistency Problems for Heath-Jarrow-Morton Interest Rate Models. *Lecture Notes in Mathematics*, 1760, 2001.

H. Fink, M. Zähle, and C. Klüppelberg. Conditional characteristic functions of processes related to fractional Brownian motion. *Not published yet*, 2010.

P.V. Gapeev. On arbitrage and Markovian short rates in fractional bond markets. *Statistics and Probability Letters*, 70:211–222, 2004.

P. Guasoni. Optimal investment with transaction costs and without semimartingales. *The Annals of Applied Probability*, 12:1227–1246, 2002.

P. Guasoni. No Arbitrage under Transaction Costs, with Fractional Brownian Motion and Beyond. *Mathematical Finance*, 16:569–582, 2006.

M. Hairer and A. Ohashi. Ergodic Theory for SDEs with extrinsic memory. *The Annals of Probability*, 35:1950–1977, 2007.

J.M. Harrison and S.R. Pliska. Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and their Applications*, 11:215–260, 1981.

D. Heath, R. Jarrow, and A. Morton. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60:77–105, 1992.

M. Henry and P. Zaffaroni. The long-range dependence paradigm for macroeconomics and finance. In P. Doukhan, G. Oppenheim and M. Taqqu: *Long-Range Dependence.* Birkhäuser, Boston, 2003.

T.Y.S. Ho and S.B. Lee. Term structure movements and pricing interest rate contingent claims. *Journal of Finance*, 41, 1979.

J. Kallsen. Stochastik III - Stochastische Prozesse, 2007. http://www-m4.ma.tum.de/courses/WS07-08/stochprozesse/Stochastik_III_Skript.pdf.

I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, Berlin Heidelberg New York, 1988.

T. Kawada and N. Kôno. On the variation of Gaussian processes. *Lecture Notes in Mathematics*, 330:176 – 192, 1973.

A. Klenke. *Wahrscheinlichkeitstheorie*. Springer, Berlin Heidelberg New York, 2006.

V. Krvavich and S. Mishura. Differentiability of fractional integrals whose kernels contain fractional Brownian motions. *Ukrainian Mathematical Journal*, 53:35–47, 2001.

B. Mandelbrot. *Fractals and scaling in finance, discontinuity, concentration, risk*. Springer, Berlin Heidelberg New York, 1997.

J. McCarthy, R. Disario, H. Saraoglu, and H. Li. Tests of long-range dependence of interest rates using wavelets. *Quarterly Review of Economics and Finance*, 44:180–189, 2004.

T. Mikosch and R. Norvaisa. Stochastic integral equations without probability. *Bernoulli*, 6:401 – 434, 2000.

I. Norros, E. Valkeila, and J. Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 5:571–587, 1999.

A. Ohashi. Fractional Term Structure Models: No Arbitrage and Consistency. *The Annals of Applied Probability*, 19:1553–1580, 2009.

V. Pipiras and M.S. Taqqu. Integration questions related to fractional Brownian motion. *Probability Theory and Related Fields*, 118:251–291, 2000.

V. Pipiras and M.S. Taqqu. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli*, 7:873–897, 2001.

L.S. Pontryagin. *Ordinary differential equations*. Addison Wesley, 1962.

R. Pulch. Stochastische Differentialgleichungen: Theorie und Numerik, 2007. http://www-num.math.uni-wuppertal.de/Lectures/Stdgl_ws06/skript_stochdgl.pdf.

S. Rostek. *Option Pricing in Fractional Brownian Markets*. Springer, Berlin Heidelberg New York, 2009.

W. Rudin. *Analysis*. Oldenbourg, Munich, Germany, 2005.

S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, New York, USA, 1993.

G. Samorodnitsky and M.S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, Boca Raton, USA, 2000.

P.J. Schönbucher. *Credit Derivatives Pricing Models*. Wiley Finance, West Sussex, England, 2006.

T. Sottinen. Fractional Brownian motion, random walks and binary market models. *Finance and Stochastics*, 5:343–355, 2001.

International Swaps and Derivatives Association. Market Survey Results, 2009. http://www.isda.org/statistics/recent.html.

Murray State University. Forward rate, spot rate and par yield. http://campus.murraystate.edu/academic/faculty/larry.guin/FIN602/Forwar3.gif.

O. Vasicek. An Equilibrium Characterisation of the Term Structure. *Journal of Financial Economics*, 5:177–188, 1977.

R. Zagst. Lecture: Fixed Income Markets, 2007. http://www.mathfinance.ma.tum.de/lehre/0708/fim_mat.php.

R. Zagst. *Interest-Rate Management*. Springer Finance, Berlin Heidelberg New York, 2002.